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Zero volume boundary for extension domains from Sobolev to *BV*

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Abstract
In this note, we prove that the boundary of a $(W^{1,p}, BV)$-extension domain is of volume zero under the assumption that the domain Ω is 1-fat at almost every $x \in \partial\Omega$. Especially, the boundary of any planar $(W^{1,p}, BV)$-extension domain is of volume zero.

Keywords Extension domains · Sobolev functions · BV functions · Boundary volume

Mathematics Subject Classification 46E35

1 Introduction

To simplify the definition of extension domains, we always assume $\Omega \subset \mathbb{R}^n$ is a bounded domain. Given $1 \leq q \leq p \leq \infty$, a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is said to be a $(W^{1,p}, W^{1,q})$-extension domain if there exists a bounded extension operator

$$E\colon W^{1,p}(\Omega) \mapsto W^{1,q}(\mathbb{R}^n),$$

and is said to be a $(W^{1,p}, BV)$-extension domain if there exists a bounded extension operator

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$$E\colon W^{1,p}(\Omega) \mapsto BV(\mathbb{R}^n).$$

The theory of Sobolev extensions is of interest in several fields in analysis. Partial motivations for the study of Sobolev extensions comes from the theory of PDEs, for example, see [18]. It was proved in [2, 22] that for every Lipschitz domain in $\mathbb{R}^n$, there exists a bounded linear extension operator $E\colon W^{k,p}(\Omega) \mapsto W^{k,p}(\mathbb{R}^n)$ for each $k \in \mathbb{N}$ and $1 \le p \le \infty$. Here $W^{k,p}(\Omega)$ is the Banach space of all L^p-integrable functions whose distributional derivatives up to order k are L^p-integrable. Later, the notion of (ϵ, δ)-domains was introduced by Jones in [9], and it was proved that for every (ϵ, δ)-domain, there exists a bounded linear extension operator $E\colon W^{k,p}(\Omega) \mapsto W^{k,p}(\mathbb{R}^n)$ for every $k \in \mathbb{N}$ and $1 \le p \le \infty$.

In [26], a geometric characterization of planar $(L^{1,2}, L^{1,2})$-extension domain was given. Here $L^{k,p}(\Omega)$ denotes the homogeneous Sobolev space which contains locally integrable functions whose k-th order distributional derivative is L^p-integrable. By later results in [11, 13, 14, 21], we now have geometric characterizations of planar simply connected $(W^{1,p}, W^{1,p})$-extension domains for all $1 \le p \le \infty$. A geometric characterization is also known for planar simply connected $(L^{k,p}, L^{k,p})$-extension domains with $2 < p \le \infty$, see [23, 29, 30]. Beyond the planar simply connected case, geometric characterizations of Sobolev extension domains are still missing. However, several necessary properties have been obtained for general Sobolev extension domains.

For a measurable subset $F \subset \mathbb{R}^n$, we use $|F|$ to denote its Lebesgue measure. In [7, 8], Hajłasz, Koskela and Tuominen proved for $1 \le p < \infty$ that a $\left(W^{1,p}, W^{1,p}\right)$-extension domain $\Omega \subset \mathbb{R}^n$ must be Ahlfors regular which means that there exists a positive constant $C > 1$ such that for every $x \in \overline{\Omega}$ and $0 < r < \min\left\{1, \frac{1}{4}\operatorname{diam}\Omega\right\}$, we have

$$|B(x,r)| \le C|B(x,r)\cap\Omega|. \tag{1.1}$$

From the results in [4, 10], we know that also (BV, BV)-extension domains are Ahlfors regular. For Ahlfors regular domains, the Lebesgue differentiation theorem then easily implies $|\partial\Omega| = 0$.

In the case where Ω is a planar Jordan $\left(W^{1,p}, W^{1,p}\right)$-extension domain, Ω has to be a so-called John domain when $1 \le p \le 2$ and the complementary domain has to be John when $2 \le p < \infty$. The John condition implies that the Hausdorff dimension of $\partial\Omega$ must be strictly less than 2, see [12]. Recently, Lučić, Takanen and the first named author gave a sharp estimate on the Hausdorff dimension of $\partial\Omega$, see [17]. In general, the Hausdorff dimension of a $(W^{1,p}, W^{1,p})$-extension domain can well be n.

The outward cusp domain with a polynomial type singularity is a typical example which is not a $(W^{1,p}, W^{1,p})$-extension domain for $1 \le p < \infty$. However, it is a $(W^{1,p}, W^{1,q})$-extension domain, for some $1 \le q < p \le \infty$, see the monograph [19] and the references therein. Hence, for $1 \le q < p \le \infty$, it is not necessary for a $(W^{1,p}, W^{1,q})$-extension domain to be Ahlfors regular. In the absence of Ahlfors regularity, one has to find alternative approaches for proving $|\partial\Omega| = 0$. The first approach in [24, 25] was to generalize the Ahlfors regularity (1.1) to a Ahlfors-type estimate

$$|B(x,r)|^p \leq C\Phi^{p-q}(B(x,r))|B(x,r)\cap\Omega|^q \tag{1.2}$$

for $(W^{1,p}, W^{1,p})$-extension domains with $n < q < p < \infty$. Here Φ is a bounded and quasiadditive set function generated by the $(W^{1,p}, W^{1,q})$-extension property and defined on open sets $U \subset \mathbb{R}^n$, see Sect. 3. By differentiating Φ with respect to the Lebesgue measure, one concludes that $|\partial\Omega| = 0$ if Ω is a $(W^{1,p}, W^{1,q})$-extension domain for $n < q < p < \infty$. Recently, Koskela, Ukhlov and the second named author [15] generalized this result and proved that the boundary of a $\left(W^{1,p}, W^{1,q}\right)$-extension domain must be of volume zero for $n-1 < q < p < \infty$ (and for $1 \leq q < p < \infty$ on the plane). For $1 \leq q < n-1$ and $(n-1)q/(n-1-q) < p < \infty$, they constructed as a counterexample a $\left(W^{1,p}, W^{1,q}\right)$-extension domain $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| > 0$. For the remaining range of exponents where $1 \leq q \leq n-1$ and $q < p \leq (n-1)q/(n-1-q)$, it is still not clear whether the boundary of every $\left(W^{1,p}, W^{1,q}\right)$-extension domain must be of volume zero.

As is well-known, for every domain $\Omega \subset \mathbb{R}^n$, the space of functions of bounded variation $BV(\Omega)$ strictly contains every Sobolev space $W^{1,q}(\Omega)$ for $1 \leq q \leq \infty$. Hence, the class of $\left(W^{1,p}, BV\right)$-extension domains contains the class of $\left(W^{1,p}, W^{1,q}\right)$-extension domains for every $1 \leq q \leq p < \infty$. As a basic example to indicate that the containment is strict when $n \geq 2$, we can take the slit disk (the unit disk minus a radial segment) in the plane. The slit disk is a $\left(W^{1,p}, BV\right)$-extension domain for every $1 \leq p < \infty$, and even a (BV, BV)-extension domain; however it is not a $\left(W^{1,p}, W^{1,q}\right)$-extension domain for any $1 \leq q \leq p < \infty$. This basic example also shows that it is natural to consider the geometric properties of $\left(W^{1,p}, BV\right)$-extension domains. In this paper, we focus on the question whether the boundary of a $(W^{1,p}, BV)$-extension domain is of volume zero. Our first theorem tells us that the (BV, BV)-extension property is equivalent to the $\left(W^{1,1}, BV\right)$-extension property. Hence, a $(W^{1,1}, BV)$-extension domain is Ahlfors regular and so its boundary is of volume zero.

Theorem 1.1 *A domain $\Omega \subset \mathbb{R}^n$ is a (BV, BV)-extension domain if and only if it is a $\left(W^{1,1}, BV\right)$-extension domain.*

Since, $W^{1,1}(\Omega)$ is a proper subspace of $BV(\Omega)$ with $\|u\|_{W^{1,1}(\Omega)} = \|u\|_{BV(\Omega)}$ for every $u \in W^{1,1}(\Omega)$, (BV, BV)-extension property implies $(W^{1,1}, BV)$-extension property immediately. The other direction from $(W^{1,1}, BV)$-extension property to (BV, BV)-extension property is not as straightforward, as $W^{1,1}(\Omega)$ is only a proper subspace of $BV(\Omega)$. The essential tool here is the Whitney smoothing operator constructed by García-Bravo and the first named author in [4]. This Whitney smoothing operator maps every function in $BV(\Omega)$ to a function in $W^{1,1}(\Omega)$ with the same trace on $\partial\Omega$, so that the norm of the image in $W^{1,1}(\Omega)$ is uniformly controlled from above by the norm of the corresponding preimage in $BV(\Omega)$.

With an extra assumption that Ω is q-fat at almost every point on the boundary $\partial\Omega$, in [15] it was shown that the boundary of a $(W^{1,p}, W^{1,q})$-extension domain is of volume zero when $1 \leq q < p < \infty$. The essential point there was that the q-fatness of the domain on the boundary guarantees the continuity of a $W^{1,q}$-function on the boundary. Maybe a bit surprisingly, the assumption that the domain is 1-fat at almost every point on the boundary also guarantees that the boundary of a $(W^{1,p}, BV)$-extension domain is of volume zero. In particular, every planar domain is 1-fat at every point of the boundary. Hence, we have the following theorem.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a $(W^{1,p}, BV)$-extension domain for $1 \le p < \infty$, which is 1-fat at Lebesgue almost every $x \in \partial\Omega$. Then $|\partial\Omega| = 0$. In particular, for every planar $(W^{1,p}, BV)$-extension domain Ω with $1 \le p < \infty$, we have $|\partial\Omega| = 0$.*

In light of the results and example given in [15], the most interesting open question is what happens in the range $1 < p \le (n-1)/(n-2)$ of exponents, without the assumption of 1-fatness. For this range, we do not know whether the boundary of a $(W^{1,p}, BV)$-extension domain must be of volume zero. If a counterexample exists in this range, it might be easier to construct it in the $(W^{1,p}, BV)$-case rather than the $(W^{1,p}, W^{1,1})$-case. Hence we leave it as a question here.

Question 1.3 *For $1 < p \le (n-1)/(n-2)$, is the boundary of a $(W^{1,p}, BV)$-extension domain of volume zero?*

2 Preliminaries

For a locally integrable function $u \in L^1_{\mathrm{loc}}(\Omega)$ and a measurable subset $A \subset \Omega$ with $0 < |A| < \infty$, we define

$$u_A := \fint_A u(y)\,dy = \frac{1}{|A|}\int_A u(y)\,dy.$$

Definition 2.1 Let $\Omega \subset \mathbb{R}^n$ be a domain. For every $1 \le p \le \infty$, we define the Sobolev space $W^{1,p}(\Omega)$ to be

$$W^{1,p}(\Omega) := \left\{u \in L^p(\Omega) : \nabla u \in L^p(\Omega; \mathbb{R}^n)\right\},$$

where ∇u denotes the distributional gradient of u. It is equipped with the nonhomogeneous norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Now, let us give the definition of functions of bounded variation.

Definition 2.2 Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $u \in L^1(\Omega)$ is said to have bounded variation and denoted $u \in BV(\Omega)$ if

$$\|Du\|(\Omega) := \sup\left\{\int_\Omega f \operatorname{div}(\phi)dx : \phi \in C^1_o(\Omega; \mathbb{R}^n), |\phi| \le 1\right\} < \infty.$$

The space $BV(\Omega)$ is equipped with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

Note that $\|Du\|$ is a Radon measure defined on Ω that is defined for every set $F \subset \Omega$ as

$$\|Du\|(F) := \inf\{\|Du\|(U) : F \subset U \subset \Omega, U \text{ open}\}.$$

Definition 2.3 We say that a domain $\Omega \subset \mathbb{R}^n$ is a $\left(W^{1,p}, BV\right)$-extension domain for $1 \leq p < \infty$, if there exists a bounded extension operator $E\colon W^{1,p}(\Omega) \mapsto BV(\mathbb{R}^n)$ i.e. for every $u \in W^{1,p}(\Omega)$, we have $E(u) \in BV(\mathbb{R}^n)$ with $E(u)\big|_\Omega \equiv u$ and

$$\|E(u)\|_{BV(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for a constant $C > 1$ independent of u.

Let $U \subset \mathbb{R}^n$ be an open set and $K \subset U$ be a compact subset. The p-admissible set $\mathcal{W}_p(K; U)$ is defined by setting

$$\mathcal{W}_p(K; U) := \left\{u \in W_0^{1,p}(U) \cap C(U) : u\big|_K \geq 1\right\}.$$

Definition 2.4 Let $U \subset \mathbb{R}^n$ be an open set and $K \subset U$ be compact. The relative p-capacity $Cap_p(K; U)$ is defined by setting

$$Cap_p(K; U) := \inf_{u \in \mathcal{W}_p(K;U)} \int_U |\nabla u(x)|^p \, dx.$$

For an open subset $A \subset U$, we define the relative p-capacity $Cap_p(K; U)$ by setting

$$Cap_p(A; U) := \sup\left\{Cap_p(K; U) : K \subset A \subset U, K \text{ compact}\right\}.$$

For arbitrary Borel measurable subset $E \subset U$, we define the relative p-capacity $Cap_p(E; U)$ by setting

$$Cap_p(E; U) := \inf\left\{Cap_p(A; U) : E \subset A \subset U, A \text{ open}\right\}.$$

Following Lahti [16], we define 1-fatness below.

Definition 2.5 Let $A \subset \mathbb{R}^n$ be a measurable subset. We say that A is 1-thin at the point $x \in \mathbb{R}^n$, if

$$\lim_{r \to 0^+} r \frac{Cap_1\left(A \cap B(x, r); B(x, 2r)\right)}{|B(x, r)|} = 0.$$

If A is not 1-thin at x, we say that A is 1-fat at x. Furthermore, we say that a set U is 1-finely open, if $\mathbb{R}^n \setminus U$ is 1-thin at every $x \in U$.

By [16, Lemma 4.2], the collection of 1-finely open sets is a topology on $\mathbb{R}^n$. For a function $u \in BV(\mathbb{R}^n)$, we define the lower approximate limit $u_\star$ by setting

$$u_\star(x) := \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0^+} \frac{|B(x,r) \cap \{u < t\}|}{|B(x,r)|} = 0\right\}$$

and the upper approximate limit $u^\star$ by setting

$$u^\star(x) := \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0^+} \frac{|B(x,r) \cap \{u > t\}|}{|B(x,r)|} = 0\right\}.$$

The set

$$S_u := \left\{x \in \mathbb{R}^n : u_\star(x) < u^\star(x)\right\}$$

is called the jump set of u. By the Lebesgue differentiation theorem, $|S_u| = 0$. Using the lower and upper approximate limits, we define the precise representative $\tilde{u} := (u^\star + u_\star)/2$. The following lemma was proved in [16, Corollary 5.1].

Lemma 2.6 *Let $u \in BV(\mathbb{R}^n)$. Then $\tilde{u}$ is 1-finely continuous at $\mathcal{H}^{n-1}$-almost every $x \in \mathbb{R}^n \setminus S_u$.*

The following lemma for $u \in W^{1,1}(\mathbb{R}^n)$ was proved in [15, Lemma 2.6], which is also a corollary of a result in [6]. We generalize it to $BV(\mathbb{R}^n)$ here.

Lemma 2.7 *Let $\Omega \subset \mathbb{R}^n$ be a domain which is 1-fat at almost every point $x \in \partial\Omega$. If $u \in BV(\mathbb{R}^n)$ with $u\big|_{B(x,r)\cap\Omega} \equiv c$ for some $x \in \partial\Omega$, $0 < r < 1$ and $c \in \mathbb{R}$. Then $u(y) = c$ for almost every $y \in B(x,r) \cap \partial\Omega$.*

Proof Let $u \in BV(\mathbb{R}^n)$ satisfy the assumptions. Then the precise representative $\tilde{u}\big|_{B(x,r)\cap\Omega} \equiv c$. Since $|S_u| = 0$, by Lemma 2.6, there exists a subset $N_1 \subset \mathbb{R}^n$ with $|N_1| = 0$ such that $\tilde{u}$ is 1-finely continuous on $\mathbb{R}^n \setminus N_1$. By the assumption, there exists a measure zero set $N_2 \subset \partial\Omega$ such that Ω is 1-fat on $\partial\Omega \setminus N_2$. By Definition 2.5, one can see that $B(x,r) \cap \Omega$ is also 1-fat on $(B(x,r) \cap \partial\Omega)\setminus N_2$. For every $y \in (B(x,r) \cap \partial\Omega)\setminus(N_1 \cup N_2)$, since $\tilde{u}$ is 1-finely continuous on it and any 1-fine neighborhood of y must intersect $B(x,r) \cap \Omega$, we have $\tilde{u}(y) = c$. Hence $u(y) = c$ for almost every $y \in B(x,r) \cap \partial\Omega$. □

We say a set $E \subset \Omega$ has finite perimeter in Ω, if $\chi_E \in BV(\Omega)$, where χ_E means the characteristic function of E. We set $P(E,\Omega) := \|D\chi_E\|(\Omega)$ and call it the perimeter of E in Ω. To simplify the notation, $P(E)$ is set to be $P(E,\mathbb{R}^n)$. For every Borel subset $F \subset \Omega$, define

$$P(E,F) := \inf\left\{P(E,U) : F \subset U \subset \Omega, U \text{ open}\right\}.$$

The following coarea formula for BV functions can be found in [3, Section 5.5]. See also [4, Theorem 2.2].

Proposition 2.8 *Given a function $u \in BV(\Omega)$, the superlevel sets $u_t = \{x \in \Omega : u(x) > t\}$ have finite perimeter in Ω for almost every $t \in \mathbb{R}$ and*

$$\|Du\|(F) = \int_{-\infty}^{\infty} P(u_t, F)\, dt$$

for every Borel set $F \subset \Omega$. Conversely, if $u \in L^1(\Omega)$ and

$$\int_{-\infty}^{\infty} P(u_t, \Omega)\, dt < \infty$$

then $u \in BV(\Omega)$.

See [1, Theorem 3.44] for the proof of the following (1, 1)-Poincaré inequality for BV functions. For a cube $Q \subset \mathbb{R}^n$, we denote by $l(Q)$ its side-length.

Proposition 2.9 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a constant $C > 0$ depending on n and Ω such that for every $u \in BV(\Omega)$, we have*

$$\int_{\Omega} |u(y) - u_{\Omega}|\, dy \le C\|Du\|(\Omega).$$

In particular, there exists a constant $C > 0$ only depending on n so that if $Q, Q' \subset \mathbb{R}^n$ are two closed dyadic cubes with $\frac{1}{4} l(Q') \le l(Q) \le 4l(Q')$ and $\Omega := \operatorname{int}(Q \cup Q')$ connected, then for every $u \in BV(\Omega)$,

$$\int_{\Omega} |u(y) - u_{\Omega}|\, dy \le Cl(Q)\|Du\|(\Omega). \tag{2.1}$$

3 A set function arising from the extension

In this subsection, we introduce a set function defined on the class of open sets in $\mathbb{R}^n$ and taking nonnegative values. Our set function here is a modification of the one originally introduced by Ukhlov [24, 25]. See also [27, 28] for related set functions. The modified version of the set function we use is from [15], where it was used by Koskela, Ukhlov and the second named author to study the size of the boundary of a $(W^{1,p}, W^{1,q})$-extension domains. Let us recall that a set function Φ defined on the class of open subsets of $\mathbb{R}^n$ and taking nonnegative values is called quasiadditive (see for example [27]), if for all open sets $U_1 \subset U_2 \subset \mathbb{R}^n$, we have

$$\Phi(U_1) \le \Phi(U_2),$$

and there exists a positive constant C such that for arbitrary pairwise disjoint open sets $\{U_i\}_{i=1}^{\infty}$, we have

$$\sum_{i=1}^{\infty} \Phi(U_i) \le C\Phi\left(\bigcup_{i=1}^{\infty} U_i\right). \tag{3.1}$$

Let $\Omega \subset \mathbb{R}^n$ be a $(W^{1,p}, BV)$-extension domain for some $1 < p < \infty$. For every open set $U \subset \mathbb{R}^n$ with $U \cap \Omega \neq \emptyset$, we define

$$W_0^p(U, \Omega) := \left\{ u \in W^{1,p}(\Omega) \cap C(\Omega) : u \equiv 0 \text{ on } \Omega \setminus U \right\}.$$

For every $u \in W_0^p(U, \Omega)$, we define

$$\Gamma(u) := \inf \left\{ \|Dv\|(U) : v \in BV(\mathbb{R}^n), v\big|_{\Omega} \equiv u \right\}.$$

Then we define the set function Φ by setting

$$\Phi(U) := \begin{cases} \sup_{u \in W_0^p(U,\Omega)} \left(\frac{\Gamma(u)}{\|u\|_{W^{1,p}(U\cap\Omega)}} \right)^k, \quad \text{with } \frac{1}{k} = 1 - \frac{1}{p}, & \text{if } U \cap \Omega \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

In [7], Hajłasz, Koskela and Tuominen proved that for an arbitrary $(W^{1,p}, W^{1,p})$-extension domain with $1 < p < \infty$, there always exists a bounded linear extension operator. For $q < p$, the existence of a bounded linear $(W^{1,p}, W^{1,q})$-extension operator is still open. However, in [15, Lemma 2.1], the authors proved that for $(W^{1,p}, W^{1,q})$-extension domains there always exists a bounded, positively homogeneous $(W^{1,p}, W^{1,q})$-extension operator. The next lemma is a version of this result in our setting of $(W^{1,p}, BV)$-extensions that follows similarly to the proof of [15, Lemma 2.1].

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a $(W^{1,p}, BV)$-extension domain. Then every bounded extension operator $E : W^{1,p}(\Omega) \to BV(\mathbb{R}^n)$ promotes to a bounded, positively homogeneous extension operator $E_h : W^{1,p}(\Omega) \to BV(\mathbb{R}^n)$ with the operator norm inequality $\|E_h\| \leq \|E\|$.*

The proof of the following lemma is almost the same as the proof of [15, Theorem 3.1]. One needs to simply replace $\|Dv\|_{L^q(U)}$ by $\|Dv\|(U)$ in the proof of [15, Theorem 3.1] and repeat the argument.

Lemma 3.2 *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded $(W^{1,p}, BV)$-extension domain. Then the set function defined in (3.2) for all open subsets of $\mathbb{R}^n$ is bounded and quasiadditive.*

The upper and lower derivatives of a quasiadditive set function Φ are defined by setting

$$\overline{D\Phi}(x) := \limsup_{r \to 0^+} \frac{\Phi(B(x,r))}{|B(x,r)|} \quad \text{and} \quad \underline{D\Phi}(x) = \liminf_{r \to 0^+} \frac{\Phi(B(x,r))}{|B(x,r)|}.$$

By [20, 27], we have the following lemma. See also [15, Lemma 3.1].

Lemma 3.3 *Let Φ be a bounded and quasiadditive set function defined on open sets $U \subset \mathbb{R}^n$. Then $\overline{D\Phi}(x) < \infty$ for almost every $x \in \mathbb{R}^n$.*

The following lemma immediately comes from the definition (3.2) for the set function Φ.

Lemma 3.4 *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded $(W^{1,p}, BV)$-extension domain. Then, for a ball $B(x,r)$ with $x \in \partial\Omega$ and every function $u \in W_0^p(B(x,r), \Omega)$, there exists a function $v \in BV(B(x,r))$ with $v\big|_{B(x,r)\cap\Omega} \equiv u$ and*

$$\|Dv\|(B(x,r)) \leq 2\Phi^{\frac{1}{k}}(B(x,r))\|u\|_{W^{1,p}(B(x,r)\cap\Omega)}, \quad \text{where } \frac{1}{k} = 1 - \frac{1}{p}. \tag{3.3}$$

4 Proofs of the results

c 1.1 and 1.2.

Proof of Theorem 1.1 Let us first assume that $\Omega \subset \mathbb{R}^n$ is a (BV, BV)-extension domain with the extension operator E. Since $W^{1,1}(\Omega) \subset BV(\Omega)$ with $\|u\|_{BV(\Omega)} = \|u\|_{W^{1,1}(\Omega)}$ for every $u \in W^{1,1}(\Omega)$, we have

$$\|E(u)\|_{BV(\Omega)} \leq C\|u\|_{BV(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}.$$

This implies that Ω is a $\left(W^{1,1}, BV\right)$-extension domain with the same operator E restricted to $W^{1,1}(\Omega)$.

Let us then prove the converse and assume that $\Omega \subset \mathbb{R}^n$ is a $(W^{1,1}, BV)$-extension domain with an extension operator E. Let $S_{\Omega,\Omega}$ be the Whitney smoothing operator defined in [4]. Then by [4, Theorem 3.1], for every $u \in BV(\Omega)$, we have $S_{\Omega,\Omega}(u) \in W^{1,1}(\Omega)$ with

$$\|S_{\Omega,\Omega}(u)\|_{W^{1,1}(\Omega)} \leq C\|u\|_{BV(\Omega)}$$

for a positive constant C independent of u, and

$$\|D(u - S_{\Omega,\Omega}(u))\|(\partial\Omega) = 0, \tag{4.1}$$

where $u - S_{\Omega,\Omega}(u)$ is understood to be defined on the whole space $\mathbb{R}^n$ via a zero-extension. Then $E(S_{\Omega,\Omega}(u)) \in BV(\mathbb{R}^n)$ with

$$\|E(S_{\Omega,\Omega}(u))\|_{BV(\mathbb{R}^n)} \leq C\|S_{\Omega,\Omega}(u)\|_{W^{1,1}(\Omega)} \leq C\|u\|_{BV(\Omega)}.$$

Now, define $T\colon BV(\Omega) \to BV(\mathbb{R}^n)$ by setting for every $u \in BV(\Omega)$

$$T(u)(x) := \begin{cases} u(x), & \text{if } x \in \Omega \\ E(S_{\Omega,\Omega}(u))(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

By (4.1), we have $T(u) \in BV(\mathbb{R}^n)$ with

$$\|T(u)\|_{BV(\mathbb{R}^n)} \leq \|E(S_{\Omega,\Omega}(u))\|_{BV(\mathbb{R}^n)} + \|u\|_{BV(\Omega)} \leq C\|u\|_{BV(\Omega)}.$$

Hence, Ω is a BV-extension domain. $\square$

Proof of Theorem 1.2 Assume towards a contradiction that $|\partial\Omega| > 0$. By the Lebesgue density point theorem and Lemma 3.3, there exists a measurable subset U of $\partial\Omega$ with $|U| = |\partial\Omega|$ such that every $x \in U$ is a Lebesgue point of $\partial\Omega$ and $\overline{D\Phi}(x) < \infty$. Fix $x \in U$. Since x is a Lebesgue point, there exists a sufficiently small $r_x > 0$, such that for every $0 < r < r_x$, we have

$$\left|B(x,r) \cap \overline{\Omega}\right| \geq \frac{1}{2^{n-1}} \left|B(x,r)\right|.$$

Let $r \in (0, r_x)$ be fixed. Since $|\partial B(x,s)| = 0$ for every $s \in (0,r)$, we have

$$\left|B\left(x, \frac{r}{4}\right) \cap \overline{\Omega}\right| \geq \frac{1}{2^{n-1}} \left|B\left(x, \frac{r}{4}\right)\right| \geq \frac{1}{2^{3n-1}} \left|B(x,r)\right| \tag{4.2}$$

and

$$\left|\left(B(x,r) \setminus B\left(x, \frac{r}{2}\right)\right) \cap \overline{\Omega}\right| \geq |B(x,r) \cap \overline{\Omega}| - \left|B\left(x, \frac{r}{2}\right)\right| \geq \frac{1}{2^n}|B(x,r)|. \tag{4.3}$$

Define a test function $u \in W^{1,p}(\Omega) \cap C(\Omega)$ by setting

$$u(y) := \begin{cases} 1, & \text{if } y \in B\left(x, \frac{r}{4}\right) \cap \Omega, \\ \frac{-4}{r}|y-x| + 2, & \text{if } y \in \left(B\left(x, \frac{r}{2}\right) \setminus B\left(x, \frac{r}{4}\right)\right) \cap \Omega, \\ 0, & \text{if } y \in \Omega \setminus B\left(x, \frac{r}{2}\right). \end{cases} \tag{4.4}$$

We have

$$\left(\int_{B(x,r)\cap\Omega} |u(y)|^p + |\nabla u(y)|^p \, dx\right)^{\frac{1}{p}} \leq \frac{C}{r} |B(x,r) \cap \Omega|^{\frac{1}{p}}. \tag{4.5}$$

Since $u \equiv 0$ on $\Omega \setminus B(x, r/2)$, we have $u \in W_0^p(B(x,r), \Omega)$. Then, by the definition (3.2) of the set function Φ and by Corollary 3.4, there exists a function $v \in BV(B(x,r))$ with $v\big|_{B(x,r)\cap\Omega} \equiv u$ and

$$\|Dv\|(B(x,r)) \leq 2\Phi^{\frac{1}{k}}(B(x,r))\|u\|_{W^{1,p}(B(x,r)\cap\Omega)}. \tag{4.6}$$

By the Poincaré inequality of BV functions stated in Proposition 2.9, we have

$$\int_{B(x,r)} |v(y) - v_{B(x,r)}| \, dy \leq Cr\|Dv\|(B(x,r)). \tag{4.7}$$

Since Ω is 1-fat on almost every $z \in \partial\Omega$, by Lemma 2.7, $v(z) = 1$ for almost every $z \in B\left(x, \frac{r}{4}\right) \cap \partial\Omega$ and $v(z) = 0$ for almost every $z \in \left(B(x,r) \setminus B\left(x, \frac{r}{2}\right)\right) \cap \partial\Omega$.

Hence, on one hand, if $v_{B(x,r)} \leq \frac{1}{2}$, we have

$$\int_{B(x,r)} |v(y) - v_{B(x,r)}|\, dy \geq \frac{1}{2} \left| B\left(x, \frac{r}{4}\right) \cap \overline{\Omega} \right| \geq c|B(x,r)|.$$

On the other hand, if $v_{B(x,r)} > \frac{1}{2}$, we have

$$\int_{B(x,r)} |v(y) - v_{B(x,r)}|\, dy \geq \frac{1}{2} \left| \left(B(x,r) \setminus B\left(x, \frac{r}{2}\right) \right) \cap \overline{\Omega} \right| > c|B(x,r)|.$$

All in all, we always have

$$\int_{B(x,r)} |v(y) - v_{B(x,r)}|\, dy \geq c|B(x,r)| \tag{4.8}$$

for a sufficiently small constant $c > 0$. Thus, by combining inequalities (4.5)–(4.8), we obtain

$$\Phi(B(x,r))^{p-1}|B(x,r) \cap \Omega| \geq c|B(x,r)|^p$$

for a sufficiently small constant $c > 0$. This gives

$$|B(x,r) \cap \partial\Omega| \leq |B(x,r)| - |B(x,r) \cap \Omega| \leq |B(x,r)| - C\frac{|B(x,r)|^p}{\Phi(B(x,r))^{p-1}}.$$

Since $\overline{D\Phi}(x) < \infty$, we have

$$\begin{aligned}
\limsup_{r \to 0^+} \frac{|B(x,r) \cap \partial\Omega|}{|B(x,r)|} &\leq \limsup_{r \to 0^+} \left(1 - \frac{|B(x,r) \cap \Omega|}{|B(x,r)|} \right) \\
&\leq \limsup_{r \to 0^+} \left(1 - \frac{|B(x,r)|^{p-1}}{\Phi(B(x,r))^{p-1}} \right) \leq 1 - c\overline{D\Phi}(x)^{1-p} < 1.
\end{aligned}$$

This contradicts the assumption that x is a Lebesgue point of $\partial\Omega$. Hence, we conclude that $|\partial\Omega| = 0$.

Let us then consider the case $\Omega \subset \mathbb{R}^2$. By [5, Theorem A.29], for every $x \in \partial\Omega$ and every $0 < r < \min\left\{1, \frac{1}{4} \operatorname{diam}(\Omega)\right\}$, we have

$$Cap_1(\Omega \cap B(x,r); B(x,2r)) \geq cr$$

for a constant $0 < c < 1$. This implies that Ω is 1-fat at every $x \in \partial\Omega$. Hence, by combining this with the first part of the theorem, we have that the boundary of any planar $(W^{1,p}, BV)$-extension domain is of volume zero. □

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