Agent-Based Time Delay Margin in Consensus of Multi-Agent Systems by an Event-Triggered Control Method: Concept and Computation

Seyed Hamid Hosseini, Mohammad Saleh Tavazoei, and Nikolay V. Kuznetsov

***Abstract*—This paper deals with defining the concept of agent-based time delay margin and computing its value in multi agent systems which are controlled by event-triggered based controllers. The agent-based time delay margin specifying the time delay tolerance of each agent for ensuring consensus in event-triggered controlled multi-agent systems can be considered as a complementary for the concept of input (network) time delay margin, which has been previously introduced in literature. In this paper, an event-triggered control method for achieving consensus in multi-agent systems with time delay is considered. It is shown that the Zeno behavior is excluded by applying this method. Then, in a multi-agent system controlled by the considered event-triggered method, the concept of agent-based time delay margin in the presence of a fixed network delay is defined. Moreover, an algorithm for computing the value of time delay margin for each agent is proposed. Numerical simulation results are also provided to verify the obtained theoretical results.**

***Index Terms*—Time delay margin, Event-triggered control, Multi-agent systems, Consensus**

I. Introduction

The area of multi-agent systems has attracted a lot of attention in recent decades due to growing need for engineered systems which are capable of performing complex tasks. One of the most important issues in such systems is to analyze whether a state agreement can be reached as a result of local inter-agent communications, which is referred as the consensus problem (See [1]-[10] for some consensus based issues in multi-agent systems and their applications).

In practice, the agents in multi-agent systems are expected to be equipped with digital microprocessors with a limited capability for computations and storage resources. This means that the agents face some limitations for handling communication and computation in the distributed consensus control. In order to overcome this challenge, the event-triggered control scheme has been proposed [11], [12], by which the numbers of inter-agent communications and controller updates can considerably be decreased. Owing to advantages of the event-triggered techniques in reducing the communication and computation costs, there has been an increasing number of studies on event-triggered control schemes in multi-agent systems [13]-[21].

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On the other hand, due to the propagation and processing of signals, time delay cannot be avoided in multi agent systems and it is quite reasonable to consider the effect of time delay in the design of consensus algorithm. For instance, in [22] an event-triggered control scheme has been designed to achieve consensus in single-integrator multi agent systems in the presence of transmission time delay (time delays between agents). Also, [15] and [23] has focused on event-triggered consensus of multi-agent systems with input time delay (time delay between the controller and actuator). Moreover, in [24], the time delay margin in an event-triggered controlled multi-agent system has been defined as the supremum of an identical input time delay of each agent for achieving consensus. Based on this definition, a method has been proposed in [24] to find the value of defined time delay margin. In [24], it has been assumed that the input delay of all agents is the same. In the present paper, we relax this assumption such that the input time delay of an agent in the event-triggered controlled multi-agent system can be different from the other agents. On the basis of this assumption relaxation, the concept of agent-based time delay margin will be defined. Also, a method will be proposed to compute the value of this delay margin in the presence of a fixed network time delay.

The paper is organized as follows. Section II is devoted to some preliminaries on under-study problem. The main results of the paper are presented in Section III. In this section, firstly an event-triggered control scheme is introduced. Then, some sufficient conditions are derived to ensure consensus by using the introduced event-triggered controller. Also, it is shown that the Zeno behavior will be excluded in such a case. Moreover, the concept of agent based delay margin is defined in the considered event-triggered controlled multi-agent system. Furthermore, a method is proposed to find the value of delay margin for each agent. Numerical simulation results to verify the obtained results are presented in Section IV. Finally, the paper is concluded in Section V.

II. Preliminaries

A. Notations

We use the following notations throughout this paper. $\boldsymbol{R}^{m\times n}$ is the set of $m \times n$ real matrices. $\mathbf{1}_n$ and $\mathbf{0}_n$ denote a $n \times 1$ column vector that all its elements are 1 and 0, respectively. The eigenvalues of matrix $A \in R^{n\times n}$ are denoted by $\lambda_i(A)$ for $i = 1, ..., n$ and det(A) denotes the determinant of this matrix. By diag$(a_1, a_2, ..., a_n)$, we denote a diagonal matrix with elements $a_i, i = 1, ..., n$ as its diagonal elements. $A \bigotimes B$ denotes the Kronecker product of matrices A and B. Also, $\|.\|$ denote the Euclidean norm operator for vectors and the induced 2-norm operator for matrices.

The communication network of the considered multi-agent system is modeled as a graph $\mathcal{G} = (\mathcal{V}, \xi, \mathcal{A})$, in which $\mathcal{V} = \{v_1, v_2, ..., v_N\}$ is a nonempty set of N agents as the graph nodes, $\xi \subseteq \mathcal{V} \times \mathcal{V}$ is the set of its edges, and $\mathcal{A} = [a_{ij}] \in \boldsymbol{R}^{N\times N}$ is the weighted adjacency matrix whose elements are defined by $a_{ii} = 0, a_{ij} > 0$ if $(v_j, v_i) \in \xi$, and $a_{ij} = 0$ otherwise. $(v_j, v_i) \in \xi$ indicates that the agent v_i can receive information from the agent v_j, v_i is an out-neighbor of v_j, and v_j is an in-neighbor of v_i. By $\mathcal{N}_i = \{j \in \mathcal{V} | (v_j, v_i) \in \xi\}$, we denote the set of in-neighbors of agent i. The elements of the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \boldsymbol{R}^{N\times N}$ are defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$. This means that $\mathcal{L} = \mathcal{D} - \mathcal{A}$. The directed communication graph $\mathcal{G}$ is said to have a spanning tree if and only if there is an agent (called the root of the spanning tree) that can reach all the other agents through the directed edges.

B. Problem Statement

In [24], the event-triggered consensus problem of multi-agent systems in the case that the all involved agents have an identical input time delay has been studied. In the aforementioned study, the concept of delay margin for the under-study multi-agent system has been defined as the supremum of the time delay for ensuring consensus. In the continuation of this research work, the present paper aims to define the agent-based delay margin in event-triggered consensus of multi-agent systems. To this aim, it is assumed that the input time delay of one of the agents is different from the others, and it is found that the supremum of such a time delay (as the agent-based delay margin) for guaranteeing consensus. In this paper, we consider a multi-agent system (consisting N agents) described by

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t, \tau_1), & i = 1, 2, ..., N, i \neq j \\ \dot{x}_j(t) = Ax_j(t) + Bu_j(t, \tau_2) \end{cases} \tag{1}$$

where $x_i(t) \in \boldsymbol{R}^n$ and $u_i(t) \in \boldsymbol{R}^p$ represent the state vector and the control input of agent i, respectively. $A \in \boldsymbol{R}^{n\times n}$ and $B \in \boldsymbol{R}^{n\times p}$ are constant matrices; $\tau_1 \geq 0$ is the input time delay of all agents except agent j and $\tau_2 \geq 0$ is the input time delay of agent j.

It is considered that the following assumptions are satisfied by system (1) throughout this paper.

Assumption 1: The pair (A, B) in system (1) is stabilizable.

Assumption 2: The directed communication graph associated to system (1) has a directed spanning tree. Without loss of generality, it is supposed that the node corresponding to agent 1 is the root of this spanning tree.

The aim is to find how much the value of τ_2 can be increased for a fixed τ_1 without losing the event-triggered consensus. Such a value specifies the agent j-based delay margin of the considered multi-agent system, when it reaches consensus by an event-triggered control strategy.

III. Main Results

A. Event-triggered Control Scheme

Suppose that the set of the event-triggered instants for agent i are considered as $\{t_0^i, t_1^i, ..., t_{k_i}^i, ...\}$. Inspired by the work [23], the control input for system (1) can be designed as

$$\begin{cases} u_i(t, \tau_1) = -K \sum_{m\in N_i} a_{im}(x_i(t_{k_i}^i - \tau_1) - x_m(t_{k_m}^m - \tau_1)), \\ \qquad i = 1, 2, ..., N, i \neq j, t \in [t_{k_i}^i, t_{k_i+1}^i) \\ u_j(t, \tau_2) = -K \sum_{m\in N_j} a_{jm}(x_j(t_{k_j}^j - \tau_2) - x_m(t_{k_m}^m - \tau_2)), \\ \qquad t \in [t_{k_j}^j, t_{k_j+1}^j) \end{cases} \tag{2}$$

where $K \in \boldsymbol{R}^{p\times n}$ is the feedback gain matrix, N_i is the set of in-neighbors of agent i and $t_{k_i}^i$ is the latest event-triggered instant of agent i at time t. The next event-triggered instant is determined by

$$t_{k_i+1}^i = \inf\{t > t_{k_i}^i : f_i(t) > 0\}, \tag{3}$$

where the event-triggered function $f_i(t)$ is chosen as

$$f_i(t) = \|e_i(t-\tau_m)\| - c_1 e^{-\alpha t},\ \tau_m = \begin{cases} \tau_1 & i = 1, 2, ..., N,\ i \neq j \\ \tau_2 & i = j \end{cases} \tag{4}$$

where c_1 and α are positive constants, and the measurement error $e_i(t - \tau_i)$ is defined by

$$e_i(t-\tau_i) = \begin{cases} x_i(t_{k_i}^i - \tau_1) - x_i(t - \tau_1) & i = 1, 2, ..., N,\ i \neq j \\ x_j(t_{k_j}^j - \tau_2) - x_j(t - \tau_2) & i = j \end{cases} \tag{5}$$

At the time instants $t \in \{t_0^i, t_1^i, ..., t_{k_i}^i, ...\}$, agent i broadcasts the information of its state vector (i.e., $x_i(t_{k_i}^i)$) to out-neighbors.

B. Consensus Analysis

The following theorem reveals that by applying control law (2) the agents in the free-delay version of system (1) (i.e., when $\tau_1 = \tau_2 = 0$) reach consensus.

Consider the delays in system (1) to be zero. By applying control law (2) to system (1), we will prove that the agents reach consensus.

Theorem 1: Consider the multi-agent system (1) satisfying Assumptions 1 and 2 with $\tau_1 = \tau_2 = 0$. If this system is controlled by the event-triggered control scheme (2)-(5), wherein $K = B^T P^{-1}$ and P as a symmetric positive-definite matrix is a solution for the following linear matrix inequality

$$AP + PA^T - 2BB^T < 0, \tag{6}$$

the consensus is then reached for any initial condition.

Proof: Let $\sigma_i(t) = x_i(t) - x_1(t), i = 2, 3, ..., N$. If vector $\sigma(t) = [\sigma_2^T(t), \sigma_3^T(t), ..., \sigma_N^T(t)]^T$ tends to zero, the consensus of system (1) is achieved. According to (1),

$$\dot{\sigma}_i(t) = A\sigma_i(t) + B(u_i(t) - u_1(t)), \qquad i = 2, 3, ..., N \tag{7}$$

which can be rewritten as

$$\begin{aligned}\dot{\sigma}_i(t) =& A\sigma_i(t) - d_iBK\sigma_i(t) - d_iBKe_i(t) + (d_1 + a_{i1})BKe_1(t)\\ &+ BK\sum_{m=2}^{N}(a_{im} - a_{1m})\sigma_m(t)\\ &+ BK\sum_{m=2}^{N}(a_{im} - a_{1m})e_m(t).\end{aligned} \tag{8}$$

Denote $\bar{e}(t) = [e_2^T(t), e_3^T(t), ..., e_N^T(t)]^T$ and $\bar{e}_1(t) = [e_1^T(t), ..., e_1^T(t)]^T$. Then the compact form of (8) is

$$\begin{aligned}\dot{\sigma}(t) =& (I_{N-1} \otimes A - H \otimes BK)\sigma(t)\\ &- H \otimes BK\bar{e}(t) + D_1 \otimes BK\bar{e}_1(t),\end{aligned} \tag{9}$$

where H and D_1 are as follows.

$$\begin{aligned}H &= L_{22} + 1_{N-1}a^T\\ &= \begin{bmatrix} d_2 & -a_{23} & ... & -a_{2N}\\ -a_{32} & d_3 & ... & -a_{3N}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{N2} & -a_{N3} & ... & d_N \end{bmatrix} + \begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}[a_{12}\ \ a_{13}\ \ ...\ \ a_{1N}],\\ D_1 &= \mathrm{diag}(d_1 + a_{21}, d_2 + a_{31}, ..., d_N + a_{N1}).\end{aligned} \tag{10}$$

Based on (4), $\|\bar{e}(t)\| \leq \sqrt{N-1}c_1e^{-\alpha t}$ and $\|\bar{e}_1(t)\| \leq \sqrt{N-1}c_1e^{-\alpha t}$. Thus $\|\bar{e}(t)\| \to 0$ and $\|\bar{e}_1(t)\| \to 0$ as $t \to \infty$. Therefore, stability of (9) is equivalent to stability of the following system

$$\dot{\sigma}(t) = (I_{N-1} \otimes A - H \otimes BK)\sigma(t). \tag{11}$$

By defining $S = \begin{bmatrix} 1 & 0_{N-1}^T\\ 1_{N-1} & I_{N-1}\end{bmatrix}$, we have

$$S^{-1}LS = \begin{bmatrix} 0 & -a^T\\ 0_{N-1} & H\end{bmatrix}, \tag{12}$$

where L denotes the Laplacian matrix of the communication graph. The sets of the eigenvalues of $S^{-1}LS$ and L are the same. Also, under Assumption 2, the eigenvalues of the Laplacian matrix L are given by $\lambda_1(L) = 0$, $\lambda_i(L) \neq 0, i = 2, 3, ..., N$. Thus, the eigenvalues of H are composed of the nonzero eigenvalues of L. Therefore there exists an invertible matrix T such that

$$T^{-1}HT = J = \mathrm{diag}(J_1, J_2, ..., J_q), \tag{13}$$

where $J_k, k = 1, 2, ..., q$, are upper triangular Jordan blocks, whose principal diagonal elements are composed of $\lambda_i(L), i = 2, 3, ..., N$. Hence,

$$\begin{aligned}&(T \otimes I_n)^{-1}(I_{N-1} \otimes A - H \otimes Bk)(T \otimes I_n)\\ &\quad = I_{N-1} \otimes A - J \otimes Bk.\end{aligned} \tag{14}$$

Let $\delta(t) = (T^{-1} \otimes I_n)\sigma(t)$. Then, (11) can be rewritten as

$$\dot{\delta}(t) = (I_{N-1} \otimes A - J \otimes BK)\delta(t). \tag{15}$$

Based on matrix and Kronecker product properties, we know that $I_{N-1} \otimes A - J \otimes BK$ is an upper diagonal matrix and its diagonal matrices are $A - \lambda_i(L)BK, i = 2, 3, ..., N$. Thus if all matrices $A - \lambda_i(L)BK, i = 2, 3, ..., N$ are Hurwitz, $I_{N-1} \otimes A - J \otimes BK$ is also a Hurwitz matrix. Assume that the feedback gain matrix is chosen as $K = B^TP^{-1}$, where P is a symmetric positive-definite solution of (6). Assumption 1 guarantees that (6) always has a positive-definite solution P [25]. Moreover, Assumption 2 ensures that the Laplacian matrix L has exactly one zero eigenvalue and all of the other eigenvalues are in the left half plane. Therefore, if Assumptions 1 and 2 hold and K is chosen as $K = B^TP^{-1}$, all matrices $A - \lambda_i(L)BK, i = 2, 3, ..., N$ are Hurwitz [6] and [25], which means that system (15) is stable and $\|\delta(t)\| \to 0$. Since $\delta(t) = (T^{-1} \otimes I_n)\sigma(t)$, $\|\delta(t)\| \to 0$ means that $\|\sigma(t)\| \to 0$ and consensus is reached. ■

Theorem 1 reveals the consensus in a delay-free multi-agent system with an event-triggered control strategy. The influence of the existence of the input time delay in this control system will be investigated in the next subsections.

C. Excluding Zeno Behavior

The Zeno behavior exists in an event-triggered control scheme if the number of event triggers is infinite in a finite time interval. In the following theorem, it is shown that by applying the proposed event-triggered controller the Zeno behavior is excluded.

Theorem 2: Consider the system (1) satisfying Assumptions 1 and 2 and controlled by the event-triggered control scheme (2)-(5). The Zeno behavior is excluded if the parameters of the event-triggered function (4) satisfies $c_1 > 0$ and $\alpha > 0$.

Proof: Define

$$\begin{aligned}\sigma^*(t-\tau_m) &= \begin{bmatrix}\sigma_1(t-\tau_m)\\ \sigma_2(t-\tau_m)\\ \vdots\\ \sigma_N(t-\tau_m)\end{bmatrix} = \begin{bmatrix}0\\ x_2(t-\tau_m) - x_1(t-\tau_m)\\ \vdots\\ x_N(t-\tau_m) - x_1(t-\tau_m)\end{bmatrix}\\ e^*(t-\tau_m) &= \begin{bmatrix}e_1(t-\tau_m)\\ \vdots\\ e_N(t-\tau_m)\end{bmatrix} = \begin{bmatrix}x_1(t_{k_1}^1-\tau_m) - x_1(t-\tau_m)\\ \vdots\\ x_N(t_{k_N}^N-\tau_m) - x_N(t-\tau_m)\end{bmatrix}\\ u(t,\tau) &= [u_1^T(t,\tau_1),\ ...\ ,\ u_j^T(t,\tau_2),\ ...\ ,\ u_N^T(t,\tau_1)]^T\end{aligned} \tag{16}$$

Decompose the Laplacian matrix L to $L_1 + L_2$, where L_1 and L_2 are as follows.

$$\begin{aligned}L_1 &= \begin{bmatrix} d_1 & -a_{12} & ... & -a_{1N}\\ -a_{21} & d_2 & ... & -a_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & ... & 0\\ \vdots & \vdots & \ddots & \vdots\\ -a_{N1} & -a_{N2} & ... & d_N\end{bmatrix}\\ L_2 &= \begin{bmatrix} 0 & ... & 0 & ... & 0\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ -a_{j1} & ... & d_j & ... & -a_{jN}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ 0 & ... & 0 & ... & 0\end{bmatrix}\end{aligned} \tag{17}$$

Since $L \times 1_N = 0_N$, we can conclude that

$$(L_1 \otimes BK)(\sigma^*(t-\tau_1) + e^*(t-\tau_1)) + (L_2 \otimes BK)(\sigma^*(t-\tau_2) + e^*(t-\tau_2)) = (L_1 \otimes BK) \begin{bmatrix} x_1(t^1_{k_1} - \tau_1) \\ \vdots \\ x_N(t^N_{k_N} - \tau_1) \end{bmatrix} + (L_2 \otimes BK) \begin{bmatrix} x_1(t^1_{k_1} - \tau_2) \\ \vdots \\ x_N(t^N_{k_N} - \tau_2) \end{bmatrix} = (-I_N \otimes B)u(t,\tau). \quad (18)$$

Therefore,

$$\|u(t,\tau)\| \leq \frac{\|L_1 \otimes BK\|}{\|I_N \otimes B\|}(\|\sigma^*(t-\tau_1)\| + \|e^*(t-\tau_1)\|) + \frac{\|L_2 \otimes BK\|}{\|I_N \otimes B\|}(\|\sigma^*(t-\tau_2)\| + \|e^*(t-\tau_2)\|) \quad (19)$$

Note that $\|\sigma^*(t-\tau_1)\|$, $\|\sigma^*(t-\tau_2)\|$, $\|e^*(t-\tau_1)\|$ and $\|e^*(t-\tau_2)\|$ are bounded. Hence, there is an upper bound such $\mathcal{M}$ for $\|u(t,\tau)\|$. Also, $c_2 \geq 1$, $c_3 > 0$ and $c_4 > 0$ can be found such that [23]

$$\|x_i(t)\| \leq c_2\|x_i(t_0)\|e^{c_3(t-t_0)} + c_4. \quad (20)$$

According to (5), the dynamic model of the measurement error is given by

$$\dot{e}_i(t) = -\dot{x}_i(t) = -Ax_i(t) - Bu_i(t,\tau_i). \quad (21)$$

Therefore,

$$\|\dot{e}_i(t)\| \leq \|A\|\|x_i(t)\| + \|B\|\|u_i(t,\tau_i)\|. \quad (22)$$

According to (19) and (20)

$$\|\dot{e}_i(t)\| \leq \phi(t) = c_5 e^{c_3 t} + c_6, \quad (23)$$

where $c_5 = c_2\|A\|\|x(t_0)\|e^{-c_3 t_0}$ and $c_6 = c_4 + \|B\|\mathcal{M}$. Since $\phi(t)$ is an increasing function, it is found that

$$\int_{t^i_{k_i}}^{t} \phi(s)ds \leq (t - t^i_{k_i})\phi(t) = (t - t^i_{k_i})(c_5 e^{c_3 t} + c_6). \quad (24)$$

Then, we have

$$\|e_i(t)\| = \|\int_{t^i_{k_i}}^{t} \dot{e}_i(s)ds\| \leq \int_{t^i_{k_i}}^{t} \phi(s)ds \leq (t - t^i_{k_i})(c_5 e^{c_3 t} + c_6), t \in [t^i_{k_i}, t^i_{k_i+1}). \quad (25)$$

We know that $\|e_i(t)\| \leq c_1 e^{-\alpha t}$ which means $\|e_i(t)\| \leq c_1$ for $\alpha > 0$. Hence, by substituting $t = t^i_{k_i+1}$ in (25), it is found that

$$T_{k_i} := t^i_{k_i+1} - t^i_{k_i} \geq \frac{c_1}{c_5 e^{c_3 t^i_{k_i+1}} + c_6}. \quad (26)$$

Now, suppose that the Zeno behavior occurs for agent i, which means $\sum_{k_i=0}^{\infty} T_{k_i}$ is convergent. Thus, T_{k_i} must tend to zero. According to (26) it means that $t^i_{k_i+1} \to \infty$. Since $t^i_{k_i+1} = \sum_{s=0}^{k_i} T_s$, $\sum_{k_i=0}^{\infty} T_{k_i}$ is divergent, which is a contradiction. Consequently, the Zeno behavior cannot occur. ■

D. The Concept of Agent-Based Delay Margin

In the previous subsections, it was shown that the control scheme (2)-(5) can yield in achieving consensus in the multi-agent system (1) and excluding the Zeno behavior. In this subsection, the concept of agent-based delay margin is defined in the multi-agent system (1) controlled by the event-triggered control method (2)-(5).

Definition 1: Let $\tau_1 \geq 0$ be a fixed constant. Assume that the event-triggered control method (2)-(5) results in consensus in the multi-agent system (1), where $\tau_2 = \tau_1$. In such a case, let τ_2^* be the supremum value such that the consensus is preserved for all $\tau_2 \in [\tau_1, \tau_2^*)$. The delay margin of agent j in the presence of network delay τ_1 is defined as $\mathrm{DM}^j_{\tau_1} := \tau_2^* - \tau_1$.

In the next subsection, we discuss about how to numerically calculate the values of agent-based delay margins.

E. Toward Finding Agent-Based Delay Margins

Let

$$X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix}_{(Nn)\times 1} \quad X(t_k - \tau_m) = \begin{bmatrix} x_1(t^1_{k_1} - \tau_m) \\ \vdots \\ x_N(t^N_{k_N} - \tau_m) \end{bmatrix}_{(Nn)\times 1} \quad (27)$$

where $t^i_{k_i}$ is the latest event-triggered instant of agent i. By these notations, the overall closed-loop dynamic model of system (1) with the control effort (2) can be written as follows.

$$\dot{X}(t) = I_N \otimes AX(t) - L_1 \otimes BKX(t_k - \tau_1) - L_2 \otimes BKX(t_k - \tau_2) \quad (28)$$

According to (5), $X(t_k - \tau_m) = X(t - \tau_m) + e^*(t - \tau_m)$. Thus,

$$\dot{X}(t) = I_N \otimes AX(t) - L_1 \otimes BKX(t-\tau_1) - L_1 \otimes BKe^*(t-\tau_1) - L_2 \otimes BKX(t-\tau_2) - L_2 \otimes BKe^*(t-\tau_2) \quad (29)$$

We know that as $t \to \infty$, $\|e^*(t-\tau_m)\| \to 0$. Therefore, the stability of (28) is equivalent to the stability of the following system

$$\dot{X}(t) = I_N \otimes AX(t) - L_1 \otimes BKX(t-\tau_1) - L_2 \otimes BKX(t-\tau_2). \quad (30)$$

To find the stable region in $\tau_1 - \tau_2$ plane, we can use Algorithm 1 which is borrowed from [26]. In this algorithm, the stability region of system

$$\dot{X}(t) = \mathcal{A}X(t) + \mathcal{B}_1 X(t-\tau_1) + \mathcal{B}_2 X(t-\tau_2) \quad (31)$$

in the $\tau_1 - \tau_2$ plane is obtained by assuming that this system is stable for $\tau_1 = \tau_2 = 0$.

Algorithm 1:
(1) Write the characteristic equation of the system as follows.

$$CE(s,\tau_1,\tau_2) = \det(sI - \mathcal{A} - \mathcal{B}_1 e^{-\tau_1 s} - \mathcal{B}_2 e^{-\tau_2 s}) = 0 \quad (32)$$

(2) Substitute $e^{-\tau_m s} = \frac{1-T_m s}{1+T_m s}, m = 1,2$ (Rekasius substitution) in $CE(s,\tau_1,\tau_2)$ to get $\widetilde{CE}(s,T_1,T_2)$ as

$$\widetilde{CE}(s,T_1,T_2) = CE(s,\tau_1,\tau_2)\big|_{e^{-\tau_m s} = \frac{1-T_m s}{1+T_m s}} = 0 \quad (33)$$

(3) Multiply (33) with $(1+T_1 s)^{N-1}(1+T_2 s)$ to obtain

$$\widetilde{CE}(s,T_1,T_2)\times(1+T_1 s)^{N-1}(1+T_2 s)=\overline{CE}(s,T_1,T_2)= \sum_{k=0}^{v} b_k(T_1,T_2)s^k=0, \quad (34)$$

where v is the largest power of s in $\overline{CE}(s,T_1,T_2)$ and $b_k(T_1,T_2)$ for $k=1,...,v$ are parametric functions of T_1 and T_2 (b_0 is a constant and is always grater than zero ($b_0>0$)).
(4) Use Routh's array over the characteristic equation (34) in parametric form (The obtained array is similar to that shown in Table 1).

Table 1 : Routh's array over $\overline{CE}(s,T_1,T_2)$

s^v	$b_v(T_1,T_2)$	...	...	$\begin{cases} b_1(T_1,T_2) & v \text{ odd} \\ b_0 & v \text{ even} \end{cases}$
s^{v-1}	$b_{v-1}(T_1,T_2)$	...	...	$\begin{cases} b_0 & v \text{ odd} \\ 0 & v \text{ even} \end{cases}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^2	$R_{21}(T_1,T_2)$	b_0		
s^1	$R_1(T_1,T_2)$			
s^0	b_0			

(5) Define the "core curve" as

$$\overline{R}_1(T_1,T_2)=\{[R_1(T_1,T_2)=0]\cap[R_{21}(T_1,T_2)>0]\} \quad (35)$$

based on the elements of the Routh's array (Table 1).
(6) For every T_1 and T_2 on $\overline{R}_1(T_1,T_2)$, find

$$\tau_m=\frac{2}{\omega_c}[\tan^{-1}(\omega_c T_m)+k\pi], \qquad k=0,1,... \quad (36)$$

where

$$\omega_c=\sqrt{\frac{b_0}{R_{21}(T_1,T_2)}}, \quad (37)$$

and $m=1$ and 2.
(7) For every τ_1 and τ_2 found in the previous step, use the corresponding ω_c and the following equation to generate the complete set of "offspring curves" for system (31).

$$\{\boldsymbol{\tau}\}=(\tau_1+\frac{2\pi}{\omega_c}t,\tau_2+\frac{2\pi}{\omega_c}r),\ t=0,1,...\ r=0,1,... \quad (38)$$

(8) After generating the "offspring curves", the area with $\tau_1>0, \tau_2>0$ which is in contact with the origin and is encircled by "offspring curves", is a stability region in the $\tau_1-\tau_2$ plane.
Now, by using Algorithm 1, the following algorithm can be proposed for calculating the delay margin of each agent in the multi-agent system (1) controlled by the event-triggered control method (2)-(5) in the presence of a fixed network delay.

Algorithm 2:
(1) By choosing

$$\mathcal{A}=I_N\otimes A,\ \mathcal{B}_1=-L_1\otimes BK,\ \mathcal{B}_2=-L_2\otimes BK \quad (39)$$

and using *Algorithm 1*, find the stability region which is in contact with the origin in the $\tau_1-\tau_2$ plane.

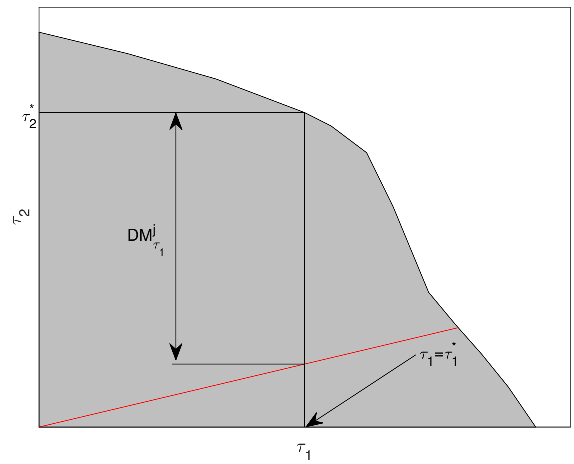

Fig. 1. Computing the agent-based time delay margin based on Algorithm 2

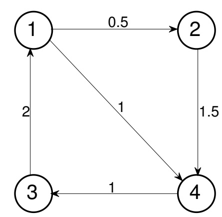

Fig. 2. Communication graph of the system

(2) For a fixed network delay $\tau_1=\tau_1^*$ (by assuming that consensus is achieved where $\tau_1=\tau_2=\tau_1^*$ in the multi-agent system (1) controlled by the event-triggered control method (2)-(5)), the vertical distance of the point $(\tau_1,\tau_2)=(\tau_1^*,\tau_1^*)$ from the boundary of the stability region found in the previous step is the delay margin of the agent j in the multi-agent system (1) controlled by the event-triggered control method (2)-(5) in the presence of the fixed network delay $\tau_1=\tau_1^*$ (See Fig. 1).

IV. A Numerical Example

Consider a group of 4 agents described by the dynamic model (1) with $A=[0\ 1\ ;\ -1\ 0]$ and $B=[1\ ;\ 1]$. The communication graph of the system is shown in Fig. 2 which is a directed weighted graph containing a directed spanning tree. By choosing the matrix P as $P=[0.6\ \ -0.2\ ;\ -0.2\ 1.4]$, the feedback gain matrix is obtained as $K=[2\ 1]$. Also, the parameters of the event-triggered function are chosen as $c_1=5$ and $\alpha=0.2$. Assume that the aim is to find the delay margin of agent 3 in the above-described multi-agent system. To this end, matrices L_1 and L_2 are obtained as follows.

$$L_1=\begin{bmatrix} 2 & 0 & -2 & 0 \\ -0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1.5 & 0 & 2.5 \end{bmatrix}, \quad L_2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By setting $\mathcal{A}$, $\mathcal{B}_1$, and $\mathcal{B}_2$ according to (39), the characteristic equation will be $\overline{CE}(s, T_1, T_2) = \sum_{k=0}^{12} b_k(T_1, T_2)s^k = 0$, where

$$b_{12} = T_1^3T_2 \qquad b_{11} = T_1^3+3T_1^2T_2-18T_1^3T_2$$

$$b_{10} = 3T_1^2-12T_1^3-24T_1^2T_2+\tfrac{433}{4}T_1^3T_2+3T_1T_2$$

$$b_9 = 3T_1+T_2-6T_1^2+\tfrac{81}{4}T_1^3-\tfrac{65}{4}T_1^2T_2-\tfrac{759}{4}T_1^3T_2+6T_1T_2$$

$$b_8 = 24T_1+12T_2+-\tfrac{401}{4}T_1^2+\tfrac{207}{4}T_1^3+\tfrac{1035}{4}T_1^2T_2+\tfrac{23}{2}T_1^3T_2-\tfrac{385}{4}T_1T_2$$

$$b_7 = \tfrac{-T_1}{4}+\tfrac{113}{4}T_2+18-\tfrac{123}{4}T_1^2+111T_1^3+\tfrac{591}{2}T_1^2T_2-186T_1^3T_2-\tfrac{465}{4}T_1T_2$$

$$b_6 = -\tfrac{1143}{4}T_1-\tfrac{99}{4}T_2+\tfrac{481}{4}-160T_1^2+90T_1^3+360T_1^2T_2-\tfrac{385}{4}385T_1^3T_2-\tfrac{539}{2}T_1T_2$$

$$b_5 = -345T_1-\tfrac{27}{2}T_2+\tfrac{1347}{4}-84T_1^2+\tfrac{375}{4}T_1^3+\tfrac{1281}{4}T_1^2T_2-\tfrac{57}{4}T_1^3T_2-210T_1T_2$$

$$b_4 = -414T_1-36T_2+466-\tfrac{263}{4}T_1^2+\tfrac{105}{4}T_1^3+\tfrac{309}{4}T_1^2T_2-\tfrac{T_1^3T_2}{2}-\tfrac{719}{4}T_1T_2$$

$$b_3 = -\tfrac{1383}{4}T_1-\tfrac{145}{4}T_2+480-\tfrac{237}{4}T_1^2+2T_1^3+\tfrac{11}{2}T_1^2T_2-\tfrac{351}{4}T_1T_2$$

$$b_2 = -\tfrac{417}{4}T_1+\tfrac{3}{4}T_2+\tfrac{1495}{4}-9T_1^2-\tfrac{19}{2}T_1T_2$$

$$b_1 = -4T_1+\tfrac{9}{2}T_2+\tfrac{645}{4} \qquad b_0 = 27.$$

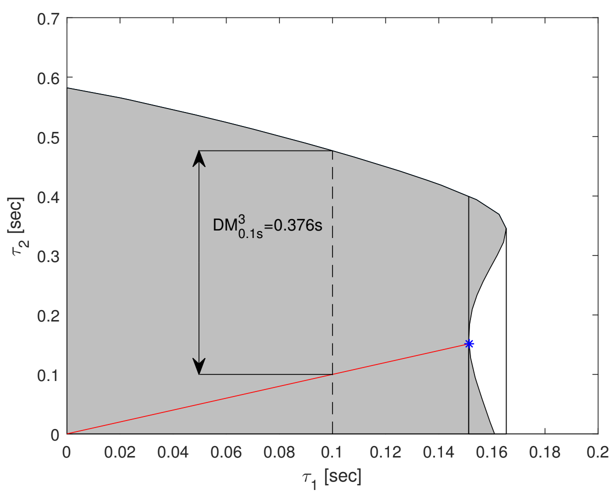

Fig. 3. Stability region in the $\tau_1 - \tau_2$ plane

Based on the above characteristic equation, the stability region in the $\tau_1 - \tau_2$ plane is obtained as that shown in Fig. 3 (The point specifying the network delay margin from the results of [24] has been shown by a blue star in this figure). For instance, it is seen that $(\tau_1, \tau_2) = (0.1, 0.1)$ is placed in the stability region, and consequently the consensus is achieved in such a case (This fact is confirmed by the numerical simulation results presented in Fig. 4). From Fig. 3, it is found that the delay margin of agent 3 in the considered event-triggered controlled multi-agent system in the presence of network delay $\tau_1 = 0.1s$ equals $\mathrm{DM}_{0.1}^3 = 0.376s$. This means that supremum of τ_2^*, for preserving the consensus for all $\tau_2 \in [\tau_1, \tau_2^*)$ is $0.476s$. To confirm this result, numerical simulation results corresponding to the cases $(\tau_1, \tau_2) = (0.1s, 0.47s)$ and $(\tau_1, \tau_2) = (0.1s, 0.48s)$ are respectively presented in Figs. 5 and 6. In the first case, the consensus is achieved, whereas in the second case we face with an unstable multi-agent system.

To compare the agents in the viewpoint of delay tolerance, the delay margin of each agent has been calculated and plotted in Fig. 7 with respect to different values of network delay. From this figure, it is deduced that the agent 2 has the maximum delay margin, whereas the minimum delay margin belong to the agent 4 among the all agents.

V. Conclusion

In this paper, the event-triggered consensus problem for linear directed multi agent systems in the presence of delay was studied. As a solution for this problem, a distributed event-based control strategy was proposed. Sufficient conditions on the control gain and parameters of the event-triggered function were derived which assure the consensus of the multi agent system and excluding the Zeno behavior. Also, the concept of agent based delay margin in the presence of a fixed network delay was defined in the multi-agent system controlled by the proposed event-triggered control method. Moreover, an algorithm was suggested to calculate the agent based delay margins.

References

[1] Y. Tian and C. Liu, "Consensus of Multi-Agent Systems With Diverse Input and Communication Delays," IEEE Transactions on Automatic Control, vol. 53, no. 9, pp. 2122-2128, 2008.

[2] J. Qin, Q. Ma, Y. Shi and L. Wang, "Recent Advances in Consensus of Multi-Agent Systems: A Brief Survey," IEEE Transactions on Industrial Electronics, vol. 64, no. 6, pp. 4972-4983, 2017.

[3] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," Systems and Control Letters, vol. 59, no. 3-4 pp. 209-217, 2010.

[4] W. Zhu, Z.-P. Jiang and G. Feng, "Event-based consensus of multi-agent systems with general linear models," Automatica,vol. 50, no. 2, pp. 552-558, 2014.

[5] J. Yu and L. Wang, "Group consensus in multi-agent systems with switching topologies and communication delays," Systems and Control Letters,vol. 59, no. 6, pp. 340-348, 2010.

[6] Z. Li, Z. Duan, G. Chen and L. Huang, "Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint," IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 57, no. 1, pp. 213-224, 2010.

[7] Z. Zhang, L. Zhang, F. Hao, and L. Wang, "Leader-following consensus for linear and Lipschitz nonlinear multiagent systems with quantized communication," IEEE Transactions Cybernetics, vol. 47, no. 8, pp. 1970–1982, 2017.

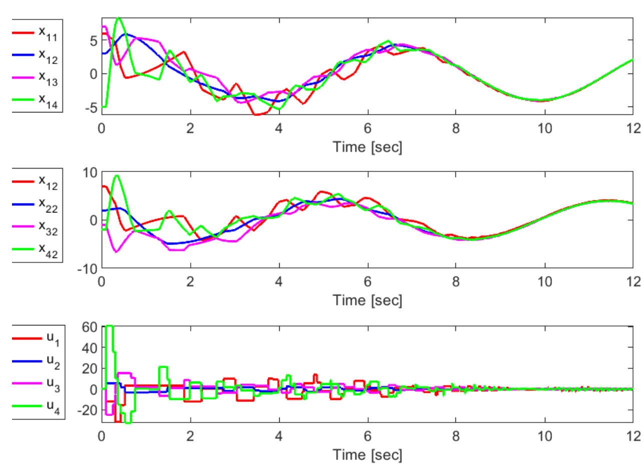

Fig. 4. States trajectories and control inputs in the considered event-triggered controlled multi-agent system, where $(\tau_1, \tau_2) = (0.1s, 0.1s)$

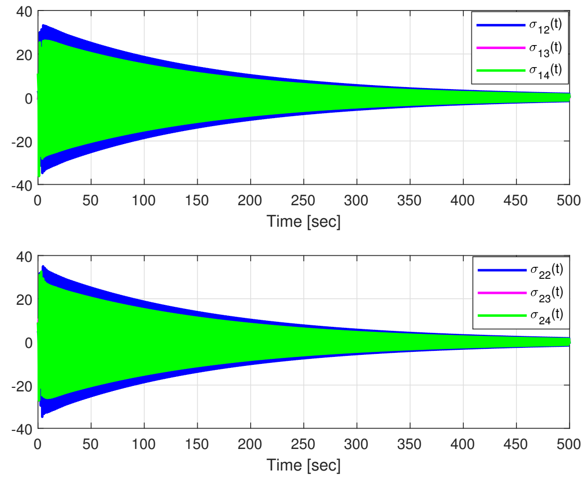

Fig. 5. $\sigma_i(t) = [\sigma_{1i}(t) \quad \sigma_{2i}(t)] := x_i(t) - x_1(t)$ $(i = 2, 3, 4)$ in the considered event-triggered controlled multi-agent system, where $(\tau_1, \tau_2) = (0.1s, 0.47s)$

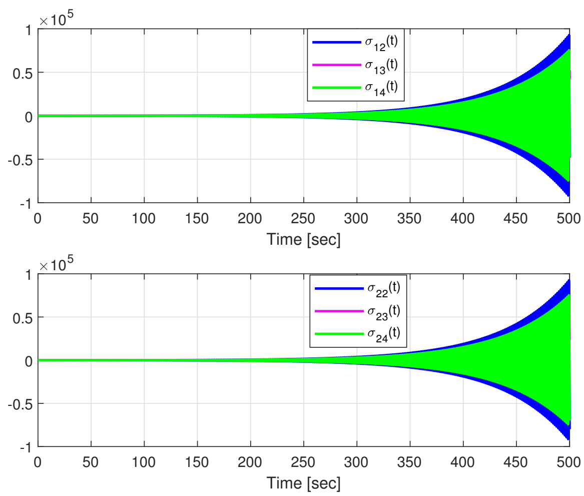

Fig. 6. $\sigma_i(t) = [\sigma_{1i}(t) \quad \sigma_{2i}(t)] := x_i(t) - x_1(t)$ $(i = 2, 3, 4)$ in the considered event-triggered controlled multi-agent system, where $(\tau_1, \tau_2) = (0.1s, 0.48s)$

[8] L. Ding, Q.-L. Han, and G. Guo, “Network-based leader-following consensus for distributed multi-agent systems,” Automatica, vol. 49, no. 7, pp. 2281–2286, 2013.
[9] F. Sun, Y. Shen, J. Kurths and W. Zhu, ”Mean-square consensus of multi-agent systems with noise and time delay via event-triggered control”, Journal of the Franklin Institute,vol. 357, no. 9, pp. 5317-5339, 2020.
[10] T. Ma, T. Yu, J. Huang, X. Yang and Zh. Gu, ”Adaptive odd impulsive consensus of multi-agent systems via comparison system method,” Nonlinear Analysis: Hybrid Systems, vol. 35, 100824, 2020,
[11] K. J. Astrom and B. M. Bernhardsson, “Comparison of Riemann and Lebesgue sampling for first order stochastic systems,” Proceedings of the 41st IEEE Conference on Decision and Control, pp. 2011–2016, 2002.
[12] B. Bernhardsson and K. J. Astrom, “Comparison of periodic and event based sampling for first-order stochastic systems,” IFAC Proceedings Volumes, vol. 32, no. 2, pp. 5006-5011, 1999.
[13] E. Garcia, Y. Cao, H. Yu, P. Antsaklis, and D. Casbeer, “Decentralised event-triggered cooperative control with limited communication,” International Journal of Control, vol. 86, no. 9, pp. 1479–1488, 2013
[14] Y. Fan, G. Feng, Y. Wang, and C. Song, “Distributed event-triggered control of multi-agent systems with combinational measurements,” Au-

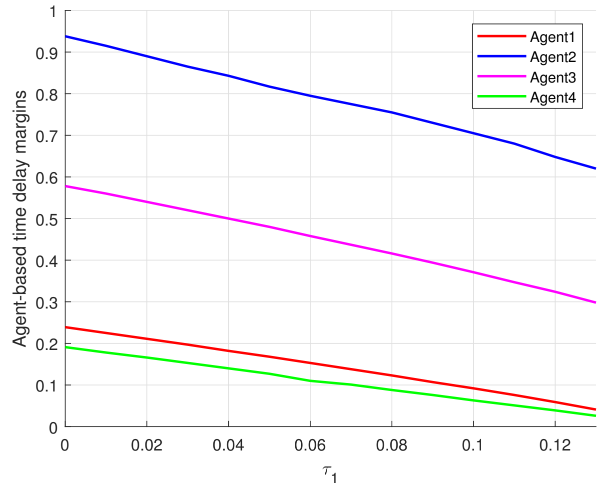

Fig. 7. The delay margin of each agent with respect to network delay τ_1

tomatica, vol. 49, no. 2, pp. 671–675, 2013
[15] W. Zhu and Z.-P. Jiang, “Event-based leader-following consensus of multi-agent systems with input time delay,” IEEE Transactions on Automatic Control, vol. 60, no. 5, pp. 1362–1367, 2015.
[16] X. Li, Y. Tang and H. R. Karimi, ”Consensus of multi-agent systems via fully distributed event-triggered control,” Automatica, vol. 116, 108898, 2020.
[17] X. Yi, K. Liu, D. V. Dimarogonas and K. H. Johansson, ”Dynamic Event-Triggered and Self-Triggered Control for Multi-agent Systems,” IEEE Transactions on Automatic Control, vol. 64, no. 8, pp. 3300-3307, 2019.
[18] X.-M. Li, Q. Zhou, P. Li, H. Li and R. Lu, ”Event-Triggered Consensus Control for Multi-Agent Systems Against False Data-Injection Attacks,” IEEE Transactions on Cybernetics, vol. 50, no. 5, pp. 1856-1866, 2020.
[19] X. Ge, Q.-L. Han, D. Ding, X.-M. Zhang, and B. Ning, “A survey on recent advances in distributed sampled-data cooperative control of multi-agent systems,” Neurocomputing, vol. 275, no. 31, pp. 1684–1701, 2018.
[20] N. Mu, X. Liao, and T. Huang, “Leader-following consensus in second-order multiagent systems via event-triggered control with nonperiodic sampled data,” IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 62, no. 10, pp. 1007–1011, 2015.
[21] J. Zhang, H. Zhang, Sh. Sun and Zh. Gao, ”Leader-follower consensus control for linear multi-agent systems by fully distributed edge-event-triggered adaptive strategies,” Information Sciences, vol. 555, pp. 314-338, 2021.
[22] L. Li, D. W. C. Ho, and J. Lu, “Event-based network consensus with communication delays,” Nonlinear Dynamics, vol. 87, no. 3, pp. 1847–1858, 2017.
[23] N. Mu, X. Liao and T. Huang, ”Event-Based Consensus Control for a Linear Directed Multi agent System With Time Delay,” IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 62, no. 3, pp. 281-285, 2015.
[24] N. Mu, Y. Wu, X. Liao and T. Huang, ”Input Time Delay Margin in Event-Triggered Consensus of Multiagent Systems,” in IEEE Transactions on Cybernetics, vol. 49, no. 5, pp. 1849-1858, 2019.
[25] D. Yang, W. Ren, X. Liu, and W. Chen, “Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs,” Automatica, vol. 69, pp. 242–249, 2016.
[26] R. Sipahi and N. Olgac, ”Complete stability robustness of third-order LTI multiple time-delay systems,” Automatica,vol. 41, no. 8, pp. 1413-1422, 2005.