ARTICLE TEMPLATE

Optimal control and genetic algorithms in modelling dynamical allocation of resources for a three-sector economy

Tatyana Alexeeva[a], Leonid Chechurin[a], Viktor Dodonov[a], Zahra Honarmand[a], Nikolay Kuznetsov[b], Pekka Neittaanmäki[b]

[a]School of Engineering Science, LUT University, Lappeenranta, Finland;
[b]Faculty of Information Technology, University of Jyväskylä, Jyväskylä, Finland

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ABSTRACT
The task of looking for optimal allocation of resources in an economy is fraught with a number of severe restrictions. This is manifested in the complexity of the technical implementation of the solution even in the case of a low dimension of the problem. In this paper, we consider two approaches, analytical and numerical, for deriving the dynamical optimal allocation of resources in a three-sector economy and show that the use of modern artificial intelligence (AI) technologies such as genetic algorithms (GA), can be useful for expanding the range of effective tools and new contributions to this problem.

KEYWORDS
Optimal control; nonlinrear dynamics; economic growth; balanced economy; genetic algorithms

1. Introduction

Studying the optimal behavior of economic systems with a multi-level structure of interaction between its elements is a cornerstone task of economic science and an important component for decision-making in a real economy. The complexity of the structure and interactions within the economy, as well as external shocks lead to appearance of complex dynamics in the economic system both locally (in industries) and at the global level — intersectoral, country-level and world-wide. Particular interest is the task of analyzing and forecasting economic growth at all mentioned levels and the associated optimal allocation of resources between sectors of the economy. The related literature discusses various aspects of this task, primarily economic mechanisms that generate economic growth and make an impact on it. Starting from the pioneering work of Solow (Nobel Prize, 1987) [1], who introduced the one-sector model of economic growth, the new growth theory is an important part of modern economics. Following main paradigm in this field, Phelps [2] and Uzawa [3] derived the optimal economic growth rule for the one-sector and two-sector Solow models, respectively. A great contribution was made by Romer [4], who, among other things, considered a three-sector economy under the influence of technological progress. Ngai et. al. [5]

CONTACT Tatyana Alexeeva. Email: tatyanalexeeva@gmail.com

explored the factors affecting structural changes in the economy, as well as the possibility of balanced economic growth in the face of such changes in a multisectoral model. Kolemayev [6,7] proposed an optimal rule for the dynamical allocation of labor and investment in a three-sector model, considering the conditions of economic equilibrium.

Economic growth tends to be uneven across different sectors of the economy. Therefore, although significant progress has been made in understanding the economic conditions and mechanisms of growth in the economy, from a technical point of view, solving the dynamical problem of optimal allocation of resources between sectors under conditions of balanced growth remains a challenging task. In this paper, we demonstrate application of analytical and numerical approaches to a dynamical problem of the optimal allocation of resources [6,8], using an example of a three-sector economy. The analytical approach makes it possible to obtain a closed-form solution to the problem that satisfies an important economic restriction — the material balance condition. At the same time, the technical complexity of solution forces us to restrict ourselves to exploring low-dimensional models and fairly simple functional forms of control variables. The usage of numerical methods contributes helps to avoid these shortcomings of the analytical procedure; however, it also leads to the material balance condition being not satisfied. The pros and cons of each approach are demonstrated using several numerical experiments with various functional forms of control variables. Thus, both analytical and numerical approaches face a number of serious limitations.

To overcome these difficulties, new tools are needed, for instance, genetic algorthms (GAs), which are not sensitive both to the dimension and computational complexity of the problem and the model functional forms. The goal of a GA is to identify the best possible solution to a problem by replicating the process of evolution in nature [9]. The algorithm searches through the space of potential solutions and stores the potential solutions in a "chromosome-like" data structure. A chromosome in GA represents a possible solution for the optimization problem [9]. The search for the best solution is iterated across generations, beginning with a population that is usually random [10]. Several chromosomes from the current generation are chosen based on their fitness and then modified depending on operators like reproduction, combination, mutation, etc. to build the new generation in each generation after the capacity of the entire population has been assessed. The solutions often evolve in a way that causes the population of solutions to "converge" towards the optimal solution with each generation. We show how GAs might be used for deriving optimal solution with a wider class of functions for the control variables and handling the difficulties associated with non-polynomial computational complexity of the problem. That allows us to refine the optimal solution and better match the model with real life.

2. Problem statement

We consider a problem of controlling three-sector economy formulated by Kolemayev [6]. The three sectors are production of raw materials, sector 0; production of investment goods, sector 1; and production of final consumption goods, sector 2. All three sectors are using labor (L_j) and capital (K_j) as inputs to production. Labor is perfectly mobile across the sectors. Total labor force is growing at the constant rate ν, and is allocated to the sector j, $j = 0, 1, 2$, at every point in time. It could be costlessly reallocated among the three sectors:

$$L_0(t) + L_1(t) + L_2(t) = L(t). \tag{1}$$

Capital, in contrast, is sector specific. Capital in a sector j accumulates due to investment I_j allocated to this sector, and depreciates with the sector-specific rate μ_j. If it is desired to increase capital in a sector, the only way is to increase investment in this sector. If less capital is needed in a sector, investment could drop below the level necessary to compensate for depreciation, allowing the stock of capital to decrease over time. Transferring already installed capital between sectors is, in contrast, impossible:

$$\dot{K}_j = -\mu_j K_j + I_j, \quad K_j(0) = K_j^0 \quad j = 0, 1, 2. \tag{2}$$

Production functions in all three sectors take the form of Cobb-Douglas function, with sector-specific productivity levels A_j and sector-specific capital intensities (exponent at the capital) α_j:

$$X_j = F_j(K_j, L_j) = A_j K_j^{\gamma_j} L_j^{1-\gamma_j}, \tag{3}$$

All production in the sector 1, X_1, is used for investment in the three sectors of the economy:

$$X_1(t) = I_0(t) + I_1(t) + I_2(t). \tag{4}$$

Usage of the raw materials produced in sector 0, X_0, is governed by the materials balance equation:

$$X_0(t) = a_0 X_0(t) + a_1 X_1(t) + a_2 X_2(t). \tag{5}$$

A share $a_0 < 1$ of the sector 0 production is used by the sector itself; the rest is demanded by the sectors 1 and 2, proportionally to their outputs. This notation is equivalent to writing the production function in each sector as the following Leontieff function $F^l = \min(X_0/a_1, X_1)$ (in the materials and capital-labor aggregate pair), with the capital-labor aggregate being the Cobb-Douglas function $X_1 = K^{\gamma_1} L^{1-\gamma_1}$ described above. It is assumed that some of the materials could remain unused and could be costlessly disposed of. The economy is controlled by affecting the shares of labor and investment allocated to each sector at every point in time. While the amount of allocated labor affects the level of production immediately, investment is reflected only in the derivative of the sector's capital, and thus of the sector's output.

Next, we formulate the problem in per capita units, where sector-specific capitals (K_j) and outputs (X_j) at each time are divided by the total amount of labor available at this time: $x_j = \frac{X_j}{L}$, $k_j = \frac{K_j}{L}$. Denoting the shares of total labor allocated to the sector j as $\theta_j = \frac{L_j}{L}$ and the shares of investment as $s_j = \frac{I_j}{X_1}$, the three static constraints connecting the shares and productions in the sectors could be written as:

$$\theta_0 + \theta_1 + \theta_2 = 1, \tag{6}$$

$$s_0 + s_1 + s_2 = 1, \tag{7}$$

$$(1-a_0)\,x_0 - a_1x_1 - a_2x_2 \geq 0. \tag{8}$$

Similarly, the capital accumulation equations and production functions are written as:

$$\dot{k_j} = -\left(\mu_j+\nu\right)k_j + s_jx_1, \quad k_j(0)=\frac{K_j^0}{L^0} \quad j=0,1,2, \tag{9}$$

$$x_j = A_jk_j^{\alpha_j}\theta_j^{1-\alpha_j}. \tag{10}$$

Here, $\mu_j+\nu$ is the depreciation rate of per capita capital. With ν positive, new workers are born at each moment in time; in order to supply them with the same amount of capital as the already living workers, extra investment equal to νk_j is needed.

With these changes in notation, the optimization problem could be written as maximizing the consumer's per capita welfare:

$$\max_{s_j,\theta_j}\int_0^T x_2(t)e^{-\delta t}dt, \quad j=0,1,2, \tag{11}$$

i.e. discounted consumption ($\delta>0$) integrated over the interval $[0,T]$, by controlling the time-varying labor and investment shares θ_j and s_j. Relations (6)–(10) are the constraints of this problem. By economic logic, all six shares must be positive. This, together with positive initial capital, guarantees non-negative values of capital and output at all points in time.

Next, we consider two ways to solve of the optimal control problem: analytical and numerical approaches. In addition, we explain advantages of GAs in overcoming a number difficulties of solving.

2.1. Analytical solution

We use the Pontryagin maximum principle. Let us write Hamiltonian:

$$\begin{aligned}\mathbb{H} &= e^{-\delta t}x_2 + \sum_{j=0}^{2}\gamma_j\left(-\left(\mu_j+\nu\right)k_j + s_jx_1\right)\\ &+\xi\left(1-(\theta_0+\theta_1+\theta_2)\right)+\eta\left(1-(s_0+s_1+s_2)\right)\\ &+\zeta\left((1-a_0)\,x_0-a_1x_1-a_2x_2\right),\end{aligned} \tag{12}$$

where γ_j are co-state variables (price of the capital in j sector), ξ, η, ζ are multipliers. From the first order conditions (FOCs):

$$\frac{\partial\mathbb{H}}{\partial\theta_j}=0, \quad \frac{\partial\mathbb{H}}{\partial s_j}=0, \quad \dot{\gamma_j}=-\frac{\partial\mathbb{H}}{\partial k_j}. \tag{13}$$

and after making the following assumptions:

$$\mu_0=\mu_1=\mu_2, \quad \theta_1=b_1t+b_0, \quad s_1=\left(b_1t+b_0\right)^{\alpha_1}, \tag{14}$$

we obtain the following:

$$k_0(t) = \frac{(1-\alpha_1)\,\alpha_0}{(1-\alpha_0)\,\alpha_1}\frac{\theta_0}{\theta_1}k_1(t), \quad k_2(t) = \frac{(1-\alpha_1)\,\alpha_2}{(1-\alpha_2)\,\alpha_1}\frac{\theta_2}{\theta_1}k_1(t), \tag{15}$$

where $k_1(t) = \left(\frac{A_1((\mu_1+\nu)(1-\alpha_1)(b_1t+b_0)-b_0)}{(\mu_1+\nu)^2(1-\alpha_1)}\right)^{\frac{1}{1-\alpha_1}}$ is solution of (9) for $j = 1$ and $k_1(0) = k_1^0$; $\gamma_0 = \gamma_1 = \gamma_2 = \gamma = \eta = \frac{(1-a_0)\frac{\partial x_2}{\partial \theta_2}\left(\frac{\partial x_0}{\partial \theta_0}+a_1\frac{\partial x_1}{\partial \theta_1}\right)}{\frac{\partial x_1}{\partial \theta_1}\left((1-a_0)\frac{\partial x_0}{\partial \theta_0}+a_2\frac{\partial x_2}{\partial \theta_2}\right)}$; $\xi = \frac{(1-a_0)\frac{\partial x_0}{\partial \theta_0}\frac{\partial x_2}{\partial \theta_2}}{(1-a_0)\frac{\partial x_0}{\partial \theta_0}+a_2\frac{\partial x_2}{\partial \theta_2}}$; $\zeta = \frac{\frac{\partial x_2}{\partial \theta_2}}{(1-a_0)\frac{\partial x_0}{\partial \theta_0}+a_2\frac{\partial x_2}{\partial \theta_2}}$; b_0, b_1 are optimal parameters. Using (6) and materials balance (8), when the LHS of (8) equals to 0, we can get the optimal allocation of labor shares:

$$\theta_2 = \frac{(1-a_0)\,A_0\Gamma_0\,(1-\theta_1)\left(\frac{k_1(t)}{\theta_1}\right)^{\alpha_0} - a_1A_1\frac{k_1(t)}{\theta_1^{\alpha_0-1}}}{(1-a_0)\,A_0\Gamma_0\left(\frac{k_1(t)}{\theta_1}\right)^{\alpha_0} + a_2A_2\Gamma_2\left(\frac{k_1(t)}{\theta_1}\right)^{\alpha_2}}, \tag{16}$$

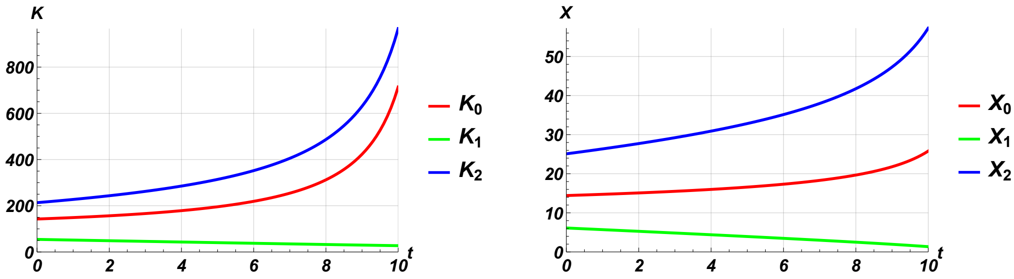
$$\theta_0 = 1-\theta_1-\theta_2, \tag{17}$$

where $\Gamma_0 = \left(\frac{(1-\alpha_1)\alpha_0}{(1-\alpha_0)\alpha_1}\right)^{\alpha_2}$, $\Gamma_2 = \left(\frac{(1-\alpha_1)\alpha_2}{(1-\alpha_2)\alpha_1}\right)^{\alpha_2}$. Using (9) for $j = 0, 2$ we can obtain the the optimal allocation of investment shares:

$$\begin{aligned} s_2 = {} & \frac{\Gamma_2^{\frac{1}{\alpha_2}}}{\theta_1\left(\Gamma_0^{\frac{1}{\alpha_0}}\left(\dot\theta_0\theta_1-\theta_0\dot\theta_1\right)+\Gamma_2^{\frac{1}{\alpha_2}}\left(\dot\theta_2\theta_1-\theta_2\dot\theta_1\right)\right)}\cdot \\ & \left[\left(\Gamma_0^{\frac{1}{\alpha_0}}\theta_2\left(\dot\theta_0\theta_1-\theta_0\dot\theta_1\right)-\left(\Gamma_0^{\frac{1}{\alpha_0}}\theta_0+\theta_1\right)\left(\dot\theta_2\theta_1-\theta_2\dot\theta_1\right)\right)s_1+\theta_1\left(\dot\theta_2\theta_1-\theta_2\dot\theta_1\right)\right], \\ s_0 = {} & 1-s_1-s_2. \end{aligned} \tag{18}$$

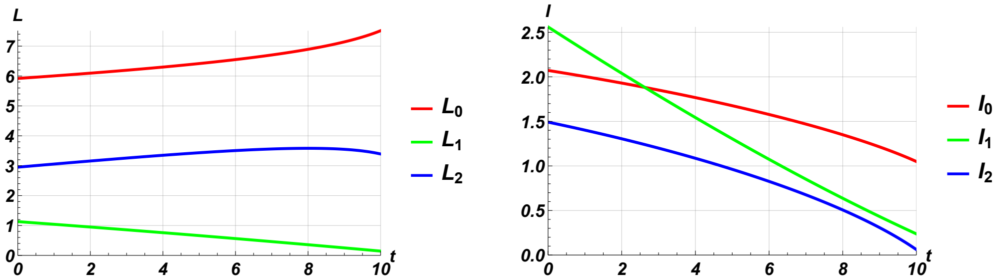
We show the obtained result (see Figs. 1-3) using the following economically justified parameter values:

$$\begin{array}{lll} \delta = 0.0015; & & \\ \nu = 0.01; & L = 10; & T = 10; \\ \mu_0 = 0.1; & \mu_1 = 0.1; & \mu_2 = 0.1; \\ K_0^0 = 142.68; & K_1^0 = 54.48; & K_2^0 = 213.46; \\ A_0 = 1.1; & A_1 = 1.15; & A_2 = 1; \\ \alpha_0 = 1/4; & \alpha_1 = 2/5; & \alpha_2 = 1/2; \\ a_0 = 0.2; & a_1 = 0.45; & a_2 = 0.35. \end{array} \tag{19}$$



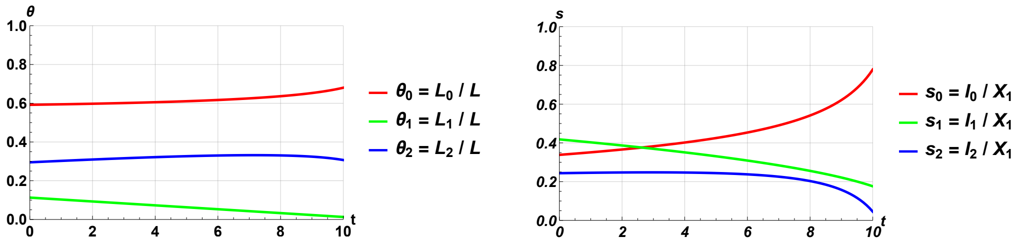
(a) Capital (K_j). (b) Output (X_j).

Figure 1. Analytical solution: capital (K_j) and output (X_j)



(a) Labor (L_j). (b) Investment (I_j).

Figure 2. Analytical solution: labor (L_j) and investment (I_j).



(a) Share of the labor (θ_j). (b) Share of the investment (s_j).

Figure 3. Analytical solution: share of the labor (θ_j) and investment (s_j).

As we can see, from a mathematical point of view, the optimal dynamical allocation of resources encounters restrictions on functional forms of control variables θ_j and s_j, caused, on the one hand, by the considered economic logic of the problem, and, on the other hand, by the admissible dimension of the economy. Next, we show how to take into account these restrictions by numerical approach.

2.2. Numerical solution

It is very difficult to completely solve an optimal problem in which all control variables are governed only by optimality conditions. In particular, this problem entails looking for a stable manifold of the saddle-type stationary point in a $6D$ space. Therefore, we resorted to a numerical solution of the problem, postulating specific functional forms of some control variables θ_j and s_j, and generally disregarding the material balance equation. We only require that the excess materials stay non-negative throughout the solution.

In order not to consider constraints (6) and (7) directly, we introduce the parametrization:

$$\begin{cases} \theta_0 = (\cos\phi_1 \sin\phi_2)^2 \\ \theta_1 = (\sin\phi_1 \sin\phi_2)^2 \\ \theta_2 = (\cos\phi_2)^2 \end{cases} \qquad \begin{cases} s_0 = (\cos\phi_3 \sin\phi_4)^2 \\ s_1 = (\sin\phi_3 \sin\phi_4)^2 \\ s_2 = (\cos\phi_4)^2 \end{cases} \tag{20}$$

Then (11) could be written as

$$\max_{\phi_j} \int_0^T x_2 e^{-\delta t} dt, \tag{21}$$

where $\phi_j \in [0, \pi/2]$, $j = \overline{1,4}$.

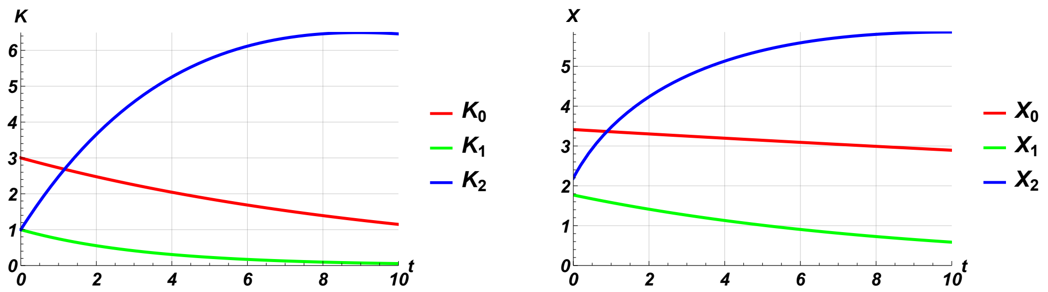
We use numerical simulations for the following economically justified parameter values:

$$\begin{array}{lll} \delta = 0.0015; & & \\ \nu = 0.01; & L = 10; & T = 10; \\ \mu_0 = 0.1; & \mu_1 = 0.3; & \mu_2 = 0.1; \\ K_0^0 = 3; & K_1^0 = 1; & K_2^0 = 1; \\ A_0 = 1.1; & A_1 = 1.15; & A_2 = 1; \\ \alpha_0 = 1/4; & \alpha_1 = 2/5; & \alpha_2 = 1/2; \\ a_0 = 0.2; & a_1 = 0.45; & a_2 = 0.35. \end{array} \tag{22}$$

Next, we demonstrate of solving the optimal control problem by two numerical experiments making some assumptions for ϕ_j.

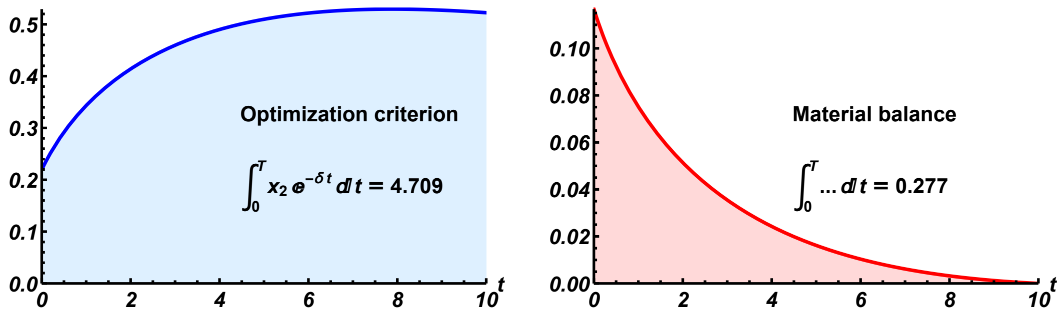
2.3. Numerical experiments

First, suppose $\phi_j = \text{const}$, $\phi_j \in [0, \pi/2]$; then θ_j and s_j are constant. In Figs. 4-5, the results of numerical simulations are presented.



(a) Capital (K_j). (b) Output (X_j).

Figure 4. First numerical experiment: capital (K_j) and output (X_j).

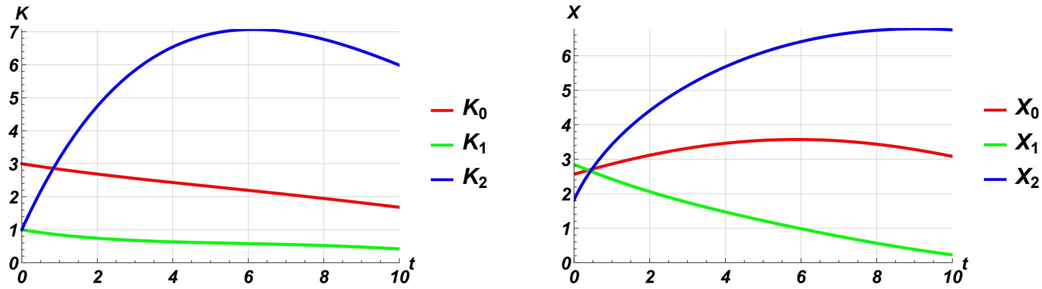


(a) Optimization criterion. (b) Material balance.

Figure 5. First numerical experiment: quality of the control.

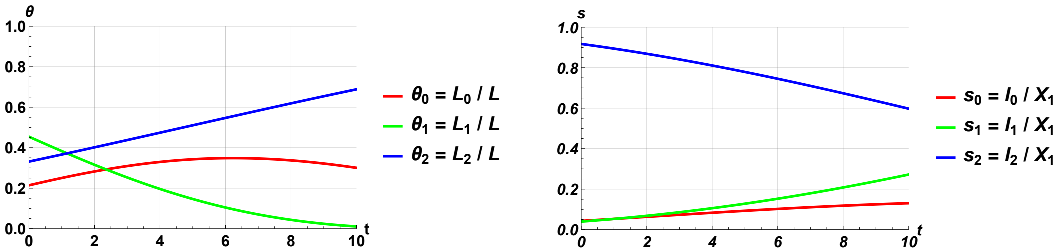
Second, suppose $\phi_j = \phi_{j1} + (\phi_{j2} - \phi_{j1})\,\frac{t}{T}$, $\phi_{ji} \in [0, \pi/2]$, $j = \overline{1,4}, i = 1, 2$; then θ_j and s_j are time-varying functions. This corresponds to a more realistic situation in the economy under consideration. The optimal capital trajectories, presented in Fig. 6, exhibit the basic feature of the solution implied by the finite nature of the problem. At time T, the remaining capital represents a pure wasted resource. That is why the total amount of capital accumulated in this economy increases from the initial value of 5 to ≈ 9.5 around $t = 6$, and declines to ≈ 6.25 at $t = T = 10$. The selected initial point that we have selected has too much K_1; the stock of capital in the sector 1, therefore, is continuously de-accumulated over time. Decline of K_1 is achieved by a significant drop of the share of labor dedicated to sector 1, θ_1 (Fig. 7(a)). Share of investment going to the sector 1, s_1, continues to grow over time (Fig. 7(b)); however, this still leads to declining production of investment goods, X_1 (Fig. 6(b)). The objective function is given as integral over discounted consumption X_2. It may explain a decline of X_2 in the latter part of the sample because a consumption at $t = T = 10$ is valued by the planner less than consumption at $t = 6$, for instance. As, in contrast to the X_1, X_2 is generally growing over time, it is not surprising that the behavior of θ_2 is opposite to the of θ_1: θ_2 is growing continuously. Investment into sector 2, on the other hand, is declining over time. This is explained by the fact that in our calibration capital intensity of the consumer goods sector is very high, leading to high capital productivity. In addition, the desire to "eat up" most of the capital by $t = T = 10$ forces the agents to continuously decrease investments into the sector 2, leading to a significant drop of K_2 by the time economy ends (Fig. 6(a)). The sector 0 of materials production plays a function of supporting consumer goods. Therefore, X_0 is generally behaving in a similar fashion as X_2, first increasing and then declining. As fluctuations in the level of X_0 are significantly lower than those in X_2, labor allocation θ_0 is also less volatile than θ_2. In Fig. 8, dynamics of the labor and investment is shown. Finally, note that the material balance is fluctuating in positive territory (Fig. 9(b)), i.e. LHS of (8) is more than 0.

It should be noted that the numerical optimization methods do not always allow solving the problem (9)–(11) in a general formulation for a balanced economy. In addition, the brute-force numerical solving of this problem is becoming very complex as we try to improve the accuracy. This can be illustrated as follows — if we approximate the function $s(t)$ piecewise-constantly, discretizing both the time and level by n intervals in $(0, T)$, then the number of combinations grows with n as power n. As the problem is non-polynomial complex, checking all possible variants takes too long time so a heuristic suboptimal strategy is to be used instead. We choose GAs that are often used in such a case. Moreover, we turn to this tool of artificial intelligence



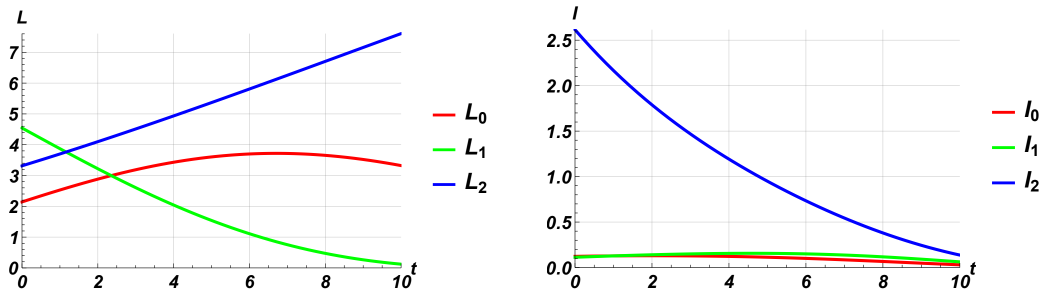
(a) Capital (K_j). (b) Output (X_j).

Figure 6. Second numerical experiment: capital (K_j) and output (X_j).



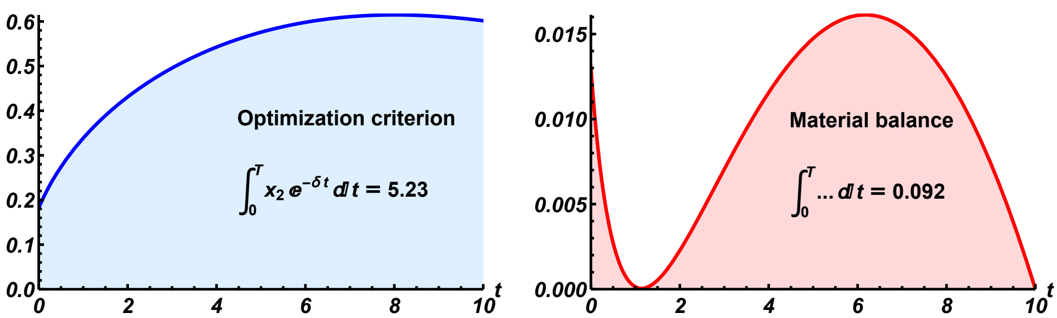
(a) Share of the labor (θ_j). (b) Share of the investment (s_j).

Figure 7. Second numerical experiment: share of the labor (θ_j) and investment (s_j).



(a) Labor (L_j). (b) Investment (I_j).

Figure 8. Second numerical experiment: labor (L_j) and investment (I_j).



(a) Optimization criterion. (b) Material balance.

Figure 9. Second numerical experiment: quality of the control.

technologies to use, among other things, additional opportunities to refine the optimal solution under binding condition of the material balance.

2.4. Genetic algorithms: additional advantages

The decision variables of the problem described in (11) are θ_j and s_j for $j = 0, 1, 2$. The choice of θ_j and s_j can be done in different ways, for example they can be assigned to be constant in the simplest case. Depending on how close we expect to come to the optimal solution, the functions θ_j and s_j can be chosen to be more and more complex, for example, in the form of polynomial expansion. Thus, we can look for the solution in the form of quadratic polynomials:

$$\begin{aligned} \theta_j(t) &= b_j + c_j t + d_j t^2, \\ s_j(t) &= e_j + f_j t + g_j t^2. \end{aligned} \tag{23}$$

Here, a real-value coded chromosome is used. When high precision is required and the search scope is large, real-value coding is frequently utilized because binary coding would be inefficient [11]. The chromosome is defined as a vector consisting of the below variables $(b_j\, c_j\, d_j\, e_j\, f_j\, g_j)$, where $j = 0, 1, 2$. There are 18 genes in the chromosome, respectively. In general, the initial solutions are chosen randomly and have a significant impact on the success of genetic algorithms. As the initial solution b_j, c_j, d_j, e_j, f_j and g_j for $j = 0, 1, 2$ are randomly selected from the range $[-\infty, +\infty]$, so the function of $\theta_j(t)$ and $s_j(t)$ will be defined by random generated genes. And then, equations (9) and (10) are integrated numerically to get the consumption as the function of time. Finally, the fitness function of the initial solution can be calculated and assessed. Normally, in genetic algorithm, the objective function is employed as a fitness function. It should be noticed that when the constraints (6), (7), or (8) are not met, the solutions are infeasible. Therefore, the following penalty functions have been introduced

$$\begin{aligned} PL(1) &= \sqrt{(1 - (\theta_0 + \theta_1 + \theta_2))^2}, \\ PL(2) &= \sqrt{(1 - (s_0 + s_1 + s_2))^2}, \\ PL(3) &= \max\{0, -((1 - a_0)\, x_0 - a_1 x_1 - a_2 x_2)\}. \end{aligned} \tag{24}$$

For each of the penalty functions, a weight value can be defined in order to highlight which of the constraints ((6), (7) or (8)) are more crucial for us. During the search process, the algorithm tries to set the penalty values equal to zero to optimize the objective function. If we denote the objective function (11) as $obj\,(\theta_j(t),\, s_j(t))$, the *fitness function* takes the following form:

$$\text{fitness}\,(\theta_j(t),\, s_j(t)) = obj\,(\theta_j(t),\, s_j(t)) - w_1 PL(1) - w_2 PL(2) - w_3 PL(3). \tag{25}$$

Depending on the fitness values, the parent chromosomes must then be chosen for the subsequent iterations so that GA operators can be applied to create the offspring. When no better solutions are discovered after a predetermined number of iterations, the iterations should be stopped.

3. Conclusion

In this paper, we consider various techniques for solving the problem of dynamical optimal allocation of resources in a three-sector economy. At the beginning, we presented analytical solution. Then, we used numerical optimization method and GAs to obtain numerical solution and to demonstrate rich opportunities of modern artificial intelligence techniques which could facilitate deriving the optimal solution. We got the following results. Firstly, an analytical solution for the problem of optimal allocation of resources in the three-sector balanced economy was obtained using the optimal control method. The analytical approach has some constraints, in particular, a special class of functions is chosen, and special assumptions are made regarding the model parameters. Secondly, a numerical experiment was carried out for two types of control functions and the optimal allocation of resources was obtained using a numerical optimization algorithm. In this case, an unbalanced economy was considered. We believe that the usage of GA can help to overcome a number of difficulties associated with the functional form of model's control variables, such as their non-linearity, dimensionality, and need to strictly adhere to the condition of material balance among economic sectors.

4. Future challenges

As already mentioned, both approaches implemented at this moment are fraught with difficulties. Solving fully the optimal control problem as $T \to \infty$ implies looking for a stable manifold of the stationary point. Numerical solution which postulates arbitrary parametrized nonlinear function as a trajectory of one or more control variables and then attempts to optimize over the parameters could miss the truly optimal trajectory, if the assumed class of function cannot approximate the optimal solution. We next plan to solve for the optimal trajectory in the optimal control problem as $T \to \infty$ using the method of backward integration [12]. At the same time, we want to utilize neural networks in order to find the best nonlinear function of the control variables on $[0, T]$ interval [13,14]. Given that neural networks are capable of fitting a very wide class of nonlinear functions and are not restricted to a few parameters, we hope that such a solution would resemble the true optimum even more. Then, by letting T in the neural network method above as large as possible, we plan to compare the optimal control solution which is valid at infinite horizon with the initial phase of the neural network solution. As turnpike theorems in economics show, the finite time problem's solution should resemble an infinite horizon solution while the model time is sufficiently far from the horizon T. Finally, to expand the range of tools and to compare the effectiveness of the techniques used, one can additionally consider applying evolutionary computation [15–18] for maximizing welfare as a function of parameters which index control functions θ_j and s_j in various classes of functions, and the optimizing over feasible indexing parameter regions.

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