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Sectorial Mertens and Mirsky formulae for imaginary quadratic number fields

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Abstract. We extend formulae of Mertens and Mirsky on the asymptotic behaviour of the usual Euler function to the Euler functions of principal rings of integers of imaginary quadratic number fields, giving versions in angular sectors and with congruences.

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1. Introduction. Let K be a number field with ring of integers $\mathscr{O}_K$, discriminant D_K, and Dedekind zeta function ζ_K. Let $\mathscr{I}_K^+$ be the semigroup of nonzero ideals of $\mathscr{O}_K$, let $\mathtt{N} : \mathscr{I}_K^+ \to \mathbb{N}$ be the norm, let $\mu_K : \mathscr{I}_K^+ \to \mathbb{Z}$ be the *Möbius function*, and let $\varphi_K : \mathscr{I}_K^+ \to \mathbb{N}$ be the Euler function of K given by[1]

$$\forall\, \mathfrak{a} \in \mathscr{I}_K^+, \quad \phi_K(\mathfrak{a}) = \operatorname{Card}\big((\mathscr{O}_K/\mathfrak{a})^\times\big) = \mathtt{N}(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\Big)$$
$$= \sum_{\mathfrak{b}|\mathfrak{a}} \mu_K(\mathfrak{b})\, \mathtt{N}(\mathfrak{a}\mathfrak{b}^{-1})\,. \tag{1}$$

For every $a \in \mathscr{O}_K \smallsetminus \{0\}$, we define $\mathtt{N}(a) = \mathtt{N}(a\mathscr{O}_K)$, $\mu_K(a) = \mu_K(a\mathscr{O}_K)$, and $\varphi_K(a) = \varphi_K(a\mathscr{O}_K)$. The functions $\mathrm{O}(\cdot)$ in this paper depend only on K. For every $\mathfrak{m} \in \mathscr{I}_K^+$, let

$$c_{\mathfrak{m}} = \mathtt{N}(\mathfrak{m}) \prod_{\mathfrak{p}|\mathfrak{m}} \Big(1 + \frac{1}{\mathtt{N}(\mathfrak{p})}\Big)\ . \tag{2}$$

The classical Mertens formula gives an asymptotic expansion on the average of $\varphi_{\mathbb{Q}}$, and has an extension to general number fields, see for instance Theorem

[1] See, for instance, [3, Sect. 2]; as usual, $\mathfrak{p}$ ranges over prime ideals in $\mathscr{I}_K^+$.

2.1. When K has a trivial class group and a finite unit group, every nonzero ideal $\mathfrak{a}$ is of the form $a\,\mathscr{O}_K$ with $|\mathscr{O}_K^\times|$ choices of $a \in \mathscr{O}_K \smallsetminus \{0\}$. The average of φ_K can then be rewritten as a sum over elements of $\mathscr{O}_K \smallsetminus \{0\}$ instead of $\mathscr{I}_K^+$. Recall that by Dirichlet's unit theorem, K has a finite unit group if and only if $K = \mathbb{Q}$ or K is imaginary quadratic. We assume in the remaining part of this introduction that K is imaginary quadratic and that $\mathscr{O}_K$ is principal. Fixing any embedding of K into $\mathbb{C}$, this allows us to state (and prove in Section 3) a uniform version in angular sectors and with congruences of the Mertens formula on the average of the Euler function. For all $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$, and $R \geq 0$, we consider the truncated angular sector

$$C(z, \theta, R) = \left\{\rho\, e^{it} z : t \in \left]-\frac{\theta}{2}, \frac{\theta}{2}\right],\ 0 < \rho \leq \frac{R}{|z|}\right\}. \tag{3}$$

Theorem 1.1. *For all $\mathfrak{m} \in \mathscr{I}_K^+$, $z \in \mathbb{C}^\times$, and $\theta \in \,]0, 2\pi]$, as $x \to +\infty$, we have*

$$\sum_{a \in \mathfrak{m} \cap C(z,\theta,x)} \varphi_K(a) = \frac{\theta}{2\sqrt{|D_K|}\,\zeta_K(2)\, c_{\mathfrak{m}}}\; x^4 + \mathrm{O}(x^3).$$

We also give a uniform asymptotic formula for the sum in angular sectors in $\mathbb{C}$ of angle θ of the products of two shifted Euler functions with congruences. When $K = \mathbb{Q}$ (the sectorial restriction is then meaningless), this formula is due to Mirsky [1, Thm. 9, Eq. (30)] without congruences, and to Fouvry [4, Appendix] with congruences. For simplicity, we give in this introduction a version without congruences and without an error term, see Section 4, Theorem 4.1 for the general statement.

Theorem 1.2. *For all $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$, and $k \in \mathscr{O}_K$, as $x \to +\infty$, we have*

$$\sum_{\substack{a \in \mathscr{O}_K \cap C(z,\theta,x)\\ a \neq -k}} \varphi_K(a)\,\varphi_K(a+k)$$

$$\sim \frac{\theta}{3\sqrt{|D_K|}} \prod_{\mathfrak{p}} \left(1 - \frac{2}{\mathtt{N}(\mathfrak{p})^2}\right) \prod_{\mathfrak{p}\,|\,k\mathscr{O}_K} \left(1 + \frac{1}{\mathtt{N}(\mathfrak{p})(\mathtt{N}(\mathfrak{p})^2 - 2)}\right) x^6.$$

Theorems 1.1 and 1.2 are used in [5] in order to study the correlations of pairs of complex logarithms of $\mathbb{Z}$-lattice points in the complex line at various scalings, when the weights are defined by the Euler function, proving the existence of pair correlation functions. We prove in op. cit. that at the linear scaling, the pair correlations exhibit level repulsion, as it sometimes occurs in statistical physics. A geometric application is given in op. cit. to the pair correlation of the lengths of common perpendicular geodesic arcs from the maximal Margulis cusp neighborhood to itself in the Bianchi orbifolds $\mathrm{PSL}_2(\mathscr{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$.

2. A Mertens formula with congruences for number fields. Let K be any number field, with degree n_K, number of real places r_1, number of complex places $2\,r_2$, regulator R_K, class number h_K, number of roots of unity ω_K, and let

$$\rho_K = \frac{2^{r_1}\,(2\pi)^{r_2}\,R_K\,h_K}{\omega_K\,\sqrt{|D_K|}}\,.$$

Let us give a version with congruences of the Mertens formula for number fields, see [3, Eq. (3)] when $\mathfrak{m} = \mathscr{O}_K$ (with a weaker error term). We provide a proof for lack of reference since its arguments will be useful for our next result. The next statement is valid when $n_K = 1$ up to replacing the error term by $\operatorname{O}(x\ln(2x))$, see [4, Lemma 4.2].

Theorem 2.1. *For every $\mathfrak{m} \in \mathscr{I}_K^+$, if $n_K \geq 2$, then as $x \to +\infty$, we have*

$$\sum_{\mathfrak{a}\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{a})\leq x,\ \mathfrak{m}|\mathfrak{a}} \varphi_K(\mathfrak{a}) = \frac{\rho_K}{2\,\zeta_K(2)\,c_{\mathfrak{m}}}\,x^2 + \operatorname{O}\left(x^{2-\frac{1}{n_K}}\right).$$

Proof. Recall (see for instance [2, Theorem 5]) that, as $y \to +\infty$, we have

$$\operatorname{Card}\{\mathfrak{a} \in \mathscr{I}_K^+ : \mathtt{N}(\mathfrak{a}) \leq y\} = \rho_K\, y + \operatorname{O}\left(y^{1-\frac{1}{n_K}}\right)\,. \tag{4}$$

By Abel's summation formula, as $y \to +\infty$, we have

$$\sum_{\mathfrak{d}\,\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{d})\leq y} \mathtt{N}(\mathfrak{d}) = \sum_{1\leq n\leq y} n\operatorname{Card}\{\mathfrak{d} \in \mathscr{I}_K^+ : \mathtt{N}(\mathfrak{d}) = n\} = \frac{\rho_K}{2}\,y^2 + \operatorname{O}\left(y^{2-\frac{1}{n_K}}\right)\,. \tag{5}$$

Furthermore, by Equation (4), we have $\operatorname{Card}\{\mathfrak{a} \in \mathscr{I}_K^+ : \mathtt{N}(\mathfrak{a}) = y\} = \operatorname{O}(y^{1-\frac{1}{n_K}})$. This formula implies, since $\mathtt{N}((\mathfrak{b},\mathfrak{m})) \leq \mathtt{N}(\mathfrak{m})$, that

$$\begin{aligned}\Big|\sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{b})\geq x} \mu_K(\mathfrak{b})\frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\Big| &= \operatorname{O}\left(\mathtt{N}(\mathfrak{m})\sum_{n\geq x}\frac{n^{1-\frac{1}{n_K}}}{n^2}\right)\\ &= \operatorname{O}\left(\mathtt{N}(\mathfrak{m})\;x^{-\frac{1}{n_K}}\right)\,.\end{aligned} \tag{6}$$

Let us denote by $S_{\mathfrak{m}}(x)$ the sum on the left hand side in the statement of Theorem 2.1. Note that by the Gauss lemma, for all $\mathfrak{m},\mathfrak{b},\mathfrak{c} \in \mathscr{I}_K^+$, we have $\mathfrak{m} \mid \mathfrak{b}\mathfrak{c}$ if and only if $\mathfrak{m}(\mathfrak{m},\mathfrak{b})^{-1} \mid \mathfrak{c}$. Then by Equation (1), by the change of variable $\mathfrak{c} = \mathfrak{m}(\mathfrak{m},\mathfrak{b})^{-1}\mathfrak{d}$, by the complete multiplicativity of the norm, by Equation (5) with $y = \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))x}{\mathtt{N}(\mathfrak{b})\,\mathtt{N}(\mathfrak{m})}$, since $\mathtt{N}((\mathfrak{b},\mathfrak{m})) \leq \mathtt{N}(\mathfrak{m})$, and by Equation (6), we have

$$\begin{aligned}S_{\mathfrak{m}}(x) &= \sum_{\mathfrak{a}\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{a})\leq x,\ \mathfrak{m}|\mathfrak{a}}\ \ \sum_{\mathfrak{b},\mathfrak{c}\,\in\mathscr{I}_K^+ :\ \mathfrak{b}\mathfrak{c}=\mathfrak{a}} \mu_K(\mathfrak{b})\ \mathtt{N}(\mathfrak{c})\\
&= \sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{b})\leq x} \mu_K(\mathfrak{b}) \sum_{\mathfrak{c}\,\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{c})\leq \frac{x}{\mathtt{N}(\mathfrak{b})},\ \mathfrak{m}|\mathfrak{b}\mathfrak{c}} \mathtt{N}(\mathfrak{c})\\
&= \sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{b})\leq x} \mu_K(\mathfrak{b}) \sum_{\mathfrak{d}\,\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{d})\leq \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))x}{\mathtt{N}(\mathfrak{b})\,\mathtt{N}(\mathfrak{m})}} \frac{\mathtt{N}(\mathfrak{m})}{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}\ \mathtt{N}(\mathfrak{d})\\
&= \left(\sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\ \mathtt{N}(\mathfrak{b})\leq x} \mu_K(\mathfrak{b})\frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\right)\frac{\rho_K}{2\ \mathtt{N}(\mathfrak{m})}\,x^2 + \operatorname{O}\left(x^{2-\frac{1}{n_K}}\right)\\
&= \left(\sum_{\mathfrak{b}\in\mathscr{I}_K^+} \mu_K(\mathfrak{b})\frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\right)\frac{\rho_K}{2\ \mathtt{N}(\mathfrak{m})}\,x^2 + \operatorname{O}\left(x^{2-\frac{1}{n_K}}\right)\,.\end{aligned} \tag{7}$$

By decomposing a nonzero integral ideal $\mathfrak{b}$ into powers of prime ideals, by the definition of the Möbius function, and by the Euler product formula for the Dedekind zeta function, we have

$$\begin{aligned}\sum_{\mathfrak{b}\in\mathscr{I}_K^+}\mu_K(\mathfrak{b})\frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2} &= \prod_{\mathfrak{p}\nmid\mathfrak{m}}\Big(1-\frac{1}{\mathtt{N}(\mathfrak{p})^2}\Big)\prod_{\mathfrak{p}\mid\mathfrak{m}}\Big(1-\frac{1}{\mathtt{N}(\mathfrak{p})}\Big)\\ &= \frac{1}{\zeta_K(2)}\prod_{\mathfrak{p}\mid\mathfrak{m}}\frac{\mathtt{N}(\mathfrak{p})}{1+\mathtt{N}(\mathfrak{p})}\,.\end{aligned}$$

Equations (7) and (2) hence imply Theorem 2.1. □

3. A sectorial Mertens formula. Assume in the remaining part of this paper that K is imaginary quadratic ($r_1=0$, $r_2=1$), seen as a subfield of $\mathbb{C}$, and that $\mathscr{O}_K$ is principal ($h_K=1$) or equivalently factorial (UFD).

Given a $\mathbb{Z}$-*lattice* $\vec{\Lambda}$ in the Euclidean space $\mathbb{C}$ (that is, a discrete (free abelian) subgroup of $(\mathbb{C},+)$ generating $\mathbb{C}$ as an $\mathbb{R}$-vector space), we denote by $\mathscr{F}_{\vec{\Lambda}}$ any fundamental parallelogram for $\vec{\Lambda}$ with smallest possible diameter, and by $\operatorname{covol}_{\vec{\Lambda}}=\operatorname{Vol}(\mathbb{C}/\vec{\Lambda})$ and $\operatorname{diam}_{\vec{\Lambda}}$ the area and diameter of $\mathscr{F}_{\vec{\Lambda}}$. Note that every element $\mathfrak{m}\in\mathscr{I}_K^+$ is a $\mathbb{Z}$-lattice in $\mathbb{C}$ with

$$\begin{aligned}\operatorname{covol}_{\mathfrak{m}} &= \mathtt{N}(\mathfrak{m})\operatorname{covol}_{\mathscr{O}_K}=\tfrac{\mathtt{N}(\mathfrak{m})\sqrt{|D_K|}}{2}\quad\text{and}\\ \operatorname{diam}_{\mathfrak{m}} &= \sqrt{\mathtt{N}(\mathfrak{m})}\,\operatorname{diam}_{\mathscr{O}_K}=\operatorname{O}\Big(\sqrt{\mathtt{N}(\mathfrak{m})}\Big)\end{aligned}\tag{8}$$

since $\mathfrak{m}$, being principal, is the image of $\mathscr{O}_K$ under a similitude of ratio $\sqrt{\mathtt{N}(\mathfrak{m})}$.

With the notation of Equation (3), note that for every $z'\in\mathbb{C}^\times$, we have

$$z'C(z,\theta,R)=C(zz',\theta,R\,|z'|)\,.\tag{9}$$

Proof of Theorem 1.1. Let $z\in\mathbb{C}^\times$, $\theta\in\,]0,2\pi]$, and $y\geq 1$. By a sector version of the Gauss counting argument,[2] since $\operatorname{Area}(C(z,\theta,y))=\frac{\theta}{2}\,y^2$ and by Formula (8), we have

$$\begin{aligned}\operatorname{Card}\big(\mathscr{O}_K\cap C(z,\theta,y)\big) &= \frac{\operatorname{Area}(C(z,\theta,y))}{\operatorname{covol}_{\mathscr{O}_K}}+\operatorname{O}\Big(\frac{\operatorname{diam}_{\mathscr{O}_K}(y+\operatorname{diam}_{\mathscr{O}_K})}{\operatorname{covol}_{\mathscr{O}_K}}\Big)\\ &= \frac{\theta}{\sqrt{|D_K|}}\,y^2+\operatorname{O}(y)\,.\end{aligned}$$

[2]This proof takes into account the varying lattices as needed in the proof of Lemma 4.5. For every $\epsilon>0$, and $A\subset\mathbb{C}$, let $\mathscr{N}_\epsilon A$ be the closed ϵ-neighbourhood of A in $\mathbb{C}$. Let $\vec{\Lambda}$ be a $\mathbb{Z}$-lattice in $\mathbb{C}$, $\lambda_0\in\mathbb{C}$, $\Lambda=\lambda_0+\vec{\Lambda}$, and $\delta=\operatorname{diam}_{\vec{\Lambda}}$. Let $C=C(z,\theta,y)$ and $\mathscr{A}_\delta=\operatorname{Area}(\mathscr{N}_\delta(\partial C))$. If $y\leq\delta$, then $\mathscr{A}_\delta=\operatorname{O}(\delta^2)$ since C is contained in a disc of radius δ. If $y\geq\delta$, then ∂C is contained in the union of a circle of radius y and two segments of length y, hence $\mathscr{A}_\delta\leq\pi(y+\delta)^2-\pi(y-\delta)^2+2(y+2\delta)(2\delta)=\operatorname{O}(\delta y)$. Therefore, if $A_{z,\theta,y}=\Lambda\cap C$ and $B_{z,\theta,y}=\bigcup_{a\in A_{z,\theta,y}}a+\mathscr{F}_{\vec{\Lambda}}$, then $\operatorname{Area}(B_{z,\theta,y})=\operatorname{Card}A_{z,\theta,y}\operatorname{Area}\mathscr{F}_{\vec{\Lambda}}$ and $\big|\operatorname{Area}(B_{z,\theta,y})-\operatorname{Area}C\big|\leq\mathscr{A}_\delta=\operatorname{O}(\delta(y+\delta))$. Thus, $\operatorname{Card}A_{z,\theta,y}=\frac{\operatorname{Area}C}{\operatorname{covol}_{\vec{\Lambda}}}+\operatorname{O}\Big(\frac{\delta\,(y+\delta)}{\operatorname{covol}_{\vec{\Lambda}}}\Big)$ for a function $\operatorname{O}(\cdot)$ that is uniform in $\vec{\Lambda}$ and λ_0.

Since the map $z' \mapsto |z'|^2 = \mathtt{N}(z')$ takes only integral values on $\mathscr{O}_K$, by Abel's summation formula, as $y \to +\infty$, we have

$$\sum_{d \in \mathscr{O}_K \cap C(z,\theta,y)} |d|^2 = \sum_{1 \le n \le y^2} n \,\mathrm{Card}\{d \in \mathscr{O}_K \cap C(z,\theta,y) : |d|^2 = n\}$$
$$= \frac{\theta}{2\sqrt{|D_K|}}\, y^4 + \mathrm{O}(y^3)\,. \tag{10}$$

For all $x \ge 1$ and $\mathfrak{b} \in \mathscr{I}_K^+$, let us fix $b, m, (b,m) \in \mathscr{O}_K \smallsetminus \{0\}$ such that $\mathfrak{b} = b\mathscr{O}_K$, $\mathfrak{m} = m\mathscr{O}_K$, and $(\mathfrak{b},\mathfrak{m}) = (b,m)\mathscr{O}_K$. Since for every $c \in \mathscr{O}_K \smallsetminus \{0\}$, we have $m \mid bc$ if and only if $\frac{m}{(b,m)} \mid c$, by the change of variable $c = \frac{m}{(b,m)}\, d$, by Equation (9) and by Equation (10) applied with $y = \frac{x|(b,m)|}{|m|\,|b|}$, if

$$S_{\mathfrak{b}} = \sum_{\mathfrak{c} \in \mathscr{I}_K^+,\, a \in \mathfrak{m} \cap C(z,\theta,x):\ \mathfrak{b}\mathfrak{c} = a\mathscr{O}_K} \mathtt{N}(\mathfrak{c}),$$

we have

$$S_{\mathfrak{b}} = \sum_{c \,\in\, \mathscr{O}_K \smallsetminus \{0\},\, a \in \mathfrak{m} \cap C(z,\theta,x)\,:\, bc = a} |c|^2 = \sum_{c \,\in\, \mathscr{O}_K \smallsetminus \{0\}\,:\, bc \,\in\, C(z,\theta,x),\ m \,|\, bc} |c|^2$$
$$= \sum_{d \,\in\, \mathscr{O}_K \smallsetminus \{0\}\,:\, d \,\in\, C\left(\frac{z(b,m)}{m\,b},\,\theta,\,\frac{x|(b,m)|}{|m|\,|b|}\right)} \frac{\mathtt{N}(\mathfrak{m})}{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}\, |d|^2$$
$$= \frac{\theta\, \mathtt{N}((\mathfrak{b},\mathfrak{m}))}{2\sqrt{|D_K|}\, \mathtt{N}(\mathfrak{m})\, \mathtt{N}(\mathfrak{b})^2}\, x^4 + \mathrm{O}\left(\frac{x^3}{\mathtt{N}(\mathfrak{b})^{3/2}}\right)\,.$$

Let us denote by $S_{\mathfrak{m},z,\theta}(x)$ the sum on the left hand side in the statement of Theorem 1.1. Then by Equation (1), we have

$$S_{\mathfrak{m},z,\theta}(x) = \sum_{a \in \mathfrak{m} \cap C(z,\theta,x)} \varphi_K(a\mathscr{O}_K) = \sum_{a \in \mathfrak{m} \cap C(z,\theta,x)} \ \sum_{\mathfrak{b},\mathfrak{c} \,\in \mathscr{I}_K^+:\ \mathfrak{b}\mathfrak{c} = a\mathscr{O}_K} \mu_K(\mathfrak{b}) \mathtt{N}(\mathfrak{c})$$
$$= \sum_{\mathfrak{b} \in \mathscr{I}_K^+:\ \mathtt{N}(\mathfrak{b}) \le x^2} \mu_K(\mathfrak{b})\ \ S_{\mathfrak{b}}$$
$$= \left(\sum_{\mathfrak{b} \in \mathscr{I}_K^+:\ \mathtt{N}(\mathfrak{b}) \le x^2} \mu_K(\mathfrak{b})\, \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\right) \frac{\theta}{2\sqrt{|D_K|}\, \mathtt{N}(\mathfrak{m})}\, x^4 + \mathrm{O}(x^3)\,.$$

The proof then proceeds exactly as in the proof of Theorem 2.1. □

4. A sectorial Mirsky formula. We keep assuming that K is imaginary quadratic with $\mathscr{O}_K$ principal. We now give a uniform asymptotic formula for the sum in angular sectors of the products of shifted Euler functions with congruences. For all $\mathfrak{m} \in \mathscr{I}_K^+$, $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$, $k \in \mathscr{O}_K$, and $x \ge 1$, let

$$S_{\mathfrak{m},z,\theta,k}(x) = \sum_{a \in \mathfrak{m} \cap C(z,\theta,x)\,:\, a \ne -k} \varphi_K(a)\, \varphi_K(a+k)\,. \tag{11}$$

Theorem 4.1. *There exists a constant $C_K > 0$ such that for all $\mathfrak{m} \in \mathscr{I}_K^+$ and $k \in \mathscr{O}_K$, there exists $c_{\mathfrak{m},k} \in \,]0,1[$ such that for all $z \in \mathbb{C}^\times$, $\theta \in \,]0,2\pi]$, and $x \geq 1$, we have*

$$\Big|\, S_{\mathfrak{m},z,\theta,k}(x) - \frac{\theta\, c_{\mathfrak{m},k}}{3\sqrt{|D_K|}}\, x^6 \,\Big| \leq C_K \Big((1+\sqrt{N(k)}\,)\, x^5 + N(k)\, x^4 + N(k) \ln(N(k))\, x^2 \ln(2x) \Big).$$

We will prove Theorem 4.1 at the end of this section after giving a number of lemmas required for the proof. We fix $k \in \mathscr{O}_K$ and $\mathfrak{m} = m\mathscr{O}_K \in \mathscr{I}_K^+$, and we define $\mathfrak{h} = k\mathscr{O}_K$, which is a possibly zero integral ideal. We start by giving the first definition and a simpler formula for the constant $c_{\mathfrak{m},k}$ that appears in the statement of Theorem 4.1. We define

$$c_{\mathfrak{m},k} = \sum_{\substack{\mathfrak{b},\mathfrak{c}\,\in \mathscr{I}_K^+ \\ (\mathfrak{b},\mathfrak{c})\,|\,\mathfrak{h},\ (\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\mathfrak{m}(\mathfrak{b},\mathfrak{c}))\,|\,\mathfrak{h}\mathfrak{b}}} \mu_K(\mathfrak{b})\,\mu_K(\mathfrak{c})\, \frac{\mathtt{N}\big((\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\, \mathfrak{m}(\mathfrak{b},\mathfrak{c}))\big)}{\mathtt{N}(\mathfrak{b})^2\,\mathtt{N}(\mathfrak{c})^2\,\mathtt{N}(\mathfrak{m})} \tag{12}$$

and $c'_{\mathfrak{m}} = \inf_{k\in\mathscr{O}_K} c_{\mathfrak{m},k}$.

Lemma 4.2. *The series in Equation (12) defining $c_{\mathfrak{m},k}$ converges absolutely. We have $c_{\mathfrak{m},k} < 1$ and $c'_{\mathfrak{m}} > 0$. Furthermore, we have*

$$c_{\mathfrak{m},k} = \frac{1}{\mathtt{N}(\mathfrak{m})} \prod_{\substack{\mathfrak{p} \\ (\mathfrak{p},\mathfrak{m})\,|\,\mathfrak{h}}} \Big(1 - \frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\Big) \prod_{\mathfrak{p}} \Big(1 - \frac{\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p})\,\kappa'_{\mathfrak{h}}(\mathfrak{p})\,\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\Big), \tag{13}$$

where

$$\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p}) = \begin{cases} \Big(1 - \frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\Big)^{-1} & \text{if } (\mathfrak{p},\mathfrak{m})\,|\,\mathfrak{h}, \\ 1 & \text{otherwise}, \end{cases} \quad \text{and } \kappa'_{\mathfrak{h}}(\mathfrak{p}) = \begin{cases} 1 - \frac{1}{\mathtt{N}(\mathfrak{p})} & \text{if } \mathfrak{p}\,|\,\mathfrak{h}, \\ 1 & \text{otherwise}. \end{cases} \tag{14}$$

In the special case $\mathfrak{m} = \mathscr{O}_K$, Equation (13) becomes

$$\begin{aligned} c_{\mathscr{O}_K,k} &= \prod_{\mathfrak{p}} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}\Big) \prod_{\mathfrak{p}\,|\,\mathfrak{h}} \left(1 - \frac{\Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}\Big)^{-1}\Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\Big)}{\mathtt{N}(\mathfrak{p})^2}\right) \\ &\qquad \prod_{\mathfrak{p}\,\nmid\,\mathfrak{h}} \left(1 - \frac{\Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}\Big)^{-1}}{\mathtt{N}(\mathfrak{p})^2}\right) \\ &= \prod_{\mathfrak{p}} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}\Big) \prod_{\mathfrak{p}\,|\,\mathfrak{h}} \Big(1 - \frac{\mathtt{N}(\mathfrak{p}) - 1}{\mathtt{N}(\mathfrak{p})(\mathtt{N}(\mathfrak{p})^2 - 1)}\Big) \\ &\qquad \times \prod_{\mathfrak{p}} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2 - 1}\Big) \prod_{\mathfrak{p}\,|\,\mathfrak{h}} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2 - 1}\Big)^{-1} \\ &= \prod_{\mathfrak{p}} \Big(1 - \frac{2}{\mathtt{N}(\mathfrak{p})^2}\Big) \prod_{\mathfrak{p}\,|\,\mathfrak{h}} \Big(1 + \frac{1}{\mathtt{N}(\mathfrak{p})(\mathtt{N}(\mathfrak{p})^2 - 2)}\Big). \end{aligned} \tag{15}$$

Theorem 1.2 in the introduction follows from Theorem 4.1 and the above computation.

Proof. Let us prove that uniformly in $x \geq 1$, we have

$$\sum_{\substack{\mathfrak{b},\mathfrak{c}\,\in \mathscr{I}_K^+\,:\,\max(\mathtt{N}(\mathfrak{b}),\mathtt{N}(\mathfrak{c}))\geq x,\\ (\mathfrak{b},\mathfrak{c})\,|\,\mathfrak{h},\ (\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\mathfrak{m}(\mathfrak{b},\mathfrak{c}))\,|\,\mathfrak{h}\mathfrak{b}}} \frac{\mathtt{N}((\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\ \mathfrak{m}(\mathfrak{b},\mathfrak{c})))}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2\ \mathtt{N}(\mathfrak{m})} = \mathrm{O}\left(\frac{1}{\sqrt{x}}\right) . \tag{16}$$

This implies, by taking $x = 1$, that the first claim of Lemma 4.2 is satisfied since the Möbius function has values in $\{0, \pm 1\}$. Let us denote by $Z_{\mathfrak{m},\mathfrak{h}}(x)$ the above sum. Since $\mathtt{N}((\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\ \mathfrak{m}(\mathfrak{b},\mathfrak{c}))) \leq \mathtt{N}(\mathfrak{m}(\mathfrak{b},\mathfrak{c}))$, Equation (16) follows from the inequalities

$$\begin{aligned}
Z_{\mathfrak{m},\mathfrak{h}}(x) &\leq \sum_{\mathfrak{b},\mathfrak{c}\,\in \mathscr{I}_K^+\,:\,\max(\mathtt{N}(\mathfrak{b}),\mathtt{N}(\mathfrak{c}))\geq x} \frac{\mathtt{N}((\mathfrak{b},\mathfrak{c}))}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2}\\
&\leq \sum_{\mathfrak{b},\mathfrak{c}\,\in \mathscr{I}_K^+\,:\,\mathtt{N}(\mathfrak{b})\geq x} \frac{\mathtt{N}((\mathfrak{b},\mathfrak{c}))}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2} + \sum_{\mathfrak{b},\mathfrak{c}\,\in \mathscr{I}_K^+\,:\,\mathtt{N}(\mathfrak{c})\geq x} \frac{\mathtt{N}((\mathfrak{b},\mathfrak{c}))}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2}\\
&= 2 \sum_{\mathfrak{b},\mathfrak{c}\,\in \mathscr{I}_K^+\,:\,\mathtt{N}(\mathfrak{b})\geq x} \frac{\mathtt{N}((\mathfrak{b},\mathfrak{c}))}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2} \leq 2 \sum_{\substack{\mathfrak{a},\mathfrak{b}',\mathfrak{c}'\in \mathscr{I}_K^+\\ \mathtt{N}(\mathfrak{a})\,\mathtt{N}(\mathfrak{b}')\geq x}} \frac{\mathtt{N}(\mathfrak{a})}{\mathtt{N}(\mathfrak{a}\mathfrak{b}')^2\ \mathtt{N}(\mathfrak{a}\mathfrak{c}')^2}\\
&= 2 \sum_{\mathfrak{c}'\in \mathscr{I}_K^+} \frac{1}{\mathtt{N}(\mathfrak{c}')^2} \sum_{\mathfrak{a}\in \mathscr{I}_K^+} \frac{1}{\mathtt{N}(\mathfrak{a})^{5/2}} \sum_{\substack{\mathfrak{b}'\in \mathscr{I}_K^+\\ \mathtt{N}(\mathfrak{a})\,\mathtt{N}(\mathfrak{b}')\geq x}} \frac{1}{\mathtt{N}(\mathfrak{b}')^{3/2}\ (\mathtt{N}(\mathfrak{a})\,\mathtt{N}(\mathfrak{b}'))^{1/2}}\\
&\leq 2\ \zeta_K(2)\,\zeta_K\left(\frac{5}{2}\right)\ \zeta_K\left(\frac{3}{2}\right)\ \frac{1}{\sqrt{x}} .
\end{aligned}$$

The proof of Equation (13) that we now give is similar to Fouvry's proof of Equation (21) in [4, Appendix].

For every $\mathfrak{b} \in \mathscr{I}_K^+$, let $\chi_{\mathfrak{b}} : \mathscr{I}_K^+ \to \{0, 1\}$ be the characteristic function of the set of elements $\mathfrak{c} \in \mathscr{I}_K^+$ such that $(\mathfrak{c}, \mathfrak{b}) \mid \mathfrak{h}$. Let us define a map $\psi_{\mathfrak{b}} : \mathscr{I}_K^+ \to \mathscr{I}_K^+$ by

$$\psi_{\mathfrak{b}} : \mathfrak{c} \mapsto \left(\mathfrak{c},\ \frac{\mathfrak{m}}{(\mathfrak{b},\mathfrak{m})}\,(\mathfrak{b},\mathfrak{c})\right) . \tag{17}$$

Note that the assertion $(\mathfrak{c}(\mathfrak{b},\mathfrak{m}), \mathfrak{m}(\mathfrak{b},\mathfrak{c})) \mid \mathfrak{b}\,\mathfrak{h}$ is equivalent to $\psi_{\mathfrak{b}}(\mathfrak{c}) \mid \frac{\mathfrak{b}}{(\mathfrak{b},\mathfrak{m})}\,\mathfrak{h}$. For every $\mathfrak{b} \in \mathscr{I}_K^+$, let $\chi^*_{\mathfrak{b}} : \mathscr{I}_K^+ \to \{0, 1\}$ be the characteristic function of the set of elements $\mathfrak{c} \in \mathscr{I}_K^+$ such that the above divisibility assertion is satisfied. Let us finally define a map $C^* : \mathscr{I}_K^+ \to \mathbb{R}$ (which depends on $\mathfrak{m}$ and $\mathfrak{h}$) by

$$C^* : \mathfrak{b} \mapsto \sum_{\mathfrak{c}\in \mathscr{I}_K^+} \frac{\mu_K(\mathfrak{c})}{\mathtt{N}(\mathfrak{c})^2}\ \chi_{\mathfrak{b}}(\mathfrak{c})\ \chi^*_{\mathfrak{b}}(\mathfrak{c})\ \mathtt{N}(\psi_{\mathfrak{b}}(\mathfrak{c})) . \tag{18}$$

By the absolute convergence property, Equation (12) then becomes

$$c_{\mathfrak{m},k} = \frac{1}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b}\in \mathscr{I}_K^+} \frac{\mu_K(\mathfrak{b})}{\mathtt{N}(\mathfrak{b})^2}\ \mathtt{N}((\mathfrak{b},\mathfrak{m}))\ C^*(\mathfrak{b}) . \tag{19}$$

In order to transform the series $C^*(\mathfrak{b})$ defined by Formula (18) into an Eulerian product and in order to analyse it, we will use the following two lemmas.

Lemma 4.3. *For every* $\mathfrak{b} \in \mathscr{I}_K^+$*, the maps* $\chi_\mathfrak{b}$*,* $\chi^*_\mathfrak{b}$*, and* $\psi_\mathfrak{b}$ *on* $\mathscr{I}_K^+$ *are multiplicative.*

Proof. We have $\psi_\mathfrak{b}(\mathscr{O}_K) = \mathscr{O}_K$ and $\chi_\mathfrak{b}(\mathscr{O}_K) = \chi^*_\mathfrak{b}(\mathscr{O}_K) = 1$. Let $I, J \in \mathscr{I}_K^+$ be coprime.

The equality $(IJ, \mathfrak{b}) = (I, \mathfrak{b})(J, \mathfrak{b})$ and the fact that $(I, \mathfrak{b})$ and $(J, \mathfrak{b})$ are coprime imply that $\chi_\mathfrak{b}(IJ) = \chi_\mathfrak{b}(I)\chi_\mathfrak{b}(J)$.

In order to prove the multiplicativity of the map $\psi_\mathfrak{b}$, we write

$$\psi_\mathfrak{b}(IJ) = \Big(IJ, \frac{\mathfrak{m}}{(\mathfrak{b}, \mathfrak{m})}\,(\mathfrak{b}, IJ)\Big) = \Big(I, \frac{\mathfrak{m}}{(\mathfrak{b}, \mathfrak{m})}\,(I, \mathfrak{b})(J, \mathfrak{b})\Big)\Big(J, \frac{\mathfrak{m}}{(\mathfrak{b}, \mathfrak{m})}\,(I, \mathfrak{b})(J, \mathfrak{b})\Big).$$

Since I is coprime to $(J, \mathfrak{b})$ and since J is coprime to $(I, \mathfrak{b})$, we obtain as wanted the equality $\psi_\mathfrak{b}(IJ) = \psi_\mathfrak{b}(I)\,\psi_\mathfrak{b}(J)$.

Finally, the multiplicativity property $\chi^*_\mathfrak{b}(IJ) = \chi^*_\mathfrak{b}(I)\chi^*_\mathfrak{b}(J)$ of the function $\chi^*_\mathfrak{b}$ is a consequence of the multiplicativity of the map $\psi_\mathfrak{b}$ and of the fact that $\psi_\mathfrak{b}(I)$ and $\psi_\mathfrak{b}(J)$ are coprime. □

Lemma 4.4. *For every prime ideal* $\mathfrak{p}$ *and every* $\mathfrak{b} \in \mathscr{I}_K^+$*, we have*

$$\psi_\mathfrak{b}(\mathfrak{p}) = \begin{cases} \mathfrak{p} \textit{ if } \mathfrak{p} \mid \mathfrak{b}, \\ (\mathfrak{p}, \mathfrak{m}) \textit{ otherwise}, \end{cases} \quad \textit{and} \quad \chi_\mathfrak{b}(\mathfrak{p})\;\chi^*_\mathfrak{b}(\mathfrak{p}) = 1 \Leftrightarrow \begin{cases} \mathfrak{p} \mid (\mathfrak{b}, \mathfrak{h}) \\ \textit{or} \\ \mathfrak{p} \nmid \mathfrak{b} \textit{ and } (\mathfrak{p}, \mathfrak{m}) \mid \mathfrak{h}. \end{cases}$$

Proof. The first formula follows from the definition of $\psi_\mathfrak{b}(\mathfrak{p})$ (see Formula (17)) by considering the three cases

- $\mathfrak{p} \mid \mathfrak{b}$,
- $\mathfrak{p} \nmid \mathfrak{b}$ and $\mathfrak{p} \mid \mathfrak{m}$, and
- $\mathfrak{p} \nmid \mathfrak{b}$ and $\mathfrak{p} \nmid \mathfrak{m}$.

The second formula follows from the first one, from the definitions of $\chi_\mathfrak{b}(\mathfrak{p})$ and $\chi^*_\mathfrak{b}(\mathfrak{p})$, and from the fact that $\chi_\mathfrak{b}(\mathfrak{p})\;\chi^*_\mathfrak{b}(\mathfrak{p}) = 1$ if and only if $\chi_\mathfrak{b}(\mathfrak{p}) = \chi^*_\mathfrak{b}(\mathfrak{p}) = 1$, by considering the two cases

- $\mathfrak{p} \mid \mathfrak{b}$ and
- $\mathfrak{p} \nmid \mathfrak{b}$. □

The arithmetic function $\mathfrak{c} \mapsto \mu_K(\mathfrak{c})\,\chi_\mathfrak{b}(\mathfrak{c})\;\chi^*_\mathfrak{b}(\mathfrak{c})\;\mathtt{N}(\psi_\mathfrak{b}(\mathfrak{c}))$ being multiplicative by Lemma 4.3 and the complete multiplicativity of the norm, and vanishing on the nontrivial powers of primes, the series defining $C^*(\mathfrak{b})$ in Formula (18) may be written as an Eulerian product

$$C^*(\mathfrak{b}) = \prod_{\mathfrak{p}} \Big(1 - \frac{\chi_\mathfrak{b}(\mathfrak{p})\;\chi^*_\mathfrak{b}(\mathfrak{p})\;\;\mathtt{N}(\psi_\mathfrak{b}(\mathfrak{p}))}{\mathtt{N}(\mathfrak{p})^2}\Big) = \prod_{\substack{\mathfrak{p} \\ \chi_\mathfrak{b}(\mathfrak{p})\;\chi^*_\mathfrak{b}(\mathfrak{p})=1}} \Big(1 - \frac{\mathtt{N}(\psi_\mathfrak{b}(\mathfrak{p}))}{\mathtt{N}(\mathfrak{p})^2}\Big)\,. \tag{20}$$

By Equations (19) and (20), and by Lemma 4.4, we have

$$c_{\mathfrak{m},k} = \frac{1}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b}\in\mathscr{I}_K^+} \frac{\mu_K(\mathfrak{b})}{\mathtt{N}(\mathfrak{b})^2}\, \mathtt{N}((\mathfrak{b},\mathfrak{m})) \prod_{\mathfrak{p}\nmid\mathfrak{b},\ (\mathfrak{p},\mathfrak{m})\,|\,\mathfrak{h}} \left(1-\frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\right) \prod_{\mathfrak{p}\,|\,(\mathfrak{b},\mathfrak{h})} \left(1-\frac{1}{\mathtt{N}(\mathfrak{p})}\right).$$

Let us define $\Gamma_{\mathfrak{m},\mathfrak{h}} = \prod_{\mathfrak{p}\,:\,(\mathfrak{p},\mathfrak{m})\,|\,\mathfrak{h}} \left(1-\frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\right)$, so that

$$c_{\mathfrak{m},k} = \frac{\Gamma_{\mathfrak{m},\mathfrak{h}}}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b}\in\mathscr{I}_K^+} \frac{1}{\mathtt{N}(\mathfrak{b})^2} \times \mu_K(\mathfrak{b})\,\mathtt{N}((\mathfrak{b},\mathfrak{m})) \prod_{\substack{\mathfrak{p} \\ \mathfrak{p}\,|\,\mathfrak{b},\ (\mathfrak{p},\mathfrak{m})\,|\,\mathfrak{h}}} \left(1-\frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\right)^{-1} \prod_{\substack{\mathfrak{p} \\ \mathfrak{p}\,|\,(\mathfrak{b},\mathfrak{h})}} \left(1-\frac{1}{\mathtt{N}(\mathfrak{p})}\right).$$

This equation writes $c_{\mathfrak{m},k}$ as a series $\frac{\Gamma_{\mathfrak{m},\mathfrak{h}}}{\mathtt{N}(\mathfrak{m})}\sum_{\mathfrak{b}\in\mathscr{I}_K^+}\frac{f(\mathfrak{b})}{N(\mathfrak{b})^2}$ where $f:\mathscr{I}_K^+\to\mathbb{R}$ is a multiplicative function, which vanishes on the nontrivial powers of prime ideals. By Eulerian product, we have therefore proved Equation (13).

Let us now prove that $0\le c_{\mathfrak{m},k}<1$. Note that for every prime ideal $\mathfrak{p}$, we have

$$1\le\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p})\le 2 \quad\text{and}\quad \frac{1}{2}\le\kappa'_{\mathfrak{h}}(\mathfrak{p})\le 1\,. \tag{21}$$

In particular, all the factors of the two products over $\mathfrak{p}$ in Equation (13) belong to $[0,1[$, hence $0\le c_{\mathfrak{m},k}<\frac{1}{\mathtt{N}(\mathfrak{m})}\le 1$.

Let us finally prove that $c'_{\mathfrak{m}}>0$. For every prime ideal $\mathfrak{p}$, let us define $w_{\mathfrak{p}} = \frac{\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p})\,\kappa'_{\mathfrak{h}}(\mathfrak{p})\,\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}$. By Formula (14), if $\mathtt{N}(\mathfrak{p})=2$, we have

$$w_{\mathfrak{p}} = \begin{cases} 1/2 & \text{if } \mathfrak{p}\mid\mathfrak{h} \text{ and } \mathfrak{p}\mid\mathfrak{m},\\ 1/6 & \text{if } \mathfrak{p}\mid\mathfrak{h} \text{ and } \mathfrak{p}\nmid\mathfrak{m},\\ 1/2 & \text{if } \mathfrak{p}\nmid\mathfrak{h} \text{ and } \mathfrak{p}\mid\mathfrak{m},\\ 1/3 & \text{if } \mathfrak{p}\nmid\mathfrak{h} \text{ and } \mathfrak{p}\nmid\mathfrak{m}.\end{cases}$$

In particular, $1-w_{\mathfrak{p}}\ge\frac{1}{2}$ if $\mathtt{N}(\mathfrak{p})=2$. By the inequalities (21) and Equation (13), we have

$$c_{\mathfrak{m},k}\ge\frac{1}{\mathtt{N}(\mathfrak{m})}\prod_{\mathfrak{p}}\left(1-\frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\right)\prod_{\mathfrak{p}:\,\mathtt{N}(\mathfrak{p})\ge 3}\left(1-\frac{2\,\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\right)\prod_{\mathfrak{p}:\,\mathtt{N}(\mathfrak{p})=2}\frac{1}{2}.$$

The term on the right hand side is a positive constant independent of k. Therefore, we have $c'_{\mathfrak{m}}=\inf_{k\in\mathscr{O}_K}c_{\mathfrak{m},k}>0$. This concludes the proof of Lemma 4.2. □

Now that we understand the constant $c_{\mathfrak{m},k}$, we continue towards the proof of Theorem 4.1 by giving an asymptotic formula (uniform in k) for the sum

$$\widetilde{S}(x)=\sum_{a\in\mathfrak{m}\cap C(z,\theta,x)\,:\,a\neq -k}\frac{\varphi_K(a)}{\mathtt{N}(a)}\,\frac{\varphi_K(a+k)}{\mathtt{N}(a+k)}\,. \tag{22}$$

Lemma 4.5. *Uniformly in* $\mathfrak{m} \in \mathscr{I}_K^+$, $k \in \mathscr{O}_K$, $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$, *and* $x \geq 1$, *we have*

$$\widetilde{S}(x) = \frac{\theta\, c_{\mathfrak{m},k}}{\sqrt{|D_K|}}\, x^2 + O(x) + O\big(\ln(2x)\ln(1+\mathtt{N}(k))\big)\ . \tag{23}$$

Proof. For all nonzero elements a and b in the factorial ring $\mathscr{O}_K$, we denote by (a,b) any fixed choice of gcd of a and b, and by $[a,b]$ any fixed choice of lcm of a and b.

By Equation (1) and as the norm $\mathtt{N}$ is completely multiplicative, for every $a \in \mathscr{O}_K \smallsetminus \{0\}$, we have

$$\frac{\varphi_K(a)}{\mathtt{N}(a)} = \frac{1}{|\mathscr{O}_K^\times|} \sum_{b \in \mathscr{O}_K \smallsetminus \{0\}\,:\, b\,|\,a} \frac{\mu_K(b)}{\mathtt{N}(b)}.$$

Let $x \geq 1$. Applying twice this equality, since $\mathtt{N}(b) \leq \mathtt{N}(a)$ when $b \mid a$ and $|c| \leq |a| + |k|$ when $c \mid a + k$, we have by Fubini's theorem

$$\begin{aligned}\widetilde{S}(x) &= \frac{1}{|\mathscr{O}_K^\times|^2} \sum_{a \in \mathfrak{m} \cap C(z,\theta,x)\,:\, a \neq -k} \;\sum_{b \in \mathscr{O}_K \smallsetminus \{0\}\,:\, b\,|\,a} \frac{\mu_K(b)}{\mathtt{N}(b)} \sum_{c \in \mathscr{O}_K \smallsetminus \{0\}\,:\, c\,|\,a+k} \frac{\mu_K(c)}{\mathtt{N}(c)} \\ &= \frac{1}{|\mathscr{O}_K^\times|^2} \sum_{b \in \mathscr{O}_K \smallsetminus \{0\}\,:\, |b| \leq x} \frac{\mu_K(b)}{\mathtt{N}(b)} \\ &\qquad \times \sum_{c \in \mathscr{O}_K \smallsetminus \{0\}\,:\, |c| \leq x + |k|} \frac{\mu_K(c)}{\mathtt{N}(c)} \sum_{\substack{a \,\in\, \mathfrak{m} \cap C(z,\theta,x)\ :\ a \neq -k \\ b\ |\ a,\ c\ |\ a+k}} 1\ . \end{aligned} \tag{24}$$

Let $b, c \in \mathscr{O}_K \smallsetminus \{0\}$. The system of congruences $\begin{cases} a \equiv 0 \bmod m \\ a \equiv 0 \bmod b \\ a \equiv -k \bmod c \end{cases}$ has a solution a in $\mathscr{O}_K \smallsetminus \{0\}$ with $|a| \leq x$ and $a \neq -k$ if and only if there exists $n \in \mathscr{O}_K \smallsetminus \{0\}$ with $a = b\,n$, $|n| \leq \frac{x}{|b|}$, $n \neq -\frac{k}{b}$, and

$$\begin{cases} bn \equiv 0 \mod m \\ bn \equiv -k \mod c\,. \end{cases} \tag{25}$$

When $(b,c) \nmid k$, no solution exists.

Assume that $(b,c) \mid k$. Since $\frac{b}{(b,c)}$ is invertible modulo $\frac{c}{(b,c)}$, we denote by $\overline{\frac{b}{(b,c)}}$ a multiplicative inverse of $\frac{b}{(b,c)}$ modulo $\frac{c}{(b,c)}$. Then the system (25) is equivalent to

$$\begin{cases} \frac{b}{(b,m)} n \equiv 0 \mod \frac{m}{(b,m)} \\ \frac{b}{(b,c)} n \equiv -\frac{k}{(b,c)} \mod \frac{c}{(b,c)} \end{cases} \Leftrightarrow \begin{cases} n \equiv 0 \mod \frac{m}{(b,m)} \\ n \equiv -\frac{k}{(b,c)}\, \overline{\frac{b}{(b,c)}} \mod \frac{c}{(b,c)}\ . \end{cases} \tag{26}$$

Recall that a system of two congruences $\begin{cases} n \equiv \alpha_0 \bmod \alpha \\ n \equiv \beta_0 \bmod \beta \end{cases}$ with unknown $n \in \mathscr{O}_K$, where $\alpha, \beta, \alpha_0, \beta_0 \in \mathscr{O}_K$ and $\alpha, \beta \neq 0$, has a solution if and only if $\alpha_0 - \beta_0 \equiv 0 \bmod (\alpha, \beta)$. Furthermore, if this congruence condition is satisfied, that is, if there exist $n_0, m_0 \in \mathscr{O}_K$ such that $\alpha_0 - \beta_0 = \beta m_0 - \alpha n_0$, then n is a solution if and only if

$$n - \alpha_0 - \alpha n_0 \in \alpha\mathscr{O}_K \cap \beta\mathscr{O}_K = [\alpha, \beta]\mathscr{O}_K \ .$$

This is equivalent to asking n to belong to the translate $\Lambda_{\alpha,\beta,\alpha_0,\beta_0} = \alpha_0 + \alpha n_0 + \vec{\Lambda}_{\alpha,\beta}$ of the $\mathbb{Z}$-lattice $\vec{\Lambda}_{\alpha,\beta} = [\alpha, \beta]\mathscr{O}_K$. Applying this with $\alpha = \frac{m}{(b,m)}$, $\beta = \frac{c}{(b,c)}$, $\alpha_0 = 0$, and $\beta_0 = -\frac{k}{(b,c)}\, \overline{\frac{b}{(b,c)}}$, since the elements $\overline{\frac{b}{(b,c)}}$ and $\frac{b}{(b,m)}$ are both coprime with $\left(\frac{m}{(b,m)}, \frac{c}{(b,c)}\right)$, the system (26) has a solution if and only if the following divisibility condition holds

$$\begin{aligned} &\left(\frac{m}{(b,m)}, \frac{c}{(b,c)}\right) \;\Big|\; \frac{k}{(b,c)}\;\overline{\frac{b}{(b,c)}} \Leftrightarrow \left(\frac{m}{(b,m)}, \frac{c}{(b,c)}\right) \;\Big|\; \frac{k}{(b,c)} \\ \Leftrightarrow\;\; &\left(\frac{m}{(b,m)}, \frac{c}{(b,c)}\right) \;\Big|\; \frac{k}{(b,c)}\;\frac{b}{(b,m)} \Leftrightarrow (m(b,c), c(b,m)) \;|\; k\, b \ . \end{aligned}$$

Thus Equation (24) becomes, using Equation (9),

$$\widetilde{S}(x) = \frac{1}{|\mathscr{O}_K^\times|^2} \sum_{\substack{b,c\,\in\mathscr{O}_K\smallsetminus\{0\}:\ |b|\le x,\ |c|\le x+|k| \\ (b,c)\ |\ k,\ (m(b,c),c(b,m))\ |\ k\,b}} \frac{\mu_K(b)\,\mu_K(c)}{\mathtt{N}(b)\ \mathtt{N}(c)} \sum_{\substack{n\in\Lambda_{\alpha,\beta,\alpha_0,\beta_0}\cap C(b^{-1}z,\,\theta,\,x/|b|) \\ n\neq -\frac{k}{b}}} 1.$$

Let b and c be as in the index of the first sum above. Using Footnote 2 with $\Lambda = \Lambda_{\alpha,\beta,\alpha_0,\beta_0}$, Formula (8) for the second equality and the equation $\mathtt{N}([\alpha,\beta]) = \frac{\mathtt{N}(\alpha)\,\mathtt{N}(\beta)}{\mathtt{N}((\alpha,\beta))}$ for the last one, we have, uniformly in $b, c, m \in \mathscr{O}_K \smallsetminus \{0\}$, $k \in \mathscr{O}_K$, $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$, and $y \geq 1$,

$$\begin{aligned} &\operatorname{Card}(\Lambda_{\alpha,\beta,\alpha_0,\beta_0} \cap C(b^{-1}z, \theta, y)) \\ &\quad = \frac{\theta}{2\ \operatorname{covol}_{\vec{\Lambda}_{\alpha,\beta}}}\, y^2 + \mathrm{O}\left(\frac{\operatorname{diam}_{\vec{\Lambda}_{\alpha,\beta}}(y + \operatorname{diam}_{\vec{\Lambda}_{\alpha,\beta}})}{\operatorname{covol}_{\vec{\Lambda}_{\alpha,\beta}}}\right) \\ &\quad = \frac{\theta}{\sqrt{|D_K|}\,\mathtt{N}\left(\left[\frac{m}{(b,m)}, \frac{c}{(b,c)}\right]\right)}\, y^2 + \mathrm{O}\left(\frac{1}{\sqrt{\mathtt{N}\left(\left[\frac{m}{(b,m)}, \frac{c}{(b,c)}\right]\right)}}\, y + 1\right) \\ &\quad = \frac{\theta\ \mathtt{N}((m(b,c), c(b,m)))}{\sqrt{|D_K|}\ \mathtt{N}(m)\ \mathtt{N}(c)}\, y^2 + \mathrm{O}\left(\frac{\mathtt{N}((m(b,c), c(b,m)))^{1/2}}{\mathtt{N}(m)^{1/2}\ \mathtt{N}(c)^{1/2}}\, y + 1\right) \ . \end{aligned}$$

Using this, we have

$$\widetilde{S}(x) = \frac{\theta\, x^2}{\sqrt{|D_K|}} \sum_{\substack{b,c\,\in\mathscr{O}_K\smallsetminus\{0\}\,:\,|b|\le x,\;|c|\le x+|k|\\ (b,c)\,|\,k,\;(m(b,c),c(b,m))\,|\,k\,b}} \frac{\mu_K(b)\,\mu_K(c)\;\mathtt{N}((m(b,c),c(b,m)))}{|\mathscr{O}_K^\times|^2\;\mathtt{N}(b)^2\;\mathtt{N}(c)^2\;\mathtt{N}(m)}$$
$$+\,\mathrm{O}\left(x \sum_{b,c\,\in\mathscr{O}_K\smallsetminus\{0\}} \frac{\mathtt{N}((m(b,c),c(b,m)))^{1/2}}{|\mathscr{O}_K^\times|^2\;\mathtt{N}(b)^{3/2}\;\mathtt{N}(c)^{3/2}\;\mathtt{N}(m)^{1/2}}\right)$$
$$+\,\mathrm{O}\left(\sum_{\substack{b,c\,\in\mathscr{O}_K\smallsetminus\{0\}\\ |b|\le x,\;|c|\le x+|k|}} \frac{1}{\mathtt{N}(b)\,\mathtt{N}(c)}\right). \tag{27}$$

By Equation (16) (replacing therein x by x^2), completing the first sum of the above equation with the indices $b,c\in\mathscr{O}_K\smallsetminus\{0\}$ such that $|b|>x$ or $|c|>x+|k|$ introduces an error of the form $\mathrm{O}(\frac{1}{x})$ (uniformly in $m\in\mathscr{O}_K\smallsetminus\{0\}$, $k\in\mathscr{O}_K$, and $x\ge 1$). A computation similar to the one done for Equation (16) gives that the second sum in Equation (27) is actually bounded by the constant $\zeta_K(\frac{3}{2})^2\,\zeta_K(\frac{5}{2})$. The third sum is[3] a $\mathrm{O}\left(\ln(2x)\ln(2x+|k|)\right)$, hence a $\mathrm{O}(x)+\mathrm{O}(\ln(2x)\,(\ln(1+\mathtt{N}(k))))$ since we have $\ln(u+v)\le\ln u+\ln v$ for all $u,v\ge 2$.

By the definition of the constant $c_{\mathfrak{m},k}$ in Equation (12), this proves Equation (23), hence concludes the proof of Lemma 4.5. □

Proof of Theorem 4.1. For all $a,k\in\mathscr{O}_K$ with $a\ne 0$, we have

$$\mathtt{N}(a+k)=\mathtt{N}(a)\Big|1+\frac{k}{a}\Big|^2\le \mathtt{N}(a)\left(1+2\sqrt{\frac{\mathtt{N}(k)}{\mathtt{N}(a)}}+\frac{\mathtt{N}(k)}{\mathtt{N}(a)}\right),$$

and similarly $\mathtt{N}(a+k)\ge\mathtt{N}(a)\left(1-2\sqrt{\frac{\mathtt{N}(k)}{\mathtt{N}(a)}}+\frac{\mathtt{N}(k)}{\mathtt{N}(a)}\right)$. Let us define the maps $f_\pm:[1,+\infty[\,\to\mathbb{R}$ by $t\mapsto t^2\pm 2\sqrt{\mathtt{N}(k)}\,t^{3/2}+\mathtt{N}(k)\,t$. Their derivatives are $f_\pm'(t)=2t\pm 3\sqrt{\mathtt{N}(k)}\,t^{1/2}+\mathtt{N}(k)$ and

$$f_-(\mathtt{N}(a))\le\mathtt{N}(a)\;\mathtt{N}(a+k)\le f_+(\mathtt{N}(a)). \tag{28}$$

Let $z\in\mathbb{C}^\times$, $\theta\in\,]0,2\pi]$, and $x\ge 1$. For all $n\in\mathbb{N}\smallsetminus\{0\}$, let

$$b_n=\sum_{a\in\mathfrak{m}\cap C(z,\theta,x)\,:\,\mathtt{N}(a)=n,\;a\ne -k}\frac{\varphi_K(a)}{\mathtt{N}(a)}\;\frac{\varphi_K(a+k)}{\mathtt{N}(a+k)},$$

so that by Equation (22), we have $\widetilde{S}(x)=\sum_{1\le n\le x^2} b_n$.

[3]Indeed, the method of proof of Equation (10) shows that for $y\ge 1$, we have $\sum_{a\in\mathscr{O}_K\smallsetminus\{0\}:|a|\le y}\mathtt{N}(a)^{-1}=\mathrm{O}(\ln(2y))$.

By the definition (11) of the sum $S_{\mathfrak{m},z,\theta,k}(x)$ and the inequalities (28), by Abel's summation formula, by applying twice Lemma 4.5, and since $c_{\mathfrak{m},k} \le 1$ by Lemma 4.2, we have

$$\begin{aligned}
S_{\mathfrak{m},z,\theta,k}(x) &\le \sum_{1\le n\le x^2} b_n\, f_+(n) \\
&= \Big(\sum_{1\le n\le x^2} b_n\Big)\ f_+(x^2) - \int_1^{x^2} \Big(\sum_{1\le n\le t} b_n\Big)\ f_+'(t)\ dt \\
&= \Big(\frac{\theta\ c_{\mathfrak{m},k}}{\sqrt{|D_K|}}\ x^2 + \mathrm{O}(x) + \mathrm{O}\left(\ln(2x)\ln(1+\mathtt{N}(k))\right)\Big) \\
&\quad \times \Big(x^4 + 2\sqrt{\mathtt{N}(k)}\,x^3 + \mathtt{N}(k)\,x^2\Big) \\
&\quad - \int_1^{x^2} \Big(\frac{\theta\ c_{\mathfrak{m},k}}{\sqrt{|D_K|}}\ t + \mathrm{O}(t^{1/2}) + \mathrm{O}\left(\ln(2t)\ln(1+\mathtt{N}(k))\right)\Big) \\
&\quad \times \Big(2t + 3\sqrt{\mathtt{N}(k)}\,t^{1/2} + \mathtt{N}(k)\Big)\ dt \\
&= \frac{\theta\ c_{\mathfrak{m},k}}{3\sqrt{|D_K|}}\ x^6 \\
&\quad + \mathrm{O}\Big((1+\sqrt{\mathtt{N}(k)}\,)\,x^5 + \mathtt{N}(k)\,x^4 + \mathtt{N}(k)\ln(\mathtt{N}(k))\,x^2\ln(2x)\Big)\ .
\end{aligned}$$

Replacing f_+ by f_- gives the same minoration to $S_{\mathfrak{m},z,\theta,k}(x)$, hence Theorem 4.1 follows. □

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