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**Author(s):** Julin, Vesa; Niinikoski, Joonas

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# Consistency of the Flat Flow Solution to the Volume Preserving Mean Curvature Flow

VESA JULIN  & JOONAS NIINIKOSKI

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## Abstract

We consider the flat flow solution, obtained via a discrete minimizing movement scheme, to the volume preserving mean curvature flow starting from  $C^{1,1}$ -regular set. We prove the consistency principle, which states that (any) flat flow solution agrees with the classical solution as long as the latter exists. In particular the flat flow solution is unique and smooth up to the first singular time. We obtain the result by proving the full regularity for the discrete time approximation of the flat flow such that the regularity estimates are stable with respect to the time discretization. Our method can also be applied in the case of the mean curvature flow and thus it provides an alternative proof, not relying on comparison principle, for the consistency between the flat flow solution and the classical solution for  $C^{1,1}$ -regular initial sets.

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## 1. Introduction

### 1.1. Statement of the Main Theorem

In this paper we consider the flat flow solution to the volume preserving mean curvature flow, which is a weak notion of solution obtained via discrete minimizing movement scheme. Our main goal is to prove the full regularity of the flat flow up to the first singular time when the initial set is  $C^{1,1}$ -regular. As a corollary we obtain the consistency principle between the flat flow and the classical solution.

Let us begin by recalling that a smooth family of sets  $(E_t)_{t \in [0, T]} \subset \mathbb{R}^{n+1}$ , for some  $T > 0$ , is a solution to the volume preserving mean curvature flow if it satisfies

$$V_t = -(H_{E_t} - \bar{H}_{E_t}), \quad (1.1)$$

where  $V_t$  denotes the normal velocity,  $H_{E_t}$  the mean curvature and  $\bar{H}_{E_t} := \int_{\partial E_t} H_{E_t} \, d\mathcal{H}^n$  the integral average of the mean curvature of the evolving boundary  $\partial E_t$ . An important feature is that (1.1) can be seen as a  $L^2$ -gradient flow of the surface area. Since it also preserves the volume, it can be regarded as the evolutionary counterpart to the isoperimetric problem.

If the initial set  $E_0$  is regular enough, e.g. it satisfies interior and exterior ball conditions, the equation (1.1) has a unique smooth solution for a short interval of time [19]. The classical result by Huisken [28] states that for convex initial sets the classical solution exists for all times and converges exponentially fast to a sphere. Similarly, it follows from [19, 44] that if the initial set is close to a local minimum of the isoperimetric problem, the equation (1.1) does not develop singularities and converges exponentially fast. However, for generic initial sets the equation (1.1) may develop singularities in finite time [40, 41]. In fact, unlike the standard mean curvature flow, (1.1) may develop singularities even in the plane and the boundary may also collapse such that the curvature of the evolving boundary stays uniformly bounded up to the singular time. It is therefore natural to find a proper notion of weak solution for (1.1) which is defined for all times even if the flow develops singularities. The crucial difference between (1.1) and the mean curvature flow is that the former is nonlocal and does not satisfy the comparison principle. Therefore we cannot directly use the notion of viscosity solution to define the level-set solution via the methods introduced by Chen-Giga-Goto [15] and Evans-Spruck [20], although in [33] Kim-Kwon are able to find a viscosity solution for (1.1) for star-shaped sets. Instead, we may use the gradient flow structure to obtain a weak solution called *flat flow* via discrete minimizing movement scheme as first introduced by Almgren-Taylor-Wang [3] and Luckhaus-Stürzenhecker [36] for the mean curvature flow, and then implemented to the volume preserving setting (1.1) by Mugnai-Seis-Spadaro [43]. We give the precise definition in Section 3. The existence of the flat flow solution of (1.1) is proven in [43] and the recent results [16, 23, 31, 32, 42] indicate that it has the expected asymptotic behavior. Indeed, it is proven in [31] that in the plane any flat flow solution of (1.1), starting from any set of finite perimeter, converges exponentially fast to a union of equisize disks.

One of the main issues with the flat flow solution is that it has a priori very low regularity. The second issue is that it is not clear if the procedure provides a

solution to the equation (1.1) in some weak sense. The first issue is related to the regularity and the second one is the problem of consistency, and it is rather clear that these are closely related to each other. Indeed, the flat flow is obtained as a limit of a discrete minimizing scheme, in the spirit of the Euler implicit method, where the time discretization is led to zero. If the flow remains smooth enough, as the time discretization goes to zero, then one can show that the limiting flat flow provides a solution to the equation (1.1). However, the only case when this seems to be known is the case when the initial set is convex. In this case the construction in [8], which however is slightly different than [43], provides a flow of sets which remains convex and thus gives a solution to (1.1). One may also define a distributional solution to (1.1) (see [43]) and in a recent work Laux [34] proves that this notion of solution, and in fact any gradient-flow calibration, agrees with the classical solution as long as the latter exists (see also [26]).

The issue with regularity and consistency is better understood in the case of the standard mean curvature flow. It is proven in [3] that the flat flow for the mean curvature equation agrees with the classical solution as long as the latter exists. If we are in a situation where the level-set solution is unique, i.e., it does not develop fattening, then due to the result by Chambolle [12] we know that the flat flow coincides with the level-set solution, see also [13, 14]. We may then use the result in [21] to conclude that the flat flow is a 'subsolution' to the mean curvature flow in the sense of Brakke and has the partial regularity proven in [9]. Thus we have the consistency and partial regularity for the mean curvature flow when the flow does not develop fattening. In addition, due to the recent result by DePhilippis-Laux [17] together with the classical result in [36], we know that the flat flow is a distributional solution to the mean curvature flow equation when the initial set is mean convex.

As we mentioned above, here we study the regularity of the flat flow solution of (1.1) when the initial set is  $C^{1,1}$ -regular, which is the same as to say that the set satisfies interior and exterior ball conditions. Throughout the paper we will say that an open set  $E \subset \mathbb{R}^{n+1}$  satisfies *uniform ball condition* (which we refer as UBC) with radius  $r > 0$  if it satisfies interior and exterior ball condition with radius  $r > 0$ . If we do not want to emphasize the radius  $r$ , we simply say that  $E$  satisfies UBC. Our main theorem reads as follows:

**Theorem 1.1.** *Assume that  $E_0 \subset \mathbb{R}^{n+1}$  is an open and bounded set which satisfies UBC with radius  $r_0$ . There is time  $T_0 > 0$ , which depends on  $r_0$  and  $n$ , such that any flat flow solution  $(E_t)_{t \geq 0}$  of (1.1) starting from  $E_0$  satisfies UBC with radius  $r_0/2$  for all  $t \leq T_0$ . This condition is open in the sense that if  $(E_t)_{t \geq 0}$  satisfies UBC with radius  $r$  for all  $t \leq T$ , then there is  $\delta > 0$  such that it satisfies UBC with radius  $r/2$  for all  $t < T + \delta$ .*

*Moreover, the flat flow  $(E_t)_{t \geq 0}$  becomes instantaneously smooth and remains smooth as long as it satisfies UBC. To be more precise, if  $(E_t)_{t \geq 0}$  satisfies UBC with radius  $r$  for all  $t \leq T$ , then for every  $k \in \mathbb{N}$  it holds that*

$$\sup_{t \in (0, T]} (t^k \|H_{E_t}\|_{H^k(\partial E_t)}^2) \leq C_k, \quad (1.2)$$

where  $C_k$  depends on  $T$ ,  $n$ ,  $k$ ,  $r$  and  $|E_0|$ .

In fact, we obtain even stronger result since we prove UBC and the estimate (1.2) directly for the discrete approximative flat flow  $(E_t^h)_{t \geq 0}$  such that the estimates hold for all  $h \leq h_0$  for constants independent of  $h$ . However, we choose to state the regularity result only for the limiting flow since the precise statement, which can be found in Theorem 4.7 and Theorem 5.2, is rather technical. The first part of the theorem is related to the result by Swartz-Yip [47], where the authors prove curvature bounds for the Merriman-Bence-Osher thresholding algorithm for the mean curvature flow.

It is well-known that we have uniqueness among smooth solutions of (1.1). Therefore an important consequence of Theorem 1.1 is the consistency between the notion of flat flow solution and the classical solution of (1.1) when the initial set is  $C^{1,1}$ -regular.

**Corollary 1.2.** *Assume that  $E_0 \subset \mathbb{R}^{n+1}$  is an open and bounded set which satisfies UBC. Let  $(\hat{E}_t)_{t \in [0, T]} \subset \mathbb{R}^{n+1}$  be the classical solution of (1.1) starting from  $E_0$ , where  $T > 0$  is the maximal time of existence, and let  $(E_t)_{t \geq 0} \subset \mathbb{R}^{n+1}$  be a flat flow solution of (1.1) starting from  $E_0$ . Then*

$$\hat{E}_t = E_t \quad \text{for all } t \in [0, T].$$

Let us next briefly comment on the regularity estimate (1.2). The first part of Theorem 1.1 (see Theorem 4.7 in Section 4) provides a bound on UBC for a short time  $[0, T_0]$  and the proof of Theorem 4.7 also provides an estimate how the curvature grows in time for the approximative flat flow  $(E_t^h)_{t \geq 0}$ . However, without higher order regularity bounds we are not able to pass these growth-estimates to the limit as  $h \rightarrow 0$ . Therefore the results of Section 4 only imply the consistency for a short time interval  $[0, T_0]$  (see the discussion at the end of Sect. 5). Our main motivation to prove (1.2) is to pass the previously mentioned curvature estimates to the limit as  $h \rightarrow 0$  by Ascoli-Arzelà theorem, and deduce that UBC is, in fact, an open condition and therefore the flat flow agrees with the classical solution over the whole maximal time of existence. Of course, in addition to that, (1.2) quantifies the smoothing effect of the equation in a sharp way.

### 1.2. An Overview of the Proof

The proof of Theorem 1.1 is divided in three sections and therefore we give here a short overview. We recall that in the minimizing movements scheme, for a fixed time discretization step  $h > 0$ , we obtain a sequence of sets  $E_k^h$  such that  $E_0^h = E_0$  is the initial set and  $E_{k+1}^h$  is defined inductively as a minimizer of the functional

$$\mathcal{F}_h(E, E_k^h) = P(E) + \frac{1}{h} \int_E d_{E_k^h} dx + \frac{1}{\sqrt{h}} \left| |E| - m_0 \right|,$$

where  $d_{E_k^h}$  denotes the signed distance function and  $m_0 = |E_0|$ . A flat flow is then defined as any cluster point of the discrete flow as  $h \rightarrow 0$ . We first prove

in Proposition 3.1 via energy comparison argument, that if  $E_k^h$  satisfies UBC with radius  $r_0$  then the subsequent set  $E_{k+1}^h$  satisfies the distance estimate

$$|d_{E_k^h}| \leq \frac{C}{r_0} h \quad \text{on } \partial E_{k+1}^h.$$

The above estimate is crucial as it implies that the speed of the discrete flow is sublinear. It also implies a bound for the mean curvature and the regularity of  $E_{k+1}^h$  by applying the Allard's regularity theory [2]. The most crucial part of the proof of the main theorem is then to show that the subsequent set  $E_{k+1}^h$  also satisfies UBC with a quantified radius.

We solve this problem by adopting the two-point function method due to Huisken [27] to the discrete setting (see also the works by Andrews [4] and Brendle [10] for an overview of the topic). The idea is to double the variables and to study the maximum and minimum values of the function

$$S_{E_k^h}(x, y) = \frac{(x - y) \cdot \nu(x)}{|x - y|^2}$$

for  $x \neq y \in \partial E_k^h$ . The point is that the extremal values of  $S_{E_k^h}$  are related to the maximal UBC radius of the set  $E_k^h$  (see Lemma 4.1). We use the maximum principle to prove the following familiar inequality (see Lemma 4.6):

$$\frac{\|S_{E_{k+1}^h}\|_{L^\infty} - \|S_{E_k^h}\|_{L^\infty}}{h} \leq C \|S_{E_k^h}\|_{L^\infty}^3.$$

By iterating the above estimate, we obtain that the sets  $E_k^h$  satisfy UBC for all  $k \leq T_0 h^{-1}$ , where the constant  $T_0$  is related to the UBC of the initial set. This implies the first part of Theorem 1.1 (see Theorem 4.7). An important technical part in this argument is the discrete version of the formula for  $\frac{d}{dt} \nu_{E_t}$  which we derive in Lemma 4.4.

The formula in Lemma 4.4 is, in fact, so simple that we are able to differentiate it multiple times and obtain in Proposition 5.1 a discrete analog for the formula

$$\frac{d}{dt} \Delta^k H_{E_t} = \Delta^{k+1} H_{E_t} + \text{lower order terms}, \quad (1.3)$$

where  $\Delta$  denotes the Laplace-Beltrami operator (see e.g. [38]). The lower order terms are due to the nonlinearity of the equation (1.1) and we need the notation and tools from differential geometry in order to control them. We stress that this is the only part in the paper where we need to introduce higher order covariant derivatives. After we have obtained the discrete version of the formula (1.3) and bounded the lower order error terms, we may adopt the argument from [22] to the discrete setting and obtain the full regularity of the flow. Finally, we point out that the argument can be adopted to the case of the mean curvature flow essentially without any modifications.

## 2. Notation and Preliminary Results

Throughout this paper,  $C_n \in \mathbb{R}_+$  stands for a generic dimensional constant which may change from line to line. We denote the open ball with radius  $r$  centered at  $x$  by  $B_r(x) \subset \mathbb{R}^{n+1}$  and by  $B_r$  if it is centered at the origin. We denote by  $\mathbf{C}(x, r, R) \subset \mathbb{R}^{n+1}$  the open cylinder

$$\mathbf{C}(x, r, R) := B_r^n(x') \times (-R + x_{n+1}, R + x_{n+1}),$$

where  $B_r^n \subset \mathbb{R}^n$  denotes the  $n$ -dimensional ball and  $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ . For a given set  $E \subset \mathbb{R}^{n+1}$  and a radius  $r \in \mathbb{R}_+$  we set its  $r$ -enlargement  $\mathcal{N}_r(E) = \{x \in \mathbb{R}^{n+1} : \text{dist}(x, E) < r\}$ . Note that we may alternatively write this as the Minkowski sum  $E + B_r$ . The notation  $\nabla^k F$  stands for  $k$ :th order differential of a vector field  $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$ . For a matrix  $\mathcal{A} \in \mathbb{R}^k \otimes \mathbb{R}^k$  we denote by  $|\mathcal{A}|$  its Frobenius norm  $\sqrt{\text{Tr}(\mathcal{A}^T \mathcal{A})}$  and by  $|\mathcal{A}|_{\text{op}}$  its operator norm  $\max\{|\mathcal{A}\xi| : \xi \in \mathbb{R}^k, |\xi| = 1\}$ .

If a set  $S \subset \mathbb{R}^k$  is Lebesgue-measurable, we denote its  $k$ -dimensional Lebesgue measure (or volume) by  $|S|$ . Given a non-empty set  $E \subset \mathbb{R}^{n+1}$  we denote the distance function by  $\text{dist}_E(x) := \inf_{y \in E} |x - y|$  and the *signed distance function* by  $d_E : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , which is defined as

$$d_E(x) := \begin{cases} \text{dist}_E(x), & \text{for } x \in \mathbb{R}^{n+1} \setminus E \\ -\text{dist}_{\mathbb{R}^n \setminus E}(x), & \text{for } x \in E. \end{cases} \tag{2.1}$$

Then clearly it holds that  $\text{dist}_{\partial E} = |d_E|$ . If for a given point  $x \in \mathbb{R}^{n+1}$  there is a unique distance minimizer  $y_x$  on  $\partial E$  (that is  $|x - y_x| = \text{dist}_{\partial E}(x)$ ), we denote  $y_x$  by  $\pi_{\partial E}(x)$  and call it the projection of  $x$  onto  $\partial E$ . For a set of finite perimeter  $E \subset \mathbb{R}^{n+1}$  we denote its reduced boundary by  $\partial^* E$ . Then  $P(E; F) = \mathcal{H}^n(\partial^* E \cap F)$  for every Borel set  $F \subset \mathbb{R}^{n+1}$  and  $P(E) = \mathcal{H}^n(\partial^* E)$ .

### 2.1. Regular Sets and Tangential Differentiation

We will mostly deal with regular and bounded sets  $E \subset \mathbb{R}^{n+1}$ . As usual, a bounded set  $E \subset \mathbb{R}^{n+1}$  is said to be  $C^{k,\alpha}$ -regular, with  $k \geq 1$  and  $0 \leq \alpha \leq 1$ , if for every  $x \in \partial E$  we find a cylinder  $\mathbf{C}(x, r, R)$  and a function  $f \in C^{k,\alpha}(B_r^n(x'))$  with  $|f - x_{n+1}| < R$  such that, up to rotating the coordinates, we may write

$$\text{int}(E) \cap \mathbf{C}(x, r, R) = \{y \in \mathbf{C}(x, r, R) : y_{n+1} < f(y')\}.$$

In particular,  $\partial E$  is a compact and embedded  $C^{k,\alpha}$ -hypersurface. Again, if  $\alpha = 0$ , we say that  $E$  is  $C^k$ -regular and if  $k = \infty$ , we say that  $E$  is smooth. If  $r$  and  $R$  are independent of the choice of  $x$  and the  $C^{k,\alpha}$ -norm of  $g$  has a bound, also independent of  $x$ , then we say that  $E$  is uniformly  $C^{k,\alpha}$ -regular. We denote the outer unit normal by  $\nu_E$ , or simply  $\nu$  if the meaning is clear from the context. Note that  $\nu_E \in C^{k-1,\alpha}(\partial E; \partial B_1)$ . We always assume that the orientation of  $\partial E$  is induced by  $\nu_E$ . We define the matrix field  $P_{\partial E} : \partial E \rightarrow \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$  by setting  $P_{\partial E} = I - \nu_E \otimes \nu_E$ . For a given point  $x \in \partial E$  the map  $P_{\partial E}(x)$  is the orthogonal projection onto the *geometric tangent plane*  $G_x \partial E := \langle \nu_E(x) \rangle^\perp$ .

For given a vector field  $F \in C^l(\mathbb{R}^{n+1}; \mathbb{R}^m)$  with  $1 \leq l \leq k$  we define its *tangential differential* along  $\Sigma = \partial E$  as a matrix field  $\nabla_{\tau_E} F : \partial E \rightarrow \mathbb{R}^m \otimes \mathbb{R}^{n+1}$  by setting

$$\nabla_{\tau_E} F = \nabla F P_{\partial E} = \nabla F - (\nabla F \nu_E) \otimes \nu_E. \tag{2.2}$$

When the meaning is clear from the context, we abbreviate  $E$  from the notation and write simply  $\nabla_{\tau} F$ . In the case  $m = n + 1$ , the *tangential divergence* of  $F$  is defined as  $\operatorname{div}_{\tau} F = \operatorname{Tr}(\nabla_{\tau} F)$  and the *tangential Jacobian*  $J_{\tau} F$  of  $F$  is defined on  $\partial E$  as

$$J_{\tau} F = \sqrt{\det((\nabla_{\tau} F \circ \iota_{\tau})^T (\nabla_{\tau} F \circ \iota_{\tau}))}, \tag{2.3}$$

where  $\iota_{\tau}(x)$  at  $x \in \partial E$  is the inclusion  $G_x \partial E \hookrightarrow \mathbb{R}^{n+1}$ . In the case  $m = 1$ , the notation  $\nabla_{\tau} F$  also stands for the *tangential gradient*  $P_{\partial E} \nabla F$ . Note that  $\nabla_{\tau} F$  is  $C^{l-1}$ -regular and independent of how  $F$  is extended beyond  $\partial E$ . On the other hand, every  $G \in C^l(\partial E, \mathbb{R}^m)$ , with  $1 \leq l \leq k$ , admits a  $C^l$ -extension  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  so we may extend the concept of tangential differential to concern  $G$  simply by setting  $\nabla_{\tau} G = \nabla_{\tau} F$  and further define the other introduced concepts in a similar manner.

If  $E$  is  $C^k$ -regular for  $k \geq 2$ , we may define its *second fundamental form*, with respect to the orientation  $\nu_E$ , as a matrix field  $B_E : \partial E \rightarrow \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$  given by

$$B_E(x) = \sum_i \lambda_i(x) \kappa_i(x) \otimes \kappa_i(x),$$

where the (unit) principal directions  $\kappa_1(x), \dots, \kappa_n(x) \in \langle \nu_E(x) \rangle^{\perp}$  and the principal curvatures  $\lambda_1(x), \dots, \lambda_n(x)$  at  $x \in \partial E$  are given by the orientation  $\nu_E$ . The corresponding (scalar) mean curvature field  $H_E$  is then given pointwise as the sum of the principal curvatures, i.e.,  $H_E = \operatorname{Tr}(B_E)$ . Note that we may simply write

$$B_E = \nabla_{\tau} \nu_E \quad \text{and} \quad H_E = \operatorname{div}_{\tau} \nu_E. \tag{2.4}$$

Finally, we define the *tangential Hessian* for given  $u \in C^2(\partial E)$  as  $\nabla_{\tau}^2 u = \nabla_{\tau}(\nabla_{\tau} u)$  and further the *tangential Laplacian* or the *Laplace-Beltrami* of  $u$  as

$$\Delta_{\tau} u = \operatorname{div}_{\tau}(\nabla_{\tau} u) = \operatorname{Tr}(\nabla_{\tau}^2 u).$$

The tangential Laplacian  $\Delta_{\tau} F$  for  $F \in C^2(\partial E; \mathbb{R}^{n+1})$  is defined as  $\sum_i \Delta_{\tau}(F \cdot e_i) e_i$ . We will need the following identities on  $\partial E$ :

$$\Delta_{\tau} \operatorname{id} = -H_E \nu_E \quad \text{and} \quad \Delta_{\tau} \nu_E = -|B_E|^2 \nu_E + \nabla_{\tau} H_E \quad \text{if } E \text{ is } C^3\text{-regular.} \tag{2.5}$$

The importance of the mean curvature  $H_E$  lies in the surface divergence theorem which states that for every  $G \in C^1(\partial E; \mathbb{R}^{n+1})$  it holds that

$$\int_{\partial E} \operatorname{div}_{\tau} G \, d\mathcal{H}^n = \int_{\partial E} H_E(G \cdot \nu_E) \, d\mathcal{H}^n. \tag{2.6}$$

The concept of mean curvature can be generalized to the setting of bounded sets of finite perimeter in the varifold sense. Indeed, for a set of finite perimeter  $E \subset \mathbb{R}^{n+1}$ , we may define the tangential divergence  $\operatorname{div}_{\tau} F$  of  $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$  along  $\partial^* E$  in the same way as in the regular case by replacing the outer unit normal



field with the measure theoretic normal field  $\partial^*E \rightarrow \partial B_1$  which we also denote by  $\nu_E$ . Then, if  $E$  is a bounded set of finite perimeter and there is  $g \in L^1(\partial^*E, \mathcal{H}^n|_{\partial^*E})$  such that

$$\int_{\partial^*E} \operatorname{div}_\tau F \, d\mathcal{H}^n = \int_{\partial^*E} g(F \cdot \nu_E) \, d\mathcal{H}^n \tag{2.7}$$

for every  $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ , we say that  $g$  is a *generalized mean curvature* of  $E$  and denote it by  $H_E$ . As mentioned, this is a concept from the context of varifold theory for which we refer to [46] as a standard introduction. Since  $\partial^*E$  is  $\mathcal{H}^n$ -rectifiable set, one may treat the pair  $(\partial^*E, \mathcal{H}^n|_{\partial^*E})$  as an rectifiable integral varifold of multiplicity one.

### 2.2. Riemannian Geometry

We need the notation related to Riemannian geometry and as an introduction to the topic we refer to [35]. Let us assume that  $E \subset \mathbb{R}^{n+1}$  is a smooth and bounded set and denote  $\Sigma = \partial E$ . Since  $\Sigma$  is embedded in  $\mathbb{R}^{n+1}$  it has natural metric  $g$  induced by the Euclidean metric. Then  $(\Sigma, g)$  is a Riemannian manifold and we denote the inner product on each tangent space  $X, Y \in T_x \Sigma$  by  $\langle X, Y \rangle$ , which we may write in local coordinates as

$$\langle X, Y \rangle = g(X, Y) = g_{ij} X^i Y^j.$$

We extend the inner product in a natural way for tensors. Note that  $x \cdot y$  denotes the inner product of two vectors in  $\mathbb{R}^{n+1}$ . We denote smooth vector fields on  $\Sigma$  by  $\mathcal{T}(\Sigma)$  and by a slight abuse of notation we denote smooth  $k$ :th order tensor fields on  $\Sigma$  by  $\mathcal{T}^k(\Sigma)$ . We write  $X^i$  for vectors and  $Z_i$  for covectors in local coordinates. We denote the Riemannian connection on  $\Sigma$  by  $\tilde{\nabla}$  and recall that for a function  $u \in C^\infty(\Sigma)$  the covariant derivative  $\tilde{\nabla}u$  is a 1-tensor field defined for  $X \in \mathcal{T}(\Sigma)$  as

$$\tilde{\nabla}u(X) = \tilde{\nabla}_X u = Xu,$$

i.e., the derivative of  $u$  in the direction of  $X$ . The covariant derivative of a smooth  $k$ -tensor field  $F \in \mathcal{T}^k(\Sigma)$ , denoted by  $\tilde{\nabla}F$ , is a  $(k + 1)$ -tensor field and for  $Y_1, \dots, Y_k, X \in \mathcal{T}(\Sigma)$  we have the recursive formula

$$\tilde{\nabla}F(Y_1, \dots, Y_k, X) = (\tilde{\nabla}_X F)(Y_1, \dots, Y_k), \tag{2.8}$$

where

$$(\tilde{\nabla}_X F)(Y_1, \dots, Y_k) = XF(Y_1, \dots, Y_k) - \sum_{i=1}^k F(Y_1, \dots, \tilde{\nabla}_X Y_i, \dots, Y_k).$$

Here  $\tilde{\nabla}_X Y$  is the covariant derivative of  $Y$  in the direction of  $X$  (see [35]) and since  $\tilde{\nabla}$  is the Riemannian connection it holds that  $\tilde{\nabla}_X Y = \tilde{\nabla}_Y X + [X, Y]$  for every  $X, Y \in \mathcal{T}(\Sigma)$ . We denote the  $k$ :th order covariant derivative of a function  $u$  on  $\Sigma$  by  $\tilde{\nabla}^k u \in \mathcal{T}^k(\Sigma)$  and the Laplace-Beltrami operator by  $\Delta$ . Note that for functions it holds that  $\Delta u = \Delta_\tau u$ . The notation  $\tilde{\nabla}_{i_k} \dots \tilde{\nabla}_{i_1} u$  means a coefficient

of  $\tilde{\nabla}^k u$  in local coordinates. We may raise the index of  $\tilde{\nabla}_i u$  by using the inverse of the metric tensor  $g^{ij}$  as  $\tilde{\nabla}^i u = g^{ij} \tilde{\nabla}_j u$ . We note that the tangential gradient of  $u : \Sigma \rightarrow \mathbb{R}$  is equivalent to its covariant derivative in the sense that for every vector field  $X \in \mathcal{T}(\Sigma)$  we find a unique vector field  $\tilde{X} : \Sigma \rightarrow \mathbb{R}^{n+1}$  which satisfies  $\tilde{X} \cdot \nu_E = 0$  and

$$\tilde{\nabla}_X u = \nabla_\tau u \cdot \tilde{X}.$$

Similarly it holds that  $\tilde{\nabla}^2 u(X, Y) = \nabla_\tau^2 u \tilde{X} \cdot \tilde{Y}$ . Finally we recall that the notation  $\nabla^k$  always stands for the standard Euclidean  $k$ :th order differential for an ambient function.

We define the Riemann curvature tensor  $R \in \mathcal{T}^4(\Sigma)$  [35, 39] via interchange of covariant derivatives of a vector field  $Y^i$  and a covector field  $Z_i$  as

$$\begin{aligned} \tilde{\nabla}_i \tilde{\nabla}_j Y^s - \tilde{\nabla}_j \tilde{\nabla}_i Y^s &= R_{ijkl} g^{ks} Y^l, \\ \tilde{\nabla}_i \tilde{\nabla}_j Z_k - \tilde{\nabla}_j \tilde{\nabla}_i Z_k &= R_{ijkl} g^{ls} Z_s, \end{aligned} \tag{2.9}$$

where we have used the Einstein summation convention. We may write the Riemann tensor in local coordinates by using the second fundamental form  $B$ , which in the Riemannian setting is understood to be 2-form, as

$$R_{ijkl} = B_{ik} B_{jl} - B_{il} B_{jk}. \tag{2.10}$$

We will also need Simon’s identity, which reads as

$$\Delta B_{ij} = \tilde{\nabla}_i \tilde{\nabla}_j H + H B_{il} g^{ls} B_{sj} - |B|^2 B_{ij}. \tag{2.11}$$

Let us next fix our notation for the function spaces. We define the Sobolev space  $W^{l,p}(\Sigma)$  in a standard way for  $p \in [1, \infty]$ , see e.g. [6], denote the Hilbert space  $H^l(\Sigma) = W^{l,2}(\Sigma)$  and define the associated norm for  $u \in W^{l,p}(\Sigma)$  as

$$\|u\|_{W^{l,p}(\Sigma)}^p = \sum_{k=0}^l \int_{\Sigma} |\tilde{\nabla}^k u|^p d\mathcal{H}^n,$$

and, for  $p = \infty$ ,

$$\|u\|_{W^{l,\infty}(\Sigma)} = \sum_{k=0}^l \sup_{x \in \Sigma} |\tilde{\nabla}^k u|.$$

The above definition extends naturally for tensor fields. We adopt the convention that  $\|u\|_{H^0(\Sigma)} = \|u\|_{L^2(\Sigma)}$  and denote  $\|u\|_{C^m(\Sigma)} = \|u\|_{W^{m,\infty}(\Sigma)}$ . We remark that we may define the  $k$ :th order covariant derivative of a function  $u \in C^k(\Sigma)$  and the space  $W^{k,p}(\Sigma)$  for  $k \geq 2$  as above assuming only that  $\Sigma$  (i.e. the set  $E$  for which  $\Sigma = \partial E$ ) is  $C^k$ -regular.

Finally we adopt the notation  $S \star T$  from [25, 38] to denote a tensor formed by contracting some indexes of tensors  $S$  and  $T$  using the coefficients of the metric tensor  $g_{ij}$ . This notation is useful as it implies

$$|S \star T| \leq C |S| |T|,$$

where the constant  $C$  depends on the ‘structure’ of  $S \star T$ .

### 2.3. Functional and Geometric Inequalities

We will need standard interpolation inequalities on smooth hypersurfaces. Since we will apply them on the moving boundary given by the flow, we need to control the constants in the inequalities. We begin with a simple interpolation on Hölder norms.

**Lemma 2.1.** *Let  $\Omega \subset \mathbb{R}^k$  be an open set and let  $u \in C^1(\Omega)$ , then for every  $\alpha \in (0, 1)$*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq 3\|u\|_{L^\infty(\Omega)}^{1-\alpha} \|u\|_{C^1(\Omega)}^\alpha.$$

*Proof.* The inequality follows from

$$\frac{|u(y) - u(x)|}{|y - x|^\alpha} \leq |u(y) - u(x)|^{1-\alpha} \left( \frac{|u(y) - u(x)|}{|y - x|} \right)^\alpha \leq 2\|u\|_{L^\infty(\Omega)}^{1-\alpha} \|u\|_{C^1(\Omega)}^\alpha.$$

□

We continue to introduce functional and geometric inequalities that we need in order to prove the higher order regularity estimates stated at the end of Theorem 1.1. As we already mentioned we do not need any deep results from differential geometry in order to prove the estimate for UBC stated in the beginning of Theorem 1.1. It is only when we deal with higher order derivatives, i.e., higher than two, we need the notation of covariant derivatives. Recall that we always assume that  $\Sigma = \partial E$  for a bounded set  $E \subset \mathbb{R}^{n+1}$ .

Let us first recall the interpolation inequality with Sobolev-norms on embedded surfaces. We use the result from [38, Proposition 6.5] which states that under curvature bound the standard interpolation inequality holds for a uniform constant.

**Proposition 2.2.** *Assume  $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C_0$  and  $\Sigma$  is  $C^m$ -regular for  $m \geq 2$ . Then for integers  $0 \leq k \leq l \leq m$  and numbers  $p, q, r \in [1, \infty)$ , there is  $\theta \in [k/l, 1]$  such that for every  $C^l$ -regular covariant tensor field  $T$  on  $\Sigma$  it holds*

$$\|\tilde{\nabla}^k T\|_{L^p(\Sigma)} \leq C \|T\|_{W^{l,q}(\Sigma)}^\theta \|T\|_{L^r(\Sigma)}^{1-\theta}$$

for a constant  $C = C(k, l, n, p, q, r, \theta, C_0) \in \mathbb{R}_+$  provided that the following compatibility condition is satisfied

$$\frac{1}{p} = \frac{k}{n} + \theta \left( \frac{1}{q} - \frac{l}{n} \right) + \frac{1}{r} (1 - \theta).$$

We denote an index vector by  $\alpha \in \mathbb{N}^k$ , i.e.,  $\alpha = (\alpha_1, \dots, \alpha_k)$  where  $\alpha_i \in \mathbb{N}$ , and define its norm by

$$|\alpha| = \sum_{i=1}^k \alpha_i.$$

The following inequality is well-known but we prove it for the reader's convenience:

**Proposition 2.3.** Assume  $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C$  and  $\Sigma$  is  $C^m$ -regular for  $m \geq 2$ . Assume  $u_1, \dots, u_l$  are  $C^m$ -regular functions such that  $\|u_i\|_{L^\infty} \leq C$ . Then for an index vector  $\alpha \in \mathbb{N}^l$  with  $|\alpha| \leq k \leq m$  and  $p \in (1, \infty)$  it holds that

$$\| |\tilde{\nabla}^{\alpha_1} u_1| \cdots |\tilde{\nabla}^{\alpha_l} u_l| \|_{L^p(\Sigma)} \leq C_k \sum_{i=1}^k \|u_i\|_{W^{k,p}(\Sigma)}.$$

*Proof.* Without loss of generality we may assume that  $|\alpha| = k$ . We first use Hölder’s inequality to get that

$$\| |\tilde{\nabla}^{\alpha_1} u_1| \cdots |\tilde{\nabla}^{\alpha_l} u_l| \|_{L^p(\Sigma)} \leq \| \tilde{\nabla}^{\alpha_1} u_1 \|_{L^{\frac{pk}{\alpha_1}}} \cdots \| \tilde{\nabla}^{\alpha_l} u_l \|_{L^{\frac{pk}{\alpha_l}}}.$$

By the interpolation inequality in Proposition 2.2 and by  $\|u_i\|_{L^\infty} \leq C$  it holds that

$$\| \tilde{\nabla}^{\alpha_i} u_i \|_{L^{\frac{pk}{\alpha_i}}} \leq C \|u_i\|_{W^{k,p}}^{\frac{\alpha_i}{k}} \|u_i\|_{L^\infty}^{1-\frac{\alpha_i}{k}} \leq C \|u_i\|_{W^{k,p}}^{\frac{\alpha_i}{k}}.$$

Hence we have

$$\| |\tilde{\nabla}^{\alpha_1} u_1| \cdots |\tilde{\nabla}^{\alpha_l} u_l| \|_{L^p(\Sigma)} \leq C_k \|u_1\|_{W^{k,p}}^{\frac{\alpha_1}{k}} \cdots \|u_l\|_{W^{k,p}}^{\frac{\alpha_l}{k}}.$$

Since  $\alpha_1 + \dots + \alpha_l = |\alpha| = k$ , the claim follows, from Young’s inequality.  $\square$

If  $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is a regular function then its restriction on  $\Sigma$  is also regular. In the next lemma we bound the covariant derivatives of  $u$  on  $\Sigma$  with the Euclidean ones. The statement of the lemma is not optimal but it is sharp enough for our purpose. In the proof we will repeatedly use the fact that the  $k$ :th order derivative of the composition  $f \circ h$  and the product  $f \cdot g$  of functions  $f, g : \mathbb{R}^m \rightarrow \mathbb{R}^k$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be written as

$$\begin{aligned} \nabla^k(f \circ h) &= \sum_{|\alpha| \leq k-1} \nabla^{1+\alpha_1} h \star \cdots \star \nabla^{1+\alpha_k} h \star \nabla^{1+\alpha_{k+1}} f \\ \nabla^k(f \cdot g) &= \sum_{i+j=k} \nabla^i f \star \nabla^j g. \end{aligned} \tag{2.12}$$

**Lemma 2.4.** Assume  $\Sigma$  is  $C^{k+2}$ -regular and  $u \in C^{k+1}(\mathbb{R}^{n+1})$ . Then it holds for all  $x \in \Sigma$  that

$$|\tilde{\nabla}^{k+1} u(x)| \leq C_k \sum_{|\alpha| \leq k} (1 + |\tilde{\nabla}^{\alpha_1} B_E(x)| \cdots |\tilde{\nabla}^{\alpha_k} B_E(x)|) |\nabla^{1+\alpha_{k+1}} u(x)|.$$

Recall that  $\tilde{\nabla}^k$  denotes the  $k$ :th order covariant derivative on  $\Sigma$  while  $\nabla^k$  is the  $k$ :th order Euclidean derivative.

*Proof.* The proof follows from basic theory of differential geometry and we merely sketch it. Let us fix  $x \in \Sigma$  and choose the coordinates such that  $x = 0$  and  $\nu_E(0) = e_{n+1}$ . Since  $\Sigma$  is  $C^{k+2}$ -regular hypersurface we may write it locally as a graph of  $f \in C^{k+2}(\mathbb{R}^n)$ , i.e.,  $\Sigma \cap B_r(0) \subset \{(x, f(x)) : x \in \mathbb{R}^n\}$ . Note that since  $\nu_E(0) = e_{n+1}$  then  $\nabla_{\mathbb{R}^n} f(0) = 0$ .

We consider the graph coordinates  $\Phi^{-1} : B_r^n \rightarrow \Phi^{-1}(B_r^n) \subset \Sigma$ ,  $\Phi^{-1}(x) = (x, f(x))$ . We denote the points on  $\mathbb{R}^n$  by  $x$ , the points on  $\Sigma$  by  $p$ ,  $\Phi(p) = (x^1(p), \dots, x^n(p))$  and  $U = \Phi^{-1}(B_r^n)$ . Then the chart  $(U, (x^i))$  determines coordinate vector fields which we denote by  $\frac{\partial}{\partial x^i} \Big|_p$  and recall that they act on smooth functions  $v : U \rightarrow \mathbb{R}$  at  $p = \Phi(x)$  as

$$\frac{\partial}{\partial x^i} \Big|_p v = \tilde{\nabla} v \left( \frac{\partial}{\partial x^i} \right) (p) = \partial_i (v \circ \Phi^{-1})(x),$$

where  $\partial_i$  denotes the standard partial derivative in  $\mathbb{R}^n$ . It holds for the metric tensor and for the Christoffel symbol  $\Gamma_{jk}^i$  (see [35]) for  $x \in B_r^n$  that

$$g_{ij}(x) = \delta_{ij} + \partial_i f(x) \partial_j f(x) \quad \text{and} \quad \Gamma_{jk}^i(x) = g^{il}(x) \partial_{jk}^2 f(x) \partial_l f(x).$$

Moreover by the recursive formula (2.8) we may write the  $(k+1)$ :th order covariant derivative of  $u$  iteratively (see [35, Lemma 4.8]) as

$$\begin{aligned} \tilde{\nabla}^{k+1} u \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}, \frac{\partial}{\partial x^j} \right) &= \partial_j \left( \tilde{\nabla}^k u \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \right) \\ &- \sum_{m=1}^k \tilde{\nabla}^k u \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^l}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \Gamma_{j i_m}^l. \end{aligned} \tag{2.13}$$

Recall that  $\tilde{\nabla} u \left( \frac{\partial}{\partial x^i} \right) (p) = \frac{\partial}{\partial x^i} \Big|_p u$ .

Using (2.12) we have

$$|\nabla_{\mathbb{R}^n}^{k+1} (u \circ \Phi^{-1})(0)| \leq C_k \sum_{|\alpha| \leq k} (1 + |\nabla_{\mathbb{R}^n}^{1+\alpha_1} f(0)| \cdots |\nabla_{\mathbb{R}^n}^{1+\alpha_k} f(0)|) |\nabla^{1+\alpha_{k+1}} u(0)|.$$

We use (2.13) and (2.12), and obtain after long but straightforward calculation that

$$|\tilde{\nabla}^{k+1} u(0)| \leq C_k \sum_{|\alpha| \leq k} (1 + |\nabla_{\mathbb{R}^n}^{1+\alpha_1} f(0)| \cdots |\nabla_{\mathbb{R}^n}^{1+\alpha_k} f(0)|) |\nabla^{1+\alpha_{k+1}} u(0)|.$$

Note that  $\nu_E \circ \Phi^{-1} = \frac{(-\nabla_{\mathbb{R}^n} f, 1)}{\sqrt{1+|\nabla_{\mathbb{R}^n} f|^2}}$ . We thus obtain by (2.12) that

$$\begin{aligned} |\nabla_{\mathbb{R}^n}^{l+1} f(0)| &\leq C_l \sum_{|\beta| \leq l} (1 + |\nabla^{\beta_1} (\nu_E \circ \Phi^{-1})| \cdots |\nabla^{\beta_l} (\nu_E \circ \Phi^{-1})|) \\ &\leq C_l \sum_{|\beta| \leq l-1} (1 + |\tilde{\nabla}^{\beta_1} B_E| \cdots |\tilde{\nabla}^{\beta_l} B_E|) \end{aligned} \tag{2.14}$$

and the claim follows.  $\square$

Next we turn our focus on geometric inequalities on compact hypersurfaces. Recall that by classical results e.g. from [6] it holds that  $\|u\|_{H^2(\Sigma)} \leq C(\|\Delta u\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)})$  and e.g. in [22] it is proven that  $\|u\|_{H^{2k}(\Sigma)} \leq C(\|\Delta u\|_{H^k(\Sigma)} + \|u\|_{L^2(\Sigma)})$ . We need these results with a quantitative control on the constant.

**Lemma 2.5.** *Assume  $\Sigma$  is  $C^{2k+2}$ -regular and  $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C$ . Then for all  $u \in C^{2k+1}(\Sigma)$  it holds*

$$\|u\|_{H^{2k}(\Sigma)} \leq C_k(\|\Delta^k u\|_{L^2(\Sigma)} + (1 + \|B_\Sigma\|_{H^{2k-1}(\Sigma)})\|u\|_{L^\infty(\Sigma)})$$

and

$$\|u\|_{H^{2k+1}(\Sigma)} \leq C_k(\|\tilde{\nabla} \Delta^k u\|_{L^2(\Sigma)} + (1 + \|B_\Sigma\|_{H^{2k}(\Sigma)})\|u\|_{L^\infty(\Sigma)}).$$

*Proof.* We only prove the first inequality since the second follows from the same argument. The proof is similar to [30, Proposition 2.11] but we sketch it for the reader’s convenience. Denote  $l = 2k$ . We begin by noticing that we may interchange the derivatives of the  $(l + 1)$ :th order covariant derivative of  $u$  by using (2.9), (2.10), (2.13) and the curvature bound  $\|B_\Sigma\|_{L^\infty} \leq C$  (see also [38, Proof of Lemma 7.3])

$$\begin{aligned} & |\tilde{\nabla}_{i_{l+1}} \cdots \tilde{\nabla}_{i_{m+1}} \tilde{\nabla}_{i_m} \cdots \tilde{\nabla}_{i_1} u - \tilde{\nabla}_{i_{l+1}} \cdots \tilde{\nabla}_{i_m} \tilde{\nabla}_{i_{m+1}} \cdots \tilde{\nabla}_{i_1} u| \\ & \leq C_l \sum_{|\alpha| \leq l-1} (1 + |\tilde{\nabla}^{\alpha_1} B_\Sigma| \cdots |\tilde{\nabla}^{\alpha_{l-1}} B_\Sigma|) |\tilde{\nabla}^{\alpha_l} u|. \end{aligned}$$

We leave the details for the reader. This holds pointwise on  $\Sigma$  and we use it without further mentioning. Let us denote  $F = \tilde{\nabla}^{2k-2} u$  and denote its components simply by  $F_\beta$ , where  $\beta = (i_1, \dots, i_{2k-2})$ . Then it holds by divergence theorem, by interchanging the derivatives and by Proposition 2.3

$$\begin{aligned} & \int_\Sigma |\tilde{\nabla}^{2k} u|^2 d\mathcal{H}^n = \int_\Sigma |\tilde{\nabla}^2 F|^2 d\mathcal{H}^n \\ & = \int_\Sigma \tilde{\nabla}_i \tilde{\nabla}_j F_\beta \tilde{\nabla}^i \tilde{\nabla}^j F^\beta d\mathcal{H}^n = - \int_\Sigma \tilde{\nabla}_j F_\beta \tilde{\nabla}_i \tilde{\nabla}^i \tilde{\nabla}^j F^\beta d\mathcal{H}^n \\ & \leq - \int_\Sigma \tilde{\nabla}_j F_\beta \tilde{\nabla}^j \tilde{\nabla}_i \tilde{\nabla}^i F^\beta d\mathcal{H}^n \\ & \quad + C_k \sum_{|\alpha| \leq l-1} \int_\Sigma (1 + |\tilde{\nabla}^{\alpha_1} B_\Sigma|^2 \cdots |\tilde{\nabla}^{\alpha_{l-1}} B_\Sigma|^2) |\tilde{\nabla}^{\alpha_l} u|^2 d\mathcal{H}^n \\ & \leq \int_\Sigma \tilde{\nabla}^j \tilde{\nabla}_j F_\beta \tilde{\nabla}_i \tilde{\nabla}^i F^\beta d\mathcal{H}^n + C_k(\|u\|_{H^{l-1}(\Sigma)}^2 + \|u\|_{L^\infty(\Sigma)} \|B_\Sigma\|_{H^{l-1}(\Sigma)}^2) \\ & = \int_\Sigma |\Delta \tilde{\nabla}^{2k-2} u|^2 d\mathcal{H}^n + C_k(\|u\|_{H^{2k-1}(\Sigma)}^2 + \|u\|_{L^\infty(\Sigma)} \|B_\Sigma\|_{H^{2k-1}(\Sigma)}^2). \end{aligned}$$

By interchanging the derivatives and arguing as above we obtain

$$\begin{aligned} & \int_\Sigma |\Delta \tilde{\nabla}^{2k-2} u|^2 d\mathcal{H}^n \leq \int_\Sigma |\tilde{\nabla}^{2k-2} \Delta u|^2 d\mathcal{H}^n \\ & \quad + C_k(\|u\|_{H^{2k-1}(\Sigma)}^2 + \|u\|_{L^\infty(\Sigma)} \|B_\Sigma\|_{H^{2k-1}(\Sigma)}^2). \end{aligned}$$

By repeating the argument by replacing  $u$  with  $\Delta^j u$ , for  $j = 1, \dots, k - 1$ , we deduce that

$$\int_{\Sigma} |\tilde{\nabla}^{2k} u|^2 d\mathcal{H}^n \leq \int_{\Sigma} |\Delta^k u|^2 d\mathcal{H}^n + C_k (\|u\|_{H^{2k-1}(\Sigma)}^2 + \|u\|_{L^\infty(\Sigma)}^2 \|B_\Sigma\|_{H^{2k-1}(\Sigma)}^2).$$

The claim follows from interpolation inequality (Proposition 2.2) as for  $\theta \in (0, 1)$  it holds that

$$\|u\|_{H^{2k-1}(\Sigma)}^2 \leq \|u\|_{H^{2k}(\Sigma)}^{2\theta} \|u\|_{L^\infty(\Sigma)}^{2(1-\theta)} \leq \varepsilon \|u\|_{H^{2k}(\Sigma)}^2 + C_\varepsilon \|u\|_{L^\infty(\Sigma)}^2,$$

where the last inequality follows from Young’s inequality.  $\square$

Lemma 2.5 together with Simon’s identity (2.11), imply

**Proposition 2.6.** *Assume that  $\Sigma$  is  $C^{2k+3}$ -regular and that  $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C$ . Then it holds that*

$$\|B_\Sigma\|_{H^{2k}(\Sigma)} \leq C_k (1 + \|\Delta^k H_\Sigma\|_{L^2(\Sigma)})$$

and

$$\|B_\Sigma\|_{H^{2k+1}(\Sigma)} \leq C_k (1 + \|\tilde{\nabla} \Delta^k H_\Sigma\|_{L^2(\Sigma)}).$$

#### 2.4. Uniform Ball Condition and Signed Distance Function

In this subsection, we recall some properties related to sets which satisfy UBC as well as properties of signed distance function defined in (2.1). Most of them can be found e.g. in [5, 7] while others are more difficult to find. We recall that a set  $E \subset \mathbb{R}^{n+1}$  satisfies UBC with given a radius  $r \in \mathbb{R}_+$ , if it simultaneously satisfies the exterior and interior ball condition with radius  $r$  at every boundary point. That is, for every  $x \in \partial E$  there are balls  $B_r(x_+)$  and  $B_r(x_-)$  such that

$$B_r(x_+) \subset \mathbb{R}^{n+1} \setminus E, \quad B_r(x_-) \subset E \quad \text{and} \quad x \in \partial B_r(x_+) \cap \partial B_r(x_-).$$

It is well known, for the experts at least, that UBC for a set implies its boundary being a uniformly  $C^{1,1}$ -regular hypersurface. We need this property in a quantitative form which states that if  $E \subset \mathbb{R}^{n+1}$  satisfies UBC with radius  $r$ , then it can be written locally in a cylinder of width  $r/2$  as a graph of a  $C^{1,1}$ -function. Since this result is not easy to find in the literature, we state it and provide a proof here.

**Proposition 2.7.** *Assume  $E \subset \mathbb{R}^{n+1}$  satisfies UBC with radius  $r > 0$ . Then for every point  $x \in \partial E$  we may, by rotating the coordinates if necessary, write the interior of the set locally as a subgraph of a function  $g : B_{r/2}^n(x') \rightarrow \mathbb{R}$ , i.e.,*

$$\begin{aligned} \text{int}(E) \cap C(x, r/2, r) &= \{(y', y_{n+1}) \in C(x, r/2, r/2) : y_{n+1} < g(y')\} \quad \text{and} \\ \partial E \cap C(x, r/2, r) &= \{(y', g(y')) : y' \in B_{r/2}^n(x')\}. \end{aligned}$$

The function  $g$  is  $C^{1,1}$ -regular and it holds for all  $y' \in B_{r/2}^n(x')$  and  $s \in (0, r/2]$

$$|g(y') - g(x')| \leq \frac{|y' - x'|^2}{r + \sqrt{r^2 - |y' - x'|^2}},$$

$$|\nabla g(y')| \leq \frac{|y' - x'|}{r} \left( 1 - \left( \frac{|y' - x'|}{r} \right)^2 \right)^{-\frac{1}{2}} \quad \text{and}$$

$$\sup_{\substack{y'_1, y'_2 \in B_s^n(x') \\ y'_1 \neq y'_2}} \frac{|\nabla g(y'_2) - \nabla g(y'_1)|}{|y'_2 - y'_1|} \leq \frac{1}{r} \left( 1 - \left( \frac{s}{r} \right)^2 \right)^{-\frac{3}{2}}.$$

Moreover, the outer unit normal  $\nu_E$  on  $\partial E$  is  $1/r$ -Lipschitz continuous in Euclidean metric.

*Remark 2.8.* We remark, that the converse of Proposition 2.7 also holds true. That is, if  $E \subset \mathbb{R}^{n+1}$  is a set such that for every  $x \in \partial E$ , we may write its boundary locally, by rotating the coordinates, as  $\partial E \cap \mathbf{C}(x, r, 2r) \subset \{(y', g(y')) : y' \in B_r^n(x')\}$  with  $\|g\|_{C^{1,1}(B_r^n(x'))} \leq C/r$ , then  $E$  satisfies UBC with radius  $cr$ , for a constant  $c > 0$  which depends on  $n$  and  $C$ . This is fairly straightforward to show and we leave it to the reader.

*Proof of Proposition 2.7.* We remark that UBC with  $r$  implies for every  $x \in \partial E$  an existence of a unique unit vector  $\nu_E(x)$  such that  $B_r(x - r\nu_E(x)) \subset E$  and  $B_r(x + r\nu_E(x)) \subset \mathbb{R}^{n+1} \setminus E$ . Therefore, we have a vector field  $\nu_E : \partial E \rightarrow \partial B_1$  which later turns out to be the outer unit normal field of  $E$ . We first show that  $\nu_E$  is  $1/r$ -Lipschitz continuous with respect to Euclidean distance. To this end, fix  $x, y \in \partial E$ . By the previous observation  $B_r(x + r\nu_E(x)) \subset \mathbb{R}^{n+1} \setminus E$  and  $B_r(y - r\nu_E(x)) \subset E$  so the balls are disjoint. Similarly, the balls  $B_r(x - r\nu_E(x))$  and  $B_r(y + r\nu_E(y))$  are disjoint. Hence the distances between the corresponding centerpoints are at least  $2r$  and we obtain the inequalities

$$4r^2 \leq |x - y + r(\nu_E(x) + \nu_E(y))|^2 \quad \text{and}$$

$$4r^2 \leq |x - y - r(\nu_E(x) + \nu_E(y))|^2.$$

By summing the above inequalities gives us  $8r^2 \leq 2|x - y|^2 + 4r^2(1 + \nu_E(x) \cdot \nu_E(y))$  and, again, by subtracting and dividing terms we further obtain

$$1 - \frac{|x - y|^2}{2r^2} \leq \nu_E(x) \cdot \nu_E(y) \quad \text{or equivalently} \quad |\nu_E(x) - \nu_E(y)|^2 \leq \frac{|x - y|^2}{r^2}. \tag{2.15}$$

In particular,  $\nu_E$  is  $1/r$ -Lipschitz.

For given a point  $x \in \partial E$ , we show the existence of  $g$  as claimed. Without loss of generality we may assume  $x = 0$  and  $\nu_E(0) = e_{n+1}$ . Then it holds  $B_r(-re_{n+1}) \subset E$  and  $B_r(re_{n+1}) \subset \mathbb{R}^{n+1} \setminus E$ . Thus, for every  $y' \in B_{r/2}^n$  there is a number  $t_{y'}$  such that  $(y', t_{y'}) \in \partial E$  and

$$|t_{y'}| \leq r - \sqrt{r^2 - |y'|^2} = \frac{|y'|^2}{r + \sqrt{r^2 - |y'|^2}}. \tag{2.16}$$



In particular,  $|t_{y'}| < |y'|$ . Combining (2.15) and (2.16) yields

$$v_E(y', t_{y'}) \cdot e_{n+1} \geq \sqrt{1 - \left(\frac{|y'|}{r}\right)^2}. \tag{2.17}$$

Let us show that such a number  $t_{y'}$  is unique.

We suppose by contradiction there is  $s_{y'} \in (-r, r) \setminus \{t_{y'}\}$  such that  $(y', s_{y'}) \in \partial E$ . Without loss of generality, we may assume  $s_{y'} > t_{y'}$ . Since  $B_r((y', t_{y'}) + rv_E(y', t_{y'})) \subset \mathbb{R}^{n+1} \setminus E$  and  $(y', s_{y'}) \in \partial E$ , then the point  $(y', s_{y'})$  is not in the ball  $B_r((y', t_{y'}) + rv_E(y', t_{y'}))$ . Hence, we obtain

$$\begin{aligned} r^2 &\leq |(y', s_{y'}) - ((y', t_{y'}) + rv_E(y', t_{y'}))|^2 \\ &= (s_{y'} - t_{y'})^2 - 2r(s_{y'} - t_{y'})v_E(y', t_{y'}) \cdot e_{n+1} + r^2. \end{aligned}$$

We first subtract  $r^2$ , then divide by  $s_{y'} - t_{y'}$  and finally use the estimates (2.16), (2.17) as well as  $|y'| < r/2$  to deduce

$$s_{y'} \geq t_{y'} + 2rv_E(y', t_{y'}) \cdot e_{n+1} \geq -r + 3\sqrt{r^2 - |y'|^2} > r - \sqrt{r^2 - |y'|^2}.$$

This implies together with  $s_{y'} < r$  that  $(y', s_{y'}) \in B_r(re_{n+1}) \subset \mathbb{R}^{n+1} \setminus E$  which, in turn, contradicts  $(y', s_{y'}) \in \partial E$  and, hence,  $t_{y'}$  is a unique value in  $(-r, r)$  satisfying  $(y', t_{y'}) \in \partial E$ .

Thus, the function  $g : B_{r/2}^n \rightarrow \mathbb{R}$ , given by the relation  $g(y') = t_{y'}$ , satisfies

$$\begin{aligned} \text{int}(E) \cap \mathbf{C}(0, r/2, r/2) &= \{(y', y_{n+1}) \in \mathbf{C}(0, r/2, r/2) : y_{n+1} < g(y')\} \text{ and} \\ \partial E \cap \mathbf{C}(0, r/2, r/2) &= \{(y', g(y')) : y' \in B_{r/2}^n\}. \end{aligned} \tag{2.18}$$

Again, (2.16) gives us the bound on  $|g(y')|$  as claimed. The condition (2.17) implies that for every  $y' \in B_{r/2}^n$  there are open sets  $y' \in V \subset B_{r/2}^n$ ,  $(y', g(y')) \in U \subset \mathbf{C}(0, r/2, r/2)$  and functions  $\psi_+, \psi_- \in C^\infty(V)$  such that  $\partial B_r((y', g(y')) \pm rv_E(y', g(y'))) \cap U$  are the graphs of  $\psi_\pm$  respectively. Then  $\psi_- \leq g \leq \psi_+$  in  $V$  and  $\psi_-(w) = g(w) = \psi_+(w)$  implying the differentiability of  $g$  at  $y'$  with  $\nabla g(y') = \nabla \psi_\pm(y')$ . Moreover, we deduce that  $v_E(y', g(y'))$  is the outer unit normal of  $\{(z', z_{n+1}) \in V \times \mathbb{R} : z_{n+1} > \psi_+(z')\}$  at  $(y', g(y'))$  and thus

$$v_E(y', g(y')) = \frac{(-\nabla \psi_+(y'), 1)}{\sqrt{1 + |\nabla \psi_+(y')|^2}} = \frac{(-\nabla g(y'), 1)}{\sqrt{1 + |\nabla g(y')|^2}}. \tag{2.19}$$

Since now  $g$  and  $v_E$  are continuous, (2.19) implies that  $\nabla g$  is continuous too. Thus,  $E$  is  $C^1$ -regular and  $v_E$  is the actual outer unit normal of  $E$ . We combine (2.17) and (2.19) to observe

$$|\nabla g(y')| \leq \frac{|y'|}{r} \left(1 - \left(\frac{|y'|}{r}\right)^2\right)^{-\frac{1}{2}}. \tag{2.20}$$

To conclude the Lipschitz estimate, if  $y'_1, y'_2 \in B_s^n$  for given  $s \in (0, r/2]$ , then the uniform ball condition implies that  $(y'_1, g(y'_1)) \notin B_r((y'_2, g(y'_2)) \pm rv_E(y'_2, g(y'_2)))$

and  $(y'_2, g(y'_2)) \notin B_r((y'_1, g(y'_1)) \pm r\nu_E(y'_1, g(y'_1)))$ . Hence, using (2.19), we obtain the estimates

$$r^2 \leq \left| (y'_2, g(y'_2)) \pm r \frac{(-\nabla g(y'_2), 1)}{\sqrt{1 + |\nabla g(y'_2)|^2}} - (y'_1, g(y'_1)) \right|^2 \quad \text{and}$$

$$r^2 \leq \left| (y'_1, g(y'_1)) \pm r \frac{(-\nabla g(y'_1), 1)}{\sqrt{1 + |\nabla g(y'_1)|^2}} - (y'_2, g(y'_2)) \right|^2.$$

By summing these inequalities and simplifying, we have

$$\begin{aligned} & \pm(y'_2 - y'_1) \cdot (\nabla g(y'_2) - \nabla g(y'_1)) \\ & \leq \frac{\sqrt{1 + |\nabla g(y'_2)|^2} + \sqrt{1 + |\nabla g(y'_1)|^2}}{2r} \left( |y'_2 - y'_1|^2 + (g(y'_2) - g(y'_1))^2 \right). \end{aligned}$$

Thus, by recalling, (2.20) we further estimate that

$$\begin{aligned} & |(y'_2 - y'_1) \cdot (\nabla g(y'_2) - \nabla g(y'_1))| \\ & \leq \frac{\sqrt{1 + |\nabla g(y'_2)|^2} + \sqrt{1 + |\nabla g(y'_1)|^2}}{2r} \\ & \quad \left( |y'_2 - y'_1|^2 + (g(y'_2) - g(y'_1))^2 \right) \\ & \leq \frac{\sqrt{1 + \sup_{B_s^n} |\nabla g|^2}}{r} \left( 1 + \sup_{B_s^n} |\nabla g|^2 \right) |y'_2 - y'_1|^2 \\ & \leq \frac{1}{r} \left( 1 - \left( \frac{s}{r} \right)^2 \right)^{-\frac{3}{2}} |y'_2 - y'_1|^2. \end{aligned} \tag{2.21}$$

The desired estimate then follows from (2.21) via a standard mollification argument.  $\square$

We recall that a signed distance function  $d_E$  of a non-empty set  $E \subset \mathbb{R}^{n+1}$  is always 1-Lipschitz and it is differentiable at  $x \in \mathbb{R}^n \setminus \partial E$  exactly when the projection  $\pi_{\partial E}(x)$  exists on  $\partial E$ . Again, UBC for  $E$  means the differentiability of  $d_E$  in a tubular neighborhood. Indeed, one may show that for a non-empty open set  $E \subset \mathbb{R}^{n+1}$  and  $r \in \mathbb{R}_+$  the conditions

- (i)  $d_E$  is differentiable in  $\mathcal{N}_r(\partial E)$  and
- (ii)  $E$  satisfies UBC with radius  $r$

are equivalent. In such a case, the projection  $\pi_{\partial E}$  onto  $\partial E$  is defined in  $\mathcal{N}_r(\partial E)$  as a continuous map and the following fundamental identities hold in  $\mathcal{N}_r(\partial E)$ :

$$\pi_{\partial E} = \text{id} - d_E \nabla d_E \quad \text{and} \quad \nabla d_E = \nu_E \circ \pi_{\partial E}. \tag{2.22}$$

In particular,  $d_E \in C^1(\mathcal{N}_r(\partial E))$ . Further, it is fairly simple to conclude that for every  $t \in (-r, r)$  the sublevel set  $E_t = \{x \in \mathbb{R}^{n+1} : d_E(x) < t\}$  has the level set

$\{x \in \mathbb{R}^{n+1} : d_E(x) = t\}$  as the boundary and satisfies UBC with radius  $r - |t|$ . Moreover, it holds that

$$d_{E_t} = d_E - t \quad \text{and} \quad \pi_{\partial E_t} = \pi_{\partial E} + t\nu_E \circ \pi_{\partial E} \quad \text{in } \mathcal{N}_{r-|t|}(\partial E_t). \tag{2.23}$$

We may then improve the regularity by showing  $\nabla d_E$  and  $\pi_{\partial E}$  are locally Lipschitz continuous in  $\mathcal{N}_r(\partial E)$  and obtain quantitative estimates for the Lipschitz constants in smaller tubes.

**Lemma 2.9.** *Assume  $E \subset \mathbb{R}^{n+1}$  satisfies UBC with radius  $r > 0$ . Then for every  $0 < \rho < r$  and  $x, y \in \overline{\mathcal{N}_\rho(\partial E)}$  it holds that*

$$|\pi_{\partial E}(x) - \pi_{\partial E}(y)| \leq \frac{r}{r - \rho} |x - y| \quad \text{and} \quad |\nabla d_E(x) - \nabla d_E(y)| \leq \frac{1}{r - \rho} |x - y|.$$

*Proof.* It is enough to prove the first estimate, since the second estimate follows from the first via Proposition 2.7 and the second identity of (2.22). We first show that the estimates hold locally, i.e., for every  $x \in \mathcal{N}_r(\partial E)$ ,

$$\text{Lip}(\pi_{\partial E}, x) \leq \frac{r}{r - |d_E(x)|} \tag{2.24}$$

To this end, we show that, for every  $x \in \partial E$  and  $y \in B_{r/4}(x)$ , it holds that

$$|\pi_{\partial E}(y) - x|^2 \leq \left(1 + \frac{4}{r - |d_E(y)|} |d_E(y)|\right) |y - x|^2. \tag{2.25}$$

We may assume that  $x = 0$ ,  $\nu_E(0) = e_{n+1}$  and  $y \notin E$ . Let  $g : B_{r/2}^n \rightarrow \mathbb{R}$  be as in Proposition 2.7. Since  $|y| < r/4$ , then  $y \in \mathbf{C}(r/2, r/2, 0)$  implying  $|d_E(y)| \leq |y_{n+1} - g_n(y')|$  and, hence, we make a technical observation

$$d_E^2(y) \leq 2d_E(y)(y_{n+1} - g(y')). \tag{2.26}$$

Thus, using Proposition 2.7, (2.22), (2.26) and Young’s inequality, we estimate that

$$\begin{aligned} |\pi_{\partial E}(y)|^2 &= |y|^2 - 2d_E(y) y \cdot \nabla d_E(y) + d_E^2(y) \\ &= |y|^2 - 2d_E(y)y_{n+1} + d_E^2(y) - 2d_E(y) y \cdot (\nabla d_E(y) - e_{n+1}) \\ &\leq |y|^2 - 2d_E(y)g(y') - 2d_E(y) y \cdot (\nu_E(\pi_{\partial E}(y)) - \nu_E(0)) \\ &\leq |y|^2 + 2\frac{|d_E(y)|}{r} |y'|^2 + 2\frac{d_E(y)}{r} |y| |\pi_{\partial E}(y)| \\ &\leq |y|^2 + 2\frac{|d_E(y)|}{r} |y|^2 + \frac{|d_E(y)|}{r} |y|^2 + \frac{|d_E(y)|}{r} |\pi_{\partial E}(y)|^2, \end{aligned}$$

and (2.25) follows. Suppose next  $y_1, y_2 \in B_\rho(x)$  for given  $x \in \partial E$  and  $0 < \rho < r/9$ . The sublevel set  $E_t$ , for  $t = d_E(y_2)$ , satisfies UBC with radius  $r - \rho$  and  $y_2 \in \partial E_t$ . Since  $|y_1 - y_2| < 2\rho \leq (r - \rho)/4$ , then by applying (2.25) for  $\partial E_t$  we have

$$|\pi_{\partial E_t}(y_1) - y_2| \leq \left(1 + \frac{8\rho}{r - 2\rho}\right)^{\frac{1}{2}} |y_1 - y_2|. \tag{2.27}$$

On the other hand, first recalling the second identity in (2.23) and then applying Proposition 2.7 gives us

$$|\pi_{\partial E_t}(y_1) - y_2| = |\pi_{\partial E_t}(y_1) - \pi_{\partial E_t}(y_2)| \geq \left(1 - \frac{\rho}{r}\right) |\pi_{\partial E}(y_1) - \pi_{\partial E}(y_2)|,$$

so, by combining the estimate above with (2.27) yields  $\text{Lip}(x, \pi_{\partial E}) = 1$ . Hence, we deduce that

$$\text{Lip}(x, \pi_{\partial E_t}) = 1 \tag{2.28}$$

for every  $t \in (-r, r)$  and  $x \in \partial E_t$ . By using (2.23) and Proposition 2.7 similarly as previous, we infer (2.24) from (2.28).

Finally, for the first estimate of the claim, we may assume  $x, y \in \mathcal{N}_\rho(\partial E)$ . Let  $J_{yx} := \{tx + (1 - t)y : t \in [0, 1]\}$  be the line segment between them. If  $J_{yx} \subset \mathcal{N}_\rho(\partial E)$ , then the first estimate of the claim follows from (2.24). Otherwise, there are  $0 < t_1 \leq t_2 < 1$  such that  $tx + (1 - t)y \in \mathcal{N}_\rho(\partial E)$  for every  $t \in [0, t_1] \cup (t_2, 1]$  and  $z_i = t_i x + (1 - t_i)y \in \partial \mathcal{N}_\rho(\partial E)$  for  $i = 1, 2$ . Since  $d_E(z_1) = \rho = d_E(z_2)$ , then Proposition 2.7 and (2.22) imply

$$|\pi_{\partial E}(z_1) - \pi_{\partial E}(z_2)| \leq \frac{r}{r - \rho} |z_1 - z_2|.$$

On the other hand, due to (2.24) we have

$$|\pi_{\partial E}(x) - \pi_{\partial E}(z_1)| \leq \frac{r}{r - \rho} |x - z_1| \text{ and } |\pi_{\partial E}(z_2) - \pi_{\partial E}(y)| \leq \frac{r}{r - \rho} |z_2 - y|$$

and we conclude the proof.  $\square$

If  $E$  is  $C^{k,\alpha}$ -regular, with  $k \geq 2$  and  $0 \leq \alpha \leq 1$ , then  $d_E \in C^{k,\alpha}(\mathcal{N}_r(\partial E))$  and  $\pi_{\partial E} \in C^{k-1,\alpha}(\mathcal{N}_r(\partial E); \mathbb{R}^{n+1})$ . In particular, (2.22) holds everywhere in  $\mathcal{N}_r(\partial E)$ . Then it holds

$$\nabla^2 d_E = B_E \text{ and } \Delta d_E = H_E \text{ on } \partial E. \tag{2.29}$$

In particular, we deduce from Lemma 2.9 and (2.29) that

$$\|H_E\|_{L^\infty(\partial E)} \leq \frac{n}{r} \text{ and } \sup_{\partial E} |B_E|_{\text{op}} \leq \frac{1}{r}. \tag{2.30}$$

Differentiating  $\nabla d_E \cdot \nabla d_E = 1$  yields  $\nabla^2 d_E \nabla d_E = 0$  in  $\mathcal{N}_r(\partial E)$ . Again, by differentiating the first identity in (2.22) we obtain

$$\nabla \pi_{\partial E} = I - \nabla d_E \otimes \nabla d_E - d_E \nabla^2 d_E \text{ in } \mathcal{N}_r(\partial E). \tag{2.31}$$

The second identity in (2.22) says that  $\nabla d_E = \nabla d_E \circ \pi_{\partial E}$  in  $\mathcal{N}_r(\partial E)$ . Thus, by differentiating this and by using the properties of the distance function mentioned before we have

$$\nabla^2 d_E = (\nabla^2 d_E)^T = \nabla \pi_{\partial E} (\nabla^2 d_E \circ \pi_{\partial E}) = (I - d_E \nabla^2 d_E)(B_E \circ \pi_{\partial E}) \text{ in } \mathcal{N}_r(\partial E). \tag{2.32}$$

We write this as

$$\nabla^2 d_E (I + d_E (B_E \circ \pi_{\partial E})) = B_E \circ \pi_{\partial E}.$$

It follows from (2.30) that the matrix field  $I + d_E(B_E \circ \pi_{\partial E})$  is invertible in  $\mathcal{N}_r(\partial E)$ . Therefore, we have

$$\nabla^2 d_E = (B_E \circ \pi_{\partial E})(I + d_E(B_E \circ \pi_{\partial E}))^{-1} \text{ in } \mathcal{N}_r(\partial E). \tag{2.33}$$

By combining (2.22), (2.31), (2.29) and (2.33), we may decompose  $\nabla \pi_{\partial E}$  as

$$\nabla \pi_{\partial E} = I - \nu_E \circ \pi_{\partial E} \otimes \nu_E \circ \pi_{\partial E} - d_E(B_E \circ \pi_{\partial E})(I + d_E(B_E \circ \pi_{\partial E}))^{-1} \text{ in } \mathcal{N}_r(\partial E). \tag{2.34}$$

By using a fairly standard calibration argument (see e.g. [1, Lemma 4.1]) we conclude that UBC implies so called  $\Lambda$ -*minimizer* condition.

**Lemma 2.10.** *Assume that  $E \subset \mathbb{R}^{n+1}$  is an open and bounded set which satisfies UBC with radius  $r > 0$ . Then for every set of finite perimeter  $F$  it holds that*

$$P(E \cap F) \leq P(F) + \frac{n+1}{r} |F \setminus E| \text{ and}$$

$$P(E \cup F) \leq P(F) + \frac{n+1}{r} |E \setminus F|.$$

In particular,  $P(E) \leq \frac{n+1}{r} |E|$ .

*Proof.* The argument is a quantitative version of [1, Lemma 4.1]. We will prove that for every set of finite perimeter  $F$  it holds that

$$P(E) \leq P(F) + \frac{n+1}{r} |F \Delta E|. \tag{2.35}$$

Then the two inequalities in the statement follow by using (2.35) with  $E \cup F$  and  $E \cap F$  in place of  $F$  and using the fact [37, Lemma 12.22] that

$$P(E \cup F) + P(E \cap F) \leq P(E) + P(F).$$

The third inequality follows by using (2.35) with  $F = \emptyset$ .

By a standard approximation argument for the sets of finite perimeter [37, Thm 13.8] we may assume that  $F$  is smooth. In turn, we may approximate also  $E$  by a sequence of smooth sets  $E_k$  in the  $C^1$ -sense such that  $E_k$  satisfies UBC with radius  $r_k$  such that  $r_k \rightarrow r$ . Therefore, by simplicity we assume that also  $E$  is smooth.

For each  $k \in \mathbb{N}$  we construct a vector-field  $X_k \in C^{0,1}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$  such that

- (i)  $X_k = \nu_E$  on  $\partial E$ ,
- (ii)  $|X_k| \leq 1$  in  $\mathbb{R}^{n+1}$  and
- (iii)  $\|\operatorname{div} X_k\|_{L^\infty(\mathbb{R}^{n+1})} \leq (n+1+k^{-1})/r$ .

To this aim, we first define  $\eta_k : \mathbb{R} \rightarrow \mathbb{R}$  by setting  $\eta_k(t) = \max\{0, 1 - (1 + 1/k)|t|/r\}$  and then set  $X_k = (\eta_k \circ d_E) \nabla d_E$ . Clearly  $X_k$  is a Lipschitz continuous vector field supported in  $\mathcal{N}_{r/(1+k^{-1})}(\partial E)$  and satisfies the properties (i) and (ii). We further compute that

$$\operatorname{div} X_k = (\eta_k \circ d_E) \Delta d_E + \eta'_k \circ d_E \text{ in } \mathcal{N}_{r/(1+k^{-1})}(\partial E) \setminus \partial E.$$

Hence, it follows from Lemma 2.9,  $\nabla^2 d_E \nabla d_E = 0$  in  $\mathcal{N}_r(\partial E)$  as well as the definition of  $\eta_k$  that  $|\operatorname{div} X_k| \leq (n+1+k^{-1})/r$  in  $\mathcal{N}_{r/(1+k^{-1})}(\partial E) \setminus \partial E$ . Since the sets  $\overline{\mathcal{N}_{r/(1+k^{-1})}(\partial E)}$  and  $\mathcal{N}_{r/(1+k^{-1})}(\partial E) \setminus \partial E$  agree in the  $L^1$ -sense we infer that  $X_k$  satisfies (iii). By using the properties (i) – (iii) for  $X_k$  as well as the divergence theorem we estimate

$$\begin{aligned} P(E) - P(F) &\leq \int_{\partial E} X_k \cdot \nu_E \, d\mathcal{H}^n - \int_{\partial F} X_k \cdot \nu_F \, d\mathcal{H}^n \\ &= \int_E \operatorname{div} X_k \, dx - \int_F \operatorname{div} X_k \, dx \leq \int_{E \Delta F} |\operatorname{div} X_k| \, dx \\ &\leq \frac{n+1+k^{-1}}{r} |F \Delta E|, \end{aligned}$$

and thus, by letting  $k \rightarrow \infty$ , the above yields (2.35).  $\square$

Suppose that  $E'$  is a connected component of a set  $E$  which satisfies UBC with  $r$ . If  $|E'| < \infty$ , then  $E'$  is bounded and we may control its diameter in terms of  $r$  and  $|E'|$ . Indeed, by the above approximation we may assume that  $E$  is smooth. Then by (2.29) we have  $|H_E| \leq n/r$  on  $\partial E$ . Thus, combining Lemma 2.10 and Topping's generalization [48] of Simon's diameter control [45] gives us the estimate

$$\operatorname{diam}(E') \leq C_n \int_{\partial E'} |H_{E'}|^{n-1} \, d\mathcal{H}^n \leq \frac{C_n}{r^n} |E'| \quad (2.36)$$

for a dimensional constant  $C_n \in \mathbb{R}_+$ . Finally, we need the following interpolation result:

**Lemma 2.11.** *Assume  $E \subset \mathbb{R}^{n+1}$  is an open and bounded set which satisfies UBC with radius  $r > 0$ . If  $U$  is an open set containing  $\partial E$  and  $u \in C^2(U)$ , then*

$$\|\nabla_\tau u\|_{L^\infty(\partial E)}^2 \leq 4\|u\|_{L^\infty(\partial E)} \left( \sup_{\partial E} |\nabla^2 u|_{\text{op}} + \frac{\|\nabla_\tau u\|_{L^\infty(\partial E)}}{r} \right).$$

*Proof.* By the above approximation argument we may assume that  $E$  is smooth.

We first observe that for a bounded function  $f \in C^2(\mathbb{R})$  it holds

$$\|f'\|_{L^\infty(\mathbb{R})}^2 \leq 4\|f\|_{L^\infty(\mathbb{R})} \|f''\|_{L^\infty(\mathbb{R})}. \quad (2.37)$$

Indeed, let us fix a  $t \in \mathbb{R}_+$ . We may assume that  $f'(t) > 0$ , since otherwise we consider the function  $-f$  instead of  $f$ . Let  $I$  be a maximal open interval containing  $t$  such that  $f' > 0$  in  $I$  so  $f$  is strictly increasing there. Then there is a decreasing sequence  $(\tilde{t}_i)_i \in (\inf I, t)$  converging to  $\inf I$  such that  $f'(\tilde{t}_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Since  $f$  is strictly increasing in  $I$ , it is invertible there. Hence, we may compute, for every  $i \in \mathbb{N}$ ,

$$\begin{aligned} |f'(t)|^2 - |f'(\tilde{t}_i)|^2 &= \int_{\tilde{t}_i}^t \frac{d}{ds} (f'(s))^2 \, ds \\ &= 2 \int_{\tilde{t}_i}^t f''(s) f'(s) \, ds = 2 \int_{\tilde{t}_i}^t f''(f^{-1}(f(s))) f'(s) \, ds \end{aligned}$$

$$= 2 \int_{f(\tilde{t}_i)}^{f(t)} f''(f^{-1}(\tau))d\tau \leq 4\|f\|_{L^\infty(\mathbb{R})}\|f''\|_{L^\infty(\mathbb{R})},$$

and thus, by letting  $i \rightarrow \infty$ , we obtain  $|f'(t)|^2 \leq 4\|f\|_{L^\infty(\mathbb{R})}\|f''\|_{L^\infty(\mathbb{R})}$ , and (2.37) follows.

Since  $\partial E$  is compact we find  $x \in \partial E$  such that  $|\nabla_\tau u(x)| = \|\nabla_\tau u\|_{L^\infty(\partial E)}$ . We may assume that  $|\nabla_\tau u(x)| > 0$ . The connected component of  $\partial E$  containing  $x$  is geodesically complete and, hence, we find a smooth unit speed geodesic curve  $\gamma : \mathbb{R} \rightarrow \partial E$  satisfying  $\gamma(0) = x$  and  $\gamma'(0) = \nabla_\tau u(x)/|\nabla_\tau u(x)|$ . Then we define a  $C^2$ -regular function  $f = u \circ \gamma$ . Note that  $f'(0) = \|\nabla_\tau u\|_{L^\infty(\partial E)}$  and that

$$f'' = \gamma' \cdot (\nabla^2 u \circ \gamma)\gamma' + \gamma'' \cdot (\nabla_\tau u \circ \gamma). \tag{2.38}$$

By differentiating the identity  $0 = d_E \circ \gamma$  twice and recalling the identities (2.22) and (2.29) we obtain  $0 = \gamma' \cdot (B_E \circ \gamma)\gamma' + \gamma'' \cdot (v_E \circ \gamma)$ . Since  $\gamma$  is a geodesic curve, then  $|\gamma'' \cdot (v_E \circ \gamma)| = |\gamma''|$  and hence we infer from the previous that  $|\gamma''| \leq |B_E \circ \gamma|_{\text{op}}$ . By combing this with (2.38) and using (2.30) gives us

$$|f''| \leq \left( |\nabla^2 u \circ \gamma|_{\text{op}} + |B_E \circ \gamma|_{\text{op}}|\nabla_\tau u \circ \gamma| \right) \leq \left( \sup_{\partial E} |\nabla^2 u|_{\text{op}} + \frac{\|\nabla_\tau u\|_{L^\infty(\partial E)}}{r} \right).$$

Thus, by observing  $\|f\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^\infty(\partial E)}$ , the claim follows from (2.37).  $\square$

### 3. Definition of the Flat Flow and the First Regularity Estimates

Let us begin by recalling the definition of the minimizing movements scheme and the flat flow solution of (1.1) from [43]. Assume that  $E_0 \subset \mathbb{R}^{n+1}$  is a bounded set of finite perimeter. For given a time step  $h \in \mathbb{R}_+$  we construct a parametrized family  $(E_t^h)_{t \geq 0}^\infty$  of sets of finite perimeter by an iterative minimizing procedure called minimizing movements, where

$$\begin{aligned} E_t^h &= E_0 \text{ for every } 0 \leq t < h \text{ and} \\ E_t^h &= E_{h \lfloor t/h \rfloor}^h \text{ is a minimizer of the functional } \mathcal{F}_h(\cdot, E_{t-h}^h) \text{ for every } t \geq h. \end{aligned} \tag{3.1}$$

Here for a generic bounded set of finite perimeter  $E \subset \mathbb{R}^{n+1}$  the functional  $\mathcal{F}_h(\cdot, E)$ , in the class of the bounded set of finite perimeter, is defined as

$$\mathcal{F}_h(F, E) = P(F) + \frac{1}{h} \int_F d_E \, dx + \frac{1}{\sqrt{h}} \left| |F| - m_0 \right|, \tag{3.2}$$

for  $m_0 = |E_0|$ . We call the family  $(E_t^h)_{t \geq 0}^\infty$ , defined in (3.1), an *approximative flat flow solution* of (1.1) starting from  $E_0$ . We note that there is always a minimizer for (3.2) but it might not be unique. By [43] we know that there is a subsequence of approximative flat flows  $(E_t^{h_l})_{t \geq 0}$  which converges to a parametrized family  $(E_t)_{t \geq 0}$  for a.e.  $t$  in the  $L^1$ -sense, where for every  $t > 0$  the set  $E_t$  is a set of finite perimeter with  $|E_t| = |E_0|$ . Any such limit is called a flat flow solution of (1.1) starting from  $E_0$ .

Let us turn our focus back on a generic minimizer of (3.2), where we assume that  $|E| = m_0$ . We then simply denote any minimizer for  $\mathcal{F}_h(\cdot, E)$  by  $E_{\min}^h$ . One has to be careful in the definition of the functional in (3.2), since the sets of finite perimeter are only defined up to measure zero. We avoid this issue by modifying a set of finite perimeter in a  $L^1(\mathbb{R}^{n+1})$ -negligible set and choose as in [37, Rmk 15.3] a representative which topological boundary agrees with the closure of its measure theoretical boundary. Thus, we always use the convention  $\partial F = \overline{\partial^* F}$  for the initial set and the minimizers. We also remark that if  $E$  is empty, then we use the convention  $d_E = \infty$  everywhere to ensure that  $E_{\min}^h$  is empty too.

Next, we recall some basic properties regarding the minimizers. First, it is easy to conclude  $P(E_{\min}^h) \leq P(E)$ . Moreover,  $E_{\min}^h$  satisfies the distance property

$$\sup_{E_{\min}^h \Delta E} |d_E| \leq \gamma_n \sqrt{h} \tag{3.3}$$

for a dimensional constant  $\gamma_n \in \mathbb{R}_+$ , see [43, Prop 3.2]. Second,  $E_{\min}^h$  has a generalized mean curvature satisfying the Euler-Lagrange equation

$$\frac{d_E}{h} = -H_{E_{\min}^h} + \lambda^h \tag{3.4}$$

in the distributional sense (2.7) on  $\partial^* E_{\min}^h$ , where the Lagrange multiplier satisfies  $|\lambda^h| = 1/\sqrt{h}$  in the case  $|E_{\min}^h| \neq m_0$ , see [43, Lemma 3.7]. Third, it is easy to see that  $E_{\min}^h$  is always a so called  $(\Lambda, r)$ -minimizer with suitable  $\Lambda, r \in \mathbb{R}_+$  satisfying  $\Lambda r \leq 1$  (see [37] for the definition). Thus, by the standard regularity theory [37, Thm 26.5 and Thm 28.1] the reduced boundary  $\partial^* E_{\min}^h$  is relatively open in  $\partial E_{\min}^h$  and an embedded  $C^{1,\alpha}$ -regular hypersurface with any  $0 < \alpha < 1/2$ , and the Hausdorff dimension of the singular part  $\partial E_{\min}^h \setminus \partial^* E_{\min}^h$  is at most  $n - 7$ . Thus, by standard Schauder estimates one may show that  $\partial^* E_{\min}^h$  is in fact  $C^{2,\alpha}$ -regular and (3.4) holds in the classical sense on  $\partial^* E_{\min}^h$ . Consequently, we may always consider  $E_{\min}^h$  as an open set.

We may improve the distance estimate (3.3) as well as regularity properties of  $E_{\min}^h$ , if we impose more regularity on  $E$ . We divide our approach into two steps. The first result states that if  $E$  is bounded and satisfies UBC with radius  $r_0 > 0$  and  $h$  is sufficiently small, then the left hand side of (3.3) is bounded linearly in  $h$ , the Lagrange multiplier  $\lambda^h$  is bounded, the generalized mean curvature  $H_{E_{\min}^h}$  is bounded in the  $L^\infty$ -sense and  $E_{\min}^h$  has the volume  $m_0$ .

**Proposition 3.1.** *Assume  $E \subset \mathbb{R}^{n+1}$  is an open and bounded set of volume  $m_0$  which satisfies UBC with radius  $r_0$ . There are positive numbers  $h_0 = h_0(n, m_0, r_0)$  and  $C_0 = C_0(n, m_0, r_0)$  and a dimensional constant  $C_n \in \mathbb{R}_+$  such that if  $h \leq h_0$ , then*

$$\sup_{E_{\min}^h \Delta E} |d_E| \leq \frac{C_n}{r_0} h, \quad \|H_{E_{\min}^h}\|_{L^\infty} + |\lambda^h| \leq C_0 \quad \text{and} \quad |E_{\min}^h| = m_0.$$



*Proof.* We prove first part of the claim, i.e., the distance estimate. If  $|E_{\min}^h \Delta E| = 0$ , then it follows from the openness of  $E_{\min}^h$  and  $E$  as well as the property  $\partial E_{\min}^h = \overline{\partial^* E_{\min}^h}$  and  $\partial E = \overline{\partial^* E}$  that  $E_{\min}^h \Delta E = \emptyset$  and there is nothing to prove. Thus, we may assume that  $|E_{\min}^h \Delta E| > 0$  and further set that

$$d_+ = \sup_{E_{\min}^h \Delta E} d_E \quad \text{and} \quad d_- = \inf_{E_{\min}^h \Delta E} d_E.$$

To conclude the first part of the claim, we show, under the assumption  $|E_{\min}^h \Delta E| > 0$ , the validity of the implication

$$\sqrt{h} \leq \frac{r_0}{\max\{n + 1, 8\gamma_n\}} \implies d_- < 0 < d_+ \quad \text{and} \quad d_+ - d_- \leq \frac{4(n + 1)h}{r_0}. \tag{3.5}$$

Thus, (3.5) and our earlier observation gives us the implication

$$\sqrt{h} \leq \frac{r_0}{\max\{n + 1, 8\gamma_n\}} \implies \sup_{E_{\min}^h \Delta E} |d_E| \leq \frac{4(n + 1)h}{r_0}. \tag{3.6}$$

To prove (3.5), we assume by contradiction that  $d_- \geq 0$  which implies  $E \subset E_{\min}^h$  due to the openness of  $E$  and, hence,  $|E_{\min}^h \setminus E| = |E_{\min}^h \Delta E| > 0$ . Using (2.35) with  $r = r_0$ , the previous observation,  $|E| = m_0$ , and the assumption on  $h$  yields

$$\begin{aligned} \mathcal{F}_h(E, E) &\leq P(E_{\min}^h) + \frac{1}{h} \int_E d_E \, dx + \frac{n + 1}{r_0} |E_{\min}^h \setminus E| \\ &< P(E_{\min}^h) + \frac{1}{h} \int_{E_{\min}^h} d_E \, dx + \frac{n + 1}{r_0} |E_{\min}^h \setminus E| \\ &= \mathcal{F}_h(E_{\min}^h, E) + \left( \frac{n + 1}{r_0} - \frac{1}{\sqrt{h}} \right) |E_{\min}^h \setminus E| \leq \mathcal{F}_h(E_{\min}^h, E), \end{aligned}$$

contradicting the minimality of  $E_{\min}^h$  and, hence,  $d_- < 0$ . Similarly we obtain  $d_+ > 0$ .

On the other hand,  $\sqrt{h} \leq r_0/(8\gamma_n)$  implies via (3.3) that  $E_{\min}^h \Delta E \subset \subset \mathcal{N}_{r_0/4}(\partial E)$ . In particular,  $-r_0/2 < d_- < 0 < d_+ < r_0/2$  and for every  $t \in (d_-, d_+)$  the sub-level set  $E_t = \{x : d_E(x) < t\}$  satisfies UBC with  $r_0/2$  and  $|E_{\min}^h \setminus E_t|, |E_t \setminus E_{\min}^h| > 0$ . By using a suitable continuity argument, we infer from the previous that for every  $t < d_+$ , sufficiently close to  $d_+$ , there is  $\tilde{t} \in (d_-, r_+)$  such that  $|E_{\min}^h \setminus E_t| = |E_{\tilde{t}} \setminus E_{\min}^h| > 0$  and  $\tilde{t} \rightarrow d_-$  as  $t \rightarrow d_+$ . For such a pair  $(t, \tilde{t})$  we set

$$F = (E_t \cap E_{\min}^h) \cup E_{\tilde{t}}.$$

Clearly,  $F$  is a bounded set of finite perimeter and  $|F| = |E_{\min}^h|$ . Thus, using  $F$  as a competitor against  $E_{\min}^h$  with respect to  $\mathcal{F}_h(\cdot, E)$  we obtain

$$P(E_{\min}^h) \leq P(F) + \frac{1}{h} \int_{E_{\tilde{t}} \setminus E_{\min}^h} d_E \, dx - \frac{1}{h} \int_{E_{\min}^h \setminus E_t} d_E \, dx$$

$$\begin{aligned}
 &\leq P(F) + \frac{\tilde{t}}{h} |E_{\tilde{t}} \setminus E_{\min}^h| - \frac{t}{h} |E_{\min}^h \setminus E_t| \\
 &= P(F) + \frac{\tilde{t} - t}{h} |E_{\tilde{t}} \setminus E_{\min}^h|. \tag{3.7}
 \end{aligned}$$

In turn, applying Lemma 2.10 to  $E_t$  and  $E_{\tilde{t}}$  gives us

$$\begin{aligned}
 P(F) &= P((E_t \cap E_{\min}^h) \cup E_{\tilde{t}}) \\
 &\leq P(E_t \cap E_{\min}^h) + \frac{n+1}{r_0/2} |E_{\tilde{t}} \setminus E_{\min}^h| \\
 &\leq P(E_{\min}^h) + \frac{n+1}{r_0/2} |E_{\min}^h \setminus E_t| + \frac{n+1}{r_0/2} |E_{\tilde{t}} \setminus E_{\min}^h| \\
 &= P(E_{\min}^h) + \frac{4(n+1)}{r_0} |E_{\tilde{t}} \setminus E_{\min}^h|. \tag{3.8}
 \end{aligned}$$

We combine (3.7) and (3.8) and recall  $|E_{\tilde{t}} \setminus E_{\min}^h| > 0$  to observe that

$$\frac{t - \tilde{t}}{h} \leq \frac{4(n+1)}{r_0}.$$

Thus, by letting  $t \rightarrow d_+$ , we obtain the second estimate in (3.5).

To prove the second part of the claim, we denote by  $C$  a generic positive constant which may change its value from the line to line but depends only on  $n, m_0$  and  $r_0$ . We fix any connected component  $E^i$  of  $E$ . By Lemma 2.10 and (2.36) we have  $\text{diam}(E^i) \leq C$  and  $P(E) \leq C$ . If  $E^j$  is a connected component of  $E$  distinct to  $E^i$ , then UBC with  $r_0$  guarantees  $\text{dist}(E^i, E^j) \geq r_0$ . Assuming that  $\sqrt{h} \leq r_0/\max\{n+1, 8\gamma_n\}$  we have, by (3.3), (3.6), openness of  $E_{\min}^h$  and  $\partial^* E_{\min}^h = \partial E_{\min}^h$ , that  $E_{\min}^h \Delta E \subset\subset \mathcal{N}_{r_0/4}(\partial E)$  and  $|d_E/h| \leq 4(n+1)/r_0$  on  $\partial^* E_{\min}^h$ . Again, we infer from the previous observations that for the intersection  $\tilde{E}^i = E_{\min}^h \cap (E^i + B_{r_0/4})$  it holds  $\partial^* \tilde{E}^i = \partial^* E_{\min}^h \cap (E^i + B_{r_0/4})$ ,  $H_{\tilde{E}^i} = H_{E_{\min}^h}|_{\partial^* \tilde{E}^i}$ ,  $\text{diam}(\tilde{E}^i) \leq C + r_0/2 \leq C$  and  $|\tilde{E}^i| \geq |B_{r_0/2}|$ . Using the divergence theorems and the Euler-Lagrange equation (3.4), which holds in the sense of (2.7) on  $\partial^* \tilde{E}^i$ , we compute that

$$\begin{aligned}
 \lambda^h(n+1)|\tilde{E}^i| &= \int_{\partial^* \tilde{E}^i} \lambda^h(\text{id} \cdot \nu_{\tilde{E}^i}) \, d\mathcal{H}^n = \int_{\partial^* \tilde{E}^i} \left( H_{\tilde{E}^i} + \frac{d_E}{h} \right) (\text{id} \cdot \nu_{\tilde{E}^i}) \, d\mathcal{H}^n \\
 &= nP(\tilde{E}^i) + \int_{\partial^* \tilde{E}^i} \frac{d_E}{h} (\text{id} \cdot \nu_{\tilde{E}^i}) \, d\mathcal{H}^n.
 \end{aligned}$$

By translating the coordinates, we may assume  $0 \in \tilde{E}^i$  so  $|\text{id}| \leq \text{diam}(\tilde{E}^i) \leq C$  on  $\partial^* \tilde{E}^i$ . Since we also have  $P(\tilde{E}^i) \leq P(E_{\min}^h) \leq P(E) \leq C$ ,  $|\tilde{E}^i| \geq |B_{r_0/2}|$  and  $|d_E/h| \leq 4(n+1)/r_0$  on  $\partial^* \tilde{E}^i$ , we infer from the previous computation  $|\lambda^h| \leq C_0$  for  $C_0 = C_0(n, m_0, r_0) \in \mathbb{R}_+$ . Therefore, using the Euler-Lagrange equation (3.4) and the first estimate again we have, by possibly increasing  $C_0$ , that  $\|H_{E_{\min}^h}\|_{L^\infty(\partial^* E_{\min}^h)} + |\lambda^h| \leq C_0$ . Finally, if  $|E_{\min}^h| \neq m_0$ , then  $|\lambda^h| = 1/\sqrt{h}$ . Thus, assuming  $h \leq (2C_0)^{-2}$  excludes this possibility and hence it must hold  $|E_{\min}^h| = m_0$ .  $\square$

Proposition 3.1, allows us to deduce, via Allard’s regularity theorem, that the singular set of minimizer is in fact empty. Further, standard Schauder estimates gives us a quantitative, albeit non-sharp, UBC for a minimizer.

**Lemma 3.2.** *Assume  $E \subset \mathbb{R}^{n+1}$  is an open and bounded set of volume  $m_0$  which satisfies UBC with radius  $r_0$ . There are positive numbers  $h_0 = h_0(n, m_0, r_0)$  and  $c_0 = c_0(n, m_0, r_0)$  such that if  $h \leq h_0$ , then  $\partial E \setminus \partial E^* = \emptyset$ ,  $E_{\min}^h$  is  $C^{3,\alpha}$ -regular with any  $0 < \alpha < 1$  and  $E_{\min}^h$  satisfies UBC with radius  $c_0 h^{1/3}$ . In particular, (3.4) is satisfied in the classical sense on  $\partial E_{\min}^h$ . Moreover, if  $E$  is  $C^k$ -regular, with  $k \geq 2$ , then  $E_{\min}^h$  is  $C^{k+2}$ -regular.*

*Proof.* We divide the proof into two steps. Recall that we may assume  $E_{\min}^h$  to be open. In the proof,  $C$  denotes a generic positive constant which may change its value from line to line but it depends only on  $n, m_0$  and  $r_0$ .

**Step 1:** By using Allard’s regularity theorem we show that the topological boundary  $\partial E_{\min}^h$  agrees with the reduced boundary  $\partial^* E_{\min}^h$  when  $h$  is sufficiently small. To be more precise, we show that there exist positive numbers  $\rho = \rho(n, m_0, r_0)$  and  $h_1 = h_1(n, m_0, r_0, \rho)$  such that if  $h \leq h_1$  and  $x \in \partial E_{\min}^h$ , then, by possibly rotating the coordinates, there is a function  $f \in C^{1,1/3}(B_\rho^n(x'))$  such that

$$C(x, \rho, 2\rho) \cap E_{\min}^h = \{y \in C(x, \rho, 2\rho) : y_{n+1} < f(y)\} \tag{3.9}$$

and  $f$  satisfies the estimates

$$\|\nabla f\|_{L^\infty(B_\rho^n(x'))} \leq 1 \quad \text{and} \quad \|\nabla f\|_{C^{0,1/3}(B_\rho^n(x'))} \leq C. \tag{3.10}$$

In particular, (3.9) implies that  $\partial^* E = \partial E$  and hence, by our earlier discussion, we conclude that  $E_{\min}^h$  is  $C^{2,\alpha}$ -regular with any  $0 < \alpha < 1/2$ . We may assume that  $h_1$  is chosen so small that via Proposition 3.1 the boundary  $\partial E_{\min}^h$  is contained in  $\mathcal{N}_{r_0/2}(\partial E)$ . Since  $d_E \in C^{1,1}(\mathcal{N}_{r_0/2}(\partial E))$ , then recalling the Euler-Lagrange equation (3.4) we may write the generalized mean curvature of  $E_{\min}^h$  as a restriction of a  $C^{1,1}$ -function to  $\partial E_{\min}^h$ . Therefore, by using standard Schauder estimates, one may show that  $E_{\min}^h$  is actually  $C^{3,\alpha}$ -regular with any  $0 < \alpha < 1$ . Also, the same method gives us  $C^{k+2,\alpha}$ -regularity for any  $k \geq 2$ , if  $E$  is already known to be  $C^{k,\alpha}$ -regular. This is well-known procedure and we leave it to the reader.

The claim of Step 1 follows essentially from [46, Thm 2.5.2], if we prove that for every  $x \in \partial E_{\min}^h$  and  $\varepsilon \in \mathbb{R}_+$  there are positive numbers  $\rho = \rho(n, m_0, r_0, \varepsilon)$  and  $\tilde{h} = \tilde{h}(n, m_0, r_0, \rho, \varepsilon)$  such that if  $h \leq \tilde{h}$ , then

$$\frac{\mathcal{H}^n(B_\rho(x) \cap \partial^* E_{\min}^h)}{|B_\rho^n|} \leq 1 + \varepsilon \quad \text{and} \tag{3.11}$$

$$\rho^{1/3} \left( \int_{B_\rho(x) \cap \partial^* E_{\min}^h} |H_{E_{\min}^h}|^{3n/2} d\mathcal{H}^n \right)^{2/3n} \leq \varepsilon. \tag{3.12}$$

We fix  $\varepsilon > 0$  and initially assume  $h \leq h_0$ , where  $h_0$  is from Proposition 3.1. It follows from Proposition 3.1 and the fact  $\partial E_{\min}^h = \overline{\partial^* E_{\min}^h}$  that

$$(\overline{E_{\min}^h} \cup \overline{E}) \setminus (E_{\min}^h \cap E) \subset \mathcal{N}_{Ch}(\partial E). \tag{3.13}$$

Thus, we may assume that  $(\overline{E_{\min}^h} \cup \overline{E}) \setminus (E_{\min}^h \cap E) \subset \mathcal{N}_{r_0/2}(\partial E)$  where the projection  $\pi_{\partial E}$  is well-defined. Proposition 3.1 also gives us  $|E_{\min}^h| = m_0$ . Next, we fix  $x \in \partial E_{\min}^h$ . Without loss of generality, we may assume  $\pi_{\partial E}(x) = 0$  and  $\nu_E(0) = e_{n+1}$ . Then it follows from Proposition 2.7 that there is  $g \in C^{1,1}(B_{r_0/2}^n)$  such that  $|g(y')| < |y'|^2/r_0$ ,  $|\nabla g(y')| < 2|y'|/r_0$  for every  $y' \in B_{r_0/2}^n$  and

$$\mathbf{C}(0, r_0/2, r_0/2) \cap E = \{y \in \mathbf{C}(0, r_0/2, r_0/2) : y_{n+1} < g(y')\}.$$

We have, for every  $0 < \rho < r_0/4$ , a density bound

$$P(E; \mathbf{C}(0, \rho, r_0/2)) = \int_{B_\rho^n} \sqrt{1 + |\nabla g|^2} \, dy' \leq (1 + C\rho^2)|B_\rho^n|. \tag{3.14}$$

Suppose that  $y \in \mathbf{C}(0, \rho, r_0/2) \cap ((\overline{E_{\min}^h} \cup \overline{E}) \setminus (E_{\min}^h \cap E))$  for  $0 < \rho < r_0/4$ . Recalling (3.13), we may assume that  $\pi_{\partial E}(y) \in \mathbf{C}(0, r_0/2, r_0/2)$  and since  $|\nabla g| \leq C$  in  $B_{r_0/2}^n$  we estimate that

$$\begin{aligned} |y_{n+1} - g(y')| &\leq |y - \pi_{\partial E}(y)| + |\pi_{\partial E}(y) - (y', g(y'))| \\ &\leq |y - \pi_{\partial E}(y)| + C|(\pi_{\partial E}(y))' - y'| \leq Ch. \end{aligned}$$

It follows then from Fubini's theorem that

$$\left| \mathbf{C}(0, \rho, r_0/2) \cap \left( (\overline{E_{\min}^h} \cup \overline{E}) \setminus (E_{\min}^h \cap E) \right) \right| \leq C\rho^n h \text{ and} \tag{3.15}$$

$$\mathcal{H}^n \left( \partial \mathbf{C}(0, \rho, r_0/2) \cap \left( (\overline{E_{\min}^h} \cup \overline{E}) \setminus (E_{\min}^h \cap E) \right) \right) \leq C\rho^{n-1} h \tag{3.16}$$

for  $0 < \rho < r_0/4$ . We define for such  $\rho$  a comparison set  $F_\rho$  by setting

$$F_\rho = (E_{\min}^h \setminus \mathbf{C}(0, \rho, r_0/2)) \cup (E \cap \mathbf{C}(0, \rho, r_0/2)),$$

and we make the following technical observations: first, since  $E_{\min}^h \cap E$  is open and contained in  $F_\rho$ , then  $\mathcal{H}^n(\partial^* F_\rho \cap (E_{\min}^h \cap E)) = 0$ . Second,  $\partial^* F_\rho \subset \overline{E_{\min}^h} \cup \overline{E}$ . With help of these, (3.14) and (3.16) we estimate

$$\begin{aligned} P(F_\rho) &= P(F_\rho; \mathbf{C}(0, \rho, r_0/2)) + P(F_\rho; \partial \mathbf{C}(0, \rho, r_0/2)) \\ &\quad + P(F_\rho; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) \\ &= P(E; \mathbf{C}(0, \rho, r_0/2)) + \mathcal{H}^n(\partial^* F_\rho \cap \partial \mathbf{C}(0, \rho, r_0/2)) \\ &\quad + P(E_{\min}^h; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) \\ &\leq P(E; \mathbf{C}(0, \rho, r_0/2)) + P(E_{\min}^h; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) \\ &\quad + \mathcal{H}^n \left( \partial \mathbf{C}(0, \rho, r_0/2) \cap \left( (\overline{E_{\min}^h} \cup \overline{E}) \setminus (E_{\min}^h \cap E) \right) \right) \end{aligned}$$

$$\leq (1 + C\rho^2)|B_\rho^n| + P(E_{\min}^h; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) + C\rho^{n-1}h.$$

Thus, the inequality  $\mathcal{F}_h(E_{\min}^h, E) \leq \mathcal{F}_h(F_\rho, E)$ , (3.13), (3.15),  $|E_{\min}^h| = m_0$  and the definition of  $F_\rho$  yield

$$\begin{aligned} & P(E_{\min}^h; \mathbf{C}(0, \rho, r_0/2)) + \frac{1}{h} \int_{\mathbf{C}(0, \rho, r_0/2) \cap (E_{\min}^h \Delta E)} |d_E| \, dx \\ & \leq (1 + C\rho^2)|B_\rho^n| + \frac{1}{\sqrt{h}} ||F_\rho| - m_0| + C\rho^{n-1}h \\ & \leq (1 + C\rho^2)|B_\rho^n| + \frac{1}{\sqrt{h}} |\mathbf{C}(0, \rho, r_0/2) \cap (E_{\min}^h \Delta E)| + C\rho^{n-1}h \\ & \leq (1 + C\rho^2)|B_\rho^n| + C(\rho^n\sqrt{h} + \rho^{n-1}h). \end{aligned}$$

Recall that for the fixed point  $x \in \partial E_{\min}^h$  it holds  $x = d_E(x)e_{n+1}$  with  $|d_E(x)| \leq Ch$ . Thus we may assume  $B_\rho(x) \subset \mathbf{C}(0, \rho, r_0/2)$  for  $0 < \rho < r_0/4$ . Hence, the above estimate yields

$$P(E_{\min}^h; B_\rho(x)) \leq (1 + C\rho^2)|B_\rho^n| + C(\rho^n\sqrt{h} + \rho^{n-1}h). \tag{3.17}$$

Moreover, it holds  $\|H_{E_{\min}^h}\|_{L^\infty(\partial^* E_{\min}^h)} \leq C$  by Proposition 3.1,  $P(E_{\min}^h) \leq P(E)$  and  $P(E) \leq C$  by Lemma 2.10. Therefore,

$$\rho^{\frac{1}{3}} \left( \int_{B_\rho(x) \cap \partial^* E_{\min}^h} |H_{E_{\min}^h}|^{\frac{3n}{2}} \, d\mathcal{H}^n \right)^{\frac{2}{3n}} \leq C\rho^{\frac{1}{3}}.$$

We infer from the previous estimate and (3.17) the existence of numbers  $\tilde{h}$  and  $\rho$  satisfying (3.11) and (3.12).

**Step 2:** We assume that  $h \leq h_1$  and fix  $x \in \partial E_{\min}^h$ . We may assume that  $x = 0$  and  $\nu_{E_{\min}^h}(0) = e_{n+1}$ . According to Step 1, up to a possible rotation of the coordinates, there is  $f \in C^3(B_{\rho_1}^n(x'))$  with  $f(0) = \nabla f(0) = 0$  satisfying (3.9) and (3.10). We use Schauder estimate in a quantitative manner to prove there is a positive  $h_0 = h_0(n, m_0, r_0) \leq h_1$  such that  $h \leq h_0$  implies

$$\|\nabla^2 f\|_{L^\infty(B_{\rho/2}^n)} \leq Ch^{-\frac{1}{3}}. \tag{3.18}$$

Once we have proven (3.18) then the claim that  $E_{\min}^h$  satisfies UBC with radius  $c_0h^{1/3}$  follows in a straightforward manner as we discussed in Remark 2.8.

Thus, we are left to prove (3.18). We may write  $H_{E_{\min}^h}$  in local coordinates as the mean curvature of the subgraph  $\{(y', y_{n+1} : y' \in B_\rho^n, y_{n+1} < f(y')\}$ , that is,

$$H_{E_{\min}^h}(y', f(y')) = -\operatorname{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right)(y') = -\operatorname{Tr} \left( \mathcal{A}(y') \nabla^2 f(y') \right). \tag{3.19}$$

It follows from (3.10) that  $\mathcal{A}$  is uniformly elliptic and bounded in the  $C^{0,1/3}$ -sense. To be more precise, we have

$$\inf_{y' \in B_\rho^n} \min_{\xi \in \partial B_1^n} \mathcal{A}(y')\xi \cdot \xi \geq 1/C \text{ and } \max_{ij} \|[\mathcal{A}]_{ij}\|_{C^{0,1/3}(B_\rho^n)} \leq C.$$

Thus, by using standard Schauder interior estimate [24], (3.10) and (3.19), we obtain

$$\begin{aligned} \|\nabla^2 f\|_{C^{0,1/3}(B_{\rho/2}^n)} &\leq C \left( \|u\|_{C^{0,1/3}(B_\rho^n)} + \|f\|_{L^\infty(B_\rho^n)} \right) \\ &\leq C \left( \|u\|_{C^{0,1/3}(B_\rho^n)} + 1 \right), \end{aligned} \tag{3.20}$$

where  $u : B_{\rho_1}^n \rightarrow \mathbb{R}^n$  is given by  $u(y') = H_{E_{\min}^h}(y', f(y'))$ . We may assume  $h$  is chosen sufficiently small so that via Proposition 3.1 we have  $\|u\|_{L^\infty(B_\rho^n)} \leq C$ . Again, (3.10) implies  $|\nabla u(y')| \leq C|\nabla_\tau H_{E_{\min}^h}(y', f(y'))|$  for every  $y' \in B_\rho^n$ . On the other hand, by (tangentially) differentiating the Euler-Lagrange equality (3.4) we obtain  $|\nabla_\tau H_{E_{\min}^h}(y', f(y'))| \leq 1/h$  for every  $y' \in B_\rho^n$ . Hence,  $\|\nabla u\|_{L^\infty(B_\rho^n)} \leq C/h$  and since  $\|u\|_{L^\infty(B_\rho^n)} \leq C$ , assuming  $h \leq 1$  yields  $\|u\|_{C^1(B_\rho^n)} \leq C/h$ . Again, Lemma 2.1 yields  $\|u\|_{C^{0,1/3}(B_\rho^n)} \leq Ch^{-1/3}$  and hence, by recalling (3.20), we conclude the existence of  $h_0 = h_0(n, m_0, r_0)$  satisfying (3.18) for all  $h \leq h_0$ .  $\square$

*Remark 3.3.* We may replace the exponent  $1/3$  with a generic  $0 < \alpha < 1$  in the proof of Lemma 3.2. Then, naturally,  $h_0$  and  $c_0$  also depend on  $\alpha$ . UBC with radius  $r_0$  for  $E$  and UBC with radius  $c_0h^{1/3}$  for  $E_{\min}^h$  imply together with the distance estimate of Proposition 3.1 and (2.22) that there is  $h_0 = h_0(n, m_0, r_0)$  such that if  $h \leq h_0$ , then  $\nabla d_E \cdot \nu_{E_{\min}^h} > 0$  on  $\partial E_{\min}^h$  and the projection  $\pi_{\partial E}$  is injective on  $\partial E_{\min}^h$ .

### 4. Uniform Ball Condition for Short-Time

In this section, we adopt the two-point function method to prove that if the initial set  $E_0$  satisfies UBC with radius  $r_0$ , then there are positive numbers  $h_0$  and  $T_0$  such that

$$h \leq h_0 \implies E_t^h \text{ satisfies UBC with radius } r_0/2 \text{ for } 0 \leq t \leq T_0, \tag{4.1}$$

where the approximative flow  $(E_t^h)_{t \geq 0}$  starting from  $E_0$  is defined as in (3.1). For more precise statement, see Theorem 4.7 at the end of the section. As we have seen in Lemma 3.2, UBC for an initial set is crucial, as it guarantees that the corresponding minimizer of the energy (3.2) has improved regularity and an initial quantitative bound on UBC although the latter depends on  $h$ . In this section, we improve the previous non-sharp estimate on UBC for the minimizer by showing the minimizer satisfies almost the same UBC as the initial set.

The original idea of the two-point function goes back to [27], where it is used to study the regularity of the classical solution to the mean curvature flow. We

refer to [10] for a comprehensive overview of the topic and mention also the works [4, 11, 18] which have inspired us. Here we will show that the method can be applied to the approximative flat flow at the level of discrete time scale. We will assume that the approximative flat flow is related to the volume preserving mean curvature flow but the arguments hold with essentially no modifications also in the case of the mean curvature flow.

#### 4.1. Two-Point Function Method

The main idea is to double the variables and, given a set  $E \subset \mathbb{R}^{n+1}$  satisfying UBC, to study the function  $S_E$  defined for  $(x, y) \in \partial E \times \partial E$  with  $x \neq y$  as

$$S_E(x, y) := \frac{(x - y) \cdot \nu_E(x)}{|x - y|^2}. \tag{4.2}$$

It is known, but we will include the proof below, that the maximum value of  $|S_E|$  is explicitly related to the maximal UBC for  $E$ . In other words, doubling the variables allows us to quantify the maximal UBC via the function  $S_E$ . It is interesting that the idea of doubling the variables is also used in [29] to study regularity of solutions of nonlinear PDEs.

For the next lemma we note that if a set  $E$  satisfies UBC with radius  $r$ , then it satisfies UBC with every  $0 < \rho < r$ . We define  $r_E$  to be the supremum of such radii and recalling our previous discussion we may write this as

$$r_E = \sup\{r > 0 : d_E \text{ is differentiable in } \mathcal{N}_r(\partial E)\}. \tag{4.3}$$

Note that  $r_E > 0$ . We use the abbreviation  $\|S_E\|_{L^\infty} := \sup\{|S_E(x, y)| : x, y \in \partial E, x \neq y\}$ .

**Lemma 4.1.** *Let  $E \subset \mathbb{R}^{n+1}$  be an open and bounded set satisfying UBC. Then it holds that*

$$2\|S_E\|_{L^\infty} = \frac{1}{r_E} \text{ and } \frac{|v(x) - v(y)|}{|x - y|} \leq 2\|S_E\|_{L^\infty} \text{ for every } x, y \in \partial E \text{ with } x \neq y,$$

where  $r_E$  is defined in (4.3). In the case  $E$  is  $C^2$ -regular, we also have  $|H_E|, |B_E| \leq 2n\|S_E\|_{L^\infty}$  on  $\partial E$ .

*Proof.* Let us first show  $2\|S_E\|_{L^\infty} \geq 1/r_E$ . First of all, we infer from the boundedness of  $E$  that  $r_E < \infty$ . Since  $E$  does not satisfy UBC with given a radius  $r \in (r_E, \infty)$ , there is  $z \in \mathcal{N}_r(\partial E)$  such that  $d_E$  is not differentiable at  $z$ . Hence, there are distinct points  $x, y \in \partial E$  such that  $|z - x| = |d_E(z)| = |z - y|$ . Without loss of generality, we may assume  $z = 0$  which implies  $|x| = |y| < r$  and  $v_E(x) = \pm x/|x|$ . Thus,

$$\begin{aligned} |S_E(x, y)| &= \left| \frac{\langle x/|x|, x - y \rangle}{|x - y|^2} \right| = \frac{1}{|x|} \left| \frac{|x|^2 - \langle x, y \rangle}{|x - y|^2} \right| \\ &= \frac{1}{2|x|} \left| \frac{|x|^2 - 2\langle x, y \rangle + |y|^2}{|x - y|^2} \right| \end{aligned}$$

$$= \frac{1}{2|d_E(x)|} > \frac{1}{2r}$$

and we conclude the inequality  $2\|S_E\|_{L^\infty} \geq 1/r_E$ .

To conclude the opposite estimate, we choose  $0 < r < r_E$ . Let  $x, y \in \partial E$  be distinct points. Since  $E$  satisfies UBC with  $r$ , we have  $|d_E(x \pm rv_E(x))| = r$  and, hence,

$$r^2 \leq |x \pm rv_E(x) - y|^2 = r^2 \pm 2r\langle v_E(x), x - y \rangle + |x - y|^2.$$

By subtracting and dividing terms we obtain  $\pm 2S_E(x, y) \leq 1/r$ . We let  $r \rightarrow r_E$  to obtain  $2\|S_E\|_{L^\infty} \leq 1/r_E$ . Thus,  $2\|S_E\|_{L^\infty} = 1/r_E$ . The rest of the claim is a direct consequence of the previous identity, (2.30) and Proposition 2.7.  $\square$

An obvious consequence of Lemma 4.1 is that for every open and bounded set  $E \subset \mathbb{R}^{n+1}$  it holds

$$\|S_E\|_{L^\infty} \geq c_0 \quad (4.4)$$

for a positive constant  $c_0 = c_0(n, |E|)$ .

We will also use the regularized version of  $S_E$ , which we define for any  $\varepsilon \in \mathbb{R}_+$  as  $S_{E,\varepsilon} : \partial E \times \partial E \rightarrow \mathbb{R}$ ,

$$S_{E,\varepsilon}(x, y) := \frac{(x - y) \cdot v_E(x)}{|x - y|^2 + \varepsilon}. \quad (4.5)$$

As in the case of  $S_E$ , we use the abbreviation  $\|S_{E,\varepsilon}\|_{L^\infty} = \max\{|S_{E,\varepsilon}(x, y)| : (x, y) \in \partial E \times \partial E\}$ . The idea behind considering  $S_{E,\varepsilon}$  instead of  $S_E$  is that, on the one hand,  $S_{E,\varepsilon} \rightarrow S$  pointwise in  $\partial E \times \partial E \setminus \{(x, x) : x \in \partial E\}$  as  $\varepsilon$  tends to zero (in particular,  $\|S_{E,\varepsilon}\|_{L^\infty} \uparrow \|S_E\|_{L^\infty}$ ) and, on the other hand, we may differentiate  $S_{E,\varepsilon}$  on the product  $\partial E \times \partial E$  provided that  $E$  is sufficiently regular. The following calculations are similar to [4, 18] but we give them in order to be self-consistent.

Let us first differentiate  $S_{E,\varepsilon}$  in the case  $E$  is  $C^2$ -regular. In the computations, the notations  $\nabla_\tau^x$  and  $\nabla_\tau^y$  stand for the tangential differentiation along  $\partial E$  with respect to  $x$  and  $y$ -variables respectively. Recalling the basic identities (2.4) as well as observing  $B_E v_E = 0$  and  $\nabla_\tau \text{id} = P_{\partial E}$  on  $\partial E$  we compute

$$\begin{aligned} \nabla_\tau^x S_{E,\varepsilon}(x, y) &= \frac{\nabla_\tau^x((x - y) \cdot v_E(x))}{|x - y|^2 + \varepsilon} - \frac{(x - y) \cdot v_E(x)}{(|x - y|^2 + \varepsilon)^2} \nabla_\tau^x |x - y|^2 \\ &= \frac{B_E(x)(x - y) - 2S_{E,\varepsilon}(x, y) P_{\partial E}(x)(x - y)}{|x - y|^2 + \varepsilon}. \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} \nabla_\tau^y S_{E,\varepsilon}(x, y) &= \frac{\nabla_\tau^y((x - y) \cdot v_E(x))}{|x - y|^2 + \varepsilon} - \frac{(x - y) \cdot v_E(x)}{(|x - y|^2 + \varepsilon)^2} \nabla_\tau^y |x - y|^2 \\ &= \frac{P_{\partial E}(y)(-v_E(x) + 2S_{E,\varepsilon}(x, y)(x - y))}{|x - y|^2 + \varepsilon} \end{aligned} \quad (4.7)$$

for every  $(x, y) \in \partial E \times \partial E$ . We immediately obtain the following identities at critical points:



**Lemma 4.2.** *Let  $E \subset \mathbb{R}^{n+1}$  be a bounded and  $C^2$ -regular set. Assume  $(x, y) \in \partial E \times \partial E$  is a local maximum or a local minimum point of  $S_{E,\varepsilon}$  defined in (4.5). Then it holds that*

$$B_E(x)(x - y) = 2S_{E,\varepsilon}(x, y)P_{\partial E}(x)(x - y) \text{ and} \tag{4.8}$$

$$P_{\partial E}(y)v_E(x) = 2S_{E,\varepsilon}(x, y)P_{\partial E}(y)(x - y). \tag{4.9}$$

Moreover, the condition  $r_E > \sqrt{\varepsilon}$  implies

$$v_E(y) = \frac{v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y)}{(v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y)) \cdot v_E(y)}. \tag{4.10}$$

*Proof.* Since  $(x, y)$  is a critical point for the functions  $S_{E,\varepsilon}(x, \cdot)$  and  $S_{E,\varepsilon}(\cdot, y)$ , then the equality (4.8) follows from (4.6) and the equality (4.9) follows from (4.7). Using  $P_{\partial E}(y) = I - v_E(y) \otimes v_E(y)$  and (4.9) we have

$$v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y) = [(v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y)) \cdot v_E(y)] v_E(y).$$

The equality (4.10) thus follows once we show that

$$v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y) \neq 0. \tag{4.11}$$

We argue by contradiction and assume  $v_E(x) = 2S_{E,\varepsilon}(x, y)(x - y)$ . Then it holds  $S_{E,\varepsilon}(x, y) \neq 0$  and the definition of  $S_{E,\varepsilon}(x, y)$  implies

$$S_{E,\varepsilon}(x, y) = \frac{(x - y) \cdot v_E(x)}{|x - y|^2 + \varepsilon} = 2S_{E,\varepsilon}(x, y) \frac{|x - y|^2}{|x - y|^2 + \varepsilon}.$$

Therefore, we have  $|x - y| = \sqrt{\varepsilon}$ . On the other hand, the contradiction assumption, the definition of  $S_{E,\varepsilon}$  and Lemma 4.1 together yield that

$$1 = |v_E(x)| = 2|S_{E,\varepsilon}(x, y)| |x - y| = 2|S_{E,\varepsilon}(x, y)| \sqrt{\varepsilon} \leq 2\|S_E\|_{L^\infty} \sqrt{\varepsilon} = \frac{\sqrt{\varepsilon}}{r_E},$$

which is impossible, by the assumption that  $r_E > \sqrt{\varepsilon}$ .  $\square$

If  $E$  has higher regularity and  $\varepsilon$  is sufficiently small, we may naturally extract more information at local extreme points. Indeed, if  $E$  is  $C^3$ -regular, then by maximum principle at a local maximum (minimum) point  $(x, y) \in \partial E \times \partial E$  of  $S_{E,\varepsilon}$  it holds that

$$\Delta_\tau^x S_{E,\varepsilon}(x, y) + 2 \operatorname{div}_\tau^x \nabla_\tau^y S_{E,\varepsilon}(x, y) + \Delta_\tau^y S_{E,\varepsilon}(x, y) \stackrel{(\geq)}{\leq} 0. \tag{4.12}$$

We calculate the LHS of (4.12) in the next lemma.

**Lemma 4.3.** *Let  $E \subset \mathbb{R}^{n+1}$  be a bounded and  $C^3$ -regular set with  $r_E > \sqrt{\varepsilon}$ . At a local maximum (minimum) point  $(x, y) \in \partial E \times \partial E$  of  $S_{E,\varepsilon}$  it holds that*

$$\begin{aligned} & \frac{\nabla_\tau H_E(x) \cdot (x - y)}{|x - y|^2 + \varepsilon} + \frac{(v_E(x) \cdot v_E(y)) H_E(y) - H_E(x)}{|x - y|^2 + \varepsilon} \\ & \stackrel{(\geq)}{\leq} |B_E(x)|^2 S_{E,\varepsilon}(x, y) - 2H_E(x) S_{E,\varepsilon}^2(x, y) - 2H_E(y) S_{E,\varepsilon}(y, x) S_{E,\varepsilon}(x, y). \end{aligned}$$

*Proof.* First, we compute the terms on the LHS of (4.12) by taking tangential divergences of (4.6) and (4.7) with respect to  $x$  and  $y$ -variables. In the computations, we use the identities (2.5) and the fact that the gradients  $\nabla_\tau^x S_{E,\varepsilon}(x, y)$  and  $\nabla_\tau^y S_{E,\varepsilon}(x, y)$  vanish. Omitting all the details we obtain by straightforward calculation

$$\begin{aligned} \Delta_\tau^x S_{E,\varepsilon}(x, y) &= \operatorname{div}_\tau^x (\nabla_\tau^x S_{E,\varepsilon}(x, y)) \\ &= \operatorname{div}_\tau^x \left( \frac{B_E(x)(x - y) - 2S_{E,\varepsilon}(x, y) P_{\partial E}(x)(x - y)}{|x - y|^2 + \varepsilon} \right) \\ &= \frac{\nabla_\tau H_E(x) \cdot (x - y)}{|x - y|^2 + \varepsilon} + \frac{H_E(x)}{|x - y|^2 + \varepsilon} - 2S_{E,\varepsilon}(x, y) \frac{n}{|x - y|^2 + \varepsilon} \\ &\quad - |B_E|^2 S_{E,\varepsilon}(x, y) + 2S_{E,\varepsilon}^2(x, y) H_E(x), \end{aligned}$$

$$\begin{aligned} \Delta_\tau^y S_{E,\varepsilon}(x, y) &= \operatorname{div}_\tau^y (\nabla_\tau^y S_{E,\varepsilon}(x, y)) \\ &= \operatorname{div}_\tau^y \left( -\frac{P_{\partial E}(y)v_E(x) + 2S_{E,\varepsilon}(x, y) P_{\partial E}(y)(x - y)}{|x - y|^2 + \varepsilon} \right) \\ &= \frac{(v_E(x) \cdot v_E(y)) H_E(y)}{|x - y|^2 + \varepsilon} - 2S_{E,\varepsilon}(x, y) \frac{n}{|x - y|^2 + \varepsilon} \\ &\quad + 2S_{E,\varepsilon}(x, y) S_{E,\varepsilon}(y, x) H_E(y) \end{aligned}$$

and

$$\begin{aligned} \operatorname{div}_\tau^x \nabla_\tau^y S_{E,\varepsilon}(x, y) &= \operatorname{div}_\tau^x \left( -\frac{P_{\partial E}(y)v_E(x) + 2S_{E,\varepsilon}(x, y) P_{\partial E}(y)(x - y)}{|x - y|^2 + \varepsilon} \right) \\ &= -\frac{H_E(x)}{|x - y|^2 + \varepsilon} + \frac{(B_E(x)v_E(y)) \cdot v_E(y)}{|x - y|^2 + \varepsilon} \\ &\quad + 2S_{E,\varepsilon}(x, y) \frac{n}{|x - y|^2 + \varepsilon} \\ &\quad - 2S_{E,\varepsilon}(x, y) \frac{(P_{\partial E}(x)v_E(y)) \cdot v_E(y)}{|x - y|^2 + \varepsilon}. \end{aligned}$$

Collecting the terms and applying the inequality (4.12), we obtain that at a local maximum (minimum) point it holds that

$$\begin{aligned} 0 &\stackrel{(\leq)}{\geq} \frac{\nabla_\tau H_E(x) \cdot (x - y)}{|x - y|^2 + \varepsilon} + \frac{(v_E(x) \cdot v_E(y)) H_E(y) - H_E(x)}{|x - y|^2 + \varepsilon} \\ &\quad - |B_E|^2 S_{E,\varepsilon}(x, y) + 2S_{E,\varepsilon}^2(x, y) H_E(x) + 2S_{E,\varepsilon}(x, y) S_{E,\varepsilon}(y, x) H_E(y) \\ &\quad + 2 \frac{(B_E(x)v_E(y)) \cdot v_E(y)}{|x - y|^2 + \varepsilon} - 4S_{E,\varepsilon}(x, y) \frac{(P_{\partial E}(x)v_E(y)) \cdot v_E(y)}{|x - y|^2 + \varepsilon}. \end{aligned}$$

The claim follows once we show that the last line above vanishes, i.e., that

$$(B_E(x)v_E(y)) \cdot v_E(y) = 2S_{E,\varepsilon}(x, y)(P_{\partial E}(x)v_E(y)) \cdot v_E(y). \tag{4.13}$$

Since  $r_E > \sqrt{\varepsilon}$ , this follows by first applying the equalities (4.8) and (4.10) in Lemma 4.2 and recalling  $B_E(x)v_E(x) = 0$ :

$$\begin{aligned} B_E(x)v_E(y) &= -2S_{E,\varepsilon}(x, y) \frac{B_E(x)(x - y)}{(v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y)) \cdot v_E(y)} \\ &= -4S_{E,\varepsilon}^2(x, y) \frac{P_{\partial E}(x)(x - y)}{(v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y)) \cdot v_E(y)}. \end{aligned}$$

Then we use (4.10) to deduce that

$$P_{\partial E}(x)v_E(y) = -2S_{E,\varepsilon}(x, y) \frac{P_{\partial E}(x)(x - y)}{(v_E(x) - 2S_{E,\varepsilon}(x, y)(x - y)) \cdot v_E(y)},$$

and (4.13) follows.  $\square$

In conclusion, by combining Lemma 4.1 and Lemma 4.3, we obtain that if a bounded  $C^3$ -regular set  $E \subset \mathbb{R}^{n+1}$  satisfies  $r_E > \sqrt{\varepsilon}$ , then at a local maximum (minimum) point  $(x, y) \in \partial E \times \partial E$  of  $S_{E,\varepsilon}$  it holds that

$$\stackrel{+}{(-)} \left( \frac{\nabla_\tau H_E(x) \cdot (x - y)}{|x - y|^2 + \varepsilon} + \frac{(v_E(x) \cdot v_E(y)) H_E(y) - H_E(x)}{|x - y|^2 + \varepsilon} \right) \leq C_n \|S_E\|_{L^\infty}^3. \tag{4.14}$$

### 4.2. Short-Time Uniform Ball Estimate

Let us turn our focus on how to prove (4.1) for an approximative flat flow solution  $(E_t^h)_{t \geq 0}$  defined in (3.1) when the initial set  $E_0$  satisfies UBC with given a radius  $r_0$ . Assuming we may control the evolution of the quantity  $\|S_{E_t^h}\|_{L^\infty}$ , then thanks to Lemma 4.1 we also control (from below) the maximal UBC for  $E_t^h$ .

We motivate ourselves by considering first the continuous and embedded setting. Assume  $(E_t)_t$  is a smooth flow and let  $v_t$  and  $V_t$  denote the outer unit normal of  $E_t$  and the normal velocity of the flow on  $\partial E_t$  respectively. Then one may use the fact that for fixed  $t$  there is a smooth normal parametrization  $(\Phi_s^t)_s$  of the flow such that  $\Phi_0^t = \text{id}$  and  $\partial_s \Phi_s^t = [V_s v_s] \circ \Phi_s^t$ . This follows essentially from [5, Thm 8]. It is straightforward to calculate that for such a parametrization

$$\frac{d}{ds} \Phi_{t+s}^t \Big|_{s=0} = V_t v_t \quad \text{and} \quad \frac{d}{ds} (v_{E_{t+s}} \circ \Phi_{t+s}^t) \Big|_{s=0} = -\nabla_\tau V_t \quad \text{on } \partial E_t. \tag{4.15}$$

In the case of volume preserving mean curvature flow, we have  $V_s = -(H_s - \bar{H}_s)$ , where  $H_s$  is the scalar mean curvature on  $\partial E_s$  and  $\bar{H}_s$  its integral average over  $\partial E_s$ . If  $x$  and  $y$  are distinct points on  $\partial E_t$ , then by using (4.15) and the previous identity, we may compute

$$\begin{aligned} \frac{d}{ds} S_{E_{t+s}}(\Phi_s^t(x), \Phi_s^t(y)) \Big|_{s=0} &= \frac{\nabla_\tau H_E(x) \cdot (x - y)}{|x - y|^2} + \frac{(v_E(x) \cdot v_E(y)) H_E(y) - H_E(x)}{|x - y|^2} \\ &\quad + R_t(x, y), \end{aligned} \tag{4.16}$$

where the remainder term  $R_t(x, y)$  has a bound  $|R_t(x, y)| \leq C_n \|S_{E_t}\|_{L^\infty}^3$ . Suppose that  $\|S_{E_t}\|_{L^\infty} = \pm S_{E_t}(x, y)$  and the function  $s \mapsto \|S_{E_{t+s}}\|_{L^\infty}$  is differentiable at  $s = 0$ , then we deduce

$$\frac{d}{ds} \|S_{E_{t+s}}\|_{L^\infty} \Big|_{s=0} = \pm \frac{d}{ds} S_{E_{t+s}}(\Phi'_s(x), \Phi'_s(y)) \Big|_{s=0}.$$

Again, the estimate (4.14) also holds for  $S_E$  when the points are distinct. Thus, by possibly increasing  $C_n$ , we infer from above and (4.16) that

$$\frac{\|S_{E_{t+s}}\|_{L^\infty} - \|S_{E_t}\|_{L^\infty}}{s} \leq C_n \|S_{E_t}\|_{L^\infty}^3, \tag{4.17}$$

provided that  $s \neq 0$  is sufficiently small.

The idea is to mimic the previous argument in the discrete setting for an approximative flat flow  $(E_t^h)_{t \geq 0}$ . To this end, we need to approximate the two-point functional by its  $\varepsilon$ -regularized version. We consider the element  $E_t^h$  and its consequent set  $E_{t+s}^h$ . For sake of brevity, we use the shorthand notations  $E_1 = E_t^h$  and  $E_2 = E_{t+s}^h$  for the rest of the subsection. First, we want to find a discrete version of the equalities in (4.15). Suppose that an element  $E_1$  satisfies UBC and  $h$  is so small that by the discussion of the previous section we have that  $E_2$  is  $C^1$ -regular set,  $\partial E_2 \subset \mathcal{N}_{r_{E_1}}(\partial E_1)$  and  $\nabla d_{E_1} \cdot \nu_{E_2} > 0$  on  $\partial E_2$  are satisfied.

Then it is natural to project the boundary  $\partial E_2$  to  $\partial E_1$  by the projection  $\pi_{\partial E_1}$  and, hence, using the identities in (2.22) we have

$$\frac{\text{id} - \pi_{\partial E_1}}{h} = \frac{d_{E_1}}{h} (\nu_{E_2} \circ \pi_{\partial E_1}) \quad \text{on } \partial E_2,$$

which can be seen as a discrete time counterpart of the first identity in (4.15). In the next simple but crucial lemma, we derive a relation between  $\nu_{E_2}$  and  $\nu_{E_1} \circ \pi_{\partial E_1}$  for  $x \in \partial E_2$ .

**Lemma 4.4.** *Assume that  $E_1 \subset \mathbb{R}^{n+1}$  is an open set satisfying UBC,  $E_2$  is a  $C^1$ -regular set such that  $\partial E_2 \subset \mathcal{N}_{r_{E_1}}(\partial E_1)$  and  $\nabla d_{E_1} \cdot \nu_{E_2} > 0$  on  $\partial E_2$ . Then*

$$\nu_{E_1} \circ \pi_{\partial E_1} = \nabla_{\tau_2} d_{E_1} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} \nu_{E_2} \quad \text{on } \partial E_2.$$

*Proof.* By using the second identity of (2.22) for  $d_{E_1}$ , as well as the definition of a tangential gradient, the following holds on  $\partial E_2$ :

$$\nu_{E_1} \circ \pi_{\partial E_1} = \nabla d_{E_1} = P_{\partial E_2} \nabla d_{E_1} + (\nabla d_{E_1} \cdot \nu_{E_2}) \nu_{E_2} = \nabla_{\tau_2} d_{E_1} + (\nabla d_{E_1} \cdot \nu_{E_2}) \nu_{E_2}.$$

Since  $|\nu_{E_1} \circ \pi_{\partial E_1}| = 1 = |\nu_{E_2}|$  and  $\nabla_{\tau_2} d_{E_1} \cdot \nu_{E_2} = 0$ , then the previous decomposition implies  $|\nabla d_{E_1} \cdot \nu_{E_2}| = \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}$ . Thus, the claim follows from the assumption  $\nabla d_{E_1} \cdot \nu_{E_2} > 0$  on  $\partial E_2$ .  $\square$

The equality in the statement of Lemma 4.4 gives us a discrete analog for the second equality in (4.15) as

$$\nu_{E_2} - \nu_{E_1} \circ \pi_{\partial E_1} = -\nabla_{\tau_2} d_{E_1} + \frac{|\nabla_{\tau_2} d_{E_1}|^2}{1 + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}} \nu_{E_2} \text{ on } \partial E_2. \quad (4.18)$$

or equivalently

$$\begin{aligned} \nu_{E_2} - \nu_{E_1} \circ \pi_{\partial E_1} = & - \left( \frac{1}{\sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}} \right) \nabla_{\tau_2} d_{E_1} \\ & + \frac{|\nabla_{\tau_2} d_{E_1}|^2}{\sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} + 1 - |\nabla_{\tau_2} d_{E_1}|^2} \nu_{E_1} \circ \pi_{\partial E_1} \text{ on } \partial E_2, \end{aligned} \quad (4.19)$$

which will be useful later. We need yet one technical lemma related to the projection  $\pi_{\partial E_1}$  on the consequent boundary  $\partial E_2$ .

**Lemma 4.5.** *Let  $E_1, E_2 \subset \mathbb{R}^{n+1}$  be open and bounded sets satisfying UBC. If  $\partial E_2 \subset \mathcal{N}_{r_{E_1}/2}(\partial E_1)$ , then for any  $x, y \in \partial E_2$  satisfying  $\pi_{\partial E_1}(x) \neq \pi_{\partial E_1}(y)$  it holds that*

$$\begin{aligned} & \left| |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x - y|^2 \right| \\ & \leq C_0 \|d_{E_1}\|_{L^\infty(\partial E_2)} \left( \|S_{E_1}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty} + \|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_2}\|_{L^\infty}^2 \right) |x - y|^2, \end{aligned}$$

where  $C_0 \geq 1$  is a universal constant.

*Proof.* First, we obtain from (2.22) and the definition of  $S_{E_1}$  that

$$\begin{aligned} & \left| |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x - y|^2 \right| \\ & = -2d_{E_1}(x)S_{E_1}(\pi_{\partial E_1}(x), \pi_{\partial E_1}(y))|\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ & \quad - 2d_{E_1}(y)S_{E_1}(\pi_{\partial E_1}(y), \pi_{\partial E_1}(x))|\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ & \quad - |d_{E_1}(x)(\nu_{E_1} \circ \pi_{\partial E_1})(x) - d_{E_1}(y)(\nu_{E_1} \circ \pi_{\partial E_1})(y)|^2. \end{aligned}$$

Thus,

$$\begin{aligned} & \left| |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x - y|^2 \right| \\ & \leq 4\|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_1}\|_{L^\infty} |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ & \quad + 2|d_{E_1}(x)|^2 |(\nu_{E_1} \circ \pi_{\partial E_1})(x) - (\nu_{E_1} \circ \pi_{\partial E_1})(y)|^2 + 2|d_{E_1}(x) - d_{E_1}(y)|^2 \\ & \leq 4\|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_1}\|_{L^\infty} |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ & \quad + 2\|d_{E_1}\|_{L^\infty(\partial E_2)}^2 |(\nu_{E_1} \circ \pi_{\partial E_1})(x) - (\nu_{E_1} \circ \pi_{\partial E_1})(y)|^2 \\ & \quad + 2|d_{E_1}(x) - d_{E_1}(y)|^2. \end{aligned}$$

The normal  $\nu_{E_1}$  is  $1/r_{E_1}$ -Lipschitz continuous by Proposition 2.7 and  $\pi_{\partial E_1}$  is 2-Lipschitz continuous in  $\mathcal{N}_{r_{E_1}/2}(\partial E_1)$  by Lemma 2.9. On the other hand, recalling

Lemma 4.1 we conclude  $\|d_{E_1}\|_{L^\infty(\partial E_2)}\|S_{E_1}\|_{L^\infty} \leq 1/4$ . Hence, we infer from the previous estimate that

$$\left| |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x - y|^2 \right| \leq 24\|d_{E_1}\|_{L^\infty(\partial E_2)}\|S_{E_1}\|_{L^\infty} |x - y|^2 + 2|d_{E_1}(x) - d_{E_1}(y)|^2. \tag{4.20}$$

Thus, we are left to estimate the term  $|d_{E_1}(x) - d_{E_1}(y)|^2$  on the boundary  $\partial E_2$ .

We divide this into two cases. First, suppose that  $|x - y| \geq r_{E_2}/2$ . Then using Lemma 4.1 we obtain

$$|d_{E_1}(x) - d_{E_1}(y)|^2 \leq \frac{4\|d_{E_1}\|_{L^\infty(\partial E_2)}^2}{r_{E_2}^2} |x - y|^2 \leq 16\|d_{E_1}\|_{L^\infty(\partial E_2)}^2\|S_{E_2}\|_{L^\infty}^2 |x - y|^2. \tag{4.21}$$

Suppose then  $|x - y| < r_{E_1}/2$ . We define a  $C^1$ -extension  $\tilde{d}_{E_1} : \mathcal{N}_{r_{E_2}}(\partial E_2) \rightarrow \mathbb{R}$  of the restriction  $d_{E_1}|_{\partial E_2}$  by setting  $\tilde{d}_{E_1} = d_{E_1} \circ \pi_{\partial E_2}$ . Then  $\nabla \tilde{d}_{E_1} = \nabla \pi_{\partial E_2} \nabla_{\tau_2} d_{E_1} \circ \pi_{\partial E_2}$  and by Lemma 2.9  $|\nabla \pi_{\partial E_2}|_{\text{op}} \leq 2$  in  $\mathcal{N}_{r_{E_2}/2}(\partial E_2)$  so that  $|\nabla \tilde{d}_{E_1}| \leq 2\|\nabla_{\tau_2} \tilde{d}_{E_1}\|_{L^\infty(\partial E_2)}$ . Since the line segment  $J_{yx}$  belongs to  $\mathcal{N}_{r_{E_2}/2}(\partial E_2)$ , we have

$$|d_{E_1}(x) - d_{E_1}(y)|^2 \leq 4\|\nabla_{\tau_2} d_{E_1}\|_{L^\infty(\partial E_2)}^2 |x - y|^2. \tag{4.22}$$

By Lemma 2.9 we have  $|\nabla^2 d_{E_1}|_{\text{op}} \leq 2/r_{E_1}$  in  $\mathcal{N}_{r_{E_1}}(\partial E_1)$ . Therefore, by using Lemma 2.11 and Lemma 4.1 we get an estimate

$$\begin{aligned} & \|\nabla_{\tau_2} d_{E_1}\|_{L^\infty(\partial E_2)}^2 \\ & \leq 4\|d_{E_1}\|_{L^\infty(\partial E_2)} \left( \sup_{\partial E_2} |\nabla^2 d_{E_1}|_{\text{op}} + \frac{\|\nabla_{\tau} d_{E_1}\|_{L^\infty(\partial E_2)}}{r_{E_2}} \right) \\ & \leq 16\|d_{E_1}\|_{L^\infty(\partial E_2)} (\|S_{E_1}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty}). \end{aligned} \tag{4.23}$$

Thus, we gather the estimate as claimed from (4.20), (4.21), (4.22) and the estimate above.  $\square$

We are now ready to prove an analogous estimate to (4.17) in the discrete setting.

**Lemma 4.6.** *Assume that  $E_1 \subset \mathbb{R}^{n+1}$  is an open and bounded set, with  $|E_1| = m_0$ , which satisfies UBC with radius  $r_0 \in \mathbb{R}_+$ . Let  $E_2$  be any minimizer of the energy  $\mathcal{F}_h(\cdot; E_1)$  defined in (3.2). Then there is  $h_0 = h_0(n, m_0, r_0)$  such that for  $h \leq h_0$   $E_2$  is  $C^3$ -regular and*

$$\frac{\|S_{E_2}\|_{L^\infty} - \|S_{E_1}\|_{L^\infty}}{h} \leq C_n \|S_{E_1}\|_{L^\infty}^3.$$

*If in addition  $E_1$  is  $C^k$ -regular, then  $E_2$  is  $C^{k+2}$ -regular.*

*Proof.* As previously,  $C = C(n, m_0, r_0) > 0$  may change from line to line. We find  $h_0 = h_0(n, m_0, r_0) \in \mathbb{R}_+$  such that assuming  $h \leq h_0$  implies that the conclusions of Proposition 3.1, Lemma 3.2 and Remark 3.3 are valid. Let us quickly summarize what we have achieved so far. First,  $E_2$  is open and bounded,  $C^3$ -regular set, or

$C^{k+2}$ -regular set provided that  $E_1$  is  $C^k$ -regular, and it satisfies UBC with radius  $c_0 h^{1/3}$  for a constant  $c_0 = c_0(n, m_0, r_0) > 0$ . Hence, by Lemma 4.1 we have a priori estimate

$$\|S_{E_2}\|_{L^\infty} \leq Ch^{-\frac{1}{3}}. \tag{4.24}$$

Second,  $\partial E_2$  is “close” to  $\partial E_1$ . To be more precise, we have  $\|d_{E_1}\|_{L^\infty(E_2)} \leq C_n h/r_0$  and we may assume that  $\partial E_2 \subset \mathcal{N}_{r_0/2}(\partial E_1)$ . Moreover, it holds that  $\nabla d_{E_1} \cdot \nu_{E_2} > 0$  on  $\partial E_2$  and  $\pi_{\partial E_1}$  is injective on  $\partial E_2$ . Third, we have the Euler-Lagrange equation (3.4) on  $\partial E_2$  in the classical sense.

Thus, we assume that  $h \leq h_0$ . We might need to shrink  $h_0$  but always in a way that we preserve the dependency  $h_0 = h_0(n, m_0, r_0)$ . By combining the estimate  $\|d_{E_1}\|_{L^\infty(E_2)} \leq C_n h/r_0$  from Proposition 3.1 with Lemma 4.1 and (4.24) and by possibly shrinking  $h_0$  we obtain

$$\frac{\|d_{E_1}\|_{L^\infty(E_2)}}{h} \leq C_n \|S_{E_1}\|_{L^\infty} \quad \text{and} \quad \|S_{E_2}\|_{L^\infty} \|d_{E_1}\|_{L^\infty(E_2)} \leq 1. \tag{4.25}$$

Then, by (3.4), Lemma 4.1 and the first estimate in (4.25), the Lagrange multiplier  $\lambda^h$  can be controlled as

$$|\lambda^h| \leq \frac{\|d_{E_1}\|_{L^\infty(E_2)}}{h} + \|H_{E_2}\|_{L^\infty(\partial E_2)} \leq C_n (\|S_{E_1}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty}). \tag{4.26}$$

The claim follows once we show

$$\frac{\|S_{E_2}\|_{L^\infty} - \|S_{E_1}\|_{L^\infty}}{h} \leq C_n \left( \|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3 \right). \tag{4.27}$$

Indeed, assuming the above holds true we have by Lemma 4.1 and (4.24)

$$\|S_{E_2}\|_{L^\infty} - \|S_{E_1}\|_{L^\infty} \leq C_n r_0^{-3} h + Ch^{\frac{1}{3}} \|S_{E_2}\|_{L^\infty}$$

and, hence, recalling (4.4) and shrinking  $h_0$ , if necessary, we obtain  $\|S_{E_2}\|_{L^\infty} \leq 2\|S_{E_1}\|_{L^\infty}$ . Thus, reiterating the previous inequality via (4.27) yields the claim.

To prove (4.27), we initially fix any  $\varepsilon < r_{E_2}^2$  and choose  $(x, y) \in \partial E_2 \times \partial E_2$  such that  $|S_{E_2, \varepsilon}(x, y)| = \|S_{E_2, \varepsilon}\|_{L^\infty}$ . Since  $\|S_{E_2, \varepsilon}\|_{L^\infty} > 0$ , then  $x \neq y$  and, hence, the injectivity of  $\pi_{\partial E_1}$  on  $\partial E_2$  ensures that  $\pi_{\partial E_1}(x) \neq \pi_{\partial E_1}(y)$ . In order to simplify our notations, we write  $\pi = \pi_{\partial E_1}$  and  $H_2 = H_{E_2}$  for short. By using the definition in (4.5), the identities (2.22) and (4.18) as well as the Euler-Lagrange equation we may decompose the difference quotient as

$$\begin{aligned} & \frac{1}{h} (S_{E_2, \varepsilon}(x, y) - S_{E_1, \varepsilon}(\pi(x), \pi(y))) \\ &= \frac{(x - y) \cdot \nabla_{\tau_2} H_2(x)}{|x - y|^2 + \varepsilon} \\ &+ \frac{(\nu_{E_1}(x) \cdot \nu_{E_2}(y)) H_2(y) - H_2(x)}{|x - y|^2 + \varepsilon} \\ &+ \frac{1}{h} \frac{|\nabla_{\tau_2} d_{E_1}(x)|^2}{1 + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}(x)|^2}} S_{E_2, \varepsilon}(x, y) \end{aligned}$$

$$\begin{aligned}
 & + \left( \lambda^h - \frac{d_{E_1}(y)}{2h} \right) \frac{|v_{E_1}(x) - v_{E_1}(y)|^2}{|x - y|^2 + \varepsilon} \\
 & + \frac{d_{E_1}(y)}{2h} \frac{|v_{E_1}(\pi(x)) - v_{E_1}(\pi(y))|^2}{|x - y|^2 + \varepsilon} \\
 & + \frac{1}{h} \left( \frac{|\pi(x) - \pi(y)|^2 - |x - y|^2}{|x - y|^2 + \varepsilon} \right) S_{E_1, \varepsilon}(\pi(x), \pi(y)). \tag{4.28}
 \end{aligned}$$

Next, we estimate the last four terms on the RHS. First, since  $\partial E_2 \subset \mathcal{N}_{r_0/2}(\partial E) \subset \mathcal{N}_{r_{E_1}/2}(\partial E_1)$ , we have the estimate (4.23) for  $\|\nabla_{\tau_2} d_{E_1}\|_{L^\infty(\partial E_2)}^2$  and, hence, recalling the first estimate in (4.25) we have

$$\begin{aligned}
 & \left| \frac{1}{h} \frac{|\nabla_{\tau_2} d_{E_1}(x)|^2}{1 + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}(x)|^2}} S_{E_2, \varepsilon}(x, y) \right| \\
 & \leq \frac{16 \|d_{E_1}\|_{\partial E_2}}{h} \left( \|S_{E_1}\|_{L^\infty} \|S_{E_2}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty}^2 \right) \tag{4.29} \\
 & \leq C_n \left( \|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3 \right).
 \end{aligned}$$

For the next term, we use Lemma 4.1, the first estimate in (4.25) and (4.26) to obtain

$$\begin{aligned}
 \left| \left( \lambda^h - \frac{d_{E_1}(y)}{2h} \right) \frac{|v_{E_1}(x) - v_{E_1}(y)|^2}{|x - y|^2 + \varepsilon} \right| & \leq C_n \left( |\lambda^h| + \|S_{E_1}\|_{L^\infty(\partial E_1)} \right) \|S_{E_2}\|_{L^\infty(\partial E_2)}^2 \\
 & \leq C_n \left( \|S_{E_1}\|_{L^\infty(\partial E_1)}^3 + \|S_{E_2}\|_{L^\infty(\partial E_2)}^3 \right). \tag{4.30}
 \end{aligned}$$

By Proposition 2.7  $v_{E_1}$  is  $1/r_0$ -Lipschitz and by Lemma 2.9  $\pi$  is 2-Lipschitz continuous in  $\mathcal{N}_{r_0/2}(\partial E_1)$ . Thus, by Lemma 4.1 and the first inequality in (4.25), we estimate the second last term as

$$\begin{aligned}
 \left| \frac{d_{E_1}(y)}{2h} \frac{|v_{E_1}(\pi(x)) - v_{E_1}(\pi(y))|^2}{|x - y|^2 + \varepsilon} \right| & \leq C_n \|S_{E_1}\|_{L^\infty} \frac{1}{r_0^2} \frac{|\pi(x) - \pi(y)|^2}{|x - y|^2} \tag{4.31} \\
 & \leq C_n \|S_{E_1}\|_{L^\infty}^3.
 \end{aligned}$$

Finally, by using Lemma 4.5 and the identities in (4.25), we have

$$\begin{aligned}
 & \left| \frac{1}{h} \left( \frac{|\pi(x) - \pi(y)|^2 - |x - y|^2}{|x - y|^2 + \varepsilon} \right) S_{E_1, \varepsilon}(\pi(x), \pi(y)) \right| \\
 & \leq C_n \frac{\|d_{E_1}\|_{L^\infty(\partial E_2)}}{h} \left( \|S_{E_1}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty} + \|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_2}\|_{L^\infty}^2 \right) \|S_{E_1}\|_{L^\infty} \\
 & \leq C_n \left( \|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3 \right). \tag{4.32}
 \end{aligned}$$

We infer from (4.28), (4.29), (4.30), (4.31) and (4.32) the expression

$$\frac{S_{E_2, \varepsilon}(x, y) - S_{E_1, \varepsilon}(\pi(x), \pi(y))}{h}$$



$$= \frac{(x - y) \cdot \nabla_{\tau_2} H_2(x)}{|x - y|^2 + \varepsilon} + \frac{(v_{E_1}(x) \cdot v_{E_2}(y)) H_2(y) - H_2(x)}{|x - y|^2 + \varepsilon} + R,$$

where for the remainder term it holds  $|R| \leq C_n (\|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3)$ . Since  $(x, y)$  is a maximum (or minimum) point for  $S_{E_2, \varepsilon}$ , then we conclude from (4.14)

$$\frac{\|S_{E_2, \varepsilon}\|_{L^\infty} - \|S_{E_1, \varepsilon}\|_{L^\infty}}{h} \leq C_n (\|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3).$$

Since now  $\|S_{E_i, \varepsilon}\|_{L^\infty} \uparrow \|S_{E_i, \varepsilon}\|_{L^\infty}$  for  $i = 1, 2$  as  $\varepsilon$  tends to zero, the above yields (4.27) and we conclude the proof.  $\square$

We may now prove the main result of this section which is the UBC estimate for the approximative flat flow.

**Theorem 4.7.** *Let  $E_0 \subset \mathbb{R}^{n+1}$  be an open and bounded set which satisfies UBC with radius  $r_0 \in \mathbb{R}_+$  and let  $m_0$  denote its volume. There are  $h_0 = h_0(n, m_0, r_0) \in \mathbb{R}_+$  and  $T_0 = T_0(n, r_0) \in \mathbb{R}_+$  such that if  $h \leq h_0$ , then any approximative flat flow  $(E_t^h)_{t \geq 0}$  of (1.1) starting from  $E_0$  satisfies UBC with radius  $r_0/2$  for all  $t \leq T_0$ . Moreover,  $E_t^h$  is  $C^{1+2\lfloor t/h \rfloor}$ -regular for every  $0 \leq t \leq T_0$ .*

*Proof.* By a slight abuse of notation, we set  $h_0$  to be as in Lemma 4.6 for the parameters  $n, m_0$  and  $r_0/2$ . Then we choose

$$T_0 = \frac{r_0^2}{4C_n}, \tag{4.33}$$

where the dimensional constant is the same as in Lemma 4.6. We assume that  $h \leq h_0$  and consider an approximative flat flow  $(E_t^h)_{t \geq 0}$  starting from  $E_0$  obtained via the minimizing movements scheme (3.1). We may assume  $h \leq T_0$ , since otherwise the proof is trivial. Since  $E_0$  satisfies UBC with radius  $r_0$ , we have by Lemma 4.1 that  $\|S_{E_0}\|_{L^\infty} = 1/(2r_0)$ . Then we set

$$K = \sup \left\{ k \in \mathbb{N} : E_t^h \text{ satisfies UBC with } \|S_{E_{lh}^h}\|_{L^\infty} \leq \frac{1}{r_0} \text{ for } 0 \leq l \leq k \right\}.$$

Note that if  $E_{kh}^k$  is a bounded set satisfying UBC with  $\|S_{E_k^h}\|_{L^\infty} \leq 1/r_0$ , then thanks to Lemma 4.1 we know that it satisfies UBC with radius  $r_0/2$ . Thus, it follows from the construction of  $(E_t^h)_{t \geq 0}$ , the choice of  $h_0$ , and Lemma 4.6 that  $E_{(k+1)h}^h$  is a bounded  $C^3$ -regular set satisfying

$$\|S_{E_{(k+1)h}^h}\|_{L^\infty} \leq \|S_{E_k^h}\|_{L^\infty} + C_n h \|S_{E_k^h}\|_{L^\infty}^3 \leq \|S_{E_k^h}\|_{L^\infty} + C_n r_0^{-3} h.$$

Since  $h_0 \leq T_0$ , then the choices in (4.33) imply that  $K$  is well-defined. By summing the above from  $k = 0$  to  $k = K$  we obtain

$$\frac{1}{r_0} \leq \|S_{E_{(K+1)h}^h}\|_{L^\infty} \leq \|S_{E_0}\|_{L^\infty} + \frac{C_n}{r_0^3} (K + 1)h = \frac{1}{2r_0} + \frac{C_n}{r_0^3} (K + 1)h.$$

This yields  $K \geq \lfloor T_0/h \rfloor$  and, hence, it follows from the construction (3.1) that  $E_t^h$  satisfies UBC with radius  $r_0/2$  for every  $0 \leq t \leq T_0$ . The last claim then follows directly from Lemma 4.6.  $\square$

## 5. Higher Regularity

In this section we utilize the short-time UBC from previous section and prove the full regularity of the flat flow solution of (1.1). It is well known that the classical solution for the mean curvature flow is well defined as long as the second fundamental form stays bounded [39]. For the volume preserving flow this is not enough as the flow may develop singularities even if it stays regular [40,41]. However, if the flow in addition satisfies UBC then these singularities do not occur. In this section we show that the approximative flat flow becomes instantaneously smooth and stays smooth as long as it satisfies UBC. We will prove this via energy estimates.

Our starting point is the formula in Lemma 4.4, which for sets  $E_1$  and  $E_2$  as in the lemma, gives the formula which relates their normals as

$$\nu_{E_1} \circ \pi_{\partial E_1} = \nabla_{\tau_2} d_{E_1} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} \nu_{E_2} \quad \text{on } \partial E_2.$$

Recall that  $\nabla_{\tau_2}$  denotes the tangential gradient on  $\partial E_2$ . Assume now further that  $E_2$  is a minimizer of the functional  $\mathcal{F}_h(\cdot, E_1)$  defined in (3.2). We may use the Euler-Lagrange equation (3.4) and have

$$\nu_{E_1} \circ \pi_{\partial E_1} = -h \nabla_{\tau_2} H_{E_2} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} \nu_{E_2} \quad \text{on } \partial E_2. \quad (5.1)$$

This identity is simple enough for us to differentiate multiple times and this in turn gives us formula which is the discrete analog of the identity (see e.g. [38, Lemma 3.5])

$$\frac{d}{dt} \Delta^k H_{E_t} = \Delta^{k+1} H_{E_t} + \text{lower order terms}. \quad (5.2)$$

Let us, for the sake of clarification, show how we obtain the discrete version of (5.2) for  $k = 0$  from (5.1), which reads as follows

$$\begin{aligned} & \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} H_{E_2} - H_{E_1} \circ \pi_{\partial E_1} \\ &= h \Delta_{\tau_2} H_{E_2} + h^2 A_2(\cdot) \nabla_{\tau_2} H_{E_2} \cdot \nabla_{\tau_2} H_{E_2} + a_1(\cdot) d_{E_1} \quad \text{on } \partial E_2, \end{aligned} \quad (5.3)$$

where the function  $a_1(\cdot)$  and the matrix field  $A_2(\cdot)$  depend smoothly on  $d_{E_1}$ ,  $\nu_{E_1} \circ \pi_{\partial E_1}$ ,  $\nu_{E_2}$ ,  $B_{E_1} \circ \pi_{\partial E_1}$  and  $B_{E_2}$ . In particular, since  $E_1$  and  $E_2$  satisfy UBC with radius  $r_0/2$ , then  $a_1(\cdot)$  and  $A_2(\cdot)$  are uniformly bounded.

Indeed, by applying the tangential divergence on (5.1) we have

$$\operatorname{div}_{\tau_2} (\nu_{E_1} \circ \pi_{\partial E_1}) = -h \Delta_{\tau_2} H_{E_2} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} H_{E_2} \quad \text{on } \partial E_2.$$

In order to calculate the LHS, we use (2.22), (2.32) and (2.33) to obtain

$$\begin{aligned} \nabla(\nu_{E_1} \circ \pi_{\partial E_1}) &= \nabla^2 d_{E_1} = B_{E_1} \circ \pi_{\partial E_1} (I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} \\ &= B_{E_1} \circ \pi_{\partial E_1} - d_{E_1} (I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} (B_{E_1} \circ \pi_{\partial E_1})^2 \end{aligned}$$

which holds in the tubular neighborhood  $\mathcal{N}_{r_0}(\partial E_1)$ , where we also used the fact

$$(B_{E_1} \circ \pi_{\partial E_1})(I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} = (I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} (B_{E_1} \circ \pi_{\partial E_1}).$$

Again from (5.1) we have

$$v_{E_2} = \frac{1}{\sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}} (h \nabla_{\tau_2} H_{E_2} + v_{E_1} \circ \pi_{\partial E_1})$$

on  $\partial E_2$ . Using the above identities and the fact  $B_{E_1} v_{E_1} = 0$  on  $\partial E_1$ , we have the following equality on  $\partial E_2$

$$\begin{aligned} \operatorname{div}_{\tau_2} (v_{E_1} \circ \pi_{\partial E_1}) &= \operatorname{Tr}((I - v_{E_2} \otimes v_{E_2}) \nabla^2 d_{E_1}) \\ &= H_{E_1} \circ \pi_{\partial E_1} - d_{E_1} \operatorname{Tr}((I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} (B_{E_1} \circ \pi_{\partial E_1})^2) \\ &\quad - \frac{h^2}{1 - |\nabla_{\tau_2} d_{E_1}|^2} ((I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} (B_{E_1} \circ \pi_{\partial E_1})) \nabla_{\tau_2} H_{E_2} \cdot \nabla_{\tau_2} H_{E_2}. \end{aligned}$$

The equation (5.3) then follows from the previous calculations and from the identity

$$(v_{E_1} \circ \pi_{\partial E_1}) \cdot v_{E_2} = \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} \quad \text{on } \partial E_2, \tag{5.4}$$

which is a direct consequence of Lemma 4.4.

We may differentiate the equality (5.3) further and obtain a discrete version of (5.2) for every order  $k$ . This will produce several nonlinear error terms which have rather complicated structure. However, by introducing sufficiently efficient notation we are able to identify the structure of these error terms and by using UBC and the interpolation inequality from Proposition 2.2 we are able to reproduce the argument from [22] in the discrete setting. The following proposition is the core of the proof for the higher order regularity.

**Proposition 5.1.** *Assume that  $E_1 \subset \mathbb{R}^{n+1}$  is an open and bounded set, with  $|E_1| = m_0$ , which satisfies UBC with radius  $r_0$  and let  $E_2$  be any minimizer of  $\mathcal{F}_h(\cdot, E_1)$  defined in (3.2). There is  $h_0 = h_0(n, m_0, r_0)$  such that if  $h \leq h_0$  and  $E_1$  is  $C^{2m+3}$ -regular for  $m = 0, 1, 2, \dots$  then*

$$\begin{aligned} \Delta_{\tau_2}^m H_{E_2} - (\Delta_{\tau_1}^m H_{E_1}) \circ \pi_{\partial E_1} &= h \Delta_{\tau_2}^{m+1} H_{E_2} + h R_{2m} \quad \text{and} \\ \nabla_{\tau_2} \Delta_{\tau_2}^m H_{E_2} - (\nabla_{\tau_1} \Delta_{\tau_1}^m H_{E_1}) \circ \pi_{\partial E_1} \\ &= h \nabla_{\tau_2} \Delta_{\tau_2}^{m+1} H_{E_2} - \partial_{v_{E_2}} (\Delta_{\tau_1}^m H_{E_1} \circ \pi_{\partial E_1}) v_{E_2} + h R_{2m+1} \end{aligned}$$

on  $\partial E_2$  and the error term  $R_l$  for  $l = 0, 1, 2, \dots$  satisfies the estimate

$$\|R_l\|_{L^2(\partial E_2)}^2 \leq C_l \left( 1 + \|B_{E_2}\|_{H^{l+1}(\partial E_2)}^2 + \|B_{E_1}\|_{H^l(\partial E_1)}^2 \right),$$

where  $C_l = C_l(l, n, m_0, r_0)$ .

We note that so far we have not used any results from differential geometry. In fact, we need the notation from geometry only to prove Proposition 5.1. Therefore, instead of giving the proof of Proposition 5.1, which is technically challenging, we show first how we may use it to obtain the regularity estimate (1.2) in the statement of Theorem 1.1. Here is the main result of this section.

**Theorem 5.2.** *Let  $E_0$  be an open and bounded set, with  $|E_0| = m_0$ , and let  $(E_t^h)_{t \geq 0}$  be an approximative flat flow starting from  $E_0$  defined in (3.1). For given  $r_0 \in \mathbb{R}_+$  there is  $h_0 = h_0(n, m_0, r_0) \in \mathbb{R}_+$  such that if  $h \leq h_0$ ,  $E_t^h$  satisfies UBC with radius  $r_0$  in  $[0, T]$  and if  $(l + 2)h \leq T$  for a given  $l \in \mathbb{N} \cup \{0\}$ , then we have*

$$\sup_{t \in [(l+2)h, T]} \left( (t - lh)^l \|H_{E_t^h}\|_{H^l(\partial E_t^h)}^2 \right) + \int_{(l+2)h}^T (t - lh)^l \|H_{E_t^h}\|_{H^{l+1}(\partial E_t^h)}^2 dt \leq C,$$

for a constant  $C = C(l, n, m_0, r_0, T)$ .

*Proof.* In the proof,  $C$  and  $C_m$  denote a positive real number which may change their values but always in a manner that we have the dependencies  $C = C(n, m_0, r_0)$  and  $C_m = C_m(m, n, m_0, r_0, T)$ . We use the abbreviation  $E_k = E_{kh}^h$  for  $k = 0, 1, 2, \dots$

First, by Proposition 3.1, Lemma 3.2, Remark 3.3 and Theorem 4.7, we find  $h_0 = h_0(n, m_0, r_0) > 0$  such that if  $h \leq h_0$  and  $E_k$  is  $C^{2k+1}$ -regular, bounded set of volume  $m_0$ , which satisfies uniform ball condition with radius  $r_0$ , then the consequent set  $E_{k+1}$  is  $C^{2k+3}$ -regular, bounded and of volume  $m_0$ , with

$$\|d_{E_k}\|_{L^\infty(\partial E_{k+1})} \leq Ch < r_0/2.$$

Moreover,  $E_{k+1}$  satisfies UBC with radius  $r_0/2$  and the projection  $\pi_{\partial E_k} : \partial E_{k+1} \rightarrow \partial E_k$  is injective. We may then prove that, for  $k \geq 1$ ,  $\pi_{\partial E_k} : \partial E_{k+1} \rightarrow \partial E_k$  is a diffeomorphism with

$$J_{\tau_{k+1}} \pi_{\partial E_k} \geq 1 - Ch > 0 \text{ on } \partial E_{k+1}, \tag{5.5}$$

where the tangential Jacobian  $J_{\tau_{k+1}} \pi_{\partial E_k}$  of  $\pi_{\partial E_k}$  on  $\partial E_{k+1}$  is defined in (2.3). Indeed, since  $\partial E_{k+1} \subset \mathcal{N}_{r_0/2}(\partial E_k)$ , then  $\pi_{\partial E_k}$  is  $C^1$ -regular map on  $\partial E_{k+1}$ . Recalling the injectivity of the projection we are remain to prove (5.5). By (2.31) we may write

$$\nabla \pi_{\partial E_{k+1}} = I - \nabla d_{E_k} \otimes \nabla d_{E_k} - d_{E_k} \nabla^2 d_{E_k} \text{ on } \partial E_{k+1}.$$

Thus, it follows from the definition in (2.3) and  $\nabla^2 d_{E_k} \nabla d_{E_k} = 0$  in  $\mathcal{N}_{r_0}(\partial E_k)$  that for given a point  $x \in \partial E_{k+1}$  there is an orthonormal basis  $v_1, \dots, v_n$  of  $G_x \partial E_{k+1}$  such that

$$\begin{aligned} J_{\tau_{k+1}} \pi_{\partial E_k}(x) &= \prod_{i=1}^n \left| \left( I - \nabla d_{E_k}(x) \otimes \nabla d_{E_k}(x) - d_{E_k}(x) \nabla^2 d_{E_k}(x) \right) v_i \right| \\ &= \prod_{i=1}^n \left( 1 - (\nabla d_{E_k}(x) \cdot v_i)^2 - 2d_{E_k}(x) \nabla^2 d_{E_k}(x) v_i \cdot v_i + |d_{E_k}(x)|^2 |\nabla^2 d_{E_k}(x) v_i|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since  $\partial E_{k+1} \subset \mathcal{N}_{r_0/2}(\partial E_k)$ , then Lemma 2.9 yields  $\sup_{\partial E_{k+1}} |\nabla^2 d_{E_k}|_{\text{op}} \leq C$ . Further, since  $E_{k+1}$  satisfies UBC with radius  $r_0/2$ , then by Lemma 2.11 and by the previous estimates we deduce

$$|\nabla d_{E_k}(x) \cdot v_i|^2 \leq |\nabla_{\tau_2} d_{E_k}(x)|^2$$

$$\begin{aligned} &\leq 4\|d_{E_k}\|_{L^\infty(\partial E_{k+1})} \left( \sup_{\partial E_{k+1}} |\nabla^2 d_{E_k}|_{\text{op}} + \frac{\|\nabla d_{E_k}\|_{L^\infty(\partial E_{k+1})}}{r_0/2} \right) \\ &\leq Ch. \end{aligned}$$

Therefore, by combining the previous observations and shrinking  $h_0$ , if needed, we obtain (5.5). Again, by possibly shrinking  $h_0$ , we may assume that the implications of Proposition 5.1 hold true for the parameters  $m_0$  and  $r_0/2$ .

Let us from now on assume that the sets  $E_t^h$  satisfy UBC with radius  $r_0$  for every  $t \in [0, T]$ . Let us denote  $K = \lfloor T/h \rfloor$ . Then the previous discussion holds for every  $E_k$  and  $k = 0, 1, 2, \dots, K$ . For the sake of presentation, we use abbreviations  $\|B_{E_k}\|_{L^2} = \|B_{E_k}\|_{L^2(\partial E_k)}$ ,  $\|B_{E_k}\|_{H^{2m}} = \|B_{E_k}\|_{H^{2m}(\partial E_k)}$  etc.

After the initialization, we prove the claim by induction and to this aim we begin by proving the main regularity estimates. We claim that for every  $m = 0, 1, 2, \dots$ , with  $m \leq K - 2$ , and every  $k = m + 1, m + 2, \dots, K$  it holds that

$$\|\Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 \leq (1 + C_m h) \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 - h \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 + C_m h \tag{5.6}$$

and

$$\|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 \leq (1 + C_m h) \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 - h \|\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}}\|_{L^2}^2 + C_m h. \tag{5.7}$$

We first prove (5.6) and fix  $m$ . Recall that for  $k \geq m + 1$  the set  $E_k$  is  $C^{2m+3}$ -regular. Therefore, by Proposition 5.1, it holds for every  $k = m + 1, m + 2, \dots, K$  that

$$\Delta_{\tau_{k+1}}^m H_{E_{k+1}} - (\Delta_{\tau_k}^m H_{E_k}) \circ \pi_{\partial E_k} = h \Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}} + h R_{2m,k} \text{ on } \partial E_{k+1},$$

where the remainder term  $R_{2m,k}$  satisfies

$$\|R_{2m,k}\|_{L^2}^2 \leq C_m (1 + \|B_{E_{k+1}}\|_{H^{2m+1}}^2 + \|B_{E_k}\|_{H^{2m}}^2).$$

Again, since  $E_k$  and  $E_{k+1}$  satisfy UBC with radius  $r_0/2$  and  $|E_k| = m_0 = |E_{k+1}|$ , then  $\|B_{E_k}\|_{L^\infty}, \|B_{E_k}\|_{L^\infty} \leq C$  by (2.30) and  $P(E_k), P(E_{k+1}) \leq C$  by Lemma 2.10. Therefore, we may use Proposition 2.6 and Young's inequality to deduce that

$$\begin{aligned} \|B_{E_k}\|_{H^{2m}}^2 &\leq C_m \left( 1 + \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \right) \quad \text{and} \\ \|B_{E_{k+1}}\|_{H^{2m+1}}^2 &\leq C_m \left( 1 + \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 \right). \end{aligned} \tag{5.8}$$

We also observe that  $\|\Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 \leq C_m \|B_{E_{k+1}}\|_{H^{2m}}$ . Let  $\varepsilon \in (0, 1)$  be a number which we will choose later. By using the previous observations, (5.5), Young’s inequality, and integration by parts we estimate as follows:

$$\begin{aligned} & \|\Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 - \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ & \leq \int_{\partial E_{k+1}} |\Delta_{\tau_{k+1}}^m H_{E_{k+1}}|^2 - |\Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}|^2 d\mathcal{H}^n + Ch \int_{\partial E_{k+1}} |\Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}|^2 d\mathcal{H}^n \\ & \leq \int_{\partial E_{k+1}} |\Delta_{\tau_{k+1}}^m H_{E_{k+1}}|^2 - |\Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}|^2 d\mathcal{H}^n + \frac{Ch}{1-Ch} \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ & \leq 2 \int_{\partial E_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}} (\Delta_{\tau_{k+1}}^m H_{E_{k+1}} - \Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}) d\mathcal{H}^n + Ch \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ & = 2h \int_{\partial E_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}} (\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}} + R_{2m,k}) d\mathcal{H}^n + Ch \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ & \leq -2h \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 + \varepsilon h \|R_{2m,k}\|_{L^2}^2 + \frac{h}{\varepsilon} \|\Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 + Ch \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ & \leq -2h \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 \\ & \quad + C_m h \left( \varepsilon \|B_{E_{k+1}}\|_{H^{2m+1}}^2 + \frac{1}{\varepsilon} \|B_{E_{k+1}}\|_{H^{2m}}^2 \right) + C_m h (1 + \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2) \\ & \leq -2h \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 + C_m h \left( \varepsilon \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 + \frac{1}{\varepsilon} \|B_{E_{k+1}}\|_{H^{2m}}^2 \right) \\ & \quad + C_m h (1 + \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2). \end{aligned}$$

By choosing  $\varepsilon = (1 + C_m)^{-1}/2$ , the previous estimate yields

$$\begin{aligned} & \|\Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 - \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ & \leq -\frac{3h}{2} \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2}^2 + C_m h \|B_{E_{k+1}}\|_{H^{2m}}^2 + C_m h (1 + \|\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2). \end{aligned} \tag{5.9}$$

Since  $\|B_{E_{k+1}}\|_{L^\infty}, P(E_{k+1}) \leq C$ , we may use Proposition 2.2 to find  $\theta = \theta(m, n) \in (0, 1)$  such that

$$\|B_{E_{k+1}}\|_{H^{2m}}^2 \leq C_m \|B_{E_{k+1}}\|_{H^{2m+1}}^{2\theta} \|B_{E_{k+1}}\|_{L^\infty}^{2(1-\theta)} \leq C_m \varepsilon \|B_{E_{k+1}}\|_{H^{2m+1}}^2 + C_m \varepsilon^{-\frac{\theta}{1-\theta}}$$

for any  $\varepsilon \in (0, 1)$ , where the last inequality follows from Young’s inequality and the curvature bound. Thus, by combing the above with (5.9) and (5.8) the estimate (5.6) follows with a suitable choice of  $\varepsilon$ .

Let us then prove (5.7). The argument is similar than above and we only point out the main differences. Now Proposition 5.1 gives for every  $k = m + 1, m + 2, \dots, K$  the formula

$$\begin{aligned} & \nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}} - (\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}) \circ \pi_{\partial E_k} \\ & = h \nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}} - \partial_{\nu_{E_{k+1}}} (\Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}) \nu_{E_{k+1}} + h R_{2m+1,k} \quad \text{on } \partial E_{k+1}, \end{aligned}$$

where

$$\|R_{2m+1,k}\|_{L^2}^2 \leq C_m \left( 1 + \|B_{E_{k+1}}\|_{H^{2m+2}}^2 + \|B_{E_k}\|_{H^{2m+1}}^2 \right),$$

and, again, by using Proposition 2.6 and Young’s inequality we have estimates

$$\begin{aligned} \|B_{E_k}\|_{H^{2m+1}}^2 &\leq C_m(1 + \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2) \quad \text{and} \\ \|B_{E_{k+1}}\|_{H^{2m+2}}^2 &\leq C_m \left(1 + \|\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}}\|_{L^2}^2\right). \end{aligned}$$

We use the previous observations, the Cauchy-Schwarz inequality, the estimate  $\|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}\|_{L^2} \leq C_m \|B_{E_{k+1}}\|_{H^{2m+1}}^2$  and argue as in proving (5.6) to deduce that

$$\begin{aligned} &\|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_k}\|_{L^2}^2 - \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ &\leq \int_{\partial E_{k+1}} |\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}}|^2 - |\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}|^2 d\mathcal{H}^n + Ch \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ &\leq 2 \int_{\partial E_{k+1}} \nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}} \cdot (\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E_{k+1}} - \nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k} \circ \pi_{\partial E_k}) d\mathcal{H}^n \\ &\quad + Ch \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ &\leq -2h \|\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}}\|_{L^2} + \varepsilon h \|R_{2m+1,k}\|_{L^2(\partial E_{k+1})}^2 + \frac{C_m}{\varepsilon} h \|B_{E_{k+1}}\|_{H^{2m+1}}^2 \\ &\quad + Ch \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ &\leq -2h \|\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}}\|_{L^2} \\ &\quad + C_m h \left( \varepsilon \|\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}}\|_{L^2}^2 + \frac{1}{\varepsilon} \|B_{E_{k+1}}\|_{H^{2m+1}}^2 \right) + C_m h (1 + \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2) \\ &\leq -\frac{3h}{2} \|\Delta_{\tau_{k+1}}^{m+1} H_{E_{k+1}}\|_{L^2} + C_m h \|B_{E_{k+1}}\|_{H^{2m+1}}^2 + C_m h (1 + \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2). \end{aligned}$$

Again, Proposition 2.2 implies that there is  $\theta = \theta(m, n) \in (0, 1)$  such that

$$\|B_{E_{k+1}}\|_{H^{2m+1}}^2 \leq C_m \|B_{E_{k+1}}\|_{H^{2m+2}}^{2\theta} \|B_{E_{k+1}}\|_{L^\infty}^{2(1-\theta)}$$

and we may proceed as previously to obtain (5.7).

Let us then prove the claim by induction. To be more precise, under the assumption  $h \leq h_0$ , we claim that, for every  $l \in \mathbb{N} \cup \{0\}$  it holds that

$$\max_{l+2 \leq k \leq K} ((k - (l + 1))h)^l \|H_{E_k}\|_{H^l}^2 + \sum_{k=l+2}^K h((k - (l + 1))h)^l \|H_{E_k}\|_{H^{l+1}}^2 dt \leq C_l \tag{5.10}$$

for  $C_l = C_l(l, n, m_0, r_0, T)$ , provided that  $(l + 2)h \leq T$ . Since  $t - lh \leq 3\lfloor t/h \rfloor h - 3(l + 1)h$  for every  $t \geq (l + 2)h$ , then by multiplying (5.10) by  $3^l$  and recalling the definition for the approximative solution in (3.1), we obtain the statement of the theorem.

Let us consider first the case  $l = 0$ . Since  $P(E_k), \|B_{E_k}\|_{L^\infty} \leq C$ , then  $\|H_{E_k}\|_{L^2} \leq C$  for every  $k = 0, 1, \dots, K$ . By combining this with (5.6) gives us that for every  $k = 1, 2, \dots, K - 1$

$$\|H_{E_{k+1}}\|_{L^2}^2 - \|H_{E_k}\|_{L^2}^2 \leq -h \|\nabla_{\tau_{k+1}} H_{E_{k+1}}\|_{L^2}^2 + Ch.$$

We sum over  $k = 1, 2, \dots, K - 1$  and use  $\|H_{E_k}\|_{L^2} \leq C$  as well as  $Kh \leq T$  to obtain

$$\|H_{E_K}\|_{L^2}^2 + \sum_{k=1}^{K-1} h \|\nabla_{\tau_{k+1}} H_{E_{k+1}}\|_{L^2}^2 \leq \|H_{E_1}\|_{L^2}^2 + CKh \leq CT.$$

Thus, we conclude that (5.10) holds in the case  $l = 0$ .

Let us then assume that (5.10) holds for  $l - 1$ , where  $l \in \mathbb{N}$ . We assume that  $(l + 2)h \leq T$  and prove (5.10) for  $l$ . To this aim, we denote  $K' = K - l$  and  $E'_k = E_{k+l}$ . Again, let  $\tau_k$  denote the tangential differentiation along  $\partial E'_k$ . Thus, the induction assumption reads as

$$\max_{1 \leq k \leq K'} (kh)^{l-1} \|H_{E'_k}\|_{H^{l-1}}^2 + \sum_{k=1}^{K'} h(kh)^{l-1} \|H_{E'_k}\|_{H^l(\partial E'_k)}^2 \leq C_{l-1}. \tag{5.11}$$

We divide the argument into two cases depending whether  $l$  is even or odd.

Let us first assume that  $l$  is even and thus is of the form  $l = 2m$  for  $m = 1, 2, \dots$ . By binomial expansion it holds  $(k + 1)^{2m} - k^{2m} \leq 2m(k + 1)^{2m-1}$ . Therefore, by multiplying (5.6) by  $k^{2m}h^{2m}$  we deduce, for every  $k = 0, 1, 2, \dots, K'$ ,

$$\begin{aligned} & (k + 1)^{2m}h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 - k^{2m}h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \\ &= ((k + 1)^{2m} - k^{2m})h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 \\ & \quad + k^{2m}h^{2m} (\|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 - \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2) \\ & \leq 2m(k + 1)^{2m-1}h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 + C_m k^{2m}h^{2m+1} \left(1 + \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2\right) \\ & \quad - k^{2m}h^{2m+1} \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2. \end{aligned}$$

Fix any  $j = 2, \dots, K'$ . Summing the previous estimate from  $k = 0$  to  $k = j - 1$  and using the fact  $K'h \leq T$  yields

$$\begin{aligned} & j^{2m}h^{2m} \|\Delta_{\tau_j}^m H_{E'_j}\|_{L^2}^2 \\ & \leq C_m \sum_{k=0}^{j-1} (k + 1)^{2m-1}h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 \\ & \quad + C_m \sum_{k=0}^{j-1} (kh)k^{2m-1}h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 + C_m \sum_{k=0}^{j-1} h k^{2m}h^{2m} \\ & \quad - \sum_{k=0}^{j-1} k^{2m}h^{2m+1} \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 \\ & \leq C_m(1 + T) \sum_{k=1}^j k^{2m-1}h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \\ & \quad + C_m \int_0^{K'h} s^{2m} ds - \sum_{k=1}^j h(k - 1)^{2m}h^{2m} \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \end{aligned}$$



$$\begin{aligned} &\leq C_m(1+T) \sum_{k=1}^{K'} h k^{2m-1} h^{2m-1} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \\ &\quad + C_m T^{2m+1} - \sum_{k=1}^j h (k-1)^{2m} h^{2m} \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2. \end{aligned}$$

Thus, reordering the previous estimate and using the induction assumption (5.11) gives us

$$\begin{aligned} &(j-1)^{2m} h^{2m} \|\Delta_{\tau_j}^m H_{E'_j}\|_{L^2}^2 + \sum_{k=1}^j h (k-1)^{2m} h^{2m} \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \\ &\leq C_m(1+T) \sum_{k=1}^{K'} k^{2m-1} h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 + C_m T^{2m+1} \\ &\leq C_m(1+T) \sum_{k=1}^{K'} h(kh)^{l-1} \|H_{E'_k}\|_{H^l}^2 + C_m T^{2m+1} \\ &\leq C_m C_{l-1} + C_m T^{2m+1}. \end{aligned}$$

After substituting  $E'_k = E_{k+l}$  and reindexing we have for every  $j = l + 2, \dots, K$

$$\begin{aligned} &((j - (l + 1))h)^l \|\Delta_{\tau_j}^m H_{E_k}\|_{L^2}^2 + \sum_{k=l+1}^j h ((k - (l + 1))h)^l \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2 \\ &\leq C_m C_{l-1} + C_m T^{2m+1}. \end{aligned}$$

Since we have  $\|B_{E_k}\|_{L^\infty}, P(E_k) \leq C$  for every  $k \leq K$ , then by combining the estimates of Proposition 2.6 with the previous estimate and using  $Kh \leq T$  we obtain (5.10).

The case when  $l$  is odd is similar. In this case, we have  $l = 2m + 1$  for some  $m \in \mathbb{N} \cup \{0\}$ . Thus, by using (5.7) in the place of (5.6) we may proceed as in the previous case.  $\square$

Let us then focus on Proposition 5.1. We will begin by proving two technical lemmas which involve high order derivatives of  $d_E$  and  $\pi_{\partial E}$ . To overcome the technicalities we adopt the notation where  $A_i$  denotes a generic tensor field, which depends on the distance function, the normal and the second fundamental form in a smooth way, i.e.,

$$A_i = A_i(d_E, \nu_E \circ \pi_{\partial E}, B_E \circ \pi_{\partial E}) \text{ in } \mathcal{N}_{r/2}(\partial E). \tag{5.12}$$

We also adopt here the notation  $S \star T$  to denote a tensor formed by contraction on some indexes of tensors  $S$  and  $T$ . If the set  $E$  satisfies UBC, then the quantities  $d_E, \nu_E$  and  $B_E \circ \pi_{\partial E}$  are uniformly bounded in  $\mathcal{N}_{r/2}(\partial E)$ , we may treat  $A_i$  in (5.12) as a bounded coefficient.

It is immediate that it holds for  $x \in \partial E$  and  $u \in C^2(\partial E)$  that

$$\nabla(u \circ \pi_{\partial E})(x) = \nabla_\tau u(x) \quad \text{and} \quad \Delta_{\mathbb{R}^{n+1}}(u \circ \pi_{\partial E})(x) = \Delta_\tau u(x).$$

Let us then derive related formulas for points  $x \in \mathcal{N}_{r/2}(\partial E)$  outside  $\partial E$ .

**Lemma 5.3.** *Assume  $E \subset \mathbb{R}^{n+1}$ , with  $\Sigma = \partial E$ , is bounded and  $C^3$ -regular set which satisfies UBC with radius  $r$ . Then it holds for  $u \in C^2(\partial E)$  in  $\mathcal{N}_{r/2}(\partial E)$  that*

$$\nabla(u \circ \pi_{\partial E}) = \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E} - d_E \nabla^2 d_E \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E}$$

and

$$\begin{aligned} \nabla^2(u \circ \pi_{\partial E}) &= (P_{\partial E} \circ \pi_{\partial E})(\nabla^2(u \circ \pi_{\partial E}) \circ \pi_{\partial E}) \\ &\quad - \nabla d_E \otimes \nabla^2 d_E \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E} \\ &\quad + d_E A_1 \star \nabla^2(u \circ \pi_{\partial E}) \circ \pi_{\partial E} \\ &\quad + d_E A_2 \star \nabla(B_E \circ \pi_{\partial E}) \circ \pi_{\partial E} \star \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E} \end{aligned}$$

where  $A_1, A_2$  are tensor fields as in (5.12). Moreover, if  $\Sigma$  is in addition  $C^{k+2}$ -regular and  $u \in C^k(\Sigma)$  for  $k \in \mathbb{N}$ , then for all  $x \in \mathcal{N}_{r/2}(\partial E)$  we may estimate

$$\begin{aligned} &|\nabla^k(u \circ \pi_{\partial E})(x)| \\ &\leq C_k \sum_{|\alpha| \leq k} (1 + |\tilde{\nabla}_\Sigma^{\alpha_1} B_E(\pi_{\partial E}(x))| \cdots |\tilde{\nabla}_\Sigma^{\alpha_{k-1}} B_E(\pi_{\partial E}(x))| |\tilde{\nabla}_\Sigma^{\alpha_k} u(\pi_{\partial E}(x))|). \end{aligned}$$

Here  $\tilde{\nabla}_\Sigma$  denotes the covariant derivative on  $\Sigma$ .

*Proof.* Let us denote  $\hat{u} = u \circ \pi_{\partial E}$  and  $\pi = \pi_{\partial E}$  for short. Since  $\pi$  is projection it holds

$$\hat{u}(x) = \hat{u}(\pi(x))$$

for all  $x \in \mathcal{N}_{r/2}(\partial E)$ . By differentiating this we obtain

$$\nabla \hat{u}(x) = \nabla \pi(x) \nabla \hat{u}(\pi(x)).$$

The first claim then follows from (2.31) and from  $\nabla \hat{u} \cdot (v_E \circ \pi) = 0$ . The second claim follows by differentiating the first and by writing  $\nabla^2 d_E(x)$ ,  $\nabla^3 d_E(x)$  and  $\nabla \pi$  in a geometric way by using (2.33) and (2.34).

In order to prove the third claim we observe that we may write the second equality simply as

$$\nabla^2 \hat{u}(x) = A_1(x) \star \nabla^2 \hat{u}(\pi(x)) + A_2(x) \star \nabla(B_E \circ \pi)(\pi_{\partial E}(x)) \star \nabla \hat{u}(\pi(x)).$$

By differentiating this  $(k - 2)$ -times and by using (2.12) and (2.34) we deduce that

$$|\nabla^k \hat{u}(x)| \leq C_k \sum_{|\alpha| \leq k} C(1 + |\nabla^{\alpha_1}(B_E \circ \pi)(\pi(x))| \cdots |\nabla^{\alpha_{k-1}}(B_E \circ \pi)(\pi(x))| |\nabla^{\alpha_k} \hat{u}(\pi(x))|).$$

The claim follows once we show that for all  $y \in \Sigma$  it holds that

$$|\nabla^l \hat{u}(y)| \leq C_l \sum_{|\beta| \leq l} (1 + |\tilde{\nabla}^{\beta_1} B_E(y)| \cdots |\tilde{\nabla}^{\beta_l} B_E(y)|) |\tilde{\nabla}^{\beta_{l+1}} u(y)|, \quad (5.13)$$

which is the opposite estimate as to that of Lemma 2.4.

We argue as in the proof of Lemma 2.4 and assume  $y = 0$ ,  $v_E(0) = e_{n+1}$  and write the surface  $\Sigma$  locally as a graph of  $f$ , i.e.,  $\Sigma \cap B_r \subset \{(x', f(x')) : x' \in \mathbb{R}^n\}$  and extended  $f$  to  $\mathbb{R}^{n+1}$  trivially as  $f(x', x_{n+1}) = f(x')$ . We may then write the metric tensor and the Christoffel symbols in coordinates as

$$g_{ij}(x') = \delta_{ij} + \partial_i f(x') \partial_j f(x') \quad \text{and} \quad \Gamma_{jk}^i(x') = g^{il}(x') \partial_{jk}^2 f(x') \partial_l f(x').$$

Since  $v_E = \frac{(-\nabla_{\mathbb{R}^n} f, 1)}{\sqrt{1 + |\nabla_{\mathbb{R}^n} f|^2}}$  and  $\nabla \hat{u} \cdot (v_E \circ \pi) = 0$ , we have

$$\partial_{n+1} \hat{u}(y) = \sum_{i=1}^n \partial_i f(\pi(y)) \cdot \partial_i \hat{u}(y). \quad (5.14)$$

Let us denote the  $l$ th order differential of the function  $x' \rightarrow \hat{u}(x', 0)$  as  $\nabla_{\mathbb{R}^n}^l \hat{u}$ . Then by applying first (5.14) and (2.12), and then (2.14) we deduce that

$$\begin{aligned} |\nabla^l \hat{u}(0)| &\leq C_l \sum_{|\beta| \leq l-1} (1 + |\nabla^{\beta_1} (\nabla f \circ \pi)(0)| \cdots |\nabla^{\beta_{l-1}} (\nabla f \circ \pi)(0)|) |\nabla_{\mathbb{R}^n}^{1+\beta_l} \hat{u}(0)| \\ &\leq C_l \sum_{|\gamma| \leq l-1} (1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_{l-1}} B_E(0)|) |\nabla_{\mathbb{R}^n}^{1+\gamma_l} \hat{u}(0)|. \end{aligned}$$

Denote the local chart given by the coordinate parametrization by  $\Phi$ , i.e.,  $\Phi^{-1}(x') = (x', f(x'))$  and note that  $\hat{u}(\Phi^{-1}(x')) = u(\Phi^{-1}(x'))$ . Fix an index vector  $\beta = (\beta_1, \dots, \beta_n, 0)$  with  $|\beta| = m$ . Then by (2.12) and (2.14) we obtain after straightforward calculations

$$\begin{aligned} |\nabla^\beta (u \circ \Phi^{-1})(0)| &\geq |\nabla^\beta \hat{u}(0)| - C_m \sum_{|\gamma| \leq m-1} (1 + |\nabla^{1+\gamma_1} f(0)| \cdots |\nabla^{1+\gamma_{m-1}} f(0)|) |\nabla^\gamma \hat{u}(0)| \\ &\geq |\nabla^\beta \hat{u}(0)| - C_m \sum_{|\gamma| \leq m-1} (1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_{m-1}} B_E(0)|) |\nabla^\gamma \hat{u}(0)|. \end{aligned}$$

From here we deduce by an inductive argument that

$$|\nabla^\beta \hat{u}(0)| \leq C_m \sum_{|\gamma| \leq m} (1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_m} B_E(0)|) |\nabla^{\gamma_{m+1}} (u \circ \Phi^{-1})(0)|.$$

Finally using the definition of the covariant derivative and the expression of the Christoffel symbols we obtain arguing as in the proof of Lemma 2.4 that

$$|\nabla^m (u \circ \Phi^{-1})(0)| \leq C_m \sum_{|\gamma| \leq m} (1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_m} B_E(0)|) |\tilde{\nabla}^{\gamma_{m+1}} u(0)|.$$

Hence, we have (5.13) and the third claim follows.  $\square$

Let us from now on assume  $E_1, E_2 \subset \mathbb{R}^{n+1}$  are as in Proposition 5.1. We write the equality (5.3) by using the Euler-Lagrange equation (3.4) as

$$H_{E_2} - H_{E_1} \circ \pi_{\partial E_1} = h \Delta_{\tau_2} H_{E_2} + h \rho_0(\cdot) \quad (5.15)$$

on  $\partial E_2$ , where the error function is of the form

$$\rho_0(x) = A_1(x) + h A_2(x) \star \nabla_{\tau_2} H_{E_2}(x) \star \nabla_{\tau_2} H_{E_2}(x). \quad (5.16)$$

Here and in the rest of the section  $A_i(\cdot)$  denotes a tensor field which depends smoothly on  $d_{E_1}, \nu_{E_1} \circ \pi_{\partial E_1}, \nu_{E_2}, B_{E_1} \circ \pi_{\partial E_1}$  and on  $B_{E_2}$ . i.e.,

$$A_i(x) = A_i(d_{E_1}(x), \nu_{E_1}(\pi_{\partial E_1}(x)), \nu_{E_2}(x), B_{E_1}(\pi_{\partial E_1}(x)), B_{E_2}(x)). \quad (5.17)$$

The following lemma is a consequence of Lemma 5.3.

**Lemma 5.4.** *Assume that the sets  $E_1, E_2 \subset \mathbb{R}^{n+1}$  are as in Proposition 5.1. Then it holds for  $u \in C^2(\partial E_1)$  on  $\partial E_2$*

$$\begin{aligned} \Delta_{\tau_2}(u \circ \pi_{\partial E_1}) &= \Delta_{\tau_1} u \circ \pi_{\partial E_1} + h A_1 \star \nabla^2(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ &\quad + h^2 A_2 \star \nabla^2(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\ &\quad + h A_3 \star \nabla(B_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ &\quad + h A_4 \star \nabla_{\tau_2} B_{E_2} \star \nabla(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}. \end{aligned}$$

*Proof.* Let us denote  $\hat{u} = u \circ \pi_{\partial E_1}$  and  $\pi = \pi_{\partial E_1}$  for short. Recall that we may write the Laplace-Beltrami on  $\partial E_2$  as

$$\Delta_{\tau_2} \hat{u} = \Delta_{\mathbb{R}^{n+1}} \hat{u} - (\nabla^2 \hat{u} \nu_{E_2} \cdot \nu_{E_2}) - H_{E_2} \partial_{\nu_{E_2}} \hat{u}, \quad (5.18)$$

where  $\Delta_{\mathbb{R}^{n+1}} \hat{u} = \text{Tr}(\nabla^2 \hat{u})$  denotes the Euclidean Laplacian. Recall that  $P_{\partial E_1} = I - \nu_{E_1} \otimes \nu_{E_1}$  stands for the projection on the (geometric) tangent space. We deduce by applying the trace on the second equality in Lemma 5.3, by  $\nabla^2 d_{E_1} \nabla d_{E_1} = 0$ , and by the Euler-Lagrange equation (3.4) that it holds on  $\partial E_2$

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}} \hat{u} &= \text{Tr}(\nabla^2 \hat{u}) = \Delta_{\tau_1} u \circ \pi + h A_1 \star (\nabla^2 \hat{u} \circ \pi) \\ &\quad + h A_3 \star \nabla(B_{E_1} \circ \pi) \circ \pi \star (\nabla \hat{u} \circ \pi). \end{aligned} \quad (5.19)$$

Similarly, we have

$$\begin{aligned} (\nabla^2 \hat{u} \nu_{E_2}) \cdot \nu_{E_2} &= ((P_{\partial E_1} \circ \pi)(\nabla^2 \hat{u} \circ \pi) \nu_{E_2}) \cdot \nu_{E_2} \\ &\quad - (\nabla d_{E_1} \cdot \nu_{E_2})(\nabla^2 d_{E_1}(\nabla \hat{u} \circ \pi) \cdot \nu_{E_2}) \\ &\quad + h \tilde{A}_1 \star (\nabla^2 \hat{u} \circ \pi) + h \tilde{A}_3 \star \nabla(B_{E_1} \circ \pi) \circ \pi \star (\nabla \hat{u} \circ \pi). \end{aligned} \quad (5.20)$$

We write

$$\begin{aligned} &((P_{\partial E_1} \circ \pi)(\nabla^2 \hat{u} \circ \pi) \nu_{E_2}) \cdot \nu_{E_2} \\ &= ((P_{\partial E_1} \circ \pi)(\nabla^2 \hat{u} \circ \pi) (\nu_{E_2} - \nu_{E_1} \circ \pi)) \cdot (\nu_{E_2} - \nu_{E_1} \circ \pi) \end{aligned}$$

and

$$\begin{aligned} & (\nabla d_{E_1} \cdot \nu_{E_2})(\nabla^2 d_{E_1}(\nabla \hat{u} \circ \pi) \cdot \nu_{E_2}) \\ &= (\nabla d_{E_1} \cdot \nu_{E_2})(\nabla^2 d_{E_1}(\nabla \hat{u} \circ \pi) \cdot (\nu_{E_2} - \nu_{E_1} \circ \pi)). \end{aligned}$$

We then use (4.19) to write  $\nu_{E_2} - \nu_{E_1} \circ \pi$  as

$$\nu_{E_2} - \nu_{E_1} \circ \pi = a_1 \nabla_{\tau_2} d_{E_1} + a_2 (\nu_{E_1} \circ \pi)$$

for functions  $a_1$  and  $a_2$  which depend on  $|\nabla_{\tau_2} d_{E_1}(x)|^2$ . Therefore we may write (5.20) by the Euler-Lagrange equation (3.4) as

$$\begin{aligned} (\nabla^2 \hat{u} \nu_{E_2}) \cdot \nu_{E_2} &= h A_1 \star (\nabla^2 \hat{u} \circ \pi) \\ &+ h^2 A_2 \star (\nabla^2 \hat{u} \circ \pi) \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\ &+ h A_3 \star \nabla(B_{E_1} \circ \pi) \circ \pi \star (\nabla \hat{u} \circ \pi) \\ &+ h A_4 \star \nabla_{\tau_2} H_{E_2} \star (\nabla \hat{u} \circ \pi). \end{aligned} \tag{5.21}$$

We use the first equality in Lemma 5.3, (4.19) and the Euler-Lagrange equation (3.4) to write on  $\partial E_2$

$$\begin{aligned} \partial_{\nu_{E_2}} \hat{u} &= (\nabla \hat{u} \circ \pi) \cdot \nu_{E_2} + h A_3 \star (\nabla \hat{u} \circ \pi) \\ &= (\nabla \hat{u} \circ \pi) \cdot (\nu_{E_2} - \nu_{E_1} \circ \pi) + h A_3 \star (\nabla \hat{u} \circ \pi) \\ &= h A_4 \star \nabla_{\tau_2} H_{E_2} \star (\nabla \hat{u} \circ \pi) + h A_3 \star (\nabla \hat{u} \circ \pi). \end{aligned} \tag{5.22}$$

The claim then follows from (5.18), (5.19), (5.21) and (5.22).  $\square$

We may now prove Proposition 5.1.

*Proof of Proposition 5.1.* We prove only the first equality since the second follows by differentiating the first. We point out that since  $E_1$  is  $C^{2m+3}$ -regular, then by Lemma 3.2 the set  $E_2$  is  $C^{2m+3}$ -regular. In particular, we have the necessary regularity for the proceeding calculations. To that aim we recall that by (5.15) it holds

$$H_{E_2} - H_{E_1} \circ \pi_{\partial E_1} = h \Delta_{\tau_2} H_{E_2} + h \rho_0 \quad \text{on } \partial E_2, \tag{5.23}$$

where

$$\rho_0(x) = A_1(x) + h A_2(x) \star \nabla_{\tau_2} H_{E_2}(x) \star \nabla_{\tau_2} H_{E_2}(x).$$

We differentiate (5.23), use Lemma 5.4 and have on  $\partial E_2$

$$\Delta_{\tau_2} H_{E_2}(x) - \Delta_{\tau_1} H_{E_1} \circ \pi_{\partial E_1} = h \Delta_{\tau_2}^2 H_{E_2} + h \rho_2 + h \Delta_{\tau_2} \rho_0,$$

where

$$\begin{aligned} \rho_2 &= A_1 \star \nabla^2(H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ &+ h A_2 \star \nabla^2(H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\ &+ A_3 \star \nabla(B_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla(H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ &+ A_4 \star \nabla_{\tau_2} B_{E_2} \star \nabla(H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}. \end{aligned}$$

We continue and deduce by an iterative argument that it holds on  $\partial E_2$

$$\Delta_{\tau_2}^m H_{E_2} - \Delta_{\tau_1} H_{E_1}^m \circ \pi_{\partial E_1} = h \Delta_{\tau_2}^{m+1} H_{E_2} + h \sum_{k=0}^m \Delta_{\tau_2}^{m-k} \rho_{2k},$$

where  $\rho_0$  is defined in (5.16) and  $\rho_{2k}$  for  $k \geq 1$  is

$$\begin{aligned} \rho_{2k} = & A_1 \star \nabla^2 (\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ & + h A_2 \star \nabla^2 (\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\ & + A_3 \star \nabla (B_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla (\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ & + A_4 \star \nabla_{\tau_2} B_{E_2} \star \nabla (\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}. \end{aligned}$$

We have thus derived a formula for the error terms in the statement of Proposition 5.1, i.e., we have

$$R_{2m}(x) = \sum_{k=0}^m \Delta_{\tau_2}^{m-k} \rho_{2k}(x).$$

We need to estimate the norm  $\|R_{2m}\|_{L^2(\Sigma_2)}$ , where  $\Sigma_2 = \partial E_2$ . The idea is that the total amount of derivatives acting on the curvature terms in  $\Delta_{\tau_2}^{m-k} \rho_{2k}$  is for most of the terms at most  $2m$ . The only difference is the second row in the definition of  $\rho_{2k}$ , which total amount of derivatives is higher but it has an extra  $h$  as a coefficient. Therefore we need to treat this term more carefully.

Recall that the tensor fields  $A_i(\cdot)$  depend on  $d_{E_1}, \nu_{E_1} \circ \pi_{\partial E_1}, \nu_{E_2}, B_{E_1} \circ \pi_{\partial E_1}$  and on  $B_{E_2}$  as stated in (5.17). Denote  $\pi = \pi_{\partial E_1}$  for short. We use repeatedly (2.12), Lemma 2.4 and the last inequality in Lemma 5.3 and obtain after long but straightforward calculations the following pointwise estimate for all  $x \in \partial E_2$ :

$$\begin{aligned} |\Delta_{\tau_2}^{m-k} \rho_{2k}(x)| \leq & C + C \sum_{|\alpha| \leq 2m} |\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)| \\ & + C \sum_{|\alpha| \leq 2m} |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))| \\ & + Ch \sum_{|\alpha| \leq 2m} (|\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))|) \cdots \\ & (|\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))|) \cdots \\ & \cdots |\tilde{\nabla}^{1+\alpha_{2m+1}} H_{E_2}(x)| |\tilde{\nabla}^{1+\alpha_{2m+2}} H_{E_2}(x)|. \end{aligned} \tag{5.24}$$

We use the uniform curvature bounds  $\|B_{\Sigma_1}\|_{L^\infty}, \|B_{\Sigma_2}\|_{L^\infty} \leq C$  and Proposition 2.3 to estimate

$$\sum_{|\alpha| \leq 2m} \|\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)|\|_{L^2(\Sigma_2)} \leq C \|B_{\Sigma_2}\|_{H^{2m}(\Sigma_2)}$$

and

$$\begin{aligned} & \sum_{|\alpha| \leq 2m} \| |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))| \|_{L^2(\Sigma_2)} \\ & \leq C \sum_{|\alpha| \leq 2m} \| |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(y)| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(y)| \|_{L^2(\Sigma_1)} \leq C \|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)}. \end{aligned}$$

We are left with the last term in (5.24). As we already mentioned, this term has different scaling with respect to  $h$ . We use the Euler-Lagrange equation (3.4), (4.23) and  $\|d_{E_1}\|_{L^\infty(\partial E_2)} \leq Ch$  from Proposition 3.1 to deduce that

$$\|\tilde{\nabla} H_{E_2}\|_{L^\infty(\Sigma_2)}^2 \leq \frac{C}{h}.$$

Therefore we have, by Proposition 2.3, that

$$\begin{aligned} & h \sum_{|\alpha| \leq 2m} \| (|\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))|) \cdots \\ & \quad (|\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))|) \cdots \\ & \quad \cdots |\tilde{\nabla}^{1+\alpha_{2m+1}} H_{E_2}(x)| |\tilde{\nabla}^{1+\alpha_{2m+2}} H_{E_2}(x)| \|_{L^2(\Sigma_2)} \\ & \leq Ch \|\tilde{\nabla} H_{E_2}\|_{L^\infty(\Sigma_2)}^2 (\|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)} + \|B_{\Sigma_2}\|_{H^{2m}(\Sigma_2)}) \\ & \quad + Ch \|\tilde{\nabla} H_{E_2}\|_{L^\infty(\Sigma_2)} \|H_{E_2}\|_{H^{2m+1}(\Sigma_2)} \\ & \leq C \|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)} + C \|B_{\Sigma_2}\|_{H^{2m}(\Sigma_2)} + C\sqrt{h} \|H_{E_2}\|_{H^{2m+1}(\Sigma_2)} \\ & \leq C \|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)} + C \|B_{\Sigma_2}\|_{H^{2m+1}(\Sigma_2)} \end{aligned}$$

when  $h \leq 1$ , and the claim follows.  $\square$

Let us conclude this section by discussing briefly how we obtain Theorem 1.1 and Corollary 1.2 from the results in Sects. 4 and 5. We obtain first from Lemma 4.6 and from Theorem 4.7 that the approximative flow  $(E_t^h)_k$  satisfies UBC with radius  $r_0/2$  for  $t \leq T_0$  and we have

$$\frac{\|S_{E_{t+h}^h}\|_{L^\infty} - \|S_{E_t^h}\|_{L^\infty}}{h} \leq C_n \|S_{E_t^h}\|_{L^\infty}^3. \tag{5.25}$$

Then we use Theorem 5.2 to deduce that for  $t \in [\delta, T_0]$  the sets  $E_t^h$  are uniformly  $C^3$ -regular when  $h$  is small enough. By Ascoli-Arzelà theorem we may pass the estimate (5.25) to the limit as  $h \rightarrow 0$  and conclude that the function  $t \mapsto \sup_{s \leq t} \|S_{E_s}\|_{L^\infty}$  is locally Lipschitz continuous and satisfies

$$\frac{d}{dt} \left( \sup_{s \leq t} \|S_{E_s}\|_{L^\infty} \right) \leq C_n \left( \sup_{s \leq t} \|S_{E_s}\|_{L^\infty} \right)^3 \tag{5.26}$$

for almost every  $t \geq 0$  as long as  $\sup_{s \leq t} \|S_{E_s}\|_{L^\infty}$  remains bounded. The inequality (5.26) implies that UBC is an open condition in time. To be more precise if the flat

flow  $(E_t^h)_t$ , starting from  $E_0$ , satisfies  $\sup_{t \leq T} \|S_{E_t}\|_{L^\infty} \leq C$ , then by (5.26) there is  $\delta > 0$  such that

$$\sup_{t \leq T+\delta} \|S_{E_t}\|_{L^\infty} \leq 2C.$$

This together with the estimate in Theorem 5.2 implies Theorem 1.1.

The consistency principle follows from the regularity in a rather straightforward way. Indeed, we obtain by the uniform regularity of the approximate flat flow  $(E_t^h)_{t \in [0, T]}$  and by the Euler-Lagrange equation (3.4) that the signed distance function satisfies

$$\partial_t d_{E_t}(x) = \Delta_{\mathbb{R}^{n+1}} d_{E_t}(\pi_{\partial E_t}(x)) + f(t)$$

for  $t \leq T$  and for  $x$  in a neighborhood of  $\partial E_t$ , where  $f(t)$  is a bounded function of time. From here we may conclude that the flat flow satisfies

$$V_t = -H_{E_t} + f(t).$$

Since the flat flow preserves the volume then necessarily  $f(t) = \int_{\partial E_t} H_{E_t} \, d\mathcal{H}^n$  and thus it is a solution to (1.1).

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## Declarations

**Conflicts of interest** The authors declare that they have no conflicts of interest.

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VESA JULIN

University of Jyväskylä: Jyvaskylan Yliopisto,  
Jyvaskyla  
Finland.

e-mail: vesa.julin@jyu.fi

and

JOONAS NIINIKOSKI

Charles University, Faculty of Mathematics and Physics,  
Prague

Czech Republic.

e-mail: niinikoski@karlin.mff.cuni.cz

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