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On the reflexivity properties of Banach bundles and Banach modules

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Abstract

In this paper, we investigate some reflexivity-type properties of separable measurable Banach bundles over a σ-finite measure space. Our two main results are the following:

- The fibers of a bundle are uniformly convex (with a common modulus of convexity) if and only if the space of its L^p-sections is uniformly convex for every $p \in (1, \infty)$.
- The fibers of a bundle are reflexive if and only if the space of its L^p-sections is reflexive for every $p \in (1, \infty)$.

They generalise well-known results for Lebesgue–Bochner spaces.

Keywords Banach bundle · Normed module · Section of a Banach bundle · Uniform convexity · Reflexivity

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1 Introduction

General overview

In this paper, we focus on the theory of measurable Banach bundles over a given σ-finite measure space $(\mathrm{X}, \Sigma, \mathfrak{m})$. Our primary aim is to understand whether some important properties of the fibers of a measurable Banach bundle (such as Hilbertianity, uniform convexity, and reflexivity) carry over to the space of its L^p-sections, and vice versa.

Given an 'ambient' Banach space $\mathbb{B}$, a weakly measurable multivalued map $\mathbf{E}\colon \mathrm{X} \twoheadrightarrow \mathbb{B}$ is said to be a *Banach* $\mathbb{B}$*-bundle* on X if $\mathbf{E}(x)$ is a closed linear subspace of $\mathbb{B}$ for every $x \in \mathrm{X}$. A strongly $\mathfrak{m}$-measurable map $v\colon \mathrm{X} \to \mathbb{B}$, such that $v(x) \in \mathbf{E}(x)$ for every $x \in \mathrm{X}$ is called a *section* of $\mathbf{E}$. For any exponent $p \in (1, \infty)$, we denote by $\Gamma_p(\mathbf{E})$ the space of (equivalence classes, up to $\mathfrak{m}$-a.e. equality, of) those sections of $\mathbf{E}$ for which $\mathrm{X} \ni x \mapsto \|v(x)\|_{\mathbb{B}} \in \mathbb{R}$ belongs to $L^p(\mathfrak{m})$. It is worth pointing out that the well-known concept of *Lebesgue–Bochner space* $L^p(\mathfrak{m}; \mathbb{B})$ is a particular instance of a section space, corresponding to the bundle constantly equal to $\mathbb{B}$.

The space $\Gamma_p(\mathbf{E})$ naturally comes with a pointwise multiplication by $L^\infty(\mathfrak{m})$-functions and with a *pointwise norm* operator $|\cdot|\colon \Gamma_p(\mathbf{E}) \to L^p(\mathfrak{m})$, given by $|v| := \|v(\cdot)\|_{\mathbb{B}}$. The function $\Gamma_p(\mathbf{E}) \ni v \mapsto \|v\|_{\Gamma_p(\mathbf{E})} := \big\||v|\big\|_{L^p(\mathfrak{m})}$ defines a complete norm on $\Gamma_p(\mathbf{E})$. All in all, $\Gamma_p(\mathbf{E})$ is an $L^p(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module*, in the sense of Gigli [7]. We remark that, more surprisingly, in the case of separable normed modules the converse implication holds as well: every separable $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ is isomorphic to $\Gamma_p(\mathbf{E})$, for some measurable Banach $\mathbb{B}$-bundle $\mathbf{E}$ on X, where $\mathbb{B}$ is a separable Banach space. This representation result—first obtained in [20] for 'locally finitely generated' modules and later generalised in [4] to all separable modules—in fact strongly motivates our interest towards the language of measurable Banach bundles.

The theory of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules was introduced by Gigli in [7]—as already mentioned—and refined further in [6]. The main purpose was to provide a robust functional–analytic framework, suitable for constructing effective notions of 1*-forms* and *vector fields* in the setting of metric measure spaces. The key object introduced in [7] is the *cotangent module* $L^2(T^*\mathrm{X})$, which is obtained, roughly speaking, as the completion of the $L^\infty(\mathfrak{m})$-linear combinations of the 'formal differentials' $\mathrm{d}f$ of Sobolev functions $f \in W^{1,2}(\mathrm{X})$. It is evident that it is not sufficient to consider only the Banach space structure of $L^2(T^*\mathrm{X})$, but instead one has to keep track also of the 'pointwise' behaviour of the elements of $L^2(T^*\mathrm{X})$, which is encoded into the $L^\infty(\mathfrak{m})$-module structure and the pointwise norm. Due to this reason, $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules were the correct class of spaces to take into account. In this regard, an enlightening side result—which is not strictly needed for the purposes of this paper, but that we report for future reference—will be discussed in Appendix 1. More precisely, in Theorem A.2 we will characterise those complete norms over a given $L^\infty(\mathfrak{m})$-module that are induced by an $L^p(\mathfrak{m})$-valued pointwise norm operator via integration.

Prior to the development of L^p-normed L^∞-modules on metric measure spaces, some strictly related notions were already well-established in the literature: for instance, *random normed modules* (or *RN modules*, or *randomly normed spaces*) were introduced by Guo [10] (after Schweizer and Sklar [23]) and by Haydon et al. [18], independently. Typically, random normed modules are formulated over a probability measure space. In view of this fact, we will work in the general framework of normed modules over a σ-finite measure space (and not only over a metric measure space). The notion of a random normed module is an important concept in random metric theory, which is derived from the investigation of probabilistic metric spaces. A key construction in this theory is that of a random conjugate space. The random metric theory has applications in finance optimisation problems, and it is connected with the study of conditional and dynamic risk measures. See [15] and the references therein.

Statement of results

Let us now describe more in details the main results that we will achieve in this paper. Fix a σ-finite measure space $(\mathrm{X}, \Sigma, \mathfrak{m})$, a separable Banach space $\mathbb{B}$, and a measurable Banach $\mathbb{B}$-bundle $\mathbf{E}$ on X. Then, we will prove the following statements:

(a) $\mathbf{E}(x)$ *is Hilbert for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *if and only if* $\Gamma_2(\mathbf{E})$ *is Hilbert.* See Theorem 3.1.
(b) $\mathbf{E}(x)$ *is uniformly convex for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *(and with modulus of convexity independent of* x*) if and only if* $\Gamma_p(\mathbf{E})$ *is uniformly convex for all* $p \in (1, \infty)$. See Theorem 3.5. Its proof is more involved than the one for the Hilbertian case, and relies upon some previous results about *random uniform convexity* by Guo and Zeng [16, 17]. The corresponding statement for Lebesgue–Bochner spaces can be found, e.g., in [3].
(c) $\mathbf{E}(x)$ *is reflexive for* $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ *if and only if* $\Gamma_p(\mathbf{E})$ *is reflexive for all* $p \in (1, \infty)$. See Theorem 3.12.

The above results are well-known in the special case of Lebesgue–Bochner spaces. We point out that the implication '$L^p(\mathfrak{m}; \mathbb{B})$ reflexive implies $\mathbb{B}$ reflexive' can be easily proved: assuming $\mathfrak{m}(\mathrm{X}) = 1$ for simplicity, one can realise $\mathbb{B}$ as a closed linear subspace of $L^p(\mathfrak{m}; \mathbb{B})$ (by sending each $v \in \mathbb{B}$ to the section constantly equal to v). However, the corresponding implication '$\Gamma_p(\mathbf{E})$ reflexive implies $\mathbf{E}(x)$ reflexive for $\mathfrak{m}$-a.e. x' will require a much more difficult proof.

We also mention that, along the way to prove item c), we will obtain a result of independent interest: shortly said, given a measurable Banach $\mathbb{B}$-bundle $\mathbf{E}$ (with $\mathbb{B}$ not necessarily separable), the dual of $\Gamma_p(\mathbf{E})$ as a normed module can be identified with the space of q-integrable *weakly* measurable* sections of the dual bundle $\mathrm{X} \ni x \mapsto \mathbf{E}(x)'$, where $\frac{1}{p} + \frac{1}{q} = 1$. See Sect. 3.2 for the precise formulation, as well as Theorem 3.8 for the relevant equivalence result. The corresponding statement for Lebesgue–Bochner spaces, stating that $L^p(\mathfrak{m}; \mathbb{B})'$ can be identified with the space $L^q_{w^*}(\mathfrak{m}; \mathbb{B}')$ of q-integrable 'weakly* measurable' maps from $(\mathrm{X}, \Sigma, \mathfrak{m})$ to $\mathbb{B}'$, was previously known (see [13]). We also point out that a variant of the statement in c) for normed modules has been recently obtained in [8, Theorems 3.9 and 4.17]. However, in general neither the results of [8] imply c), nor the vice versa.

Addendum

While in a previous version of this manuscript only one of the two implications in (c) was obtained (namely, that 'reflexive fibers implies reflexive section space'), in the current version the full equivalence is proved. This is due to the fact that an anonymous colleague kindly pointed out to us the result [18, Theorem 6.19], which is the analogue of (c) in the setting of *direct integrals*. However, we do not obtain the implication 'reflexive section space implies reflexive fibers' as a consequence of [18, Theorem 6.19], but we rather follow the same proof strategy; see Remark 3.13 for more comments on this. It would be very interesting—but outside the scopes of this manuscript—to investigate the relation between our notion of Banach bundle and the theory of direct integrals considered in [18].

2 Preliminaries

To begin with, we fix some general terminology, which we will use throughout the entire paper. For any $p \in [1, \infty]$, we tacitly denote by $q \in [1, \infty]$ its **conjugate exponent**, that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Given a σ-finite measure space $(\mathrm{X}, \Sigma, \mathfrak{m})$, we denote by $\mathcal{L}^0_{\mathrm{ext}}(\Sigma)$ the space of all measurable functions from X to $\mathbb{R} \cup \{\pm\infty\}$, while $L^0_{\mathrm{ext}}(\mathfrak{m})$ stands for the quotient of $\mathcal{L}^0_{\mathrm{ext}}(\Sigma)$ up to $\mathfrak{m}$-a.e. equality. We call $\pi_{\mathfrak{m}} : \mathcal{L}^0_{\mathrm{ext}}(\Sigma) \to L^0_{\mathrm{ext}}(\mathfrak{m})$ the usual projection map on the quotient. Moreover, we define

$$\mathcal{L}^0(\Sigma) := \left\{ f \in \mathcal{L}^0_{\mathrm{ext}}(\Sigma) \;\middle|\; f(\mathrm{X}) \subseteq \mathbb{R} \right\}$$

and $L^0(\mathfrak{m}) := \pi_{\mathfrak{m}}\big(\mathcal{L}^0(\Sigma)\big)$. During this paper, we will use two different notions of 'essential supremum/infimum', namely:

- If $f \in \mathcal{L}^0_{\mathrm{ext}}(\Sigma)$ and $E \in \Sigma$, we define $\operatorname{ess\,sup}_E f, \operatorname{ess\,inf}_E f \in \mathbb{R} \cup \{\pm\infty\}$ respectively as

$$\operatorname*{ess\,sup}_E f := \inf \left\{ \lambda \in \mathbb{R} \cup \{\pm\infty\} \;\middle|\; f \le \lambda, \text{ holds } \mathfrak{m}\text{-a.e. on } E \right\},$$
$$\operatorname*{ess\,inf}_E f := \sup \left\{ \lambda \in \mathbb{R} \cup \{\pm\infty\} \;\middle|\; f \ge \lambda, \text{ holds } \mathfrak{m}\text{-a.e. on } E \right\}.$$

- Given a (possibly uncountable) family $\{f_i\}_{i \in I} \subseteq \mathcal{L}^0_{\mathrm{ext}}(\Sigma)$, we define $\bigvee_{i \in I} f_i \in L^0_{\mathrm{ext}}(\mathfrak{m})$ as the unique $f \in L^0_{\mathrm{ext}}(\mathfrak{m})$, such that $f_i \le f$ $\mathfrak{m}$-a.e. for every $i \in I$ and satisfying

$$g \in L^0_{\mathrm{ext}}(\mathfrak{m}), \quad f_i \le g \ \ \mathfrak{m}\text{-a.e. for every } i \in I \quad \Longrightarrow \quad f \le g \ \ \mathfrak{m}\text{-a.e.}$$

Similarly, $\bigwedge_{i\in I} f_i \in L^0_{\rm ext}(\mathfrak{m})$ is the unique element f of $L^0_{\rm ext}(\mathfrak{m})$, such that $f_i \geq f$ $\mathfrak{m}$-a.e. for every $i \in I$ and satisfying

$$g \in L^0_{\rm ext}(\mathfrak{m}), \quad f_i \geq g \ \ \mathfrak{m}\text{-a.e. for every } i \in I \quad \Longrightarrow \quad f \geq g \ \ \mathfrak{m}\text{-a.e.}$$

Notice that the above notions of essential supremum/infimum are invariant under modifications of the functions f and f_i on an $\mathfrak{m}$-negligible set, thus accordingly we can unambiguously consider $\operatorname{ess\,sup}_E f$, $\operatorname{ess\,inf}_E f$, $\bigvee_{i\in I} f_i$, $\bigwedge_{i\in I} f_i$ whenever f and f_i are elements of $L^0_{\rm ext}(\mathfrak{m})$.

The **Lebesgue spaces** are defined in the usual way: given any $p \in [1,\infty)$, we define

$$\mathcal{L}^p(\mathfrak{m}) := \left\{ f \in \mathcal{L}^0(\Sigma) \;\middle|\; \int |f|^p \,\mathrm{d}\mathfrak{m} < +\infty \right\},$$
$$\mathcal{L}^\infty(\mathfrak{m}) := \left\{ f \in \mathcal{L}^0(\Sigma) \;\middle|\; \sup_{\mathrm{X}} |f| < +\infty \right\}.$$

We also consider the spaces $L^p(\mathfrak{m}) := \pi_{\mathfrak{m}}\big(\mathcal{L}^p(\mathfrak{m})\big)$ and $L^\infty(\mathfrak{m}) := \pi_{\mathfrak{m}}\big(\mathcal{L}^\infty(\mathfrak{m})\big)$, which are Banach spaces if endowed with the usual pointwise operations and with the norms

$$\|f\|_{L^p(\mathfrak{m})} := \left(\int |f|^p \,\mathrm{d}\mathfrak{m} \right)^{1/p}, \qquad \|f\|_{L^\infty(\mathfrak{m})} := \operatorname*{ess\,sup}_{\mathrm{X}} |f|.$$

Recall that $L^q(\mathfrak{m})$ is isomorphic as a Banach space to the dual of $L^p(\mathfrak{m})$.

It is worth recalling that, given an arbitrary σ-finite measure space $(\mathrm{X},\Sigma,\mathfrak{m})$ and any exponent $p \in [1,\infty)$, the Lebesgue space $L^p(\mathfrak{m})$ is not necessarily separable. In fact, it holds that

$$(\mathrm{X},\Sigma,\mathfrak{m}) \text{ is separable} \iff L^p(\mathfrak{m}) \text{ is separable for every } p \in [1,\infty), \tag{2.1}$$

where $(\mathrm{X},\Sigma,\mathfrak{m})$ is said to be **separable** provided there exists a countable family $\mathcal{C} \subseteq \Sigma$ for which the following property holds: given any set $E \in \Sigma$ with $\mathfrak{m}(E) < +\infty$ and $\varepsilon > 0$, there exists $F \in \mathcal{C}$, such that $\mathfrak{m}(E \Delta F) < \varepsilon$. The equivalence stated in (2.1) is well-known; it follows, for instance, from [4, Lemma 2.14]. We also point out that if (X,d) is a complete and separable metric space, Σ is the Borel σ-algebra of X, and $\mathfrak{m}$ is a boundedly-finite Borel measure on X, then $(\mathrm{X},\Sigma,\mathfrak{m})$ is a separable measure space.

2.1 Banach spaces

Let us begin by fixing some basic terminology about Banach spaces. Given a Banach space $\mathbb{B}$, we denote by $\mathbb{B}'$ its (continuous) dual space. Moreover, we denote by $B_{\mathbb{B}}$ and $\mathbb{S}_{\mathbb{B}}$ the **closed unit ball** and the **unit sphere** of $\mathbb{B}$, respectively. Namely, we set

$$B_{\mathbb{B}} := \big\{ v \in \mathbb{B} \;\big|\; \|v\|_{\mathbb{B}} \leq 1 \big\}, \qquad \mathbb{S}_{\mathbb{B}} := \big\{ v \in \mathbb{B} \;\big|\; \|v\|_{\mathbb{B}} = 1 \big\}.$$

In this paper we are mostly concerned with Hilbert, uniformly convex, and reflexive spaces. We recall the notion of uniform convexity, just to fix a notation for the modulus of convexity.

Definition 2.1 (*Uniform convexity*) Let $\mathbb{B}$ be a Banach space. Let us define the **modulus of convexity** $\delta_{\mathbb{B}} : (0, 2) \to [0, 1]$ of the space $\mathbb{B}$ as follows: given any $\varepsilon \in (0, 2)$, we set

$$\delta_{\mathbb{B}}(\varepsilon) := \inf \left\{ 1 - \left\| \frac{v + w}{2} \right\|_{\mathbb{B}} \,\middle|\, v, w \in \mathbb{S}_{\mathbb{B}},\ \|v - w\|_{\mathbb{B}} \geq \varepsilon \right\}.$$

Then we say that $\mathbb{B}$ is **uniformly convex** if and only if $\delta_{\mathbb{B}}(\varepsilon) > 0$ holds for every $\varepsilon \in (0, 2)$.

It is well-known that the following implications are verified:

$$\mathbb{B} \text{ is Hilbert} \quad \Longrightarrow \quad \mathbb{B} \text{ is uniformly convex} \quad \Longrightarrow \quad \mathbb{B} \text{ is reflexive.}$$

The following elementary observation will play a rôle during the proof of Theorem 3.5.

Remark 2.2 The uniform convexity condition can be checked on a dense set. Namely, given any dense subset D of $\mathbb{S}_{\mathbb{B}}$, one has that for every $\varepsilon \in (0, 2)$ it holds that

$$\delta_{\mathbb{B}}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{v + w}{2} \right\|_{\mathbb{B}} \,\middle|\, v, w \in D,\ \|v - w\|_{\mathbb{B}} > \varepsilon \right\}.$$

This claim can be easily proved via a standard approximation argument.

Let us briefly recall the basic theory of Lebesgue–Bochner spaces; for a detailed account we refer, e.g., to [19]. Fix a σ-finite measure space $(\mathrm{X}, \Sigma, \mathfrak{m})$ and a Banach space $\mathbb{B}$. Then, a given map $v : \mathrm{X} \to \mathbb{B}$ is said to be **strongly** $\mathfrak{m}$**-measurable** if it is $\mathfrak{m}$-measurable and **essentially separably valued**, i.e., there exists an $\mathfrak{m}$-null set $N \in \Sigma$ such that the image $v(\mathrm{X} \backslash N) \subseteq \mathbb{B}$ is separable. It holds that a given map $v : \mathrm{X} \to \mathbb{B}$ is strongly $\mathfrak{m}$-measurable if and only if it is both essentially separably valued and **weakly** $\mathfrak{m}$**-measurable**, that is

$$\mathrm{X} \ni x \mapsto \langle \omega, v(x) \rangle \in \mathbb{R}, \quad \text{is } \mathfrak{m}\text{-measurable for every } \omega \in \mathbb{B}'.$$

We denote by $\mathcal{L}^0(\mathfrak{m}; \mathbb{B})$ the space of all strongly $\mathfrak{m}$-measurable maps from X to $\mathbb{B}$, while for any given exponent $p \in [1, \infty)$ we define

$$\mathcal{L}^p(\mathfrak{m}; \mathbb{B}) := \left\{ v \in \mathcal{L}^0(\mathfrak{m}; \mathbb{B}) \,\middle|\, \int \|v(\cdot)\|_{\mathbb{B}}^p \, \mathrm{d}\mathfrak{m} < +\infty \right\},$$
$$\mathcal{L}^\infty(\mathfrak{m}; \mathbb{B}) := \left\{ v \in \mathcal{L}^0(\mathfrak{m}; \mathbb{B}) \,\middle|\, \sup_{\mathrm{X}} \|v(\cdot)\|_{\mathbb{B}} < +\infty \right\}.$$

These definitions are well-posed, since $\|v(\cdot)\|_{\mathbb{B}}$ is $\mathfrak{m}$-measurable thanks to the $\mathfrak{m}$-measurability of v and the continuity of $\|\cdot\|_{\mathbb{B}}$. We introduce an equivalence relation on $\mathcal{L}^0(\mathfrak{m};\mathbb{B})$: given any $v, w \in \mathcal{L}^0(\mathfrak{m};\mathbb{B})$, we declare that $v \sim w$ if and only if $v(x) = w(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. Then, we define

$$L^0(\mathfrak{m};\mathbb{B}) := \mathcal{L}^0(\mathfrak{m};\mathbb{B})/\sim,$$

while $\pi_{\mathfrak{m}} \colon \mathcal{L}^0(\mathfrak{m};\mathbb{B}) \to L^0(\mathfrak{m};\mathbb{B})$ stands for the projection map on the quotient. Moreover, we define $L^p(\mathfrak{m};\mathbb{B}) := \pi_{\mathfrak{m}}\big(\mathcal{L}^p(\mathfrak{m};\mathbb{B})\big)$ for every $p \in [1,\infty]$. The linear space $L^p(\mathfrak{m};\mathbb{B})$ becomes a Banach space if it is endowed with the norm $\|v\|_{L^p(\mathfrak{m};\mathbb{B})} := \big\| \|v(\cdot)\|_{\mathbb{B}} \big\|_{L^p(\mathfrak{m})}$. The spaces $L^p(\mathfrak{m};\mathbb{B})$ are called the **Lebesgue–Bochner spaces**. Note also that $L^p(\mathfrak{m};\mathbb{R}) = L^p(\mathfrak{m})$.

2.2 Banach bundles

Aim of this section is to recall the notion of Banach bundle introduced in [4] and its main properties. Let us fix a measurable space (X, Σ) and a Banach space $\mathbb{B}$. By $\boldsymbol{\varphi} \colon \mathrm{X} \twoheadrightarrow \mathbb{B}$ we denote a **multivalued map**, i.e., a map from X to the power set of $\mathbb{B}$. Following [1], we say that $\boldsymbol{\varphi}$ is **weakly measurable** provided $\big\{x \in \mathrm{X} \,:\, \boldsymbol{\varphi}(x) \cap U \neq \varnothing\big\} \in \Sigma$ holds for every open set $U \subseteq \mathbb{B}$. The following definition is taken from [4, Definition 4.1] (cf. also with [21, Definition 2.15] for the case of a non-separable ambient space $\mathbb{B}$):

Definition 2.3 (*Banach bundle*) Let (X, Σ) be a measurable space and $\mathbb{B}$ a Banach space. Then a given weakly measurable multivalued map $\mathbf{E} \colon \mathrm{X} \twoheadrightarrow \mathbb{B}$ is said to be a **Banach $\mathbb{B}$-bundle** on X provided $\mathbf{E}(x)$ is a closed linear subspace of $\mathbb{B}$ for every $x \in \mathrm{X}$.

We define the **support** of a Banach $\mathbb{B}$-bundle $\mathbf{E}$ as follows:

$$H(\mathbf{E}) := \big\{x \in \mathrm{X} \;\big|\; \mathbf{E}(x) \neq \{0_{\mathbb{B}}\}\big\}.$$

Notice that $H(\mathbf{E}) = \mathrm{X} \backslash \big\{x \in \mathrm{X} \,:\, \mathbf{E}(x) \cap (\mathbb{B}\backslash\{0_{\mathbb{B}}\}) \neq \varnothing\big\} \in \Sigma$.

Let us also introduce the following subclasses of Banach bundles, which will be studied in details in Sects. 3.1 and 3.3.

Definition 2.4 (*Hilbert, uniformly convex, reflexive bundles*) Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a measure space, $\mathbb{B}$ a Banach space, and $\mathbf{E}$ a Banach $\mathbb{B}$-bundle over X. Then, we say that:

(i) $\mathbf{E}$ is **Hilbert** if $\mathbf{E}(x)$ is Hilbert for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.
(ii) $\mathbf{E}$ is **uniformly convex** if $\mathbf{E}(x)$ is uniformly convex for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.
(iii) $\mathbf{E}$ is **reflexive** if $\mathbf{E}(x)$ is reflexive for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.

Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space. By a **section** of a Banach $\mathbb{B}$-bundle $\mathbf{E}$ over X we mean a measurable selector of $\mathbf{E}$, i.e., a strongly $\mathfrak{m}$-measurable map $v \colon \mathrm{X} \to \mathbb{B}$ with $v(x) \in \mathbf{E}(x)$ for all $x \in \mathrm{X}$. We denote by $\bar{\Gamma}_0(\mathbf{E})$ the family of all sections of $\mathbf{E}$. We introduce an equivalence relation on $\bar{\Gamma}_0(\mathbf{E})$: given $v, w \in \bar{\Gamma}_0(\mathbf{E})$, we declare that $v \sim w$ if and only if $v(x) = w(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. We then define

$$\Gamma_0(\mathbf{E}) := \bar{\Gamma}_0(\mathbf{E})/\sim,$$

while $\pi_{\mathfrak{m}} \colon \bar{\Gamma}_0(\mathbf{E}) \to \Gamma_0(\mathbf{E})$ stands for the projection map on the quotient. By analogy with the case of Lebesgue–Bochner spaces, for any given exponent $p \in [1, \infty]$ we define

$$\bar{\Gamma}_p(\mathbf{E}) := \left\{ v \in \bar{\Gamma}_0(\mathbf{E}) \;\middle|\; \|v(\cdot)\|_{\mathbb{B}} \in \mathcal{L}^p(\mathfrak{m}) \right\}, \qquad \Gamma_p(\mathbf{E}) := \pi_{\mathfrak{m}}\big(\bar{\Gamma}_p(\mathbf{E})\big).$$

The previous definitions are well-posed, since $\mathrm{X} \ni x \mapsto \|v(x)\|_{\mathbb{B}} \in \mathbb{R}$ is $\mathfrak{m}$-measurable thanks to the strong $\mathfrak{m}$-measurability of v and the continuity of $\|\cdot\|_{\mathbb{B}}$. One can readily check that $\Gamma_p(\mathbf{E})$ is a Banach space if endowed with the pointwise operations and with the norm

$$\|v\|_{\Gamma_p(\mathbf{E})} := \big\| \|v(\cdot)\|_{\mathbb{B}} \big\|_{L^p(\mathfrak{m})}, \quad \text{for every } v \in \Gamma_p(\mathbf{E}).$$

This is de facto a generalisation of Lebesgue–Bochner spaces: calling $\mathbf{E}_{\mathbb{B}}$ the Banach $\mathbb{B}$-bundle whose fibers are constantly equal to the space $\mathbb{B}$, it holds that $\bar{\Gamma}_p(\mathbf{E}_{\mathbb{B}}) = \mathcal{L}^p(\mathfrak{m}; \mathbb{B})$ and $\Gamma_p(\mathbf{E}_{\mathbb{B}}) = L^p(\mathfrak{m}; \mathbb{B})$.

Remark 2.5 Consistently with the case of Lebesgue spaces, the space of sections $\Gamma_p(\mathbf{E})$ of a given Banach $\mathbb{B}$-bundle $\mathbf{E}$ over X needs not be separable, even if $\mathbb{B}$ is separable, $\mathfrak{m}$ is σ-finite, and $p \in [1, \infty)$. In fact, under the assumption that the ambient space $\mathbb{B}$ is separable, it holds that

$$\Gamma_p(\mathbf{E}) \text{ is separable for all } p \in [1, \infty) \quad \Longleftrightarrow \quad \big(\mathrm{X}, \Sigma, \mathfrak{m}|_{H(\mathbf{E})}\big) \text{ is separable.}$$

We omit the proof, similar to the one of (2.1).

Hereafter, we shall focus on σ-finite measure spaces and Banach $\mathbb{B}$-bundles $\mathbf{E}$ over X, where the space $\mathbb{B}$ is separable. We will need the following result, taken from [4, Proposition 4.4], whose proof we sketch here for the reader's usefulness.

Proposition 2.6 *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a* σ*-finite measure space. Let* $\mathbb{B}$ *be a separable Banach space and let* $\mathbf{E}$ *be a Banach* $\mathbb{B}$*-bundle over* X. *Let* $p \in [1, \infty)$ *be given. Then there exists a countable* $\mathbb{Q}$*-linear subspace* $\mathcal{C}$ *of* $\bar{\Gamma}_p(\mathbf{E})$*, such that* $\mathbf{E}(x) = \mathrm{cl}_{\mathbb{B}}\big\{v(x) : v \in \mathcal{C}\big\}$ *for every* $x \in \mathrm{X}$.

Proof Fix $(w_n)_{n \in \mathbb{N}}$ dense in $\mathbb{B}$. For any $k, n \in \mathbb{N}$, we define $\boldsymbol{\varphi}_{nk} : \mathrm{X} \twoheadrightarrow \mathbb{B}$ as

$$\boldsymbol{\varphi}_{nk}(x) := \begin{cases} \big\{ w \in \mathbf{E}(x) : \|w - w_n\|_{\mathbb{B}} \leq 1/k \big\}, & \text{if such set is not empty,} \\ \{0_{\mathbb{B}}\}, & \text{otherwise.} \end{cases}$$

One can readily check that the multivalued map $\boldsymbol{\varphi}_{nk}$ is weakly measurable. Therefore, an application of the Kuratowski–Ryll-Nardzewski Selection theorem (see, e.g., [1, Theorem 18.13]) ensures the existence of a section $v_{nk} \in \bar{\Gamma}_p(\mathbf{E})$, such that $v_{nk} \in$

$\boldsymbol{\varphi}_{nk}(x)$ for every $x \in \mathrm{X}$. This implies that $\{v_{nk} : n, k \in \mathbb{N}\}$ is dense in $\mathbf{E}(x)$ for every $x \in \mathrm{X}$. Consequently, the $\mathbb{Q}$-linear subspace of $\bar{\Gamma}_p(\mathbf{E})$ generated by $\{v_{nk} : n, k \in \mathbb{N}\}$ fulfils the requirements. □

We will need also the following easy consequence of Proposition 2.6:

Corollary 2.7 *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a* σ*-finite measure space. Let* $\mathbb{B}$ *be a separable Banach space and let* $\mathbf{E}$ *be a Banach* $\mathbb{B}$*-bundle over* X*. Then, there exists a countable subset* $\mathcal{D}$ *of* $\bar{\Gamma}_\infty(\mathbf{E})$*, such that*

$$\begin{aligned} \big\|v(x)\big\|_{\mathbb{B}} &\in \{0, 1\}, \quad \text{for every } v \in \mathcal{D} \text{ and } x \in \mathrm{X}, \\ \mathbb{S}_{\mathbf{E}(x)} &\subseteq \mathrm{cl}_{\mathbb{B}}\big\{v(x) \bigm| v \in \mathcal{D}\big\}, \quad \text{for every } x \in \mathrm{X}. \end{aligned}$$

Proof Take any family $\mathcal{C}$ as in Proposition 2.6 and define

$$\mathcal{D} := \big\{\mathbb{1}_{\{\|v(\cdot)\|_{\mathbb{B}}>0\}}\|v(\cdot)\|_{\mathbb{B}}^{-1}v(\cdot) \bigm| v \in \mathcal{C}\big\} \subseteq \bar{\Gamma}_\infty(\mathbf{E}).$$

It is then immediate to check that $\mathcal{D}$ has the desired properties. □

2.3 Banach modules

In this section, we recall the basics of the theory of Banach modules. We begin by introducing the notion of L^p-normed L^∞-module proposed by N. Gigli [7].

Definition 2.8 (L^p*-normed* L^∞*-module*) Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space and let $p \in (1, \infty)$ be a given exponent. Let $\mathscr{M}$ be a module over $L^\infty(\mathfrak{m})$. Then, a map $|\cdot|\colon \mathscr{M} \to L^p(\mathfrak{m})$ is said to be an $L^p(\mathfrak{m})$- **pointwise norm** on $\mathscr{M}$ provided it verifies the following conditions:

$$|v| \geq 0, \quad \text{for every } v \in \mathscr{M}, \text{ with } |v| = 0 \iff v = 0, \tag{2.2a}$$
$$|v + w| \leq |v| + |w|, \quad \text{for every } v, w \in \mathscr{M}, \tag{2.2b}$$
$$|f \cdot v| = |f||v|, \quad \text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathscr{M}, \tag{2.2c}$$

where equalities and inequalities are intended in the $\mathfrak{m}$-a.e. sense. The couple $(\mathscr{M}, |\cdot|)$ is called an $L^p(\mathfrak{m})$-**normed** $L^\infty(\mathfrak{m})$-**module**. We endow $\mathscr{M}$ with the norm $\|\cdot\|_{\mathscr{M}}$, given by

$$\|v\|_{\mathscr{M}} := \big\||v|\big\|_{L^p(\mathfrak{m})}, \quad \text{for every } v \in \mathscr{M}.$$

When the norm $\|\cdot\|_{\mathscr{M}}$ is complete, we say that $(\mathscr{M}, |\cdot|)$ is an $L^p(\mathfrak{m})$-**Banach** $L^\infty(\mathfrak{m})$-**module**.

The **support** of an $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ is defined as the $\mathfrak{m}$-a.e. uniquely determined set $H(\mathscr{M}) \in \Sigma$, such that

$$\mathbb{1}_{H(\mathscr{M})} = \bigvee\big\{\mathbb{1}_{\{|v|>0\}} \bigm| v \in \mathscr{M}\big\}.$$

A prototypical example of $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module is the space of $L^p(\mathfrak{m})$-sections of a Banach $\mathbb{B}$-bundle $\mathbf{E}$ over X, the $L^p(\mathfrak{m})$-pointwise norm on $\Gamma_p(\mathbf{E})$ being $|v| := \|v(\cdot)\|_{\mathbf{E}(\cdot)}$. If the space $\mathbb{B}$ is separable, then $H(\Gamma_p(\mathbf{E}))$ is ($\mathfrak{m}$-a.e. equivalent to) the support $H(\mathbf{E})$ of $\mathbf{E}$ for every $p \in (1, \infty)$.

An important class of $L^2(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules is that of Hilbert modules. Following [7, Definition 1.2.20], an $L^2(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module $\mathscr{H}$ is said to be **Hilbert** provided it is Hilbert when viewed as a Banach space. It is shown in [7, Proposition 1.2.21] that $\mathscr{H}$ is a Hilbert module if and only if the pointwise parallelogram identity holds:

$$|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2 \quad \mathfrak{m}\text{-a.e.}, \quad \text{for every } v, w \in \mathscr{H}.$$

An operator between two $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules is called a **homomorphism of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules** provided it is $L^\infty(\mathfrak{m})$-linear and continuous. The **dual** of an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ is given by the space $\mathscr{M}^*$ of all $L^\infty(\mathfrak{m})$-linear and continuous maps from $\mathscr{M}$ to $L^1(\mathfrak{m})$. It holds that $\mathscr{M}^*$ is an $L^q(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module if endowed with the $L^q(\mathfrak{m})$-pointwise norm operator $|\cdot|\colon \mathscr{M}^* \to L^q(\mathfrak{m})$, which is defined as

$$|\omega| := \bigvee \left\{ \omega(v) \;\middle|\; v \in \mathscr{M},\ |v| \le 1 \text{ holds } \mathfrak{m}\text{-a.e.} \right\}, \quad \text{for every } \omega \in \mathscr{M}^*.$$

We thus have a natural duality pairing $\langle \cdot, \cdot \rangle \colon \mathscr{M}^* \times \mathscr{M} \to L^1(\mathfrak{m})$, which is given by $\langle \omega, v \rangle := \omega(v)$ for every $\omega \in \mathscr{M}^*$ and $v \in \mathscr{M}$. Notice that $\langle \cdot, \cdot \rangle$ is a 'pointwise' duality pairing, taking values in $L^1(\mathfrak{m})$ (and not in $\mathbb{R}$). The Hahn–Banach theorem for normed modules ([11, 14], also [22, Theorem 3.30]) implies that if $\mathscr{N}$ is an $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-submodule of $\mathscr{M}$, then each $\omega \in \mathscr{N}^*$ can be extended to an element $\bar{\omega} \in \mathscr{M}^*$ satisfying $|\bar{\omega}| = |\omega|$. We denote by $J_{\mathscr{M}} \colon \mathscr{M} \to \mathscr{M}^{**}$ the **James' embedding** of $\mathscr{M}$ into its bidual, i.e., the unique homomorphism of $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules satisfying

$$\langle J_{\mathscr{M}}(v), \omega \rangle = \langle \omega, v \rangle, \quad \text{for every } v \in \mathscr{M} \text{ and } \omega \in \mathscr{M}^*. \tag{2.3}$$

We have that $|J_{\mathscr{M}}(v)| = |v|$ holds $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$ (as a consequence of the Hahn–Banach theorem for normed modules). Then, $\mathscr{M}$ is said to be **reflexive (as a module)** provided $J_{\mathscr{M}}$ is surjective (and thus an isomorphism). According to [7, Corollary 1.2.18], $\mathscr{M}$ is reflexive if and only if it is reflexive as a Banach space.

Let us also recall the notion of **adjoint operator**: given a homomorphism $\varphi \colon \mathscr{M} \to \mathscr{N}$ between two $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules $\mathscr{M}$ and $\mathscr{N}$, we denote by $\varphi^{\mathrm{ad}} \colon \mathscr{N}^* \to \mathscr{M}^*$ the unique homomorphism of $L^q(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules, such that

$$\langle \varphi^{\mathrm{ad}}(\omega), v \rangle = \langle \omega, \varphi(v) \rangle, \quad \text{for every } \omega \in \mathscr{N}^* \text{ and } v \in \mathscr{M}. \tag{2.4}$$

It holds that φ^{ad} is an isomorphism if and only if φ is an isomorphism.

In the theory of Banach modules, it is often convenient to drop the integrability assumption:

Definition 2.9 (L^0-*normed* L^0-*module*) Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space. Let $\mathscr{M}$ be a module over $L^0(\mathfrak{m})$. Then, a map $|\cdot|\colon \mathscr{M} \to L^0(\mathfrak{m})$ is said to be an $L^0(\mathfrak{m})$-**pointwise norm** on $\mathscr{M}$ provided it verifies (2.2a), (2.2b), (2.2c), but replacing $L^\infty(\mathfrak{m})$ with $L^0(\mathfrak{m})$ in (2.2c). The couple $(\mathscr{M}, |\cdot|)$ is called an $L^0(\mathfrak{m})$-**normed** $L^0(\mathfrak{m})$-**module**, or a **random normed module over** $\mathbb{R}$ **with base** $(\mathrm{X}, \Sigma, \mathfrak{m})$ in the case where $\mathfrak{m}$ is a probability measure. We also endow $\mathscr{M}$ with the distance $\mathsf{d}_{\mathscr{M}}$, given by

$$\mathsf{d}_{\mathscr{M}}(v, w) := \int \min\{|v - w|, 1\}\, \mathrm{d}\mathfrak{m}', \qquad \text{for every } v, w \in \mathscr{M}, \tag{2.5}$$

where $\mathfrak{m}'$ is any given probability measure on Σ satisfying $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. When $\mathsf{d}_{\mathscr{M}}$ is complete, $(\mathscr{M}, |\cdot|)$ is called an $L^0(\mathfrak{m})$-**Banach** $L^0(\mathfrak{m})$-**module**.

A key example of $L^0(\mathfrak{m})$-Banach $L^0(\mathfrak{m})$-module is the space $\Gamma_0(\mathbf{E})$, where $\mathbf{E}$ is a Banach $\mathbb{B}$-bundle over X, and as an $L^0(\mathfrak{m})$-pointwise norm on $\Gamma_0(\mathbf{E})$ we consider $|v| := \|v(\cdot)\|_{\mathbf{E}(\cdot)}$.

It is worth pointing out that a random normed module is complete with respect to the distance introduced in (2.5) if and only if it is complete in the sense of [16, 17], i.e., with respect to the so-called (ϵ, λ)-topology. Indeed, both the topology induced by the L^0-distance and the (ϵ, λ)-topology coincide with the one of 'convergence in measure', cf. with [9] and [12].

An operator between two $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules is called a **homomorphism of** $L^0(\mathfrak{m})$-**normed** $L^0(\mathfrak{m})$-**modules** provided it is $L^0(\mathfrak{m})$-linear and continuous.

The relation between $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules and $L^0(\mathfrak{m})$-Banach $L^0(\mathfrak{m})$-modules can be expressed by the following result, which is taken from [6, Theorem/Definition 2.7] and follows also from [13, Theorem 3.1].

Proposition 2.10 (L^0-completion) *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a* σ*-finite measure space. Let* $\mathscr{M}$ *be an* $L^p(\mathfrak{m})$*-Banach* $L^\infty(\mathfrak{m})$*-module, for some exponent* $p \in (1, \infty)$*. Then, there exists a unique couple* $(\mathscr{M}^0, \iota)$*, where* $\mathscr{M}^0$ *is an* $L^0(\mathfrak{m})$*-Banach* $L^0(\mathfrak{m})$*-module, while* $\iota\colon \mathscr{M} \to \mathscr{M}^0$ *is a linear operator which preserves the pointwise norm and has dense image. Uniqueness is intended up to unique isomorphism: given another couple* $(\mathscr{N}^0, \iota')$ *having the same properties, there exists a unique isomorphism of* $L^0(\mathfrak{m})$*-Banach* $L^0(\mathfrak{m})$*-modules* $\Phi\colon \mathscr{M}^0 \to \mathscr{N}^0$*, such that* $\iota' = \Phi \circ \iota$*. The space* $\mathscr{M}^0$ *is called the* $L^0(\mathfrak{m})$*-**completion** of* $\mathscr{M}$*.*

3 Main results

3.1 Hilbertian and uniformly convex bundles/modules

In this section, we prove that a given separable Banach bundle is Hilbert (resp. uniformly convex) if and only if its space of sections is Hilbert (resp. uniformly convex). Let us begin with Hilbert bundles/modules.

Theorem 3.1 (Hilbert bundles/modules) *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a* σ*-finite measure space. Let* $\mathbb{B}$ *be a separable Banach space and let* $\mathbf{E}$ *be a Banach* $\mathbb{B}$*-bundle over* X*. Then,* $\mathbf{E}$

is a Hilbert bundle if and only if $\Gamma_2(\mathbf{E})$ *is a Hilbert space. Necessity holds also when* $\mathbb{B}$ *is non-separable.*

Proof Suppose $\mathbb{B}$ is an arbitrary Banach space and $\mathbf{E}$ is a Hilbert bundle. Fix $v, w \in \Gamma_2(\mathbf{E})$. Then,

$$\|v(x) + w(x)\|^2_{\mathbf{E}(x)} + \|v(x) - w(x)\|^2_{\mathbf{E}(x)} = 2\|v(x)\|^2_{\mathbf{E}(x)} + 2\|w(x)\|^2_{\mathbf{E}(x)}$$

for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. By integrating it over X, we obtain

$$\|v + w\|^2_{\Gamma_2(\mathbf{E})} + \|v - w\|^2_{\Gamma_2(\mathbf{E})} = 2\|v\|^2_{\Gamma_2(\mathbf{E})} + 2\|w\|^2_{\Gamma_2(\mathbf{E})},$$

whence it follows that $\Gamma_2(\mathbf{E})$ is a Hilbert module.

Conversely, suppose $\mathbb{B}$ is separable and $\Gamma_2(\mathbf{E})$ is a Hilbert module. Thanks to Proposition 2.6, we can find a $\mathbb{Q}$-linear space $(v_n)_{n\in\mathbb{N}} \subseteq \Gamma_2(\mathbf{E})$, such that $\{v_n(x) : n \in \mathbb{N}\}$ is dense in $\mathbf{E}(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. We argue by contradiction: suppose there exists $P' \in \Sigma$ with $\mathfrak{m}(P') > 0$, such that $\mathbf{E}(x)$ is non-Hilbert for every $x \in P'$. Hence, there must exist $n, m \in \mathbb{N}$ and $P \in \Sigma$, with $P \subseteq P'$ and $\mathfrak{m}(P) > 0$, such that for $\mathfrak{m}$-a.e. $x \in P$ it holds

$$\|v_n(x) + v_m(x)\|^2_{\mathbf{E}(x)} + \|v_n(x) - v_m(x)\|^2_{\mathbf{E}(x)} < 2\|v_n(x)\|^2_{\mathbf{E}(x)} + 2\|v_m(x)\|^2_{\mathbf{E}(x)}.$$

By integrating the above inequality over P, we conclude that

$$\begin{aligned}&\|\mathbb{1}_P \cdot v_n + \mathbb{1}_P \cdot v_m\|^2_{\Gamma_2(\mathbf{E})} + \|\mathbb{1}_P \cdot v_n - \mathbb{1}_P \cdot v_m\|^2_{\Gamma_2(\mathbf{E})}\\ &\quad < 2\|\mathbb{1}_P \cdot v_n\|^2_{\Gamma_2(\mathbf{E})} + 2\|\mathbb{1}_P \cdot v_m\|^2_{\Gamma_2(\mathbf{E})},\end{aligned}$$

thus leading to a contradiction with the assumption that $\Gamma_2(\mathbf{E})$ is a Hilbert module. □

Next, we aim at obtaining the analogue of Theorem 3.1, but with the term ‘Hilbert’ replaced by ‘uniformly convex’. Its proof, which is more involved than the one for the Hilbertian case, requires some auxiliary notions and results. More precisely, we have to work with the concept of ‘pointwise uniform convexity’, see Definition 3.2. This notion was proposed and studied by Guo and Zeng in [16, 17], where it is called ‘random uniform convexity’.

Definition 3.2 (*Pointwise uniform convexity*) Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space and let $\mathscr{M}$ be an $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module for some exponent $p \in (1, \infty)$. We denote by $|\cdot|$ the $L^p(\mathfrak{m})$-pointwise norm on $\mathscr{M}$. Then, the **pointwise modulus of convexity** $\delta^{\mathrm{pw}}_{\mathscr{M}} \colon (0, 2) \to L^\infty(\mathfrak{m})$ of $\mathscr{M}$ is defined as

$$\delta^{\mathrm{pw}}_{\mathscr{M}}(\varepsilon) := \bigwedge \left\{ \mathbb{1}_{H(\mathscr{M})} - \mathbb{1}_{H(\mathscr{M})}\left|\frac{v + w}{2}\right| \;\middle|\; \begin{array}{l} v, w \in \mathscr{M}^0,\ |v| = |w| = 1 \text{ and} \\ |v - w| > \varepsilon \ \mathfrak{m}\text{-a.e. on } H(\mathscr{M}) \end{array} \right\},$$

for every $\varepsilon \in (0,2)$, where $\mathscr{M}^0$ denotes the $L^0(\mathfrak{m})$-completion of $\mathscr{M}$. Moreover, we say that $\mathscr{M}$ is **pointwise uniformly convex** if and only if

$$\operatorname*{ess\,inf}_{H(\mathscr{M})} \delta^{\mathrm{pw}}_{\mathscr{M}}(\varepsilon) > 0, \quad \text{for every } \varepsilon \in (0,2).$$

When $\mathfrak{m}$ is a probability measure, the notion of pointwise uniform convexity in Definition 3.2 coincides with the one of *random uniform convexity* (see [16, Definition 4.1] and [24]). Notice also that, given a Banach $\mathbb{B}$-bundle $\mathbf{E}$ on X and two exponents $p, p' \in (1,\infty)$, one has that $\delta^{\mathrm{pw}}_{\Gamma_p(\mathbf{E})} = \delta^{\mathrm{pw}}_{\Gamma_{p'}(\mathbf{E})}$.

The following theorem, which states that the pointwise uniform convexity of a Banach module is equivalent to its uniform convexity as a Banach space, is a beautiful result obtained by Guo and Zeng in [16, 17]; they consider probability measures, but it is easy to see that it holds for σ-finite measures.

Theorem 3.3 *Let* $(\mathrm{X},\Sigma,\mathfrak{m})$ *be a σ-finite measure space and let $\mathscr{M}$ be an $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-module for some $p \in (1,\infty)$. Then, $\mathscr{M}$ is pointwise uniformly convex if and only if it is uniformly convex as a Banach space.*

Before stating the main result of this section, we need to check the following technical fact:

Remark 3.4 Let $(\mathrm{X},\Sigma,\mathfrak{m})$ be a σ-finite measure space, $\mathbb{B}$ a separable Banach space, $\mathbf{E}$ a Banach $\mathbb{B}$-bundle over X. Then, the function $\mathrm{X} \ni x \mapsto \delta_{\mathbf{E}(x)}(\varepsilon)$ is $\mathfrak{m}$-measurable for every $\varepsilon \in (0,2)$. Indeed, given any $\mathcal{D}$ as in Corollary 2.7, we know from Remark 2.2 that the function $\delta_{\mathbf{E}(\cdot)}(\varepsilon)$ can be written as

$$\inf_{v,w\in\mathcal{D}} \left(1 - \mathbb{1}_{\{\|v(\cdot)\|_{\mathbb{B}}=1\}}\mathbb{1}_{\{\|w(\cdot)\|_{\mathbb{B}}=1\}}\mathbb{1}_{\{\|v(\cdot)-w(\cdot)\|_{\mathbb{B}}>\varepsilon\}}\left\|\frac{v(\cdot)+w(\cdot)}{2}\right\|_{\mathbb{B}}\right). \tag{3.1}$$

In particular, the function $\delta_{\mathbf{E}(\cdot)}(\varepsilon)$ can be expressed as a countable infimum of $\mathfrak{m}$-measurable functions, thus it is $\mathfrak{m}$-measurable. This yields the claim.

Finally, we are in a position to prove the equivalence between the uniform convexity of a given separable Banach bundle and the uniform convexity of its space of sections.

Theorem 3.5 (Uniformly convex bundles/modules) *Let* $(\mathrm{X},\Sigma,\mathfrak{m})$ *be a σ-finite measure space. Let $\mathbb{B}$ be a separable Banach space and let $\mathbf{E}$ be a Banach $\mathbb{B}$-bundle over* X. *Then, it holds that*

$$\delta_{\mathbf{E}(x)}(\varepsilon) = \delta^{\mathrm{pw}}_{\Gamma_p(\mathbf{E})}(\varepsilon)(x), \quad \textit{for every } \varepsilon \in (0,2) \textit{ and } \mathfrak{m}\textit{-a.e. } x \in H(\mathbf{E}). \tag{3.2}$$

In particular, the following two conditions are equivalent:

(i) $\mathbf{E}$ *is a uniformly convex bundle and* $\operatorname{ess\,inf}_{H(\mathbf{E})}\delta_{\mathbf{E}(\cdot)}(\varepsilon) > 0$ *for all* $\varepsilon > 0$.
(ii) $\Gamma_p(\mathbf{E})$ *is uniformly convex for every (or, equivalently, for some) exponent* $p \in (1,\infty)$.

Proof Fix $\varepsilon \in (0,2)$. We aim to show that $\pi_{\mathfrak{m}}\big(\delta_{\mathbf{E}(\cdot)}(\varepsilon)\big) = \delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon)$ on $H(\mathbf{E})$. Take any $\mathcal{D}$ as in Corollary 2.7 and recall that $\delta_{\mathbf{E}(\cdot)}(\varepsilon)$ can be written as in (3.1). Fix auxiliary elements $\bar{v}, \bar{w} \in \bar{\Gamma}_\infty(\mathbf{E})$ satisfying $\|\bar{v}(x)\|_{\mathbb{B}} = \|\bar{w}(x)\|_{\mathbb{B}} = 1$ and $\|\bar{v}(x) - \bar{w}(x)\|_{\mathbb{B}} > \varepsilon$ for every $x \in H(\mathbf{E})$. Given any $v, w \in \mathcal{D}$, we denote

$$A_{vw} := \Big\{x \in H(\mathbf{E}) \;\Big|\; \|v(x)\|_{\mathbb{B}} = \|w(x)\|_{\mathbb{B}} = 1,\ \|v(x) - w(x)\|_{\mathbb{B}} > \varepsilon\Big\} \in \Sigma.$$

Letting $\tilde{v} := \mathbb{1}_{H(\mathbf{E})\setminus A_{vw}} \cdot \bar{v} + \mathbb{1}_{A_{vw}} \cdot v$ and $\tilde{w} := \mathbb{1}_{H(\mathbf{E})\setminus A_{vw}} \cdot \bar{w} + \mathbb{1}_{A_{vw}} \cdot w$, we have that $\delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon) \le \mathbb{1}_{H(\mathbf{E})} - \mathbb{1}_{H(\mathbf{E})}\big|\frac{\tilde{v}+\tilde{w}}{2}\big|$ $\mathfrak{m}$-a.e., whence it follows that

$$\mathbb{1}_{A_{vw}}\, \delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon) \le 1 - \mathbb{1}_{A_{vw}} \left\| \frac{v(\cdot) + w(\cdot)}{2} \right\|_{\mathbb{B}}, \quad \text{holds } \mathfrak{m}\text{-a.e. on } H(\mathbf{E}).$$

Since the right-hand side in the above inequality equals 1 on $H(\mathbf{E})\setminus A_{vw}$, and $\delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon) \le 1$, we deduce that $\delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon) \le 1 - \mathbb{1}_{A_{vw}}\big\|\frac{v(\cdot)+w(\cdot)}{2}\big\|_{\mathbb{B}}$ $\mathfrak{m}$-a.e. on $H(\mathbf{E})$. The arbitrariness of $v, w \in \mathcal{D}$ gives $\delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon) \le \pi_{\mathfrak{m}}\big(\delta_{\mathbf{E}(\cdot)}(\varepsilon)\big)$ on $H(\mathbf{E})$.

Let us now pass to the converse inequality. Fix any $\lambda > 0$ and two elements $u, w \in \Gamma_\infty(\mathbf{E})$ with $|u| = |z| = 1$ and $|u - z| > \varepsilon$ $\mathfrak{m}$-a.e. on $H(\mathbf{E})$. Therefore, we can find a partition $\{B_{vw}\}_{(v,w)\in\mathcal{D}\times\mathcal{D}} \subseteq \Sigma$ of $H(\mathbf{E})$, such that

$$\big|\pi_{\mathfrak{m}}(v) - u\big| + \big|\pi_{\mathfrak{m}}(w) - z\big| \le \big(|u - z| - \varepsilon\big) \wedge \lambda, \quad \text{holds } \mathfrak{m}\text{-a.e. in } B_{vw}$$

for every $(v,w) \in \mathcal{D}\times\mathcal{D}$. It follows that $B_{vw} \subseteq A_{vw}$ up to $\mathfrak{m}$-negligible sets. Moreover, for every $(v,w) \in \mathcal{D}\times\mathcal{D}$ and $\mathfrak{m}$-a.e. $x \in B_{vw}$ we can estimate

$$1 - \mathbb{1}_{A_{vw}}(x)\left\|\frac{v(x) + w(x)}{2}\right\|_{\mathbb{B}} \le \mathbb{1}_{H(\mathbf{E})}(x) - \mathbb{1}_{H(\mathbf{E})}(x)\left|\frac{u+z}{2}\right|(x) + \lambda,$$

whence it follows that $\mathbb{1}_{B_{vw}}\delta_{\mathbf{E}(\cdot)}(\varepsilon) \le \mathbb{1}_{H(\mathbf{E})} - \mathbb{1}_{H(\mathbf{E})}\big|\frac{u+z}{2}\big| + \lambda$ holds $\mathfrak{m}$-a.e. for every $(v,w) \in \mathcal{D}\times\mathcal{D}$. Since $\{B_{vw}\}_{(v,w)\in\mathcal{D}\times\mathcal{D}}$ is a partition of $H(\mathbf{E})$, we deduce that $\pi_{\mathfrak{m}}\big(\delta_{\mathbf{E}(\cdot)}(\varepsilon)\big) \le \mathbb{1}_{H(\mathbf{E})} - \mathbb{1}_{H(\mathbf{E})}\big|\frac{u+z}{2}\big| + \lambda$ holds $\mathfrak{m}$-a.e. on $H(\mathbf{E})$. Being λ, u, and z arbitrary, we conclude that $\pi_{\mathfrak{m}}\big(\delta_{\mathbf{E}(\cdot)}(\varepsilon)\big) \le \delta^{\rm pw}_{\Gamma_p(\mathbf{E})}(\varepsilon)$ $\mathfrak{m}$-a.e. on $H(\mathbf{E})$. This proves (3.2), whence the equivalence (i) $\Longleftrightarrow$ (ii) follows. □

3.2 Characterisation of the dual of a section space

Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space. Let $\mathbb{B}$ be a Banach space and $\mathbf{E}$ a Banach $\mathbb{B}$-bundle over X. Then, we define $\bar{\Gamma}_0(\mathbf{E}'_{w^*})$ as the space of all those maps $\bar{\omega}\colon \mathrm{X} \to \bigsqcup_{x\in\mathrm{X}} \mathbf{E}(x)'$, such that $\bar{\omega}(x) \in \mathbf{E}(x)'$ for every $x \in \mathrm{X}$ and

$$\mathrm{X} \ni x \longmapsto \langle\bar{\omega}(x), \bar{v}(x)\rangle \in \mathbb{R} \quad \text{is } \mathfrak{m}\text{-measurable}, \quad \text{for every } \bar{v} \in \bar{\Gamma}_0(\mathbf{E}).$$

We introduce an equivalence relation on $\bar{\Gamma}_0(\mathbf{E}'_{w^*})$: given any $\bar{\omega}, \bar{\eta} \in \bar{\Gamma}_0(\mathbf{E}'_{w^*})$, we declare that $\bar{\omega} \sim \bar{\eta}$ if for any $\bar{v} \in \bar{\Gamma}_0(\mathbf{E})$ it holds $\langle\bar{\omega}(x) - \bar{\eta}(x), \bar{v}(x)\rangle = 0$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.

Remark 3.6 When the ambient Banach space $\mathbb{B}$ is separable, it holds that

$$\bar{\omega} \sim \bar{\eta} \quad \Longleftrightarrow \quad \bar{\omega}(x) = \bar{\eta}(x), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

On arbitrary Banach spaces, this needs not necessarily be the case.

We denote the associated quotient space by

$$\Gamma_0(\mathbf{E}'_{w^*}) := \bar{\Gamma}_0(\mathbf{E}'_{w^*})/\sim,$$

while $\pi_{\mathfrak{m}} \colon \bar{\Gamma}_0(\mathbf{E}'_{w^*}) \to \Gamma_0(\mathbf{E}'_{w^*})$ stands for the projection map. Then, the space $\Gamma_0(\mathbf{E}'_{w^*})$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module if endowed with the following $L^0(\mathfrak{m})$-pointwise norm operator:

$$|\omega| := \bigvee \left\{ \langle \bar{\omega}(\cdot), \bar{v}(\cdot) \rangle \;\middle|\; \bar{v} \in \bar{\Gamma}_0(\mathbf{E}), \ |\bar{v}| \leq 1 \right\}, \quad \forall \omega = \pi_{\mathfrak{m}}(\bar{\omega}) \in \Gamma_0(\mathbf{E}'_{w^*}).$$

Remark 3.7 When $\mathbb{B}$ is separable, $|\omega|(x) = \|\bar{\omega}(x)\|_{\mathbf{E}(x)'}$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.

In particular, for any exponent $q \in (1, \infty)$ we can consider the space

$$\Gamma_q(\mathbf{E}'_{w^*}) := \left\{ \omega \in \Gamma_0(\mathbf{E}'_{w^*}) \;\middle|\; |\omega| \in L^q(\mathfrak{m}) \right\},$$

which inherits an $L^q(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module structure.

Our interest towards the space $\Gamma_q(\mathbf{E}'_{w^*})$ is motivated by the following result, which states that the module dual of the section space $\Gamma_p(\mathbf{E})$ can be identified with $\Gamma_q(\mathbf{E}'_{w^*})$ itself. When $\mathbf{E}(x) := \mathbb{B}$ for every $x \in \mathrm{X}$ (i.e., $\Gamma_p(\mathbf{E})$ is the Lebesgue–Bochner space $L^p(\mathfrak{m}; \mathbb{B})$), this result was proved in [13].

Theorem 3.8 (Dual of a section space) *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a* σ*-finite measure space. Fix any exponent* $p \in (1, \infty)$. *Let* $\mathbb{B}$ *be a Banach space and* $\mathbf{E}$ *a Banach* $\mathbb{B}$*-bundle over* X. *Then*

$$\Gamma_q(\mathbf{E}'_{w^*}) \cong \Gamma_p(\mathbf{E})^*.$$

An isomorphism $\mathrm{I} \colon \Gamma_q(\mathbf{E}'_{w^*}) \to \Gamma_p(\mathbf{E})^*$ *of* $L^q(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-modules is*

$$\langle \mathrm{I}(\omega), v \rangle := \pi_{\mathfrak{m}}\big(\langle \bar{\omega}(\cdot), \bar{v}(\cdot) \rangle\big), \quad \forall \omega = \pi_{\mathfrak{m}}(\bar{\omega}) \in \Gamma_q(\mathbf{E}'_{w^*}), \ v = \pi_{\mathfrak{m}}(\bar{v}) \in \Gamma_p(\mathbf{E}). \tag{3.3}$$

In particular, the space $\Gamma_q(\mathbf{E}'_{w^*})$ *is an* $L^q(\mathfrak{m})$*-Banach* $L^\infty(\mathfrak{m})$*-module.*

Proof The validity of the $\mathfrak{m}$-a.e. inequality $\left|\pi_{\mathfrak{m}}\big(\langle \bar{\omega}(\cdot), \bar{v}(\cdot) \rangle\big)\right| \leq |\omega||v|$ implies that I is a well-defined homomorphism of $L^q(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules satisfying $|\mathrm{I}(\omega)| \leq |\omega|$ $\mathfrak{m}$-a.e. for every $\omega \in \Gamma_q(\mathbf{E}'_{w^*})$. To conclude, it only remains to prove that the map I is surjective and satisfies $|\mathrm{I}(\omega)| \geq |\omega|$ $\mathfrak{m}$-a.e. for every $\omega \in \Gamma_q(\mathbf{E}'_{w^*})$. To this aim, let $T \in \Gamma_p(\mathbf{E})^*$ be fixed. Since $\Gamma_p(\mathbf{E})$ is an $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-submodule of $L^p(\mathfrak{m};\mathbb{B})$, the Hahn–Banach extension theorem for normed modules gives an element $\bar{T} \in L^p(\mathfrak{m};\mathbb{B})^*$, such that $\bar{T}|_{\Gamma_p(\mathbf{E})} = T$ and $|\bar{T}| = |T|$. Now, recall that, thanks to [13], the statement holds for the constant bundle $\mathbf{E}_\mathbb{B}$ given by $\mathbf{E}_\mathbb{B}(x) := \mathbb{B}$ for all $x \in \mathrm{X}$. Hence, denoting $L^q_{w^*}(\mathfrak{m};\mathbb{B}') := \Gamma_q((\mathbf{E}_\mathbb{B})'_{w^*})$ for consistency of notation, we have the isomorphism $\bar{\mathrm{I}}\colon L^q_{w^*}(\mathfrak{m};\mathbb{B}') \to L^p(\mathfrak{m};\mathbb{B})^*$. Define $\eta := \bar{\mathrm{I}}^{-1}(\bar{T}) \in L^q_{w^*}(\mathfrak{m};\mathbb{B}')$ and take a representative $\bar{\eta} \in \mathcal{L}^0_{w^*}(\mathfrak{m};\mathbb{B}') := \bar{\Gamma}_0((\mathbf{E}_\mathbb{B})'_{w^*})$ of η. Letting $\bar{\omega}$ be given by $\bar{\omega}(x) := \bar{\eta}(x)|_{\mathbf{E}(x)} \in \mathbf{E}(x)'$ for every $x \in \mathrm{X}$, we have that $\bar{\omega} \in \bar{\Gamma}_0(\mathbf{E}'_{w^*})$ (since $\bar{\Gamma}_0(\mathbf{E}) \subseteq \mathcal{L}^0(\mathfrak{m};\mathbb{B})$). Finally, we define $\omega := \pi_\mathfrak{m}(\bar{\omega}) \in \Gamma_0(\mathbf{E}'_{w^*})$. Given any element $\bar{v} \in \bar{\Gamma}_0(\mathbf{E})$, we have that $\langle \bar{\omega}(\cdot), \bar{v}(\cdot)\rangle = \langle \bar{\eta}(\cdot), \bar{v}(\cdot)\rangle$, whence it follows that $\pi_\mathfrak{m}\big(\langle \bar{\omega}(\cdot), \bar{v}(\cdot)\rangle\big) \le |\eta||\pi_\mathfrak{m}(v)|$. We thus conclude that $\omega \in \Gamma_q(\mathbf{E}'_{w^*})$, $\mathrm{I}(\omega) = T$, and $|\omega| \le |\eta| = |\bar{T}| = |T| = |\mathrm{I}(\omega)|$. The proof is complete. □

We point out that the previous result is obtained as a consequence of the corresponding result for Lebesgue–Bochner spaces, studied in [13]. This proof was kindly suggested to us by the referee. In a previous version of the paper, Theorem 3.8 was proved in a more involved way, making use of a generalised form of the Lebesgue differentiation theorem [21]. We also point out that results in the same spirit were obtained in [8, Theorem 3.9] and [8, Theorem 4.17], but in that case it is not sufficient to apply the result for Lebesgue–Bochner spaces (cf. with the discussion in [21, Remark 4.6]).

3.3 Reflexive bundles/modules

In this section, we will prove that the section space of a separable Banach bundle is reflexive if and only if (almost all) its fibers are reflexive. Before stating the main theorem, we need to discuss a few auxiliary results.

Remark 3.9 Let us recall a standard fact in Banach space theory. Let $\mathbb{B}$ be a Banach space whose dual $\mathbb{B}'$ is separable. Let $(v_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}$ and $(\omega_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}'$ be given sequences satisfying $\|v_n\|_\mathbb{B} = \|\omega_n\|_{\mathbb{B}'} = \langle \omega_n, v_n\rangle = 1$ for every $n \in \mathbb{N}$. Suppose $(\omega_n)_{n\in\mathbb{N}}$ is dense in the unit sphere $\mathbb{S}_{\mathbb{B}'}$. Then, the $\mathbb{Q}$-linear subspace of $\mathbb{B}$ generated by $(v_n)_{n\in\mathbb{N}}$ is dense in $\mathbb{B}$.

Proposition 3.10 *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a* σ*-finite measure space and* $p \in (1, \infty)$ *a given exponent. Let* $\mathbb{B}$ *be a separable Banach space and let* $\mathbf{E}$ *be a reflexive Banach* $\mathbb{B}$*-bundle over* X*. Let* $\theta\colon \Gamma_p(\mathbf{E}) \to \Gamma_q(\mathbf{E}'_{w^*})^*$ *be given by*

$$\langle \theta(v), \omega\rangle := \pi_\mathfrak{m}\big(\langle \bar{\omega}(\cdot), \bar{v}(\cdot)\rangle\big), \quad \forall v = \pi_\mathfrak{m}(\bar{v}) \in \Gamma_p(\mathbf{E}),\ \omega = \pi_\mathfrak{m}(\bar{\omega}) \in \Gamma_q(\mathbf{E}'_{w^*}). \tag{3.4}$$

Then the operator θ *is an isomorphism of* $L^p(\mathfrak{m})$*-Banach* $L^\infty(\mathfrak{m})$*-modules.*

Proof
Step 1. The $\mathfrak{m}$-a.e. inequality $\big|\pi_\mathfrak{m}(\langle \bar{\omega}(\cdot), \bar{v}(\cdot)\rangle)\big| \le |v||\omega|$ ensures that θ is a well-defined homomorphism of $L^p(\mathfrak{m})$-Banach $L^\infty(\mathfrak{m})$-modules satisfying $|\theta(v)| \le |v|$

for every $v \in \Gamma_p(\mathbf{E})$.

Step 2. It remains to prove that θ is surjective and satisfies $|\theta(v)| \geq |v|$ for every $v \in \Gamma_p(\mathbf{E})$. To this aim, fix any $L \in \Gamma_q(\mathbf{E}'_{w^*})^*$. Pick a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \Gamma_p(\mathbf{E})$ with $|v_n|(x) \in \{0, 1\}$ for every $n \in \mathbb{N}$ and $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, and such that

$$\{v_n(x) : n \in \mathbb{N}\} \backslash \{0_{\mathbf{E}(x)}\} \text{ is dense in } \mathbb{S}_{\mathbf{E}(x)}, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

Given any $n \in \mathbb{N}$, thanks to [7, Corollary 1.2.16] we can find an element $\tilde{\omega}_n \in \Gamma_p(\mathbf{E})^*$, such that $|\tilde{\omega}_n| = |v_n| = \langle \tilde{\omega}_n, v_n \rangle$ holds $\mathfrak{m}$-a.e. on X. Now define $\omega_n := \mathrm{I}^{-1}(\tilde{\omega}_n) \in \Gamma_q(\mathbf{E}'_{w^*})$, where $\mathrm{I} : \Gamma_q(\mathbf{E}'_{w^*}) \to \Gamma_p(\mathbf{E})^*$ stands for the isomorphism provided by Theorem 3.8. Let us denote by $\mathcal{V}$ the $\mathbb{Q}$-linear subspace of $\Gamma_q(\mathbf{E}'_{w^*})$ generated by $(\omega_n)_{n\in\mathbb{N}}$. Notice that $\mathcal{V}$ is a countable family by construction. Given $n \in \mathbb{N}$ and $\omega \in \mathcal{V}$, fix representatives $\bar{v}_n$, $\bar{\omega}$, $\overline{L(\omega)}$, and $\overline{|L|}$ of v_n, ω, $L(\omega)$, and $|L|$, respectively. By Remark 3.7, there exists $N \in \Sigma$ with $\mathfrak{m}(N) = 0$, such that for any $x \in \mathrm{X} \backslash N$ it holds that

$$\begin{aligned}
\mathbf{E}(x), &\quad \text{is reflexive,} && \text{(3.5a)}\\
\{\bar{v}_m(x) : m \in \mathbb{N}\} \backslash \{0_{\mathbf{E}(x)}\}, &\quad \text{is dense in } \mathbb{S}_{\mathbf{E}(x)}, && \text{(3.5b)}\\
\|\bar{\omega}_n(x)\|_{\mathbf{E}(x)'} &= \|\bar{v}_n(x)\|_{\mathbf{E}(x)} = \langle \bar{\omega}_n(x), \bar{v}_n(x) \rangle, && \text{(3.5c)}\\
\overline{(\omega + \eta)}(x) &= \bar{\omega}(x) + \bar{\eta}(x), && \text{(3.5d)}\\
\overline{(\lambda\, \omega)}(x) &= \lambda\, \bar{\omega}(x), && \text{(3.5e)}\\
\overline{L(\omega + \eta)}(x) &= \overline{L(\omega)}(x) + \overline{L(\eta)}(x), && \text{(3.5f)}\\
\overline{L(\lambda\, \omega)}(x) &= \lambda\, \overline{L(\omega)}(x), && \text{(3.5g)}\\
\overline{L(\omega)}(x) &\leq \overline{|L|}(x) \|\bar{\omega}(x)\|_{\mathbf{E}(x)'}, && \text{(3.5h)}
\end{aligned}$$

for every $n \in \mathbb{N}$, $\omega, \eta \in \mathcal{V}$, and $\lambda \in \mathbb{Q}$. Given any $x \in \mathrm{X} \backslash N$, let us consider the countable, $\mathbb{Q}$-linear subspace $\mathcal{V}_x := \{\bar{\omega}(x) : \omega \in \mathcal{V}\}$ of $\mathbf{E}(x)'$. The fact that $\mathcal{V}_x$ is a $\mathbb{Q}$-linear space is granted by (3.5d) and (3.5e). By taking (3.5a), (3.5b), (3.5c), and Remark 3.9 into account, we deduce that $\mathcal{V}_x$ is dense in $\mathbf{E}(x)'$. Now we define the function $\varphi_x : \mathcal{V}_x \to \mathbb{R}$ as

$$\varphi_x\big(\bar{\omega}(x)\big) := \overline{L(\omega)}(x), \quad \text{for every } \omega \in \mathcal{V}.$$

The well-posedness of φ_x stems from the observation that for any $\omega, \eta \in \mathcal{V}$ it holds that

$$\begin{aligned}
\left|\overline{L(\omega)}(x) - \overline{L(\eta)}(x)\right| &\overset{(3.5f)}{=} \left|\overline{L(\omega - \eta)}(x)\right| \overset{(3.5h)}{\leq} \overline{|L|}(x) \left\|\overline{(\omega - \eta)}(x)\right\|_{\mathbf{E}(x)'}\\
&\overset{(3.5g)}{=} \overline{|L|}(x) \left\|\bar{\omega}(x) - \bar{\eta}(x)\right\|_{\mathbf{E}(x)'}.
\end{aligned}$$

The $\mathbb{Q}$-linearity of φ_x is a consequence of (3.5f) and (3.5g). Moreover, (3.5h) grants the validity of the inequality $\left|\varphi_x\big(\bar{\omega}(x)\big)\right| \leq \overline{|L|}(x) \|\bar{\omega}(x)\|_{\mathbf{E}(x)'}$ for every $\omega \in \mathcal{V}$, whence the continuity of the function φ_x follows. Therefore, there exists a unique element

$\bar{v}(x) \in \mathbf{E}(x) \cong \mathbf{E}(x)''$, such that $\langle \bar{\omega}(x), \bar{v}(x) \rangle = \overline{L(\omega)}(x)$ holds for every $\omega \in \mathcal{V}$ and $\|\bar{v}(x)\|_{\mathbf{E}(x)} \leq \overline{|L|}(x)$. Finally, for any point $x \in N$ we define $\bar{v}(x) := 0_{\mathbf{E}(x)}$.

Step 3. Next we claim that the resulting map $\bar{v}$ belongs to $\bar{\Gamma}_0(\mathbf{E})$. By virtue of the separability of $\mathbb{B}$, it is sufficient to prove that $\bar{v}\colon \mathrm{X} \to \mathbb{B}$ is weakly $\mathfrak{m}$-measurable. To this aim, fix any $\eta_0 \in \mathbb{B}'$. Define $\bar{\eta}(x) := \eta_0|_{\mathbf{E}(x)} \in \mathbf{E}(x)'$ for every $x \in \mathrm{X}$. For any $\omega \in \mathcal{V}$, one has

$$\big\|\bar{\eta}(x) - \bar{\omega}(x)\big\|_{\mathbf{E}(x)'} = \sup_{n\in\mathbb{N}} \big\langle \eta_0 - \bar{\omega}(x), \bar{v}_n(x) \big\rangle, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \tag{3.6}$$

Since the function $\mathrm{X} \ni x \mapsto \big\|\bar{\eta}(x) - \bar{\omega}(x)\big\|_{\mathbf{E}(x)'}$ is measurable for every $\omega \in \mathcal{V}$ thanks to (3.6) and the space $\mathcal{V}_x$ is dense in $\mathbf{E}(x)'$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, we deduce that for any $k \in \mathbb{N}$ we can find a partition $\{A^k_\omega\}_{\omega\in\mathcal{V}} \subseteq \Sigma$ of X (up to $\mathfrak{m}$-null sets), such that $\big\|\bar{\eta}(x) - \bar{\eta}_k(x)\big\|_{\mathbf{E}(x)'} \leq 1/k$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, where we set $\bar{\eta}_k := \sum_{\omega\in\mathcal{V}} \mathbb{1}_{A^k_\omega} \bar{\omega}$. Therefore, for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ we can express

$$\begin{aligned}\langle \eta_0, \bar{v}(x) \rangle = \langle \bar{\eta}(x), \bar{v}(x) \rangle &= \lim_{k\to\infty} \langle \bar{\eta}_k(x), \bar{v}(x) \rangle = \lim_{k\to\infty} \sum_{\omega\in\mathcal{V}} \mathbb{1}_{A^k_\omega}(x)\, \langle \bar{\omega}(x), \bar{v}(x) \rangle \\ &= \lim_{k\to\infty} \sum_{\omega\in\mathcal{V}} \mathbb{1}_{A^k_\omega}(x)\, \overline{L(\omega)}(x),\end{aligned}$$

thus accordingly $\langle \eta_0, \bar{v}(\cdot) \rangle$ is measurable. By arbitrariness of $\eta_0 \in \mathbb{B}'$, we conclude that $\bar{v}$ is weakly $\mathfrak{m}$-measurable (thus, strongly $\mathfrak{m}$-measurable). Let us then define $v := \pi_{\mathfrak{m}}(\bar{v}) \in \Gamma_0(\mathbf{E})$.

Step 4. To conclude, it only remains to show that $\theta(v) = L$ and $|v| \leq |L|$ in the $\mathfrak{m}$-a.e. sense. Fix any $\eta \in \Gamma_q(\mathbf{E}'_{w^*})$, with representative $\bar{\eta} \in \bar{\Gamma}_0(\mathbf{E}'_{w^*})$. By arguing as we did in **Step 3**, we can construct a sequence $(\bar{\eta}_k)_{k\in\mathbb{N}} \subseteq \bar{\Gamma}_0(\mathbf{E}'_{w^*})$ of the form $\bar{\eta}_k = \sum_{\omega\in\mathcal{V}} \mathbb{1}_{A^k_\omega} \bar{\omega}$, such that

$$\lim_k \big\|\bar{\eta}_k(x) - \bar{\eta}(x)\big\|_{\mathbf{E}(x)'} = 0, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

Therefore, for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ it holds that

$$\begin{aligned}\langle \theta(v), \eta \rangle(x) = \langle \bar{\eta}(x), \bar{v}(x) \rangle &= \lim_{k\to\infty} \langle \bar{\eta}_k(x), \bar{v}(x) \rangle \\ &= \lim_{k\to\infty} \sum_{\omega\in\mathcal{V}} \mathbb{1}_{A^k_\omega}(x)\, \langle \bar{\omega}(x), \bar{v}(x) \rangle = \lim_{k\to\infty} \sum_{\omega\in\mathcal{V}} \mathbb{1}_{A^k_\omega}(x)\, \overline{L(\omega)}(x) \\ &= \lim_{k\to\infty} L\big(\pi_{\mathfrak{m}}(\bar{\eta}_k)\big)(x) = L(\eta)(x).\end{aligned}$$

By arbitrariness of $\eta \in \Gamma_q(\mathbf{E}'_{w^*})$, it follows that $\theta(v) = L$. Finally, since $\|\bar{v}(x)\|_{\mathbf{E}(x)} \leq \overline{|L|}(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, we obtain the $\mathfrak{m}$-a.e. inequality $|v| \leq |L|$. Hence, the statement is achieved. □

Proposition 3.10 will play a key role in proving one implication of the main result of this section, namely, Theorem 3.12. To prove the converse implication, we need the alternative—more ‘quantitative’—characterisation of reflexivity that we report in Lemma 3.11. Before passing to its statement, it is convenient to introduce some additional notation.

We denote by $\bigoplus_{\mathbb{N}} \mathbb{Q}$ the set of all those sequences $\boldsymbol{q} = (q_i)_{i\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ that satisfy $q_i = 0$ for all but finitely many $i \in \mathbb{N}$. We define

$$\Delta := \left\{ \boldsymbol{q} \in \bigoplus_{\mathbb{N}} \mathbb{Q} \cap [0,1]^{\mathbb{N}} \;\middle|\; \sum_{i\in\mathbb{N}} q_i = 1 \right\}.$$

Given any $\boldsymbol{q}, \boldsymbol{r} \in \Delta$, we declare that $\boldsymbol{q} \prec \boldsymbol{r}$ if

$$\max\{i \in \mathbb{N} \,|\, q_i \neq 0\} < \min\{j \in \mathbb{N} \,|\, r_j \neq 0\}.$$

Finally, we define $\mathscr{F}$ as

$$\mathscr{F} := \big\{ (\boldsymbol{q}, \boldsymbol{r}) \in \Delta \times \Delta \;\big|\; \boldsymbol{q} \prec \boldsymbol{r} \big\}.$$

Lemma 3.11 *Let $\mathbb{B}$ be a Banach space. Then, the following two conditions are equivalent:*

(i) *$\mathbb{B}$ is not reflexive.*
(ii) *Given any $\lambda \in (0,1)$, there exists a sequence $(v_i)_{i\in\mathbb{N}} \subseteq B_{\mathbb{B}}$, such that*

$$\left\| \sum_{i\in\mathbb{N}} q_i v_i - \sum_{i\in\mathbb{N}} r_i v_i \right\|_{\mathbb{B}} \geq \lambda, \quad \textit{for every } (\boldsymbol{q}, \boldsymbol{r}) \in \mathscr{F}. \tag{3.7}$$

Proof The Eberlein–Šmulian Theorem (see, e.g., [2, Theorems 3.18 and 3.19]) says that $\mathbb{B}$ is reflexive if and only if every sequence in $B_{\mathbb{B}}$ admits a weakly converging subsequence. Then:

(i) $\Longrightarrow$ (ii). It readily follows, e.g., from [5, Theorem 3.132].

(ii) $\Longrightarrow$ (i). Let $(v_i)_{i\in\mathbb{N}} \subseteq B_{\mathbb{B}}$ satisfy (3.7). We argue by contradiction: suppose $\mathbb{B}$ is reflexive. Then, Mazur’s Lemma (see, e.g., [2, Corollary 3.8]) yields an element $v \in B_{\mathbb{B}}$ and a sequence $(\boldsymbol{q}^j)_{j\in\mathbb{N}} \subseteq \Delta$ such that $\boldsymbol{q}^j \prec \boldsymbol{q}^{j+1}$ for every $j \in \mathbb{N}$ and $\sum_{i\in\mathbb{N}} q_i^j v_i \to v$ strongly in $\mathbb{B}$ as $j \to \infty$. In particular, $\big\| \sum_{i\in\mathbb{N}} q_i^j v_i - \sum_{i\in\mathbb{N}} q_i^{j+1} v_i \big\|_{\mathbb{B}} < \lambda$ for $j \in \mathbb{N}$ big enough, contradicting ii). □

Combining Proposition 3.10 with Lemma 3.11, we obtain the main result of this section:

Theorem 3.12 (Reflexive bundles/modules) *Let $(\mathrm{X}, \Sigma, \mathfrak{m})$ be a σ-finite measure space, $\mathbb{B}$ a separable Banach space, and $\mathbf{E}$ a Banach $\mathbb{B}$-bundle over X. Then, $\mathbf{E}$ is a reflexive bundle if and only if $\Gamma_p(\mathbf{E})$ is a reflexive Banach space for every (or, equivalently, for some) $p \in (1, \infty)$.*

Proof

Necessity. Suppose $\mathbf{E}$ is a reflexive bundle and fix any exponent $p \in (1, \infty)$. We call $\mathrm{I}\colon \Gamma_q(\mathbf{E}'_{w^*}) \to \Gamma_p(\mathbf{E})^*$ the isomorphism provided by Theorem 3.8. We denote by $\mathrm{J}\colon \Gamma_p(\mathbf{E})^* \to \Gamma_q(\mathbf{E}'_{w^*})$ its inverse and consider the adjoint $\mathrm{J}^{\mathrm{ad}}\colon \Gamma_q(\mathbf{E}'_{w^*})^* \to \Gamma_p(\mathbf{E})^{**}$ of the isomorphism J. Let $\theta\colon \Gamma_p(\mathbf{E}) \to \Gamma_q(\mathbf{E}'_{w^*})^*$ be the isomorphism given by Proposition 3.10. By unwrapping the various definitions, it can be readily checked that

$$\begin{tikzcd} \Gamma_p(\mathbf{E}) \arrow[r, "\theta"] \arrow[dr, "J_{\Gamma_p(\mathbf{E})}"'] & \Gamma_q(\mathbf{E}'_{w^*})^* \arrow[d, "\mathrm{J}^{\mathrm{ad}}"] \\ & \Gamma_p(\mathbf{E})^{**} \end{tikzcd}$$

is a commutative diagram. Indeed, let us fix any $v = \pi_{\mathfrak{m}}(\bar{v}) \in \Gamma_p(\mathbf{E})$ and $T \in \Gamma_p(\mathbf{E})^*$. Also, define $\omega := \mathrm{J}(T) \in \Gamma_q(\mathbf{E}'_{w^*})$ and pick a representative $\bar{\omega} \in \bar{\Gamma}_0(\mathbf{E}'_{w^*})$ of ω. Then, we have that

$$\begin{aligned} \langle(\mathrm{J}^{\mathrm{ad}} \circ \theta)(v), T\rangle &\overset{(2.4)}{=} \langle\theta(v), \mathrm{J}(T)\rangle = \langle\theta(v), \omega\rangle \overset{(3.4)}{=} \pi_{\mathfrak{m}}\big(\langle\bar{\omega}(\cdot), \bar{v}(\cdot)\rangle\big) \\ &\overset{(3.3)}{=} \langle\mathrm{I}(\omega), v\rangle = \langle T, v\rangle \overset{(2.3)}{=} \langle J_{\Gamma_p(\mathbf{E})}(v), T\rangle, \end{aligned}$$

yielding $\mathrm{J}^{\mathrm{ad}} \circ \theta = J_{\Gamma_p(\mathbf{E})}$. Therefore, $J_{\Gamma_p(\mathbf{E})}$ is an isomorphism and thus $\Gamma_p(\mathbf{E})$ is reflexive.

Sufficiency. Suppose that $\Gamma_p(\mathbf{E})$ is reflexive for some $p \in (1, \infty)$. By Proposition 2.6, there is a countable family $Z \subseteq \bar{\Gamma}_\infty(\mathbf{E})$, such that $\|v(x)\|_{\mathbb{B}} \leq 1$ for every $(v, x) \in Z \times \mathrm{X}$ and

$$\big\{v(x) \bigm| v \in Z\big\} \text{ is dense in } B_{\mathbf{E}(x)}, \qquad \text{for every } x \in \mathrm{X}.$$

We equip Z with the discrete topology and $Z^{\mathbb{N}}$ with the product topology. Then, $Z^{\mathbb{N}}$ is a Polish space (i.e., a metrisable space whose topology is induced by a complete, separable distance), homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ (see [1, Section 3.14]). We define $\boldsymbol{\varphi}\colon \mathrm{X} \twoheadrightarrow Z^{\mathbb{N}}$ as $\boldsymbol{\varphi}(x) := \big\{\boldsymbol{v} \in Z^{\mathbb{N}} \bigm| (\boldsymbol{v}, x) \in H\big\}$ for every $x \in \mathrm{X}$, where we set

$$H := \bigcap_{(\boldsymbol{q},\boldsymbol{r})\in\mathscr{F}} \left\{(\boldsymbol{v}, x) \in Z^{\mathbb{N}} \times \mathrm{X} \;\middle|\; \left\|\sum_{i\in\mathbb{N}} q_i v_i(x) - \sum_{i\in\mathbb{N}} r_i v_i(x)\right\|_{\mathbb{B}} \geq \frac{1}{2}\right\}.$$

Recalling that a base for the topology of $Z^{\mathbb{N}}$ is given by those sets of the form

$$\{v_1\} \times \cdots \times \{v_n\} \times Z \times Z \times \cdots, \qquad \text{with } n \in \mathbb{N} \text{ and } v_1, \ldots, v_n \in Z,$$

one can readily check that $\boldsymbol{\varphi}$ is a weakly measurable map from X to $Z^{\mathbb{N}}$ having closed values.

We now argue by contradiction: suppose that there exists $P \in \Sigma$ with $0 < \mathfrak{m}(P) < +\infty$, such that $\mathbf{E}(x)$ is not reflexive for every $x \in P$. Applying Lemma 3.11 to each $\mathbf{E}(x)$ with $x \in P$, we deduce that $\boldsymbol{\varphi}(x) \neq \varnothing$ for every $x \in P$. Thanks to the

Kuratowski–Ryll-Nardzewski Selection Theorem (see, e.g., [1, Theorem 18.13]), we can find a measurable mapping $V\colon P \to Z^{\mathbb{N}}$, such that $V(x) \in \boldsymbol{\varphi}(x)$ for every $x \in P$. For any $i \in \mathbb{N}$, we denote by $\pi_i\colon Z^{\mathbb{N}} \to Z$ the projection onto the i-th component, which is continuous by definition of the product topology. Then, $\pi_i \circ V\colon P \to Z$ is measurable, so that $P^i_v := (\pi_i \circ V)^{-1}(\{v\}) \in \Sigma$ for every $v \in Z$ and $(P^i_v)_{v \in Z}$ is a partition of P. Given any $i \in \mathbb{N}$, we define $\bar{v}_i\colon \mathrm{X} \to \mathbb{B}$ as

$$\bar{v}_i(x) := \frac{(\pi_i \circ V)(x)(x)}{\mathfrak{m}(P)^{1/p}} \in \mathbf{E}(x), \quad \text{for every } x \in P,$$

and $\bar{v}_i(x) := 0_{\mathbb{B}}$ for all $x \in \mathrm{X} \setminus P$. Since $\bar{v}_i(x) = \sum_{v \in Z} \mathfrak{m}(P)^{-1/p} \mathbb{1}_{P^i_v}(x) v(x)$ for all $x \in P$, we have that $\bar{v}_i \in \bar{\Gamma}_\infty(\mathbf{E}) \cap \bar{\Gamma}_p(\mathbf{E})$ and $\|\pi_{\mathfrak{m}}(\bar{v}_i)\|_{\Gamma_p(\mathbf{E})} \leq 1$. Observe also that it holds

$$\Big\| \sum_{i \in \mathbb{N}} q_i \bar{v}_i(x) - \sum_{i \in \mathbb{N}} r_i \bar{v}_i(x) \Big\|_{\mathbb{B}} \geq \frac{1}{2\,\mathfrak{m}(P)^{1/p}}, \quad \forall (\boldsymbol{q}, \boldsymbol{r}) \in \mathscr{F},\ x \in P. \tag{3.8}$$

Hence, denoting by $v_i \in \Gamma_p(\mathbf{E})$ the equivalence class of $\bar{v}_i$, for any $(\boldsymbol{q}, \boldsymbol{r}) \in \mathscr{F}$ we can estimate

$$\begin{aligned}\Big\| \sum_{i \in \mathbb{N}} q_i v_i - \sum_{i \in \mathbb{N}} r_i v_i \Big\|_{\Gamma_p(\mathbf{E})} &= \bigg(\int_P \Big\| \sum_{i \in \mathbb{N}} q_i \bar{v}_i(x) - \sum_{i \in \mathbb{N}} r_i \bar{v}_i(x) \Big\|_{\mathbb{B}}^p \mathrm{d}\mathfrak{m}(x) \bigg)^{1/p} \\ &\overset{(3.8)}{\geq} \frac{1}{2}.\end{aligned}$$

Using Lemma 3.11 again, we deduce that $\Gamma_p(\mathbf{E})$ is not reflexive, leading to a contradiction. □

Remark 3.13 As we already mentioned in the Introduction, the proof of the sufficiency part of Theorem 3.12 follows along the lines sketched in the proof of [18, Theorem 6.19]. On the other hand, the proof of the necessity part is different from the one of [18, Theorem 6.19], and in particular it avoids the use of Rosenthal's ℓ^1-Theorem.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A. A criterion to detect Banach modules

Aim of this appendix is to address the following problem: *given a module $\mathscr{M}$ over $L^\infty(\mathfrak{m})$, can we characterise those complete norms on $\mathscr{M}$ that come from an $L^p(\mathfrak{m})$-pointwise norm?* We will provide a positive answer to this question in Theorem A.2 below.

First, we recall a well-known, elementary result concerning Radon–Nikodým derivatives. We report its proof for the reader's usefulness.

Lemma A.1 *Let (X, Σ) be a measurable space. Let $\mathfrak{m}, \mu_1, \mu_2, \mu_3$ be σ-finite measures on Σ, such that $\mu_1, \mu_2, \mu_3 \ll \mathfrak{m}$. Let $\alpha \in (0, +\infty)$ be given. Suppose*

$$\mu_1(E)^\alpha \le \mu_2(E)^\alpha + \mu_3(E)^\alpha, \quad \textit{for every } E \in \Sigma. \tag{A.1}$$

Then it holds that

$$\left(\frac{d\mu_1}{d\mathfrak{m}}\right)^\alpha \le \left(\frac{d\mu_2}{d\mathfrak{m}}\right)^\alpha + \left(\frac{d\mu_3}{d\mathfrak{m}}\right)^\alpha, \quad \textit{in the } \mathfrak{m}\textit{-a.e. sense.} \tag{A.2}$$

Proof Let us denote $f_j := \frac{\mathrm{d}\mu_j}{\mathrm{d}\mathfrak{m}}$ for $j = 1, 2, 3$. Let $k \in \mathbb{N}$ be fixed. By using the σ-finiteness of $\mathfrak{m}$, we can find a partition $(E_i)_{i\in\mathbb{N}} \subseteq \Sigma$, such that $0 < \mathfrak{m}(E_i) < +\infty$ for every $i \in \mathbb{N}$ and

$$\big|f_j(x) - f_j(y)\big| \le \frac{1}{k}, \quad \text{for all } i \in \mathbb{N},\ j = 1, 2, 3,\ \text{and } \mathfrak{m}\text{-a.e. } x, y \in E_i. \tag{A.3}$$

Define $\lambda_{ij} := \frac{1}{\mathfrak{m}(E_i)} \int_{E_i} f_j \,\mathrm{d}\mathfrak{m}$ for all $i \in \mathbb{N}$ and $j = 1, 2, 3$. Observe that (A.3) ensures that

$$\big|f_j(x) - \lambda_{ij}\big| \le \frac{1}{k}, \quad \text{for all } i \in \mathbb{N},\ j = 1, 2, 3,\ \text{and } \mathfrak{m}\text{-a.e. } x \in E_i. \tag{A.4}$$

Given that $\int_{E_i} f_j \,\mathrm{d}\mathfrak{m} = \int_{E_i} \frac{\mathrm{d}\mu_j}{\mathrm{d}\mathfrak{m}} \,\mathrm{d}\mathfrak{m} = \mu_j(E_i)$, we deduce that

$$\lambda_{i1}^\alpha = \frac{\mu_1(E_i)^\alpha}{\mathfrak{m}(E_i)^\alpha} \overset{(A.1)}{\le} \frac{\mu_2(E_i)^\alpha}{\mathfrak{m}(E_i)^\alpha} + \frac{\mu_3(E_i)^\alpha}{\mathfrak{m}(E_i)^\alpha} = \lambda_{i2}^\alpha + \lambda_{i3}^\alpha, \quad \forall i \in \mathbb{N}. \tag{A.5}$$

Hence, combining (A.4) with (A.5), for every $i \in \mathbb{N}$ and $\mathfrak{m}$-a.e. $x \in E_i$ we get

$$\left(f_1(x) - \frac{1}{k}\right)^\alpha \le \lambda_{i1}^\alpha \le \lambda_{i2}^\alpha + \lambda_{i3}^\alpha \le \left(f_2(x) + \frac{1}{k}\right)^\alpha + \left(f_3(x) + \frac{1}{k}\right)^\alpha.$$

By arbitrariness of $i, k \in \mathbb{N}$, we conclude that $f_1^\alpha \le f_2^\alpha + f_3^\alpha$ holds $\mathfrak{m}$-a.e., yielding (A.2). □

We are in a position to characterise which complete norms $\|\cdot\|$ on an $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ are induced by an $L^p(\mathfrak{m})$-pointwise norm. Roughly speaking, the required compatibility between the norm and the module structure is expressed via two conditions, labelled 2a) and 2b): the former relates the given norm with the multiplication by $L^\infty(\mathfrak{m})$-functions and the chosen exponent p, while the latter is a weak continuity assumption on the multiplication operator.

Theorem A.2 (When a norm is induced by a pointwise norm) *Let* $(\mathrm{X}, \Sigma, \mathfrak{m})$ *be a σ-finite measure space. Let $\mathscr{M}$ be a module over the ring $L^\infty(\mathfrak{m})$ and $\|\cdot\|$ a complete norm on $\mathscr{M}$. Let $p \in [1, \infty)$ be a given exponent. Then, the following two conditions are equivalent:*

(1) *There exists an $L^p(\mathfrak{m})$-pointwise norm operator $|\cdot| \colon \mathscr{M} \to L^p(\mathfrak{m})$ on $\mathscr{M}$, such that*

$$\|v\| = \big\||v|\big\|_{L^p(\mathfrak{m})}, \quad \textit{for every } v \in \mathscr{M}.$$

(2) *The following two properties are satisfied:*

(2a) *It holds $\|\mathbb{1}_E \cdot v\|^p + \|\mathbb{1}_{\mathrm{X}\setminus E} \cdot v\|^p = \|v\|^p$ for every $E \in \Sigma$ and $v \in \mathscr{M}$.*

(2b) *It holds $\lim_{n\to\infty} \|f_n \cdot v\| = 0$ for every $v \in \mathscr{M}$ and for every $(f_n)_{n\in\mathbb{N}} \subseteq L^\infty(\mathfrak{m})$, such that $|f_n| \rightharpoonup 0$ weakly* in $L^\infty(\mathfrak{m})$ as $n \to \infty$.*

Proof

(1) $\Longrightarrow$ (2). Suppose (1) holds. Let us prove (2a). Fix $E \in \Sigma$ and $v \in \mathscr{M}$. Then

$$\|\mathbb{1}_E \cdot v\|^p + \|\mathbb{1}_{\mathrm{X}\setminus E} \cdot v\|^p = \int_E |v|^p \,\mathrm{d}\mathfrak{m} + \int_{\mathrm{X}\setminus E} |v|^p \,\mathrm{d}\mathfrak{m} = \int |v|^p \,\mathrm{d}\mathfrak{m} = \|v\|^p,$$

thus (2a) holds. To prove (2b), fix any sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^\infty(\mathfrak{m})$, such that $|f_n| \rightharpoonup 0$ weakly* in $L^\infty(\mathfrak{m})$. This yields $M := \sup_n \|f_n\|_{L^\infty(\mathfrak{m})} < +\infty$ thanks to the Uniform Boundedness Principle. Therefore, since $|v|^p \in L^1(\mathfrak{m})$,

$$\begin{aligned}\varlimsup_{n\to\infty} \|f_n \cdot v\| &= \varlimsup_{n\to\infty} \left(\int |f_n \cdot v|^p \,\mathrm{d}\mathfrak{m}\right)^{1/p} \\ &\le M^{(p-1)/p} \lim_{n\to\infty} \left(\int |f_n||v|^p \,\mathrm{d}\mathfrak{m}\right)^{1/p} = 0,\end{aligned}$$

thus (2b) holds. All in all, (2) is proven.
(2) $\Longrightarrow$ (1). Suppose (2) holds. First, we claim that for any $v \in \mathscr{M}$ one has

$$\|\mathbb{1}_E \cdot v\|^p = \sum_{n\in\mathbb{N}} \|\mathbb{1}_{E_n} \cdot v\|^p, \quad \text{if } (E_n)_{n\in\mathbb{N}} \subseteq \Sigma \text{ is a partition of } E. \tag{A.6}$$

To prove it, denote $E'_n := \bigcup_{i=1}^n E_i$ for every $n \in \mathbb{N}$ and notice that $\mathbb{1}_{E\setminus E'_n} \rightharpoonup 0$ weakly* in $L^\infty(\mathfrak{m})$ as $n \to \infty$. By repeatedly applying (2a), we obtain for any $n \in \mathbb{N}$ that

$$\|\mathbb{1}_E \cdot v\|^p = \|\mathbb{1}_{E_1} \cdot v\|^p + \|\mathbb{1}_{E\setminus E_1} \cdot v\|^p = \cdots = \sum_{i=1}^n \|\mathbb{1}_{E_i} \cdot v\|^p + \|\mathbb{1}_{E\setminus E'_n} \cdot v\|^p,$$

whence by letting $n \to \infty$ and using (2b) we conclude that (A.6) holds.

Given any $v \in \mathscr{M}$, we define the set-function $\mu_v \colon \Sigma \to [0, +\infty]$ as

$$\mu_v(E) := \|\mathbb{1}_E \cdot v\|^p, \quad \text{for every } E \in \Sigma.$$

It follows from (A.6) that μ_v is σ-additive. Given any $N \in \Sigma$ with $\mathfrak{m}(N) = 0$, it holds $\mathbb{1}_N = 0$ as elements of $L^\infty(\mathfrak{m})$, thus $\mu_v(N) = \|0 \cdot v\|^p = 0$. Moreover, $\mu_v(\mathrm{X}) = \|v\|^p < +\infty$. All in all, we have proven that μ_v is a finite measure on Σ satisfying $\mu_v \ll \mathfrak{m}$. Hence, we can define

$$|v| := \left(\frac{\mathrm{d}\mu_v}{\mathrm{d}\mathfrak{m}}\right)^{1/p} \in L^p(\mathfrak{m}), \quad \text{for every } v \in \mathscr{M}.$$

Observe that $\int |v|^p \,\mathrm{d}\mathfrak{m} = \mu_v(\mathrm{X}) = \|v\|^p$, thus to conclude it only remains to show that $|\cdot| \colon \mathscr{M} \to L^p(\mathfrak{m})$ is a pointwise norm operator. Trivially, $|v| = 0$ holds $\mathfrak{m}$-a.e. if and only if $v = 0$. The $\mathfrak{m}$-a.e. inequality $|v + w| \le |v| + |w|$ stems from Lemma A.1: for $E \in \Sigma$ we have

$$\begin{aligned}\mu_{v+w}(E)^{1/p} &= \left\|\mathbb{1}_E \cdot (v + w)\right\| = \left\|\mathbb{1}_E \cdot v + \mathbb{1}_E \cdot w\right\| \le \|\mathbb{1}_E \cdot v\| + \|\mathbb{1}_E \cdot w\| \\ &= \mu_v(E)^{1/p} + \mu_w(E)^{1/p},\end{aligned}$$

thus Lemma A.1 ensures that $|v+w| \le |v|+|w|$ holds $\mathfrak{m}$-a.e. on X. Finally, we claim that

$$|f \cdot v| = |f||v|, \quad \text{holds } \mathfrak{m}\text{-a.e. on X} \tag{A.7}$$

for every $f \in L^\infty(\mathfrak{m})$ and $v \in \mathscr{M}$. Let us first prove it in the case where f is a simple function, namely, $f = \sum_{i=1}^n \lambda_i \mathbb{1}_{E_i}$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and pairwise disjoint sets $E_1, \ldots, E_n \in \Sigma$. To this aim, notice that for any set $F \in \Sigma$ the following identities are satisfied:

$$\int_F |f \cdot v|^p \,\mathrm{d}\mathfrak{m} = \sum_{i=1}^n \int_{F\cap E_i} |f \cdot v|^p \,\mathrm{d}\mathfrak{m} = \sum_{i=1}^n \mu_{f\cdot v}(F \cap E_i)$$

$$\begin{aligned}
&= \sum_{i=1}^{n} \left\| \mathbb{1}_{F\cap E_i} \cdot (f \cdot v) \right\|^p = \sum_{i=1}^{n} \left\| \lambda_i (\mathbb{1}_{F\cap E_i} \cdot v) \right\|^p \\
&= \sum_{i=1}^{n} |\lambda_i|^p \|\mathbb{1}_{F\cap E_i} \cdot v\|^p = \sum_{i=1}^{n} |\lambda_i|^p \int_F \mathbb{1}_{E_i} |v|^p \, \mathrm{d}\mathfrak{m} \\
&= \int_F |f|^p |v|^p \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

By arbitrariness of F, we deduce that (A.7) holds whenever f is a simple function. The general case follows by approximation: given any $f \in L^\infty(\mathfrak{m})$, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions, such that $f_n \to f$ strongly in $L^\infty(\mathfrak{m})$ as $n \to \infty$. Then, $|f_n - f| \rightharpoonup 0$ weakly* in $L^\infty(\mathfrak{m})$, thus 2b) yields

$$\int \left| |f_n \cdot v| - |f \cdot v| \right|^p \mathrm{d}\mathfrak{m} \le \int \left| (f_n - f) \cdot v \right|^p \mathrm{d}\mathfrak{m} = \left\| (f_n - f) \cdot v \right\|^p \longrightarrow 0.$$

Moreover, since $|f_n| \to |f|$ in $L^\infty(\mathfrak{m})$, we have $|f_n||v| \to |f||v|$ in $L^p(\mathfrak{m})$. Since we already know that $|f_n \cdot v| = |f_n||v|$ for all $n \in \mathbb{N}$, we conclude that $|f \cdot v| = \lim_n |f_n \cdot v| = \lim_n |f_n||v| = |f||v|$ strongly in $L^p(\mathfrak{m})$, proving (A.7). Therefore, $|\cdot|$ is a pointwise norm, whence (1) follows. □

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