Application of the stochastic formalism for spectator scalars during inflation

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Abstract

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The quantum fluctuations of light and energetically subdominant scalar fields grow during inflation to scales larger than the Hubble horizon becoming effectively classical perturbations. These perturbations could influence on structure formation in the universe, dark matter abundance or for example generate primordial blackholes. Scalar field fluctuations can be modelled by means of stochastic formalism where the field is divided into long-wavelength (IR) field outside the horizon and short-wavelength (UV) field inside the horizon. IR-field can then be treated effectively classical stochastic quantity. Two-point correlation functions for some function of a scalar field φ can in stochastic formalism be expressed as a spectral expansion. The terms for this expansion are obtained from a Schrödinger-like eigenvalue equation. In this thesis, we introduce the relevant parts of stochastic inflation formalism and apply it to a $\lambda\varphi^4$ potential with a running coupling $\lambda(\varphi)$, resulting in the formation of metastable minima. We then analyse the influence of these metastable minima on the two-point correlators of the scalar field with numerical methods. This type of field is particularly interesting since the Higgs potential may have qualitatively similar metastable minima and the Higgs field could have acted as a spectator during inflation. In the presence of metastable minima, we find close to scale invariant slightly blue tilted spectrum. The change in the spectral index can be up to 80% compared to the case with constant λ.

Keywords: Stochastic formalism, Inflation, Spectator field

Tiivistelmä

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Kevyiden ja energiatiheydeltään pienten spektaattorikenttien kvanttifluktuaatiot kasvavat inflaation aikana Hubblen horisontin ulkopuolisilla skaaloilla ja käyttäytyvät efektiivisesti kuten klassiset kentän häiriöt. Nämä inflaation aikana syntyneet häiriöt voivat potentiaalisesti vaikuttaa maailmankaikkeuden rakenteen muodostumiseen, pimeän aineen määrään tai esimerkiksi primordiaalisten mustien aukkojen syntyyn. Häiriöitä voidaan mallintaa stokastisen formalismin avulla, jossa kenttä jaetaan horisontin ulkopuoliseen, efektiivisesti klassiseen (IR) kenttään ja horisontin sisäpuolella olevaan kvanttikenttään (UV). IR-kenttää voidaan johtavana approksimaationa kuvata klassisena stokastisena suureena. Stokastisessa formalismissa minkä tahansa kentän φ funktion 2-pistekorrelaattori horisonttia suuremmilla skaaloilla voidaan esittää spektraalihajotelmana, jonka termit saadaan Schrödinger-tyyppisen ominaisarvoyhtälön ratkaisuina. Tässä tutkielmassa esitellään stokastisen formalismin perusteita ja sovelletaan sitä $\lambda\varphi^4$-potentiaalille juoksevalla kytkennällä $\lambda(\varphi)$. Numeerisen analyysin avulla työssä tarkastellaan miten kytkennän juoksusta kentän potentiaaliin muodostuvat metastabiilit minimit vaikuttavat sen 2-pistekorrelattoriin. Tällainen kenttä on erityisen mielenkiintoinen, koska Higgsin potentiaalissa voi olla kvalitatiivisesti saman tyyppisiä metstabiileja minimejä ja Higgs on voinut toimia spektaattorina inflaation aikana. Metastabiilien minimien vaikutuksesta spektri on lähes skaalainvariantti ja taittaa siniseen. Spektri-indeksi pienenee jopa 80 % verrattuna tapaukseen, missä λ pysyy vakiona.

Avainsanat: Stokastinen formalismi, Inflaatio, Spektaattorikenttä

Contents

1 Introduction

Observations have shown that at scales larger than 100 Mpc universe appears approximately homogeneous and isotropic [1–3]. Thus, we assume spatially homogeneous and isotropic universe on large scales, meaning that the universe looks the same everywhere and in every direction. This assumption is known as the cosmological principle.

In the beginning of the twentieth century, it was thought that the cosmological principle applies to both, time and space (perfect cosmological principle), meaning that our universe is static and not evolving in time. This model was called into question after Vesto Slipher discovered the galactic redshift in 1912 [4]. A heated debate regarding the static universe model commenced, lasting nearly two decades until Edwin Hubble's publication of the distance-velocity relation in 1929 [5]. Hubble presented observational evidence that the universe was not static, but in fact expanding.

Theoretical basis for the expanding universe model was founded in 1915 when Einstein published his theory of general relativity [6]. Later in 1922 Alexander Friedmann presented solutions to the Einstein field equations in a spatially homogeneous and isotropic universe [7]. In 1927, Georges Lemaître independently obtained similar results [8], which would then lead to the hot Big Bang model. Together, the Friedmann equations, with the spatially homogeneous and isotropic Robertson-Walker (RW) metric [9, 10], defines the Friedmann-Lemaître-Robertson-Walker (FLRW) universe.

The hot Big Bang is a consequence of the expanding universe model, and according to it the early universe was extremely hot and dense, and dominated by radiation. The most important achievements of this model are explaining the cosmic microwave background (CMB) [11] and the abundance of light elements in agreement with observations [12, 13]. The hot Big Bang alone would however require an extreme fine-tuning in early universe, e.g. the flatness problem and the horizon problem.

The flatness problem arises from an observation that the universe is spatially flat [14]. However, this is not a stable solution and would require extremely fine-tuned initial values in the early universe, a topic we will discuss in more detail in section 2.2.

The observation that the cosmic microwave background is remarkably homogeneous [15], yet containing regions that were causally disconnected from each other at recombination brings us to the horizon problem. We would expect to see uncorrelated patches at scales surpassing the causal horizon, but this is not what we observe. Instead, assuming that the early universe was radiation-dominated, patches that have never been in causal contact share the same average temperature.

In 1981 Alan Guth proposed that a period of accelerated expansion, called inflation, would solve the horizon problem and the flatness problem of the hot Big Bang model [16]. A year later it was discovered that inflation could also explain the origin of the primordial perturbations that would arise from quantum fluctuations stretched to macroscopic scales during the inflationary period [17, 18].

The study of inflation has come far from those pioneering days, and it has become an active research field. A general description of inflation is that the expansion is driven by the potential of a scalar field (inflaton) while the field is slowly rolling towards a minimum of the potential. The potential energy behaves essentially like vacuum energy, causing the nearly exponential expansion of the universe. This model is known as the slow-roll inflation. The idea of the inflationary phase in the early universe is widely accepted and supported by observations, as the high degree of homogenity and isotropy, and nearly scale invariant spectrum of fluctuations in CMB [19]. However, there is still no absolute certainty about inflation and the microscopic nature behind it is yet unknown.

One interesting aspect connected to inflation are so-called spectator fields. Spectator fields refer to energetically subdominant matter fields during inflation which do not directly affect the inflationary dynamics [20]. However, they may play a significant role in subsequent stages, potentially influencing CMB anisotropies and structure formation [21], production of dark matter [22] or even generate primordial black holes [23, 24]. The standard model (SM) Higgs could also have acted as a spectator during inflation [25, 26]. During inflation, quantum fluctuations of light scalar fields grow to macroscopic (superhorizon) scales, transitioning effectively into classical fluctuations. These fluctuations can be studied with the approximative stochastic formalism of inflation, where the superhorizon part of the scalar field is modelled as a classical stochastic quantity, while the subhorizon part is treated as stochastic white noise [27]. The dynamics of the superhorizon field can then be described with a stochastic Langevin equation. The stochastic formalism offers a convenient non-perturbative tool to study the large-scale dynamics of scalar fields where perturbative computations suffer from IR-divergences [28]. In this formalism, correlators of any local function of a spectator scalar field can be conveniently expressed as a spectral expansion with coefficients obtained from a Schrödinger-like eigenvalue equation.

The aim of this thesis is to study light spectator scalars during inflation using the stochatic formalism. We introduce the relevant components of the stochastic formalism, deriving the Langevin equation, the corresponding Fokker-Planck equation and the spectral expansion of its solution. We then numerically apply this formalism to a spectator scalar with a Higgs-like potential with metastable minima generated by a running coupling.

This thesis is structured as follows: In chapter 2 we review the basic results of Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology and inflation. Quantization of a free scalar field in both flat and de Sitter space-time is described in chapter 3. Next, we will turn to the stochastic inflation formalism in chapter 4, including a derivation of the Langevin equation for a scalar field. Then we introduce the Fokker-Planck equation and find a solution for the equilibrium distribution. In chapter 5, we present the spectral expansion method for the calculation of two-point functions. Finally, in chapter 6 we apply these methods to numerically calculate the spectral expansion for a $\lambda\phi^4$ potential with a running coupling.

Throughout this thesis, unless stated otherwise, we use the natural units with $c = \hbar = 1$ and the metric signature (-,+,+,+).

2 Introduction to cosmology

In this chapter, we give a brief overview of cosmology, covering topics from general relativity to the Friedmann-Lemaître-Robertson-Walker (FLRW) universe. Followed by an introduction to the basic ingredients of cosmic inflation.

2.1 FLRW-universe

As mentioned in the previous section, the FLRW-universe is defined by the Einstein equations in a spatially homogeneous and isotropic universe. Here we will present the most relevant results, starting from the derivation of the Friedman equations. More detailed approach can be found in many textbooks for example in [29, 30].

We start by introducing the Einstein equations, which relate the geometry of spacetime to its matter distribution [29]

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor describing the curvature of the spacetime and $T_{\mu\nu}$ is the energy-momentum tensor representing the matter content within. The Einstein tensor is defined by the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.2}$$

Both $R_{\mu\nu}$ and R are obtained as contractions of the Riemann tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.3}$$

$$R_{\sigma\nu} = R^{\mu}{}_{\sigma\mu\nu} \tag{2.4}$$

$$R = R^{\mu}{}_{\mu} \tag{2.5}$$

where $\Gamma^{\rho}_{\mu\nu}$ is the connection coefficient (or Christoffel symbol)

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right). \tag{2.6}$$

g_{μ_ν} appearing in the connection coefficient is the metric tensor and it determines the geometry of a spacetime and specifies the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.7}$$

In order to solve the Einstein equations (2.1) we need to find the form of our metric tensor. Following the cosmological principle, we take spatially isotropic and homogeneous universe, which can be described as a spacetime that is divided into maximally symmetric spacelike slices as

$$ds^2 = -dt^2 + a^2(t) d\boldsymbol{\sigma}^2, \tag{2.8}$$

where the scale factor $a(t)$ tells the size of the spacelike slice at a given time t. The RW-metric describes this type of spacetime and in spherical coordinates it has the form [10]

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right), \tag{2.9}$$

where the constant K determines the curvature, such that $K > 0$ corresponds to positive curvature (closed universe), $K = 0$ corresponds to no curvature (flat universe) and $K < 0$ corresponds to negative curvature (open universe).

In cosmology, matter is generally described as a perfect fluid, for which the energy-momentum tensor has the form [29]

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{2.10}$$

where ρ is the energy density, p is pressure and u_μ is the four-velocity. The fluid is at rest in comoving coordinates and thus its four-velocity is given by

$$u_\mu = (1{,}0{,}0{,}0). \tag{2.11}$$

The non-vanishing components of $T_{\mu\nu}$ then are

$$
\begin{aligned}
T_{00} &= \rho \\
T_{11} &= \frac{a^2}{1 - Kr^2}p \\
T_{22} &= a^2 r^2 p \\
T_{33} &= a^2 r^2 sin^2\theta p
\end{aligned}
\tag{2.12}
$$

and together with the non-vanishing components of the Einstein tensor

$$
\begin{aligned}
G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \\
G_{11} &= -\frac{2a\ddot{a} + \dot{a}^2 + K}{a - Kr^2} \\
G_{22} &= -r^2\left(2a\ddot{a} + a^2 + K\right) \\
G_{33} &= -r^2\sin^{\theta}\left(2a\ddot{a} + a^2 + K\right)
\end{aligned}
$$

we find the Friedmann equations

$$
H^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2} \tag{2.13}
$$

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{2.14}
$$

with Hubble parameter $H = \dot{a}/a$ [29].

It can be seen that the scale factor depends on the energy density, ρ, and pressure, p. These two are connected, as perfect fluid obeys the equation of state

$$
p = w\rho \tag{2.15}
$$

with a parameter w often taken as a constant. The energy-momentum tensor obeys the continuity equation [29]

$$
\nabla_\mu T^\mu{}_\nu = 0. \tag{2.16}
$$

From the zero component we obtain

$$\nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \tag{2.17}$$
$$= -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{2.18}$$

and together with eq. (2.15) we get

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1 + w)\rho. \tag{2.19}$$

When the fluid consists of more than one non-interacting species, the total energy density can be written as a sum of these components

$$\rho = \sum_i \rho_i \tag{2.20}$$

each of them satisfying the continuity equation.

For constant w, eq. (2.19) then gives

$$\rho_i = \rho_{i0} \left(\frac{a}{a_0}\right)^{-3(1+w_i)}. \tag{2.21}$$

The parameter w_i takes different values depending on the type of matter under consideration. For non-relativistic matter, often referred as dust, $w = 0$. In the case of relativistic matter (i.e. radiation) $w = 1/3$, and for vacuum energy $w = -1$ [29].

2.2 Inflation

In the beginning of this chapter we discussed the problems of the hot Big Bang model and how inflation, a period of exponential expansion of space, was proposed as a possible solution. In this section, we first present the flatness problem and then focus on single scalar field driven slow-roll inflation.

2.2.1 The flatness problem

Let us start by introducing the density parameter, Ω, which describes the geometry of our universe and tells us the sign of the curvature parameter, K, defined as [29]

$$\Omega = \frac{8\pi G}{3H^2}\rho \tag{2.22}$$

and can be further expressed together with eq. (2.13) as

$$\Omega - 1 = K/a^2H^2. \tag{2.23}$$

We can find the sign of K by determining the value of the density parameter. If $K = 0$, then $\Omega = 1$ and stays constant giving that our universe is perfectly flat, otherwise the density parameter evolves in time. The behaviour of Ω can be found by using eqs. (2.21) and (2.13), giving

$$\rho \propto a^{-3(1+w)} \quad \text{and} \quad H \propto a^{-3(1+w)/2}, \tag{2.24}$$

then plugging these into eq. (2.23) yields

$$\Omega - 1 \propto a^{1+3w}. \tag{2.25}$$

According to the hot Big Bang model, in the early universe there was a radiation dominated era followed by matter dominance [29]. During these eras $|\Omega - 1|$ grows, since $w = 1/3$ for radiation, giving $\Omega - 1 \propto a^2$ and $w = 0$ for matter, giving $\Omega - 1 \propto a$. The current value of Ω is observed to be very close to unity [14], which would require extremely fine-tuned initial values in the early universe.

Considering a case that in very early times before radiation dominance took place, the universe was instead dominated by the vacuum energy, for which $w = -1$ and thus $\rho \propto a^0$. In this case, the curvature term K/a^2 in eq. (2.13) decreases much faster than the energy density term $8\pi G\rho/3$, leading to

$$H^2 = \frac{8\pi G\rho}{3}. \tag{2.26}$$

This makes the Hubble parameter $H = \dot{a}/a$ a constant, since ρ is a constant during this period. For a vacuum energy dominated universe, we then have the de Sitter solution with a scale factor

$$a \propto e^{Ht}. \tag{2.27}$$

The second Friedmann equation (2.14) now reads

$$\frac{\ddot{a}}{a} = \frac{8\pi G\rho}{3} \tag{2.28}$$

and we find that with a vacuum dominated universe we get an accelerated expansion as $\ddot{a} > 0$. This also solves the flatness problem as $\Omega - 1 \propto a^{-2}$ drives Ω exponentially small during vacuum dominance and thus removes the need for fine tuning. Thus, we expect that inflation is driven by something that behaves like a vacuum energy and that is converted into matter and radiation after the inflationary period [31]. Again, from the second Friedmann equation (2.14) we see that for $\ddot{a} > 0$ it is sufficient that $p < -\rho/3$, implying negative pressure as we require that $\rho > 0$.

2.2.2 Inflaton field and the slow-roll approximation

Inflation can be described by a hypothetical scalar field called inflaton, the expansion is then driven by its potential [32]. It will be described in section 3.2 that in RW-metric a scalar field obeys the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2}\phi + V'(\phi) = 0, \tag{2.29}$$

where the prime denotes a derivative with respect to ϕ and where we have assumed that $K = 0$ in eq. (2.13). Even if we would have $K \neq 0$, the curvature term in the Friedmann equation (2.13) will be redshifted away during the inflationary phase as we saw in section 2.2.1. In the case of a homogeneous field, the spatial derivatives are zero and thus eq. (2.29) is reduced to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2.30}$$

Energy density and pressure for a homogeneous scalar field are [32]

$$\begin{aligned}\rho =&\frac{1}{2}\dot{\phi}^2 + V(\phi)\\ p =&\frac{1}{2}\dot{\phi}^2 - V(\phi).\end{aligned} \tag{2.31}$$

Plugging this expression for ρ into eq. (2.26) then gives

$$H^2 =\frac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{2.32}$$

Here we have introduced the reduced Planck mass $M_p = 1/\sqrt{8\pi G}$. Differentiating the above expression and using eq. (2.30) gives

$$\dot{H} = -\frac{1}{2M_p^2}\dot{\phi}^2. \tag{2.33}$$

Recalling that we need the condition $\rho+3p < 0$ in order to achieve the inflationary period ($\ddot{a} > 0$), which can be attained if the potential energy dominates the kinetic energy, as seen with eq. (2.31).

Nearly exponential expansion is achieved if the kinetic term is negligible compared to the potential term $\dot{\phi}^2 \ll V$ giving $p \simeq -\rho$, as seen from eq. (2.31). Furthermore, we assume the field varies slowly enough for the condition $\dot{\phi}^2 \ll V$ to be maintained long enough, yielding the slow-roll conditions [33]

$$\dot{\phi}^2 \ll V \tag{2.34}$$

$$|\ddot{\phi}| \ll |3H\dot{\phi}|,|V'|. \tag{2.35}$$

Subsequently the Friedmann equation (2.32) and the equation of motion (2.30) can be approximated as

$$\begin{aligned}H^2 \simeq&\frac{V}{3M_p^2}\\ 3H\dot{\phi} \simeq& - V'.\end{aligned} \tag{2.36}$$

The first condition in eq. (2.34) together with eqs. (2.36) and (2.33) then implies

$$-\frac{\dot{H}}{H^2} \ll 1, \tag{2.37}$$

meaning approximately constant H and therefore a quasi-de Sitter solution. The slow-roll conditions can be expressed with the so called slow-roll parameters [33]

$$\begin{aligned}\epsilon =& -\frac{\dot{H}}{H^2} = \frac{1}{2}M_p^2\left(\frac{V'}{V}\right)^2 \\ \eta =& M_p^2\frac{V''}{V}.\end{aligned} \tag{2.38}$$

The conditions (2.34) then translates to $\epsilon, |\eta| \ll 1$.

In this section we focused on a single field slow-roll inflation, but it's important to note that this represents just one of many inflationary models (for a review of inflation models, see for example [33, 34]). While slow-roll is commonly assumed in various models of inflation, the use of slow-roll conditions is not mandatory and there are models that do not rely on the slow-roll approximation, see for example ref. [35]. Likewise, in addition to the single-field inflation there are plenty of different models involving more than one field, for early works see refs. [36] and [37].

One particular case that is discussed in this thesis are energetically subdominant scalar fields, often referred to as spectator fields. Spectator fields do not influence the dynamics during inflation but may become important at a later stage [20]. The quantum fluctuations of spectator fields, when stretched to superhorizon scales during inflation, may give rise to primordial perturbations for example through a curvaton mechanism [21, 38–41] or modulated reheating [42, 43]. For instance, the SM Higgs may have acted as a spectator field during inflation [25, 26, 44].

3 Quantization of a scalar field

As noted in the previous chapter the primordial perturbations can be explained by quantum fluctuations generated during inflation. Therefore, we need to study how scalar fields are quantized. In this chapter we first discuss the quantization process of free fields in flat Minkowski spacetime and then generalize the process into de Sitter space that corresponds to inflation.

3.1 A free scalar field in Minkowski spacetime

In the Minkowski case we mainly follow the approach of [45]. Consider a free Klein-Gordon scalar field in Minkowski spacetime where the metric is of the form

$$ds^2 = -dt^2 + (d\boldsymbol{x})^2. \tag{3.1}$$

The Lagrangian density of a free scalar ϕ is given by

$$\mathcal{L} = \sqrt{-g}(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2), \tag{3.2}$$

where g is the metric determinant. From this we obtain the equation of motion

$$\Box\phi - m^2\phi = 0, \tag{3.3}$$

which is also known as the Klein-Gordon equation and where $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$. Writing the field as a Fourier transformation gives

$$\phi(x) = \int d^4kNe^{ik\cdot x}\tilde{\varphi}(k) \tag{3.4}$$

where $k \cdot x$ is a dot product between $k = (k^0, \boldsymbol{k})$, $x = (t,\boldsymbol{x})$ and N is a constant. Substituting (3.4) into the Klein-Gordon equation (3.3), we obtain

$$\int d^4kN\left((k^0)^2 - |\boldsymbol{k}|^2 - m^2\right)e^{ik\cdot x}\tilde{\varphi}(k) = 0. \tag{3.5}$$

For nonvanishing $\tilde{\varphi}(k)$ we find that $(k^0)^2 = m^2 + |\boldsymbol{k}|^2 \equiv E_k$, which yields $k^0 = \pm E_k$. We can then define $\tilde{\varphi}(k) = \delta\left((k^0)^2 - E_k^2\right) b(k)$, which upon substitution into eq. (3.5) and dividing into positive ($k^0 = E_k$) and negative energy ($k^0 = -E_k$) solutions gives

$$\begin{aligned}\phi(x) &= \int d^4k \frac{N}{2E_k} \Big(\delta\left(k^0 - E_k\right) + \delta\left(k^0 + E_k\right)\Big) e^{ik\cdot x} b(k) \\ &= \int d^3k \frac{N}{2E_k}\left(e^{ik\cdot x} b(k) + e^{-ik\cdot x} b(-k)\right) \end{aligned} \tag{3.6}$$

where we have used the property $\delta\left((k^0)^2 - E_k^2\right) = \frac{1}{2E_k}\left(\delta\left(k^0 - E_k\right) + \delta\left(k^0 + E_k\right)\right)$, and where $k = (E_k, \boldsymbol{k})$. Since $\phi(x)$ is real $\phi(x) = \phi^\dagger(x)$ we have that

$$\int d^3k \frac{N}{2E_k}\left(e^{ik\cdot x} b(k) + e^{-ik\cdot x} b(-k)\right) = \int d^3k \frac{N}{2E_k}\left(e^{-ik\cdot x} b^\dagger(k) + e^{ik\cdot x} b^\dagger(-k)\right), \tag{3.7}$$

which gives $b^\dagger(k) = b(-k)$. From the dispersion relation $E_k = \sqrt{m^2 + |\boldsymbol{k}|^2}$, it follows that $E_k = E_k(|\boldsymbol{k}|)$ and thus b(k) can be written as a function of $\boldsymbol{k}$. We rewrite $b(k)$ and $b(k)^\dagger$ as

$$b(k) \equiv \sqrt{2E_k} a(\boldsymbol{k}) \quad \text{and} \quad b^\dagger(k) \equiv \sqrt{2E_k} a^\dagger(\boldsymbol{k}). \tag{3.8}$$

In order to quantize the field

$$\phi(x) = \int d^3k \frac{N}{\sqrt{2E_k}}\left(e^{ik\cdot x} a(\boldsymbol{k}) + e^{-ik\cdot x} a^\dagger(\boldsymbol{k})\right) \tag{3.9}$$

and its conjugate momentum

$$\begin{aligned}\pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \\ &= i\int d^3k N \sqrt{\frac{E_k}{2}}\left(e^{-ik\cdot x} a^\dagger(\boldsymbol{k}) - e^{ik\cdot x} a(\boldsymbol{k})\right)\end{aligned} \tag{3.10}$$

both fields are promoted into field operators satisfying the commutation relations on equal-time hypersurfaces

$$\begin{aligned}&[\hat{\phi}(t,\boldsymbol{x}),\hat{\phi}(t,\boldsymbol{y})] = 0 \\ &[\hat{\pi}(t,\boldsymbol{x}),\hat{\pi}(t,\boldsymbol{y})] = 0 \\ &[\hat{\phi}(t,\boldsymbol{x}),\hat{\pi}(t,\boldsymbol{y})] = i\delta^3\left(\boldsymbol{x} - \boldsymbol{y}\right).\end{aligned} \tag{3.11}$$

Commutation relations for creation and annihilation operators $\hat{a}^\dagger$, $\hat{a}$, and the value of the constant N, follow directly from above commutation relations

$$\begin{aligned}
[\hat{a}(\boldsymbol{k}),\hat{a}(\boldsymbol{p})] &= [\hat{a}^\dagger(\boldsymbol{k}),\hat{a}^\dagger(\boldsymbol{p})] = 0 \\
[\hat{a}(\boldsymbol{k}),\hat{a}^\dagger(\boldsymbol{p})] &= \delta^3(\boldsymbol{k}-\boldsymbol{p}) \\
N &= \frac{1}{(2\pi)^{3/2}}.
\end{aligned} \tag{3.12}$$

Thus the quantized field operator and its conjugate momentum can be written as

$$\begin{aligned}
\hat{\phi}(x) &= \int \frac{d^3k}{(2\pi)^{3/2}} \left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \varphi_k(t) \hat{a}_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \varphi_k^*(t) \hat{a}_{\boldsymbol{k}}^\dagger \right) \\
\hat{\pi}(x) &= \int \frac{d^3k}{(2\pi)^{3/2}} \left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\varphi}_k(t) \hat{a}_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\varphi}_k^*(t) \hat{a}_{\boldsymbol{k}}^\dagger \right),
\end{aligned} \tag{3.13}$$

where $\varphi_k(t) = \frac{1}{\sqrt{2E_k}} e^{-iE_k t}$ and from here on $k = |\boldsymbol{k}|$.

3.2 A free scalar field in de Sitter space

Quantization of a scalar field in de Sitter space follows similar steps as the quantization in Minkowski spacetime. The de Sitter metric is given by

$$ds^2 = -dt^2 + a(t)^2 d\boldsymbol{x}, \tag{3.14}$$

where $a(t) = e^{Ht}$ and the Hubble parameter H is constant.

The Lagrangian density for a scalar field in a general curved spacetime reads [29]

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \tag{3.15}$$

$$\tag{3.16}$$

where the partial derivatives ∂_μ of the previous section are now replaced with the covariant derivatives ∇_μ, which can be seen as a generalization of the partial derivatives to curved spacetimes. For example, the covariant derivative of a dual vector V_μ is defined by $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu} V_\sigma$, where $\Gamma^\sigma_{\mu\nu}$ is the Christoffel symbol eq. (2.6). This reduces to $\partial_\mu V_\nu$ in Minkowski spacetime, where $\Gamma^\sigma_{\mu\nu} = 0$ in cartesian coordinates. The scalar field ϕ satisfies the following equation of motion

$$\nabla^\mu \nabla_\mu \phi - V'(\phi) = 0. \tag{3.17}$$

In order to compute $\nabla^\mu \nabla_\mu \phi$ we need the Christoffel symbols in de Sitter spacetime. Straightforward calculation yields the only non-zero Christoffel symbols as

$$\begin{aligned} \Gamma^0_{ij} &= a\dot{a}\delta^i_j \\ \Gamma^i_{j0} &= \frac{\dot{a}}{a}\delta^i_j. \end{aligned} \tag{3.18}$$

Thus we get

$$\begin{aligned} \nabla^\mu \nabla_\mu \phi &= g^{\mu\nu}\nabla_\nu \nabla_\mu \phi \\ &= g^{\mu\nu}(\partial_\nu \partial_\mu \phi - \Gamma^\sigma_{\mu\nu}\partial_\sigma \phi) \\ &= g^{00}\partial_0\partial_0\phi + g^{ij}(\partial_j\partial_i\phi - \Gamma^0_{ij}\partial_0\phi) + g^{j0}(\partial_j\partial_0\phi - \Gamma^i_{j0}\partial_i\phi) \\ &= -\partial_0^2\phi + a^{-2}\partial_i^2\phi - \frac{\dot{a}}{a}\delta^i_i\partial_0\phi \\ &= (-\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t} + \frac{\nabla^2}{a^2})\phi \end{aligned} \tag{3.19}$$

and the equation of motion (3.17) takes the form

$$(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} - \frac{\nabla^2}{a^2})\phi + V'(\phi) = 0. \tag{3.20}$$

For a free scalar field with $V(\phi) = \frac{1}{2}m^2\phi^2$, this reads

$$(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} - \frac{\nabla^2}{a^2} + m^2)\phi = 0. \tag{3.21}$$

As in the Minkowski case, the field and its conjugate momentum are promoted into operators, which can be expanded in Fourier modes

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}} \left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \varphi_k(t) \hat{a}_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \varphi_k^*(t) \hat{a}_{\boldsymbol{k}}^\dagger \right) \\
&\equiv \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{\varphi}_{\boldsymbol{k}}(t) \\
\pi(x) &= \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} \\
&\equiv a^3 \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\tilde{\varphi}}_{\boldsymbol{k}}(t),
\end{aligned} \tag{3.22}
$$

where $\tilde{\varphi}_{\boldsymbol{k}}(t) = \varphi_k(t)\hat{a}_{\boldsymbol{k}} + \varphi_k^*(t)\hat{a}_{-\boldsymbol{k}}^\dagger$. By inserting this into the equation of motion (3.21) we find that the Fourier modes satisfy the following equation

$$
\ddot{\varphi}_{\boldsymbol{k}}(t) + 3H\dot{\varphi}_{\boldsymbol{k}}(t) + \left(\frac{|\boldsymbol{k}|^2}{a^2} + m^2 \right) \varphi_{\boldsymbol{k}}(t) = 0. \tag{3.23}
$$

For now on, it is more convenient to use conformal time τ defined by $ad\tau \equiv dt$, as it simplifies the following calculations. We also introduce a rescaled field variable $\chi \equiv a\phi$. In the conformal time the metric (3.14) takes the form $ds^2 = a^2(-d\tau^2 + d\boldsymbol{x}^2)$. The conformal time τ in de Sitter can be solved from the definition above as

$$
\begin{aligned}
\int_{\tau_0}^{\tau} d\tau &= \int_{t_0}^{t} e^{-Ht} dt \\
\tau &= -\frac{1}{Ha} + \frac{1}{Ha_0} + \tau_0,
\end{aligned} \tag{3.24}
$$

and by choosing $a_0 = -1/(H\tau_0)$ we have $\tau = -1/(Ha)$.

By imposing commutation relations for creation and annihilation operators (3.12) we find the canonical commutation relations for the rescaled field and its conjugate momentum

$$
\begin{aligned}
[\chi(\tau,\boldsymbol{x}), \chi(\tau,\boldsymbol{y})] &= [\chi'(\tau,\boldsymbol{x}), \chi'(\tau,\boldsymbol{y})] = 0 \\
[\chi(\tau,\boldsymbol{x}), \chi'(\tau,\boldsymbol{x})] &= i\delta^3(\boldsymbol{x} - \boldsymbol{y})
\end{aligned} \tag{3.25}
$$

and the normalization condition $\chi_k \chi_k'^* - \chi_k^* \chi_k' = i$.

Then writing the first and the second time derivatives of the mode function φ_k

in terms of the rescaled field variable χ_k yields

$$\dot{\varphi}_k = \frac{\chi_k'}{a^2} - \frac{\chi_k}{a}H$$
$$\ddot{\varphi}_k = \frac{1}{a^2}\Big(\frac{\chi_k''}{a} - 3H\chi_k' + H^2 a\chi_k\Big) \tag{3.26}$$

and inserting these into (3.23) gives the equation of motion for χ_k

$$\chi_k'' + \Big(k^2 + \frac{1}{\tau^2}\Big(\frac{m^2}{H^2} - 2\Big)\Big)\chi_k = 0. \tag{3.27}$$

This equation can be rewritten in the form of a Bessel differential equation by defining $\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$

$$\chi_k'' + \Big(k^2 - \frac{1}{\tau^2}\Big(\nu^2 - \frac{1}{4}\Big)\Big)\chi_k = 0. \tag{3.28}$$

Here ν is assumed to be real, that is $\frac{m}{H} < \frac{3}{2}$. Solution to this differential equation can be written as a linear combination of Hankel functions of the first and the second kind $H_\upsilon^{(1,2)}$ [46] as

$$\chi_k = \sqrt{-\tau}\Big(C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\Big) \tag{3.29}$$

where $C_{1,2}(k)$ are some k dependent constants. The Hankel functions have asymptotic expansions for large and small arguments [46] For subhorizon modes i.e. $k \gg aH$, or $-k\tau \gg 1$, we have the following expansions

$$H_\nu^{(1)}(-k\tau) \simeq \sqrt{\frac{-2}{\pi k\tau}}e^{-ik\tau}e^{-\theta_\nu}$$
$$H_\nu^{(2)}(-k\tau) \simeq \sqrt{\frac{-2}{\pi k\tau}}e^{ik\tau}e^{\theta_\nu} \tag{3.30}$$

where $\theta_\nu = \frac{i\pi}{2}(\nu + \frac{1}{2})$. Similarly for superhorizon modes i.e. $k \ll aH$ or $-k\tau \ll 1$ we have

$$H_\nu^{(1)}(-k\tau) \simeq -\frac{i2^\nu\Gamma(\nu)}{\pi}(-k\tau)^{-\nu}$$
$$H_\nu^{(2)}(-k\tau) \simeq \frac{i2^\nu\Gamma(\nu)}{\pi}(-k\tau)^{-\nu}. \tag{3.31}$$

In order to solve the constants $C_{1,2}(k)$, we need to define the vacuum state that is annihilated by $\hat{a}_{\boldsymbol{k}}$, for all $\boldsymbol{k}$

$$\hat{a}_{\boldsymbol{k}}|0>=0. \tag{3.32}$$

We choose the vacuum such that in the subhorizon limit ($k \gg aH$) we recover the Minkowski result $\chi_k \to e^{-ik\tau}$ this is obtained by choosing $C_2 = 0$. This is known as the Bunch-Davies vacuum [47].

The constant $C_1(k)$ is then determined by the normalization condition $\chi_k \chi_k'^* - \chi_k^* \chi_k' = i$, which yields $C_1(k) = \frac{\sqrt{\pi}}{2}$. Therefore, we obtain the solutions for the mode functions χ_k and φ_k as

$$\begin{aligned}\chi_k &= \frac{\sqrt{-\tau\pi}}{2} H_\nu^{(1)}(-k\tau) \\ \varphi_k &= (-\tau)^{3/2} \frac{H\sqrt{\pi}}{2} H_\nu^{(1)}(-k\tau).\end{aligned} \tag{3.33}$$

4 Stochastic formalism

During inflation the quantum fluctuations of a scalar field are stretched to superhorizon scales affecting the large-scale dynamics of this same field [27, 48]. In stochastic approach, presented by Starobinsky [27], the field is split in two parts, coarse-grained field containing super horizon modes and quantum part consisting of subhorizon modes. It turns out that we can focus on the superhorizon part only, treating it as classical stochastic field characterized by a Langevin equation. Langevin equation is a stochastic differential equation where a random noise term is added to the macroscopic equation of motion [49, 50].

In this chapter, we introduce the stochastic inflation formalism starting with the derivation of the Langevin equation for a scalar field. Subsequently, in section 4.2 we obtain the corresponding Fokker-Planck equation describing the time evolution of the probability distribution function. Finally, in section 4.3 we find the equilibrium distribution and consider two concrete examples.

4.1 Langevin equation for a scalar field

In section 3.2 we saw that a scalar field has a very different behaviour on superhorizon scales ($k \ll aH$) eq. (3.30) and subhorizon scales ($k \gg aH$) eq. (3.31), giving us the base for stochastic inflation. Here we will consider a scalar field $\phi(\boldsymbol{x},t)$ in de Sitter background (i.e. $a = e^{Ht}$ and $H =$ const.) and derive a stochastic Langevin equation by roughly following the derivation in [48]. We begin by splitting the field ϕ and its conjugate momentum $\dot{\phi}$ into two parts, a coarse-grained long-wavelength part $\bar{\varphi}$,$\bar{\pi}$ and a short-wavelength part φ_s, π_s as [48]

$$\begin{aligned}\phi(\boldsymbol{x}, t) &= \bar{\varphi}(\boldsymbol{x},t) + \varphi_s(\boldsymbol{x},t)\\ \dot{\phi}(\boldsymbol{x}, t) &= \bar{\pi}(\boldsymbol{x},t) + \pi_s(\boldsymbol{x},t).\end{aligned} \tag{4.1}$$

Where the short-wavelength part is defined as

$$\begin{aligned}\varphi_s(\boldsymbol{x},t) &= \int \frac{d^3k}{(2\pi)^{3/2}} \theta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{\varphi}_{\boldsymbol{k}}(t) \\ \pi_s(\boldsymbol{x},t) &= \int \frac{d^3k}{(2\pi)^{3/2}} \theta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\tilde{\varphi}}_{\boldsymbol{k}}(t) \\ \tilde{\varphi}_{\boldsymbol{k}}(t) &= \varphi_k(t) a_{\boldsymbol{k}} + \varphi_k^*(t) a_{-\boldsymbol{k}}^\dagger, \end{aligned} \tag{4.2}$$

with $\theta(z)$ being the Heaviside step function and $\sigma < 1$ a constant parameter defining the splitting scale. Therefore, the coarse-grained part contains all the wavelengths larger than the Horizon size i.e. $k < \sigma aH < aH$. Furthermore, the mode functions $\varphi_{\boldsymbol{k}}(t)$ satisfies the equation of motion (3.23) with $V''(\bar{\varphi}) = m^2$ [51].

Plugging eq. (4.1) into the equation of motion (3.20) and expanding to first order in φ_s gives

$$\left(\frac{\partial^2}{\partial t^2} + 3H \frac{\partial}{\partial t} - \frac{\nabla^2}{a^2} \right) \phi(\boldsymbol{x},t) + V'(\bar{\varphi}) + V''(\bar{\varphi}) \varphi_s(\boldsymbol{x},t) = 0. \tag{4.3}$$

Together with eqs. (4.1) and (4.2) we can write $\dot{\bar{\varphi}}$ as

$$\begin{aligned}\dot{\bar{\varphi}}(\boldsymbol{x},t) =& \bar{\pi}(\boldsymbol{x},t) + \int \frac{d^3k}{(2\pi)^{3/2}} \theta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\tilde{\varphi}}_{\boldsymbol{k}}(t) \\ &- \int \frac{d^3k}{(2\pi)^{3/2}} \theta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\tilde{\varphi}}_{\boldsymbol{k}}(t) \\ &+ \sigma a(t) H^2 \int \frac{d^3k}{(2\pi)^{3/2}} \delta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{\varphi}_{\boldsymbol{k}}(t) \\ =& \bar{\pi}(\boldsymbol{x},t) + \sigma a(t) H^2 \int \frac{d^3k}{(2\pi)^{3/2}} \delta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{\varphi}_{\boldsymbol{k}}(t). \end{aligned} \tag{4.4}$$

Expression for $\dot{\bar{\pi}}$ is obtained by plugging the definition (4.2) into eq. (4.3), yielding

$$\begin{aligned}\dot{\bar{\pi}}(\boldsymbol{x},t) =& -3H\bar{\pi}(\boldsymbol{x},t) + \frac{\nabla^2}{a^2} \bar{\varphi}(\boldsymbol{x},t) - V'(\bar{\varphi}) \\ &+ \sigma a(t) H^2 \int \frac{d^3k}{(2\pi)^{3/2}} \delta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \dot{\tilde{\varphi}}_{\boldsymbol{k}}(t) \\ &- \int \frac{d^3k}{(2\pi)^{3/2}} \theta(k - \sigma a(t)H) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \Big(\frac{\partial^2}{\partial t^2} + 3H \frac{\partial}{\partial t} + \frac{k^2}{a^2} + V''(\bar{\varphi}) \Big) \tilde{\varphi}_{\boldsymbol{k}}(t), \end{aligned} \tag{4.5}$$

where the last line vanishes according to eq. (3.23).

Hence we get following coupled equations for the long-wavelength modes

$$\begin{aligned}\dot{\bar{\varphi}} &= \bar{\pi} + g \\ \dot{\bar{\pi}} &= -3H\bar{\pi} + \frac{\nabla^2}{a^2}\bar{\varphi} - V'(\bar{\varphi}) + h.\end{aligned} \tag{4.6}$$

where g and h are defined by

$$\begin{aligned}g(\boldsymbol{x},t) &\equiv \sigma a(t)H^2 \int \frac{d^3k}{(2\pi)^{3/2}}\delta(k - \sigma a(t)H)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\tilde{\varphi}_{\boldsymbol{k}}(t) \\ h(\boldsymbol{x},t) &\equiv \sigma a(t)H^2 \int \frac{d^3k}{(2\pi)^{3/2}}\delta(k - \sigma a(t)H)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\dot{\tilde{\varphi}}_{\boldsymbol{k}}(t).\end{aligned} \tag{4.7}$$

Given that g and h are quantum operators, it follows that both φ and π also have a quantum nature. However, as we will see later in this section, if the splitting scale is set much larger than the Hubble horizon, the quantum nature becomes negligible and we obtain essentially classical Langevin equation for the long-wavelength field.

Since the mode functions φ_k can be considered as Gaussian to a good approximation [32], the noise terms g and h are also Gaussian and their properties are fully specified by their averages $\langle g \rangle = \langle h \rangle = 0$ and their two-point correlation functions.

As shown in appendix A, the two-point functions for the noise terms are given by

$$\begin{aligned}\langle 0|\, g(\boldsymbol{x},t)g(\boldsymbol{x}',t')\,|0\rangle &\simeq \sigma^{2m^2/3H^2}\frac{H^3}{4\pi^2}\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x}'|) \\ \langle 0|\, h(\boldsymbol{x},t)h(\boldsymbol{x}',t')\,|0\rangle &\simeq \sigma^{2m^2/3H^2}\frac{H^5}{4\pi^2}\left(\frac{m^2}{3H^2}+\sigma^2\right)^2\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x}'|) \\ \langle 0|\, g(\boldsymbol{x},t)h(\boldsymbol{x}',t')\,|0\rangle &\simeq -\sigma^{2m^2/3H^2}\frac{H^4}{4\pi^2}\left(\frac{m^2}{3H^2}+\sigma^2\right)\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x}'|).\end{aligned} \tag{4.8}$$

Commutation relations for $g(x)$ and $h(x)$ can be found by imposing commutation relations for creation and annihilation operators eq. (3.12)

$$[g(x),g(x')] = [h(x),h(x')] = 0. \tag{4.9}$$

With $[g(x),h(x')]$ we notice that it is proportional to the commutator between $\tilde{\varphi}_{\boldsymbol{k}}(t)$

and $\dot{\tilde{\varphi}}_{\boldsymbol{k'}}(t')$ for which we have

$$\begin{aligned}[\tilde{\varphi}_{\boldsymbol{k}}(t),\dot{\tilde{\varphi}}_{\boldsymbol{k'}}(t')] =&[(\varphi_k(t)a_{\boldsymbol{k}} + \varphi_k^*(t)a_{-\boldsymbol{k}}^\dagger),(\dot{\varphi}_{k'}(t')a_{\boldsymbol{k'}} + \dot{\varphi}_{k'}^*(t')a_{-\boldsymbol{k'}}^\dagger)] \\ =&\varphi_k(t)\dot{\varphi}_{k'}^*[a_{\boldsymbol{k}},a_{-\boldsymbol{k'}}^\dagger] + \varphi_k^*(t)\dot{\varphi}_{k'}[a_{-\boldsymbol{k}}^\dagger,a_{\boldsymbol{k'}}] \\ =&\left(\varphi_k(t)\dot{\varphi}_{k'}^* - \varphi_k^*(t)\dot{\varphi}_{k'}\right)\delta^3(\boldsymbol{k}+\boldsymbol{k'}). \end{aligned} \tag{4.10}$$

Then recalling that $\chi = a\varphi$ and using the normalization condition $\chi_k\chi_k'^* - \chi^*\chi_k' = i$ we get

$$[g(x),h(x')] =i\sigma^3\frac{H^4}{2\pi^2}\delta(t-t')j_0(\sigma aH|\boldsymbol{x}-\boldsymbol{x'}|). \tag{4.11}$$

If we choose σ, which again determines the splitting scale for the coarse-grained field, so that it falls within a range

$$\exp(-\frac{3H^2}{|m^2|}) \ll \sigma^2 \ll \frac{|m^2|}{3H^2}, \tag{4.12}$$

the explicit σ-dependencies in the two-point functions (4.8) vanishes. In addition, we find that $[g(x),h(x')] \sim 0$ and thus the quantum nature of g and h becomes negligible, and

$$h = -\frac{m^2}{3H}g. \tag{4.13}$$

Then, by plugging this into eq. (4.6), we arrive at the classical coupled Langevin equations

$$\begin{aligned}\dot{\bar{\varphi}} =&\bar{\pi} + g \\ \dot{\bar{\pi}} =& -3H\bar{\pi} - V'(\bar{\varphi}) - \frac{m^2}{3H}g \end{aligned} \tag{4.14}$$

where the gradient term has been neglected since it is σ-suppressed for the long-wavelength modes

$$\frac{k^2}{a^2} < \sigma^2H^2, \text{ when } k < \sigma aH. \tag{4.15}$$

The stochastic noise term g is fully characterized by its two-point function

$$\langle 0| \, g(x)g(x') \, |0\rangle = \frac{H^3}{4\pi^2}\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x'}|). \tag{4.16}$$

From the $\delta(t-t')$ term in the two-point function we can see that this stochastic process has no memory. That means the future of this process depends only on its current state and has no dependence on its past. The process is thus Markovian.

Then in the slow-roll approximation introduced in section 2.2 where $|\ddot{\phi}| \ll |3H\dot{\phi}|$, we have

$$3H\bar{\pi} = -V'(\bar{\varphi}) \tag{4.17}$$

and the Langevin equations (4.14) takes the final form

$$\dot{\bar{\varphi}} = -\frac{1}{3H}V'(\bar{\varphi}) + g. \tag{4.18}$$

Although we are treating $\dot{\bar{\varphi}}$ as a classical stochastic quantity it should be noted that the stochastic noise g arises from initially small-scale quantum fluctuations, which as a result of inflation, are stretched over the horizon eventually becoming a part of the coarse-grained superhorizon field.

For now on we will omit the bar from long-wavelength field so that $\varphi = \bar{\varphi}$.

4.2 Fokker-Planck equation for a scalar field

In appendix B it is shown, that for Langevin equation of the form

$$\begin{aligned} \dot{y} &= A(y) + f(t), \\ \langle f(t)f(t')\rangle &= \Gamma\delta(t-t') \end{aligned} \tag{4.19}$$

the corresponding Fokker-Planck equation is found as

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left(A(y)P(y,t)\right) + \frac{\Gamma}{2}\frac{\partial^2}{\partial y^2}P(y,t). \tag{4.20}$$

By noticing that eq. (4.18) has the same form as eq. (4.19) and that for the coarse-grained field φ, the Γ-term in eq. (4.20) is given by the correlator eq. (4.16) at $\boldsymbol{x} = \boldsymbol{x}'$. We find the Fokker-Planck equation for the PDF $P(\varphi,t)$, as

$$\frac{\partial P(\varphi,t)}{\partial t} = \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(V'(\varphi)P(\varphi,t)\right) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}P(\varphi,t). \tag{4.21}$$

4.3 Equilibrium distribution

One particularly interesting case is the equilibrium (or steady-state) distribution. As we will see later in section 5.2 in the long time limit $P(\varphi,t)$ approaches the equilibrium distribution where $P_{\text{eq}}(\varphi)$ is independent of time and hence

$$\frac{\partial}{\partial\varphi}\left[\frac{1}{3H}\left(V'(\varphi)P_{eq}(\varphi)\right) + \frac{H^3}{8\pi^2}\frac{\partial}{\partial\varphi}P_{eq}(\varphi)\right] = 0. \tag{4.22}$$

As $P_{\text{eq}}(\varphi)$ satisfies the normalization condition $\int d\varphi P_{\text{eq}}(\varphi) = 1$, $P_{\text{eq}}(\varphi)$ must decay at infinity strictly faster than φ^{-1}. Therefore, both P_{eq} and $\frac{\partial}{\partial\varphi}P_{\text{eq}}$ are 0 at infinity and thus the term in brackets in eq. (4.22) is 0 everywhere, and we get

$$\frac{1}{3H}V'(\varphi)P_{\text{eq}}(\varphi) = -\frac{H^3}{8\pi^2}\frac{\partial}{\partial\varphi}P_{\text{eq}}(\varphi). \tag{4.23}$$

It is then straightforward to find the solution for the equilibrium distribution

$$P_{\text{eq}}(\varphi) = N\exp\left[-\frac{8\pi^2}{3H^4}V(\varphi)\right] \tag{4.24}$$

where N is a constant, which is solved from the normalization condition.

Next we discuss two examples: a massive free field and a massles interacting field. These results are well known and discussed for example in refs. [52–55].

4.3.1 Massive free field

Let us consider a massive free scalar field with a potential of the form

$$V(\varphi) = \frac{1}{2}m^2\varphi^2. \tag{4.25}$$

The equilibrium distribution then takes the form

$$P_{\text{eq}}(\varphi) = N\exp\left[-\frac{4\pi^2 m^2}{3H^4}\varphi^2\right] \tag{4.26}$$

and the normalization factor N can be found by applying the condition

$$\int_{-\infty}^{\infty} d\varphi P_{\mathrm{eq}}(\varphi) = \int_{-\infty}^{\infty} d\varphi N \exp\left[-\frac{4\pi^2 m^2}{3H^4}\varphi^2\right] = 1. \tag{4.27}$$

This is a Gaussian integral of the form $\int_0^\infty dx \exp\left[-ax^2\right]$ and has the solution

$$\int_0^\infty dx\, e^{-ax^2} = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}. \tag{4.28}$$

Therefore we obtain

$$N = \sqrt{\frac{\pi}{3}}\frac{2m}{H^2} \tag{4.29}$$

and find the equilibrium distribution

$$P_{\mathrm{eq}}(\varphi) = \sqrt{\frac{\pi}{3}}\frac{2m}{H^2}\exp\left[-\frac{4\pi^2 m^2}{3H^4}\varphi^2\right]. \tag{4.30}$$

It is straightforward to calculate the moments of the field φ

$$\langle\varphi^n\rangle = \int_{-\infty}^{\infty} d\varphi\, \varphi^n P_{\mathrm{eq}}(\varphi) \tag{4.31}$$

as they also become simple Gaussian integrals.

We notice that all odd moments including the average of the field vanishes $\langle\varphi^{2n+1}\rangle = 0$.

The second moment of the field is obtained as

$$\langle\varphi^2\rangle = \frac{3H^4}{8\pi^2 m^2}, \tag{4.32}$$

this result is the same as the leading term given by the quantum field theory [52]. The average potential is given by

$$\begin{aligned}\langle V(\varphi)\rangle &= \frac{m^2}{2}\langle\varphi^2\rangle \\ &= \frac{3H^4}{16\pi^2}.\end{aligned} \tag{4.33}$$

4.3.2 Massless interacting field

As a second example we have a massles interacting case with a quartic potential

$$V(\varphi) = \frac{1}{4}\lambda\varphi^4. \tag{4.34}$$

The equilibrium distribution has the form

$$P_{\rm eq}(\varphi) = N \exp\left[-\frac{2\pi^2\lambda}{3H^4}\varphi^4\right] \tag{4.35}$$

and the normalization factor N is obtained by requiring

$$N\int_{-\infty}^{\infty} d\varphi \exp\left[-\frac{2\pi^2\lambda}{3H^4}\varphi^4\right] = 1. \tag{4.36}$$

We can rewrite the integral by substituting $x = \frac{2\pi^2\lambda}{3H^4}\varphi^4$

$$\begin{aligned}\int_{-\infty}^{\infty} d\varphi \exp\left[-\frac{2\pi^2\lambda}{3H^4}\varphi^4\right] &= \frac{1}{4}\left(\frac{3H^4}{2\pi^2\lambda}\right)^{1/4}\int_{-\infty}^{\infty} dx\, x^{-3/4}\, e^{-x} \\ &= \frac{1}{2}\left(\frac{3H^4}{2\pi^2\lambda}\right)^{1/4}\int_{0}^{\infty} dx\, x^{-3/4}\, e^{-x} \\ &= \left(\frac{3}{32\pi^2\lambda}\right)^{1/4}\Gamma(\tfrac{1}{4})H \end{aligned} \tag{4.37}$$

where on the last line we have identified the integral as the Gamma function $\Gamma(z) = \int_0^\infty dt\, t^{z-1}e^{-t}$. The equilibrium PDF then reads

$$P_{\rm eq}(\varphi) = \left(\frac{32\pi^2\lambda}{3}\right)^{1/4}\frac{1}{\Gamma(\frac{1}{4})H}\exp\left[-\frac{2\pi^2\lambda}{3H^4}\varphi^4\right]. \tag{4.38}$$

Similarly as in the previous case with the potential $V(\varphi) = \frac{1}{2}m^2\varphi^2$ all odd moments of the field vanishes. The second moment can be solved by making the same substitution as before $x = \frac{2\pi^2\lambda}{3H^4}\varphi^4$

$$\begin{aligned}\langle\varphi^2\rangle &= \sqrt{\frac{3}{2\lambda}}\frac{H^2}{\pi}\frac{1}{\Gamma(\frac{1}{4})}\int_0^\infty dx\, x^{-1/4}\, e^{-x} \\ &= \sqrt{\frac{3}{2\lambda}}\frac{H^2}{\pi}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \approx 0.132\frac{H^2}{\sqrt{\lambda}}. \end{aligned} \tag{4.39}$$

In a similar manner we find the fourth moment

$$\begin{aligned}\langle\varphi^4\rangle &= \frac{3H^4}{2\pi^2\lambda}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})} \\ &= \frac{3H^4}{8\pi^2\lambda}\end{aligned} \tag{4.40}$$

where we have used the relation $\Gamma(\frac{5}{4}) = \Gamma(\frac{1}{4})/4$. The equilibrium distribution is clearly non-Gaussian and Wick's theorem for Gaussian distribution $\langle\varphi^4\rangle = 3\langle\varphi^2\rangle^2$ does not hold. The result for $\langle\varphi^2\rangle$ can be compared to the QFT result in the Hartee-Fock approximation yielding $\langle\varphi^2\rangle \sim H^2/\pi\sqrt{8\lambda} \approx 0.113H^2/\sqrt{\lambda}$ [53, 54]. The Hartee-Fock result differs $\sim 14.6\%$ from our stochastic solution.

Again, the average potential is found as

$$\begin{aligned}\langle V(\varphi)\rangle &= \frac{\lambda}{4}\langle\varphi^4\rangle \\ &= \frac{3H^4}{32\pi^2}.\end{aligned} \tag{4.41}$$

5 Spectral expansion method

In stochastic formalism, the two-point correlation functions of any function $f(\varphi)$ of a light scalar field can be expressed as a spectral expansion. The coefficients in this expansion are specified by eigenvalues and eigenfunctions obtained from a Schrödinger-like equation. Here we will discuss the method for obtaining the mentioned eigenvalue equation as well as the spectral expansion.

As we will later see, in the equilibrium state the general two-point function $G_f(\boldsymbol{x}_1,\boldsymbol{x}_2;t_1,t_2)$ in de Sitter space can be written in terms of the temporal correlator $G_f(t)$ [54]. The temporal correlation function gives the correlations at equal spatial points i.e. $\boldsymbol{x_1} = \boldsymbol{x_2}$. Therefore, it is enough to find an expression for the temporal correlation function given by

$$\begin{aligned} G_f(t_1,t_2) &= \langle f(\varphi_1(t_1))f(\varphi_2(t_2))\rangle \\ &= \int d\varphi_1 d\varphi_2 f(\varphi_1(t_1))f(\varphi_2(t_2))P(\varphi_1,t_1;\varphi_2,t_2), \end{aligned} \tag{5.1}$$

where $P(\varphi_1,t_1;\varphi_2,t_2)$ is the two-point (or joint) PDF.

We start this chapter by presenting the one-point probability distribution, which allows us to calculate the expectation value for any function of the scalar field φ. Subsequently, in section 5.2 we introduce the conditional probability distribution at equal spatial points, denoted as $P(\varphi_2, t_2|\varphi_1, t_1)$. This conditional PDF expresses the probability for the field to take the value φ_2 at t_2 provided that at time t_1 it has a value φ_1.

Following the definition for conditional PDF, the one-point distribution, together with the conditional distribution, allows us to obtain the two-point PDF $P(\varphi_1,t_1;\varphi_2,t_2)$ as

$$\begin{aligned} P(\varphi_1,t_1;\varphi_2,t_2) =&P(\varphi_2,t_2|\varphi_1,t_1)P(\varphi_1,t_1)\theta(t_2-t_1) \\ &+ P(\varphi_1,t_1|\varphi_2,t_2)P(\varphi_2,t_2)\theta(t_1-t_2). \end{aligned} \tag{5.2}$$

Additionally, we can expresse the one-point PDF $P(\varphi_2, t_2)$, for $t_2 > t_1$ as

$$P(\varphi_2, t_2) = \int d\varphi_1 P(\varphi_2, t_2|\varphi_1, t_1)P(\varphi_1,t_1). \tag{5.3}$$

Finally, in section 5.3.2 we introduce the spectral expansion for the two-point correlation functions.

5.1 One-point probability distribution

In section 4.3 we found one special case for the one-point PDF, where the distribution was independent of time. However, in order to calculate the two-point correlators, it is necessary to find the general time-dependent solution. This general solution also allows us to calculate the expectation value for any function $f(\varphi)$ of the field as

$$\langle f(\varphi)\rangle = \int d\varphi f(\varphi)P(\varphi,t). \tag{5.4}$$

Here we follow the approach by Starobinsky and Yokoyama [54], similar calculations can be found in [55].

We begin by finding a self-adjoint form of the Fokker-planck equation. In order to achieve that we define

$$\tilde{P}(\varphi,t) = e^{\frac{4\pi^2 V(\varphi)}{3H^4}} P(\varphi,t). \tag{5.5}$$

Computing the partial derivatives $\frac{\partial}{\partial\varphi}$ and $\frac{\partial^2}{\partial\varphi^2}$ for $P(\varphi,t)$ gives

$$\begin{aligned}
\frac{\partial P(\varphi,t)}{\partial\varphi} =& e^{-\frac{4\pi^2 V(\varphi)}{3H^4}} \left(\frac{\partial}{\partial\varphi} - \frac{4\pi^2}{3H^4}V'(\varphi)\right)\tilde{P}(\varphi,t)\\
\frac{\partial^2 P(\varphi,t)}{\partial\varphi^2} =& e^{-\frac{4\pi^2 V(\varphi)}{3H^4}} \left(\frac{\partial^2}{\partial\varphi^2} - \frac{8\pi^2}{3H^4}V'(\varphi)\frac{\partial}{\partial\varphi}\right.\\
&\left. - \frac{4\pi^2}{3H^4}V''(\varphi) + \left(\frac{4\pi^2}{3H^4}V'(\varphi)\right)^2\right)\tilde{P}(\varphi,t)
\end{aligned} \tag{5.6}$$

and by plugging them into the Fokker-Planck equation for $P(\varphi, t)$ eq. (4.21) we obtain

$$\frac{\partial\tilde{P}(\varphi,t)}{\partial t} = \left[\frac{1}{2}\frac{1}{3H}\left(V''(\varphi) - \frac{4\pi^2}{3H^4}V'(\varphi)^2\right) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\right]\tilde{P}(\varphi,t)$$

Equation (5.7) is a linear partial differential equation, and it can be solved by the separation of variables method. We write an ansatz for independent solutions $\tilde{P}_n(\varphi,t)$ as a product of functions of one variable, that is

$$\tilde{P}_n(\varphi,t) = T_n(t)\psi_n(\varphi). \tag{5.7}$$

Then solving eq. (5.7) for $T_n(t)\psi_n(\varphi)$ gives

$$\frac{T_n'(t)}{T_n(t)} = \frac{H^3}{8\pi^2}\frac{\psi_n''(\varphi)}{\psi_n(\varphi)} + \frac{1}{2}\frac{1}{3H}\left(V''(\varphi) - \frac{4\pi^2}{3H^4}V'(\varphi)^2\right) \tag{5.8}$$

Since the left side depends only on t and the right sides depends only on φ both sides must be equal to some constant, which we denote $-\Lambda_n$. The partial differential equation can thus be split in two ordinary differential equations. Time-dependent part is then solved as

$$\begin{aligned} T_n'(t) &= -\Lambda_n T_n(t) \\ T_n(t) &= Ce^{-\Lambda_n t} \end{aligned} \tag{5.9}$$

with some constant C. And $\psi_n(\varphi)$ are found as solutions for the following Schrödinger-type eigenvalue equation

$$D_\varphi \psi_n(\varphi) = -\frac{4\pi^2\Lambda_n}{H^3}\psi_n(\varphi). \tag{5.10}$$

Here we have defined the differential operator D_φ as

$$\begin{aligned} D_\varphi &\equiv \frac{1}{2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{2}\left(v'(\varphi)^2 - v''(\varphi)\right) \\ v(\varphi) &= \frac{4\pi^2}{3H^4}V(\varphi). \end{aligned} \tag{5.11}$$

This operator can be recasted as

$$-D_\varphi\psi_n(\varphi) = \frac{1}{2}\left(-\frac{\partial}{\partial\varphi} + v'(\varphi)\right)\left(-\frac{\partial}{\partial\varphi} + v'(\varphi)\right)^\dagger \psi_n(\varphi), \tag{5.12}$$

from which we can see that it is clearly self-adjoint, meaning $D_\varphi = D_\varphi^\dagger$, thus all eigenvalues Λ_n must be non-negative. And the eigenfunctions ψ_n form an orthonormal and complete set, that is

$$\int d\varphi \psi_n(\varphi)\psi_m(\varphi) = \delta_{n,m} \tag{5.13}$$

$$\sum_n \psi_n(\varphi)\psi_n(\varphi') = \delta(\varphi - \varphi'). \tag{5.14}$$

The lowest eigenfunction $\psi_0(\varphi)$ is obtained from the eigenvalue equation (5.10) by setting $\Lambda_0 = 0$ and using the definition for D_φ eq. (5.11). It follows that

$$\frac{\partial^2 \psi_0(\varphi)}{\partial \varphi^2} = (v'(\varphi)^2 - v''(\varphi))\psi_0(\varphi) \tag{5.15}$$

$$\psi_0(\varphi) = A \exp\left[-\frac{4\pi^2}{3H^4} V(\varphi)\right]. \tag{5.16}$$

Comparing this to the equilibrium solution eq. (4.24) together with the normalization conditions for both P_{eq} and ψ_n, we find that

$$P_{\text{eq}}(\varphi) = \psi_0^2(\varphi). \tag{5.17}$$

Furthermore, the solution for $P(\varphi,t) = e^{-\frac{4\pi^2 V(\varphi)}{3H^4}} \tilde{P}(\varphi,t)$ can be written as a sum of independent solutions as

$$P(\varphi,t) = \psi_0(\varphi) \sum_{n=0}^{\infty} a_n e^{-\Lambda_n t} \psi_n(\varphi), \tag{5.18}$$

where a_n are constant coefficients and will be solved in the next section together with the two-point PDF.

5.2 Conditional and two-point probability distributions

Now we turn to the conditional PDF at equal spatial points, which, together with the one-point PDF, allows us to write the two-point distribution in eq. (5.2).

By definition, the conditional PDF satisfies the Fokker-Planck equation

$$\frac{\partial P(\varphi_2, t_2|\varphi_1, t_1)}{\partial t_2} = \frac{1}{3H} \frac{\partial}{\partial \varphi_2} \left(V'(\varphi_2) P(\varphi_2, t_2|\varphi_1, t_1)\right) + \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \varphi_2^2} P(\varphi_2, t_2|\varphi_1, t_1). \tag{5.19}$$

By defining

$$\begin{aligned} P(\varphi_2, t_2|\varphi_1, t_1) \equiv & e^{-\frac{4\pi^2}{3H^4}V(\varphi_2)}\tilde{P}(\varphi_2, t_2|\varphi_1, t_1)e^{\frac{4\pi^2}{3H^4}V(\varphi_1)} \\ = & \psi_0(\varphi_2)\tilde{P}(\varphi_2, t_2|\varphi_1, t_1)\psi_0^{-1}(\varphi_1), \end{aligned} \tag{5.20}$$

we can solve for $\tilde{P}(\varphi_2, t_2|\varphi_1, t_1)$ by following similar steps as with the one-point PDF in the previous section, and find

$$\tilde{P}(\varphi_2, t_2|\varphi_1, t_1) = \sum_{n=0}^{\infty} C_n e^{-\Lambda_n t_2} T_n(t_1)\psi_n(\varphi_2)\psi_n(\varphi_1), \tag{5.21}$$

where C_n are constants.

We can find $T_n(t_1)$ by using the initial condition

$$P(\varphi_2, t|\varphi_1, t) = \tilde{P}(\varphi_2, t|\varphi_1, t) = \delta(\varphi_2 - \varphi_1) \tag{5.22}$$

together with the completeness relation for ψ_n eq. (5.14). Thus we get

$$\tilde{P}(\varphi_2, t_2|\varphi_1, t_1) = \sum_{n=0}^{\infty} e^{-\Lambda_n \Delta t}\psi_n(\varphi_2)\psi_n(\varphi_1), \tag{5.23}$$

with $\Delta t = t_2 - t_1$.

Recalling eq. (5.3), we can now find an expression for the one-point PDF $P(\varphi,t)$. Substituting eqs. (5.20) and (5.23) into eq. (5.3) with $\varphi_2,t_2 \to \varphi, t$ and $\varphi_1,t_1 \to \varphi_0, t_0 = 0$, gives

$$\begin{aligned} P(\varphi, t) = & \int d\varphi_0 P(\varphi, t|\varphi_0, 0)P(\varphi_0,0) \\ = & \int d\varphi_0\, e^{-\frac{4\pi^2}{3H^4}V(\varphi)} \sum_n e^{-\Lambda_n t}\psi_n(\varphi)\psi_n(\varphi_0)e^{\frac{4\pi^2}{3H^4}V(\varphi_0)} P_{\text{eq}}(\varphi_0) \\ = & \psi_0^2(\varphi) + \psi_0(\varphi)\sum_{n=1}^{\infty} a_n e^{-\Lambda_n t}\psi_n(\varphi) \end{aligned} \tag{5.24}$$

where between the second and the third line we have used eqs. (5.17), (5.16) and (5.13). And the coefficients a_n are defined as

$$a_n \equiv \int d\varphi_0 \psi_n(\varphi_0)\psi_0(\varphi_0). \tag{5.25}$$

Moreover, since $\psi_0^2(\varphi) = P_{\mathrm{eq}}(\varphi)$, we find that $P(\varphi,t)$ asymptotically approaches the equilibrium solution.

By plugging eqs. (5.20) and (5.23) in eq. (5.2) we finally obtain the equilibrium solution for the two-point PDF

$$P(\varphi,t;\varphi_0,0) = \psi_0(\varphi)\sum_{n=0}^{\infty} e^{-\Lambda_n t}\psi_n(\varphi)\psi_n(\varphi_0)\psi_0(\varphi_0). \tag{5.26}$$

5.3 Two-point correlation functions

As mentioned in the beginning of this chapter, in the equilibrium state the general two-point correlation function $G_f(\boldsymbol{x}_1,\boldsymbol{x}_2;t_1,t_2)$ can be written in terms of the temporal correlator $G_f(t)$, as shown in ref. [54], by using the de Sitter invariant function

$$y = \cosh H(t_1 - t_2) - \frac{H^2}{2}e^{H(t_1+t_2)}|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2. \tag{5.27}$$

This function is connected to the geodesic interval s with $y = 1 + s^2H^2/2$ [54]. For large space-like and time-like separations, that is $|y| \gg 1$, the general two-point function can be written as [54]

$$G_f(\boldsymbol{x}_1,\boldsymbol{x}_2;t_1,t_2) = G_f(H^{-1}\ln|2y-1|). \tag{5.28}$$

It should be noted that this relation holds only in de Sitter space.

We therefore start by introducing the temporal correlation function. Then we write the two-point correlator for spatially separated points by using the relation between general correlation function and temporal correlation function.

5.3.1 Temporal correlation function

In the beginning of this chapter, we introduced the temporal two-point correlation function for an arbitrary function $f(\varphi)$ given by eq. (5.1). In equilibrium this correlator is obtained by plugging in the equilibrium two-point PDF eq. (5.26). We get

$$\begin{aligned} G_f(t) &= \int d\varphi d\varphi_0 f(\varphi) f(\varphi_0) P(\varphi,t;\varphi_0,0) \\ &= \sum_{n=0}^{\infty} \int d\varphi d\varphi_0 \psi_0(\varphi) f(\varphi) \psi_n(\varphi) e^{-\Lambda_n t} \psi_n(\varphi_0) f(\varphi_0) \psi_0(\varphi_0). \end{aligned} \tag{5.29}$$

By defining

$$f_n \equiv \int d\varphi \psi_0(\varphi) f(\varphi) \psi_n(\varphi) \tag{5.30}$$

we can rewrite eq. (5.29) as a spectral expansion

$$G_f(t) = \sum_n f_n^2 e^{-\Lambda_n t}. \tag{5.31}$$

In the case of an even potential, that is $V(-\varphi) = V(\varphi)$, all eigenfunctions $\psi_n(\varphi)$ are either even or odd. Since $\psi_0(\varphi)$ is even, ψ_n is even/odd for even/odd n respectively. From the definition of the spectral coefficient f_n one can see that for even f only the even eigenvalues Λ_n contributes to the two-point correlator. Similarly for odd f only the odd eigenvalues contributes.

5.3.2 Spatial correlation function

The spatial correlator is then obtained as

$$\begin{aligned} G_f(\boldsymbol{x};0) =& G_f\left(\frac{2}{H}\ln\left(|\boldsymbol{x}|aH\right)\right) \\ =& \sum_n f_n^2 e^{-\Lambda_n \frac{2}{H}\ln|\boldsymbol{x}|aH} \\ =& \sum_n f_n^2 \left(|\boldsymbol{x}|aH\right)^{-2\Lambda_n/H}. \end{aligned} \tag{5.32}$$

Spatial correlations are typically described by the power spectrum $\mathcal{P}_f(k)$. It is defined as a Fourier transform of the spatial correlation function

$$\mathcal{P}_f(k) = \frac{k^3}{2\pi^2}\int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} G_f(\boldsymbol{x};0). \tag{5.33}$$

Inserting (5.32) into (5.33) gives

$$\begin{aligned}\mathcal{P}_f(k) =& \sum_n \frac{k^3}{2\pi^2} \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f_n^2 \left(|\boldsymbol{x}|aH\right)^{-2\Lambda_n/H} \\ =& \sum_n \frac{2}{\pi} f_n^2 \,\Gamma\left(2 - \frac{2\Lambda_n}{H}\right) \sin\left(\frac{\pi\Lambda_n}{H}\right) \left(\frac{k}{aH}\right)^{2\Lambda_n/H} .\end{aligned} \tag{5.34}$$

If the spatial correlator is dominated by a single n=d term the asymptotic behaviour is given by a simple power-law form

$$G_f(\boldsymbol{x}; 0) \sim f_d^2 \left(|\boldsymbol{x}|aH\right)^{-2\Lambda_d/H} , \tag{5.35}$$

and the power spectrum is given by

$$\mathcal{P}_f(k) \sim f_d^2 \left(\frac{k}{aH}\right)^{2\Lambda_d/H} . \tag{5.36}$$

Scale dependence of the power spectrum is characterized by spectral index $n_f - 1$

$$n_f - 1 \equiv \frac{d \ln \mathcal{P}_f(k)}{d \ln k}. \tag{5.37}$$

$n_f - 1 = 0$ implying perfect scale invariant spectrum, while $n_f - 1 > 0$ is called blue-tilted spectrum and $n_f - 1 < 0$ red-tilted spectrum. If the spectrtal index is dominated by a single n=d term, $n_f - 1$ takes a constant value

$$n_f - 1 = 2\Lambda_n/H. \tag{5.38}$$

On the other hand, if the spectral-index changes with k we say it is running and define the running of the spectral index as

$$\alpha_f(k) = \frac{dn_f(k)}{d \ln k}. \tag{5.39}$$

5.4 Case of a massive non-interacting scalar field

As an example, we consider a quadratic potential

$$V = \frac{1}{2} m^2 \varphi^2 \tag{5.40}$$

for which the eigenvalue equation (5.10) takes the form

$$\left[\frac{\partial^2}{\partial \varphi^2} - \left(\frac{4\pi^2}{3H^4} \right)^2 m^4 \varphi^2 + \frac{4\pi^2}{3H^4} m^2 \right] \psi_n(\varphi) = -\frac{8\pi^2 \Lambda_n}{H^3} \psi_n(\varphi). \tag{5.41}$$

In this particular case, the eigenvalues and eigenfunctions can be solved analytically. By defining dimensionless parameters

$$x \equiv \frac{2\pi m}{\sqrt{3} H^2} \varphi\,, \quad \tilde{\Lambda}_n \equiv \frac{6H}{m^2} \Lambda_n + 1, \tag{5.42}$$

we can rewrite the eigenvalue equation (5.41) as

$$\frac{\partial^2}{\partial x^2} \psi_n(x) = \left(x^2 - \tilde{\Lambda}_n \right) \psi_n(x). \tag{5.43}$$

This differential equation can be solved with a power series method, which is well known and discussed for example in ref. [56]. We make an ansatz in a form of a power series

$$\begin{aligned} \psi_n(x) &= h(x) e^{-x^2/2} \\ h(x) &= \sum_{j=0}^{\infty} a_j x^j \end{aligned} \tag{5.44}$$

Substituting this ansatz into eq. (5.43) then gives

$$\sum_{j=0}^{\infty} \left((j+2)(j+1) a_{j+2} - (2j + 1 - \tilde{\Lambda}_n) a_j \right) x^j = 0. \tag{5.45}$$

From which we can obtain the following recursion formula for the coefficients a_j

$$a_{j+2} = \frac{2j + 1 - \tilde{\Lambda}_n}{(j+1)(j+2)} a_j. \tag{5.46}$$

Since all even coefficients are generated by a_0 and all odd coefficients are generated by a_1, the series h can be separated into even and odd solutions.

We require the solutions $\psi_n(x) = e^{-x^2/2}h(x)$ to be normalizable, that is $\psi_n(x) \to 0$ as $x \to \infty$. To check this requirement we look at the behaviour of $h(x)$ for large j. In this limit, the recursion formula can be approximated as

$$a_{j+2} \approx \frac{2}{j} a_j \tag{5.47}$$

and by comparing two succesive terms we find a ratio

$$\frac{a_{j+2}x^{j+2}}{a_j x^j} \sim \frac{2}{j} x^2. \tag{5.48}$$

On the other hand, the series expansion for e^{x^2} reads

$$e^{x^2} = \sum_{j=even}^{\infty} \frac{x^j}{(j/2)!}. \tag{5.49}$$

Again, with $j \gg 1$ the ratio between two succesive terms can be approximated as

$$\frac{x^{j+2}(j/2)!}{x^j((j+2)/2)!} \sim \frac{2}{j} x^2. \tag{5.50}$$

Therefore, in a large j limit the behaviour of $h(x)$ is asymptotic to e^{x^2} yielding diverging and thus not normalizable result $\psi \sim e^{x^2/2}$.

In order to find normalizable solutions, the power series must terminate for some $j = n$. This can be achieved by choosing

$$\tilde{\Lambda}_n = 2n + 1. \tag{5.51}$$

After transforming back from $\tilde{\Lambda}_n$ to Λ_n in eq. (5.42), we obtain the eigenvalues

$$\Lambda_n = \frac{m^2 n}{3H}. \tag{5.52}$$

Moreover, the power series $h(x)$ can be written in terms of Hermite polynomials, given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{5.53}$$

Plugging this to eq. (5.44) and using the normalization eq. (5.13) yields the normalized eigenfunctions

$$\psi_n(\varphi) = \frac{\sqrt{m}}{H}\left(\frac{4\pi}{3}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{2\pi m}{\sqrt{3}H^2}\varphi\right)\exp\left[-\frac{2\pi^2 m^2 \varphi^2}{3H^4}\right]. \tag{5.54}$$

6 Numerical analysis of a massless self-interacting field

In the previous chapter we discussed probability distribution functions of a light scalar field and derived the spectral expansion for two-point correlators. This method allows one to straightforwardly calculate the two-point correlators for any function of the scalar field. Moreover, the computation of the two-point correlators is essentially reduced into finding eigenvalues and eigenfunctions from a Scröhdinger-like equation.

In this chapter, we will apply the spectral expansion method and calculate spatial two-point correlators eq. (5.32) of a self-interacting spectator scalar field. We recall that the spectator field is light $V''(\varphi) \ll H^2$ and energetically subdominant $V < 3H^2M_p^2$ to inflaton field.

We consider a scalar field with a qualitatively similar potential to the SM Higgs potential in the large field limit, that is [53]

$$V(\varphi) = \frac{1}{4}\lambda(\varphi)\varphi^4. \tag{6.1}$$

In Higgs potential, the coupling $\lambda(\varphi)$ is a renormalization group improved running coupling [53]. We parametrize the running phenomenologically with an expansion

$$\lambda(\varphi) = \lambda_0 \left(1 + \beta_1 \ln\left(\frac{\varphi^2}{\varphi_0^2}\right) + \beta_2 \ln^2\left(\frac{\varphi^2}{\varphi_0^2}\right)\right). \tag{6.2}$$

Here φ_0, λ_0, $\beta_1 < 0$ and $\beta_2 > 0$ are arbitrary constants. The running of $\lambda(\varphi)$ influences the form of the potential leading to the formation of two metastable minima. Since $\beta_1 < 0$, the first logarithmic term causes the potential to decrease at a scale defined by φ_0 forming the metastable minima. While the second term with β_2 makes sure that the potential starts to increase again as φ increases.

We will numerically solve the eigenvalue equation (5.10) and calculate the power spectrum eq. (5.34) and the spectral index eq. (5.37) for a spatial two-point correlator $G_\varphi(\boldsymbol{x}; 0)$ of the scalar field φ.

6.1 Numerical methods

To numerically solve the eigenvalue equation (5.10) we first convert it into dimensionless form

$$\frac{\partial^2}{\partial x^2}\psi_n(x) = [U(x) - \Lambda_n]\,\psi_n(x). \tag{6.3}$$

This equation is then divided into two first-order differential equations

$$\begin{aligned}\psi_n'(x) &= \phi_n(x)\\ \phi_n'(x) &= [U(x) - \Lambda_n]\,\psi_n(x).\end{aligned} \tag{6.4}$$

We apply the boundary conditions $\psi_n(\infty) = \psi_n(-\infty) = 0$, so that the solutions are normalizable.

As discussed earlier in section 5.3.1, for a symmetric potential the eigenfunctions are either even or odd. By utilising this property, we can convert this boundary value problem into an initial value problem at $x_0 = 0$. Here, the derivative of even function vanishes and odd function changes its sign. Since the solutions will be normalized, the remaining conditions $\psi_n(0)$ for even n and $\psi_n'(0)$ for odd n, are irrelevant and we can set them equal to unity. Thus, we have

$$\begin{aligned}\psi_n(0) = 1,\ \psi_n'(0) = 0 \quad &\text{for even } n\\ \psi_n(0) = 0,\ \psi_n'(0) = 1 \quad &\text{for odd } n.\end{aligned} \tag{6.5}$$

We are left with only one unknown variable, the eigenvalue Λ_n which is then solved with the so-called shooting method. This method is well known and covered in many textbooks, for example in [56, 57].

As discussed in section 5.4 with the case of a massive free field, the eigenfunction diverges at the integration boundaries unless we use the correct eigenvalue. If we use an eigenvalue that is slightly larger than the correct one and another one that is slightly smaller, both solutions will blow up at large x, but in different directions. By bisecting this interval, we can search the eigenvalue for which the solution is closest to zero.

Keeping in mind that numerical methods come with limited accuracy, we choose the integration range $x \in [0,L]$ so that L is where the eigenfunction is closest to zero before diverging. The complete solutions $\psi_n(x)$ for $x \in [-L,L]$ are then obtained based on parity as $\psi_n(-L) = \psi_n(L)$ for even n and $\psi_n(-L) = -\psi_n(L)$ for odd n.

Since the eigenfunction for the lowest eigenvalue $\Lambda_0 = 0$ can be solved analytically with eq. (5.16). We compute the spectral coefficients f_n in eq. (5.30) by using a numerical result for ψ_n and analytical result ψ_0.

6.2 φ^4 potential

Before turning to the case of a running coupling eq. (6.2) let us first consider a simpler case with the potential

$$V(\varphi) = \frac{1}{4}\lambda\varphi^4, \tag{6.6}$$

where $\lambda = 0.013$ is constant. We convert the eigenvalue equation (5.10) into a dimensionless form by defining

$$x \equiv \frac{\varphi}{H}, \quad \tilde{\Lambda}_n \equiv \frac{8\pi^2}{H}\Lambda_n. \tag{6.7}$$

This gives

$$\frac{\partial^2}{\partial x^2}\psi_n(x) = \left[U(x) - \tilde{\Lambda}_n\right]\psi_n(x), \tag{6.8}$$

with

$$U(x) = \left(\frac{4\pi^2}{3}\right)^2 \lambda^2 x^6 - 4\pi^2\lambda x^2. \tag{6.9}$$

Our goal here is to compute the spatial two-point correlator $G_\varphi(\boldsymbol{x}; 0)$ in eq. (5.32) for the scalar field φ. First, we solve the eigenvalue equation (6.8) numerically and then compute the spectral coefficients φ_n in eq. (5.30) for the spatial two-point correlator. In addition, we compute the spectral coefficients φ_n^2 which can be used to calculate the two-point functions for φ^2.

The first four non-zero eigenvalues and the spectral coefficients φ_n^1, φ_n^2 are given in table 1 and the numerically solved eigenfunctions are presented in figure 1. These results are in agreement with the ones obtained in ref. [55].

As mentioned in the previous section, the eigenfunction for the lowest eigenvalue

$\Lambda_0 = 0$ can be solved analytically with eq. (5.16) and has the form

$$\psi_0(x) = \left(\frac{32\pi^2\lambda}{3}\right)^{1/8} \frac{1}{\sqrt{\Gamma\left(\frac{1}{4}\right) H}} \exp\left[-\frac{\pi^2\lambda}{3}x^4\right]. \tag{6.10}$$

n	Λ_n/H	$\lvert\varphi_n^1\rvert/H$	$\lvert\varphi_n^2\rvert/H^2$
1	0.01014	1.06662	0
2	0.03299	0	1.2216
3	0.06119	0.13253	0
4	0.09452	0	0.30125

Table 1. The $n \in [1,4]$ eigenvalues Λ_n and spectral coefficients $|\varphi_n^j|$ for correlators of $f(\varphi) = \varphi$ and $f(\varphi) = \varphi^2$.

The spatial correlations of the scalar field φ are described by the power spectrum eq. (5.34). Here, $\mathcal{P}_\varphi(k)$ takes the form

$$\mathcal{P}_\varphi(k) = 0.02287H^2(\frac{k}{aH})^{0.02028} + 0.00204H^2(\frac{k}{aH})^{0.12238} + ... \tag{6.11}$$

and is dominated by the $n = 1$ term at long distances ($k \ll aH$). This behaviour can be seen in fig. 2 where we have plotted the power spectrum and the spectral index truncated to $n = 1$,3 and 5. Similarly, the spectral index defined by eq. (5.37) approaches a constant value at long distances

$$n_\varphi - 1 \approx \frac{2\Lambda_1}{H} \approx 0.02028. \tag{6.12}$$

6.3 φ^4 potential with running coupling

For φ^4 potential with a running coupling $\lambda(\varphi)$ eq. (6.2), the eigenvalue equation can be written in dimensionless form eq. (6.8) with x and $\tilde{\Lambda}_n$ defined in eq. (6.7), and where the effective potential takes the form

$$\begin{aligned} U(x) = &\left(\frac{4\pi^2}{3}\right)^2 \lambda_0^2 x^6 \left[1 + \beta_1\left(\frac{1}{2} + \ln\frac{x^2}{x_0^2}\right) + \beta_2\left(\ln\frac{x^2}{x_0^2} + \ln^2\frac{x^2}{x_0^2}\right)\right]^2 \\ &- \frac{4\pi^2}{3}\lambda_0 x^2 \left[3 + \beta_1\left(3\ln\frac{x^2}{x_0^2} + \frac{7}{2}\right) + \beta_2\left(3\ln^2\frac{x^2}{x_0^2} + 7\ln\frac{x^2}{x_0^2} + 2\right)\right]. \end{aligned} \tag{6.13}$$

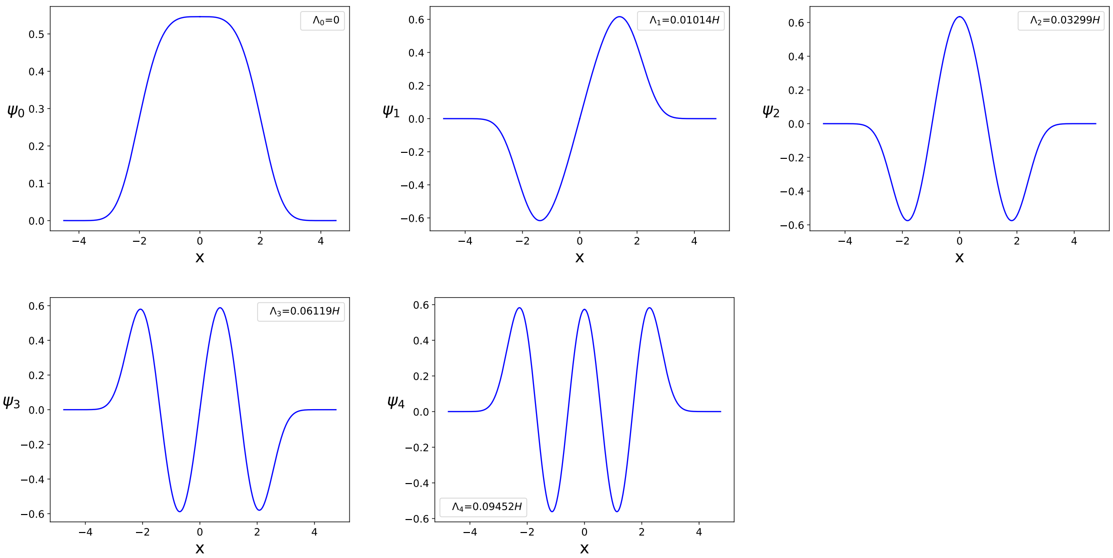

Figure 1. The first five numerically solved eigenfunctions $\psi_n(x)$ for the quartic potential (6.6).

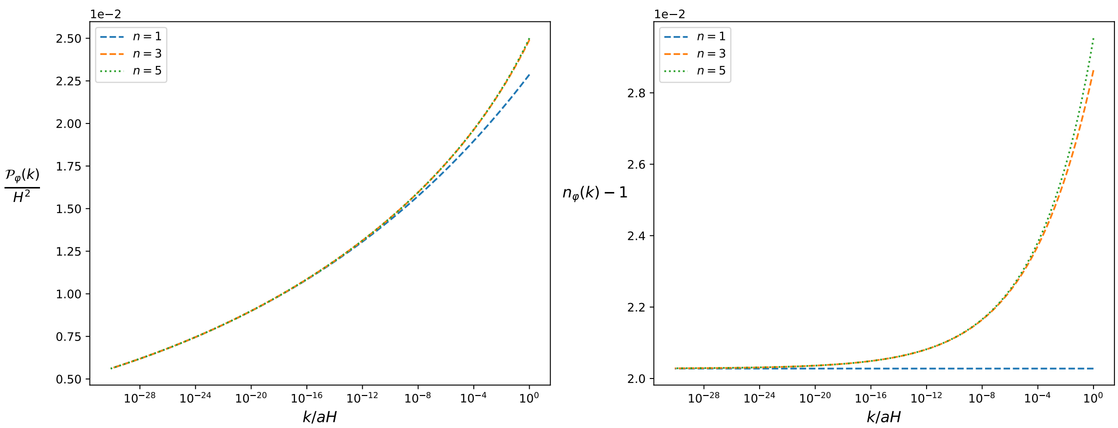

Figure 2. Power spectrum and spectral index for quartic potential (6.6) truncated to $n = 1$,3,5.

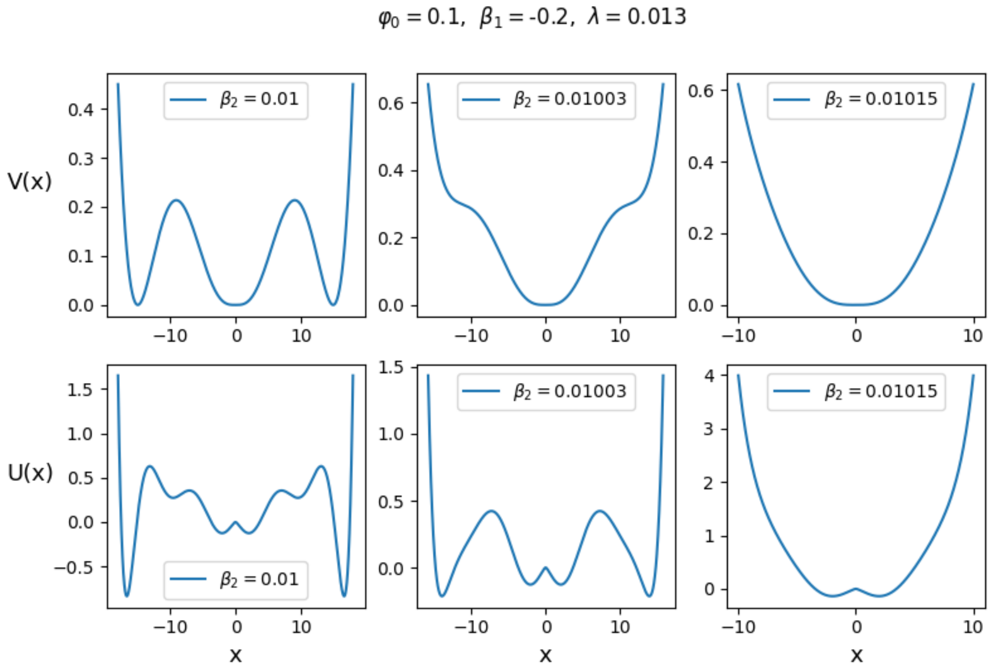

Figure 3. Potentials $V(x)$ and $U(x)$ eqs. (6.1) and (6.13) in units $H = 1$.

We fix $\lambda_0 = 0.013$, $\beta_1 = -0.2$ and $x_0 = \varphi_0/H = 0.1$ and vary β_2. These values are chosen so that the metastable minima are present in the potential $V(x)$ and therefore it resembles the Higgs potential during inflation, while the eigenvalue problem stays numerically stable. The potentials $V(x)$ and $U(x)$ are presented in fig. 3 and the behaviour of $\lambda(x)$ can be seen in fig. 4.

The $n \leq 4$ eigenfunctions are presented in fig. 5. When $\beta_2 = 0.01$ the potential $V(x)$ has three degenerate minima, and the lowest eigenfunction ψ_0 has three peaks localised in all three minima. The two peaks, located at $\varphi \neq 0$, are clearly narrower than the one at $\varphi = 0$. The behaviour of ψ_2 is quite similar to ψ_0, while ψ_1 is entirely localised at the two $\varphi \neq 0$ minima. On the other hand, the eigenfunctions ψ_3 and ψ_4 starts to resemble the $n = 1$ and $n = 2$ eigenfunctions for the $\lambda\varphi^4$ potential.

As β_2 grows, there is one stable minimum located at $\varphi = 0$ while the other two become metastable eventually disappearing completely at $\beta_2 \sim 0.01025$. Here, the two peaks of the $n = 0$ eigenfunction, which are located at $\varphi \neq 0$, get shorter with growing β_2 and disapears as $\beta_2 > 0.01003$. Meanwhile, the $n = 1$ and $n = 2$ eigenfunctions are mainly concentrated around the two metastable minima. However, $n \geq 1$ states are very short-lived, which can be seen in the temporal correlator eq. (5.31). In the upper limit of β_2, the potential $V(\varphi)$ has only one minimum located

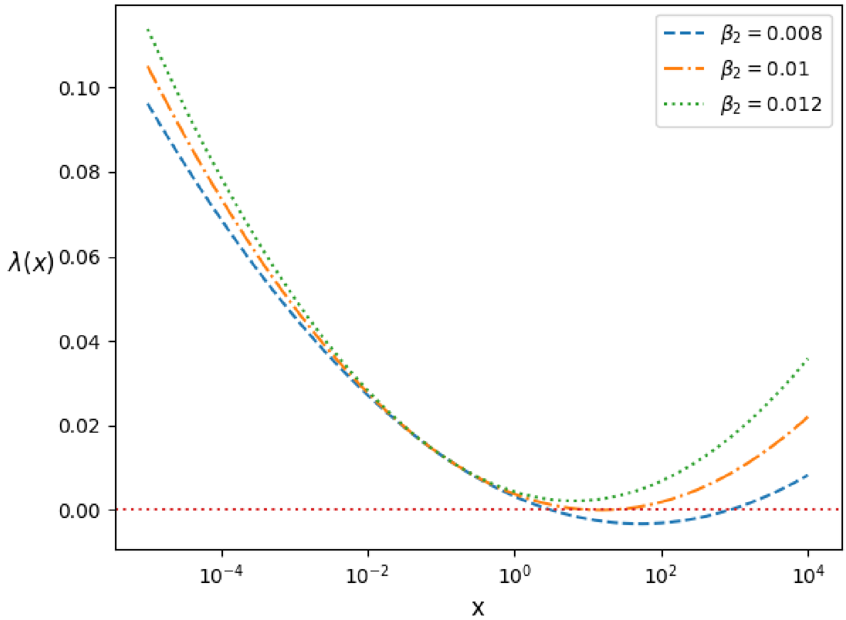

Figure 4. Running coupling $\lambda(x)$ defined in eq. (6.2) with $\beta_2 = 0.008, 0.01, 0.012$

at $\varphi = 0$. Here, the eigenfunctions starts to resemble the corresponding quartic eigenfunctions. For additional figures, refer to appendix C.

The first four non-zero numerically solved eigenvalues together with the spectral coefficients $|\varphi_1|$ and $|\varphi_3|$ for the correlator $G_\varphi(\boldsymbol{x}; 0)$ eq. (5.32) are plotted in fig. 6. The spectral coefficients eq. (5.30) are solved by using numerically solved eigenfunctions ψ_n and the analytic solution for ψ_0 eq. (5.16).

There is a steep drop in eigenvalues Λ_1 and Λ_2 when the metastable minima starts to form. Both eigenvalues becoming almost degenerate with Λ_0 at the three degenerate minima. In a vicinity of the two metastable minima Λ_3 stays almost constant slightly decreasing with decreasing β_2.

While the $n < 4$ eigenvalues are decreasing as the two minima at $\varphi \neq 0$ becomes degenerate, the $n \geq 4$ eigenvalues instead tend to grow. Therefore, the $n \geq 4$ solutions near the degenerate minima might be spurious due to yet unknown numerical issues. Fortunately, the spectral coefficients $|\varphi_5|$ and $|\varphi_7|$ have maximum values ~ 0.3 and are clearly subleading to $|\varphi_1|$ and $|\varphi_3|$, thus the main contribution to the power spectrum $\mathcal{P}_\varphi$ comes from leading and next to leading order terms.

Since ψ_1 has minimal overlap with ψ_0 in the presence of metastable minima the coefficient $|\varphi_3|$ dominates over $|\varphi_1|$ in this range. Therefore, the power spectrum $\mathcal{P}_\varphi$

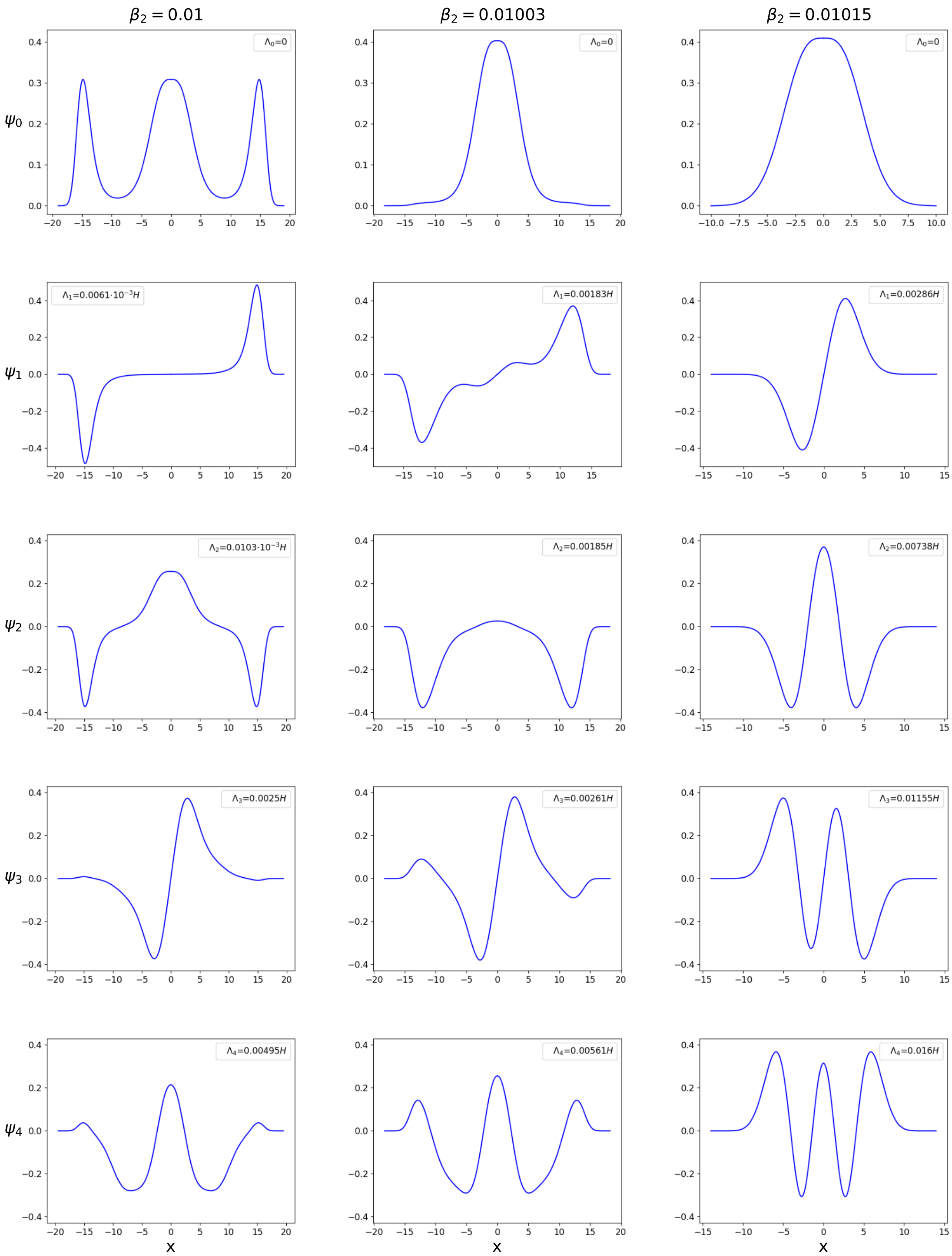

Figure 5. Eigenfunctions ψ_n for potential (6.1) with $\beta_2 = 0.01,\ 0.01003,\ 0.01015$.

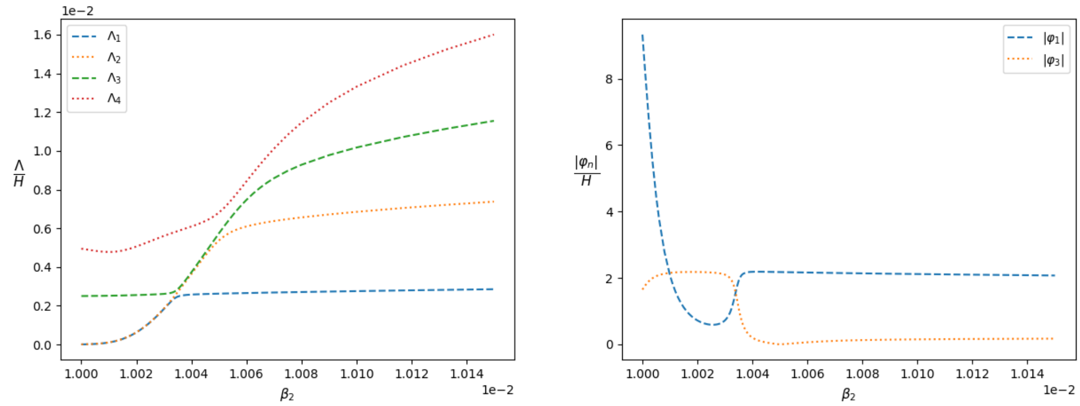

Figure 6. The first four non-zero eigenvalues (left) and the first two non-zero spectral coefficients of the spectral expansion of the two-point correlator $G_\varphi(\boldsymbol{x};0)$ (right) for the potential (6.1) as a function of β_2.

defined by eq. (5.34) is also dominated here by the $n=3$ term fig. (7). The spectral index $n_\varphi - 1$ defined by eq. (5.37) has similar behaviour. The $n=1$ term starts clearly dominate the power spectrum and the spectral index for $\beta_2 \gtrsim 0.010034$, that is when the potential V starts to resemble the quartic case.

The metastable minima have little effect on the power spectrum which stays roughly constant while β_2 is varied with slight decrease for increasing β_2 (disappearing minima). But there is a sharp drop in $\mathcal{P}_\varphi(k)$ at degenerate minima.

For $k/aH = 1$ and $\beta_2 \gtrsim 0.010006$ the power spectrum values are approximately same as in the quartic case, but the quartic $\mathcal{P}_\varphi$ is decreasing faster with decreasing k fig. 2. For example, when $k/aH = 10^{-30}$ the quartic power spectrum has a value $\mathcal{P}_\varphi/H^2 \approx 0.06 \cdot 10^{-2}$ while in the case of the running coupling we have $\mathcal{P}_\varphi/H^2 \approx 1.7 \cdot 10^{-2}$. Also, the spectral index $n_\varphi - 1$ indicates that the spectrum is closer to scale invariance than the quartic spectrum.

Since the $n=2$ eigenfunction also has a minimal overlap with the $n=0$ eigenfunction in the vicinity of metastable minima, we can expect that the coefficients for $f = \varphi^2$ has similar behaviour to the two-point correlators where the next-to-leading order term dominates.

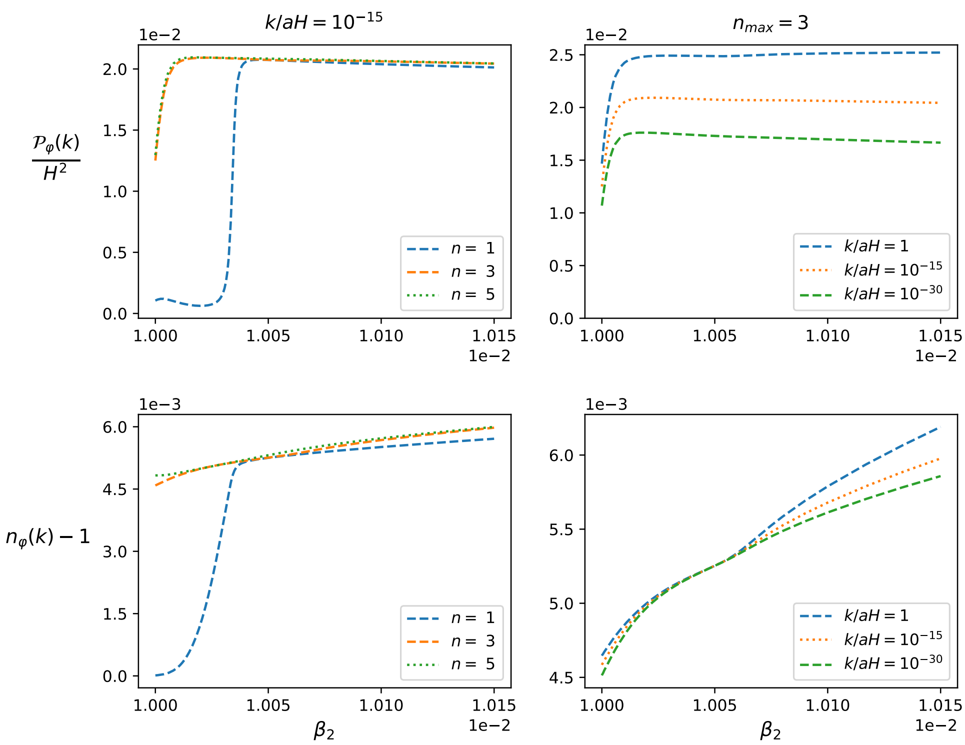

Figure 7. Power spectrum $\mathcal{P}_\varphi$ and the spectral index $n_\varphi - 1$ truncated to $n = 1$,3 and 5 with $k/aH = 10^{-15}$ (left) and truncated to $n = 3$ with $k/aH = 1$,10^{-15} and 10^{-30} (right).

7 Summary

In this thesis we have studied energetically subdominant light spectator scalars during inflation using the stochastic formalism. The spectator fields gain superhorizon perturbations during inflation which, depending on the setup, could have influenced e.g. primordial perturbations [21] or dark matter [22]. During inflation there is a constant flow of UV modes crossing the horizon and thus impacting the dynamics of the superhorizon field as the quantum fluctuations "freeze in", becoming effectively classical perturbations. Perturbative computation of spectator scalar correlators encounters IR-divergences [28] and therefore, non-perturbative methods are required. The stochastic formalism [27] offers a powerful non-perturbative tool to study the superhorizon dynamics of light scalar fields during inflation.

In the stochastic formalism, a scalar field is split in the IR and the UV parts, at a splitting scale slightly larger than the Hubble horizon size. The coarse-grained IR field can then be treated as an essentially classical stochastic quantity and its dynamics can be approximatively described by a Langevin equation with a random noise term arising from the UV fluctuations crossing the horizon.

In this work, we presented the derivation of the Langevin equation in the slow-roll approximation, and the corresponding Fokker-Planck equation, which controls the time evolution of the probability distribution function of a scalar field. The one-point PDF of a scalar field can be expressed as a spectral expansion that in de Sitter approaches the equilibrium distribution if inflation has lasted long enough.

The field correlations between two spatially separated points are characterized by the two-point correlation functions. We presented the spectral expansion form for these two-point correlators on scales larger than the horizon size. The terms in this expansion are given by eigenvalues and eigenfunctions of a Schrödinger-like equation. With this method, a two-point function for any function of a scalar field $f(\varphi)$ can be easily obtained.

In the final part of this thesis, we applied the stochastic inflation to numerically investigate the two-point function of a spectator scalar with $\lambda(\varphi)\varphi^4$ potential where the coupling is running. The running of the coupling can lead to the formation of two metastable minima in the potential qualitatively resembling the large field behaviour of the SM Higgs potential. Analysing this type of potential is particularly interesting since the SM Higgs may have acted as a spectator during inflation. However, it should be noted that we are not considering the SM Higgs here, but just a potential akin to it with a phenomenological parametrization of the running coupling. We first considered a quartic potential with constant coupling λ, followed by a running coupling generating metastable minima.

The parameters in the coupling were chosen so that, $\lambda(\varphi) \geq 0$ and has one minimum. We then varied one of the parameters in such a way that the location of this minimum changes while keeping $\lambda(\varphi)$ positive. As a result, the shape of the potential varies from three degenerate minima to two of them becoming metastable eventually disappearing completely, as shown in fig. 3.

We calculated the power spectrum $\mathcal{P}_\varphi(k)$ of this aforementioned spectator field, which is defined as a Fourier transform of the two-point correlator. The terms in the spectral expansion were obtained by solving numerically the Schrödinger-like eigenvalue equation.

When the potential has two metastable minima, $\mathcal{P}_\varphi(k)$ as well as the spectral index $n_\varphi(k) - 1$ are dominated by the next-to-leading order $n = 3$ term in the spectral expansion. But after metastable minima disappears and the potential starts to resemble the quartic potential the leading $n = 1$ term becomes dominant. The dominance of the next-to-leading order term is due to minimal overlap of the eigenfunctions ψ_1 with ψ_0 making the corresponding leading order spectral coefficient $|\varphi_1|$ subdominant to the next-to-leading order spectral coefficient $|\varphi_3|$. Metastable minima have little effect on the power spectrum at constant k and it stays approximately constant, except a sharp drop occuring when all three minima becomes degenerate. For example, for $k/aH = 10^{-15}$, at degenerate minima $\mathcal{P}_\varphi(k) \approx 0.013$, while elsewhere $\mathcal{P}_\varphi(k) \approx 0.02$.

When compared to the quartic case, at $k/aH = 1$, the power spectrum for running coupling is very close (differing by approximately 1%) to the quartic spectrum. However when $k/aH < 1$ the difference becomes more significant. For example, for $k/aH = 10^{-15}$, the difference is already around 80%. This can also be seen by comparing the spectral index $n_\varphi(k) - 1$ in these two cases. For the running coupling $n_\varphi(k) - 1 \sim \mathcal{O}(10^{-3})$ when in the quartic case $n_\varphi - 1 \sim \mathcal{O}(10^{-2})$, the overall change between constant and running coupling being around $70 - 80\%$.

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A Noise correlators for Langevin equation

In this appendix, we calculate the two-point functions for the Langevin noise terms g and h. We begin by presenting the two-point functions for $\tilde{\varphi}_{\boldsymbol{k}}$ and $\dot{\tilde{\varphi}}_{\boldsymbol{k}}$, which are obtained by using the commutation relations for $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{\dagger}$ (3.12) as

$$\begin{aligned}
\langle 0|\, \tilde{\varphi}_{\boldsymbol{k}}(t)\tilde{\varphi}_{\boldsymbol{k}'}(t')\, |0\rangle &= \varphi_k(t)\varphi^*_{k'}(t')\delta^3(\boldsymbol{k}+\boldsymbol{k}') \\
\langle 0|\, \tilde{\varphi}_{\boldsymbol{k}}(t)\dot{\tilde{\varphi}}_{\boldsymbol{k}'}(t')\, |0\rangle &= \varphi_k(t)\dot{\varphi}^*_{k'}(t')\delta^3(\boldsymbol{k}+\boldsymbol{k}') \\
\langle 0|\, \dot{\tilde{\varphi}}_{\boldsymbol{k}}(t)\dot{\tilde{\varphi}}_{\boldsymbol{k}'}(t')\, |0\rangle &= \dot{\varphi}_k(t)\dot{\varphi}^*_{k'}(t')\delta^3(\boldsymbol{k}+\boldsymbol{k}').
\end{aligned} \tag{A.1}$$

By using these solutions, the two-point function for g defined in eq. (4.7) can be written as

$$\begin{aligned}
\langle 0|\, g(\boldsymbol{x},t)g(\boldsymbol{x}',t')\, |0\rangle =& \sigma^2 a a' H^4 \int \frac{d^3k}{(2\pi)^3}\delta(k-\sigma a H)\delta(k-\sigma a' H)\varphi_k(t)\varphi_k^*(t')e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\
=& \frac{\sigma^3}{2\pi^2}a^3H^4\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x}'|)\Big(\varphi_k(t)\varphi_k^*(t')\Big)_{k=\sigma a H},
\end{aligned} \tag{A.2}$$

where the integral is evaluated in spherical coordinates with $d^3k = k^2 d\phi d(\cos\theta)dk$ and θ being the angle between $\boldsymbol{k}$ and $\boldsymbol{x}-\boldsymbol{x}'$ so that $\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}') = k|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta$. The $\cos\theta$ integral gives

$$\int_{-1}^{1} d(\cos\theta)e^{ik|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta} = 2\sin(k|\boldsymbol{x}-\boldsymbol{x}'|)/k|\boldsymbol{x}-\boldsymbol{x}'| \tag{A.3}$$

which is then written in terms of the zeroth order spherical Bessel function $j_0(x) = \sin x/x$. Similarly, we obtain

$$\langle 0|\, h(\boldsymbol{x},t)h(\boldsymbol{x}',t')\, |0\rangle = \frac{\sigma^3}{2\pi^2}a^3H^4\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x}'|)\Big(\dot{\varphi}_k(t)\dot{\varphi}_k^*(t')\Big)_{k=\sigma a H} \tag{A.4}$$

$$\langle 0|\, g(\boldsymbol{x},t)h(\boldsymbol{x}',t')\, |0\rangle = \frac{\sigma^3}{2\pi^2}a^3H^4\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x}'|)\Big(\varphi_k(t)\dot{\varphi}_k^*(t')\Big)_{k=\sigma a H}. \tag{A.5}$$

Next we need to find an expression for the mode functions in eqs. (A.2), (A.4) and (A.5). In section 3.2 it was shown that the mode functions φ_k can be written as

$$\varphi_k = (-\tau)^{3/2}\frac{H\sqrt{\pi}}{2}H_\nu^{(1)}(-k\tau). \tag{A.6}$$

The solution for the time derivative $\dot{\varphi}$ is found by using the relation $zf'_\nu(z) = \lambda q z^q f_{\nu-1}(z) + (p-\nu q)f_\nu(z)$ for the Hankel functions of the form $f_\nu(z) = z^p H_\nu^{(1)}(\lambda z^q)$ [46] and by noticing that $\frac{d}{dt} = -\tau H\frac{d}{d\tau}$. And φ_k takes the following form

$$\begin{aligned}\dot{\varphi}_k =& \frac{\sqrt{\pi}}{2}H^2(-\tau)\frac{d}{d\tau}\left((-\tau)^{3/2}H_\nu^{(1)}(-k\tau)\right)\\ =&-\frac{\sqrt{\pi}}{2}H^2\left((-k\tau)H_{\nu-1}^{(1)}(-k\tau) + \left(\frac{3}{2}-\nu\right)H_\nu^{(1)}(-k\tau)\right)(-\tau)^{3/2}.\end{aligned} \tag{A.7}$$

Together with eqs. (A.6) and (A.7), and recalling that $-\tau = 1/aH$ we find

$$\begin{aligned}\left(\varphi_k(t)\varphi_k^*(t)\right)_{k=\sigma aH} =& \frac{\pi}{4}\frac{1}{a^3H}|H_\nu^{(1)}(\sigma)|^2\\ \left(\dot{\varphi}_k(t)\dot{\varphi}_k^*(t)\right)_{k=\sigma aH} =& \frac{\pi}{4}\frac{H}{a^3}\Big[\left(\frac{3}{2}-\nu\right)^2|H_\nu^{(1)}(\sigma)|^2 + \sigma^2|H_{\nu-1}^{(1)}(\sigma)|^2\\ &+\sigma\left(\frac{3}{2}-\nu\right)\left(H_\nu^{(1)}(\sigma)H_{\nu-1}^{*(1)}(\sigma) + h.c.\right)\Big]\\ \left(\varphi_k(t)\dot{\varphi}_k^*(t)\right)_{k=\sigma aH} =& -\frac{\pi}{4}\frac{1}{a^3}\Big[\left(\frac{3}{2}-\nu\right)|H_\nu^{(1)}(\sigma)|^2 + \sigma H_\nu^{(1)}(\sigma)H_{\nu-1}^{*(1)}(\sigma)\Big].\end{aligned} \tag{A.8}$$

Since $\sigma < 1$ we can use the asymptotic form for the Hankel functions eq. (3.31) and obtain

$$\begin{aligned}\left(\varphi_k(t)\varphi_k^*(t)\right)_{k=\sigma aH} =& \frac{2^{2\nu}}{4\pi}\frac{1}{a^3H}\Gamma^2(\nu)\sigma^{-2\nu}\\ \left(\dot{\varphi}_k(t)\dot{\varphi}_k^*(t)\right)_{k=\sigma aH} =& \frac{2^{2\nu}}{4\pi}\frac{H}{a^3}\sigma^{-2\nu}\Big[\left(\frac{3}{2}-\nu\right)^2\Gamma^2(\nu) + \frac{\sigma^4}{4}\Gamma^2(\nu-1)\\ &+\sigma^2\left(\frac{3}{2}-\nu\right)\Gamma(\nu)\Gamma(\nu-1)\Big]\\ \left(\varphi_k(t)\dot{\varphi}_k^*(t)\right)_{k=\sigma aH} =& -\frac{2^{2\nu}}{4\pi}\frac{1}{a^3}\sigma^{-2\nu}\Big[\left(\frac{3}{2}-\nu\right)\Gamma^2(\nu) + \frac{\sigma^2}{2}\Gamma(\nu)\Gamma(\nu-1)\Big].\end{aligned} \tag{A.9}$$

Recall that $\nu = \sqrt{9/4 - m^2/H^2}$. By assuming that $|m^2| \ll H^2$ we can approximate that $\nu \simeq 3/2 - m^2/3H^2$ and then expanding around $\nu = 3/2$ gives

$$\begin{aligned}
2^{2\nu}\Gamma^2(\nu) \simeq& 2\pi - 2^2\pi\left(\psi_0\left(\frac{3}{2}\right) + \log 2\right)\frac{m^2}{3H^2} \\
2^{2\nu}\Gamma^2(\nu-1) \simeq& 2^3\pi - 2^4\pi\left(\psi_0\left(\frac{1}{2}\right) + \log 2\right)\frac{m^2}{3H^2} \\
\left(\frac{3}{2}-\nu\right)^n 2^{2\nu}\Gamma^2(\nu) \simeq& 2\pi\left(\frac{m^2}{3H^2}\right)^n \\
\left(\frac{3}{2}-\nu\right)^n 2^{2\nu}\Gamma(\nu)\Gamma(\nu-1) \simeq& \left(\frac{m^2}{3H^2}\right)^n\left(4\pi + 4\pi\left(\log 4 + \psi_0\left(\frac{1}{2}\right) + \psi_0\left(\frac{3}{2}\right)\right)\frac{m^2}{3H^2}\right)
\end{aligned} \tag{A.10}$$

where ψ_0 is the digamma function.

By inserting these results into eqs. (A.2),A.4,(A.5) we finally obtain the two-point functions

$$\begin{aligned}
\langle 0|\, g(\boldsymbol{x},t)g(\boldsymbol{x'},t')\,|0\rangle \simeq& \sigma^{2m^2/3H^2}\frac{H^3}{4\pi^2}\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x'}|) \\
\langle 0|\, h(\boldsymbol{x},t)h(\boldsymbol{x'},t')\,|0\rangle \simeq& \sigma^{2m^2/3H^2}\frac{H^5}{4\pi^2}\left(\frac{m^2}{3H^2}+\sigma^2\right)^2\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x'}|) \\
\langle 0|\, g(\boldsymbol{x},t)h(\boldsymbol{x'},t')\,|0\rangle \simeq& -\sigma^{2m^2/3H^2}\frac{H^4}{4\pi^2}\left(\frac{m^2}{3H^2}+\sigma^2\right)\delta(t-t')j_0(\sigma a H|\boldsymbol{x}-\boldsymbol{x'}|).
\end{aligned} \tag{A.11}$$

In addition, we find

$$\langle 0|\, g(\boldsymbol{x},t)h(\boldsymbol{x'},t') + h(\boldsymbol{x'},t')g(\boldsymbol{x},t)\,|0\rangle = 2\,\langle 0|\, g(\boldsymbol{x},t)h(\boldsymbol{x'},t')\,|0\rangle\,. \tag{A.12}$$

B Langevin and Fokker-Planck equation

Langevin equation is a stochastic differential equation where a random force term $f(t)$ is added to the macroscopic equation of motion

$$\dot{y} = A(y) + f(t). \tag{B.1}$$

This equation was originally proposed by Paul Langevin in 1908 to describe Brownian motion [49]. The random force $f(t)$ is Gaussian white noise and its stochastic properties are given by its average and two-point function

$$\begin{aligned} \langle f(t) \rangle &= 0 \\ \langle f(t) f(t') \rangle &= \Gamma \delta(t - t') \end{aligned} \tag{B.2}$$

where Γ is a constant.

This process is Markovian i.e. has no memory of earlier times and thus the conditional probability distribution (PDF) for $y(t)$ can be expressed as

$$P(y_n,t_n|y_1,t_1,...,y_{n-1},tn-1) = P(y_n,t_n|y_{n-1},tn-1), \tag{B.3}$$

and it obeys the master equation [50]

$$\frac{\partial}{\partial t}P(y,t) = \int dy' \left[W_t(y|y')P(y',t) - W_t(y'|y)P(y,t)\right]. \tag{B.4}$$

Here $W_t(y|y')$ is the transition probability per unit time from y to y'. It should be noted, that the master equation is, in fact, an equation for the conditional PDF $P(y,t|y_0,t_0)$. However, with the concept of extracting the subensemble, it can be written for $P(y,t)$ [50].

Considering a stationary Markov process described by $P(y)$ and $P(y,t|y_0,t_0)$. By preparing the system in a certain non-equilibrium state $P(y_0)$, we can extract a non-stationary subensemble for $t \geq t_0$ as

$$P^*(y,t) = \int dy_0 P(y,t|y_0,t_0)P(y_0). \tag{B.5}$$

In addition, after a long time the system is expected to return to equilibrium

$$P^*(y,t) \to P(y)\,, \text{ as } t \to \infty. \tag{B.6}$$

Through Kramers-Moyal expansion of the master equation, we can obtain the Fokker-Planck equation as a special case, written as [50]

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left(b^{(1)}(y,t)P(y,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(b^{(2)}(y,t)P(y,t)\right). \tag{B.7}$$

where the jump moments $b^{(m)}(y,t)$ are defined by [58]

$$b^{(m)}(y,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle (y(t+\Delta t) - y(t))^m \right\rangle. \tag{B.8}$$

In order to find the jump moments, we first cast eq.(B.1) into an integral equation

$$y(t+\Delta t) - y(t) = \int_t^{t+\Delta t} dt' A(y(t')) + \int_t^{t+\Delta t} dt' f(t') \tag{B.9}$$

and expand $A(y(t'))$ around t to get

$$y(t+\Delta t) - y(t) = A(y(t))\Delta t + \int_t^{t+\Delta t} dt' f(t') + \mathcal{O}((\Delta t)^2). \tag{B.10}$$

Averaging with fixed $y = y(t)$ and utilizing eq. (B.2), we find

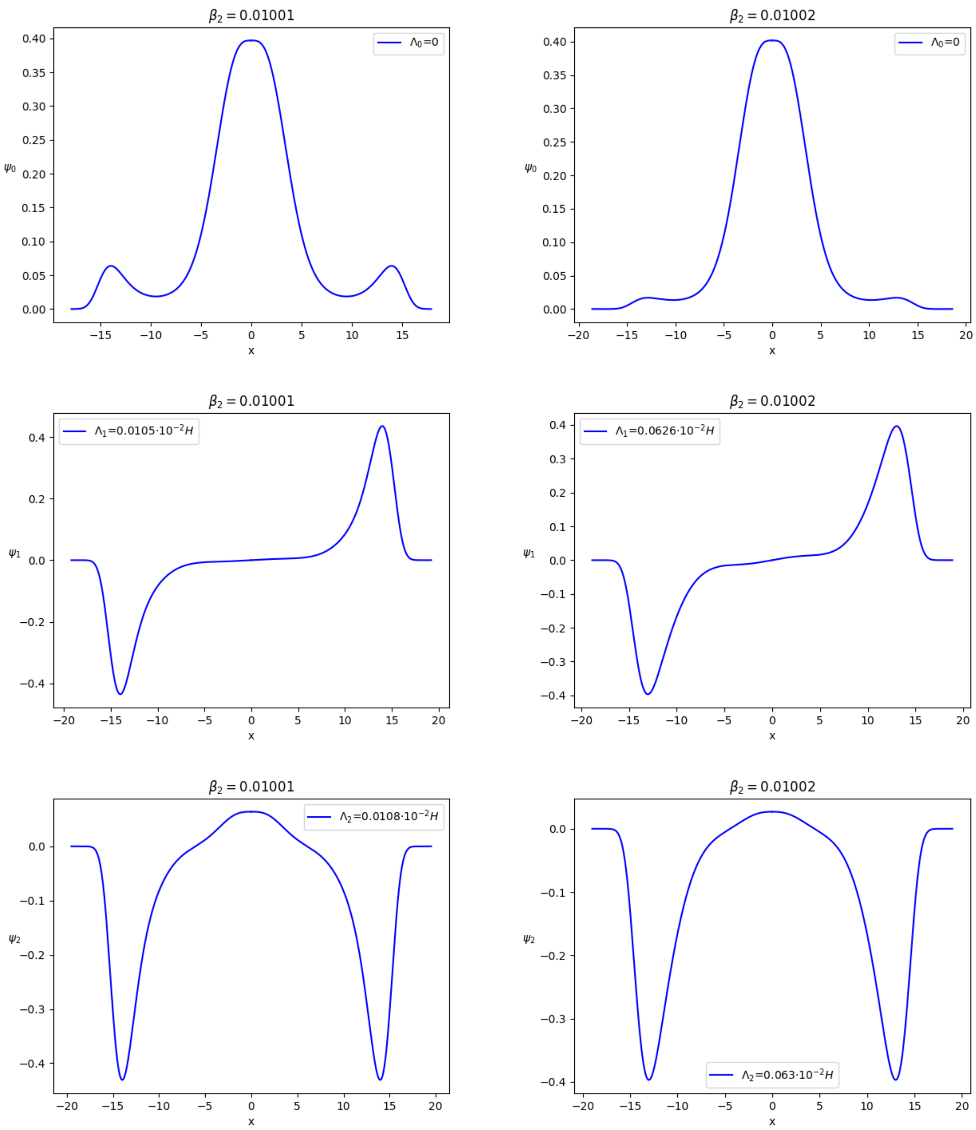
$$\begin{aligned}
\langle y(t+\Delta t) - y \rangle &= A(y)\Delta t + \mathcal{O}((\Delta t)^2) \\
\langle (y(t+\Delta t) - y)^2 \rangle &= \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \langle f(t') f(t'') \rangle + \mathcal{O}((\Delta t)^2) \\
&= \Gamma \Delta t.
\end{aligned} \tag{B.11}$$

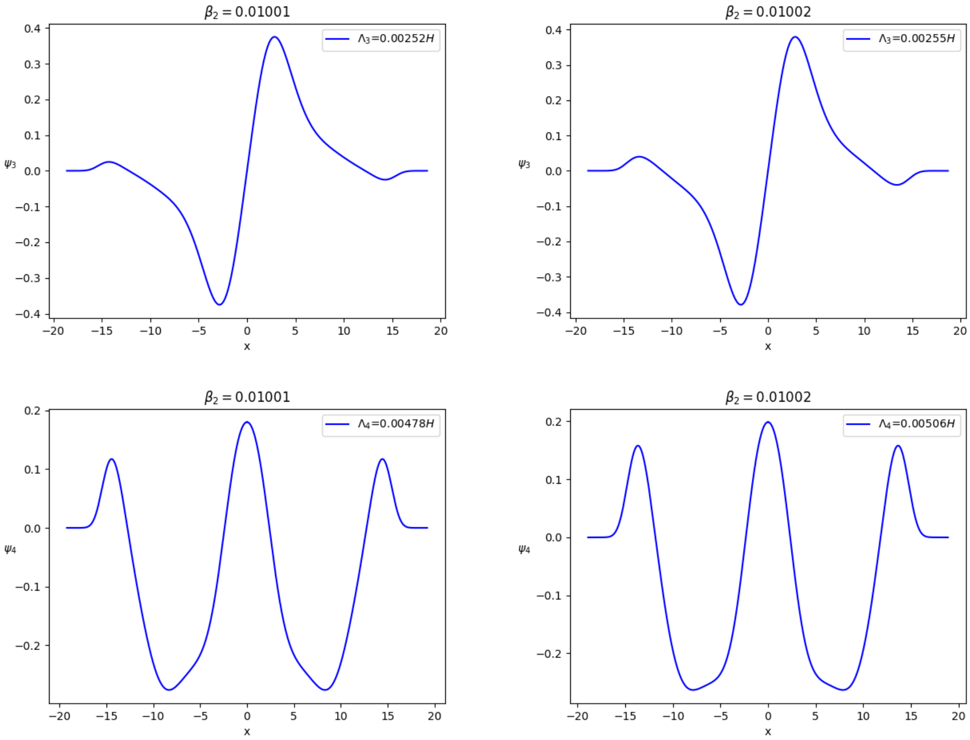
Dividing by Δt and taking a limit $\Delta t \to 0$ we find the jump moments $b^{(1)}(y,t)$ and $b^{(2)}(y,t)$ and thus the corresponding Fokker-Planck equation has the form

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left(A(y)P(y,t)\right) + \frac{\Gamma}{2}\frac{\partial^2}{\partial y^2}P(y,t). \tag{B.12}$$

C Eigenfunction figures

Eigenfunctions ψ_n for potential 6.1 with $\beta_2 = 0.01001$,0.01002.




β2 = 0.01001
Λ3=0.00252H
ψ3
x
β2 = 0.01002
Λ3=0.00255H
ψ3
x
β2 = 0.01001
Λ4=0.00478H
ψ4
x
β2 = 0.01002
Λ4=0.00506H
ψ4
x