Inclusive and semi-inclusive deeply inelastic lepton-hadron scattering in next-to-leading order perturbative QCD

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Abstract

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The structure functions and cross sections of inclusive and semi-inclusive deeply inelastic lepton-hadron scattering are calculated at next-to leading order perturbative Quantum Chromodynamics (QCD). Calculations are done using dimensional regularization and renormalization in the $\overline{\text{MS}}$ scheme. These cross sections are then used along with several available experimentally determined parton density function sets and fragmentation function sets to numerically calculate multiplicities for charged pions produced in electron-proton scattering. These numerical multiplicities are compared with those measured at the HERMES experiment. The calculated multiplicities of the different fragmentation function sets agree with the HERMES results to a varying degree.

Keywords: DIS, SIDIS, NLO pQCD, PDF, FF, dimensional regularization

Tiivistelmä

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Inklusiivinen ja semi-inklusiivinen syvä epäelastinen leptoni-hadroni sironta QCD:n häiriöteorian alinta seuraavassa kertaluvussa
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Työssä lasketaan rakennefunktiot ja vaikutusalat inklusiiviselle ja semi-inklusiiviselle syvälle epäelastiselle sironnalle Kvanttiväridynamiikan (QCD) häiriöteorian alinta seuraavassa kertaluvussa. Laskennassa käytetään dimensionaalista regularisointia ja renormalisaatio tehdään $\overline{\text{MS}}$ skeeman mukaan. Laskettuja vaikutusaloja ja muutamaa saatavilla olevaa kokeellisesti määritettyä partonitiheysfunktio- ja fragmentaatiofunktiokokoelmaa käytetään sitten elektroni-protoni sironnassa syntyvien varattujen pionien multiplisiteettien numeeriseen laskemiseen. Saatuja tuloksia verrataan HERMES-kokeen tuloksiin. Eri fragmentaatiofunktiokokoelmille lasketut multiplisiteetit vastaavat vaihtelevalla menestyksellä HERMES-kokeen tuloksia.

Avainsanat: DIS, SIDIS, NLO pQCD, PDF, FF, dimensionaalinen regularisointi

Contents

1 Introduction

Quantum Chromodynamics (QCD) is a theory of the strong interaction. Phenomena described by the theory are rich and colorful, but in many occasions non-perturbative by nature. However, thanks to the asymptotic freedom of QCD [1, 2], processes with interactions happening at high energy scales, larger than 1 GeV, or over sufficiently small distances become perturbative, as the strong interactions become effectively weaker [3]. This makes it possible to do perturbative QCD (pQCD), at least to some extent.

Another analytically simplifying property is the existence of factorization theorems for high energy QCD processes [4–10]. Factorization theorems state that the non-perturbative and the perturbative parts separate. Ultimately they do so in such a way that the singular terms in the perturbative part can be factored to the non-perturbative part order by order [4]. For example, in the case of deeply inelastic lepton-hadron scattering (DIS) $\ell + H \to \ell + X$ this factorization means that the cross section of the whole process is a convolution of the part describing the inner structure of the hadron H and the part describing the interaction between the lepton ℓ and one constituent particle of the hadron (parton). Factorization theorems provide the factorization as a Taylor series of the inverse of the energy scale of the process. Hence, they are applied mostly to high energy processes without the need for higher twist corrections.

The idea behind factorization theorems was first introduced with the parton model [11–13]. Partons are usually quarks, antiquarks and gluons. The non-perturbative parts of the hadron-probing processes manifested into parton density functions (PDFs). Shortly after, similar analysis lead to introduction of the parton fragmentation functions (FFs) [14, 15]. PDFs are used to describe the inner structure of a hadron as a sort of probability density function to find a certain type of particle inside of the hadron. FFs on the other hand describe the hadronization process of partons into hadrons as a sort of probability density to find a certain hadron from the jet produced by the parton. The inner structure of the hadron or the hadronization of the parton should be intrinsic to the particle at hand and not depend on the context it is observed in. Hence PDFs and FFs are considered to be universal in the sense that the same function should be applicable for any high energy process.

Study of PDFs, FFs and their generalizations sheds light on the non-perturbative properties of QCD. Some processes used for studying these functions are DIS $\ell + H \to \ell + X$, Drell-Yan process $H_1 + H_2 \to \ell + \bar{\ell} + X$, single-inclusive annihilation (SIA) $\ell + \bar{\ell} \to h + X$, semi-inclusive deeply inelastic scattering (SIDIS) $\ell + H \to h + X$ and single-inclusive hadron production in hadron-hadron or hadron-antihadron collisions. The first two are for studying PDFs and the last three mainly for FFs.

In this thesis we focus only on DIS and SIDIS. We calculate the cross sections for these processes in next-to-leading order (NLO) pQCD as was done in Reference [16] for DIS and in Reference [17] for SIDIS. Calculations are done with massless quarks and using dimensional regularization [18] with the $\overline{\text{MS}}$ scheme. The theoretical background and the results for both calculations are summarized in Reference [10]. This thesis is a continuation to our review where the LO calculation and phenomena of DIS and SIDIS was considered [19].

This thesis is organized as follows. In Section 2 we review the lowest order (LO) pQCD results for DIS and SIDIS and describe the changes in the background of the calculations when moving to NLO. In Section 3 we present the NLO calculation for the DIS cross section and in Section 4 for SIDIS. In Section 5 we present the results of the numerical calculation of multiplicities for pions $\pi^{\pm}$ done using the results of the other sections and some available experimentally determined PDF and FF sets. The numerical results are then compared with the multiplicity results of the HERMES experiment [20].

2 Inclusive and semi-inclusive deeply inelastic scattering

In this chapter we review the deeply inelastic scattering (DIS) and semi-inclusive deeply inelastic scattering (SIDIS) cross sections parametrized with structure functions and calculate the structure functions in the lowest order (LO) perturbative QCD (pQCD) parton model. The LO calculation is done with the dimensional regularization procedure to illuminate the process even though no regularization or renormalization is needed in the LO. In the last subsection we describe schematically how the next-to-leading order (NLO) hadronic tensors are computed from the graphs contributing in the DIS and SIDIS. This forms the basis for the NLO calculations done in Sections 3 and 4.

The parametrization of the DIS cross section with structure functions, the method for calculating structure functions using contractions of the hadronic tensor and the LO parton model results are described for example in References [16, 17, 21, 22]. We also reviewed the LO DIS and SIDIS in Reference [19].

2.1 LO DIS

The Feynman graph of the fully inclusive photon mediated DIS $H(P) + \ell(p_\ell) \rightarrow \ell(p'_\ell) + X$ is given in Figure 1. Note the momentum assignments given here in the parentheses. The inclusive differential cross section of this process is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{M}{S}\frac{\alpha_{em}^2}{Q^4}\frac{2\pi y}{x}L^{\mu\nu}W_{\mu\nu} \tag{1}$$

$$= \frac{4\pi\alpha_{em}^2}{xQ^4}\left(xy^2F_1\left(x,Q^2\right) + (1-y)\,F_2\left(x,Q^2\right)\right). \tag{2}$$

Here M is the mass of the hadron, $S = (P + p_\ell)^2$ a Mandelstam variable of the process, $L^{\mu\nu}$ the leptonic tensor, $W_{\mu\nu}$ the hadronic tensor and α_{em} the electromagnetic coupling constant. Leptonic and hadronic tensors represent the lepton and hadron

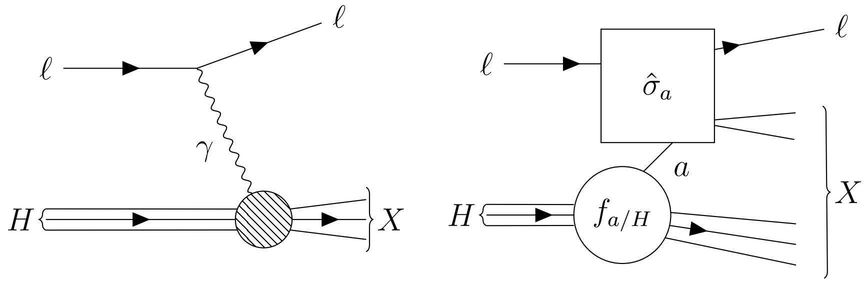

Figure 1. Feynman diagram of the inclusive DIS (left) and schematic representation of the parton model cross section factorization (right). Parton a is either a quark, an antiquark or a gluon. Here $\hat{\sigma}_a$ is the cross section of the parton-level subprocess.

currents on the cross section. The three DIS variables are

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P \cdot q}, \quad \text{and} \quad y = \frac{P \cdot q}{P \cdot p_\ell}, \tag{3}$$

where q is the photon 4-momentum. These three are not fully independent, but $Q^2 = Sxy$.

The most general form of the (electromagnetic) hadronic tensor $W_{\mu\nu}$ relates to the structure functions F_j of the hadron H as

$$W_{\mu\nu} = \frac{-F_1}{M}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{F_2}{MP \cdot q}\left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P \cdot q}{q^2} q_\nu\right). \tag{4}$$

In parton model calculations we aim to find the structure functions given by the model. Because of this we will next present a way to calculate these straight out of some contractions of the hadronic tensor. We will do this in N spacetime dimensions similarly to the NLO calculation. In this LO calculation $N = 4$ suffices but with NLO some terms need to be regularized with $N \neq 4$ to get finite results after the renormalization. Details of the implications that this choice of dimensions has to our calculations are given in the appendices. Most of the implications needed here are covered in Appendix C.

First the contraction by the metric tensor:

$$g^{\mu\nu} MW_{\mu\nu} = -F_1 (N-1) + \frac{F_2}{P\cdot q}\left(P^2 - \frac{(P\cdot q)^2}{q^2}\right) \tag{5}$$

$$= -F_1 (N-1) + \frac{1}{2x}F_2. \tag{6}$$

On the second equality we discarded the hadron mass P^2 and substituted the variable x. Next we contract the hadronic tensor twice by hadron N-momentum P. This operation gives us

$$P^\mu P^\nu MW_{\mu\nu} = -F_1 \left(P^2 - \frac{(P\cdot q)^2}{q^2}\right) + \frac{F_2}{P\cdot q}\left(P^2 - \frac{(P\cdot q)^2}{q^2}\right)^2 \tag{7}$$

$$= \frac{Q^2}{4x^2}\left(-F_1 + \frac{1}{2x}F_2\right). \tag{8}$$

Here again P^2 was discarded and the variables x and $Q^2 = -q^2$ were substituted. From Equations (6) and (8) we can then solve

$$\frac{1}{x}F_2 = \frac{2}{N-2}\left(-g^{\mu\nu}MW_{\mu\nu} + (N-1)\frac{4x^2}{Q^2}P^\mu P^\nu MW_{\mu\nu}\right) \tag{9}$$

$$F_1 = \frac{1}{2x}F_2 - \frac{4x^2}{Q^2}P^\mu P^\nu MW_{\mu\nu}. \tag{10}$$

Next we will demonstrate how these results are used to calculate the structure functions in the LO parton model.

In the parton model the cross section (2) is factorized as

$$\sigma = \sum_a \int_0^1 \mathrm{d}\xi f_{a/H}(\xi)\hat{\sigma}_a, \tag{11}$$

where the summation goes over every possible parton of H (in LO over quarks and antiquarks), $f_{a/H}$ is the process independent parton density function (PDF) of parton a in H and $\hat{\sigma}_a$ is the cross section of the parton-level subprocess $a(p)+\ell(p_\ell) \to a(p')+\ell(p'_\ell)$. Note that the momentum assignments are shown in the parentheses. This factorization is visualized on the right side of Figure 1. In the parton-level subprocess the 4-momentum of the parton a is taken to be $p = \xi P$. [4, 10]

We already introduce here the parameter

$$w = \frac{x}{\xi} = \frac{Q^2}{2p \cdot q}, \tag{12}$$

which is more convenient than ξ in the NLO calculation. Note that w is the analog of x for the parton-level subprocess and

$$\frac{\mathrm{d}\xi}{\xi} = \frac{\mathrm{d}w}{w}. \tag{13}$$

The leptonic tensors on both σ and $\hat{\sigma}_a$ are equal and the parton-level cross section also separates similarly to Equation (1). Due to this the factorization (11) can be written in terms of the hadronic tensor

$$W_{\mu\nu} = \sum_a \int_x^1 \frac{\mathrm{d}\xi}{\xi} f_{a/H}(\xi) W^a_{\mu\nu}. \tag{14}$$

The division by ξ is a consequence of both σ and $\hat{\sigma}_a$ being inversely proportional to their Mandelstam variable s with $\hat{s} = \xi S$. The lower limit of the integration is due to the conservation of the 4-momentum on the subprocess and $p = \xi P$. Note that on with these values of ξ we also have $w \in [x,1]$. Equation (14) shows that the effect of a parton a to the hadronic tensor is defined by the parton density function $f_{a/H}$ and the parton's contribution $W^a_{\mu\nu}$ to the hadronic tensor.

The hadronic tensor contribution for the parton a is defined in LO as

$$4\pi M W^a_{\mu\nu} = \int \frac{\mathrm{d}^{N-1}\vec{p'}}{2(2\pi)^{N-1} p'^0} (2\pi)^N \delta^{(N)}(p + q - p')\, C_a^2 \mathcal{W}_{\mu\nu}, \tag{15}$$

with $\delta^{(N)}$ the N-dimensional Dirac delta function, C_a the fractional charge of the parton a and

$$\mathcal{W}_{\mu\nu} = \frac{1}{2} \mathrm{tr}\left(\not{p}\gamma_\mu \not{p}'\gamma_\nu\right) \tag{16}$$

$$= 2\left(p_\mu p'_\nu + p_\nu p'_\mu - p \cdot p' g_{\mu\nu}\right). \tag{17}$$

The second equality follows from the N-dimensional results for gamma matrices given in C. Because $p = \xi P$ we see that $P^\mu P^\nu \mathcal{W}_{\mu\nu}$ is proportional to P^2 and hence is zero. We also find that

$$g^{\mu\nu}\mathcal{W}_{\mu\nu} = (2-N)\,2p\cdot p' \tag{18}$$

$$= (N-2)\,q^2, \tag{19}$$

because $q = p' - p$ and the partons are considered to be massless. The phase space integral in Equation (15) can be simplified as described in Appendix F and we get

$$4\pi M W^a_{\mu\nu} = \frac{2\pi C_a^2}{Q^2}\delta\left(1-\frac{\xi}{x}\right)\mathcal{W}_{\mu\nu}\bigg|_{p'=p+q}. \tag{20}$$

Using the contraction given in Equation (19)

$$-g^{\mu\nu}MW_{\mu\nu} = \frac{N-2}{2}\sum_a\int_x^1\frac{\mathrm{d}\xi}{\xi}f_{a/H}(\xi)C_a^2\delta\left(1-\frac{\xi}{x}\right) \tag{21}$$

$$= \frac{N-2}{2}\sum_a C_a^2 f_{a/H}(x), \tag{22}$$

and as was deducted before Equation (19) ,

$$P^\mu P^\nu W_{\mu\nu} = 0. \tag{23}$$

Using the solutions for the structure functions (9) and (10) we conclude with the classical LO results

$$\frac{1}{x}F_2 = \sum_a C_a^2 f_{a/H}(x) \tag{24}$$

$$F_1 = \frac{1}{2x}F_2. \tag{25}$$

2.2 LO SIDIS

The Feynman graph of the SIDIS $H(P) + \ell(p_\ell) \to \ell(p'_\ell) + h(p_h) + X$ is given in Figure 2. Note the momentum assingments shown here in the parentheses. Note the momentum assignments in the parentheses of the equation describing the process. The results given in Equations (1) through (10) apply here with just a little tweaking. The structure functions F_j^h and the hadronic tensor are now also functions of the SIDIS variable z and depend on the tracked outgoing hadron h. The SIDIS cross section σ^h is then also made differential with respect to z too. The SIDIS variable z

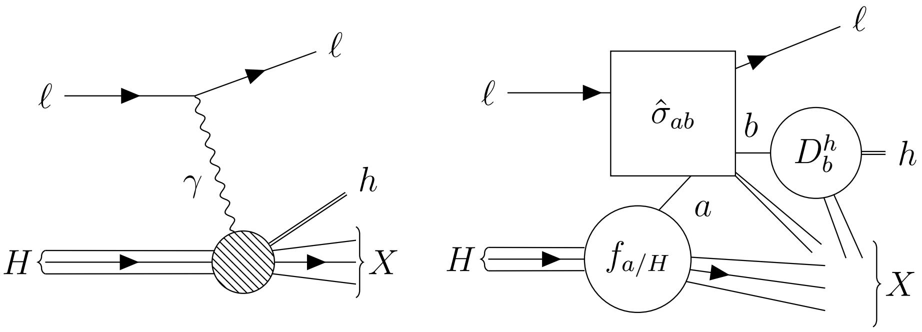

Figure 2. Feynman diagram of the SIDIS (left) and schematic representation of the parton level factorization (right). Here a and b are quarks, antiquarks or gluons and $\hat{\sigma}_{ab}$ is the cross section of the parton-level subprocess.

is defined as

$$z = \frac{P \cdot p_h}{P \cdot q}, \tag{26}$$

which is the Lorentz invariant version of the fraction of photon energy that the hadron h carries in the Target Rest Frame (TRF).

The factorization applied here is [10]

$$\sigma^h = \sum_{a,b} \int_0^1 \mathrm{d}\xi \int_0^1 \mathrm{d}\zeta f_{a/H}(\xi)\, D_b^h(\zeta)\, \hat{\sigma}_{ab} \tag{27}$$

and it is visualized in Figure 2. The summation in the equation above goes over every possible parton a in H and over every possible elementary particle b that can produce the hadron h. In our calculations the possible particles are quarks, antiquarks and gluons. The function D_b^h is the process-independent fragmentation function (FF) that gives the number of hadrons h produced out of parton b with 4-momentum p_h being ζ times that of parton b. The cross section $\hat{\sigma}_{ab}$ is that of the subprocess $a(p) + \ell(p_\ell) \to b(p') + \ell(p'_\ell)$. Here in the parentheses are shown the momentum assignments.

Similarly to Section 2.1 the factorization (27) can be written in terms of the hadronic tensor as

$$W^h_{\mu\nu} = \sum_{a,b} \int_x^1 \frac{\mathrm{d}\xi}{\xi} \int_z^1 \mathrm{d}\zeta f_{a/H}\left(\xi\right) D^h_b\left(\zeta\right) W^{ab}_{\mu\nu}. \tag{28}$$

The lower limits of both the ζ and the ξ integrals are again a consequence of the conservation of the 4-momentum. Comparing to the DIS calculation the LO N-dimensional calculation proceeds with SIDIS almost identically. The differences appearing mainly concern the definition of $W^{ab}_{\mu\nu}$ by $\mathcal{W}_{\mu\nu}$. As the hadronic tensor has an additional degree of freedom z so should have $W^{ab}_{\mu\nu}$. In the definition of $W_{\mu\nu}$ that degree of freedom is from the phase space integral of p_h and hence in $W^{ab}_{\mu\nu}$ the degree of freedom should be extracted from the phase space integral of the outgoing parton b. Because of this, in $W^{ab}_{\mu\nu}$ there is a delta function

$$\delta\left(z-\zeta\right) \tag{29}$$

in the phase space integral with respect to p', as the definition of z can be written here

$$z = \frac{p \cdot p_h}{p \cdot q} = \zeta \frac{p \cdot p'}{p \cdot q} = \zeta \frac{p \cdot (p+q)}{p \cdot q} = \zeta. \tag{30}$$

Note that the partons are taken to be massless. In LO the subprocess can only have $a = b$ so there is also a Kronecker delta δ_{ab} in $W^{ab}_{\mu\nu}$. Consequently

$$4\pi M W^{ab}_{\mu\nu} = \frac{2\pi C_a^2}{Q^2} \delta\left(1 - \frac{\xi}{x}\right) \delta\left(z-\zeta\right) \delta_{ab} \mathcal{W}_{\mu\nu}\bigg|_{p'=p+q}, \tag{31}$$

where $\mathcal{W}_{\mu\nu}$ is the same as in Section 2.1. Therefore

$$P^\mu P^\nu W_{\mu\nu} = 0 \tag{32}$$

and

$$-g^{\mu\nu} M W_{\mu\nu} = \frac{N-2}{2} \sum_a \int_x^1 \frac{\mathrm{d}\xi}{\xi} \int_z^1 \mathrm{d}\zeta f_{a/H}(\xi) D^h_a\left(\zeta\right) C_a^2 \delta\left(1 - \frac{\xi}{x}\right) \delta\left(z-\zeta\right) \tag{33}$$

$$= \frac{N-2}{2} \sum_a C_a^2 f_{a/H}(x) D^h_a\left(z\right). \tag{34}$$

This gives us the LO SIDIS structure functions

$$\frac{1}{x}F_2^h = \sum_a C_a^2 f_{a/H}(x) D_a^h(z) \tag{35}$$

$$F_1^h = \frac{1}{2x}F_2. \tag{36}$$

2.3 NLO DIS and SIDIS

In Figure 3 are shown the possible NLO graphs contributing in DIS and SIDIS at the parton-level. It happens, however, that the quark self-energy correction graphs A_{S1} and A_{S2} do not contribute to the total cross section in our calculations in the end. This phenomenon is further addressed in Section 3.2. We will next describe how to calculate from these graphs the hadronic tensors in NLO for DIS and SIDIS. Note that when calculating the hadronic tensor in parton-level from these graphs the initial state photon (DIS virtual photon) and the elementary charge of the quark-photon vertex are left out as they are already included into the total cross section as can be seen in Equation (1).

In the DIS process the six contributing graphs form three indistinguishable pairs: LO+Vertex correction, two gluon radiation (final state gluon) graphs and the two initial gluon graphs. For each quark and antiquark a the graphs combine into the relevant squared amplitude of the parton subprocess cross section schematically as

$$\left|A_{a(LO)}\right|^2 + 2\mathrm{Re}\left(A_{a(LO)}\left(A_{a(V)}\right)^\dagger\right) + \left|A_{a(C1)} + A_{a(C2)}\right|^2. \tag{37}$$

Note that the terms of the form $A_{a(V)}^2$ would be of the order 2 in the powers of the strong coupling constant and hence they are not included into the NLO calculation. The matrix element of the initial state gluon is of the form

$$\sum_a \left|A_{ig1(a)} + A_{ig2(a)}\right|^2, \tag{38}$$

where the summing goes over quarks.

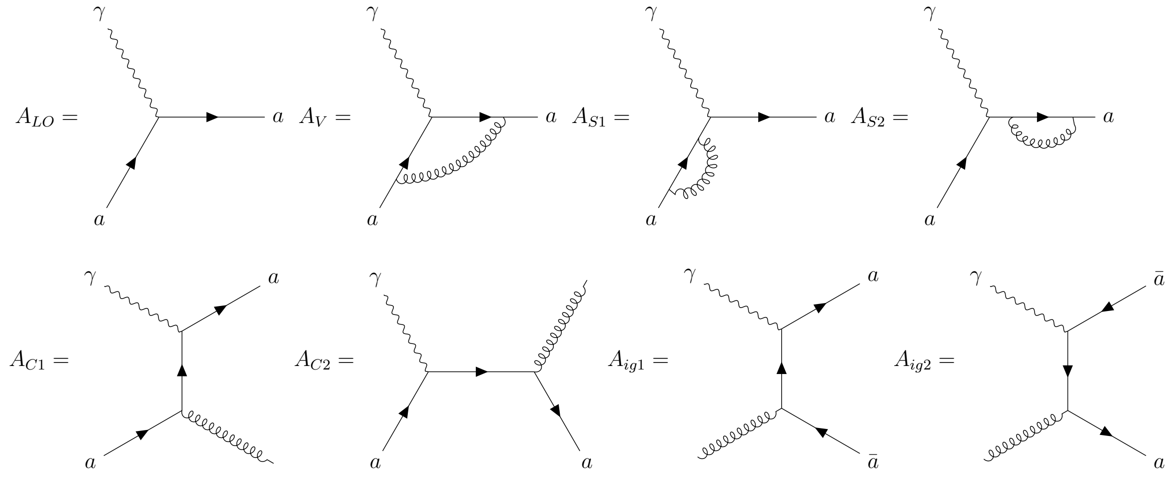

Figure 3. The graphs contributing into the parton-level NLO cross section. On the first row, from left to right, are LO, virtual vertex correction and two quark self-energy correction graphs. On the second row, from left to right, are two gluon radiation (final state gluon) and two initial state gluon graphs.

The full hadronic tensor factorizes as in Equation (14), but with more subprocesses as in the LO:

$$W_{\mu\nu} = \sum_a \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right) \left(\left(W_{a(LO)}\right)_{\mu\nu} + \left(W_{a(V)}\right)_{\mu\nu} + \left(W_{a(C)}\right)_{\mu\nu}\right) + \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right) \left(W_{ig}\right)_{\mu\nu} . \tag{39}$$

The variable w was defined in Equation (12). The LO contribution to the hadronic tensor is given in the previous section 2.1 and

$$4\pi M \left(W_{a(V)}\right)_{\mu\nu} = PS_{p'}\left[C_a^2 \left(\mathcal{W}_{a(V)}\right)_{\mu\nu}\right], \tag{40}$$

$$4\pi M \left(W_{a(C)}\right)_{\mu\nu} = PS_{p',k}\left[C_a^2 \left(\mathcal{W}_{a(C)}\right)_{\mu\nu}\right], \tag{41}$$

$$4\pi M \left(W_{ig}\right)_{\mu\nu} = PS_{p',k}\left[\left(\mathcal{W}_{ig}\right)_{\mu\nu}\right]. \tag{42}$$

Here $PS_{(p_j)}[f]$ denotes the N-dimensional phase space integral over the 4-momenta (p_j) of the function f with the appropriate 4-momentum conserving delta functions. In the case of the vertex correction and the gluon radiation p' is the 4-momentum of the outgoing quark, for the gluon radiation k is the 4-momentum of the outgoing gluon and for the initial state gluon p' and k are the 4-momenta of the outgoing quark and antiquark. The phase space integrals are processed in Appendix F. The terms in Equations (37) and (38) form the "partonic tensors" $\mathcal{W}$ after averaging over initial quantum numbers and summing over the final state quantum numbers as

$$2\text{Re}\left(A_{a(LO)}\left(A_{a(V)}\right)^{\dagger}\right) \rightarrow C_a^2 \left(\mathcal{W}_{a(V)}\right)_{\mu\nu}, \tag{43}$$

$$\left|A_{a(C1)} + A_{a(C2)}\right|^2 \rightarrow C_a^2 \left(\mathcal{W}_{a(C)}\right)_{\mu\nu}, \tag{44}$$

$$\sum_a \left|A_{ig1(a)} + A_{ig2(a)}\right|^2 \rightarrow \left(\mathcal{W}_{ig}\right)_{\mu\nu}, \tag{45}$$

where again the sum in the gluon partonic tensor goes over each quark.

Next, we will consider the SIDIS hadronic tensor. The relevant subprocess graphs are the same as in DIS. Difference to DIS was the extra degree of freedom in hadronic tensor and that the outgoing partons are also subject to the factorization in the process. In the factorization (27) the outgoing parton indexing tells us from which parton b we take the hadron h to appear.

Now for a quark or an antiquark a the subprocess graphs combine schematically into the squared amplitude in $\hat{\sigma}_{ab}$ as

$$\left|A_{a(LO)}\right|^2 \delta_{ab} + 2\text{Re}\left(A_{a(LO)}\left(A_{a(V)}\right)^{\dagger}\right)\delta_{ab} + \left|A_{a(C1)} + A_{a(C2)}\right|^2 \left(\delta_{ab} + \delta_{bg}\right) \tag{46}$$

and for the gluon a as

$$\left|A_{ig1(b)} + A_{ig2(b)}\right|^2 \left(\delta_{b\mathtt{q}} + \delta_{b\bar{\mathtt{q}}}\right). \tag{47}$$

Here we use notation $\delta_{b\mathtt{q}} = 1$ when b is a quark, $\delta_{b\bar{\mathtt{q}}} = 1$ when b is an antiquark and both are zero otherwise. These terms define the SIDIS versions of the partonic tensors $\mathcal{W}_{ab(V)}$, $\mathcal{W}_{ab(C)}$ and $\mathcal{W}_{ig(b)}$ similarly to Equations (43) to (45). Note that the Kronecker deltas in these terms are to be included into the definitions and that now the initial state gluon partonic tensor does not include the summation over the quark flavors.

With these graphs and the SIDIS hadronic tensor factorization (28) our hadronic tensor can be written as

$$W^h_{\mu\nu} = \sum_{a,b} \int_x^1 \frac{\mathrm{d}\xi}{\xi} \int_z^1 \mathrm{d}\zeta f_{a/H}(\xi)\, D^h_b(\zeta) \left(\left(W_{ab(LO)}\right)_{\mu\nu} + \left(W_{ab(V)}\right)_{\mu\nu} + \left(W_{ab(C)}\right)_{\mu\nu} \right)$$
$$+ \sum_b \int_x^1 \frac{\mathrm{d}\xi}{\xi} \int_z^1 \mathrm{d}\zeta f_{g/H}(\xi)\, D^h_b(\zeta) \left(W_{ig(b)}\right)_{\mu\nu}, \tag{48}$$

where the a sum goes over all quarks and antiquarks and b over quarks, antiquarks and the gluon. Each graph contribution here is defined similarly to Equations (40)-(42) but with SIDIS versions of the partonic tensors. Also, the SIDIS hadronic tensor has an extra degree of freedom in the form of the variable z when comparing to the DIS tensor. This degree of freedom is realized as a delta function

$$\delta\left(z - \frac{p \cdot p_h}{p \cdot q}\right) \tag{49}$$

in the phase space integrals in the definitions of the SIDIS hadronic tensor contributions for each graph. The definition of the variable z inside of the delta function can be a little different with different processes and different particles b. We will finish this chapter by explaining how each situation is handled.

In LO and vertex correction graphs we have only one outgoing parton-level particle with 4-momentum $p' = p + q$. Therefore, we can only have $p_h = \zeta p'$ and

$$\frac{p \cdot p_h}{p \cdot q} = \zeta. \tag{50}$$

In the gluon radiation and initial state gluon graphs the hadron h can be produced from either of the outgoing particles: the one with 4-momentum p' or the one with 4-momentum k. Hence we can have either $p_h = \zeta p'$ or $p_h = \zeta k$ while $p + q = k + p'$. The arguments of the delta functions can in these cases be simplified with the variable

$$1 - v = \frac{p \cdot k}{p \cdot q} \tag{51}$$

that is used to parametrize the phase space integrals of these two processes in Appendix F. In the first case we have

$$\frac{p \cdot p_h}{p \cdot q} = \zeta \frac{p \cdot (q - k)}{p \cdot q} = \zeta v \tag{52}$$

and in the second case

$$\frac{p \cdot p_h}{p \cdot q} = \zeta \frac{p \cdot k}{p \cdot q} = \zeta \left(1 - v\right). \tag{53}$$

From these relations the restriction $\zeta \geq z$ can be easily seen as $v \in [0,1]$.

So there is a delta function

$$\delta \left(z - \zeta\right) \tag{54}$$

in LO and the vertex correction phase space integrals. In the gluon radiation and initial state gluon phase space integrals there is

$$\delta \left(z - \zeta v\right) \tag{55}$$

in those terms that describe h produced from the particle with 4-momentum p' and

$$\delta \left(z - \zeta \left(1 - v\right)\right) \tag{56}$$

in those where h is produced from the particle with 4-momentum k.

3 NLO DIS calculation

In this section we present the calculations for NLO structure functions with the hadronic tensor projection method described in 2.1. The NLO calculations are done with dimensional regularization [18] with spacetime dimensions N. The regularization prescription uses $N = 4 - 2\epsilon$ to regularize the divergences on the limit $N \to 4$. Regulated divergences are renormalized according to the $\overline{\text{MS}}$ scheme.

The parton-level graphs that contribute to the NLO cross section and how they combine into the DIS hadronic tensor were presented in Section 2.3. We calculate here in distinct subsections the contributions of the virtual vertex correction, gluon radiation (final state gluon) and initial state gluon graphs to projections $g^{\mu\nu}W_{\mu\nu}$ and $P^{\mu}P^{\nu}W_{\mu\nu}$. In Section 3.2 we consider briefly why the quark self-energy corrections do not contribute into our results. In Section 3.5 we combine the results of the preceding sections into full structure functions F_j, renormalize our parton density functions and present the differential NLO DIS cross section.

The calculation of this section bases on Reference [16], but with different renormalization scheme. I was introduced into the calculation by Reference [23]. Summary of the theoretical background, the $\overline{\text{MS}}$ scheme and the results can be found in Reference [10]. DIS and the NLO calculation with details are also covered in many books, for example in References [21, 22, 24]. The Section 3.2 is based on books [21, 25].

3.1 Vertex correction

The graph $A_{a(V)}$ of the vertex correction and the notation used in this subsection are presented in Figure 4. When a is a quark the Feynman rules in N-dimensions

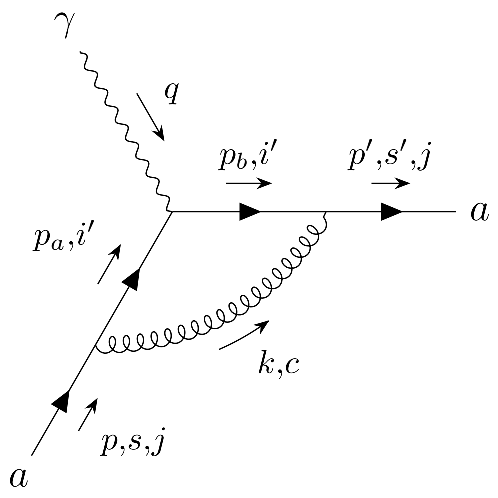

Figure 4. Virtual vertex correction graph $A_{a(V)}$. For the incoming and outgoing quark the triplet of symbols next to the momentum arrow denote 4-momentum, spin component and color. For the virtual quarks and the gluon the two symbols denote 4-momentum and color.

given in Appendix A imply that

$$e^{(N)}\left(A_{a(V)}\right)_{\nu}^{s'sj}=\int\frac{\mathrm{d}^Nk}{(2\pi)^N}\overline{u}\left(p',s'\right)\left(-ig_S^{(N)}\left(t^c\right)_{ji'}\gamma^\beta\right)\frac{i\not p_b}{p_b^2+i\varepsilon}\left(-iC_ae^{(N)}\gamma_\nu\right)$$
$$\times\frac{i\not p_a}{p_a^2+i\varepsilon}\left(-ig_S^{(N)}\left(t^c\right)_{i'j}\gamma^\alpha\right)u\left(p,s\right)\frac{-i}{k^2+i\varepsilon}\left(g_{\alpha\beta}+\eta\frac{k_\alpha k_\beta}{k^2}\right). \quad (57)$$

Note that the repeated indices j, i', c, α and β are summed over and the quarks are considered massless. The incoming photon propagator is excluded and the elementary charge is separated from the rest of the graph as they are not included in the hadronic tensor or the partonic tensor definitions. The ε prescription of the propagators is in this section left explicit to see how it affects the loop integral sign in the end. To calculate the vertex correction we also need [19]

$$e^{(N)}\left(A_{a(LO)}\right)_{\mu}^{s'sj}=-ie^{(N)}C_a\overline{u}\left(p',s'\right)\gamma_\mu u\left(p,s\right). \quad (58)$$

As described in Section 2.3

$$C_a^2\left(\mathcal{W}_{a(V)}\right)_{\mu\nu}=\frac{1}{2}\frac{1}{3}\sum_{s,s',j,c}2\mathrm{Re}\left(\left(A_{a(LO)}\right)_{\mu}^{s'sj}\left(\left(A_{a(V)}\right)_{\nu}^{s'sj}\right)^\dagger\right), \quad (59)$$

where the average is taken over the two spin states s and three colors j. With spinor completeness relations and the fact that $\sum_c \mathrm{tr}\,(t^c t^c) = 4$ (see Appendix A) we get

$$C_a^2 \left(\mathcal{W}_{a(V)}\right)_{\mu\nu} = \frac{4C_a^2}{3}\left(g_S^{(N)}\right)^2 \mathrm{Re}\left(i\int \frac{\mathrm{d}^N k}{(2\pi)^N}\frac{\mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\gamma_\alpha\not{p}_a\gamma_\nu\not{p}_b\gamma_\beta\right)}{\left(p_a^2-i\varepsilon\right)\left(p_b^2-i\varepsilon\right)\left(k^2-i\varepsilon\right)}\right) \times\left(g_{\alpha\beta}+\eta\frac{k_\alpha k_\beta}{k^2}\right). \tag{60}$$

Here we can note two points. First, this expression would be the same with antiquarks. Through similar steps the trace would read

$$\mathrm{tr}\left(\not{p}\gamma_\mu\not{p}'\gamma_\beta\not{p}_b\gamma_\nu\not{p}_a\gamma_\alpha\right). \tag{61}$$

But with square matrices A,B $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ and when computing the trace of a sequence of gamma matrices one can invert the order of the matrices without changing the value of the trace. Hence

$$\mathrm{tr}\left(\not{p}\gamma_\mu\not{p}'\gamma_\alpha\not{p}_b\gamma_\nu\not{p}_a\gamma_\beta\right) = \mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\gamma_\beta\not{p}_a\gamma_\nu\not{p}_b\gamma_\alpha\right). \tag{62}$$

The trace is only different from that of Equation (60) by the change of indices $\alpha \leftrightarrow \beta$ and those indices are contracted by a symmetric tensor.

The second point is that the loop integral part proportional to η is

$$\int \frac{\mathrm{d}^N k}{(2\pi)^N}\frac{\mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\not{k}\not{p}_a\gamma_\nu\not{p}_b\not{k}\right)}{\left(p_a^2-i\varepsilon\right)\left(p_b^2-i\varepsilon\right)\left(k^2-i\varepsilon\right)^2} = \int \frac{\mathrm{d}^N k}{(2\pi)^N}\frac{\mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\not{p}_a\not{p}_a\gamma_\nu\not{p}_b\not{p}_b\right)}{\left(p_a^2-i\varepsilon\right)\left(p_b^2-i\varepsilon\right)\left(k^2-i\varepsilon\right)^2} \tag{63}$$

$$= \int \frac{\mathrm{d}^N k}{(2\pi)^N}\frac{p_a^2 p_b^2\mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\gamma_\nu\right)}{\left(p_a^2-i\varepsilon\right)\left(p_b^2-i\varepsilon\right)\left(k^2-i\varepsilon\right)^2} \tag{64}$$

$$= \mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\gamma_\nu\right)\int \frac{\mathrm{d}^N k}{(2\pi)^N}\frac{1}{\left(k^2-i\varepsilon\right)^2}. \tag{65}$$

This is achieved by gamma matrix properties given in Appendix C and with $p_a = p-k$ and $p_b = p'-k$. In dimensional regularization it is consistent to set the remaining integral to zero [25]. Therefore, the partonic tensor of Equation (60) is independent of the used gauge parameter η.

Next we substitute Equation (60) without the η term into Equation (40) and use the phase space integral results of Appendix F to get

$$4\pi M \left(W_{a(V)}\right)_{\mu\nu} = \frac{2\pi}{Q^2}\delta\left(1-w\right)\frac{4C_a^2}{3}\left(g_S^{(N)}\right)^2 \times \mathrm{Re}\left(i\int\frac{\mathrm{d}^N k}{\left(2\pi\right)^N}\frac{\mathrm{tr}\left(\not{p}'\gamma_\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\nu\not{p}_b\gamma_\alpha\right)}{\left(p_a^2-i\varepsilon\right)\left(p_b^2-i\varepsilon\right)\left(k^2-i\varepsilon\right)}\right). \tag{66}$$

The contraction of the hadronic tensor contribution (66) by $P^\mu P^\nu = \xi^{-2}p^\mu p^\nu$ is zero as the quarks are considered massless and with gamma matrix results given in A we have

$$\mathrm{tr}\left(\not{p}'\not{p}\not{p}\gamma^\alpha\not{p}_a\not{p}\not{p}_b\gamma_\alpha\right) = p^2\mathrm{tr}\left(\not{p}'\gamma^\alpha\not{p}_a\not{p}\not{p}_b\gamma_\alpha\right) = 0. \tag{67}$$

For the contraction with the metric tensor, we need the value of the fully contracted trace and then we need to perform the loop integration. The trace is given in Appendix C and it is

$$\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\mu\not{p}_b\gamma_\alpha\right) = -2q^2\left(N-2\right)\mathcal{N}, \tag{68}$$

where

$$\mathcal{N} = 2q^2 + 4\left(p+p'\right)\cdot k + \frac{8}{q^2}p\cdot k\, p'\cdot k - \left(4-N\right)k^2. \tag{69}$$

We can then write

$$g^{\mu\nu}M\left(W_{a(V)}\right)_{\mu\nu} = \frac{4C_a^2}{3}\left(N-2\right)\delta\left(1-\frac{\xi}{x}\right)\left(g_S^{(N)}\right)^2\mathrm{Re}\left(i\mathcal{I}\right) \tag{70}$$

and the loop integral to be evaluated is

$$\mathcal{I} \equiv \int\frac{\mathrm{d}^N k}{\left(2\pi\right)^N}\frac{\mathcal{N}}{\left(p_a^2-i\varepsilon\right)\left(p_b^2-i\varepsilon\right)\left(k^2-i\varepsilon\right)}. \tag{71}$$

First we introduce a Feynman parametrization for the integrand using the general formula presented in Appendix B:

$$\mathcal{I} = \int\frac{\mathrm{d}^N k}{\left(2\pi\right)^N}\int_0^1\mathrm{d}x'\int_0^1\mathrm{d}y'\int_0^1\mathrm{d}z'\frac{2\delta\left(x'+y'+z'-1\right)\mathcal{N}}{\left(x'p_a^2+y'p_b^2+z'k^2-i\varepsilon\right)^3}. \tag{72}$$

The delta function can be integrated right away with the z' integral:

$$\mathcal{I} = \int \frac{\mathrm{d}^N k}{(2\pi)^N} \int\limits_0^1 \mathrm{d}x' \int\limits_0^{1-x'} \mathrm{d}y' \frac{2\mathcal{N}}{\left(x' p_a^2 + y' p_b^2 + (1 - x' - y')\, k^2 - i\varepsilon\right)^3}. \tag{73}$$

Note that the delta function also restricts the upper limit of y integral as $z' = 1 - x' - y' \in [0,1]$. Substituting $p_a = p - k$ and $p_b = p' - k$ one can rearrange the integrand as

$$\mathcal{I} = \int \frac{\mathrm{d}^N k}{(2\pi)^N} \int\limits_0^1 \mathrm{d}x' \int\limits_0^{1-x'} \mathrm{d}y' \frac{2\mathcal{N}}{\left(\ell^2 - \Delta - i\varepsilon\right)^3}, \tag{74}$$

where

$$\ell \equiv k - x' p - y' p' \qquad \text{and} \qquad \Delta \equiv 2x' y' p \cdot p' = -x' y' q^2 > 0. \tag{75}$$

We then want to perform a change of variables in the loop integral from k to ℓ. As k is a linear function of ℓ the Jacobian is unity. The numerator changes to

$$\mathcal{N} = \frac{8}{q^2} p \cdot \ell \, p' \cdot \ell + (4 - N)\, \ell^2 + 2q^2 \left(1 - x' - y' + \frac{1}{2}(N-2)\, x' y'\right) + g \cdot \ell. \tag{76}$$

Here $g = g(p,p',q^2,x',y',N)$ is some (auxiliary) N-vector. Its exact form is not relevant as in the loop integration the terms proportional to odd number of components of ℓ produce zero due to the domain of integration being symmetric respect to zero and the denominator being an even function of ℓ [26]. Due to this symmetry, we also have an equivalence

$$\ell^\mu \ell^\nu \Leftrightarrow \frac{1}{N} g^{\mu\nu} \ell^2 \tag{77}$$

for the terms inside the loop integral [26]. In our numerator $\mathcal{N}$ we can thus change

$$\frac{8}{q^2} p \cdot \ell \, p' \cdot \ell \implies \frac{8}{N q^2} p \cdot p' \ell^2 = -\frac{4}{N} \ell^2. \tag{78}$$

After the change of variables from k to ℓ is done we also change the order of the integrals from (k,x',y') to (x',y',k). To evaluate the loop integrals, we use the identity

$$\int \frac{\mathrm{d}^N \ell}{(2\pi)^N} \frac{\left(\ell^2\right)^R}{\left(\ell^2 - \Delta \pm i\varepsilon\right)^m} = \frac{\pm i\,(-1)^{m+R}}{(4\pi)^{\frac{N}{2}}} \frac{\Delta^{\frac{N}{2}+R-m}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma\left(m - \frac{N}{2} - R\right)\Gamma\left(\frac{N}{2} + R\right)}{\Gamma(m)} \tag{79}$$

given in Appendix B. This identity is well defined when $\Delta > 0$ and $\frac{N}{2} < m - R$. Idea of the dimensional regularization is that given $\frac{N}{2} < m - R$ for some dimension N the right-hand side can be taken to present the left-hand side in the sense of analytical continuation. The Gamma function allows us then to continuously vary N for different values as long as the arguments of the Gamma functions stay inside the domain of Gamma function $\mathbb{C} \setminus (-\mathbb{N}_0)$ (see Appendix B for more details).

With the above identity and the product property of the Gamma function $\Gamma(x+1) = x\Gamma(x)$ we are left with

$$\mathcal{I} = \frac{i}{(4\pi)^{\frac{N}{2}}} \int_0^1 \mathrm{d}x' \int_0^{1-x'} \mathrm{d}y' \left(\frac{4}{N} + N - 4\right) \Delta^{\frac{N}{2}-2} \frac{N}{2} \Gamma\left(2 - \frac{N}{2}\right) \tag{80}$$

$$+ 2q^2 \Delta^{\frac{N}{2}-3} \Gamma\left(3 - \frac{N}{2}\right)\left(1 - x' - y' + \frac{1}{2}(N-2)\,x'y'\right). \tag{81}$$

To evaluate the integrals over Feynman parameters we use the identity (see Appendix B)

$$\int_0^1 \mathrm{d}x' \int_0^{1-x'} \mathrm{d}y' \left(bx' + by' + c\right)\left(x'y'\right)^a = \frac{\left(\Gamma(a+1)\right)^2}{\Gamma(2a+3)} \left(c + 2b\frac{a+1}{2a+3}\right), \tag{82}$$

where $b, c \in \mathbb{R}$, $c \neq 0$ and $a > -1$. Here the evaluation is again done with N for which $a > -1$ and then it can be varied according to the domain of Gamma function. After some rearrangement and simplification, we arrive at

$$\mathcal{I} = \frac{-i}{(4\pi)^{\frac{N}{2}}} \left(-q^2\right)^{\frac{N}{2}-2} \frac{\Gamma\left(3 - \frac{N}{2}\right)}{N-3} \frac{\left(\Gamma\left(\frac{N}{2} - 1\right)\right)^2}{\Gamma(N-3)}$$

$$\times \left(\frac{4}{(N-4)(N-2)} + \frac{2N-2}{N-2} + \frac{8}{(N-4)^2}\right). \tag{83}$$

The arguments of the Gamma function in the prefactor are chosen in such a way that with $N = 4$ the argument is 1 so that they do not diverge when $N \to 4$. We then proceed to substitute $N = 4 - 2\epsilon$ and expand the rational functions of N in powers of ϵ and discard powers of order 1 or higher. Expansion can be done using geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{when } |x| < 1, \tag{84}$$

and the result is

$$\mathcal{I} = \frac{-i}{(4\pi)^{2-\epsilon}} \left(-q^2\right)^{-\epsilon} \Gamma\left(1+\epsilon\right) \frac{\left(\Gamma\left(1-\epsilon\right)\right)^2}{\Gamma\left(1-2\epsilon\right)} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8\right) + \mathcal{O}\left(\epsilon\right). \tag{85}$$

We use $\mathcal{O}(\epsilon)$ to emphasize that the equation holds up to some finite terms of the order at least one in powers of ϵ. We then substituted this result back to Equation (70) with (see Appendix A and E)

$$\left(g_S^{(N)}\right)^2 = 4\pi\alpha_S \left(m_D^2\right)^{\epsilon}, \tag{86}$$

$$\Gamma\left(1+\epsilon\right)\Gamma\left(1-\epsilon\right) = 1 + \frac{\pi^2}{6}\epsilon^2 + \mathcal{O}\left(\epsilon^4\right) \tag{87}$$

and conclude that

$$\begin{aligned} g^{\mu\nu} M \left(W_{a(V)}\right)_{\mu\nu} = {} & \frac{4C_a^2}{3}\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon} \delta\left(1-w\right)\left(1-\epsilon\right) \\ & \times \frac{\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \frac{\pi^2}{3}\right). \end{aligned} \tag{88}$$

3.2 Quark self-energy correction

In this subsection we examine the subgraph of Figure 5 and why the self-energy corrections it defines can be discarded in our calculation. We discuss the arising self-energy integral $\Sigma(p)$ without the color factors as the needed properties do not depend on them. We also take quarks to be massless, but do not assume at first that the incoming quark in the amplitude is on its mass shell. Without the color factors,

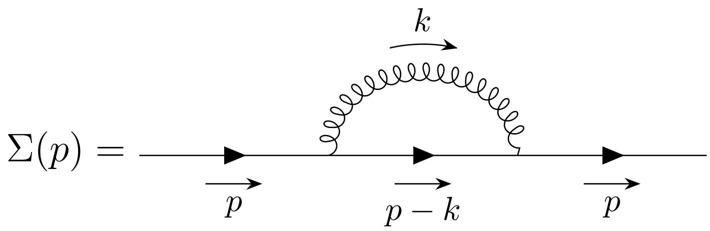

Figure 5. Graph of the one loop self-energy correction with incoming 4-momentum p.

the integral $\Sigma(p)$ becomes

$$\Sigma(p) = \int \frac{\mathrm{d}^N k}{(2\pi)^N} \frac{\gamma_\alpha \left(\not{p} - \not{k}\right) \gamma_\beta}{(p-k)^2 k^2} \left(g^{\mu\nu} + \eta \frac{k^\alpha k^\beta}{k^2}\right). \tag{89}$$

With the anticommutation relations of the gamma matrices given in Appendix A

$$\not{k}\not{p}\not{k} = 2p \cdot k \not{k} - k^2 \not{p} = k^2 \left(\not{k} - \not{p}\right) + \left(p^2 - (p-k)^2\right) \not{k} \tag{90}$$

and

$$\Sigma(p) = \int \frac{\mathrm{d}^N k}{(2\pi)^N} \left[(2-N) \frac{\not{p} - \not{k}}{(p-k)^2 k^2} + \eta \left(\frac{-\not{p}}{(p-k)^2 k^2} + \frac{p^2 \not{k}}{(p-k)^2 k^4} - \frac{\not{k}}{k^4} \right) \right]. \tag{91}$$

The last term here is an odd function of the components of the k so it produces a zero. The calculation of the loop integral proceeds as in Section 3.1. We introduce a Feynman parametrization and change the loop integral variable into $\ell = k - x'p$. By defining $\Delta = -p^2 x'(1-x')$ we get

$$\Sigma(p) = \int_0^1 \mathrm{d}x' \int \frac{\mathrm{d}^N \ell}{(2\pi)^N} \left[(2-N) \frac{(1-x')\not{p}}{(\ell^2 - \Delta)^2} - \eta \not{p} \left(\frac{1}{(\ell^2 - \Delta)^2} + \frac{2\Delta}{(\ell^2 - \Delta)^3} \right) \right]. \tag{92}$$

Note that the terms with odd number of components of ℓ produced only zeros. Again the identity (79) (given in Appendix B) is used to evaluate the loop integrals. The integrals with respect to x' fit straight into the definition of the Beta function and after that we arrive at

$$\Sigma(p) = \frac{i\not{p}}{(4\pi)^{\frac{N}{2}}} \Gamma\left(2 - \frac{N}{2}\right)\left(-p^2\right)^{\frac{N}{2}-2} \frac{\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(N-2\right)}\left(1-\eta\right). \tag{93}$$

Here we can see two things. First $\Sigma(p) = 0$ if we use gauge with $\eta = 1$ (Landau gauge). Only the vertex correction term in our cross section could have been dependent of the chosen gluon gauge, but as we showed in the previous section (after Equation (65)) the η dependent part is zero. So, we can select the right gauge and after that graphs with self-energy corrections have a value of zero independent of the value of p^2 and the gauge choice does not affect the rest of the calculations.

On the other hand, the on shell version of $\Sigma(p)$ would be regularized with $N > 4$. Then

$$\left(-p^2\right)^{\frac{N}{2}-2} = 0 \tag{94}$$

and the value of the self-energy correction graphs is again zero. In dimensional regularization it is hence consistent to set $\Sigma(p) = 0$ when considering massless quarks on mass shell with any gauge η [21].

3.3 Gluon radiation

The graphs $A_{a(C)}$ for a quark a are given in Figure 6. With the Feynman rules given in Appendix A we can evaluate the sum of these graphs as

$$\begin{aligned} e^{(N)}\left(A_{a(C)}\right)_{\mu}^{s'si'jc\lambda} &= \overline{u}\left(p',s'\right)\left(-ie^{(N)}C_a\gamma_\mu\right)\frac{i\not{p}_a}{p_a^2}\left(-ig_S^{(N)}\gamma_\alpha\left(t^c\right)_{i'j}\right)u\left(p,s\right)\left(\varepsilon^*\right)^\alpha\left(k,\lambda\right) \\ &+ \overline{u}\left(p',s'\right)\left(-ig_S^{(N)}\gamma_\alpha\left(t^c\right)_{i'j}\right)\frac{i\not{p}_b}{p_b^2}\left(-ie^{(N)}C_a\gamma_\mu\right)u\left(p,s\right)\left(\varepsilon^*\right)^\alpha\left(k,\lambda\right) \\ &= -ie^{(N)}C_a g_S^{(N)}\left(t^c\right)_{i'j}\overline{u}\left(p',s'\right)\Lambda_{\alpha\mu}u\left(p,s\right)\left(\varepsilon^*\right)^\alpha\left(k,\lambda\right). \end{aligned} \tag{95}$$

Here ε is the polarization 4-vector of the outgoing gluon. We also introduced an auxiliary tensor

$$\Lambda_{\alpha\mu} = \frac{1}{p_a^2}\gamma_\mu\not{p}_a\gamma_\alpha + \frac{1}{p_b^2}\gamma_\alpha\not{p}_b\gamma_\mu \tag{96}$$

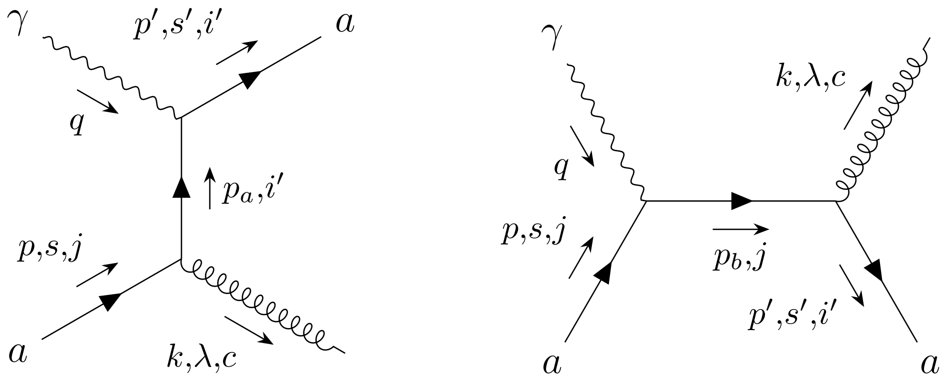

Figure 6. Gluon radiation (final state gluon) graphs $A_{a(C)}$. For the incoming and outgoing quarks the triplet of symbols next to the momentum arrow denote 4-momentum, spin component and color. For the gluon the triplet is 4-momentum, polarization and color. For the virtual quark the two symbols are 4-momentum and color.

to simplify the expression. As described in Section 2.3 we have

$$C_a^2 \left(\mathcal{W}_{a(C)}\right)_{\mu\nu} = \frac{1}{2}\frac{1}{3} \sum_{s,s',i',j,c,\lambda} \left(A_{a(C)}\right)_{\mu}^{s'si'jc\lambda} \left(\left(A_{a(C)}\right)_{\nu}^{s'si'jc\lambda}\right)^{\dagger}, \tag{97}$$

where the average is taken over the incoming spin index s (two states) and the incoming color j (three states). With the color matrix trace sum result from Appendix A and the polarization sum result [21, 26]

$$\sum_{\lambda} \varepsilon^{\alpha}(k,\lambda) (\varepsilon^{*})^{\beta} (k,\lambda) = -g^{\alpha\beta} + \frac{k^{\alpha}\bar{k}^{\beta} + k^{\beta}\bar{k}^{\alpha}}{k \cdot \bar{k}}, \tag{98}$$

where $\bar{k}$ is the same as k but with spatial components negated, we get

$$C_a^2 \left(\mathcal{W}_{a(C)}\right)_{\mu\nu} = \frac{2C_a^2}{3} \left(g_S^{(N)}\right)^2 \mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{p}\Lambda_{\nu\beta}\right) \left(-g^{\alpha\beta} + \frac{k^{\alpha}\bar{k}^{\beta} + k^{\beta}\bar{k}^{\alpha}}{k \cdot \bar{k}}\right). \tag{99}$$

If the above process was done for the antiquark $\bar{a}$ the result here would be the same with similar reasoning as was described in the Chapter 3.1. We included the k dependent part of the gluon polarization sum into this expression, but in the end it produces only p^2, p'^2 and k^2 dependent terms, which are set to zero. This property can be seen by noting that

$$\not{p}'\Lambda_{\alpha\mu}\not{p}k^{\alpha} = \not{p}'\left(\frac{1}{p_a^2}\gamma_\mu\not{p}_a\not{k} + \frac{1}{p_b^2}\not{k}\not{p}_b\gamma_\mu\right)\not{p} \tag{100}$$

$$= \not{p}'\left(\frac{1}{p_a^2}\gamma_\mu\not{p}\not{k} + \frac{1}{p_b^2}\not{k}\not{p}'\gamma_\mu\right)\not{p} + \mathcal{O}\left(k^2\right) \tag{101}$$

$$= 2\not{p}'\gamma_\mu\not{p}\left(\frac{p\cdot k}{p_a^2} + \frac{p'\cdot k}{p_b^2}\right) + \mathcal{O}\left(p^2,p'^2,k^2\right) \tag{102}$$

$$= 0 + \mathcal{O}\left(p^2,p'^2,k^2\right), \tag{103}$$

as $p_a = p - k$ and $p_b = p' + k$. By $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ we also have

$$\left(\not{p}\Lambda_{\mu\alpha}\not{p}'k^{\alpha}\right)^\dagger = \gamma^0\not{p}'\Lambda_{\alpha\mu}\not{p}k^{\alpha}\gamma^0 \tag{104}$$

$$= 0 + \mathcal{O}\left(p^2,p'^2,k^2\right). \tag{105}$$

Hence the contractions with k produce zero, when the masses p^2, p'^2 and k^2 are set to zero. So, we need only to consider the contractions by $P^\mu P^\nu$ and metric tensor for the trace

$$\mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{p}\Lambda_{\nu\beta}\right)g^{\alpha\beta}. \tag{106}$$

First of these is with results from Appendix C

$$p^\mu p^\nu g^{\alpha\beta}\mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{p}\Lambda_{\nu\beta}\right) = -16\frac{N-2}{p_a^4}\left(k\cdot p\right)^2 p\cdot p' \tag{107}$$

$$= -4\left(N-2\right)p\cdot p'. \tag{108}$$

The calculation of the second contraction of the trace is given in Appendix C and the result reads

$$\mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{p}\Lambda^{\mu\alpha}\right) = -2\left(N-2\right)\left(\left(N-2\right)\left(\frac{p_a^2}{p_b^2} + \frac{p_b^2}{p_a^2}\right) - \frac{4}{p_a^2p_b^2}q^2p\cdot p' + 2\left(N-4\right)\right). \tag{109}$$

With (99) we then get

$$C_a^2 P^\mu P^\nu \mathcal{W}_{\mu\nu} = \frac{2C_a^2}{3\xi^2}\left(g_S^{(N)}\right)^2 4\,(N-2)\,p\cdot p', \tag{110}$$

$$C_a^2 g^{\mu\nu}\mathcal{W}_{\mu\nu} = \frac{4C_a^2}{3}\left(g_S^{(N)}\right)^2 (N-2)\left((N-2)\left(\frac{p_a^2}{p_b^2}+\frac{p_b^2}{p_a^2}\right)\right.$$
$$\left. -\frac{4q^2}{p_a^2 p_b^2}p\cdot p' + 2\,(N-4)\right). \tag{111}$$

Next we will need to transform this expression into a function of the variable v used in the two dimensional phase space integral result given in Appendix F to get the hadronic tensor contribution of the gluon radiation. The other variables used are the photon virtuality Q^2 and w given in Equation (12). Variable v is defined in Appendix F and the most relevant version is the Lorentz invariant form

$$1 - v = \frac{k\cdot p}{p\cdot q}. \tag{112}$$

With these variables

$$p_a^2 = -2k\cdot p = -\frac{Q^2}{w}\,(1-v)\,, \tag{113}$$

$$p_b^2 = 2p\cdot q + q^2 = \frac{Q^2}{w}\,(1-w)\,, \tag{114}$$

$$2p\cdot p' = 2p\cdot(p+q-k) = \frac{Q^2}{w} - \frac{Q^2}{w}\,(1-v) = \frac{Q^2}{w}v. \tag{115}$$

Using the above three identities Equations (110) and (111) transform into

$$C_a^2 P^\mu P^\nu \mathcal{W}_{\mu\nu} = \frac{4C_a^2}{3\xi^2}\left(g_S^{(N)}\right)^2 (N-2)\,\frac{Q^2}{w}v, \tag{116}$$

$$C_a^2 g^{\mu\nu}\mathcal{W}_{\mu\nu} = -\frac{4C_a^2}{3}\left(g_S^{(N)}\right)^2 (N-2)\left((N-2)\left(\frac{1-v}{1-w}+\frac{1-w}{1-v}\right)\right.$$
$$\left. + \frac{4wv}{(1-w)\,(1-v)} - 2\,(N-4)\right). \tag{117}$$

These equations give the contractions of the hadronic tensor contribution (41) together with phase space integral (302) and Mandelstam variable $\hat{s} = p_b^2$:

$$P^\mu P^\nu M\left(W_{a(C)}\right)_{\mu\nu} = \frac{2C_a^2}{3\xi^2}\frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}}\frac{N-2}{\Gamma\left(\frac{N}{2}-1\right)}\left(\frac{Q^2}{w}\right)^{\frac{N}{2}-1}(1-w)^{\frac{N}{2}-2} \times \int_0^1 \mathrm{d}v\,(v)^{\frac{N}{2}-1}(1-v)^{\frac{N}{2}-2}, \tag{118}$$

$$-g^{\mu\nu} M\left(W_{a(C)}\right)_{\mu\nu} = \frac{2C_a^2}{3}\frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}}\frac{N-2}{\Gamma\left(\frac{N}{2}-1\right)}\left(\frac{Q^2}{w}(1-w)\right)^{\frac{N}{2}-2} \times \int_0^1 \mathrm{d}v v^{\frac{N}{2}-2}(1-v)^{\frac{N}{2}-2}\left((N-2)\left(\frac{1-v}{1-w}+\frac{1-w}{1-v}\right) + \frac{4wv}{(1-w)(1-v)} - 2(N-4)\right). \tag{119}$$

Now the v integrals can be evaluated with the Beta function identity

$$\int_0^1 t^a (1-t)^b \,\mathrm{d}t = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \tag{120}$$

given in Appendix B. We then substitute the definition of α_S and expand the parts with $N = 4 - 2\epsilon$ in powers of ϵ in a similar manner as in the end of the Chapter 3.1. This leaves us with

$$P^\mu P^\nu M\left(W_{a(C)}\right)_{\mu\nu} = \frac{4C_a^2}{3}\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^\epsilon (1-\epsilon)\left(\frac{w}{1-w}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{Q^2}{4x^2}w, \tag{121}$$

$$-g^{\mu\nu} M\left(W_{a(C)}\right)_{\mu\nu} = \frac{4C_a^2}{3}\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^\epsilon (1-\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\tilde{g}(w,\epsilon), \tag{122}$$

where we introduced an auxiliary function

$$\tilde{g}(w,\epsilon) = \left(\frac{w}{1-w}\right)^\epsilon \left(\frac{1-\epsilon}{2(1-w)(1-2\epsilon)} - \frac{1-\epsilon}{\epsilon}\left(1-w+\frac{2w}{1-w}\frac{1}{1-2\epsilon}\right) + \frac{2\epsilon}{1-2\epsilon}\right) \tag{123}$$

to simplify the presentation. It is defined to include terms that are divergent in the limit $\epsilon \to 0$ and $w \to 1$. We want to expand $P^\mu P^\nu M\left(W_{a(C)}\right)_{\mu\nu}$ and $\tilde{g}$ so that the divergent functions of w and ϵ can be seen explicitly. To achieve that we use the series expansion

$$r^{\epsilon}=\sum_{k=0}^{\infty}\frac{\epsilon^{k}}{k!}\left(\ln\left(r\right)\right)^{k} \tag{124}$$

and the distributional identities

$$\frac{1}{\left(1-x\right)^{1+\epsilon}}=\left(\frac{1}{1-x}\right)_{+}-\frac{1}{\epsilon}\delta\left(1-x\right) \tag{125}$$

$$\frac{1}{\epsilon\left(1-x\right)^{1+\epsilon}}=\frac{1}{\epsilon}\left(\frac{1}{1-x}\right)_{+}-\frac{1}{\epsilon^{2}}\delta\left(1-x\right)-\left(\frac{\ln\left(1-x\right)}{1-x}\right)_{+} \tag{126}$$

with the plus-distributions introduced in Appendix D. This leaves us with

$$P^{\mu}P^{\nu}M\left(W_{a(C)}\right)_{\mu\nu}=\frac{4C_{a}^{2}}{3}\frac{\alpha_{S}}{2\pi}\left(\frac{4\pi m_{D}^{2}}{Q^{2}}\right)^{\epsilon}\left(1-\epsilon\right)\frac{\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}\frac{Q^{2}}{4x^{2}}w+\mathcal{O}\left(\epsilon\right) \tag{127}$$

$$\tilde{g}(w,\epsilon)=\left(\frac{w}{1-w}\right)^{\epsilon}\left(-\frac{1}{\epsilon}\frac{1+w^{2}}{1-w}+3-w-\frac{3+7\epsilon}{2\left(1-w\right)}+\mathcal{O}\left(\epsilon\right)\right) \tag{128}$$

$$=\delta\left(1-w\right)\left(\frac{2}{\epsilon^{2}}+\frac{3}{2\epsilon}+\frac{7}{2}\right)-\left(\frac{1}{\epsilon}+\ln w\right)\frac{1+w^{2}}{\left(1-w\right)_{+}}$$
$$+\left(1+w^{2}\right)\left(\frac{\ln\left(1-w\right)}{1-w}\right)_{+}-\frac{3}{2}\left(\frac{1}{1-w}\right)_{+}+3-w+\mathcal{O}\left(\epsilon\right). \tag{129}$$

In the first term of the function $\tilde{g}$ we have substituted $(1+w^2)=2$ due to it being multiplied by a delta function with argument zero at $w=1$. Note also that $\mathcal{O}\left(\epsilon\right)$ includes only terms that are integrable with finite integrals respect to w over every subinterval $[a,b]\subset[0,1]$. That is to say only terms that are sure to be finite in our results after the limit $\epsilon\to 0$.

3.4 Initial state gluon

All the initial gluon graphs are of the types given in Figure 7 with different quarks a. We see that for each quark a one gets the two graphs from the graphs of the gluon radiation (Figure 6) by interchanging the initial state parton and final state gluon legs. This crossing symmetry allows us to find the modulus of $A_{ig(a)}$ for each quark a from that of $A_{a(C)}$ by changing $k\to -p$, $p\to -k$ and the overall sign from $+$ to $-$ [26]. The last change is due to only one of the changed lines being fermion and the

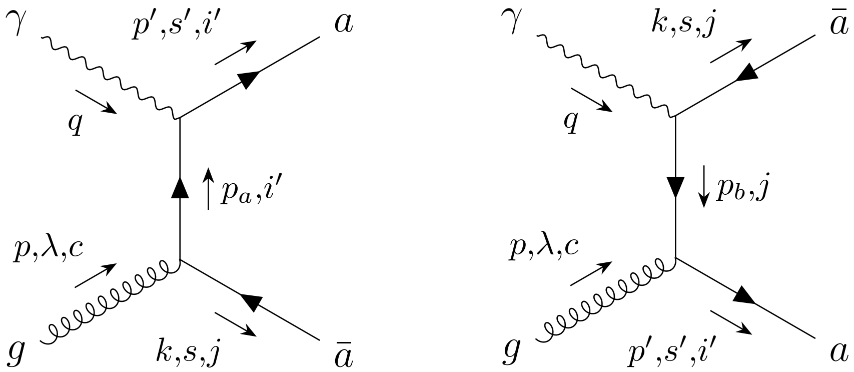

Figure 7. Initial state gluon graphs $A_{ig(a)}$. For the incoming gluon the triplet of symbols next to the momentum arrow denote 4-momentum, polarization and color. For the outgoing quark and antiquark the triplet is 4-momentum, spin component and color. For the virtual quark the two symbols are 4-momentum and color.

other a boson. Note that the crossing symmetry accounts here only for the change of the sign in k and p. The change of their places is due to us renaming the 4-momenta of the gluon and quark legs. This is done so that k is outgoing and p is incoming 4-momentum similarly to other graphs in the calculation. Also this ensures that the 4-momenta p_a and p_b can be left as they were in the results of Section 3.3.

As described in Section 2.3 we have

$$(\mathcal{W}_{ig})_{\mu\nu} = \frac{1}{8}\frac{1}{N-2}\sum_a \sum_{s,s',i',j,c,\lambda} \left(A_{ig(a)}\right)_{\mu}^{s'si'jc\lambda} \left(\left(A_{ig(a)}\right)_{\nu}^{s'si'jc\lambda}\right)^{\dagger}, \tag{130}$$

where the average is over all the initial polarizations λ and colors c. There are 8 different colors and in $N = 4-2\epsilon$ dimensions $N-2 = 2(1-\epsilon)$ gluon polarization states.

Using the crossing symmetry for graphs corresponding to each quark a as explained above we get with the results given in Section 3.3

$$(\mathcal{W}_{ig})_{\mu\nu} = -\frac{1}{8}\frac{1}{N-2}\cdot 2\cdot 3\cdot \sum_a C_a^2 \left(\mathcal{W}_{a(C)}\right)_{\mu\nu}\Big|_{k\leftrightarrow -p} \tag{131}$$

$$= -\frac{\sum_a C_a^2}{2(N-2)}\left(g_S^{(N)}\right)^2 \operatorname{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{k}\Lambda_{\nu\beta}\right) g^{\alpha\beta}. \tag{132}$$

Note that the quark summation acts here only into the squares of the fractional charges. To keep each a summation similar between graphs we change from here on the summation to go over the antiquarks as well as quarks and add a multiplier $\frac{1}{2}$. This is done to avoid confusion as the other summations over a appearing in this thesis go over both quarks and antiquarks.

Now the contraction of the partonic tensor with metric tensor can be deduced from that of in Section 3.3. The simplification process for

$$\mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{p}\Lambda^{\mu\alpha}\right) \tag{133}$$

presented in Appendix C does not change if we interchange p and $-k$ in the whole trace including p_a and p_b. Hence, we get through that interchange

$$\begin{aligned} g^{\mu\nu}\left(\mathcal{W}_{ig}\right)_{\mu\nu} = \frac{\sum_a C_a^2}{2}\left(g_S^{(N)}\right)^2 \Bigg((N-2)\left(\frac{p_a^2}{p_b^2}+\frac{p_b^2}{p_a^2}\right) \\ + \frac{8q^2}{p_a^2 p_b^2} k\cdot p' + 2(N-4)\Bigg). \end{aligned} \tag{134}$$

The other contraction needs to be calculated from the beginning, as the contraction is still done by p but the trace has p and $-k$ switched. Using the gamma matrix properties given in Appendix C we get

$$p^\mu p^\nu g^{\alpha\beta}\mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{k}\Lambda_{\nu\beta}\right) = -16k\cdot p'. \tag{135}$$

Therefore

$$P^\mu P^\nu \left(\mathcal{W}_{ig}\right)_{\mu\nu} = \frac{\sum_a C_a^2}{\xi^2 (N-2)}\left(g_S^{(N)}\right)^2 4k\cdot p'. \tag{136}$$

Next we will change this into the same variables w, v and Q^2 that were used in Chapter 3.3 and are used in the phase space results in Appendix F. Here

$$p_a^2 = -2p\cdot k = -\frac{Q^2}{w}(1-v), \tag{137}$$

$$p_b^2 = -2p\cdot p' = -\frac{Q^2}{w}v, \tag{138}$$

$$2k\cdot p' = (p+q)^2 = \frac{Q^2}{w}(1-w). \tag{139}$$

Using these identities

$$P^{\mu}P^{\nu}\left(\mathcal{W}_{ig}\right)_{\mu\nu} = \frac{2\sum_a C_a^2}{x^2\left(N-2\right)}\left(g_S^{(N)}\right)^2 Q^2 w\left(1-w\right) \tag{140}$$

$$g^{\mu\nu}\left(\mathcal{W}_{ig}\right)_{\mu\nu} = \frac{\sum_a C_a^2}{2}\left(g_S^{(N)}\right)^2\left(\left(N-2\right)\left(\frac{1-v}{v}+\frac{v}{1-v}\right)\right.$$
$$\left. -4\frac{w\left(1-w\right)}{v\left(1-v\right)}+2\left(N-4\right)\right). \tag{141}$$

We get the contractions of the hadronic tensor contributions as defined in Equation (42) by result (302). Here $\hat{s} = 2k\cdot p'$. The remaining integrals are simplified by the Beta function identity (120)

$$P^{\mu}P^{\nu}M\left(W_{ig}\right)_{\mu\nu} = \frac{\sum_a C_a^2\left(g_S^{(N)}\right)^2}{\left(4\pi\right)^{\frac{N}{2}}x^2\left(N-2\right)}\frac{\Gamma\left(\frac{N}{2}-1\right)}{\Gamma\left(N-2\right)}w^2\left(\frac{Q^2}{w}\left(1-w\right)\right)^{\frac{N}{2}-1} \tag{142}$$

$$g^{\mu\nu}M\left(W_{ig}\right)_{\mu\nu} = \frac{\sum_a C_a^2}{\left(4\pi\right)^{\frac{N}{2}}}\left(g_S^{(N)}\right)^2\left(\frac{Q^2}{w}\left(1-w\right)\right)^{\frac{N-4}{2}}\frac{\Gamma\left(\frac{N}{2}-1\right)}{\Gamma\left(N-2\right)}$$
$$\times\left(\frac{\left(N-2\right)^2}{2\left(N-4\right)}-4w\left(1-w\right)\frac{N-3}{N-4}+\frac{1}{2}\left(N-4\right)\right). \tag{143}$$

As the final step these contractions are expanded in powers of ϵ, and terms of at least linear in ϵ are discarded as was done to form Equations (127) and (129). We also substitute α_S in the place of $g_S^{(N)}$ as in the previous sections. We arrive at

$$P^{\mu}P^{\nu}M\left(W_{ig}\right)_{\mu\nu} = \frac{\sum_a C_a^2}{2}\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon}\frac{\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}\frac{Q^2}{2x^2}w\left(1-w\right)+\mathcal{O}\left(\epsilon\right) \tag{144}$$

$$-g^{\mu\nu}M\left(W_{ig}\right)_{\mu\nu} = -\frac{\sum_a C_a^2}{2}\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon}\frac{\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}$$
$$\times\left(\frac{1}{\epsilon}+\ln\left(\frac{w}{1-w}\right)\right)\left(w^2+\left(1-w\right)^2\right)+\mathcal{O}\left(\epsilon\right). \tag{145}$$

3.5 Structure functions and cross section

We can now sum the contractions for the full hadronic tensor (39) and after that the structure functions (9) and (10) and the DIS NLO cross section (2). The gluon radiation result (127) and the initial state gluon result (144) combine into

$$\frac{4x^2}{Q^2}P^\mu P^\nu M(W)_{\mu\nu} = \frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\sum_a C_a^2 \left(\int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)(1-\epsilon)\frac{4w}{3} + \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right) w(1-w)\right) \tag{146}$$

and when the rest of the terms are expanded in powers of ϵ

$$\frac{4x^2}{Q^2}P^\mu P^\nu M(W)_{\mu\nu} = \frac{\alpha_S}{2\pi}\sum_a C_a^2 \left(\int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)\frac{4w}{3} + \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right) w(1-w)\right) + \mathcal{O}(\epsilon)\,. \tag{147}$$

Note that there are no divergent terms in this contraction when $\epsilon \to 0$. The metric tensor contractions of LO contribution (22), vertex correction (88), gluon radiation (122) and (129) and initial state gluon (145) combine into

$$\begin{aligned}
-g^{\mu\nu}M(W)_{\mu\nu} = \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)\Bigg[&(1-\epsilon)\,\delta(1-w) + \frac{4}{3}\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^\epsilon \\
&\times(1-\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\Bigg(-\delta(1-w)\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8+\frac{\pi^2}{3}\right) \\
&+\delta(1-w)\left(\frac{2}{\epsilon^2}+\frac{3}{2\epsilon}+\frac{7}{2}\right) - \left(\frac{1}{\epsilon}+\ln w\right)\frac{1+w^2}{(1-w)_+} \\
&+\left(1+w^2\right)\left(\frac{\ln(1-w)}{1-w}\right)_+ - \frac{3}{2}\left(\frac{1}{1-w}\right)_+ + 3 - w\Bigg)\Bigg] \\
&-\sum_a \frac{C_a^2}{2}\int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right)\frac{\alpha_S}{2\pi}\left(\frac{4\pi m_D^2}{Q^2}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\
&\times\left(\frac{1}{\epsilon}+\ln\left(\frac{w}{1-w}\right)\right)\left(w^2+(1-w)^2\right).
\end{aligned} \tag{148}$$

Here we notice that the ϵ^{-2} poles of the vertex correction and the gluon radiation cancel each other and we are left only with simple poles. We then do rest of the expansions respect to ϵ using results of Appendix E:

$$
\begin{aligned}
-g^{\mu\nu} M\left(W\right)_{\mu\nu} = {} & (1-\epsilon)\sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)\Bigg[\delta\left(1-w\right)+\frac{4}{3}\frac{\alpha_S}{2\pi} \\
& \times\left(-\left(\frac{1}{\hat{\epsilon}}-\ln\left(\frac{Q^2}{m_D^2}\right)\right)\left(\delta\left(1-w\right)+\frac{1+w^2}{\left(1-w\right)_+}\right)\right. \\
& -\delta\left(1-w\right)\left(\frac{9}{2}+\frac{\pi^2}{3}\right)-\ln w\frac{1+w^2}{\left(1-w\right)_+} \\
& \left.+\left(1+w^2\right)\left(\frac{\ln\left(1-w\right)}{1-w}\right)_+ -\frac{3}{2}\left(\frac{1}{1-w}\right)_+ +3-w\right)\Bigg] \\
& -(1-\epsilon)\sum_a \frac{C_a^2}{2}\int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right)\frac{\alpha_S}{2\pi} \\
& \times\left(\frac{1}{\hat{\epsilon}}-\ln\left(\frac{Q^2}{m_D^2}\right)+1+\ln\left(\frac{w}{1-w}\right)\right)\left(w^2+\left(1-w\right)^2\right). \qquad (149)
\end{aligned}
$$

These contractions determine the structure functions as

$$
\begin{aligned}
\frac{1}{x}F_2\left(x,Q^2\right) = {} & \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)\Bigg[\delta\left(1-w\right) \\
& +\frac{\alpha_S}{2\pi}\left(C_{q,2}\left(w\right)+\left(\ln\left(\frac{Q^2}{m_D^2}\right)-\frac{1}{\hat{\epsilon}}\right)P_{qq}\left(w\right)\right)\Bigg] \\
& +\sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right)\frac{\alpha_S}{2\pi}\Bigg[C_{g,2}\left(w\right) \\
& +\left(\ln\left(\frac{Q^2}{m_D^2}\right)-\frac{1}{\hat{\epsilon}}\right)P_{qg}\left(w\right)\Bigg] \qquad (150) \\
2F_1\left(x,Q^2\right) = {} & \frac{1}{x}F_2\left(x,Q^2\right)-\sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)\frac{4}{3}\cdot 2w \\
& -\sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right)\frac{\alpha_S}{2\pi}2w\left(1-w\right), \qquad (151)
\end{aligned}
$$

where we introduced Altarelli-Parisi splitting functions [27]

$$
P_{qq}\left(w\right) = \frac{4}{3}\left[\delta\left(1-w\right)+\frac{1+w^2}{\left(1-w\right)_+}\right] \qquad (152)
$$

$$
P_{qg}\left(w\right) = \frac{1}{2}\left(w^2+\left(1-w\right)^2\right), \qquad (153)
$$

and coefficient functions

$$C_{q,2}(w) = \frac{4}{3}\left[\left(1+w^2\right)\left(\frac{\ln(1-w)}{1-w}\right)_+ - \frac{3}{2}\left(\frac{1}{1-w}\right)_+ - \frac{1+w^2}{(1-w)_+}\ln w \right.$$
$$\left. + 3 + 2w - \delta(1-w)\left(\frac{9}{2}+\frac{\pi^2}{3}\right)\right] \tag{154}$$
$$C_{g,2}(w) = \ln\left(\frac{1-w}{w}\right)P_{qg}(w) + \frac{1}{2}\left(8w(1-w)-1\right). \tag{155}$$

Now we renormalize the parton density functions according to the $\overline{\mathrm{MS}}$ scheme. The renormalized density functions are [10]

$$f_{a/H}\left(x,Q_f^2\right) = \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w}\right)\left[\delta(1-w) + \frac{\alpha_S}{2\pi}\left(\ln\left(\frac{Q_f^2}{m_D^2}\right) - \frac{1}{\hat{\epsilon}}\right)P_{qq}(w)\right]$$
$$+ \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w}\right)\frac{\alpha_S}{2\pi}\left(\ln\left(\frac{Q_f^2}{m_D^2}\right) - \frac{1}{\hat{\epsilon}}\right)P_{qg}(w) \tag{156}$$
$$f_{g/H}\left(x,Q_f^2\right) = f_{g/H}(x) + \mathcal{O}(\alpha_S). \tag{157}$$

We choose here the factorization scale to be $Q_f^2 = Q^2$. This leaves the expressions of the structure functions with no explicit logarithmic scale dependence, but still logarithmic Q^2 dependence is left into the density functions after this absorption.

Note also that we are doing the calculation only in the first order of the α_S and the gluon density function in Equations (150) and (151) appear in the first order correction terms. Because of this the exact redefinition of the gluon density could be determined only by going into the next-to-next-to leading order of α_S. Here we merely recognize that the gluon density has a scale dependence similarly to the quark and antiquark densities.

Using the renormalized parton densities we get our NLO structure functions in the $\overline{\text{MS}}$ scheme:

$$\begin{aligned}
\frac{1}{x}F_2\left(x,Q^2\right) &= \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w},Q^2\right)\left(\delta\left(1-w\right)+\frac{\alpha_S}{2\pi}C_{q,2}\left(w\right)\right) \\
&\quad + \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w},Q^2\right)\frac{\alpha_S}{2\pi}C_{g,2}\left(w\right) \qquad (158)\\
2F_1\left(x,Q^2\right) &= \frac{1}{x}F_2\left(x,Q^2\right) - \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w},Q^2\right)\frac{4}{3}\cdot 2w \\
&\quad - \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w},Q^2\right)\frac{\alpha_S}{2\pi}2w\left(1-w\right). \qquad (159)
\end{aligned}$$

These structure functions coincide with those given in References [10, 16]. We conclude this section with the NLO cross section determined by the above structure functions:

$$\begin{aligned}
\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha_{em}^2}{Q^4}&\left[\sum_a C_a^2 f_{a/H}\left(x,Q^2\right)\frac{\left(1-y\right)^2+1}{2}\right. \qquad (160)\\
&+ \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{a/H}\left(\frac{x}{w},Q^2\right)\frac{\alpha_S}{2\pi}\left(C_{q,2}\left(w\right)\frac{\left(1-y\right)^2+1}{2}-\frac{4}{3}wy^2\right) \\
&\left.+ \sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w} f_{g/H}\left(\frac{x}{w},Q^2\right)\frac{\alpha_S}{2\pi}\left(C_{g,2}\left(w\right)\frac{\left(1-y\right)^2+1}{2}-y^2w\left(1-w\right)\right)\right].
\end{aligned}$$

Here again the sum a runs over quarks and antiquarks and the coefficient functions $C_{q,2}\left(w\right)$ and $C_{g,2}\left(w\right)$ are given in Equations (154) and (155).

4 NLO SIDIS calculation

In this section we present the calculation of the SIDIS structure functions and the cross section in NLO. The calculation proceeds similarly to DIS NLO calculation given in Section 3 with the differences described in Section 2.3.

Virtual vertex correction, gluon radiation and initial state gluon graphs are considered on their own separate subsections. In those subsections we calculate the contractions of hadronic tensor contribution $W_{\mu\nu}^{ab}$ with the metric tensor and $P^\mu P^\nu$ for each subprocess. In the last subsection we combine the results of the preceding subsections into full SIDIS structure functions F_j^h, renormalize the parton density functions and the fragmentation functions and present the differential NLO SIDIS cross section.

We calculate here the SIDIS results given in Reference [17]. We use a different renormalization scheme as in Reference [17], though they provide information needed to compare the results in the $\overline{\text{MS}}$ scheme. A summary of the $\overline{\text{MS}}$ scheme results can also be found in Reference [10].

4.1 Vertex correction

As described in Section 2.3, the virtual correction contributes to the hadronic tensor for the initial state quark or antiquark a as

$$\left(W_{ab(V)}\right)_{\mu\nu} = \left(W_{a(V)}\right)_{\mu\nu} \delta_{ab}\delta\left(z-\zeta\right). \tag{161}$$

Here $\left(W_{a(V)}\right)_{\mu\nu}$ is the DIS virtual correction given in Section 3.1. Using the results derived there we get

$$P^\mu P^\nu M \left(W_{ab(V)}\right)_{\mu\nu} = 0 \tag{162}$$

$$g^{\mu\nu} M \left(W_{ab(V)} \right)_{\mu\nu} = \frac{4C_a^2}{3} \frac{\alpha_S}{2\pi} \left(\frac{4\pi m_D^2}{Q^2} \right)^{\epsilon} \delta \left(1 - w\right) \left(1 - \epsilon\right) \frac{1}{\Gamma \left(1 - \epsilon\right)} \times \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 \right) \delta_{ab} \delta \left(z - \zeta\right). \tag{163}$$

Here we used the relation

$$\frac{1}{\Gamma \left(1 - \epsilon\right)} = \Gamma \left(1 + \epsilon\right) \frac{\Gamma \left(1 - \epsilon\right)^2}{\Gamma \left(1 - 2\epsilon\right)} + \mathcal{O} \left(\epsilon^3\right) \tag{164}$$

presented in Appendix E. The metric contraction can be further expanded into

$$g^{\mu\nu} M \left(W_{ab(V)} \right)_{\mu\nu} = \frac{4C_a^2}{3} \frac{\alpha_S}{2\pi} \left(1 - \epsilon\right) \delta \left(1 - w\right) \delta \left(z - \zeta\right) \delta_{ab} \times \left(\frac{2}{\epsilon^2} \frac{\left(\frac{4\pi m_D^2}{Q^2} \right)^{\epsilon}}{\Gamma \left(1 - \epsilon\right)} + \frac{3}{\hat{\epsilon}} - 3 \ln \left(\frac{Q^2}{m_D^2} \right) + 8 \right). \tag{165}$$

where (see Appendix E)

$$\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} \frac{\left(4\pi\right)^{\epsilon}}{\Gamma \left(1 - \epsilon\right)}. \tag{166}$$

Note that the second order pole in Equation (165) is not expanded further as it will sum to zero with a similar term from the gluon radiation contraction similarly to the DIS calculation.

4.2 Gluon radiation

As explained in Section 2.3 we get the gluon radiation contribution to the hadronic tensor as

$$4\pi M \left(W_{ab(C)} \right)_{\mu\nu} = PS_{p',k} \left[C_a^2 \left(\mathcal{W}_{a(C)} \right)_{\mu\nu} \left(\delta_{ab} \delta \left(z - \zeta v\right) + \delta_{bg} \delta \left(z - \zeta \left(1 - v\right)\right) \right) \right] \tag{167}$$

and with the phase space result (302)

$$
\begin{aligned}
4\pi M\left(W_{ab(C)}\right)_{\mu\nu} &= (4\pi)^{1-\frac{N}{2}} \frac{C_a^2}{\Gamma\left(\frac{N}{2}-1\right)} \frac{(\hat{s})^{\frac{N-4}{2}}}{2} \int_0^1 \mathrm{d}v v^{\frac{N}{2}-2} (1-v)^{\frac{N}{2}-2} \\
&\times \left(\mathcal{W}_{a(C)}\right)_{\mu\nu} \Big(\delta_{ab}\delta\left(z-\zeta v\right)+\delta_{bg}\delta\left(z-\zeta\left(1-v\right)\right)\Big) \qquad (168)\\
&= (4\pi)^{1-\frac{N}{2}} \frac{C_a^2}{\Gamma\left(\frac{N}{2}-1\right)} \frac{(\hat{s})^{\frac{N-4}{2}}}{2\zeta} (\mathrm{v})^{\frac{N}{2}-2} (1-\mathrm{v})^{\frac{N}{2}-2} \\
&\times \left(\delta_{ab}\left(\mathcal{W}_{a(C)}\right)_{\mu\nu}\Big|_{v=\mathrm{v}} + \delta_{bg}\left(\mathcal{W}_{a(C)}\right)_{\mu\nu}\Big|_{1-v=\mathrm{v}}\right). \qquad (169)
\end{aligned}
$$

Here $\left(\mathcal{W}_{a(C)}\right)_{\mu\nu}$ is the same as in Section 3.3, $\hat{s} = Q^2(1-w)/w$ and we introduced a new variable

$$
\mathrm{v} \equiv \frac{z}{\zeta}, \qquad (170)
$$

convenient to the NLO SIDIS calculation. We get from Equations (116) and (117)

$$
\begin{aligned}
P^\mu P^\nu M\left(W_{ab(C)}\right)_{\mu\nu} &= \frac{4C_a^2}{3x^2} \frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}} \frac{(N-2)}{\Gamma\left(\frac{N}{2}-1\right)} \frac{1}{2\zeta} \left(\frac{Q^2(1-w)}{w}\mathrm{v}(1-\mathrm{v})\right)^{\frac{N-4}{2}} \\
&\times Q^2 w\Big(\delta_{ab}\mathrm{v}+\delta_{bg}(1-\mathrm{v})\Big) \qquad (171)\\
-g^{\mu\nu} M\left(W_{ab(C)}\right)_{\mu\nu} &= \frac{4C_a^2}{3} \frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}} \frac{(N-2)}{\Gamma\left(\frac{N}{2}-1\right)} \frac{1}{2\zeta} \left(\frac{Q^2(1-w)}{w}\mathrm{v}(1-\mathrm{v})\right)^{\frac{N-4}{2}} \\
&\times \Bigg[\delta_{ab}\Bigg((N-4)\left(\frac{1-\mathrm{v}}{1-w}+\frac{1-w}{1-\mathrm{v}}\right)+2\frac{w^2+\mathrm{v}^2}{(1-w)(1-\mathrm{v})} \\
&+2(6-N)\Bigg)+\delta_{bg}\Bigg((N-2)\left(\frac{\mathrm{v}}{1-w}+\frac{1-w}{\mathrm{v}}\right) \\
&+\frac{4w(1-\mathrm{v})}{\mathrm{v}(1-w)}-2(N-4)\Bigg)\Bigg] \qquad (172)\\
&= \frac{4C_a^2}{3} \frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}} \frac{(N-2)}{\Gamma\left(\frac{N}{2}-1\right)} \frac{(Q^2)^{\frac{N-4}{2}}}{4\zeta} \Big[\delta_{ab}\tilde{g}_{\mathtt{q}}(w,\mathrm{v})+\delta_{bg}\tilde{g}_g(w,\mathrm{v})\Big]. \qquad (173)
\end{aligned}
$$

The functions $\tilde{g}_{\mathtt{q}}$ and $\tilde{g}_g$ are auxiliary functions defined to simplify the presentation of the next steps.

Now we expand these contractions in powers of ϵ in $N = 4 - 2\epsilon$ using the results given in Appendices D and E. The $P^\mu P^\nu$ contraction is

$$P^\mu P^\nu M\left(W_{ab(C)}\right)_{\mu\nu} = \frac{4C_a^2}{3x^2\zeta}\frac{\alpha_S}{4\pi}\left(1-\epsilon\right)Q^2 w\Big(\delta_{ab}\mathrm{v} + \delta_{bg}\left(1-\mathrm{v}\right)\Big) \tag{174}$$

and the contraction with the metric tensor

$$-g^{\mu\nu} M\left(W_{ab(C)}\right)_{\mu\nu} = \frac{4C_a^2}{3\zeta}\frac{\alpha_S}{2\pi}\frac{1-\epsilon}{\Gamma\left(1-\epsilon\right)}\left(\frac{4\pi m_D^2}{Q^2}\right)^\epsilon \Big[\delta_{ab}\tilde{g}_{\mathtt{q}}(w,\mathrm{v}) + \delta_{bg}\tilde{g}_g(w,\mathrm{v})\Big]. \tag{175}$$

Then the auxiliary functions $\tilde{g}_{\mathtt{q}}$ and $\tilde{g}_g$. With the results of Appendix D we get

$$\begin{aligned}\frac{\left(w^2+\mathrm{v}^2\right)\left(\frac{w}{\mathrm{v}}\right)^\epsilon}{\left(1-w\right)^{1+\epsilon}\left(1-\mathrm{v}\right)^{1+\epsilon}} = {} & \frac{w^2+\mathrm{v}^2}{\left(1-w\right)_+\left(1-\mathrm{v}\right)_+} + \frac{1}{\epsilon^2}\delta\left(1-w\right)\delta\left(1-\mathrm{v}\right) \\ & + \delta\left(1-w\right)\Bigg[\left(1+\mathrm{v}^2\right)\left(\frac{\ln\left(1-\mathrm{v}\right)}{1-\mathrm{v}}\right)_+ \\ & \qquad - \left(\frac{1}{1-\mathrm{v}}\right)_+\left(1+\mathrm{v}^2\right)\left(\frac{1}{\epsilon} - \ln\mathrm{v}\right)\Bigg] \\ & + \delta\left(1-\mathrm{v}\right)\Bigg[\left(1+w^2\right)\left(\frac{\ln\left(1-w\right)}{1-w}\right)_+ \\ & \qquad - \left(\frac{1}{1-w}\right)_+\left(1+w^2\right)\left(\frac{1}{\epsilon} + \ln w\right)\Bigg]\end{aligned} \tag{176}$$

and hence

$$\tilde{g}_{\mathtt{q}} = \left(1-\mathrm{v}\right)\delta\left(1-w\right) + \left(1-w\right)\delta\left(1-\mathrm{v}\right) + \frac{\left(w^2+\mathrm{v}^2\right)\left(\frac{w}{\mathrm{v}}\right)^\epsilon}{\left(1-w\right)^{1+\epsilon}\left(1-\mathrm{v}\right)^{1+\epsilon}} + 2 \tag{177}$$

becomes

$$\tilde{g}_{\mathtt{q}} = \frac{w^2+\mathrm{v}^2}{(1-w)_+(1-\mathrm{v})_+} + \frac{1}{\epsilon^2}\delta(1-w)\,\delta(1-\mathrm{v}) + 2$$
$$+\,\delta(1-w)\left[\left(1+\mathrm{v}^2\right)\left(\frac{\ln(1-\mathrm{v})}{1-\mathrm{v}}\right)_+\right.$$
$$\left. -\left(\frac{1}{1-\mathrm{v}}\right)_+\left(1+\mathrm{v}^2\right)\left(\frac{1}{\epsilon}-\ln\mathrm{v}\right)+1-\mathrm{v}\right]$$
$$+\,\delta(1-\mathrm{v})\left[\left(1+w^2\right)\left(\frac{\ln(1-w)}{1-w}\right)_+\right.$$
$$\left. -\left(\frac{1}{1-w}\right)_+\left(1+w^2\right)\left(\frac{1}{\epsilon}+\ln w\right)+1-w\right]. \tag{178}$$

Similarly

$$\tilde{g}_g = \mathrm{v}\left(\frac{1}{1-w}\right)_+ + \mathrm{v}\delta(1-w)\left(1-\frac{1}{\epsilon}+\ln(\mathrm{v}(1-\mathrm{v}))\right)$$
$$+\frac{1-w}{\mathrm{v}}+\frac{2w(1-\mathrm{v})}{\mathrm{v}}\left(\left(\frac{1}{1-w}\right)_+ + \delta(1-w)\left(\ln(\mathrm{v}(1-\mathrm{v}))-\frac{1}{\epsilon}\right)\right) \tag{179}$$
$$= \left(\frac{1}{1-w}\right)_+\frac{2-2\mathrm{v}+\mathrm{v}^2}{\mathrm{v}} - \frac{1+w-2\mathrm{v}}{\mathrm{v}}$$
$$+\,\delta(1-w)\left(\mathrm{v}+\frac{2-2\mathrm{v}+\mathrm{v}^2}{\mathrm{v}}\left(\ln(\mathrm{v}(1-\mathrm{v}))-\frac{1}{\epsilon}\right)\right). \tag{180}$$

Note that on the second equality we used the plus-distribution property (see Appendix D)

$$\left(\frac{1}{1-x}\right)_+(1-x)\,f(x) = f(x). \tag{181}$$

The expansion of the coefficient

$$\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon} \tag{182}$$

in the metric tensor contraction is achieved by changing in functions $\tilde{g}_{\mathtt{q}}$ and $\tilde{g}_g$

$$a+b\frac{1}{\epsilon}+c\frac{1}{\epsilon^2} \to a+b\left(\frac{1}{\hat{\epsilon}}-\ln\left(\frac{Q^2}{m_D^2}\right)\right)+c\frac{1}{\epsilon^2}\frac{\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon}}{\Gamma(1-\epsilon)}. \tag{183}$$

4.3 Initial state gluon

As described in Section 2.3 we get the initial state gluon contribution to hadronic tensor as

$$4\pi M\left(W_{ig(b)}\right)_{\mu\nu}=PS_{p',k}\left[\left(\mathcal{W}_{ig(b)}\right)_{\mu\nu}\left(\delta_{b\mathsf{q}}\delta\left(z-\zeta v\right)+\delta_{b\bar{\mathsf{q}}}\delta\left(z-\zeta\left(1-v\right)\right)\right)\right], \quad (184)$$

where $\left(\mathcal{W}_{ig(b)}\right)_{\mu\nu}$ is similar to that in Equation (132) but with summation over quark flavors dropped. Each graph indexed by the outgoing quark will contribute the same amount up to the square of the fractional charge C_b^2. Because of this

$$\left(\mathcal{W}_{ig(b)}\right)_{\mu\nu}=\frac{C_b^2}{\sum_{a=\mathsf{q}}C_a^2}\left(\mathcal{W}_{ig}\right)_{\mu\nu}=\frac{2C_b^2}{\sum_{a=\mathsf{q},\bar{\mathsf{q}}}C_a^2}\left(\mathcal{W}_{ig}\right)_{\mu\nu}. \quad (185)$$

The latter version is the one we will use here as in Section 3.4 the sum was changed for the calculational reasons to go over both quarks and antiquarks.

By simplifying the phase space integral with identity (302) we get

$$4\pi M\left(W_{gb(ig)}\right)_{\mu\nu}=\frac{(4\pi)^{1-\frac{N}{2}}}{\Gamma\left(\frac{N}{2}-1\right)}\frac{(\hat{s})^{\frac{N-4}{2}}}{2\zeta}(\mathrm{v})^{\frac{N}{2}-2}(1-\mathrm{v})^{\frac{N}{2}-2} \quad (186)$$

$$\times\left(\delta_{b\mathsf{q}}\left(\mathcal{W}_{ig(b)}\right)_{\mu\nu}\Big|_{v=\mathrm{v}}+\delta_{b\bar{\mathsf{q}}}\left(\mathcal{W}_{ig(b)}\right)_{\mu\nu}\Big|_{1-v=\mathrm{v}}\right), \quad (187)$$

where again $\mathrm{v}=z/\zeta$ and $\hat{s}=Q^2\left(1-w\right)/w$. With Equations (140) and (141) we get

$$P^\mu P^\nu M\left(W_{gb(ig)}\right)_{\mu\nu}=\frac{C_b^2}{(N-2)}\frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}}\frac{1}{\Gamma\left(\frac{N}{2}-1\right)}\frac{1}{2\zeta}\left(\frac{Q^2}{w}\left(1-w\right)\mathrm{v}\left(1-\mathrm{v}\right)\right)^{\frac{N-4}{2}}$$

$$\times\frac{4Q^2}{x^2}w\left(1-w\right)\left(\delta_{b\mathsf{q}}+\delta_{b\bar{\mathsf{q}}}\right) \quad (188)$$

$$g^{\mu\nu}M\left(W_{gb(ig)}\right)_{\mu\nu}=C_b^2\frac{\left(g_S^{(N)}\right)^2}{(4\pi)^{\frac{N}{2}}}\frac{1}{\Gamma\left(\frac{N}{2}-1\right)}\frac{1}{2\zeta}\left(\frac{Q^2}{w}\left(1-w\right)\mathrm{v}\left(1-\mathrm{v}\right)\right)^{\frac{N-4}{2}}$$

$$\times\left(\delta_{b\mathsf{q}}+\delta_{b\bar{\mathsf{q}}}\right)\left((N-2)\left(\frac{1-\mathrm{v}}{\mathrm{v}}+\frac{\mathrm{v}}{1-\mathrm{v}}\right)\right.$$

$$\left.-4\frac{w\left(1-w\right)}{\mathrm{v}\left(1-\mathrm{v}\right)}+2\left(N-4\right)\right). \quad (189)$$

Again the contractions are expanded in powers of ϵ using the results given in Appendices D and E. The first contraction is

$$P^{\mu}P^{\nu}M\left(W_{gb(ig)}\right)_{\mu\nu} = C_b^2\frac{\alpha_S}{2\pi}\frac{1}{\zeta}\frac{Q^2}{2x^2}w\left(1-w\right)\left(\delta_{b\mathrm{q}}+\delta_{b\bar{\mathrm{q}}}\right). \tag{190}$$

And the second contraction

$$\begin{aligned}
g^{\mu\nu}M\left(W_{gb(ig)}\right)_{\mu\nu} = {} & C_b^2\frac{\alpha_S}{4\pi}\frac{1}{\Gamma\left(1-\epsilon\right)}\frac{1}{\zeta}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon}\left(\delta_{b\mathrm{q}}+\delta_{b\bar{\mathrm{q}}}\right)\Bigg[\frac{1-\mathrm{v}}{\mathrm{v}}-\frac{1+\mathrm{v}}{\mathrm{v}} \\
& +\left(\frac{1}{1-\mathrm{v}}\right)_{+}\frac{1-2w\left(1-w\right)}{\mathrm{v}} \\
& +\delta\left(1-\mathrm{v}\right)\left(1-2w\left(1-w\right)\right)\ln\left(\frac{1-w}{w}\right) \\
& +\delta\left(1-\mathrm{v}\right)\left(1-\frac{1}{\epsilon}\left(1-2w\left(1-w\right)\right)\right)\Bigg] \qquad (191) \\
= {} & \frac{C_b^2}{\zeta}\frac{\alpha_S}{2\pi}\frac{1}{\Gamma\left(1-\epsilon\right)}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon}\left(\delta_{b\mathrm{q}}+\delta_{b\bar{\mathrm{q}}}\right)\Bigg[-1 \\
& +\left(\frac{1}{1-\mathrm{v}}\right)_{+}\frac{P_{qg}\left(w\right)}{\mathrm{v}} \\
& +\delta\left(1-\mathrm{v}\right)P_{qg}\left(w\right)\ln\left(\frac{1-w}{w}\right) \\
& +\delta\left(1-\mathrm{v}\right)\left(\frac{1}{2}-\frac{1}{\epsilon}P_{qg}\left(w\right)\right)\Bigg]. \qquad (192)
\end{aligned}$$

Here we substituted the splitting function $P_{qg}\left(w\right)$ defined in Equation (153) and the plus-distribution property (181). We further expand the prefactors and arrive at

$$\begin{aligned}
g^{\mu\nu}M\left(W_{gb(ig)}\right)_{\mu\nu} = {} & \frac{C_b^2}{\zeta}\frac{\alpha_S}{2\pi}\left(1-\epsilon\right)\left(\delta_{b\mathrm{q}}+\delta_{b\bar{\mathrm{q}}}\right)\Bigg[-1+\left(\frac{1}{1-\mathrm{v}}\right)_{+}\frac{P_{qg}\left(w\right)}{\mathrm{v}} \\
& +\delta\left(1-\mathrm{v}\right)P_{qg}\left(w\right)\ln\left(\frac{1-w}{w}\right) \\
& -\delta\left(1-\mathrm{v}\right)\left(P_{qg}\left(w\right)\left(\frac{1}{\hat{\epsilon}}-\ln\left(\frac{Q^2}{m_D^2}\right)\right)+w\left(1-w\right)\right)\Bigg]. \qquad (193)
\end{aligned}$$

4.4 SIDIS structure functions and cross section

We can now sum the contractions of the full hadronic tensor (48) from the results of the preceding subsections. First we get from the gluon radiation result (174) and the initial state gluon result (190)

$$\begin{aligned}\frac{4x^2}{Q^2}P^\mu P^\nu MW^h_{\mu\nu} = \sum_a C_a^2\left(1-\epsilon\right)\frac{\alpha_S}{2\pi}\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}}\Bigg[&f_{a/H}\left(\frac{x}{w}\right)D_a^h\left(\frac{z}{\mathrm{v}}\right)\frac{4}{3}2w\mathrm{v}\\ &+f_{a/H}\left(\frac{x}{w}\right)D_g^h\left(\frac{z}{\mathrm{v}}\right)\frac{4}{3}2w\left(1-\mathrm{v}\right)\\ &+f_{g/H}\left(\frac{x}{w}\right)D_b^h\left(\frac{z}{\mathrm{v}}\right)2w\left(1-w\right)\Bigg],\end{aligned}\tag{194}$$

where the sums go over quarks and antiquarks. The metric tensor contraction results are given in (34), (165), (175) and (193) and they combine into

$$\begin{aligned}-g^{\mu\nu}MW^h_{\mu\nu} = \left(1-\epsilon\right)\sum_a C_a^2\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}}\Bigg[&f_{a/H}\left(\frac{x}{w}\right)D_a^h\left(\frac{z}{\mathrm{v}}\right)\Bigg(\delta\left(1-w\right)\delta\left(1-\mathrm{v}\right)\\ &+\frac{4}{3}\frac{\alpha_S}{2\pi}\Bigg[-8\delta\left(1-w\right)\delta\left(1-\mathrm{v}\right)+\frac{w^2+\mathrm{v}^2}{\left(1-w\right)_+\left(1-\mathrm{v}\right)_+}+2\\ &+\left(\ln\left(\frac{Q^2}{m_D^2}\right)-\frac{1}{\hat\epsilon}\right)\Big(\delta\left(1-w\right)P_{qq}\left(\mathrm{v}\right)+\delta\left(1-\mathrm{v}\right)P_{qq}\left(w\right)\Big)\\ &+\delta\left(1-w\right)\left(\left(1+\mathrm{v}^2\right)\left(\frac{\ln\left(1-\mathrm{v}\right)}{1-\mathrm{v}}\right)_+ +\frac{1+\mathrm{v}^2}{1-\mathrm{v}}\ln\mathrm{v}+1-\mathrm{v}\right)\\ &+\delta\left(1-\mathrm{v}\right)\left(\left(1+w^2\right)\left(\frac{\ln\left(1-w\right)}{1-w}\right)_+ -\frac{1+w^2}{1-w}\ln w+1-w\right)\Bigg]\Bigg)\\ &+f_{a/H}\left(\frac{x}{w}\right)D_g^h\left(\frac{z}{\mathrm{v}}\right)\frac{\alpha_S}{2\pi}\Bigg(P_{gq}\left(\mathrm{v}\right)\left(\frac{1}{1-w}\right)_+-\frac{1+w-2\mathrm{v}}{\mathrm{v}}\\ &+\delta\left(1-w\right)\left[\mathrm{v}+P_{gq}\left(\mathrm{v}\right)\left(\ln\left(\mathrm{v}\left(1-\mathrm{v}\right)\right)+\ln\left(\frac{Q^2}{m_D^2}\right)-\frac{1}{\hat\epsilon}\right)\right]\Bigg)\\ &+f_{g/H}\left(\frac{x}{w}\right)D_a^h\left(\frac{z}{\mathrm{v}}\right)\frac{\alpha_S}{2\pi}\Bigg(-1+\left(\frac{1}{1-\mathrm{v}}\right)_+\frac{P_{qg}\left(w\right)}{\mathrm{v}}+\delta\left(1-\mathrm{v}\right)\\ &\times\left[P_{qg}\left(w\right)\left(\ln\left(\frac{1-w}{w}\right)+\ln\left(\frac{Q^2}{m_D^2}\right)-\frac{1}{\hat\epsilon}\right)+w\left(1-w\right)\right]\Bigg)\Bigg].\end{aligned}\tag{195}$$

Here we introduced yet another Altarelli-Parisi splitting function [27]

$$P_{gq}\left(\mathrm{v}\right)=\frac{1+\left(1-\mathrm{v}\right)^{2}}{\mathrm{v}}. \tag{196}$$

Next, we would like to combine these contractions into the renormalized structure functions (9) and (10). The parton density functions are renormalized as we did in Section 3.5 with identities (156). The fragmentation functions are renormalized according to the $\overline{\mathrm{MS}}$ scheme: [10]

$$\begin{aligned} D_a^h\left(z,Q_f^2\right) = & \int_z^1 \frac{\mathrm{dv}}{\mathrm{v}} D_a^h\left(\frac{z}{\mathrm{v}}\right)\left[\delta\left(1-\mathrm{v}\right)+\frac{\alpha_S}{2\pi}P_{qq}\left(\mathrm{v}\right)\left(\ln\left(\frac{Q_f^2}{m_D^2}\right)-\frac{1}{\hat{\epsilon}}\right)\right] \\ & +\int_z^1 \frac{\mathrm{dv}}{\mathrm{v}} D_g^h\left(\frac{z}{\mathrm{v}}\right)\left[\frac{\alpha_S}{2\pi}P_{gq}\left(\mathrm{v}\right)\left(\ln\left(\frac{Q_f^2}{m_D^2}\right)-\frac{1}{\hat{\epsilon}}\right)\right] \end{aligned} \tag{197}$$

$$D_g^h\left(z,Q_f^2\right) = D_g^h\left(z\right)+\mathcal{O}\left(\alpha_S\right). \tag{198}$$

Again, explicit gluon fragmentation function redefinition is a next-to NLO correction and we only recognize here the possible scale dependence. As with the density functions we take $Q_f^2 = Q^2$.

We then get our renormalized SIDIS structure functions

$$\begin{aligned} \frac{1}{x}F_2^h\left(x,Q^2\right) = \sum_a C_a^2 \Bigg[& f_{a/H}\left(x,Q^2\right)D_a^h\left(z,Q^2\right) \\ & +\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}} f_{a/H}\left(\frac{x}{w},Q^2\right)D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}C_{qq,2}\left(w,\mathrm{v}\right) \\ & +\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}} f_{a/H}\left(\frac{x}{w},Q^2\right)D_g^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}C_{gq,2}\left(w,\mathrm{v}\right) \\ & +\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}} f_{g/H}\left(\frac{x}{w},Q^2\right)D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}C_{qg,2}\left(w,\mathrm{v}\right)\Bigg] \end{aligned} \tag{199}$$

$$
\begin{aligned}
2F_1^h\left(x,Q^2\right) = &\frac{1}{x}F_2^h\left(x,Q^2\right)\\
&-\sum_a C_a^2\Bigg[\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}}f_{a/H}\left(\frac{x}{w},Q^2\right)D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}\frac{4}{3}4w\mathrm{v}\\
&-\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}}f_{a/H}\left(\frac{x}{w},Q^2\right)D_g^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}\frac{4}{3}4w\left(1-\mathrm{v}\right)\\
&-\int_x^1\frac{\mathrm{d}w}{w}\int_z^1\frac{\mathrm{dv}}{\mathrm{v}}f_{g/H}\left(\frac{x}{w},Q^2\right)D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}4w\left(1-w\right)\Bigg]. \qquad (200)
\end{aligned}
$$

Here we have used coefficient functions

$$
\begin{aligned}
C_{qq,2}\left(w,\mathrm{v}\right) = &\frac{4}{3}\Bigg[-8\delta\left(1-w\right)\delta\left(1-\mathrm{v}\right)+\frac{w^2+\mathrm{v}^2}{\left(1-w\right)_+\left(1-\mathrm{v}\right)_+}+2+6w\mathrm{v}\\
&+\delta\left(1-w\right)\Big(L_1\left(\mathrm{v}\right)+L_2\left(\mathrm{v}\right)+1-\mathrm{v}\Big)\\
&+\delta\left(1-\mathrm{v}\right)\Big(L_1\left(w\right)-L_2\left(w\right)+1-w\Big)\Bigg] \qquad (201)\\
C_{gq,2}\left(w,\mathrm{v}\right) = &\frac{4}{3}\Bigg[P_{gq}\left(\mathrm{v}\right)\left(\frac{1}{1-w}\right)_+ +\delta\left(1-w\right)\Big(\mathrm{v}+P_{gq}\left(\mathrm{v}\right)\ln\Big(\mathrm{v}\left(1-\mathrm{v}\right)\Big)\Big)\\
&-\frac{1+w}{\mathrm{v}}+2+6w\left(1-\mathrm{v}\right)\Bigg] \qquad (202)\\
C_{qg,2}\left(w,\mathrm{v}\right) = &\left(\frac{1}{1-\mathrm{v}}\right)_+\frac{P_{qg}\left(w\right)}{\mathrm{v}}-2P_{qg}\left(w\right)+4w\left(1-w\right)\\
&+\delta\left(1-\mathrm{v}\right)\left[P_{qg}\left(w\right)\ln\left(\frac{1-w}{w}\right)+w\left(1-w\right)\right], \qquad (203)
\end{aligned}
$$

where

$$
L_1\left(w\right) = \left(1+w^2\right)\left(\frac{\ln\left(1-w\right)}{1-w}\right)_+ \qquad (204)
$$

$$
L_2\left(w\right) = \frac{1+w^2}{1-w}\ln w. \qquad (205)
$$

These structure functions coincide with the ones given in References [10, 17].[1] The apparent differences arise from the plus-distribution identities similar to those given

[1]In the printed version of Reference [10] the exponent in the non-distributional logarithm term in $C_2^{F(1)}$ is on the wrong side of the parenthesis. In the hand-written version the exponent is placed correctly.

in the end of Appendix D. We conclude with the NLO SIDIS cross section

$$\begin{aligned}
\frac{\mathrm{d}^3\sigma^h}{\mathrm{d}x\mathrm{d}Q^2\mathrm{d}z} = \frac{4\pi\alpha_{em}^2}{Q^4}\sum_a C_a^2 \int_x^1 \frac{\mathrm{d}w}{w}\int_z^1 \frac{\mathrm{d}\mathrm{v}}{\mathrm{v}}\Bigg[& f_{a/H}\left(\frac{x}{w},Q^2\right) D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right) \\
& \times \delta\left(1-w\right)\delta\left(1-\mathrm{v}\right)\frac{\left(1-y\right)^2+1}{2} \\
& + f_{a/H}\left(\frac{x}{w},Q^2\right) D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}\left(C_{qq,2}\left(w,\mathrm{v}\right)\frac{\left(1-y\right)^2+1}{2} - \frac{4}{3}2y^2 w\mathrm{v}\right) \\
& + f_{a/H}\left(\frac{x}{w},Q^2\right) D_g^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}\left(C_{gq,2}\left(w,\mathrm{v}\right)\frac{\left(1-y\right)^2+1}{2} - \frac{4}{3}2y^2 w\left(1-\mathrm{v}\right)\right) \\
& + f_{g/H}\left(\frac{x}{w},Q^2\right) D_a^h\left(\frac{z}{\mathrm{v}},Q^2\right)\frac{\alpha_S}{2\pi}\left(C_{qg,2}\left(w,\mathrm{v}\right)\frac{\left(1-y\right)^2+1}{2} - 2y^2 w\left(1-w\right)\right)\Bigg].
\end{aligned} \tag{206}$$

Again, here the sum a runs over quarks and antiquarks and the coefficient functions $C_{qq,2}\left(w,\mathrm{v}\right)$, $C_{gq,2}\left(w,\mathrm{v}\right)$ and $C_{qg,2}\left(w,\mathrm{v}\right)$ are given in Equations (201)-(203).

5 Numerical calculations

In this Section we present the results of the numerical calculations done with the results of Sections 3 and 4. We compare how different PDF and FF sets perform in comparison to the results from the HERA particle accelerator's HERMES experiment [20] for the two charged pions. We use combinations of CT14 PDF set [28] and FFs of the sets NNFF1.0 [29] and MAPFF [30]. We also use sets JAM19 [31] and JAM20 [32] which include both PDFs and FFs determined with a simultaneous fit.

The HERMES experiment measured the yields of $\pi^{\pm}$ in an electron-proton scattering and the derived quantity is the multiplicity of the particles. Multiplicity is defined as the SIDIS cross section divided by the DIS cross section. We use here the fully integrated version of the multiplicity defined as

$$M^{\pi^{\pm}}(z) = \frac{1}{\int \mathrm{d}x \int \mathrm{d}Q^2 \frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2}} \int \mathrm{d}x \int \mathrm{d}Q^2 \frac{\mathrm{d}^3\sigma^{\pi^{\pm}}}{\mathrm{d}x\mathrm{d}Q^2\mathrm{d}z}, \tag{207}$$

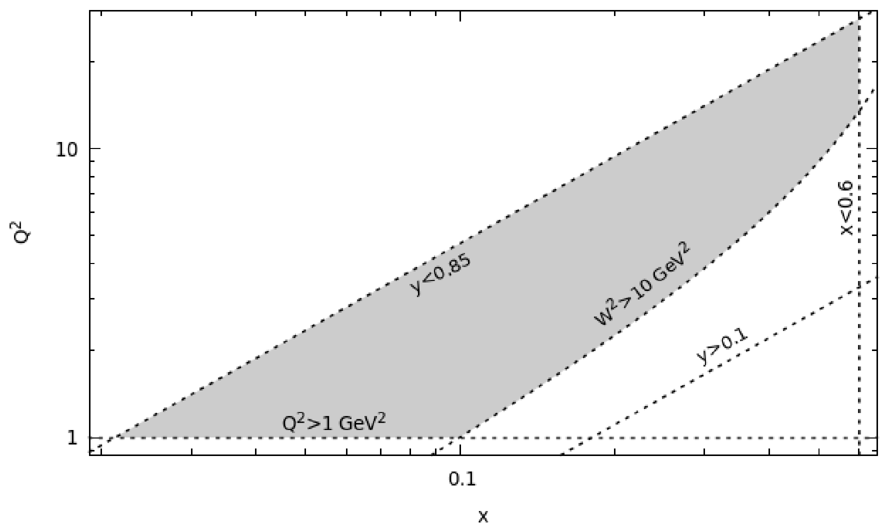

Figure 8. Kinematic acceptance of the HERMES experiment (gray) and the constraining curves, labels pointing on which side of the curve the constraint is fulfilled. Here W^2 is the total invariant mass of the photon-proton system.

where the DIS cross section is as in Equation (160) and the SIDIS cross section as in Equation (206). The integrals over x and Q^2 cover the HERMES kinematic acceptance, which is presented in Figure 8 with its constraints.

The numerical integration of the DIS and SIDIS cross section was done using GNU Scientific Library Monte Carlo Integration routines with C++. The PDF sets and the FF sets were used through the LHAPDF library [33].

The results of this calculation against the HERMES results for π^+ are given in Figure 9 and for π^- in Figure 10. We see that the numerical multiplicities reproduce the experimental results reasonably well. Only the multiplicities produced with NNFF1.0 FFs differ clearly from the experimental results. In the fitting process of NNFF1.0 and JAM20 the assumption $D_u^{\pi^+} = D_{\bar{d}}^{\pi^+}$ and the similar favored quark equality for π^- was used and in Reference [34] it is mentioned to cause a deterioration of the quality of the fit. The MAPFF fit was done without this assumption, so it used more independent flavors and seems to fit better. It should also be noted that in the

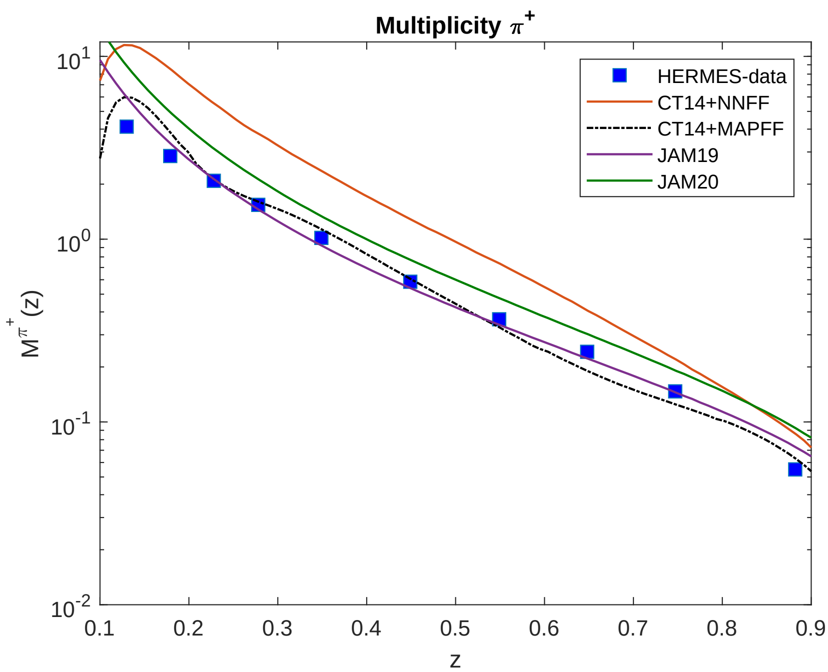

Figure 9. The multiplicities of the HERMES experiment along with numerical results for π^+ using different sets of parton density functions and fragmentation functions.

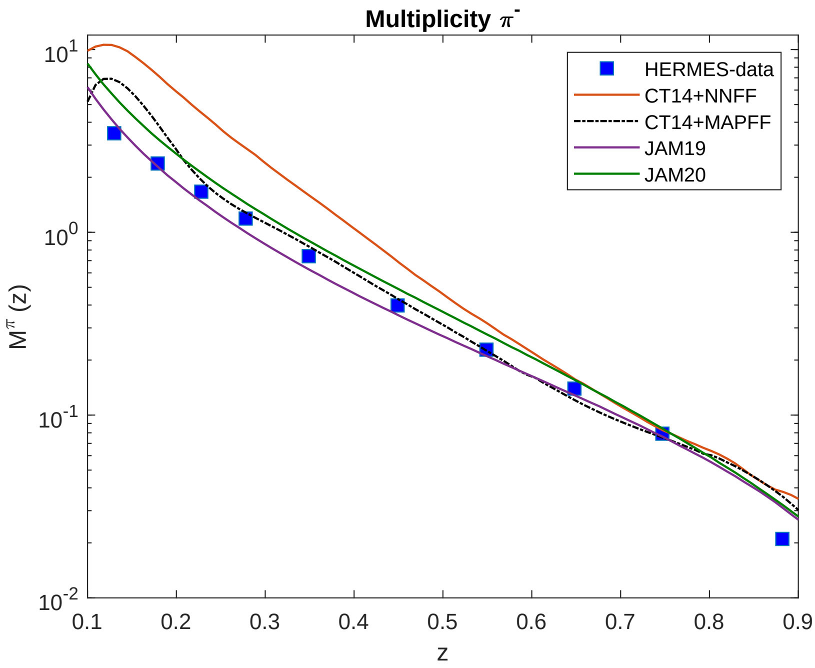

Figure 10. The multiplicities of the HERMES experiment along with numerical results for π^- using different sets of parton density functions and fragmentation functions.

fitting process some data from HERMES experiment was used in both MAPFF and JAM19 to constrain the fits, although most of the SIDIS data used by MAPFF is from the COMPASS experiment [35]. This also seems to be the case with input data of JAM19 in the interval $0.2 < z < 0.8$.

6 Summary

In this thesis we presented calculations of the NLO pQCD cross sections for DIS and SIDIS. In Section 2 we reviewed the determination of the LO contributions to the cross sections. The full NLO DIS cross section is given in Equation (160) using coefficient functions (154) and (155). The full NLO SIDIS cross section is given in Equation (206) using coefficient functions (201)-(203). The derived results agree with those found in the literature. The apparent differences between our results and those in the literature arise mainly from different expansions of the plus-distribution and rational (plus-)functions. Equivalences between the results follow from the last properties given in Appendix D or similar results.

Dimensional regularization made it compact and systematic to carry out the calculation, isolate the appearing divergences and renormalize PDFs and FFs. The similarity of DIS and SIDIS as processes simplified the calculations of the latter after the former process was handled already. Most significant differences in the calculations appear from the phase space integration onwards as in the SIDIS one more degree of freedom is left out of the phase space integral.

We then numerically calculated multiplicities for pions $\pi^{\pm}$ using a few different PDF and FF sets and compared these with the multiplicities measured in the HERMES experiment. The results depended mainly on the FFs used and different FFs reproduced the HERMES multiplicities to a varying degree. From the four sets used, two seemed to perform well, one got somewhat close and one did not perform that well. Each of the FF sets were fitted with either a different method, with different assumptions or using different datasets. Some possible factors behind the performance differences were listed, but otherwise it was left as an open question.

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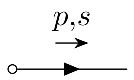
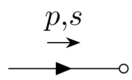
A Feynman rules and dimensional regularization

In this appendix we present the Feynman rules and the corresponding notations used in this thesis. We then explain how the usage of the dimensional regularization changes the usage of these rules and the properties of the associated mathematical objects.

The Feynman rules used in this thesis are taken from Reference [21]. The external lines of the fermions in graphs are presented using the Dirac spinors $u\,(p{,}s)$ where p is the 4-momentum and s the spin index of the fermion:



$f \;= \overline{u}\,(p{,}s)$ $\qquad$ f



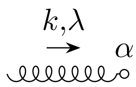
$= u\,(p{,}s)\;.$

With antifermions we replace spinor u with $\overline{v}$ and $\overline{u}$ with v. For a massless fermion the spinors fulfill the completeness relation

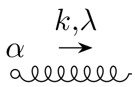
$$\sum_s u\,(p{,}s)\,\overline{u}\,(p{,}s) = \not{p} = \sum_s v\,(p{,}s)\,\overline{v}\,(p{,}s)\,. \tag{208}$$

The external lines of the gluon are represented as

g



$= \varepsilon^{\alpha}\,(k{,}\lambda)$



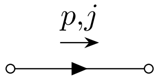
g $= (\varepsilon^{*})^{\alpha}\,(k{,}\lambda)\;\;,$

where λ is the helicity of the gluon, k its 4-momentum and ε the polarization vector. Polarization vector is contracted with the operator of the vertex it is connected to. The polarization vectors fulfill sum result [21, 26]

$$\sum_{\lambda} \varepsilon^{\alpha}\,(k{,}\lambda)\,(\varepsilon^{*})^{\beta}\,(k{,}\lambda) = -g^{\alpha\beta} + \frac{k^{\alpha}\overline{k}^{\beta} + k^{\beta}\overline{k}^{\alpha}}{k\cdot\overline{k}}, \tag{209}$$

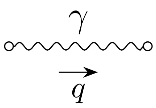
where $\overline{k}$ is the same as k but with spatial components negated, $g^{\mu\nu}$ the metric tensor and the sum runs over the physical polarizations of the gluon.

The propagator for massless fermions is given by

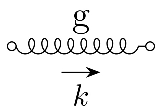


$$= i\frac{\not p}{p^2+i\varepsilon}$$

where p is the 4-momentum and j the color index. The photon and gluon propagators used are



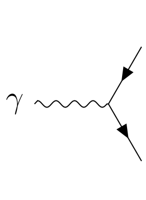
$$= -i\frac{g_{\mu\nu}}{q^2}$$



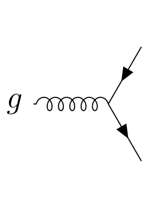
$$= \frac{-i}{k^2+i\varepsilon}\left(g_{\alpha\beta} + \eta\frac{k_\alpha k_\beta}{k^2}\right) \; .$$

The indices of these propagators are contracted by the operators of the two vertices they are connected into. The photon propagator is in the Feynman gauge and the gluon gauge parameter η is left undecided. For every closed loop we add integration over the 4-momentum of the loop with a normalization factor $(2\pi)^{-4}$.

Finally the needed vertices are



$$= -ieC_f\gamma^\mu$$



$$= -ig_S\gamma^\alpha(t^c)_{i'j} \; .$$

Here e is the elementary charge, C_f the charge of the fermion in units of the elementary charge, g_S the interaction strength of the strong interaction, γ^μ Dirac gamma matrix and t^c is the SU(3) color matrix, with c color of the gluon and i', j the two color indices of the fermions connected to the vertex. The tensor indices of the vertices are contracted by the operators of the boson lines conneted to the vertices. The color matrix identities are presented in full detail for example in Reference [21, 26]. In this thesis the only properties needed are the hermiticity and the fact that

$$\sum_c \mathrm{tr}\,(t^c t^c) = 4. \tag{210}$$

Instead of e and g_S it is more common to use the coupling constants

$$\alpha_{em} = \frac{e^2}{4\pi}, \tag{211}$$

that is also known as fine structure constant, and

$$\alpha_S = \frac{g_S^2}{4\pi}, \tag{212}$$

when presenting the results as they tend to be functions of the squares of e and g_S.

In our thesis we use the Dimensional Regularization [18] to regularize the infinities in our pQCD results. In this regularization method the calculations are done effectively first with some spacetime dimension N with which the divergent loop integral or similar divergent term is well defined and finite. The spacetime dimension N is left implicit in the resulting expressions. These expressions are then analytically continued so that the limit $N \to 4$ can be taken. Using the prescription $N = 4 - 2\epsilon$ the analytic terms are expanded respect to $\epsilon = 0$ so that the divergences related to the limit $\epsilon \to 0$ appear as poles ϵ^{-n} for some natural number n. These poles are then absorbed into some quantity (here parton density and fragmentation functions) through renormalization and the final 4-dimensional result is then the remaining convergent $\epsilon \to 0$ limit.

The change into N dimensions is effectively implemented by changing for example momentum and gluon polarization vectors into N-vectors. This change also affects phase space and loop integrals, where 4 is changed into N in the integration measure and in the exponent of the normalization factors. Also the metric $g_{\mu\nu}$ is N-dimensional so

$$g_{\mu\nu} g^{\mu\nu} = \delta^{\mu}_{\nu} = N. \tag{213}$$

The algebra of the Dirac gamma matrices is still

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2 g^{\mu\nu} I_N. \tag{214}$$

For the identity matrix I_N of the space of the Dirac gamma matrices it is chosen that

$$\mathrm{tr}\,(I_N) = 4. \tag{215}$$

The trace could be non-constant, but the only relevant property needed is that $\lim_{n\to 4} \mathrm{tr}\,(I_n) = 4$ [18, 25]. Therefore this is the preferred option. The full effects of the given changes into the calculations of the Dirac gamma matrix traces is given in Appendix C.

As the above-mentioned dimensional regularization procedure changes the spacetime dimensions it also induces changes in the dimensions of some quantities. Most relevant change is that the constants e and g_S should have dimensions of mass to the power of $\frac{1}{2}(4-N)$ so that the action of the model defined by the Lagrangian is dimensionless. This can be implemented by introducing some auxiliary mass m_D and by defining the N-dimensional versions for these constants as

$$e^{(N)} = e\,(m_D)^{\frac{1}{2}(4-N)} \qquad \text{and} \qquad g_S^{(N)} = g_S\,(m_D)^{\frac{1}{2}(4-N)}\,. \tag{216}$$

Only the change in constant g_S is explicitly needed in the calculations of this thesis as the regularization and renormalization are done for observable structure functions F_j and they are defined without the elementary charge e.

B Results for momentum loop integration

In this appendix we define some special functions needed and present the integral identities used in the loop integral calculations.

We widely employ two special functions in this thesis, the Gamma function and the Beta function. The Gamma function Γ is the generalization of the factorial in such a way that

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(0) = 1, \tag{217}$$

and it is defined on the set $\mathbb{C} \setminus (-\mathbb{N}_0)$, the set of complex numbers excluding the negatives of natural numbers and zero, or on some subset of this usually containing the non-negative reals. For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$ we use definition [36]

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t \tag{218}$$

and with this the continuation to whole $\mathbb{C} \setminus (-\mathbb{N}_0)$ is guaranteed by (217).

The Beta function is defined for $x{,}y \in \mathbb{C}$ such that $\mathrm{Re}(x), \mathrm{Re}(y) > 0$ as

$$B(x{,}y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, \mathrm{d}t \tag{219}$$

and it holds that [21, 36]

$$B(x{,}y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{220}$$

$$= \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, \mathrm{d}t. \tag{221}$$

From the relation between the Beta and the Gamma functions we see that the Beta function is at least defined as $x,y,x+y \in \mathbb{C} \setminus (-\mathbb{N}_0)$ as the Gamma function has no zeros.

We then proceed to the identities needed with the loop integrations. The Feynman parametrization formula needed here is for every $n \in \mathbb{N}$ and $a_j \in \mathbb{C}$, $j \in \{1,\dots,n\}$ [21, 26]

$$\frac{1}{a_1 \cdots a_n} = \Gamma(n) \prod_{j=1}^{n} \left(\int_0^1 \mathrm{d}x_j \right) \frac{\delta\left(1 - \sum_{k=1}^{n} x_k\right)}{\left(\sum_{k=1}^{n} x_k a_k\right)^n}. \tag{222}$$

To evaluate the loop integrals we use the equation

$$\int \frac{\mathrm{d}^N \ell}{(2\pi)^N} \frac{\left(\ell^2\right)^R}{\left(\ell^2 - \Delta \pm i\varepsilon\right)^m} = \frac{\pm i\,(-1)^{m+R}}{(4\pi)^{\frac{N}{2}}} \frac{\Delta^{\frac{N}{2}+R-m}}{\Gamma\left(\frac{N}{2}\right)} \frac{\Gamma\left(m - \frac{N}{2} - R\right) \Gamma\left(\frac{N}{2} + R\right)}{\Gamma(m)} \tag{223}$$

valid for $\Delta > 0$ and $\frac{N}{2} < m - R$. It is taken that $N,R,m > 0$. To prove this equality one needs to do Wick rotation in the manner that the ε-prescription specifies. This leaves us with the Euclidean integral which is rotationally symmetric and can hence be simplified using the presentation of the N-dimensional unit sphere area [18]

$$\int \mathrm{d}\Omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{224}$$

Finally, with some change of variables and the Beta function identity (220) we conclude the result. Similar identities can be found in Reference [21, 26].

In dimensional regularization it is enough that the condition for Equation (223) $\frac{N}{2} < m - R$ holds for some N. Then, using the analytic properties of the Gamma function, the analytic continuation of the right-hand side is used to define the effective version of the left-hand side for N that the original integral would not be well defined. In this thesis mainly to find the limit $N \to 4$. One can see that the right-hand side should be well defined as long as $m - \frac{N}{2} - R \notin -\mathbb{N}$.

Before taking the said limit we may also need to integrate the integrals introduced by the Feynman parametrization. Integration is done using the definition of the Beta function and

$$\int_0^1 \mathrm{d}x' \int_0^{1-x'} \mathrm{d}y' \left(bx' + by' + c\right) \left(x'y'\right)^a = \frac{\left(\Gamma\left(a+1\right)\right)^2}{\Gamma\left(2a+3\right)} \left(c + 2b\frac{a+1}{2a+3}\right), \tag{225}$$

where $b, c \in \mathbb{R}$, $c \neq 0$ and $a > -1$. This identity can be seen for example through standard integration by parts and the definition of the Beta function.

C Trace identities

In this appendix we deduce the N-dimensional results for two traces needed in the calculations. First is

$$\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\mu\not{p}_b\gamma_\alpha\right), \tag{226}$$

with $p_a = p - k$, $p_b = p' - k$ and both p and $p' = p + q$ massless. The second is

$$\mathrm{tr}\left(\not{p}'\Lambda_{\alpha\mu}\not{p}\Lambda_{\nu\beta}\right) g^{\alpha\beta} g^{\mu\nu}, \tag{227}$$

with $p_a = p - k$, $p_b = p' + k$,

$$\Lambda_{\alpha\mu} = \frac{1}{p_a^2}\gamma_\mu\not{p}_a\gamma_\alpha + \frac{1}{p_b^2}\gamma_\alpha\not{p}_b\gamma_\mu \tag{228}$$

and each p, p' and $k = p + q - p'$ massless.

We deduce the values for these traces by using the N-dimensional metric and Dirac gamma matrices that obey properties (213), (214) and (215). These rules imply

$$\not{a}\not{b} + \not{a}\not{b} = 2a \cdot b \tag{229}$$

$$\not{a}\not{a} = a^2 \tag{230}$$

$$\gamma^\mu\not{p}\gamma_\mu = (2 - N)\not{p} \tag{231}$$

$$\gamma^\alpha\not{p}\gamma^\mu = 2p^\alpha\gamma^\mu - 2\not{p}g_{\alpha\mu} + 2p^\mu\gamma^\alpha - \gamma^\mu\not{p}\gamma^\alpha \tag{232}$$

$$\mathrm{tr}\left(\not{a}\not{b}\right) = 4a \cdot b. \tag{233}$$

Also for every sequence of gamma matrices $(\gamma^{\mu_1},\gamma^{\mu_2},\cdots,\gamma^{\mu_n})$ it holds that

$$\mathrm{tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\right) = \mathrm{tr}\left(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\right). \tag{234}$$

With Equation (232) the first trace (226) becomes

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\mu\not{p}_b\gamma_\alpha\right) &= 2\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma_\mu\not{p}_b\not{p}_a\right) - 2\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\not{p}_a\not{p}_b\gamma_\mu\right)\\ &+ 2\mathrm{tr}\left(\not{p}'\not{p}_a\not{p}\gamma^\alpha\not{p}_b\gamma_\alpha\right) - \mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\mu\not{p}_a\gamma_\alpha\not{p}_b\gamma_\alpha\right).\end{aligned} \tag{235}$$

Using Equation (231)

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\mu\not{p}_b\gamma_\alpha\right) &= 2\left(2-N\right)\Big(\mathrm{tr}\left(\not{p}'\not{p}\not{p}_b\not{p}_a\right) - \mathrm{tr}\left(\not{p}'\not{p}\not{p}_a\not{p}_b\right)\\ &+\mathrm{tr}\left(\not{p}'\not{p}_a\not{p}\not{p}_b\right) - \frac{1}{2}\left(2-N\right)\mathrm{tr}\left(\not{p}'\not{p}\not{p}_a\not{p}_b\right)\Big)\end{aligned} \tag{236}$$

and with Equations (229), (230) and (233) we get

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\mu\not{p}_b\gamma_\alpha\right) &= 2\left(2-N\right)\big(4p\cdot p'p_a\cdot p_b + 4p\cdot p_a p'\cdot p_b\\ &-\frac{1}{2}\left(8-N\right)\mathrm{tr}\left(\not{p}'\not{p}\not{p}_a\not{p}_b\right)\big).\end{aligned} \tag{237}$$

Now substituting $p_a = p - k$, $p_b = p' - k$, $p^2 = p'^2 = 0$ and $-2p\cdot p' = q^2$ we conclude with

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\alpha\not{p}_a\gamma_\mu\not{p}_b\gamma_\alpha\right) &= 2\left(2-N\right)q^2\Bigg(2q^2 + 4\left(p+p'\right)\cdot k\\ &+\frac{8}{q^2}p\cdot k\,p'\cdot k - \left(4-N\right)k^2\Bigg).\end{aligned} \tag{238}$$

For the second trace (227) we find that

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\Lambda^{\alpha\mu}\not{p}\Lambda_{\mu\alpha}\right) &= \frac{1}{p_a^4}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}_a\gamma^\alpha\not{p}\gamma_\alpha\not{p}_a\gamma_\mu\right) + \frac{1}{p_b^4}\mathrm{tr}\left(\not{p}'\gamma^\alpha\not{p}_b\gamma^\mu\not{p}\gamma_\mu\not{p}_b\gamma_\alpha\right)\\ &+\frac{1}{p_a^2p_b^2}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}_a\gamma^\alpha\not{p}\gamma_\mu\not{p}_b\gamma_\alpha\right) + \frac{1}{p_a^2p_b^2}\mathrm{tr}\left(\not{p}'\gamma^\alpha\not{p}_b\gamma^\mu\not{p}\gamma_\alpha\not{p}_a\gamma_\mu\right).\end{aligned} \tag{239}$$

On the right-hand side the first two traces both simplify to

$$\left(2-N\right)^2\mathrm{tr}\left(\not{p}'\not{k}\not{p}\not{k}\right) \tag{240}$$

through Equations (231) and (230) and the relations of our momenta. It can further be seen that

$$\mathrm{tr}\left(\not{p}'\not{k}\not{p}\not{k}\right) = 8p\cdot kp'\cdot k \tag{241}$$

$$= -2p_a^2p_b^2. \tag{242}$$

The third and the fourth traces in Equation (239) are the same as the order of the gamma matrices in the trace can be inverted without of changing the value of the trace. Also the third trace is the same as (226) up to the interchange of p and p_a. With similar steps as with the trace (226) we then find that

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}_a\gamma^\alpha\not{p}\gamma_\mu\not{p}_b\gamma_\alpha\right) &= 2\left(2-N\right)\left(\mathrm{tr}\left(\not{p}'\not{p}_a\not{p}_b\not{p}\right) - \mathrm{tr}\left(\not{p}'\not{p}_a\not{p}\not{p}_b\right)\right.\\ &\left.+\mathrm{tr}\left(\not{p}'\not{p}\not{p}_a\not{p}_b\right) - \frac{1}{2}\left(2-N\right)\mathrm{tr}\left(\not{p}'\not{p}_a\not{p}\not{p}_b\right)\right).\end{aligned} \tag{243}$$

By substituting the definitions of p_a and p_b with $k^2 = 0$ we get

$$\mathrm{tr}\left(\not{p}'\gamma^\mu\not{p}_a\gamma^\alpha\not{p}\gamma_\mu\not{p}_b\gamma_\alpha\right) = 2\left(2-N\right)\left(8p\cdot p'p_a\cdot p_b + \frac{1}{2}\left(4-N\right)\mathrm{tr}\left(\not{p}'\not{k}\not{p}\not{k}\right)\right). \tag{244}$$

Now the only thing we need is to substitute

$$p_a\cdot p_b = -\frac{1}{2}q^2 \tag{245}$$

and we can conclude that

$$\begin{aligned}\mathrm{tr}\left(\not{p}'\Lambda^{\alpha\mu}\not{p}\Lambda_{\mu\alpha}\right) &= -2\left(N-2\right)^2\left(\frac{p_a^2}{p_b^2} + \frac{p_b^2}{p_a^2}\right) + \frac{16\left(N-2\right)}{p_a^2p_b^2}q^2p\cdot p'\\ &- 4\left(N-2\right)\left(N-4\right).\end{aligned} \tag{246}$$

D Plus-distributions

In this appendix we define a certain class of distributions called plus-distributions and present relevant identities needed in this thesis. Some of the properties and equivalent definitions are given for example in References [16, 21, 24].

For any function $q : [0,1) \to \mathbb{R}$ that is integrable on every subinterval $[0,a] \subset [0,1)$ we define a plus-distribution (or plus-function) $(q(x))_+$ that acts on a smooth function f, for which $f(x) - f(1)$ vanishes in the limit $x \to 1$ sufficiently rapidly, as

$$\int_z^1 (q(x))_+ f(x)\,\mathrm{d}x = \int_0^1 q(x)\Big(f(x) - f(1)\Big)\,\mathrm{d}x - \int_0^z q(x) f(x)\,\mathrm{d}x \tag{247}$$

$$= \int_z^1 q(x)\Big(f(x) - f(1)\Big)\,\mathrm{d}x - f(1)\int_0^z q(x)\,\mathrm{d}x. \tag{248}$$

On this thesis we will be using two functions as a plus-distribution defining function q :

$$\frac{1}{1-x} \quad \text{and} \quad \frac{\ln(1-x)}{1-x}. \tag{249}$$

For the corresponding plus-distributions it holds by Equation (248) that

$$\int_z^1 \left(\frac{1}{1-x}\right)_+ f(x)\,\mathrm{d}x = \int_z^1 \frac{f(x)-f(1)}{1-x}\,\mathrm{d}x + f(1)\ln(1-z) \tag{250}$$

$$\int_z^1 \left(\frac{\ln(1-x)}{1-x}\right)_+ f(x)\,\mathrm{d}x = \int_z^1 \frac{f(x)-f(1)}{1-x}\ln(1-x)\;\mathrm{d}x + \frac{1}{2}f(1)\ln^2(1-z)\,. \tag{251}$$

During the DIS calculation we will confront terms similar to

$$\int_z^1 \frac{f(x)}{(1-x)^{1+\epsilon}}\left(a + \frac{b}{\epsilon}\right)\mathrm{d}x \tag{252}$$

for some a,$b \in \mathbb{R}$. Note that with $\epsilon < 0$ this integral is well defined, but through analytical continuation this does not need to be the case for our results of this Appendix. These terms are to expanded respect to ϵ to take the limit $\epsilon \to 0$ and to isolate the poles presenting the divergences related to that limit. We expand next the two terms in the integrand of Equation (252) using plus-distributions, Dirac delta functions and the identity

$$\frac{1}{(1-x)^{\epsilon}} = \exp\left(-\epsilon \ln\left(1-x\right)\right) \tag{253}$$

$$= \sum_{k=0}^{\infty} \frac{(-\epsilon)^k}{k!} \ln^k\left(1-x\right). \tag{254}$$

First we have

$$\int_z^1 \frac{f(x)}{(1-x)^{1+\epsilon}} \, \mathrm{d}x = \int_z^1 \frac{f(x)-f(1)}{(1-x)^{1+\epsilon}} \, \mathrm{d}x + f(1) \int_z^1 \frac{1}{(1-x)^{1+\epsilon}} \, \mathrm{d}x \tag{255}$$

$$= \int_z^1 \frac{f(x)-f(1)}{1-x} \, \mathrm{d}x - \frac{1}{\epsilon} f(1) + f(1) \ln\left(1-z\right) + \mathcal{O}\left(\epsilon\right), \tag{256}$$

so by Equation (250) in a distributional sense

$$\frac{1}{(1-x)^{1+\epsilon}} = \left(\frac{1}{1-x}\right)_+ - \frac{1}{\epsilon}\delta\left(1-x\right) + \mathcal{O}\left(\epsilon\right). \tag{257}$$

Next we have

$$\int_z^1 \frac{f(x)}{\epsilon\,(1-x)^{1+\epsilon}} \, \mathrm{d}x = \frac{1}{\epsilon} \int_z^1 \frac{f(x)-f(1)}{(1-x)^{1+\epsilon}} \, \mathrm{d}x + \frac{f(1)}{\epsilon} \int_z^1 \frac{1}{(1-x)^{1+\epsilon}} \, \mathrm{d}x \tag{258}$$

$$= \frac{1}{\epsilon} \int_z^1 \frac{f(x)-f(1)}{1-x} \, \mathrm{d}x - \int_z^1 \frac{f(x)-f(1)}{1-x} \ln\left(1-x\right) \, \mathrm{d}x - \frac{1}{\epsilon^2} f(1) + \frac{f(1)}{\epsilon} \ln\left(1-z\right) - \frac{f(1)}{2} \ln^2\left(1-z\right) + \mathcal{O}\left(\epsilon\right). \tag{259}$$

Hence by Equations (250) and (251) we have in a distributional sense

$$\frac{1}{\epsilon\,(1-x)^{1+\epsilon}} = \frac{1}{\epsilon}\left(\frac{1}{1-x}\right)_+ - \frac{1}{\epsilon^2}\delta\left(1-x\right) - \left(\frac{\ln\left(1-x\right)}{1-x}\right)_+ + \mathcal{O}\left(\epsilon\right). \tag{260}$$

In the SIDIS calculation we will also face terms like

$$I[f] \equiv \int_z^1 \int_w^1 \frac{f(x,y)}{(1-x)^{1+\epsilon}(1-y)^{1+\epsilon}} \,\mathrm{d}y\,\mathrm{d}x. \tag{261}$$

where f is a smooth function. By Equation (260) we have

$$\frac{1}{(1-x)^{1+\epsilon}} = \left(\frac{1}{1-x}\right)_+ - \frac{1}{\epsilon}\delta(1-x) - \epsilon\left(\frac{\ln(1-x)}{1-x}\right)_+ + \mathcal{O}\left(\epsilon^2\right) \tag{262}$$

and hence up to the zeroth order in ϵ and in a distributional sense

$$\begin{aligned} \frac{1}{(1-x)^{1+\epsilon}(1-y)^{1+\epsilon}} = &\left(\frac{1}{1-x}\right)_+\left(\frac{1}{1-y}\right)_+ + \delta(1-x)\left(\frac{\ln(1-y)}{1-y}\right)_+ \\ &+ \delta(1-y)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{1}{\epsilon}\delta(1-x)\left(\frac{1}{1-y}\right)_+ \\ &- \frac{1}{\epsilon}\delta(1-y)\left(\frac{1}{1-x}\right)_+ + \frac{1}{\epsilon^2}\delta(1-x)\,\delta(1-y)\,. \end{aligned} \tag{263}$$

We finish this appendix with a frequently used plus function property and its implications. If $f(1)=0$ then for any plus function $(q(x))_+$ we have in a distributional sense

$$f(x)\,(q(x))_+ = f(x)q(x). \tag{264}$$

Because of this in a distributional sense

$$\left(\frac{1}{1-x}\right)_+(1-x) = 1, \tag{265}$$

$$\left(\frac{1}{1-x}\right)_+ x = \left(\frac{1}{1-x}\right)_+ - 1 \tag{266}$$

and

$$\left(\frac{1}{1-x}\right)_+ x^2 = \left(\frac{1}{1-x}\right)_+ - (1+x)\,. \tag{267}$$

Last of these identities also implies that

$$\frac{x^2+y^2}{(1-x)_+(1-y)_+} = \frac{2}{(1-x)_+(1-y)_+} - \frac{1+x}{(1-y)_+} - \frac{1+y}{(1-x)_+}. \tag{268}$$

E Approximations

In this appendix we present some of the approximation identities needed during this thesis.

Often in this thesis the approximations done are based on identities

$$(f(x))^{\epsilon} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \left(\ln\left(f(x)\right)\right)^k \tag{269}$$

$$\frac{1}{1-f(\epsilon)} = \sum_{k=0}^{\infty} \left(f(\epsilon)\right)^k . \tag{270}$$

Right-hand side of the latter equation is well defined only when $|f(\epsilon)| < 1$. Another needed identity is the expansion of the Gamma function $\Gamma(1+x)$ with small values of x. At most the second order terms are needed so

$$\Gamma(1+x) = 1 - x\gamma_E + \frac{x^2}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) + \mathcal{O}\left(x^3\right) \tag{271}$$

will suffice here. In this equation γ_E is Euler-Mascheroni constant for which it holds that

$$\gamma_E = \Gamma'(1) = \int_0^{\infty} e^{-t} \ln(t)\ \mathrm{d}t \approx 0.57721... \tag{272}$$

Note that the latter equality is just the derivative identity of the gamma function: for every $n \in \mathbb{N}_0$

$$\Gamma^{(n)}(x) = \int_0^{\infty} t^{x-1} e^{t} \left(\ln(t)\right)^n\ \mathrm{d}t. \tag{273}$$

Using Equations (271) and (270) it can be calculated that

$$\Gamma(1+x)\,\Gamma(1-x) = 1 + x^2\frac{\pi^2}{6} + \mathcal{O}\left(x^4\right), \tag{274}$$

$$\frac{1}{\Gamma\left(1-x\right)}=1-x\gamma_E+\frac{x^2}{2}\left(\gamma_E^2-\frac{\pi^2}{6}\right)+\mathcal{O}\left(x^3\right) \tag{275}$$

and

$$\frac{\Gamma\left(1-x\right)}{\Gamma\left(1-2x\right)}=1-x\gamma_E+\frac{x^2}{2}\left(\gamma_E^2-\frac{\pi^2}{2}\right)+\mathcal{O}\left(x^3\right). \tag{276}$$

Furthermore

$$\Gamma\left(1-x\right)\Gamma\left(1+x\right)\frac{\Gamma\left(1-x\right)^2}{\Gamma\left(1-2x\right)}=1+\mathcal{O}\left(x^3\right) \tag{277}$$

or

$$\frac{1}{\Gamma\left(1-x\right)}=\Gamma\left(1+x\right)\frac{\Gamma\left(1-x\right)^2}{\Gamma\left(1-2x\right)}+\mathcal{O}\left(x^3\right). \tag{278}$$

We will also deploy the result

$$\frac{1}{\epsilon}\left(\frac{4\pi m_D^2}{Q^2}\right)^{\epsilon}\frac{\Gamma\left(1-\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}=\frac{1}{\epsilon}+\ln\left(\frac{4\pi m_D^2}{Q^2}\right)-\gamma_E+\mathcal{O}\left(\epsilon\right) \tag{279}$$

$$=\frac{1}{\hat{\epsilon}}-\ln\left(\frac{Q^2}{m_D^2}\right)+\mathcal{O}\left(\epsilon\right), \tag{280}$$

where we introduced the notation

$$\frac{1}{\hat{\epsilon}}\equiv\frac{1}{\epsilon}+\ln\left(4\pi\right)-\gamma_E \tag{281}$$

$$=\frac{1}{\epsilon}\frac{\left(4\pi\right)^{\epsilon}}{\Gamma\left(1-\epsilon\right)}+\mathcal{O}\left(\epsilon\right). \tag{282}$$

The point with this notation is to compactly group the pole with remnant coefficients of the dimensional regularization as they all are included into the renormalization in the $\overline{\text{MS}}$ scheme.

F Phase space integration

In this appendix we calculate the phase space integrals for the 4-momentum processes $p,q \rightarrow p'$ and $p,q \rightarrow k,p'$ with $p^2 = p'^2 = k^2 = 0$ and N space-time dimensions. The results are used in the deeply inelastic parton model context given in Section 2.1 and the processes are those given in Section 2.3. We denote the corresponding phase space integrals $PS_{p'}\left[f\right]$ and $PS_{p',k}\left[f\right]$ where f is the matrix element or the partonic tensor of the process. We use here the DIS variables defined in Equation (3), the Mandelstam variable $\hat{s} = \left(p+q\right)^2$ and the variable w given in Equation (12).

In the phase space integrations we need to integrate the N-dimensional Dirac delta function. Results similar to the case $N = 4$ hold for $N \neq 4$. If $p = (p^0,\vec{p})$ is an N-vector with $p^2 = m^2$ then

$$\int \frac{\mathrm{d}^{N-1}\vec{p}}{2p^0} f(p) = \int \mathrm{d}^N p \delta_+ \left(p^2 - m^2\right) f(p) \tag{283}$$

as for a smooth function g

$$\delta\left(g\left(x\right)\right) = \sum_{x_0 \in g^{-1}(\{0\})} \frac{\delta\left(x - x_0\right)}{\left|g'\left(x_0\right)\right|}. \tag{284}$$

Using Equation (283) one finds

$$\int \frac{\mathrm{d}^{N-1}\vec{p}}{2p^0} \delta^{(N)}\left(p - c\right) f(p) = \delta_+ \left(c^2 - m^2\right) f(p)\Big|_{p=c}. \tag{285}$$

In the results above the notation δ_+ means that the normal Dirac delta function arguments are restricted to accept $p^0 > 0$ or in the latter equality $c^0 > 0$ only.

Then the phase space integrals. First we consider the process $p,q \rightarrow p'$:

$$PS_{p'}[f] = \int \frac{\mathrm{d}^{N-1}\vec{p'}}{2(2\pi)^{N-1}p'^0} \left(2\pi\right)^N \delta^{(N)}\left(p + q - p'\right) f\left(p'\right). \tag{286}$$

Using Equation (285) and discarding masses we get

$$PS_{p'}[f] = 2\pi\delta_+\left(q^2 + 2p\cdot q\right) f\left(p+q\right). \tag{287}$$

Next we use the Dirac delta property (284) regarding the zeros of its arguments to find that

$$PS_{p'}\left[f\right] = \frac{2\pi}{Q^2}\delta_+\left(1-w\right) f\left(p+q\right). \tag{288}$$

The second phase space integral is for the process $p{,}q \to p'{,}k$. Note that even though we handle p' and k differently here the process can be made similarly the other way around. We choose here to use delta functions to integrate p' right at the beginning and parametrize the now unintegrable degrees of freedom of k with variables presented above. Using equations (283) and (285) we find that

$$\begin{aligned} PS_{p',k}[f] &= \int \frac{\mathrm{d}^{N-1}\vec{p'}}{2(2\pi)^{N-1}p'^0}\int\frac{\mathrm{d}^{N-1}\vec{k}}{2(2\pi)^{N-1}k^0}\left(2\pi\right)^N\delta^{(N)}\left(p+q-p'-k\right) f\left(p'{,}k\right) && (289)\\ &= \left(2\pi\right)^{2-N}\int \mathrm{d}^N k\,\delta_+\left(\left(p+q-k\right)^2\right)\delta_+\left(k^2\right) f\left(p+q-k{,}k\right). && (290)\end{aligned}$$

Next we change the integration variables k into k^0 and $(N-1)$-dimensional spherical coordinates using $\vec{p}$ as the base, so that the first coordinate angle θ_k is the angle between $\vec{p}$ and $\vec{k}$. We also take that the function f is a constant function of every coordinate angle except θ_k. The change of variables and these assumptions change the integral as

$$\begin{aligned} \int \mathrm{d}^N k\tilde{f}\left(k\right) &= \int \mathrm{d}k^0\int \mathrm{d}\left|\vec{k}\right|\left|\vec{k}\right|^{N-2}\int_0^\pi \mathrm{d}\theta_k\sin^{N-3}\left(\theta_k\right)\int \mathrm{d}\Omega_{N-2}\tilde{f}\left(k^0,\left|\vec{k}\right|{,}\theta_k\right) && (291)\\ &= \frac{2\pi^{\frac{N}{2}-1}}{\Gamma\left(\frac{N}{2}-1\right)}\int \mathrm{d}k^0\int \mathrm{d}\left|\vec{k}\right|\left|\vec{k}\right|^{N-2}\int_0^\pi \mathrm{d}\theta_k\sin^{N-3}\left(\theta_k\right)\tilde{f}\left(k^0,\left|\vec{k}\right|{,}\theta_k\right), && (292)\end{aligned}$$

where we used the area of the N-dimensional unit sphere (224). In Equation (290) we will first use the k^0 integral to act on the second delta function and then use the $\left|\vec{k}\right|$ integral to act on the first delta function. Considering the second we use the fact that

$$\delta_+\left(k^2\right) = \frac{1}{2\left|\vec{k}\right|}\delta_+\left(k^0 - \left|\vec{k}\right|\right). \tag{293}$$

The easiest way to handle the first delta function is to go momentarily to the center-of-momentum frame (CMS, abbreviation of center-of-momentum system), where $\vec{p}+\vec{q}=0$, and then write the result in a Lorentz invariant form. Using $k^0 = \left|\vec{k}\right|$ in the CMS we have

$$\delta_+\left((p+q-k)^2\right) = \delta_+\left((p+q)^2 - 2\left(p^0+q^0\right)k^0\right) \tag{294}$$

$$= \frac{1}{2\left|p^0+q^0\right|}\delta_+\left(\left|\vec{k}\right| - \frac{p^0+q^0}{2}\right) \tag{295}$$

$$= \frac{1}{2\hat{s}^{\frac{1}{2}}}\delta\left(\left|\vec{k}\right| - \frac{1}{2}\hat{s}^{\frac{1}{2}}\right). \tag{296}$$

These steps transform (290) into

$$PS_{p',k}[f] = (16\pi)^{1-\frac{N}{2}}\frac{1}{\Gamma\left(\frac{N}{2}-1\right)}\int_0^\pi \mathrm{d}\theta_k \sin^{N-3}\left(\theta_k\right)\left(\hat{s}\right)^{\frac{N-4}{2}} f\left(p,q,\theta_k\right). \tag{297}$$

As a final step we perform a change of variable from θ_k to variable v defined as

$$v \equiv \frac{1}{2}\left(1+\cos\theta_k\right) \tag{298}$$

$$= \frac{1}{4}\frac{\sin^2\left(\theta_k\right)}{1-v}. \tag{299}$$

Because

$$\mathrm{d}v = -\frac{1}{2}\sin\theta_k \mathrm{d}\theta_k \tag{300}$$

we have

$$\int_0^\pi \mathrm{d}\theta_k \sin^{N-3}\left(\theta_k\right) f\left(\theta_k\right) = 2^{N-3}\int_0^1 \mathrm{d}v\left(v\left(1-v\right)\right)^{\frac{N}{2}-2}. \tag{301}$$

Hence

$$PS_{p',k}[f] = (4\pi)^{1-\frac{N}{2}}\frac{1}{\Gamma\left(\frac{N}{2}-1\right)}\int_0^1 \mathrm{d}v v^{\frac{N}{2}-2}\left(1-v\right)^{\frac{N}{2}-2}\frac{\left(\hat{s}\right)^{\frac{N-4}{2}}}{2} f\left(p,q,v\right). \tag{302}$$

We also note that even though v is defined in a specific frame it has a Lorentz invariant expression. In CMS $p' = (k^0, -\vec{k})$ and hence we get

$$1 - v = \frac{p \cdot k}{2k^0 p^0} = \frac{p \cdot k}{p \cdot (k + p')} = \frac{p \cdot k}{p \cdot q}. \tag{303}$$

This Lorentz invariant form is the definition used in Sections 3 and 4.