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QUANTITATIVE ALEXANDROV THEOREM AND ASYMPTOTIC BEHAVIOR OF THE VOLUME PRESERVING MEAN CURVATURE FLOW

VESA JULIN AND JOONAS NIINIKOSKI

We prove a new quantitative version of the Alexandrov theorem which states that if the mean curvature of a regular set in $\mathbb{R}^{n+1}$ is close to a constant in the L^n sense, then the set is close to a union of disjoint balls with respect to the Hausdorff distance. This result is more general than the previous quantifications of the Alexandrov theorem, and using it we are able to show that in $\mathbb{R}^2$ and $\mathbb{R}^3$ a weak solution of the volume preserving mean curvature flow starting from a set of finite perimeter asymptotically convergences to a disjoint union of equisize balls, up to possible translations. Here by a weak solution we mean a flat flow, obtained via the minimizing movements scheme.

1. Introduction

Here we study the asymptotic behavior of the weak solution of the volume preserving mean curvature flow starting from a set of finite perimeter. In the classical setting we are given a smooth set $E_0 \subset \mathbb{R}^{n+1}$ and we let it evolve into a smooth family of sets $(E_t)_t$ according to the law, where the normal velocity V_t is proportional to the mean curvature of E_t as

$$V_t = -(H_{E_t} - \overline{H}_{E_t}) \quad \text{on } \partial E_t, \tag{1-1}$$

where $\overline{H}_{E_t} = \fint_{\partial E_t} H_{E_t} \, \mathrm{d}\mathcal{H}^n$. Equations of mean curvature type are important in geometry, where one usually studies the geometric properties of ∂E_t which are inherited from ∂E_0. Equation (1-1) can also be seen as a volume preserving gradient flow of the surface area. These equations arise naturally in physical models involving surface tension; see [Taylor et al. 1992].

The main issue with (1-1) is that it may develop singularities in finite time even in the plane [Mayer 2001; Mayer and Simonett 2000]. In order to pass over the singular time one may try to do a surgery procedure and restart the flow after a singular time as in [Huisken and Sinestrari 2009] or to define a weak solution of (1-1), which is what we will consider here. For the mean curvature flow one may define a weak solution by using the varifold setting by Brakke [1978], the level set solution developed independently by Chen, Giga and Goto [Chen et al. 1989] and Evans and Spruck [1991], or by using the minimizing movements scheme developed independently by Almgren, Taylor and Wang [Almgren et al. 1993] and Luckhaus and Stürzenhecker [1995]. Since we want the solution of (1-1) to be a family of sets and since (1-1) does not satisfy the comparison principle, the natural choice is to define a weak solution via the

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minimizing movements scheme as in [Almgren et al. 1993; Luckhaus and Sturzenhecker 1995]. This solution is usually called a flat flow, and it is well defined due to [Mugnai et al. 2016] but might not be unique.

The advantage of the flat flow is that it is defined for all times for any bounded initial set with finite perimeter and we may thus study its asymptotic behavior. Heuristically, one may guess that the flat flow converges to a critical point of the static problem, which are classified in [Delgadino and Maggi 2019] as a disjoint union of balls, possibly tangent to each other. The asymptotic convergence of (1-1) has been proved for initial sets with certain geometric properties such as convexity [Huisken 1987], nearly spherical [Escher and Simonett 1998] or sets which are near a stable critical set in the flat torus in low dimensions [Niinikoski 2021]. We note that in these cases the flow does not develop singularities and is thus classically well defined for all times. The result in [Kim and Kwon 2020] shows that the convergence holds also for star-shaped sets, up to possible translations. For the mean curvature flow with forcing, the asymptotic behavior has been studied for the level set solution in [Giga et al. 2019; 2020] and for the flat flow in the plane in [Fusco et al. 2022]. The result closest to ours is the work by Morini, Ponsiglione and Spadaro [Morini et al. 2022], where the authors prove that the discrete-in-time approximation of the flat flow of (1-1) converges exponentially fast to a disjoint union of balls. Here we are able to pass the time discretization to zero and characterize the limit sets for the flat flow of (1-1) in $\mathbb{R}^2$ and $\mathbb{R}^3$. The precise definition of the flat flow is given in Section 4.

Theorem 1.1. *Assume $E_0 \subset \mathbb{R}^{n+1}$, with $n \le 2$ and $|E_0| = |B_1|$, is a bounded set of finite perimeter which is either essentially open or essentially closed, and let $(E_t)_{t\ge 0}$ be a flat flow of (1-1) starting from E_0. There is $N \in \mathbb{N}$ such that the following holds: for every $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for every $t \ge T_\varepsilon$ there are points $x_1, \dots, x_N$, which may depend on time, with $|x_i - x_j| \ge 2r$ for $i \ne j$ and $r = N^{-1/(n+1)}$ such that for $F_t = \bigcup_{i=1}^N B_r(x_i)$,*

$$\sup_{x \in E_t \Delta F_t} d_{\partial F_t}(x) \le \varepsilon.$$

Here $d_{\partial F}$ denotes the distance function. To the best of our knowledge this is the first result on the characterization of the asymptotic limit of (1-1) in $\mathbb{R}^3$. The above result holds for any limit of the approximative flat flow, and we do not need the additional assumption on the convergence of the perimeters as in [Luckhaus and Sturzenhecker 1995; Mugnai et al. 2016]. We note that the assumption on E_0 being either essentially open or closed is only needed to ensure that the flow is continuous up to time zero. It plays no role in the asymptotic analysis.

Concerning the limiting configurations, Theorem 1.1 is sharp since the flow (1-1) may converge to tangent balls as shown in [Fusco et al. 2022]. On the other hand, we believe that one may rule out the possible translations and the flow actually convergences to a disjoint union of balls. The higher dimensional case and the possible speed of convergence are also open problems.

Quantitative Alexandrov theorem. The proof of Theorem 1.1 is based on the dissipation inequality proven in [Mugnai et al. 2016] and stated in Proposition 4.1. This implies that there is a sequence of times $t_j \to \infty$ such that the mean curvatures of the evolving sets E_{t_j} are asymptotically close to a constant

with respect to the L^2-norm. Therefore, we need a quantified version of the Alexandrov theorem which enables us to conclude that the sets E_{t_j} are close to a disjoint union of balls.

There is a lot of recent research on generalizing the Alexandrov theorem [Ciraolo and Maggi 2017; Delgadino and Maggi 2019; Delgadino et al. 2018; De Rosa et al. 2020; Krummel and Maggi 2017; Magnanini and Poggesi 2020]. We refer the survey paper [Ciraolo 2021] for the state of the art. Unfortunately, none of the available results is applicable to our problem, and we are also not able to use the characterization of the critical sets in [Delgadino and Maggi 2019, Corollary 2] to identify the limit set. Indeed, even if we know that the sets E_{t_j} converge to a set of finite perimeter and their mean curvatures converge to a constant, it is not clear why the limit set is a set of finite perimeter with weak mean curvature as this class of sets is not in general closed. Our main result is the following quantification of the Alexandrov theorem, which is the main technical tool in the proof of Theorem 1.1.

Theorem 1.2. *Let $E \subset \mathbb{R}^{n+1}$ be a C^2 regular set such that $P(E) \le C_0$ and $|E| \ge 1/C_0$. There are positive constants $q = q(n) \in (0, 1]$, $C = C(C_0, n)$ and $\delta = \delta(C_0, n)$ such that if $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$ for some $\lambda \in \mathbb{R}$, then $1/C \le \lambda \le C$ and there are points $x_1, \dots, x_N$ with $|x_i - x_j| \ge 2R$, where $R = n/\lambda$, such that for $F = \bigcup_{i=1}^N B_R(x_i)$,*

$$\sup_{x \in E \Delta F} d_{\partial F}(x) \le C \|H_E - \lambda\|^q_{L^n(\partial E)}.$$

Moreover,

$$|P(E) - N(n+1)\omega_{n+1} R^n| \le C \|H_E - \lambda\|^q_{L^n(\partial E)}.$$

The main advantage of Theorem 1.2 with respect to the previous results in the literature is that we do not assume any geometric restriction on E such as mean convexity. Moreover, we assume the mean curvature to be close to a constant only in the L^n sense, which is exactly what we need for the asymptotic analysis in Theorem 1.1. This makes the proof challenging as, for example, we cannot use the estimates from Allard's regularity theory [1972].

Theorem 1.2 is sharp in the sense that $\|H_E - \lambda\|_{L^n(\partial E)}$ cannot be replaced by a weaker L^p-norm. This can be seen by considering a set which is a union of the unit ball and a ball of small radius ε located far away. On the other hand, the dissipation inequality in Proposition 4.1 controls only the L^2-norm of the mean curvature, which is the reason we cannot prove Theorem 1.1 in higher dimensions. The proof of Theorem 1.2 is done in a constructive way and we obtain an explicit bound on the exponent $q = (n+2)^{-3}$. It would be interesting to obtain the sharp bound as it might be crucial in order to obtain the possible exponential convergence of (1-1) as in [Morini et al. 2022]. In the two-dimensional case the optimal power $q = 1$ is proven in [Fusco et al. 2022].

Outline of the proof of Theorem 1.2. Since the proof of Theorem 1.2 is rather long, we outline it here. As in [Delgadino and Maggi 2019], our argument is based on the proof of the Heinze–Karcher inequality by Montiel and Ros [1991], which is an alternative for the proof in [Ros 1987]. In [Delgadino and Maggi 2019], the authors are able to generalize the Montiel–Ros argument to sets of finite perimeter with weak distributional mean curvature. Here we revisit the argument by Montiel and Ros and deduce in Proposition 3.3 that for E and R as in Theorem 1.2 and for $0 < r < R$, the volume of the set

$E_r = \{x \in E : \operatorname{dist}(x, \partial E) > r\}$ satisfies the estimate

$$\left| |E_r| - \frac{|E|}{R^{n+1}}(R-r)^{n+1} \right| \le C\|H_E - \lambda\|_{L^n(\partial E)}.$$

We use this in Step 1 of the proof of Theorem 1.2 to deduce that for r close to R the set E_r is a union of a finite number of components, or clusters, with positive distance to each other.

We note that the above inequality is not enough to conclude the proof as, e.g., the cube $Q = (-1, 1)^{n+1}$ satisfies $|Q_r| = (1-r)^{n+1}|Q|$. Therefore, we need further information from the Montiel–Ros argument and we prove in Proposition 3.3 that the Minkowski sum $E_r + B_\rho = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, E_r) < \rho\}$, with $0 < \rho < r < R$, satisfies

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1} \right| \le \frac{C}{(R-r)^{n+1}}\|H_E - \lambda\|_{L^n(\partial E)}.$$

This enables us to prove that the components of $E_r + B_\rho \subset E$, with properly chosen ρ and r, are almost spherical. In particular, if E satisfies the above estimate with $C = 0$, then it is a disjoint union of balls. This, together with the density estimate from [Topping 2008], concludes the proof.

2. Notation and preliminary results

In this section we briefly introduce our notation and recall some results from differential geometry. Given a set $E \subset \mathbb{R}^{n+1}$ the distance function $d_E : \mathbb{R}^{n+1} \to [0, \infty)$ is defined, as usual, as

$$d_E(x) := \inf_{y \in E} |x - y|,$$

and we denote the signed distance function $\bar{d}_E : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\bar{d}_E(x) := \begin{cases} -d_{\partial E}(x) & \text{for } x \in E, \\ d_{\partial E}(x) & \text{for } x \in \mathbb{R}^{n+1} \setminus E. \end{cases}$$

Then clearly $d_{\partial E} = |\bar{d}_E|$. We denote the ball with radius r centered at x by $B_r(x)$ and by B_r if it is centered at the origin. Given a set $E \subset \mathbb{R}^{n+1}$ we denote its ρ-enlargement by the Minkowski sum

$$E + B_\rho = \{x + y \in \mathbb{R}^{n+1} : x \in E,\ y \in B_\rho\} = \{x \in \mathbb{R}^{n+1} : d_E(x) < \rho\}.$$

For a measurable set $E \subset \mathbb{R}^{n+1}$ the shorthand notation $|E|$ denotes its Lebesgue measure, and we denote the k-dimensional measure of the unit ball in $\mathbb{R}^k$ by ω_k. In some cases, we may use the shorthand notation $|E|$ more generally for a measurable set $E \subset \mathbb{R}^k$ to denote its k-dimensional Lebesgue measure but this shall be clear from context.

For a set of finite perimeter $E \subset \mathbb{R}^{n+1}$ we denote its reduced boundary by $\partial^* E$ and the perimeter by $P(E)$. Recall that $P(E) = \mathcal{H}^n(\partial^* E)$ and for a regular enough set, $\partial^* E = \partial E$. The relative isoperimetric inequality states that for every set of finite perimeter E and for every ball $B_r(x)$,

$$\mathcal{H}^n(\partial^* E \cap B_r(x))^{(n+1)/n} \ge c_n \min\{|E \cap B_r(x)|, |B_r(x) \setminus E|\},$$

for a dimensional constant c_n. We refer to [Maggi 2012] for an introduction to the topic.

We define the *tangential differential* of $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^m)$ on ∂E by

$$\mathrm{D}_\tau F(x) = \mathrm{D}F(x)(I - \nu_E(x) \otimes \nu_E(x)),$$

where ν_E denotes the unit outer normal of E. For a function $f \in C^1(\mathbb{R}^{n+1}; \mathbb{R})$ we denote by $\nabla_\tau f$ its tangential gradient which is a vector in $\mathbb{R}^{n+1}$. We define the tangential divergence of $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ by $\operatorname{div}_\tau F = \operatorname{Tr}(\mathrm{D}_\tau F)$. Then the divergence theorem on manifolds generalizes to

$$\int_{\partial^* E} \operatorname{div}_\tau F \, \mathrm{d}\mathcal{H}^n = \int_{\partial^* E} H_E \langle F, \nu_E \rangle \, \mathrm{d}\mathcal{H}^n,$$

where $H_E \in L^1(\partial^* E)$ is the distributional mean curvature. When ∂E is smooth, H_E agrees with the classical definition of the mean curvature, which for us is the sum of the principal curvatures.

We begin by recalling the well-known inequality proven first by Simon [1993] in $\mathbb{R}^3$ and then by Topping [2008] in the general case.

Theorem 2.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact and connected C^2-hypersurface. Then*

$$\operatorname{diam}(\Sigma) \le C_n \int_\Sigma |H_\Sigma|^{n-1} \, \mathrm{d}\mathcal{H}^n, \tag{2-1}$$

where C_n depends only on the dimension.

We need also the Michael–Simon inequality [Michael and Simon 1973].

Theorem 2.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$, $n \ge 2$, be a compact C^2-hypersurface. Then for every nonnegative $\varphi \in C^1(\mathbb{R}^{n+1})$,*

$$\|\varphi\|_{L^{n/(n-1)}(\Sigma)} \le C_n \int_\Sigma |\nabla_\tau \varphi| + \varphi |H_\Sigma| \, \mathrm{d}\mathcal{H}^n, \tag{2-2}$$

where C_n depends only on the dimension.

The following density-type estimate is essentially proven in [Morini et al. 2022, Lemma 2.1].

Proposition 2.3. *Let $E \subset \mathbb{R}^{n+1}$ be a set of finite perimeter with $P(E) > 0$ and $0 < \beta < 1$. There is a positive constant $c = c(n, \beta)$ such that*

$$r_{E,\beta} := \sup\{r \in \mathbb{R}_+ : \textit{there exists } x \in \mathbb{R}^{n+1} \textit{ with } |B_r(x) \cap E| \ge \beta |B_r(x)|\} \ge c \frac{|E|}{P(E)}.$$

We use the previous results to prove the following lemma, which is useful when we bound the Lagrange multipliers and the number of the components of the flat flow of (1-1).

Lemma 2.4. *Let $E \subset \mathbb{R}^{n+1}$ be a bounded set of finite perimeter with a distributional mean curvature $H_E \in L^1(\partial^* E)$, $\lambda \in \mathbb{R}$ and $1 \le C_0 < \infty$. There is a positive constant $C = C(C_0, n)$ such that:*

(i) *If $P(E) \le C_0$ and $|E| \ge 1/C_0$, then*

$$1/C - C\|H_E - \lambda\|_{L^1(\partial^* E)} \le \lambda \le C + C\|H_E - \lambda\|_{L^1(\partial^* E)}.$$

(ii) *If $P(E) \le C_0$, $|E| \ge 1/C_0$ and E is C^2 regular, then the number of components of E is bounded by $C(1 + \|H_E - \lambda\|^n_{L^n(\partial E)})$ and the diameters of the components are bounded by $C(1 + \|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)})$.*

Proof. Our standing assumptions throughout the proof are $P(E) \le C_0$ and $|E| \ge 1/C_0$. The perimeter bound and the global isoperimetric inequality yield

$$|E| \le c_n P(E)^{(n+1)/n} \le c_n C_0^{(n+1)/n}.$$

By the assumptions on E and by the divergence theorems, we compute the following for any vector field $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$:

$$\begin{aligned}\lambda \int_E \operatorname{div} F \,\mathrm{d}x &= \int_{\partial^* E} \lambda \langle F, \nu_E \rangle \,\mathrm{d}\mathcal{H}^n \\ &= \int_{\partial^* E} H_E \langle F, \nu_E \rangle \,\mathrm{d}\mathcal{H}^n + \int_{\partial^* E} (\lambda - H_E) \langle F, \nu_E \rangle \,\mathrm{d}\mathcal{H}^n \\ &= \int_{\partial^* E} \operatorname{div}_\tau F \,\mathrm{d}\mathcal{H}^n + \int_{\partial^* E} (\lambda - H_E) \langle F, \nu_E \rangle \,\mathrm{d}\mathcal{H}^n. \end{aligned} \tag{2-3}$$

Our goal is to construct a suitable vector field F to obtain (i) from (2-3). To this aim, we use first the relative isoperimetric inequality, Proposition 2.3 and a suitable continuity argument to find positive $r_0 = r_0(C_0, n)$, $R_0 = R_0(C_0, n)$ and r such that $r_0 \le r \le R_0$ and, by possibly translating the coordinates, $|B_r \cap E| = \frac{1}{2}|B_r|$. Again, it follows from the relative isoperimetric inequality that $\mathcal{H}^n(\partial^* E \cap B_r) \ge c$ for some positive $c = c(C_0, n)$. Choose a decreasing C^1 function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(t) = \begin{cases} (2r)^{-1} & \text{for } t \le \frac{3}{2} r \\ t^{-1} & \text{for } t \ge \frac{5}{2} r \end{cases}$$

and for which the conditions $f(t) \le \min\{(2r)^{-1}, t^{-1}\}$ and $|f'(t)| \le (2r)^{-2}$ hold on $\left[\frac{3}{2} r, \frac{5}{2} r\right]$. We define the function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting $F(x) = f(|x|)x$. Then F is a C^1 vector field with

$$\begin{aligned} \mathrm{D}F(x) &= f(|x|) I + \frac{f'(|x|)}{|x|} x \otimes x && \text{for every } x \in \mathbb{R}^{n+1}, \\ \operatorname{div} F(x) &= (n+1) f(|x|) + f'(|x|)|x| && \text{for every } x \in \mathbb{R}^{n+1}, \\ \operatorname{div}_\tau F(x) &= n f(|x|) + f'(|x|)\left(|x| - \frac{\langle x, \nu_E \rangle^2}{|x|}\right) && \text{for every } x \in \partial^* E. \end{aligned}$$

Then $0 < \operatorname{div} F \le (n+1)(2r)^{-1}$ everywhere and $\operatorname{div} F = (n+1)(2r)^{-1}$ in B_r, so by using these and the earlier observations we obtain

$$\begin{aligned} \frac{n+1}{4R_0} |B_{r_0}| \le \frac{n+1}{4r} |B_r| &= \frac{n+1}{2r} |B_r \cap E| \\ &\le \int_E \operatorname{div} F \,\mathrm{d}x \le \frac{n+1}{2r} |E| \le \frac{c_n (n+1)}{2r_0} C_0^{(n+1)/n}. \end{aligned} \tag{2-4}$$

Again, $0 \le \operatorname{div}_\tau F \le n(2r)^{-1}$ on $\partial^* E$ and $\operatorname{div}_\tau F = n(2r)^{-1}$ on $\partial^* E \cap B_r$, and thus

$$\frac{nc}{2R_0} \le \frac{n}{2r} \mathcal{H}^n(\partial^* E \cap B_r) \le \int_{\partial^* E} \operatorname{div}_\tau F \,\mathrm{d}\mathcal{H}^n \le \frac{n P(E)}{2r} \le \frac{n C_0}{2r_0}. \tag{2-5}$$

We use (2-3), (2-4), (2-5) and $|F| \le 1$ to obtain (i).

The claim (ii) is easy to prove in the planar case and therefore we assume that $n \geq 2$. Let $E_1, E_2, \dots, E_N$ denote the connected components of E. We apply Theorem 2.2 on ∂E_i with $\varphi = 1$ and use Hölder's inequality to obtain

$$C_n^{-1} \leq \|H_{E_i}\|_{L^n(\partial E_i)} \leq \|H_{E_i} - \lambda\|_{L^n(\partial E_i)} + |\lambda| P(E_i)^{1/n},$$

from which we conclude, using (i) and Hölder's inequality, that

$$\begin{aligned} N C_n^{-n} &\leq 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^n |\lambda|^n P(E) \\ &\leq 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^{2n} C_0 C^n (1 + \|H_E - \lambda\|^n_{L^1(\partial E)}) \\ &\leq 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^{2n} C_0 C^n (1 + C_0^{n-1} \|H_E - \lambda\|^n_{L^n(\partial E)}). \end{aligned} \tag{2-6}$$

On the other hand, Theorem 2.1 together with (i) and Hölder's inequality implies

$$\begin{aligned} \sum_i \operatorname{diam}(E_i) &\leq \sum_i C_n \int_{\partial E_i} |H_{E_i}|^{n-1} \, \mathrm{d}\mathcal{H}^n \\ &\leq \sum_i 2^{n-1} C_n \left(\int_{\partial E_i} |H_{E_i} - \lambda|^{n-1} \, \mathrm{d}\mathcal{H}^n + |\lambda|^{n-1} P(E_i) \right) \\ &\leq 2^{n-1} C_n \left(\int_{\partial E} |H_E - \lambda|^{n-1} \, \mathrm{d}\mathcal{H}^n + P(E) |\lambda|^{n-1} \right) \\ &\leq 2^{n-1} C_n (\|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)} + 2^{n-1} C_0 C^n (1 + \|H_E - \lambda\|^{n-1}_{L^1(\partial E)})) \\ &\leq 2^{n-1} C_n (\|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)} + 2^{n-1} C_0 C^n (1 + C_0^{n-2} \|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)})). \end{aligned} \tag{2-7}$$

Thus, by possibly increasing C, the second claim follows from (2-6) and (2-7). □

3. Quantitative Alexandrov theorem

We split the proof of Theorem 1.2 into two parts. We first revisit the Montiel–Ros argument in Proposition 3.3 where all the technical heavy lifting is done. The idea of Proposition 3.3 is to transform the (local) information of the mean curvature of E being close to a constant into information on the ρ-enlargement of the level sets of the distance function of ∂E. We note that the statement of Proposition 3.3 is given by the sharp exponent. The proof of Theorem 1.2 is then based on purely geometric arguments.

We first state the following equivalent formulation of the theorem.

Remark 3.1. Once we prove that in Theorem 1.2 the number of component of E is bounded, the statement on the L^∞-distance is equivalent to the fact that, under the assumption $\|H_E - \lambda\|_{L^n(\partial E)} \leq \delta$, there are points $x_1, \dots, x_N$ such that

$$\bigcup_{i=1}^N B_{\rho_-}(x_i) \subset E \subset \bigcup_{i=1}^N B_{\rho_+}(x_i),$$

where we have $\rho_- = R - C\|H_E - \lambda\|^q_{L^n(\partial E)}$, $\rho_+ = R + C\|H_E - \lambda\|^q_{L^n(\partial E)}$, $R = n/\lambda$ and the balls $B_{\rho_-}(x_1), \dots, B_{\rho_-}(x_N)$ are disjoint to each other. We leave the details to the reader.

In Theorem 1.2 we assume that the mean curvature is bounded only in the L^n sense and thus the estimates from Allard's regularity theory [1972] are not available for us. Indeed, the L^n-boundedness of the mean curvature is not strong enough to give proper density estimates. Moreover, even in the three dimensional case $\mathbb{R}^3$ we cannot use the results from [Simon 1993], because we do not have a uniform bound on the Euler characteristic of the set E. However, if we know that the mean curvature is close to a constant with respect to the L^n-norm, then the following density estimate holds. The proof is based on [Topping 2008, Lemma 1.2].

Lemma 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact C^2-hypersurface and $\lambda \in \mathbb{R}_+$. There is a positive dimensional constant δ_n such that if $\|H_\Sigma - \lambda\|_{L^n(\Sigma)} \le \delta_n$, then*

$$\delta_n \le \frac{\mathcal{H}^n(B(x,r)\cap\Sigma)}{r^n}$$

for every $x \in \Sigma$ and $0 < r \le \delta_n/\lambda$.

Proof. The planar case $n = 1$ is rather obvious and we leave it to the reader. Assume $n \ge 2$. Fix $x \in \Sigma$ and define $V : [0,\infty) \to [0,\infty)$ as $V(r) = \mathcal{H}^n(B_r(x)\cap\Sigma)$. Since V is increasing, the derivative $V'(r)$ is defined for almost every $r \in [0,\infty)$, and

$$\int_{r_1}^{r_2} V'(\rho)\,\mathrm{d}\rho \le V(r_2) - V(r_1) \quad \text{whenever } 0 \le r_1 < r_2.$$

By a standard foliation argument we have that $\mathcal{H}^n(\partial B_r(x)\cap\Sigma) > 0$ for at most countably many $r \in \mathbb{R}_+$. Thus $V'(r)$ is defined and $\mathcal{H}^n(\partial B_r(x)\cap\Sigma) = 0$ for almost every $r \in [0,\infty)$. Fix such an r and choose $h \in \mathbb{R}_+$ for which $\mathcal{H}^n(\partial B_{r+h}(x)\cap\Sigma) = 0$. Define a cut-off function $f_h : \mathbb{R}^{n+1} \to \mathbb{R}$ by setting

$$f_h(y) = \begin{cases} 1, & y \in B_r(x), \\ 1 - |y-x|/h, & y \in B_{r+h}(x)\setminus B_r(x), \\ 0, & y \in \mathbb{R}^{n+1}\setminus B_{r+h}(x). \end{cases}$$

By using a suitable approximation argument combined with Theorem 2.2 we obtain

$$V(r)^{(n-1)/n} \le C_n\left(\frac{V(r+h)-V(r)}{h} + \|f_h H_\Sigma\|_{L^1(\Sigma)}\right).$$

In turn, we may choose a sequence $(h_k)_k$ such that $h_k \to 0$ and $\mathcal{H}^n(\partial B_{r+h_k}(x)\cap\Sigma) = 0$. Then by letting $k \to \infty$ the previous estimate yields

$$\begin{aligned} V(r)^{(n-1)/n} &\le C_n\left(V'(r) + \int_{\bar{B}_r(x)\cap\Sigma} |H_\Sigma|\,\mathrm{d}\mathcal{H}^n\right) \\ &\le C_n\left(V'(r) + \int_{B_r(x)\cap\Sigma} |H_\Sigma|\,\mathrm{d}\mathcal{H}^n\right) \\ &\le C_n\left(V'(r) + \int_{B_r(x)\cap\Sigma} |H_\Sigma - \lambda|\,\mathrm{d}\mathcal{H}^n + \lambda V(r)\right) \\ &\le C_n(V'(r) + \|H_\Sigma - \lambda\|_{L^n(\Sigma)} V(r)^{(n-1)/n} + \lambda V(r)). \end{aligned}$$

Thus for almost every $r \in (0, \infty)$,

$$\left(\frac{C_n^{-1} - \|H_\Sigma - \lambda\|_{L^n(\Sigma)}}{V(r)^{1/n}} - \lambda\right) V(r) \le V'(r).$$

If $\|H_\Sigma - \lambda\|_{L^n(\Sigma)} \le \delta_n$ for small δ_n, then the above inequality implies

$$\frac{1}{2C_n} V(r)^{1-1/n} - \lambda V(r) \le V'(r).$$

Fix $r < \delta_n/\lambda$. We will assume that $V(r) \le \delta_n r^n$, since otherwise the claim is trivially true. By the monotonicity of V we have

$$V(\rho)^{1/n} \le V(r)^{1/n} \le \delta_n/\lambda$$

for all $0 < \rho < r$. For δ_n small enough the above inequality then yields

$$\frac{1}{4C_n} V(\rho)^{1-1/n} \le V'(\rho)$$

for almost every $0 < \rho < r$. The claim follows by integrating this over $(0, r)$. □

Montiel–Ros argument. We recall that for $E \subset \mathbb{R}^{n+1}$ we write

$$E_r := \{x \in E : \operatorname{dist}(x, \partial E) > r\}. \tag{3-1}$$

We use the fact that E is C^2 regular and say that $x \in \partial E$ satisfies the interior ball condition with radius r if, for $y = x - r\nu_E(x)$, it holds that $B_r(y) \subset E$. For $r > 0$ we define

$$\Gamma_r := \{x \in \partial E : x \text{ satisfies the interior ball condition with radius } r\}. \tag{3-2}$$

Proposition 3.3. *Let $\lambda \in \mathbb{R}$ and suppose that a bounded and C^2 regular set $E \subset \mathbb{R}^{n+1}$ satisfies $P(E) \le C_0$ and $|E| \ge 1/C_0$ with $C_0 \in \mathbb{R}_+$. Then for $0 < r < R$ with $R = n/\lambda$,*

$$\left| |E_r| - \frac{|E|}{R^{n+1}} (R - r)^{n+1} \right| \le C \|H_E - \lambda\|_{L^n(\partial E)}$$

and

$$\mathcal{H}^n(\partial E \setminus \Gamma_r) \le \frac{C}{(R-r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)},$$

provided that $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$, where the constants C and δ depend only on C_0 and on the dimension. Moreover, under the same assumptions, for $0 < \rho < r < R$,

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}} (R - (r - \rho))^{n+1} \right| \le \frac{C}{(R-r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)}.$$

Proof. As we already mentioned the proof is based on the Montiel–Ros argument for the Heinze–Karcher inequality, which we recall shortly. To that aim, we define $\zeta : \partial E \times \mathbb{R} \to \mathbb{R}^{n+1}$ as

$$\zeta(x, t) = x - t\nu_E(x).$$

We denote the principle curvatures of ∂E at x by $k_1(x), \dots, k_n(x)$ and assume that they are pointwise ordered as $k_i(x) \le k_{i+1}(x)$. If we consider $\partial E \times \mathbb{R}$ as a hypersurface embedded in $\mathbb{R}^{n+2}$, then its tangential Jacobian is

$$J_\tau \zeta(x,t) = \prod_{i=1}^{n} |1 - t k_i(x)| \quad \text{on } \partial E \times \mathbb{R}.$$

For every bounded Borel set $M \subset \partial E \times \mathbb{R}$ we have, by the area formula,

$$\int_{\zeta(M)} \mathcal{H}^0(\zeta^{-1}(y) \cap M)\, \mathrm{d}y = \int_M J_\tau \zeta \, \mathrm{d}\mathcal{H}^{n+1}.$$

In the proof, C denotes a positive constant which may change from line to line, depending only on C_0 and on the dimension.

Step 1: In order to utilize Lemma 2.4, we choose $\delta = \delta(C_0, n)$ to be the same as in the lemma and assume $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$. Then E has N connected components with $N \le C$. We may thus prove the claim componentwise and assume that E is connected. We write

$$\Sigma := \left\{ x \in \partial E : |H_E(x) - \lambda| < \tfrac{1}{2}\lambda \right\}.$$

By Lemma 2.4 we have $\lambda \ge 1/C$, and thus by Hölder's inequality

$$\mathcal{H}^n(\partial E \setminus \Sigma) \le \frac{2}{\lambda} \int_{\partial E} |H_E(x) - \lambda| \, \mathrm{d}\mathcal{H}^n \le C \|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-3}$$

Moreover, we have

$$\begin{aligned} \frac{n}{n+1} \int_\Sigma \frac{1}{H_E} \, \mathrm{d}\mathcal{H}^n &= \frac{n}{n+1} \int_\Sigma \left(\frac{1}{\lambda} + \left(\frac{1}{H_E} - \frac{1}{\lambda} \right) \right) \mathrm{d}\mathcal{H}^n \\ &\le \frac{nP(E)}{(n+1)\lambda} + C \|H_E(x) - \lambda\|_{L^n(\partial E)}. \end{aligned}$$

Since E is connected, Lemma 2.4 yields $\operatorname{diam}(E) \le \tilde{R}$ with $\tilde{R} = \tilde{R}(C_0, n) \ge R$. Choose $x_0 \in E$. Then using (2-3) with $F(x) = x - x_0$ we obtain

$$nP(E) = (n+1)\lambda |E| + \int_{\partial E} (H_E - \lambda) \langle (x - x_0), \nu_E \rangle \, \mathrm{d}\mathcal{H}^n,$$

which in turn implies

$$\left| nP(E) - (n+1)\lambda |E| \right| \le C \|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-4}$$

Hence we deduce

$$\frac{n}{n+1} \int_\Sigma \frac{1}{H_E} \, \mathrm{d}\mathcal{H}^n \le |E| + C \|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-5}$$

Next we define

$$Z = \{ (x,t) \in \Sigma \times [0, \infty) : 0 \le t \le 1/k_n(x) \}.$$

Note that this is well defined, since $x \in \Sigma$ implies $k_n(x) \geq H_E(x)/n \geq \lambda/(2n) > 0$. We also set

$$\Sigma_1' = \{x \in \partial E \setminus \Sigma : k_n(x) \leq 1/\tilde{R}\} \quad \text{and} \quad \Sigma_2' = \{x \in \partial E \setminus \Sigma : k_n(x) > 1/\tilde{R}\},$$
$$Z_1' = \Sigma_1' \times [0, \tilde{R}] \quad \text{and} \quad Z_2' = \{(x,t) \in \Sigma_2' \times [0,\infty) : 0 \leq t \leq 1/k_n(x)\},$$

and finally

$$Z' = Z_1' \cup Z_2'.$$

Then Z and Z' are disjoint and bounded Borel sets and $E \subset \zeta(Z \cup Z')$. To see this fix $y \in E$ and let $x \in \partial E$ be such that $r = d_{\partial E}(y) = |x - y|$. Then we may write $y = x - r\nu_E(x)$, and by the maximum principle $k_n(x) \leq 1/r$. Since $\operatorname{diam}(E) \leq \tilde{R}$, we have $r \leq \tilde{R}$ and so we conclude that $(x, r) \in Z \cup Z'$ and $y = \zeta(x, r)$.

We now recall the Montiel–Ros argument. We use the fact that E is a subset of $\zeta(Z \cup Z')$, the area formula, the arithmetic geometric inequality and the fact that $1/k_n(x) \leq n/H_E(x)$ for $x \in \Sigma$ to obtain

$$\begin{aligned}
|E| \leq |\zeta(Z)| + |\zeta(Z')| &\leq \int_{\zeta(Z)} \mathcal{H}^0(\zeta^{-1}(y) \cap Z)\,\mathrm{d}y + |\zeta(Z')| \\
&= \int_Z J_\tau \zeta \,\mathrm{d}\mathcal{H}^{n+1} + |\zeta(Z')| \\
&= \int_\Sigma \int_0^{1/k_n(x)} \prod_{i=1}^n (1 - t k_i(x))\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| \\
&\leq \int_\Sigma \int_0^{1/k_n(x)} \left(1 - \frac{t}{n} H_E(x)\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| \\
&\leq \int_\Sigma \int_0^{n/H_E(x)} \left(1 - \frac{t}{n} H_E(x)\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| \\
&= \frac{n}{n+1} \int_\Sigma \frac{1}{H_E}\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')|.
\end{aligned}$$

Next we quantify the previous four inequalities. To that aim we define the nonnegative numbers R_1, R_2, R_3 and R_4 as

$$R_1 = |\zeta(Z) \setminus E|, \tag{3-6}$$

$$R_2 = \int_{\zeta(Z)} |\mathcal{H}^0(\zeta^{-1}(y) \cap Z) - 1|\,\mathrm{d}y, \tag{3-7}$$

$$R_3 = \int_\Sigma \int_0^{1/k_n(x)} \left| \left(1 - \frac{t}{n} H_E(x)\right)^n - \prod_{i=1}^n (1 - t k_i(x)) \right| \mathrm{d}t\,\mathrm{d}\mathcal{H}^n, \tag{3-8}$$

$$R_4 = \int_\Sigma \int_{1/k_n(x)}^{n/H_E(x)} \left| 1 - \frac{t}{n} H_E(x) \right|^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n. \tag{3-9}$$

Then by repeating the Montiel–Ros argument we deduce that

$$|E| \leq \frac{n}{n+1} \int_\Sigma \frac{1}{H_E}\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| - R_1 - R_2 - R_3 - R_4.$$

Therefore, by (3-5),

$$R_1 + R_2 + R_3 + R_4 \le |\zeta(Z')| + C\|H_E - \lambda\|_{L^n(\partial E)},$$

where the R_i are defined in (3-6)–(3-9).

Let us next show that

$$|\zeta(Z')| \le C\|H_E(x) - \lambda\|_{L^n(\partial E)}. \tag{3-10}$$

Indeed, by the area formula we have

$$|\zeta(Z')| \le \int_{Z'} J_\tau \zeta \, \mathrm{d}\mathcal{H}^{n+1} = \int_{\Sigma_1'} \int_0^{\tilde{R}} \prod_{i=1}^n |1 - tk_i(x)| \, \mathrm{d}t \, \mathrm{d}\mathcal{H}^n + \int_{\Sigma_2'} \int_0^{1/k_n(x)} \prod_{i=1}^n |1 - tk_i(x)| \, \mathrm{d}t \, \mathrm{d}\mathcal{H}^n. \tag{3-11}$$

By the definition of Σ_1', we have $|1 - tk_i(x)| = (1 - tk_i(x))$ for every $(x, t) \in \Sigma_1' \times [0, \tilde{R}]$, and therefore by the arithmetic-geometric inequality we may estimate

$$\prod_{i=1}^n |1 - tk_i(x)| \le C(1 + |H_E(x)|^n) \quad \text{for } (x, t) \in \Sigma_1' \times [0, \tilde{R}].$$

Similarly, we deduce that

$$\prod_{i=1}^n |1 - tk_i(x)| \le C(1 + t^n |H_E(x)|^n) \qquad \text{for } x \in \Sigma_2' \quad \text{and} \quad 0 \le t \le 1/k_n(x).$$

On the other hand, by the definition of Σ_2' we have $1/k_n(x) < \tilde{R}$. Therefore, by (3-11), $\lambda \le C$ and (3-3) we have

$$\begin{aligned}
|\zeta(Z')| &\le C \int_{\Sigma_1' \cup \Sigma_2'} \int_0^{\tilde{R}} (1 + |H_E(x)|^n) \, \mathrm{d}t \, \mathrm{d}\mathcal{H}^n \\
&= C\tilde{R} \int_{\partial E \setminus \Sigma} (1 + |H_E(x)|^n) \, \mathrm{d}\mathcal{H}^n \\
&\le C \int_{\partial E \setminus \Sigma} (1 + \lambda^n + |H_E - \lambda|^n) \, \mathrm{d}\mathcal{H}^n \\
&\le C(\mathcal{H}^n(\partial E \setminus \Sigma) + \|H_E - \lambda\|^n_{L^n(\partial E)}) \\
&\le C\|H_E - \lambda\|_{L^n(\partial E)}
\end{aligned}$$

when $\|H_E - \lambda\|_{L^n(\partial E)} \le 1$. Hence by decreasing δ, if needed, we have (3-11). In particular,

$$R_1 + R_2 + R_3 + R_4 \le C\|H_E - \lambda\|_{L^n(\partial E)}, \tag{3-12}$$

where the R_i are defined in (3-6)–(3-9).

Step 2: Here we utilize the estimate (3-12) and prove the following auxiliary result. For a Borel set $\Gamma \subset \partial E$ and $0 < r < R$,

$$|E \cap \zeta(Z \cap (\Gamma \times (r, R)))| \ge \frac{\mathcal{H}^n(\Gamma)}{(n+1)R^n}(R - r)^{n+1} - C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-13}$$

We prove (3-13) by "backtracking" the Montiel–Ros argument. By the definition of R_1, R_2, R_3, R_4 and (3-12) we may estimate

$$\begin{aligned}
|E\cap\zeta(Z\cap(\Gamma\times(r,R)))| &\geq |\zeta(Z\cap(\Gamma\times(r,R)))| - R_1\\
&\geq \int_{\zeta(Z\cap(\Gamma\times(r,R)))} \mathcal{H}^0(\zeta^{-1}(y)\cap Z\cap(\Gamma\times(r,R)))\,\mathrm{d}y - R_1 - R_2\\
&= \int_{\Gamma\cap\Sigma}\int_{\min\{r,1/k_n(x)\}}^{\min\{R,1/k_n(x)\}} \prod_{i=1}^{n}(1-tk_i(x))\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^n - R_1 - R_2\\
&\geq \int_{\Gamma\cap\Sigma}\int_{\min\{r,1/k_n(x)\}}^{\min\{R,1/k_n(x)\}} \left(1-\frac{t}{n}H_E(x)\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n - R_1 - R_2 - R_3\\
&\geq \int_{\Gamma\cap\Sigma}\int_{\min\{r,1/k_n(x)\}}^{\min\{R,n/H_E(x)\}} \left(1-\frac{t}{n}H_E\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n - R_1 - R_2 - R_3 - R_4\\
&\geq \int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}} \left(1-\frac{t}{n}H_E\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n - R_1 - R_2 - R_3 - R_4.
\end{aligned}$$

Recall that for $x\in\Sigma$, we have $\frac{1}{2}\lambda\leq H_E(x)\leq 2\lambda$ and $R=n/\lambda$. Therefore, we may estimate

$$\begin{aligned}
\int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}} \left(1-\frac{t}{n}H_E\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n &\geq \int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}} \left(1-\frac{t}{n}\lambda\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n - C\|H_E-\lambda\|_{L^n(\partial E)}\\
&\geq \int_{\Gamma\cap\Sigma}\int_r^R \left(1-\frac{t}{n}\lambda\right)^n \mathrm{d}t\,\mathrm{d}\mathcal{H}^n - C\|H_E-\lambda\|_{L^n(\partial E)}\\
&= \frac{\mathcal{H}^n(\Gamma\cap\Sigma)n}{(n+1)\lambda}\left(1-\frac{\lambda}{n}r\right)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}\\
&= \frac{\mathcal{H}^n(\Gamma\cap\Sigma)R}{(n+1)}\left(1-\frac{r}{R}\right)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}.
\end{aligned}$$

Hence we obtain (3-13) from the previous two inequalities, from (3-3) and from (3-12).

Step 3: Here we finally prove the proposition. Recall the definition of E_r in (3-1). Let us first prove that

$$|E_r| \geq \frac{P(E)}{(n+1)R^n}(R-r)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)} \tag{3-14}$$

for all $0<r<R$.

To this aim, we claim that

$$E\cap\zeta(Z\cap(\Sigma\times(r,R)))\subset E_r\cup\{y\in\zeta(Z):\mathcal{H}^0(\zeta^{-1}(y)\cap Z)\geq 2\}\cup\zeta(Z'). \tag{3-15}$$

The point of this inclusion is that almost every point which is of the form $y=x-t\nu_E(x)$, for $x\in Z$ and $t\in(r,R)$, belongs to E_r.

To this aim, let $y\in E\cap\zeta(\Sigma\times(r,R))$. Then we may write $y=x-t\nu_E(x)=\zeta(x,t)$ for some $x\in\Sigma$ and $t\in(r,R)$, with $(x,t)\in Z$. If $d_{\partial E}(y)=|y-x|$, then $y\in E_r$ because $|x-y|=t>r$. Otherwise,

$$d_{\partial E}(y)=|y-\tilde{x}|=\tilde{r}<t \quad \text{for } \tilde{x}\in\partial E,$$

so we may write $y = \tilde{x} - \tilde{r}\nu_E(x) = \zeta(\tilde{x}, \tilde{r})$ and $(\tilde{x}, \tilde{r}) \in Z \cup Z'$. Again, if $(\tilde{x}, \tilde{r}) \notin Z'$, then $(\tilde{x}, \tilde{r}) \in Z$ and thus $\mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2$. Hence we have (3-15).

Recall that by the definition of R_2 and by (3-12),

$$\begin{aligned} |\{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2\}| &\leq \int_{\zeta(Z)} |\mathcal{H}^0(\zeta^{-1}(y) \cap Z) - 1| \, \mathrm{d}y \\ &\leq C\|H_E - \lambda\|_{L^n(\partial E)}. \end{aligned} \tag{3-16}$$

We then use (3-15), (3-16), (3-10) and (3-13) with $\Gamma = \Sigma$ to deduce

$$\begin{aligned} |E_r| &\geq |E \cap \zeta(Z \cap (\Sigma \times (r, R)))| - C\|H_E - \lambda\|_{L^n(\partial E)} \\ &\geq \frac{\mathcal{H}^n(\Sigma)}{(n+1)R^n}(R - r)^{n+1} - C\|H_E - \lambda\|_{L^n(\partial E)}. \end{aligned}$$

The inequality (3-14) then follows from (3-3).

Let us next show that for all $r \in (0, R)$,

$$|E_r| \leq \frac{\mathcal{H}^n(\Gamma_r)}{(n+1)R^n}(R - r)^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}, \tag{3-17}$$

where $\Gamma_r \subset \partial E$ is defined in (3-2).

First we show

$$|E_R| \leq C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3-18}$$

This follows from an already familiar argument, so we only sketch it here. It is easy to see that

$$E_R \subset \zeta(Z') \cup \zeta(Z \cap (\Sigma \times (R, \infty))).$$

Moreover, since $\frac{1}{2}\lambda \leq H_E(x) \leq 2\lambda$ for $x \in \Sigma$,

$$J_\tau\zeta(x, t) = \prod_{i=1}^{n} |1 - tk_i(x)| \leq C(1 + |H_E(x)|^n) \leq C \quad \text{for } (x, t) \in Z \cap (\Sigma \times (R, \infty)).$$

Recall that $R = n/\lambda$. Therefore, we have

$$\begin{aligned} |\zeta(Z \cap (\Sigma \times (R, \infty)))| &\leq \int_\Sigma \int_R^{\max\{n/H_E(x), R\}} J_\tau\zeta(x, t) \, \mathrm{d}t \, \mathrm{d}\mathcal{H}^n \\ &\leq C \int_\Sigma \left| \frac{n}{H_E} - R \right| \mathrm{d}t \, \mathrm{d}\mathcal{H}^n \\ &\leq C\|H_E - \lambda\|_{L^n(\partial E)}. \end{aligned}$$

The estimate (3-18) then follows from $|E_R| \leq |\zeta(Z \cap (\Sigma \times (R, \infty)))| + |\zeta(Z')|$ and (3-10).

Note that for all $\rho \in (r, R)$ we have $\{x \in E : d_{\partial E}(x) = \rho\} = \zeta(\Gamma_\rho, \rho)$ and $\Gamma_\rho \subset \Gamma_r$. We also set $\zeta_\rho = \zeta(\cdot, \rho) : \partial E \to \mathbb{R}^{n+1}$, and thus $\{x \in E : d_{\partial E}(x) = \rho\} = \zeta_\rho(\Gamma_\rho)$ and

$$J_\tau\zeta_\rho(x) = \prod_{i=1}^{n} |1 - \rho k_i(x)| \leq \left(1 - \frac{H_E}{n}\rho\right)^n \quad \text{for } x \in \Gamma_\rho.$$

Therefore by (3-18) and by coarea and area formulas we obtain

$$\begin{aligned}
|E_r| \le |E_r| - |E_R| + C\|H_E - \lambda\|_{L^n(\partial E)} &\le \int_r^R \mathcal{H}^n(\{x \in E : d_{\partial E} = \rho\})\,\mathrm{d}\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\
&= \int_r^R \mathcal{H}^n(\zeta_\rho(\Gamma_\rho))\,\mathrm{d}\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\
&\le \int_r^R \int_{\Gamma_\rho} J_\tau \zeta_\rho(x)\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\
&\le \int_r^R \int_{\Gamma_\rho} \left(1 - \frac{H_E}{n}\rho\right)^n \mathrm{d}\mathcal{H}^n\,\mathrm{d}\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\
&\le \int_r^R \mathcal{H}^n(\Gamma_\rho)\left(1 - \frac{\lambda}{n}\rho\right)^n \mathrm{d}\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\
&\le \mathcal{H}^n(\Gamma_r) \int_r^R \left(1 - \frac{\rho}{R}\right)^n \mathrm{d}\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\
&= \frac{\mathcal{H}^n(\Gamma_r)}{(n+1)R^n}(R-r)^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}.
\end{aligned}$$

Hence we have (3-17).

The second claim of the proposition follows immediately from (3-14) and (3-17). These also imply

$$\left| |E_r| - \frac{P(E)}{(n+1)R^n}(R-r)^{n+1} \right| \le C\|H_E - \lambda\|_{L^n(\partial E)}.$$

The first claim thus follows from (3-4) and $R = n/\lambda$.

For the last claim we refine the inclusion (3-15) and show that for $0 < \rho < r < R$ and $r' \in (r, R)$,

$$E \cap \zeta(Z \cap (\Gamma_{r'} \times (r' - \rho, R))) \subset (E_r + B_\rho) \cup \{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \ge 2\} \cup \zeta(Z'). \tag{3-19}$$

Indeed, let $y \in E \cap \zeta(Z \cap (\Gamma_{r'} \times (r' - \rho, R)))$. Then we may write $y = x - t\nu_E(x)$ for some $x \in \Sigma \cap \Gamma_{r'}$ and $t \in (r' - \rho, R)$, with $(x, t) \in Z$. If $t \in (r', R)$, then by (3-15),

$$\begin{aligned}
y \in E \cap \zeta(Z \cap (\Sigma \times (r, R))) &\subset E_r \cup \{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \ge 2\} \cup \zeta(Z') \\
&\subset (E_r + B_\rho) \cup \{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \ge 2\} \cup \zeta(Z').
\end{aligned}$$

Let us then assume that $t \in (r' - \rho, r']$. We write $y = x - r'\nu_E(x) + (r' - t)\nu_E(x)$. Since $x \in \Gamma_{r'}$, i.e., ∂E satisfies the interior ball condition at x with radius $r' > r$, necessarily we have $x - r'\nu_E(x) \in E_r$. Therefore, since $0 \le r' - t < \rho$, we conclude that $y \in E_r + B_\rho$ and (3-19) follows.

We use (3-10), (3-13), (3-16) and (3-19) to conclude

$$\begin{aligned}
|E_r + B_\rho| &\ge |E \cap \zeta(Z \cap (\Gamma_{r'} \cap \times (r' - \rho, R)))| - C\|H_E - \lambda\|_{L^n(\partial E)} \\
&\ge \frac{\mathcal{H}^n(\Gamma_{r'})}{(n+1)R^n}(R - (r' - \rho))^{n+1} - C\|H_E - \lambda\|_{L^n(\partial E)}.
\end{aligned}$$

By using the second claim of the proposition and then letting $r' \to r$, we deduce

$$|E_r + B_\rho| \geq \frac{P(E)}{(n+1)R^n}(R-(r-\rho))^{n+1} - \frac{C}{(R-r)^{n+1}}\|H_E - \lambda\|_{L^n(\partial E)}.$$

On the other hand, clearly $E_r + B_\rho \subset E_{r-\rho}$. Then by (3-17) we have

$$|E_r + B_\rho| \leq |E_{r-\rho}| \leq \frac{P(E)}{(n+1)R^n}(R-(r-\rho))^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}.$$

The last claim thus follows from the two previous inequalities and (3-4). □

Proof of Theorem 1.2. Let E, λ and C_0 be as in the formulation of Theorem 1.2. Recall that we write $R = n/\lambda$. As before C denotes a constant which may change from line to line but always depends only on C_0 and n. Let us write

$$\varepsilon := \|H_E - \lambda\|_{L^n(\partial E)}.$$

If $\varepsilon = 0$, then E is a disjoint union of balls by [Delgadino and Maggi 2019]. Let us then assume that $0 < \varepsilon \leq \delta$, where δ is initially set as in Proposition 3.3. We might shrink δ several times but always in such a way that it depends only on C_0 and the dimension n. Indeed, by shrinking δ, if needed, Lemma 2.4 provides the estimates

$$1/C \leq \lambda \quad \text{and} \quad R \leq C,$$

and hence the first claim of Theorem 1.2 is clear. We will use these estimates repeatedly without further mention.

By Lemma 2.4, the number of connected components of E and their diameters are bounded by C. Thus, by applying a similar argument as in the proof of Proposition 3.3 (to obtain (3-4)) on each component and then summing these estimates we obtain

$$\big|nP(E) - (n+1)\lambda|E|\big| \leq C\varepsilon. \tag{3-20}$$

By possibly shrinking δ we have $R - \delta^{1/(n+2)} \geq \frac{1}{2}R$. Choose $r_0 = R - \varepsilon^{1/(n+2)}$. Then the volume estimates given by Proposition 3.3 read as

$$\left||E_r| - \frac{|E|}{R^{n+1}}(R-r)^{n+1}\right| \leq C\varepsilon \tag{3-21}$$

for all $0 \leq r < R$ and

$$\left||E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1}\right| \leq C\varepsilon^{1/(n+2)} \tag{3-22}$$

for all $0 \leq \rho \leq r \leq r_0$. We remark that by (3-21) we have

$$|E_{r_0}| \geq \frac{|E|}{R^{n+1}}\varepsilon^{(n+1)/(n+2)} - C\varepsilon \geq \frac{1}{C}\varepsilon^{(n+1)/(n+2)} - C\varepsilon.$$

Hence by decreasing δ, if needed, we may assume that E_{r_0} is nonempty. This implies that $E_{r'}$ is nonempty for $r' > r_0$ when $|r' - r_0|$ is small enough. Since for any $r' > r_0$ it is geometrically clear that $\Gamma_{r'} \subset \partial E_{r_0} + \overline{B}_{r_0}$,

and then by using Proposition 3.3 and $r_0 = R - \varepsilon^{1/(n+2)}$ we have

$$\mathcal{H}^n(\partial E \setminus (\bar{E}_{r_0} + \bar{B}_{r_0})) \le \mathcal{H}^n(\partial E \setminus \Gamma_{r'}) \le C \frac{\varepsilon}{(r_0 - r' + \varepsilon^{1/(n+2)})^{n+1}}.$$

Thus by letting $r' \to r_0$ the previous estimate yields

$$\mathcal{H}^n(\partial E \setminus (\bar{E}_{r_0} + \bar{B}_{r_0})) \le C\varepsilon^{1/(n+2)}. \tag{3-23}$$

As previously, we divide the proof into three steps.

Step 1: Recall that $r_0 = R - \varepsilon^{1/(n+2)} \ge \frac{1}{2}R$. We prove that there is a positive constant $d_0 = d_0(C_0, n) \le \frac{1}{4}R$ such that if $x, y \in E_{r_0}$, then either

$$|x - y| < \varepsilon^{1/(2(n+2))} \quad \text{or} \quad |x - y| \ge d_0. \tag{3-24}$$

Let us fix $x, y \in E_{r_0}$. We write $d := |x - y|$ and denote the segment from x to y by

$$J_{xy} := \{tx + (1-t)y : t \in [0, 1]\}.$$

We may assume that d is small, since otherwise the claim (3-24) is trivially true. To be more precise, we assume

$$d \le \min\left\{\tfrac{1}{4}R, 1\right\}. \tag{3-25}$$

Let us first show that

$$J_{xy} \subset E_{r_0 - R^{-1}d^2}. \tag{3-26}$$

Note that $r_0 - R^{-1}d^2 > 0$ by $r_0 \ge \frac{1}{2}R$ and (3-25), and hence $E_{r_0 - R^{-1}d^2}$ is well defined and nonempty. Choose $z \in \mathbb{R}^{n+1} \setminus E$ and $z' \in J_{xy}$ such that

$$|z - z'| = \operatorname{dist}(\mathbb{R}^{n+1} \setminus E, J_{xy}).$$

If $z' = x$ or $z' = y$, then it follows from $x, y \in E_{r_0}$ that $|z - z'| > r_0$. If not, then from the fact that z' is the closest point on J_{xy} to z, we deduce that the vector $x - z'$ is orthogonal to $z - z'$, i.e., $\langle x - z', z - z' \rangle = 0$. Note also that $\min\{|x - z'|, |y - z'|\} \le \frac{1}{2}d$ and we may thus assume that $|x - z'| \le \frac{1}{2}d$. Therefore, by the Pythagorean theorem we have

$$|x - z|^2 = |x - z'|^2 + |z - z'|^2 \le \tfrac{1}{4}d^2 + |z - z'|^2.$$

Since $|x - z| > r_0$, the previous estimate gives us

$$|z - z'|^2 > r_0^2 - \tfrac{1}{4}d^2.$$

We deduce from $r_0 \ge \frac{1}{2}R$ and (3-25) that

$$\left(r_0^2 - \tfrac{1}{4}d^2\right)^{1/2} \ge r_0 - R^{-1}d^2.$$

The previous two estimates yield $|z - z'| > r_0 - R^{-1}d^2$, and claim (3-26) follows due to the choice of z and z'.

Again, we use $r_0 \geq \frac{1}{2}R$ and (3-25) to observe that

$$r_0 - (1+R^{-1})d^2 \geq r_0 - d - R^{-1}d^2 \geq \tfrac{1}{2}R - \tfrac{1}{4}R - \tfrac{1}{16}R > 0.$$

Thus $E_{r_0-(1+R^{-1})d^2}$ is well defined and nonempty. Next, we deduce from (3-26) and $E_r + B_\rho \subset E_{r-\rho}$ that

$$J_{xy} + B_{d^2} \subset E_{r_0-R^{-1}d^2} + B_{d^2} \subset E_{r_0-(1+R^{-1})d^2}. \tag{3-27}$$

Since $J_{xy} + B_{d^2}$ contains the cylinder $J_{xy} \times B^n_{d^2}$, it is clear that

$$|J_{xy} + B_{d^2}| \geq \omega_n d^{1+2n}.$$

On the other hand, (3-21) and $\varepsilon \leq 1$ (we may assume $\delta \leq 1$) imply

$$\begin{aligned}
|E_{r_0-(1+R^{-1})d^2}| &\leq \frac{|E|}{R^{n+1}}(R - (r_0 - (1+R^{-1})d^2))^{n+1} + C\varepsilon \\
&= \frac{|E|}{R^{n+1}}(\varepsilon^{1/(n+2)} + (1+R^{-1})d^2)^{n+1} + C\varepsilon \\
&\leq \frac{|E|}{R^{n+1}}(\varepsilon^{1/(n+2)} + (1+R^{-1})d^2)^{n+1} + C\varepsilon^{(n+1)/(n+2)} \\
&\leq Cd^{2(n+1)} + C\varepsilon^{(n+1)/(n+2)}.
\end{aligned}$$

Then (3-27) yields

$$\omega_n d^{1+2n} \leq Cd^{2(n+1)} + C\varepsilon^{(n+1)/(n+2)}.$$

If $d \geq \varepsilon^{1/(2(n+2))}$, then

$$\omega_n d^{1+2n} \leq Cd^{2(n+1)}.$$

This implies $d \geq c > 0$ for some $c = c(C_0, n)$. By recalling (3-25), claim (3-24) follows.

Step 2: By (3-24) and possibly replacing δ with $\min\{\delta, (\frac{1}{8}d_0)^{2(n+2)}\}$ we may divide the set E_{r_0} into N clusters $E^1_{r_0}, \dots, E^N_{r_0}$ such that we fix a point $x_i \in E_{r_0}$ and define the corresponding cluster $E^i_{r_0}$ as

$$E^i_{r_0} = \{x \in E_{r_0} : |x - x_i| \leq \tfrac{1}{8}d_0\}.$$

By (3-24), we have $E^i_{r_0} \subset B_{\varepsilon_0}(x_i)$, where $\varepsilon_0 = \varepsilon^{1/(2(n+2))}$ and $|x_i - x_j| \geq d_0$ for $i \neq j$. Therefore, we have for every $\rho > 0$

$$\bigcup_{i=1}^N B_\rho(x_i) \subset E_{r_0} + B_\rho \subset \bigcup_{i=1}^N B_{\rho+\varepsilon_0}(x_i). \tag{3-28}$$

Since $r_0 \geq \frac{1}{2}R > \frac{1}{4}R \geq d_0$ and $|x_i - x_j| \geq d_0$ for $i \neq j$, we have that the balls $B_\rho(x_1), \dots, B_\rho(x_N)$ with $\rho = \frac{1}{4}d_0$ are disjoint and contained in E, which in turn implies there is an upper bound $N_0 = N_0(C_0, n) \in \mathbb{N}$ for the number of clusters N.

Next we improve the lower bound $|x_i - x_j| \geq d_0$ and prove that there is a positive constant $C_1 = C_1(C_0, n)$ such that

$$|x_i - x_j| \geq 2R - 2C_1\varepsilon^{1/(n+2)^2} \quad \text{for all pairs } i \neq j. \tag{3-29}$$

As a byproduct we prove the last statement of the theorem, i.e., we show

$$|P(E) - N(n+1)\omega_{n+1}R^n| \le C\varepsilon^{1/(2(n+2))}. \tag{3-30}$$

Recall that the balls $B_{d_0/4}(x_1), \dots, B_{d_0/4}(x_N)$ are disjoint. Therefore, using $N \le N_0$ and (3-28) with $\rho = \frac{1}{4}d_0$ we deduce

$$\big||E_{r_0} + B_{d_0/4}| - N\omega_{n+1}\big(\tfrac{1}{4}d_0\big)^{n+1}\big| \le C\varepsilon_0 = C\varepsilon^{1/(2(n+2))}.$$

On the other hand, we have $\frac{1}{4}d_0 \le \frac{1}{16}R < \frac{1}{2}R \le r_0$, so we may use (3-22) to obtain

$$\left||E_{r_0} + B_{d_0/4}| - \frac{|E|}{R^{n+1}}\big(\tfrac{1}{4}d_0 + \varepsilon^{1/(n+2)}\big)^{n+1}\right| \le C\varepsilon^{1/(n+2)}.$$

These two estimates and $\varepsilon \le 1$ imply

$$\big||E| - N\omega_{n+1}R^{n+1}\big| \le C\varepsilon^{1/(2(n+2))}. \tag{3-31}$$

Thus (3-20), $R = n/\lambda$ and (3-31) yield (3-30).

To obtain (3-29), let us assume that there is $0 < h < \frac{1}{2}R$ such that $|x_i - x_j| < 2R - 2h$ for some $i \ne j$. This implies that the balls $B_R(x_i)$ and $B_R(x_j)$ intersect each other such that a set enclosed by a spherical cap of height h is included in their intersection. As the volume enclosed by the spherical cap of height h has a lower bound $c_n R^{n+1} h^{(n+2)/2}$, with some dimensional constant c_n, then there is $c = c(C_0, n)$ such that

$$|B_R(x_i) \cap B_R(x_j)| \ge ch^{(n+2)/2}.$$

We use the previous estimate as well as (3-22), (3-28), (3-31), $\varepsilon \le 1$ and $N \le N_0$ to estimate

$$\begin{aligned}
N\omega_{n+1}R^{n+1} &\le |E| + C\varepsilon_0 \\
&\le |E_{r_0} + B_{r_0}| + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le \left|\bigcup_{i=1}^N B_{R+\varepsilon_0}(x_i)\right| + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le \left|\bigcup_{i=1}^N B_R(x_i)\right| + N\omega_{n+1}((R+\varepsilon_0)^{n+1} - R^{n+1}) + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le N\omega_{n+1}R^{n+1} - |B_R(x_i) \cap B_R(x_j)| + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&\le N\omega_{n+1}R^{n+1} - ch^{(n+2)/2} + C\varepsilon_0 + C\varepsilon^{1/(n+2)} \\
&= N\omega_{n+1}R^{n+1} - ch^{(n+2)/2} + C\varepsilon^{1/(2(n+2))} + C\varepsilon^{1/(n+2)} \\
&\le N\omega_{n+1}R^{n+1} - ch^{(n+2)/2} + C\varepsilon^{1/(2(n+2))}.
\end{aligned}$$

Thus $h^{(n+2)/2} \le C\varepsilon^{1/(2(n+2))}$ and (3-29) follows.

<u>Step 3</u>: Let C_1 be as in (3-29). By decreasing δ, if needed, we may assume

$$0 < R - C_1\varepsilon^{1/(n+2)^2} < R - \varepsilon^{1/(n+2)} = r_0.$$

Then by (3-28) and (3-29) we have that the balls $B_\rho(x_1), \dots, B_\rho(x_N)$, with $\rho = R - C_1\varepsilon^{1/(n+2)^2}$, are disjoint and

$$\bigcup_{i=1}^{N} B_\rho(x_i) \subset E_{r_0} + B_\rho \subset E_{r_0-\rho} \subset E. \tag{3-32}$$

This, $\varepsilon \le 1$, $N \le N_0$ and (3-31) imply

$$\left| E \setminus \bigcup_{i=1}^{N} B_\rho(x_i) \right| \le C\varepsilon^{1/(n+2)^2}. \tag{3-33}$$

Set $\varepsilon_1 = \varepsilon^{1/(n+2)^3}$. We prove

$$E \subset \bigcup_{i=1}^{N} B_\eta(x_i) \tag{3-34}$$

for $\eta = R + C_2\varepsilon_1$ with some positive $C_2 = C_2(n, C_0)$. By decreasing δ, if necessary, we deduce from (3-33) that

$$|B_{\varepsilon_1}| > \left| E \setminus \bigcup_{i=1}^{N} B_\rho(x_i) \right|.$$

Thus, if $x \in E_{\varepsilon_1}$, then $B_{\varepsilon_1}(x) \cap \bigcup_{i=1}^{N} B_\rho(x_i)$ must be nonempty. This implies

$$E_{\varepsilon_1} \subset \bigcup_{i=1}^{N} B_{\rho+\varepsilon_1}(x_i). \tag{3-35}$$

Assume that for $x \in \partial E$,

$$d_x := \operatorname{dist}(x, \bar{E}_{r_0} + \bar{B}_{r_0}) > 0.$$

Then by (3-23)

$$\mathcal{H}^n(\partial E \cap B(x, d_x)) \le C\varepsilon^{1/(n+2)}.$$

Let $\delta_n \in \mathbb{R}_+$ be as in Lemma 3.2, and set $r_x = \min\{d_x, \delta_n/\lambda\}$. Again, by possibly decreasing δ so that $\delta \le \delta_n$, Lemma 3.2 yields

$$\delta_n r_x^n \le \mathcal{H}^n(\partial E \cap B_{r_x}(x)).$$

By combining the two previous estimates we have

$$\min\left\{ d_x, \frac{\delta_n}{\lambda} \right\} \le C\varepsilon^{1/(n(n+2))}.$$

Since $\delta_n/\lambda \ge \delta_n/C$, by decreasing δ, if necessary, the previous estimate implies $r_x = d_x$ and further yields

$$d_x \le C\varepsilon^{1/(n(n+2))} \le C\varepsilon^{1/(n+2)^2}. \tag{3-36}$$

On the other hand, by (3-28),

$$\bar{E}_{r_0} + \bar{B}_{r_0} \subset E_{r_0} + B_R \subset \bigcup_{i=1}^{N} B_{R+\varepsilon_0}(x_i), \tag{3-37}$$

where $\varepsilon_0 = \varepsilon^{1/(2(n+2))} \le \varepsilon^{1/(n+2)^2}$. Thus (3-36) and (3-37) imply

$$\partial E \subset \bigcup_{i=1}^{N} B_{\tilde{\eta}}(x_i)$$

with $\tilde{\eta} = R + C\varepsilon^{1/(n+2)^2}$. By combining this observation with (3-35) we obtain (3-34).

Finally, by decreasing δ one more time, if necessary, (3-30), (3-32) and (3-34) yield

$$\bigcup_{i=1}^{N} B_{\rho_-}(x_i) \subset E \subset \bigcup_{i=1}^{N} B_{\rho_+}(x_i),$$

where $\rho_- = R - C\varepsilon^{1/(n+2)^3}$, $\rho_+ = R + C\varepsilon^{1/(n+2)^3}$, the balls $B_{\rho_-}(x_1), \dots, B_{\rho_-}(x_N)$ are mutually disjoint, for N we have

$$|P(E) - N(n+1)\omega_{n+1}R^n| \le C\varepsilon^{1/(n+2)^3}$$

and $C = C(C_0, n) \in \mathbb{R}_+$. The claim of Theorem 1.2 then follows by Remark 3.1. □

4. Asymptotic behavior of the volume preserving mean curvature flow

In this section we first define the flat flow and recall some of its basic properties. We do this in the general dimensional case $\mathbb{R}^{n+1}$ and restrict ourselves to the case $n \le 2$ only in the proof of Theorem 1.1. We begin by defining the flat flow of (1-1).

Assume that $E_0 \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter with the volume of the unit ball $|E_0| = \omega_{n+1}$. For a given $h \in \mathbb{R}_+$ we construct a sequence of sets $(E_k^h)_{k=1}^{\infty}$ by an iterative minimizing procedure called minimizing movements, where initially $E_0^h = E_0$ and E_{k+1}^h is a minimizer of the problem

$$\mathcal{F}_h(E, E_k) = P(E) + \frac{1}{h}\int_E \bar{d}_{E_k}\,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||E| - \omega_{n+1}\big|. \tag{4-1}$$

Recall that $\bar{d}_{E_k}$ is the signed distance function from E_k. We then define the approximative flat flow $(E_t^h)_{t\ge 0}$ by

$$E_t^h = E_k^h, \quad \text{for } (k-1)h \le t < kh. \tag{4-2}$$

By [Mugnai et al. 2016] we know that there is a subsequence of the approximative flat flow which converges:

$$(E_t^{h_l})_{t\ge 0} \to (E_t)_{t\ge 0},$$

where for every $t > 0$ the set E_t is a set of finite perimeter with $|E_t| = \omega_{n+1}$. Any such limit is called a flat flow of (1-1). It follows from [Mugnai et al. 2016] that when $n \le 6$ and if the perimeters of E_t^h converge, i.e., $\lim_{h\to 0} P(E_t^h) = P(E_t)$ for every $t > 0$, then the flat flow is a weak solution of the volume preserving mean curvature flow. It is not known if the flat flow coincides with the classical solution of (1-1) when the latter is well defined and smooth, but the result in [Chambolle and Novaga 2008] seems to suggest this (see also [Chambolle et al. 2015]).

Preliminary results. Let us take a more rigorous approach to the concepts heuristically introduced above. We base this mainly on [Mugnai et al. 2016], with the only difference being that the volume constraint has a different value. Obviously, this does not affect the arguments.

First, we take a closer look at the functional $\mathcal{F}_h$ given by (4-1). If $E, F \subset \mathbb{R}^{n+1}$ are bounded sets of finite perimeter, then it is easy to see that modifications of E in a set of measure zero do not affect the value $\mathcal{F}_h(E, F)$, whereas such modifications of F may lead to drastic changes of the value of $\mathcal{F}_h(E, F)$. To eliminate this issue, we use a convention that a topological boundary of a set of finite perimeter is always the support of the corresponding Gauss–Green measure. Thus, we consider $\mathcal{F}_h$ as a functional

$$X_{n+1} \times \{A \in X_{n+1} : A \neq \varnothing\} \to \mathbb{R},$$

where

$$X_{n+1} = \{E \subset \mathbb{R}^{n+1} : E \text{ is a bounded set of finite perimeter with } \partial E = \operatorname{spt} \mu_E\}.$$

We remark that if E_0 is essentially open or closed and $E_0 \in X_{n+1}$, then we may assume X_{n+1} to be open or closed, respectively.

For $F \in X_{n+1}$ nonempty, there is always a minimizer E of the functional $\mathcal{F}_h(\,\cdot\,, F)$ in the class X_{n+1} satisfying the *discrete dissipation inequality*

$$P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}\,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||E| - \omega_{n+1}\big| \le P(F) + \frac{1}{\sqrt{h}}\big||F| - \omega_{n+1}\big|; \tag{4-3}$$

see [Mugnai et al. 2016, Lemma 3.1]. Moreover, there is a dimensional constant C_n such that

$$\sup_{E\Delta F} d_{\partial F} \le C_n\sqrt{h}; \tag{4-4}$$

see [Mugnai et al. 2016, Proposition 3.2]. The minimizer E is always a (Λ, r_0)-*minimizer* in any open neighborhood of E with suitable $\Lambda, r_0 \in \mathbb{R}_+$ satisfying $\Lambda r_0 \le 1$. Thus, by the standard regularity theory [Maggi 2012, Theorem 26.5 and Theorem 28.1] $\partial^* E$ is relatively open in ∂E and $C^{1,\alpha}$ regular with any $0 < \alpha < \frac{1}{2}$ and the Hausdorff dimension of the singular part $\partial E \setminus \partial^* E$ is at most $n-7$. These imply that E can always be chosen as an open set. On the other hand, if E is nonempty, it has a Lipschitz-continuous distributional mean curvature H_E satisfying the Euler–Lagrange equation

$$\frac{\bar{d}_F}{h} = -H_E + \lambda_E, \tag{4-5}$$

where the *Lagrange multiplier* can be written in the case $|E| \neq \omega_{n+1}$ as

$$\lambda_E = \frac{1}{\sqrt{h}}\operatorname{sgn}(\omega_{n+1} - |E|), \tag{4-6}$$

see [Mugnai et al. 2016, Lemma 3.7]. Thus, using standard elliptic estimates one can show that $\partial^* E$ is in fact $C^{2,\alpha}$ regular and (4-5) holds in the classical sense on $\partial^* E$. In particular, E is $C^{2,\alpha}$ regular when $n \le 6$. Moreover, if $x \in \partial E$ satisfies the exterior or interior ball condition with any r, then it must belong to the reduced boundary of E. This is well known and follows essentially from [Delgadino and Maggi 2019, Lemma 3].

Now let us turn our attention back to flat flows. Let $E_0 \in X_{n+1}$ be a set with volume ω_{n+1} and let $0 < h < (\omega_{n+1}/P(E_0))^2$. Then we find a minimizer $E_1^h \in X_{n+1}$ for $\mathcal{F}_h(\,\cdot\,, E_0)$, and by (4-3) we have $\big||E_1^h| - \omega_{n+1}\big| \le \sqrt{h}P(E_0)$ implying, via the condition $h < (\omega_n/P(E_0))^2$, that E_1^h is nonempty. Again we find a minimizer $E_2^h \in X_{n+1}$ for $\mathcal{F}_h(\,\cdot\,, E_1)$, and using (4-3) twice we obtain $\big||E_2^h| - \omega_{n+1}\big| \le \sqrt{h}P(E_0)$ and thus E_2^h is also nonempty. By continuing the procedure we find nonempty sets $E_0^h, E_1^h, E_2^h, \ldots \in X_{n+1}$ as mentioned earlier, i.e., $E_0^h = E_0$ and E_k^h is a minimizer of $\mathcal{F}_h(\,\cdot\,, E_{k-1})$ for every $k \in \mathbb{N}$. Thus we may define an approximate flat flow $(E_t^h)_{t\ge 0}$, with the initial set E_0, defined by (4-2). Further, a flat flow as a limit is defined as before. By iterating (4-3) we obtain

$$P(E_{kh}^h) + \frac{1}{h}\sum_{j=1}^{k}\int_{E_{jh}^h \Delta E_{(j-1)h}^h} d_{\partial E_{(j-1)h}^h}\,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||E_{kh}^h| - \omega_{n+1}\big| \le P(E_0) \quad \text{for every } k \in \mathbb{N}. \tag{4-7}$$

By the earlier discussion we may assume that E_t^h is an open set, for every $t \ge h$, and ∂E_t^h is C^2 regular up to the singular part $\partial E_t^h \setminus \partial^* E_t^h$ with Hausdorff dimension at most $n-7$. We use the shorthand notation λ_t^h for the corresponding Lagrange multiplier.

Next we list some basic properties of the approximative flat flow.

Proposition 4.1. *Let $(E_t^h)_{t\ge 0}$ be an approximative flat flow starting from $E_0 \in X_{n+1}$ with volume ω_{n+1} and $P(E_0) \le C_0$. There is a positive constant $C = C(C_0, n)$ such that the following hold for every $0 < h < (\omega_n/P(E_0))^2$:*

(i) *For every s, t with $h \le s \le t - h$ we have $|E_s^h \Delta E_t^h| \le C\sqrt{t-s}$.*

(ii) *Suppose that for a given $T_1 \ge 0$ we have $|E_{T_1}^h| = \omega_{n+1}$. Then $P(E_{T_1}^h) \ge P(E_t^h)$ for every $t \ge T_1$ and*

$$\int_{T_1+h}^{T_2}\int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}t \le C(P(E_{T_1}^h) - P(E_{T_2}^h))$$

for every $T_2 \ge T_1 + h$. Moreover, for every $h \le T_1 < T_2$,

$$\int_{T_1}^{T_2}\int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2\,\mathrm{d}\mathcal{H}^n\,\mathrm{d}t \le C P(E_0).$$

(iii) *For every $T > 0$ there is $R = R(E_0, T)$ such that $E_t^h \subset B_R$ for all $0 \le t \le T$.*

(iv) *If $(h_k)_k$ is a sequence of positive numbers converging to zero, then up to a subsequence there exist approximative flat flows $((E_t^{h_k})_{t\ge 0})_k$ which converge to a flat flow $(E_t)_{t\ge 0}$ in the L^1 sense in space and pointwise in time, where $E_t \in X_{n+1}$, i.e., for every $t \ge 0$,*

$$\lim_{h_k\to 0} |E_t^{h_k} \Delta E_t| = 0.$$

The limit flow also satisfies $|E_s \Delta E_t| \le C\sqrt{t-s}$ for every $0 < s < t$ and $|E_t| = \omega_{n+1}$ for every $t \ge 0$.

(v) *If E_0 is either open or closed, then the sequence in* (iv) *converges to $(E_t)_{t\ge 0}$ in the L^1 sense in space and compactly uniformly in time, i.e., for a fixed T,*

$$\lim_{h_k\to 0} \sup_{t\in[0,T]} |E_t^{h_k} \Delta E_t| = 0.$$

Moreover, $|E_s \Delta E_t| \le C\sqrt{t-s}$ for every $0 \le s < t$.

Proof. Claims (i)–(iv) are essentially proved in [Mugnai et al. 2016]; see the proofs of Proposition 3.5, Lemma 3.6 and Theorem 2.2.

To prove (v) we first show that

$$|E_h^h \Delta E_0| \to 0 \quad \text{as } h \to 0,$$

which immediately implies via (iv) that $|E_0 \Delta E_t| \le C\sqrt{t}$ for every $t \ge 0$ and hence the second claim of (v) holds. Then the compactly uniform convergence in time is a rather direct consequence of this and (i).

To this aim, let $(h_k)_k$ be an arbitrary sequence of positive numbers converging to zero. By (iii) and by the standard compactness property of sets of finite perimeter, there is a bounded set of finite perimeter E_∞ such that, up to extracting a subsequence, $E_{h_k}^{h_k} \to E_\infty$ in the L^1 sense. In particular, by (4-7) we have $|E_\infty| = \omega_{n+1} = |E_0|$. Again, by using $|E_{h_k}^{h_k} \Delta E_\infty| \to 0$ and (4-4) we have

$$|E_\infty \setminus \{y \in \mathbb{R}^n : \bar{d}_{E_0}(y) \le j^{-1}\}| = 0 \quad \text{and} \quad |\{y \in \mathbb{R}^n : \bar{d}_{E_0}(y) \le -j^{-1}\} \setminus E_\infty| = 0$$

for every $j \in \mathbb{N}$. Thus, by letting $j \to \infty$ we obtain $|E_\infty \setminus \bar{E}_0| = 0$ and $|\mathrm{int}(E_0) \setminus E_\infty| = 0$. Since E_0 is open or closed, this means either $|E_\infty \setminus E_0| = 0$ or $|E_0 \setminus E_\infty| = 0$. But now $|E_\infty| = |E_0|$, so the previous yields $|E_\infty \Delta E_0| = 0$. Thus $|E_{h_k}^{h_k} \setminus E_0| \to 0$ up to a subsequence and since $(h_k)_k$ was arbitrarily chosen we have $|E_h^h \Delta E_0| \to 0$. □

We note that claim (v) does not hold for every bounded set of finite perimeter E_0. As an example one may construct a wild set of finite perimeter E_0 such that $|E_h^h \Delta E_0| \ge c_0 > 0$ for all $h > 0$.

By [Mugnai et al. 2016, Corollary 3.10], for a fixed time $T \ge h$, we have that the integral $\int_h^T |\lambda_t^h|^2 \, \mathrm{d}t$ is uniformly bounded in h and hence, via (4-6), that $|\{t \in (h, T) : |E_t^h| \ne \omega_{n+1}\}| \le Ch$, where C depends also on T. We may improve this by using Lemma 2.4.

Proposition 4.2. *Let $C_0 > 0$ and $E_0 \in X_{n+1}$ be a set of finite perimeter with volume ω_{n+1} and $P(E_0) \le C_0$. There are positive constants $C = C(C_0, n)$ and $h_0 = h_0(C_0, n)$ such that if $h \le h_0$ and $(E_t^h)_{t \ge 0}$ is an approximative flat flow starting from E_0, then for every $h \le T_1 \le T_2$*

$$\int_{T_1}^{T_2} |\lambda_t^h|^2 \, \mathrm{d}t \le C(T_2 - T_1 + 1) \quad \textit{and} \quad |\{t \in (T_1, T_2) : |E_t^h| \ne \omega_{n+1}\}| \le Ch(T_2 - T_1 + 1).$$

Proof. By (4-7) we may choose $h_0 = h_0(C_0, n)$ such that $|E_t^h| \ge \frac{1}{2}\omega_{n+1}$ whenever $h \le h_0$. We may also assume $C_0 > 2\omega_{n+1}$ so that $|E_t^h| \ge 1/C_0$ for $h \le h_0$. Thus, by Lemma 2.4 and $P(E_t^h) \le C_0$, we find a positive $C = C(C_0, n)$ such that for every $t \ge h$ and $h \le h_0$

$$|\lambda_t^h|^2 \le C\left(1 + \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \, \mathrm{d}\mathcal{H}^n\right),$$

and therefore

$$\int_{T_1}^{T_2} |\lambda_t^h|^2 \, \mathrm{d}t \le C(T_2 - T_1) + C \int_{T_1}^{T_2} \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \, \mathrm{d}\mathcal{H}^n \, \mathrm{d}t.$$

By Proposition 4.1 (ii) we obtain the first inequality. The first inequality implies, via (4-6), the second inequality with the same constant C. □

We need also the following comparison result for the proof.

Lemma 4.3. *Let $1 \le C_0 < \infty$. Assume $E_0 \in X_{n+1}$ is a set of finite perimeter with volume ω_{n+1} and $P(E_0) \le C_0$, and let $F = \bigcup_{i=1}^N B_r(x_i)$ with $|x_i - x_j| \ge 2r$ and $1/C_0 \le r \le C_0$. There is a positive constant $\varepsilon_0 = \varepsilon_0(C_0, n)$ such that if $(E_t^h)_{t\ge 0}$ is an approximative flat flow starting from E_0 and*

$$\sup_{x \in E_{t_0}^h \Delta F} d_{\partial F}(x) \le \varepsilon \quad \text{with } \varepsilon \le \varepsilon_0$$

for $t_0 \ge 0$, then

$$\sup_{x \in E_t^h \Delta F} d_{\partial F}(x) \le C\varepsilon^{1/9} \quad \text{for all } t_0 < t < t_0 + \sqrt{\varepsilon}$$

provided that $h \le \min\{\sqrt{\varepsilon}, h_0\}$, where $h_0 = h_0(C_0, n)$ is as in Proposition 4.2.

Proof. Our standing assumptions are

$$h \le \min\{\sqrt{\varepsilon}, h_0\} \quad \text{and} \quad \varepsilon \le \min\{1/(2C_0), 1\}.$$

As usual, C denotes a positive constant which may change from line to line but depends only on the parameters C_0 and n.

Without loss of generality we may assume $t_0 = 0$. Fix an arbitrary $x_i \in \{x_1, \dots, x_N\}$. Up to translating the coordinates we may assume that $x_i = 0$. We set for every $k = 0, 1, 2, \dots$

$$\rho_k = \inf\{|x| : x \in \mathbb{R}^{n+1} \setminus E_{kh}^h\}$$

and

$$r_k = \min\{r, \rho_0, \dots, \rho_k\}.$$

We claim that

$$r_{k+1}^2 - r_k^2 \ge -C_1(1 + |\lambda_{(k+1)h}^h|)h, \tag{4-8}$$

with some positive constant $C_1 = C_1(C_0, n)$. First, if $r_{k+1} = r_k$, the claim (4-8) is trivially true. Thus we may assume $r_{k+1} < r_k$ which implies $\rho_{k+1} = r_{k+1} < r_k \le \rho_k$. Then $\rho_k > 0$ which in turn means

$$\rho_k = \min_{\partial E_{kh}^h} |x|.$$

Since $E_{(k+1)h}^h$ is bounded and open, there is a point $x \in \mathbb{R}^{n+1} \setminus E_{(k+1)h}^h$ with $\rho_{k+1} = |x|$. Let x' be a closest point to x on ∂E_{kh}^h. Then

$$r_{k+1} + |\bar{d}_{E_{kh}^h}(x)| = |x| + |\bar{d}_{E_{kh}^h}(x)| \ge |x'| \ge \rho_k \ge r_k.$$

The condition $|x| < \rho_k$ means x exists in E_{kh}^h, so the previous estimate yields

$$r_{k+1} - r_k \ge \bar{d}_{E_{kh}^h}(x). \tag{4-9}$$

Again, $x \in E_{kh}^h \setminus E_{(k+1)h}^h$ so by Equation (4-4), $|\bar{d}_{E_{kh}^h}(x)| \le C_n\sqrt{h}$ and hence

$$r_{k+1} - r_k \ge -C_n\sqrt{h}. \tag{4-10}$$

We split the argument into two cases. First, if $r_{k+1} < C_n\sqrt{h}$, then by (4-10) we have $r_k < 2C_n\sqrt{h}$. Therefore, using (4-10) we obtain

$$r_{k+1}^2 - r_k^2 \geq -C_n(r_{k+1}+r_k)\sqrt{h} \geq -3C_n^2 h. \tag{4-11}$$

If $r_{k+1} \geq C_n\sqrt{h}$, then by (4-10) we have $r_k \leq 2r_{k+1}$. Since $r_{k+1} > 0$, we have $x \in \partial E^h_{(k+1)h}$ and $E^h_{(k+1)h}$ satisfies the interior ball condition of radius r_{k+1} at x. Thus by the discussion in Section 2, x belongs to the reduced boundary of $E^h_{(k+1)h}$ and therefore by the maximum principle $H_{E^h_{(k+1)h}}(x) \leq n/r_{k+1}$. Again, by the previous estimate, (4-9), the Euler–Lagrange equation (4-5) and $r_{k+1} \leq C_0$ we obtain

$$\begin{aligned}\frac{r_{k+1}-r_k}{h} \geq \frac{\bar{d}_{E^h_{kh}}(x)}{h} &\geq -\frac{n}{r_{k+1}} - |\lambda^h_{(k+1)h}| \\ &\geq -\frac{1}{r_{k+1}}(n + C_0|\lambda^h_{(k+1)h}|).\end{aligned}$$

Therefore

$$\frac{r_{k+1}^2 - r_k^2}{h} \geq -\left(1+\frac{r_k}{r_{k+1}}\right)(n + C_0|\lambda^h_{(k+1)h}|) \geq -3(n + C_0|\lambda^h_{(k+1)h}|). \tag{4-12}$$

Thus (4-11) and (4-12) yield the claim (4-8) in the case $r_{k+1} < r_k$.

We iterate (4-8) up to $K \in \mathbb{N}$, which is chosen so that $Kh \in (\sqrt{\varepsilon}, 2\sqrt{\varepsilon})$ (recall $h < \sqrt{\varepsilon}$), and use Proposition 4.2 to obtain

$$\begin{aligned}r_K^2 - r_0^2 &\geq -C_1 \sum_{k=0}^{K-1}(1+|\lambda^h_{(k+1)h}|)h \\ &= -C_1 Kh - C_1\int_h^{(K+1)h} |\lambda^h_t|\,\mathrm{d}t \\ &\geq -2C_1\sqrt{\varepsilon} - C_1\int_h^{3\sqrt{\varepsilon}} |\lambda^h_t|\,\mathrm{d}t \\ &\geq -2C_1\sqrt{\varepsilon} - \int_h^{3\sqrt{\varepsilon}} \varepsilon^{-1/4} + \varepsilon^{1/4}|\lambda^h_t|^2\,\mathrm{d}t \\ &\geq -C\varepsilon^{1/4}\left(1 + \int_h^{3\sqrt{\varepsilon}} |\lambda^h_t|^2\,\mathrm{d}t\right) \\ &\geq -C\varepsilon^{1/4}.\end{aligned} \tag{4-13}$$

By the assumption $\sup_{x\in E_0\Delta F} d_{\partial F}(x) \leq \varepsilon$ we have $r - \varepsilon \leq r_0$. Thus we divide $r_K^2 - r_0^2$ by $r_K + r_0$ and use $r_0 \geq r - \varepsilon \geq \frac{1}{2}r \geq 1/(2C_0)$ as well as (4-13) to find a positive constant $C_2 = C_2(C_0, n)$ such that $r_K \geq r - C_2\varepsilon^{1/4}$. This means that

$$\inf_{\mathbb{R}^{n+1}\setminus E^h_t} \bar{d}_{B_r(x_i)} \geq -C_2\varepsilon^{1/4} \quad \text{for all } t < \sqrt{\varepsilon},$$

and again due to the arbitrariness of $x_i \in \{x_1, \dots, x_N\}$, that

$$\inf_{\mathbb{R}^{n+1}\setminus E^h_t} \bar{d}_F \geq -C_2\varepsilon^{1/4} \quad \text{for all } t < \sqrt{\varepsilon}.$$

To conclude the proof, we show that there is a positive constant $\varepsilon_1 = \varepsilon_1(C_0, n)$ such that

$$\sup_{E_t^h} \bar{d}_F \le 2\varepsilon^{1/9} \quad \text{for all } t < \sqrt{\varepsilon} \tag{4-14}$$

provided that $\varepsilon \le \varepsilon_1$. To this aim we choose an arbitrary $x_0 \in \mathbb{R}^{n+1} \setminus \bar{F}$ with $\bar{d}_F(x_0) \ge 2\varepsilon^{1/9}$. For every $k = 0, 1, 2, \dots$, we set

$$\rho_k = \inf_{x \in E_{kh}^h} |x - x_0|$$

and

$$r_k = \min\{2\varepsilon^{1/9}, \rho_1, \dots, \rho_k\}.$$

In particular, $r_k \le 2C_0^{1/9}$. A slight modification of the procedure we used to obtain (4-13) yields

$$r_K^2 - r_0^2 \ge -C\varepsilon^{1/4},$$

where K is the same as described earlier. Again, the conditions $\sup_{x \in E_0 \Delta F} d_{\partial F}(x) \le \varepsilon$ and $\varepsilon \le 1$ imply $r_0 \ge 2\varepsilon^{1/9} - \varepsilon \ge \varepsilon^{1/9}$. Thus

$$r_K - r_0 \ge -C\frac{\varepsilon^{1/4}}{r_0} \ge -C\varepsilon^{5/36} = -C\varepsilon^{1/36}\varepsilon^{1/9},$$

and thus

$$r_K \ge (1 - C\varepsilon^{1/36})\varepsilon^{1/9} > \tfrac{1}{2}\varepsilon^{1/9},$$

when ε is small enough. Since x_0, with $d_F(x_0) \ge 2\varepsilon^{1/9}$, was arbitrarily chosen we deduce that

$$E_{kh}^h \subset \{x \in \mathbb{R}^{n+1} : d_F(x) \le 2\varepsilon^{1/9}\} \quad \text{for all } k = 0, \dots, K.$$

The claim (4-14) then follows from the choice of K. □

Proof of Theorem 1.1. The proof of Theorem 1.1 is based on Theorem 1.2. We first use it together with the dissipation inequality in Proposition 4.1 (ii) to deduce that there exists a sequence of times $t_j \to \infty$ such that the sets E_{t_j} are close to a disjoint union of balls. Since the perimeter of the approximative flat flow is essentially decreasing, the number of balls is also monotone. In particular, we deduce that after some time, the sets E_{t_j} are close to a fixed number, say N, of balls. We use the second statement of Theorem 1.2 to deduce that the perimeters of E_{t_j} converge to the perimeter of N balls with volume ω_{n+1} and thus the right-hand side of the dissipation inequality converges to zero. This allows us to improve our estimate and use Theorem 1.2 again to deduce that the flat flow E_t is close to a disjoint union of N balls for all large t except a set of times with small measure. The statement then finally follows from Lemma 4.3.

Proof. Assume that the initial set $E_0 \in X_{n+1}$ has the volume of the unit ball $|E_0| = \omega_{n+1}$, fix a positive C_0 with $C_0 \ge \max\{1, P(E_0)\}$ and assume $h < (C_0/\omega_{n+1})^2$. Let $(E_t)_{t\ge 0}$ be a flat flow starting from E_0 and let $(E_t^{h_l})_{t\ge 0}$ be an approximative flat flow which by Proposition 4.1 converges to $(E_t)_{t\ge 0}$ locally uniformly in L^1. We simplify the notation and denote the converging subsequence again by h. Since we are now in the dimensions 2 and 3 ($n = 1, 2$), the sets E_t^h are C^2 regular.

Step 1: Let us denote

$$\Sigma_h := \{t \in (0,\infty) : |E_t^h| \neq \omega_{n+1}\}. \tag{4-15}$$

By (4-7) and Proposition 4.2 we find a constant $h_0 = h_0(C_0, n) < 1$ such that $|E_t^h| \geq 1/C_0$ for every $t \geq 0$ and

$$|(T_1, T_2) \cap \Sigma_h| \leq \tfrac{1}{3}(T_2 - T_1)$$

for every $T_1 \geq 1$ and $T_2 \geq T_1 + 1$ provided that $h \leq h_0$. On the other hand, by Proposition 4.1 (ii) we have, for every $h \leq h_0$ and $l \in \mathbb{N}$, that

$$I_{l,h} := \fint_{l^2}^{(l+1)^2} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)}\,\mathrm{d}t \leq \frac{C}{l}.$$

By Chebysev's inequality,

$$|\{t \in (l^2, (l+1)^2) : \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} \geq 3I_{l,h}\}| \leq \tfrac{1}{3}((l+1)^2 - l^2).$$

Therefore, by choosing $T_1 = l^2$ and $T_2 = (l+1)^2$ we deduce that the set

$$\{t \in (T_1, T_2) : |E_t^h| = \omega_{n+1},\ \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} < 3I_{l,h}\}$$

is nonempty. Thus if $h \leq h_0$, then there is a sequence of times $(T_l^h)_l$, with $l^2 \leq T_l^h \leq (l+1)^2$, such that the corresponding sets satisfy $|E^h_{T_l^h}| = \omega_{n+1}$ and

$$\|H_{E^h_{T_l^h}} - \lambda^h_{T_l^h}\|_{L^2(\partial E^h_{T_l^h})} \leq Cl^{-1/2}. \tag{4-16}$$

By slight abuse of the notation we set $E_l^h := E^h_{T_l^h}$ and $\lambda_{l,h} := \lambda^h_{T_l^h}$ for $h \leq h_0$. Since the sets E_l^h are C^2 regular and bounded and thanks to $P(E_0) \leq C_0$, $|E_l^h| \geq 1/C_0$, (4-16) and Theorem 1.2, we find $l_0 = l_0(C_0, n)$ such that for every $l \geq l_0$ we have $1/C \leq \lambda_{l,h} \leq C$,

$$|P(E_l^h) - N_l^h(n+1)\omega_{n+1}(r_l^h)^n| \leq Cl^{-q/2} \quad \text{and} \quad \sup_{E_l^h \Delta F_l^h} d_{\partial F_l^h} \leq Cl^{-q/2}, \tag{4-17}$$

where $r_l^h = n/\lambda_{l,h}$ and F_l^h is a union of N_l^h pairwise disjoint (open) balls of radius $r_{l,h}$. Since we have $1/C \leq \lambda_{l,h} \leq C$, we also have $1/C \leq r_{l,h} \leq C$, which together with the perimeter estimate $P(E_l^h) \leq P(E_0) \leq C_0$ implies that there is $N_0 = N_0(C_0, n) \in \mathbb{N}$ such that $N_l^h \leq N_0$. Further, the distance estimate in (4-17), together with $1/C \leq r_{l,h} \leq C$ and $N_l^h \leq N_0$, yields

$$|E_l^h \Delta F_l^h| \leq Cl^{-q/2}.$$

Since $|E_l^h| = \omega_{n+1}$, we have that the estimate above implies $|(r_{l,h})^{n+1}N_l^h - 1| \leq Cl^{-q/2}$ and further that $|(r_{l,h})^n (N_l^h)^{n/(n+1)} - 1| \leq Cl^{-q/2}$. This inequality, the perimeter estimate in (4-17) and $N_l^h \leq N_0$ imply

$$|P(E_l^h) - (n+1)\omega_{n+1}(N_l^h)^{1/(n+1)}| \leq Cl^{-q/2}. \tag{4-18}$$

Since by Proposition 4.1 (ii) $(P(E_l^h))_{l \geq l_0}$ is nonincreasing, we have that (4-18) implies there is a positive integer $l_1 = l_1(C_0, n) \geq l_0$ for which $(N_l^h)_{l \geq l_1}$ is nonincreasing for all $h \leq h_0$.

Step 2: For $l \geq l_1$ and $h \leq h_0$ the sets E_l^h are thus close to N_l^h balls. We claim that there are $N \in \mathbb{N}$ and $l_2 \geq l_1$ such that for every integer $L \geq l_2$,

$$N_l^h = N \quad \text{for all } l_2 \leq l \leq L, \tag{4-19}$$

provided that h is small enough.

By using a standard diagonal argument and possibly passing to a subsequence we find a sequence of positive integers $(N_l)_{l \geq l_1}$, with $N_l \leq N_0$, such that $N_l^h \to N_l$ for every $l \geq l_1$. Since $(N_l^h)_{l \geq l_1}$ is nonincreasing, we have that $(N_l)_{l \geq l_1}$ is nonincreasing too and hence there are $N, l_2 \in \mathbb{N}$, $l_2 \geq l_1$, such that $N_l = N$ for every $l \geq l_2$. Hence we have (4-19) by the convergence of N_l^h to N_l.

We obtain from (4-18) and (4-19) that

$$|P(E_l^h) - (n+1)\omega_{n+1}(N)^{1/(n+1)}| \leq Cl^{-q/2} \tag{4-20}$$

for $l_2 \leq l \leq L$, provided that h is small enough. Therefore, it follows from Proposition 4.1 (ii) that

$$\int_{T_l^h + h}^{T_L^h} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} \, \mathrm{d}t \leq Cl^{-q/2}.$$

Since $h \leq 1$ and $L > 1$ was arbitrarily chosen, the above yields

$$\sup_{T \geq (l+2)^2} \left[\limsup_{h \to 0} \int_{(l+2)^2}^{T} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} \, \mathrm{d}t \right] \leq Cl^{-q/2} \tag{4-21}$$

for every $l \geq l_2$.

Step 3: Let us fix a small δ, the choice of which will be clear later. Then it follows from (4-21), (4-20) and the fact that the map $t \mapsto P(E_t^h)$ is nonincreasing in Σ_h that there is T_δ such that for every $T \geq T_\delta + 1$ there is $h_{\delta,T}$ such that

$$\int_{T_\delta}^{T} \|H_{E_t^h} - \lambda_t^h\|^2_{L^2(\partial E_t^h)} \, \mathrm{d}t \leq \delta \tag{4-22}$$

for all $h \leq h_{\delta,T}$ and

$$|P(E_t^h) - (n+1)\omega_{n+1} N^{1/(n+1)}| \leq \delta \tag{4-23}$$

for all $t \in (T_\delta, T) \setminus \Sigma_h$. On the other hand, by Proposition 4.2 and by decreasing $h_{\delta,T}$ if necessary, we deduce that

$$|\Sigma_h \cap (T_\delta, T)| \leq \delta \quad \text{for all } h \leq h_{\delta,T}. \tag{4-24}$$

Let $\varepsilon > 0$ and let us fix $t \geq T_\delta + 1$. (The time $T_\delta + 1$ will be T_ε in the claim.) We claim that, when δ is chosen small enough, we have

$$\sup_{E_t^h \Delta F_t^h} d_{\partial F_t^h} \leq \varepsilon \tag{4-25}$$

for $h \leq h_{\delta,T}$, where F_t^h is a union of N pairwise disjoint (open) balls of radius $r = N^{-1/(n+1)}$ with volume ω_{n+1}.

Fix $T \geq t+1$. Then it follows from (4-22) that

$$\int_{t-\delta^{1/4}}^{t} \|H_{E_\tau^h} - \lambda_\tau^h\|^2_{L^2(\partial E_\tau^h)}\, \mathrm{d}\tau \leq \delta,$$

and from (4-23) and (4-24) that

$$|P(E_\tau^h) - (n+1)\omega_{n+1}N^{1/(n+1)}| \leq \delta \quad \text{for all } \tau \in (t-\delta^{1/4}, t) \setminus \Sigma_h$$

and $|\Sigma_h \cap (t-\delta^{1/4}, t)| \leq \delta$. Using these estimates we deduce that there is $t_0 \in (t-\delta^{1/4}, t)$ such that $|E_{t_0}^h| = \omega_{n+1}$,

$$|P(E_{t_0}^h) - (n+1)\omega_{n+1}N^{1/(n+1)}| \leq \delta \tag{4-26}$$

and

$$\|H_{E_{t_0}^h} - \lambda_{t_0}^h\|_{L^2(\partial E_{t_0}^h)} \leq \delta^{1/4}.$$

Theorem 1.2 implies that

$$\sup_{E_{t_0}^h \Delta F_{t_0}^h} d_{\partial F_{t_0}^h} \leq C\delta^{q/4}$$

for all $h \leq h_{\delta,T}$, where $F_{t_0}^h$ is a union of $N_{t_0,h}$ pairwise disjoint (open) balls of radius $r_{t_0,h}$ with volume ω_{n+1}, and

$$|P(E_{t_0}^h) - N_{t_0,h}(n+1)\omega_{n+1}r_{t_0,h}^n| \leq C\delta^{q/4}.$$

Since $1/C \leq r_{t_0,h} \leq C$, as in Step 1 we deduce from the previous two estimates that $|E_{t_0}^h \Delta F_{t_0}^h| \leq C\delta^{q/4}$. Then by (4-26) and $|F_{t_0}^h| = \omega_{n+1}$ we further conclude that $N_{t_0,h} = N$, i.e., $F_{t_0}^h$ is a union of N pairwise disjoint (open) balls with volume ω_{n+1} and radius $r = N^{-1/(n+1)}$.

By Lemma 4.3,

$$\sup_{E_\tau^h \Delta F_{t_0}^h} d_{\partial F_{t_0}^h} \leq C\delta^{q/36} \quad \text{for all } t_0 < \tau < t_0 + \delta^{q/8}$$

and $h \leq h_{\delta,T}$. In particular, since $\delta^{q/8} > \delta^{1/4}$ the above inequality holds for t. This proves (4-25) by choosing $F_t^h = F_{t_0}^h$ and δ small enough. The claim follows by letting $h \to 0$. Note that by Proposition 4.1 (iii) there is $R > 0$ such that $F_t^h \subset B_R$ for all $h \leq h_{\delta,T}$. Therefore, by passing to another subsequence if necessary, we have that $F_t^h \to F_t$, where F_t is a union of N pairwise disjoint (open) balls with volume ω_{n+1}, and by (4-25),

$$\sup_{E_t \Delta F_t} d_{\partial F_t} \leq \varepsilon. \qquad \square$$

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ANALYSIS & PDE

Volume 16 No. 3 2023

Global regularity for the nonlinear wave equation with slightly supercritical power 613
MARIA COLOMBO and SILJA HAFFTER

Subelliptic wave equations are never observable 643
CYRIL LETROUIT

Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow 679
VESA JULIN and JOONAS NIINIKOSKI

A simple nuclear C^*-algebra with an internal asymmetry 711
ILAN HIRSHBERG and N. CHRISTOPHER PHILLIPS

Partial regularity of Leray–Hopf weak solutions to the incompressible Navier–Stokes equations with hyperdissipation 747
WOJCIECH S. OŻAŃSKI

The Peskin problem with viscosity contrast 785
EDUARDO GARCÍA-JUÁREZ, YOICHIRO MORI and ROBERT M. STRAIN

Solution of the qc Yamabe equation on a 3-Sasakian manifold and the quaternionic Heisenberg group 839
STEFAN IVANOV, IVAN MINCHEV and DIMITER VASSILEV

Garland's method with Banach coefficients 861
IZHAR OPPENHEIM