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A note on Kakeya sets of horizontal and *SL*(2) lines

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Abstract

We consider unions of SL(2) lines in \mathbb{R}^3 . These are lines of the form

 $L = (a, b, 0) + \operatorname{span}(c, d, 1),$

where ad - bc = 1. We show that if \mathcal{L} is a Kakeya set of SL(2) lines, then the union $\cup \mathcal{L}$ has Hausdorff dimension 3. This answers a question of Wang and Zahl. The SL(2) lines can be identified with *horizontal lines* in the first Heisenberg group, and we obtain the main result as a corollary of a more general statement concerning unions of horizontal lines. This statement is established via a point-line duality principle between horizontal and *conical* lines in \mathbb{R}^3 , combined with recent work on *restricted families of projections to planes*, due to Gan, Guo, Guth, Harris, Maldague and Wang. Our result also has a corollary for Nikodym sets associated with horizontal lines, which answers a special case of a question of Kim.

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1 | INTRODUCTION

The purpose of this note is to study the Hausdorff dimension of unions of SL(2) lines in \mathbb{R}^3 . Here is the definition of SL(2) lines, following [10].

Definition 1.1 ($\mathcal{L}_{SL(2)}$). The family $\mathcal{L}_{SL(2)}$ consists of the following lines $L \subset \mathbb{R}^3$. Either *L* is a line contained in the *xy*-plane, and $0 \in L$, or then

$$L := L_{\alpha,\beta,\gamma,\delta} := (\alpha,\beta,0) + \operatorname{span}(\gamma,\delta,1),$$

where $\alpha \delta - \beta \gamma = 1$.

We also use the following notation. If \mathcal{L} is any family of lines in \mathbb{R}^3 , we write dir(\mathcal{L}) := { $e \in S^2$: $\ell \parallel \text{span}(e)$ for some $\ell \in \mathcal{L}$ }. Here is the main result of the note.

Theorem 1.2. Let $\mathcal{L} \subset \mathcal{L}_{SL(2)}$ be a set with $\mathcal{H}^2(\operatorname{dir}(\mathcal{L})) > 0$. Then

$$\dim_{\mathrm{H}}(\cup \mathcal{L}) = 3$$

Here 'dim_H($\cup \mathcal{L}$)' is the Euclidean Hausdorff dimension of the union $\cup \mathcal{L} := \bigcup_{\ell \in \mathcal{L}} \ell$.

Remark 1.3. Theorem 1.2 answers a question posed by Wang and Zahl in [10, Section 1.2]. This question was motivated by earlier work of Katz and Zahl [5]. Theorem 1.2 continues to hold if the full lines in \mathcal{L} are replaced by line segments of positive length. We will discuss this briefly below (3.2).

Katz, Wu and Zahl [4] also proved Theorem 1.2 independently, using a different method.

The *SL*(2) lines are essentially (up to a change in coordinates) the same as *horizontal lines* in the first Heisenberg group $\mathbb{H} = (\mathbb{R}^3, *)$, viewed as subsets of \mathbb{R}^3 (see Proposition 2.1). We will infer Theorem 1.2 from a more general statement concerning unions of these horizontal lines, Theorem 1.5 below. We first need to define the concepts properly.

The family of all horizontal lines is denoted by $\mathcal{L}(\mathbb{H})$. The 'Heisenberg' definition of these lines is the following. Let $\Pi_0 := \{(x, y, 0) : x, y \in \mathbb{R}\}$ be the *xy*-plane, and for $p \in \mathbb{R}^3$, let $\Pi_p := p * H_0$ be the *left translate* of Π_0 by the Heisenberg group product

$$(x, y, t) * (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)\right).$$

Then, $\mathcal{L}(\mathbb{H})$ consists of all the lines in Π_p (for every $p \in \mathbb{R}^3$) which contain the point p.

The family $\mathcal{L}(\mathbb{H})$ is a three-dimensional submanifold of the full (four-dimensional) family of lines in \mathbb{R}^3 . In fact, the definition above of horizontal lines will not be used in the note: rather, we

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focus attention on the following parameterised subset of $\mathcal{L}(\mathbb{H})$:

$$\mathcal{L}'(\mathbb{H}) = \{ \mathcal{\ell}_{(a,b,c)} : (a,b,c) \in \mathbb{R}^3 \},\$$

where

$$\ell_{(a,b,c)} = \left\{ (as+b,s,\tfrac{b}{2}s+c) : s \in \mathbb{R} \right\}.$$

The subset $\mathcal{L}'(\mathbb{H})$ consists of all elements of $\mathcal{L}(\mathbb{H})$, except for those contained in some translate of the plane $\mathbb{W}_0 := \{(x, 0, t) : x, t \in \mathbb{R}\}$. By definition, every set $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$ can be written as

$$\mathcal{L} = \ell(P) := \{\ell_{(a,b,c)} : (a,b,c) \in P\}$$

for some set $P \subset \mathbb{R}^3$. This identification of $\mathcal{L}'(\mathbb{H})$ with \mathbb{R}^3 allows us to transport notions like 'Borel set' and 'dimension' from \mathbb{R}^3 to corresponding notions for subsets of $\mathcal{L}'(\mathbb{H})$.

Definition 1.4. Let $\mathcal{L} = \ell(P) \subset \mathcal{L}'(\mathbb{H})$. We say that \mathcal{L} is a Borel set if $P \subset \mathbb{R}^3$ is a Borel set. We define dim_H $\mathcal{L} := \dim_H P$, where 'dim_H P' refers to the Euclidean Hausdorff dimension of $P \subset \mathbb{R}^3$.

Now we can state our main result about unions of horizontal lines.

Theorem 1.5. Let $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$. Then,

$$\dim_{\mathrm{H}}(\cup \mathcal{L}) = \min\{\dim_{\mathrm{H}}\mathcal{L} + 1, 3\}.$$

The following corollary implies Theorem 1.2, as we will verify in Section 2.

Corollary 1.6. Let $\mathcal{L} \subset \mathcal{L}(\mathbb{H})$ with $\mathcal{H}^2(\operatorname{dir}(\mathcal{L})) > 0$. Then,

 $\dim_{\mathrm{H}}(\cup\mathcal{L})=3.$

Remark 1.7. Theorem 1.5 and Corollary 1.6 continue to hold if full lines are replaced by line segments of positive length, see the discussion below (3.2). Thus, if $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$, and every line $\ell \in \mathcal{L}$ contains a segment $I(\ell) \subset \ell$ of positive length, then

$$\dim_{\mathrm{H}}\left(\bigcup_{\ell\in\mathcal{L}}I(\ell)\right) = \min\{\dim_{\mathrm{H}}\mathcal{L}+1,3\}.$$
(1.8)

1.1 | Nikodym sets associated with horizontal lines

Theorem 1.5 easily yields information about the dimension of *Nikodym sets* associated with horizontal lines. A set $N \subset \mathbb{R}^3$ is called an $\mathcal{L}(\mathbb{H})$ -*Nikodym set* if for every $p \in \mathbb{R}^3$ (or more generally every $p \in \mathbb{R}^3$ in a measurable set of positive measure $\Omega \subset \mathbb{R}^3$), there exists a line $\mathscr{C}_p \in \mathcal{L}(\mathbb{H})$ containing p such that N contains a line segment $I_p \subset \mathscr{C}_p$ of positive length.

Corollary 1.9. *Every* $\mathcal{L}(\mathbb{H})$ *-Nikodym set* $N \subset \mathbb{R}^3$ *has* $\dim_{\mathbb{H}} N = 3$.

It is well known that bounds for Kakeya sets yield bounds for Nikodym sets: we only repeat the standard details below for the reader's convenience. For a similar argument in the case of classical Kakeya and Nikodym sets, see [9, Section 11.3].

Proof of Corollary 1.9. We may assume without loss of generality that all the lines $\ell_p \in \mathcal{L}(\mathbb{H})$ appearing in the definition of 'N' lie in $\mathcal{L}'(\mathbb{H})$. Namely, if this is true for a positive measure subset of the points $p \in \Omega$, we simply replace Ω by that subset. If this fails for Lebesgue almost every point $p \in \Omega$, then we apply a rotation R of, say, 10° around the *t*-axis to the objects Ω , N, and the lines ℓ_p , $p \in \Omega$. Rotations around the *t*-axis preserve $\mathcal{L}(\mathbb{H})$, and the measure and dimension of Ω and N. After this procedure, we moreover have $\ell_p \in \mathcal{L}'(\mathbb{H})$ for a.e. $p \in R(\Omega)$.

Using Fubini's theorem, start by picking $y_0 \in \mathbb{R}$ such that $\mathcal{H}^2(\Omega \cap \mathbb{W}_{y_0}) > 0$. Here, $\mathbb{W}_y = \{(x, y, t) : x, t \in \mathbb{R}^3\}$ for $y \in \mathbb{R}$. By assumption, for every $p = (x, y_0, t) \in \Omega \cap \mathbb{W}_{y_0}$, there exists a line

$$\ell_p := \ell_{(a(p), b(p), c(p))} \in \mathcal{L}'(\mathbb{H})$$

containing *p* such that *N* contains a line segment $I_p \subset \ell_p$ of positive length.

Now, note that the map $(a, b, c) \mapsto \Psi(a, b, c) = (ay_0 + b, y_0, \frac{b}{2}y_0 + c)$ is Lipschitz, and

 $\Omega \cap \mathbb{W}_{\gamma_0} \subset \Psi(\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{\gamma_0}\}).$

(This is because the lines ℓ_p contain the points $p \in \Omega \cap \mathbb{W}_{v_0}$.) Therefore,

$$\dim_{\mathrm{H}}\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\} \ge \dim_{\mathrm{H}}(\Omega \cap \mathbb{W}_{y_0}) = 2.$$

In particular, the set of lines $\mathcal{L} := \{\ell_p : p \in \Omega\} \subset \mathcal{L}'(\mathbb{H})$ has $\dim_{\mathrm{H}} \mathcal{L} \ge 2$ by definition. Therefore, it follows from Theorem 1.5, or to be precise (1.8), that

$$\dim_{\mathrm{H}} N \ge \dim_{\mathrm{H}} \left(\bigcup_{p \in \Omega} I_p \right) = 3$$

This completes the proof.

Remark 1.10. Nikodym set for 'restricted' families of lines was earlier considered by Kim [6]. Corollary 1.9 answers (a special case of) a question raised on [6, p. 478]. We elaborate on this a little further. The paper [6] considered general families of 2-planes $p \mapsto \Pi_{\mathfrak{a}}(p) \subset \mathbb{R}^3$, where $p \mapsto \mathfrak{a}(p)$ is a non-vanishing measurable vector field, and

 $p \in \Pi_{\mathfrak{a}}(p)$ and $\operatorname{span}(\mathfrak{a}(p)) = \Pi_{\mathfrak{a}}(p)^{\perp}$.

One can associate Nikodym sets $N \subset \mathbb{R}^3$ to such a plane family, as follows: for every $p \in \mathbb{R}^3$, the requirement is that there exists a line $\ell_p \subset \mathbb{R}^3$ satisfying

$$p \in \ell_p \subset \Pi_{\mathfrak{a}}(p),$$

and a non-trivial segment $I_p \subset N \cap \ell_p$. How small can such a Nikodym set $N \subset \mathbb{R}^3$ be? In [6], Kim approached the question via maximal function estimates, and his results depend on the properties of the vector field \mathfrak{a} . Kim considered vector fields \mathfrak{a} of the form

$$\mathfrak{a}(p) = (a_{11}p_1 + a_{21}p_2, a_{12}p_1 + a_{22}p_2, -1), \qquad p = (p_1, p_2, p_3) \in \mathbb{R}^3,$$

and defined the 'discriminant' $D_a = (a_{12} + a_{21})^2 - 4a_{11}a_{22}$. In [6, Corollary 1, p. 478], it was shown that the dimension of N equals 3 if $D_a \neq 0$. Right after the corollary, the question is raised, what happens in the situation $D_a = 0$.

Now, recall the definition of horizontal lines $\mathcal{L}(\mathbb{H})$: these were the lines contained in the planes $\Pi_p = p * \Pi_0$, and passing through p. The planes Π_p fit in the framework of [6], choosing $\mathfrak{a}(p) = (-p_2/2, p_1/2, -1)$, or $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$. In particular, $D_{\mathfrak{a}} = 0$. Also, the $\mathcal{L}(\mathbb{H})$ -Nikodym sets defined above Corollary 1.9 are the same as the Nikodym sets of [6] associated with the planes $\Pi_p = p * \Pi_0$. Thus, Corollary 1.9 covers the special case $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ of the problem raised on [6, p. 478].

1.2 | Ingredients of the proof

The proof of Theorem 1.5 is based on two ingredients. The first one is a *point-line duality* between horizontal lines and *conical lines* in \mathbb{R}^3 , namely translates of lines contained in the light cone $\{(x, y, t) : t^2 = x^2 + y^2\}$. This duality was formalised in our paper [2], although it was already implicit in the work [7] of Liu. Using this point-line duality, Kakeya-type problems for horizontal lines can be transformed into projection problems in \mathbb{R}^3 . These projection problems concern 'restricted' families of projections to planes in \mathbb{R}^3 . Sharp results for such families were recently established by Gan, Guo, Guth, Harris, Maldague and Wang [3]. This is the second key component in the proof of Theorem 1.5.

2 | PROOFS CONCERNING SL(2) LINES

In this section, we formalise the connection between SL(2) lines and horizontal lines. We also deduce our main result, Theorem 1.2, from Corollary 1.6.

Recall the *SL*(2) lines from Definition 1.1. We write $\mathcal{L}'_{SL(2)}$ for all the lines in $\mathcal{L}_{SL(2)}$, except for the *x*-axis, and lines of the form $L_{\alpha,\beta,\gamma,\delta}$ with $\delta = 0$. The difference between $\mathcal{L}_{SL(2)}$ and $\mathcal{L}'_{SL(2)}$ is the same as the difference between $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}'(\mathbb{H})$. Consider the map

$$\Xi(x, y, t) := (x, y, t/2).$$

We claim that Ξ maps the *SL*(2) lines to horizontal lines. More precisely:

Proposition 2.1. If $L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}'_{SL(2)}$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$, then

$$\Xi(L_{\alpha,\beta,\gamma,\delta}) = \ell_{(a,b,c)} \in \mathcal{L}'(\mathbb{H}), \tag{2.2}$$

where

$$\begin{cases} a = \gamma/\delta, \\ b = 1/\delta, \\ c = -\beta/(2\delta) \end{cases}$$

Proof. Fix $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$. Write $L_{\alpha,\beta,\gamma,\delta}(s) = (\alpha, \beta, 0) + (s\gamma, s\delta, s)$. It is a straightforward computation to check that

$$\Xi(L_{\alpha,\beta,\gamma,\delta}(s)) = \ell_{(a,b,c)}(\beta + s\delta), \qquad s \in \mathbb{R}$$

Since $\delta \neq 0$ by assumption, this completes the proof.

We are then prepared to prove Theorem 1.2.

Proof of Theorem 1.2. We may assume that $\mathcal{L} \subset \mathcal{L}'_{SL(2)}$, since the directions of the lines in $\mathcal{L}_{SL(2)} \setminus \mathcal{L}'_{SL(2)}$ are contained in the \mathcal{H}^2 null set $S^2 \cap \{(x, 0, t) : x, t \in \mathbb{R}\}$. Similarly, we may assume that \mathcal{L} contains no lines in the *xy*-plane; thus, every $L \in \mathcal{L}$ has the form $L = L_{\alpha,\beta,\gamma,\delta}$ for some $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$.

Since $\mathcal{L} \subset \mathcal{L}'_{SL(2)}$, we infer from Proposition 2.1 that $\Xi(\mathcal{L}) := \{\Xi(\ell) : \ell \in \mathcal{L}\} \subset \mathcal{L}'(\mathbb{H})$. We claim that

$$\mathcal{H}^2(\operatorname{dir}(\Xi(\mathcal{L}))) > 0. \tag{2.3}$$

According to Corollary 1.6, this will imply that

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) = \dim_{\mathrm{H}}\Xi(\cup\mathcal{L}) = \dim_{\mathrm{H}}(\cup\Xi(\mathcal{L})) = 3,$$

and complete the proof.

To verify (2.3), fix $L = L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}$. Then, by (2.2), we have $\Xi(L) = \ell_{(a,b,c)}$ with

$$\begin{cases} a = \gamma/\delta, \\ b = 1/\delta \\ c = -\beta/(2\delta). \end{cases}$$

We will use this information in the form of the following inclusion: writing $F(\gamma, \delta) := (\gamma/\delta, 1/\delta)$, we have

$$\{(a,b) \in \mathbb{R}^2 : \ell_{(a,b,c)} \in \Xi(\mathcal{L})\} \supset \{F(\gamma,\delta) : L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}\}.$$
(2.4)

Since $\mathcal{H}^2(\operatorname{dir}(\mathcal{L})) > 0$, and the direction of $L_{\alpha,\beta,\gamma,\delta} = (\alpha,\beta,0) + \operatorname{span}(\gamma,\delta,1)$ is determined by γ and δ , we know that

$$\mathcal{H}^{2}(\{(\gamma, \delta) \in \mathbb{R}^{2} : L_{\alpha, \beta, \gamma, \delta} \in \mathcal{L}\}) > 0.$$

It now follows from (2.4), and the fact that *F* is locally bilipschitz in the set $\mathbb{R}^2 \setminus \{(\gamma, \delta) : \delta = 0\}$ (since $|\det DF(\gamma, \delta)| = 1/\delta^3$), that also

$$\mathcal{H}^{2}(\{(a,b)\in\mathbb{R}^{2}:\ell_{(a,b,c)}\in\Xi(\mathcal{L})\}) \geq \mathcal{H}^{2}(\{F(\gamma,\delta):L_{\alpha,\beta,\gamma,\delta}\in\mathcal{L}\}) > 0.$$

Since the direction of $\ell_{(a,b,c)} = (b,0,c) + \operatorname{span}(a,1,b/2)$ is determined by (a,b), we may now infer that $\mathcal{H}^2(\operatorname{dir}(\Xi(\mathcal{L}))) > 0$, as claimed in (2.3).

3 | PROOFS CONCERNING HORIZONTAL LINES

We start by proving Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality, we may assume that $\mathcal{L} = \ell(P)$ is a Borel set of lines, that is, $P \subset \mathbb{R}^3$ is a Borel set. For the full details of this reduction, see [7, Section 3] or [1, Theorem 7.9]. The idea is that we can first replace $\cup \mathcal{L}$ by a G_{δ} -set $G \supset \cup \mathcal{L}$ without affecting $\dim_{\mathrm{H}}(\cup \mathcal{L})$. Then, it is easy to check that the set of parameters $P' := \{p \in \mathbb{R}^3 : \ell(p) \subset G\}$ is a Borel set with $P' \supset P$, in particular $\dim_{\mathrm{H}} P' \ge \dim_{\mathrm{H}} P$. Finally, writing $\mathcal{L}' := \ell(P')$, we have

$$\dim_{\mathrm{H}}(\cup \mathcal{L}) = \dim_{\mathrm{H}} G \ge \dim_{\mathrm{H}}(\cup \mathcal{L}').$$

So, if the result is known for Borel sets of lines, it follows for \mathcal{L} .

Write $\mathcal{L} := \ell(P)$, where $P \subset \mathbb{R}^3$ is Borel. Write also

$$K_{y} := \left\{ \left(ay + b, \frac{b}{2}y + c \right) : (a, b, c) \in P \right\}, \qquad y \in \mathbb{R},$$

and note that K_v is a 'slice' of $\cup \mathcal{L}$ with the plane $\mathbb{W}_v := \{(x, y, t) : x, t \in \mathbb{R}\}$:

 $(\cup \mathcal{L}) \cap \mathbb{W}_{v} \cong K_{v},$

where ' \cong ' refers to the isometry $\iota_v : \mathbb{R}^2 \to \mathbb{W}_v$, defined by $\iota_v(x, t) = (x, y, t)$. In order to prove that

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) \ge \min\{\dim_{\mathrm{H}}\mathcal{L}+1,3\},\tag{3.1}$$

we now claim that

$$\dim_{\mathrm{H}} K_{v} = \min\{\dim_{\mathrm{H}} P, 2\} \quad \text{for a.e. } y \in \mathbb{R}.$$
(3.2)

If \mathcal{L} consisted of line segments of positive length, and not full lines, then we would have to modify (3.2) as follows: for every $\epsilon > 0$, there exists an interval $I \subset \mathbb{R}$ of positive length such that $\dim_{\mathrm{H}} K_{y} \ge \min\{\dim_{\mathrm{H}} P - \epsilon, 2\}$ for a.e. $y \in I$. This interval would (be chosen to) consist of points $y \in \mathbb{R}$ with the property that the plane \mathbb{W}_{y} intersects a family of segments corresponding to a $(\dim_{\mathrm{H}} P - \epsilon)$ -dimensional Borel subset $P' \subset P$. We refer the reader to [7, Section 3] for a very similar argument.

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Clearly, (3.1) follows from (3.2) by the 'Fubini inequality' for Hausdorff measures (hence dimension), see [1, Theorem 5.8] or [8, Theorem 7.7]. To prove (3.2), we define

$$v(y) := (y, 1, 0)$$
 and $w(y) := (0, y/2, 1), y \in \mathbb{R}$.

Then, we note that for $y \in \mathbb{R}$ fixed, K_y can be expressed as

$$K_{y} = \{(\langle p, v(y) \rangle, \langle p, w(y) \rangle) : p \in P\}$$
$$= \{(\langle \pi_{V_{y}}(p), v(y) \rangle, \langle \pi_{V_{y}}(p), w(y) \rangle) : p \in P\},$$
(3.3)

where ' $\langle \cdot, \cdot \rangle$ ' is the Euclidean dot product and π_{V_y} the Euclidean orthogonal projection from \mathbb{R}^3 onto the plane

$$V_{v} := \operatorname{span}(\{v(y), w(y)\})$$

It is then easy to see that

$$\dim_{\mathrm{H}} K_{y} = \dim_{\mathrm{H}} \pi_{V_{y}}(P), \quad y \in \mathbb{R}.$$
(3.4)

Indeed, expression (3.3) shows that K_y can be written as the image of $\pi_{V_y}(P)$ under the linear map

$$M_{v}: V_{v} \to \mathbb{R}^{2}, \quad M_{v}(q) = (\langle q, v(y) \rangle, \langle q, w(y) \rangle),$$

and thus, $\dim_{\mathrm{H}} K_{y} = \dim_{\mathrm{H}} M_{y}(\pi_{V_{y}}(P))$. Moreover, $\dim_{\mathrm{H}} M_{y}(\pi_{V_{y}}(P)) = \dim_{\mathrm{H}} \pi_{V_{y}}(P)$ holds as the linear map M_{y} is invertible by the linear independence of v(y) and w(y). Hence, (3.4) holds as desired.

To complete the proof, we claim that

$$\dim_{\mathrm{H}} \pi_{V_{\mathrm{u}}}(P) = \min\{\dim_{\mathrm{H}} P, 2\} \quad \text{for a.e. } y \in \mathbb{R}.$$
(3.5)

The idea is that $\{\pi_{V_y}\}_{y \in \mathbb{R}}$ is a one-parameter family of orthogonal projections to planes in \mathbb{R}^3 which satisfies the hypotheses of [3, Corollary 1].

Which planes are the planes V_{v} ? Note that

$$v(y) \times w(y) = (1, -y, y^2/2) =: e_y.$$

Thus, $V_y = e_y^{\perp}$. Moreover, the lines $\ell_y := \text{span}(e_y)$ are all contained in a 45° rotated copy of the light cone

$$C := \{ (x, y, t) \in \mathbb{R}^3 : t^2 = x^2 + y^2 \},\$$

see [2, Section 2.2] for the details. This implies that the projections $\{\pi_{V_y}\}_{y \in \mathbb{R}}$ satisfy the curvature condition [3, (1)]. In fact, up to the rotation by 45°, this family of projections is precisely the 'model

example' mentioned just below [3, (1)]. Therefore, (3.5) follows from [3, Corollary 1], and the proof is complete.

We conclude the paper by proving Corollary 1.6.

Proof of Corollary 1.6. Firstly, note that $\mathcal{H}^2(\operatorname{dir}(\mathcal{L} \cap \mathcal{L}'(\mathbb{H}))) > 0$. This is because $\operatorname{dir}(\mathcal{L}'(\mathbb{H}))$ contains all the directions on S^2 , except for those contained in the null set $\{(x, 0, t) : x, t \in \mathbb{R}\}$. Therefore, we may assume that $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$.

Write $\mathcal{L} = \ell(P)$, where $P \subset \mathbb{R}^3$. Recall that

$$\mathcal{L} = \ell(P) = \left\{ \left(as + b, s, \frac{b}{2}s + c \right) : s \in \mathbb{R}, (a, b, c) \in P \right\}$$
$$= \left\{ (b, 0, c) + \operatorname{span}\left(a, 1, \frac{b}{2}\right) : (a, b, c) \in P \right\}.$$

Since $\mathcal{H}^2(\operatorname{dir}(\mathcal{L})) > 0$ by assumption, we see that

$$\mathcal{H}^{2}\left(\left\{\left(a,\frac{b}{2}\right):(a,b,c)\in P\right\}\right)>0,$$

and consequently, dim_H $P \ge 2$. The claim now follows from Theorem 1.5.

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