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RESEARCH ARTICLE

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A note on Kakeya sets of horizontal and $SL(2)$ lines

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Abstract

We consider unions of $SL(2)$ lines in $\mathbb{R}^3$. These are lines of the form

$$L = (a, b, 0) + \operatorname{span}(c, d, 1),$$

where $ad - bc = 1$. We show that if $\mathcal{L}$ is a Kakeya set of $SL(2)$ lines, then the union $\cup\mathcal{L}$ has Hausdorff dimension 3. This answers a question of Wang and Zahl. The $SL(2)$ lines can be identified with *horizontal lines* in the first Heisenberg group, and we obtain the main result as a corollary of a more general statement concerning unions of horizontal lines. This statement is established via a point-line duality principle between horizontal and *conical* lines in $\mathbb{R}^3$, combined with recent work on *restricted families of projections to planes*, due to Gan, Guo, Guth, Harris, Maldague and Wang. Our result also has a corollary for Nikodym sets associated with horizontal lines, which answers a special case of a question of Kim.

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Contents

1 | INTRODUCTION

The purpose of this note is to study the Hausdorff dimension of unions of $SL(2)$ *lines* in $\mathbb{R}^3$. Here is the definition of $SL(2)$ lines, following [10].

Definition 1.1 ($\mathcal{L}_{SL(2)}$). The family $\mathcal{L}_{SL(2)}$ consists of the following lines $L \subset \mathbb{R}^3$. Either L is a line contained in the xy-plane, and $0 \in L$, or then

$$L := L_{\alpha,\beta,\gamma,\delta} := (\alpha, \beta, 0) + \mathrm{span}(\gamma, \delta, 1),$$

where $\alpha\delta - \beta\gamma = 1$.

We also use the following notation. If $\mathcal{L}$ is any family of lines in $\mathbb{R}^3$, we write $\mathrm{dir}(\mathcal{L}) := \{e \in S^2 : \ell \,\|\, \mathrm{span}(e) \text{ for some } \ell \in \mathcal{L}\}$. Here is the main result of the note.

Theorem 1.2. *Let $\mathcal{L} \subset \mathcal{L}_{SL(2)}$ be a set with $\mathcal{H}^2(\mathrm{dir}(\mathcal{L})) > 0$. Then*

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) = 3.$$

Here '$\dim_{\mathrm{H}}(\cup\mathcal{L})$' is the Euclidean Hausdorff dimension of the union $\cup\mathcal{L} := \bigcup_{\ell\in\mathcal{L}} \ell$.

Remark 1.3. Theorem 1.2 answers a question posed by Wang and Zahl in [10, Section 1.2]. This question was motivated by earlier work of Katz and Zahl [5]. Theorem 1.2 continues to hold if the full lines in $\mathcal{L}$ are replaced by line segments of positive length. We will discuss this briefly below (3.2).

Katz, Wu and Zahl [4] also proved Theorem 1.2 independently, using a different method.

The $SL(2)$ lines are essentially (up to a change in coordinates) the same as *horizontal lines in the first Heisenberg group* $\mathbb{H} = (\mathbb{R}^3, *)$, viewed as subsets of $\mathbb{R}^3$ (see Proposition 2.1). We will infer Theorem 1.2 from a more general statement concerning unions of these horizontal lines, Theorem 1.5 below. We first need to define the concepts properly.

The family of all horizontal lines is denoted by $\mathcal{L}(\mathbb{H})$. The 'Heisenberg' definition of these lines is the following. Let $\Pi_0 := \{(x, y, 0) : x, y \in \mathbb{R}\}$ be the xy-plane, and for $p \in \mathbb{R}^3$, let $\Pi_p := p * H_0$ be the *left translate* of Π_0 by the Heisenberg group product

$$(x, y, t) * (x', y', t') = \left(x + x', y + y', t + t' + \tfrac{1}{2}(xy' - x'y)\right).$$

Then, $\mathcal{L}(\mathbb{H})$ consists of all the lines in Π_p (for every $p \in \mathbb{R}^3$) which contain the point p.

The family $\mathcal{L}(\mathbb{H})$ is a three-dimensional submanifold of the full (four-dimensional) family of lines in $\mathbb{R}^3$. In fact, the definition above of horizontal lines will not be used in the note: rather, we focus attention on the following parameterised subset of $\mathcal{L}(\mathbb{H})$:

$$\mathcal{L}'(\mathbb{H}) = \{\ell_{(a,b,c)} : (a,b,c) \in \mathbb{R}^3\},$$

where

$$\ell_{(a,b,c)} = \left\{(as+b, s, \tfrac{b}{2}s + c) : s \in \mathbb{R}\right\}.$$

The subset $\mathcal{L}'(\mathbb{H})$ consists of all elements of $\mathcal{L}(\mathbb{H})$, except for those contained in some translate of the plane $\mathbb{W}_0 := \{(x,0,t) : x,t \in \mathbb{R}\}$. By definition, every set $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$ can be written as

$$\mathcal{L} = \ell(P) := \{\ell_{(a,b,c)} : (a,b,c) \in P\}$$

for some set $P \subset \mathbb{R}^3$. This identification of $\mathcal{L}'(\mathbb{H})$ with $\mathbb{R}^3$ allows us to transport notions like 'Borel set' and 'dimension' from $\mathbb{R}^3$ to corresponding notions for subsets of $\mathcal{L}'(\mathbb{H})$.

Definition 1.4. Let $\mathcal{L} = \ell(P) \subset \mathcal{L}'(\mathbb{H})$. We say that $\mathcal{L}$ is a Borel set if $P \subset \mathbb{R}^3$ is a Borel set. We define $\dim_{\mathrm{H}} \mathcal{L} := \dim_{\mathrm{H}} P$, where '$\dim_{\mathrm{H}} P$' refers to the Euclidean Hausdorff dimension of $P \subset \mathbb{R}^3$.

Now we can state our main result about unions of horizontal lines.

Theorem 1.5. *Let $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$. Then,*

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) = \min\{\dim_{\mathrm{H}} \mathcal{L} + 1, 3\}.$$

The following corollary implies Theorem 1.2, as we will verify in Section 2.

Corollary 1.6. *Let $\mathcal{L} \subset \mathcal{L}(\mathbb{H})$ with $\mathcal{H}^2(\mathrm{dir}(\mathcal{L})) > 0$. Then,*

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) = 3.$$

Remark 1.7. Theorem 1.5 and Corollary 1.6 continue to hold if full lines are replaced by line segments of positive length, see the discussion below (3.2). Thus, if $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$, and every line $\ell \in \mathcal{L}$ contains a segment $I(\ell) \subset \ell$ of positive length, then

$$\dim_{\mathrm{H}}\left(\bigcup_{\ell\in\mathcal{L}} I(\ell)\right) = \min\{\dim_{\mathrm{H}} \mathcal{L} + 1, 3\}. \tag{1.8}$$

1.1 | Nikodym sets associated with horizontal lines

Theorem 1.5 easily yields information about the dimension of *Nikodym sets* associated with horizontal lines. A set $N \subset \mathbb{R}^3$ is called an $\mathcal{L}(\mathbb{H})$*-Nikodym set* if for every $p \in \mathbb{R}^3$ (or more generally every $p \in \mathbb{R}^3$ in a measurable set of positive measure $\Omega \subset \mathbb{R}^3$), there exists a line $\ell_p \in \mathcal{L}(\mathbb{H})$ containing p such that N contains a line segment $I_p \subset \ell_p$ of positive length.

Corollary 1.9. *Every $\mathcal{L}(\mathbb{H})$-Nikodym set $N \subset \mathbb{R}^3$ has* $\dim_{\mathrm{H}} N = 3$.

It is well known that bounds for Kakeya sets yield bounds for Nikodym sets: we only repeat the standard details below for the reader's convenience. For a similar argument in the case of classical Kakeya and Nikodym sets, see [9, Section 11.3].

Proof of Corollary 1.9. We may assume without loss of generality that all the lines $\ell_p \in \mathcal{L}(\mathbb{H})$ appearing in the definition of 'N' lie in $\mathcal{L}'(\mathbb{H})$. Namely, if this is true for a positive measure subset of the points $p \in \Omega$, we simply replace Ω by that subset. If this fails for Lebesgue almost every point $p \in \Omega$, then we apply a rotation R of, say, 10° around the t-axis to the objects Ω, N, and the lines ℓ_p, $p \in \Omega$. Rotations around the t-axis preserve $\mathcal{L}(\mathbb{H})$, and the measure and dimension of Ω and N. After this procedure, we moreover have $\ell_p \in \mathcal{L}'(\mathbb{H})$ for a.e. $p \in R(\Omega)$.

Using Fubini's theorem, start by picking $y_0 \in \mathbb{R}$ such that $\mathcal{H}^2(\Omega \cap \mathbb{W}_{y_0}) > 0$. Here, $\mathbb{W}_y = \{(x, y, t) : x, t \in \mathbb{R}^3\}$ for $y \in \mathbb{R}$. By assumption, for every $p = (x, y_0, t) \in \Omega \cap \mathbb{W}_{y_0}$, there exists a line

$$\ell_p := \ell_{(a(p),b(p),c(p))} \in \mathcal{L}'(\mathbb{H})$$

containing p such that N contains a line segment $I_p \subset \ell_p$ of positive length.

Now, note that the map $(a, b, c) \mapsto \Psi(a, b, c) = (ay_0 + b, y_0, \frac{b}{2}y_0 + c)$ is Lipschitz, and

$$\Omega \cap \mathbb{W}_{y_0} \subset \Psi(\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\}).$$

(This is because the lines ℓ_p contain the points $p \in \Omega \cap \mathbb{W}_{y_0}$.) Therefore,

$$\dim_{\mathrm{H}}\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\} \geqslant \dim_{\mathrm{H}}(\Omega \cap \mathbb{W}_{y_0}) = 2.$$

In particular, the set of lines $\mathcal{L} := \{\ell_p : p \in \Omega\} \subset \mathcal{L}'(\mathbb{H})$ has $\dim_{\mathrm{H}} \mathcal{L} \geqslant 2$ by definition. Therefore, it follows from Theorem 1.5, or to be precise (1.8), that

$$\dim_{\mathrm{H}} N \geqslant \dim_{\mathrm{H}}\left(\bigcup_{p\in\Omega} I_p\right) = 3.$$

This completes the proof. □

Remark 1.10. Nikodym set for 'restricted' families of lines was earlier considered by Kim [6]. Corollary 1.9 answers (a special case of) a question raised on [6, p. 478]. We elaborate on this a little further. The paper [6] considered general families of 2-planes $p \mapsto \Pi_{\mathfrak{a}}(p) \subset \mathbb{R}^3$, where $p \mapsto \mathfrak{a}(p)$ is a non-vanishing measurable vector field, and

$$p \in \Pi_{\mathfrak{a}}(p) \quad \text{and} \quad \operatorname{span}(\mathfrak{a}(p)) = \Pi_{\mathfrak{a}}(p)^{\perp}.$$

One can associate Nikodym sets $N \subset \mathbb{R}^3$ to such a plane family, as follows: for every $p \in \mathbb{R}^3$, the requirement is that there exists a line $\ell_p \subset \mathbb{R}^3$ satisfying

$$p \in \ell_p \subset \Pi_{\mathfrak{a}}(p),$$

and a non-trivial segment $I_p \subset N \cap \ell_p$. How small can such a Nikodym set $N \subset \mathbb{R}^3$ be? In [6], Kim approached the question via maximal function estimates, and his results depend on the properties of the vector field $\mathfrak{a}$. Kim considered vector fields $\mathfrak{a}$ of the form

$$\mathfrak{a}(p) = (a_{11}p_1 + a_{21}p_2, a_{12}p_1 + a_{22}p_2, -1), \qquad p = (p_1, p_2, p_3) \in \mathbb{R}^3,$$

and defined the 'discriminant' $D_{\mathfrak{a}} = (a_{12} + a_{21})^2 - 4a_{11}a_{22}$. In [6, Corollary 1, p. 478], it was shown that the dimension of N equals 3 if $D_{\mathfrak{a}} \neq 0$. Right after the corollary, the question is raised, what happens in the situation $D_{\mathfrak{a}} = 0$.

Now, recall the definition of horizontal lines $\mathcal{L}(\mathbb{H})$: these were the lines contained in the planes $\Pi_p = p * \Pi_0$, and passing through p. The planes Π_p fit in the framework of [6], choosing $\mathfrak{a}(p) = (-p_2/2, p_1/2, -1)$, or $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$. In particular, $D_{\mathfrak{a}} = 0$. Also, the $\mathcal{L}(\mathbb{H})$-Nikodym sets defined above Corollary 1.9 are the same as the Nikodym sets of [6] associated with the planes $\Pi_p = p * \Pi_0$. Thus, Corollary 1.9 covers the special case $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ of the problem raised on [6, p. 478].

1.2 | Ingredients of the proof

The proof of Theorem 1.5 is based on two ingredients. The first one is a *point-line duality* between horizontal lines and *conical lines* in $\mathbb{R}^3$, namely translates of lines contained in the light cone $\{(x, y, t) : t^2 = x^2 + y^2\}$. This duality was formalised in our paper [2], although it was already implicit in the work [7] of Liu. Using this point-line duality, Kakeya-type problems for horizontal lines can be transformed into projection problems in $\mathbb{R}^3$. These projection problems concern 'restricted' families of projections to planes in $\mathbb{R}^3$. Sharp results for such families were recently established by Gan, Guo, Guth, Harris, Maldague and Wang [3]. This is the second key component in the proof of Theorem 1.5.

2 | PROOFS CONCERNING $SL(2)$ LINES

In this section, we formalise the connection between $SL(2)$ lines and horizontal lines. We also deduce our main result, Theorem 1.2, from Corollary 1.6.

Recall the $SL(2)$ lines from Definition 1.1. We write $\mathcal{L}'_{SL(2)}$ for all the lines in $\mathcal{L}_{SL(2)}$, except for the x-axis, and lines of the form $L_{\alpha,\beta,\gamma,\delta}$ with $\delta = 0$. The difference between $\mathcal{L}_{SL(2)}$ and $\mathcal{L}'_{SL(2)}$ is the same as the difference between $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}'(\mathbb{H})$. Consider the map

$$\Xi(x, y, t) := (x, y, t/2).$$

We claim that Ξ maps the $SL(2)$ lines to horizontal lines. More precisely:

Proposition 2.1. *If* $L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}'_{SL(2)}$ *with* $\delta \neq 0$ *and* $\alpha\delta - \beta\gamma = 1$, *then*

$$\Xi(L_{\alpha,\beta,\gamma,\delta}) = \ell_{(a,b,c)} \in \mathcal{L}'(\mathbb{H}), \tag{2.2}$$

where

$$\begin{cases} a = \gamma/\delta, \\ b = 1/\delta, \\ c = -\beta/(2\delta). \end{cases}$$

Proof. Fix $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$. Write $L_{\alpha,\beta,\gamma,\delta}(s) = (\alpha, \beta, 0) + (s\gamma, s\delta, s)$. It is a straightforward computation to check that

$$\Xi(L_{\alpha,\beta,\gamma,\delta}(s)) = \ell_{(a,b,c)}(\beta + s\delta), \qquad s \in \mathbb{R}.$$

Since $\delta \neq 0$ by assumption, this completes the proof. □

We are then prepared to prove Theorem 1.2.

Proof of Theorem 1.2. We may assume that $\mathcal{L} \subset \mathcal{L}'_{SL(2)}$, since the directions of the lines in $\mathcal{L}_{SL(2)} \setminus \mathcal{L}'_{SL(2)}$ are contained in the $\mathcal{H}^2$ null set $S^2 \cap \{(x, 0, t) : x, t \in \mathbb{R}\}$. Similarly, we may assume that $\mathcal{L}$ contains no lines in the xy-plane; thus, every $L \in \mathcal{L}$ has the form $L = L_{\alpha,\beta,\gamma,\delta}$ for some $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$.

Since $\mathcal{L} \subset \mathcal{L}'_{SL(2)}$, we infer from Proposition 2.1 that $\Xi(\mathcal{L}) := \{\Xi(\ell) : \ell \in \mathcal{L}\} \subset \mathcal{L}'(\mathbb{H})$. We claim that

$$\mathcal{H}^2(\mathrm{dir}(\Xi(\mathcal{L}))) > 0. \tag{2.3}$$

According to Corollary 1.6, this will imply that

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) = \dim_{\mathrm{H}} \Xi(\cup\mathcal{L}) = \dim_{\mathrm{H}}(\cup\Xi(\mathcal{L})) = 3,$$

and complete the proof.

To verify (2.3), fix $L = L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}$. Then, by (2.2), we have $\Xi(L) = \ell_{(a,b,c)}$ with

$$\begin{cases} a = \gamma/\delta, \\ b = 1/\delta \\ c = -\beta/(2\delta). \end{cases}$$

We will use this information in the form of the following inclusion: writing $F(\gamma, \delta) := (\gamma/\delta, 1/\delta)$, we have

$$\{(a, b) \in \mathbb{R}^2 : \ell_{(a,b,c)} \in \Xi(\mathcal{L})\} \supset \{F(\gamma, \delta) : L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}\}. \tag{2.4}$$

Since $\mathcal{H}^2(\mathrm{dir}(\mathcal{L})) > 0$, and the direction of $L_{\alpha,\beta,\gamma,\delta} = (\alpha, \beta, 0) + \mathrm{span}(\gamma, \delta, 1)$ is determined by γ and δ, we know that

$$\mathcal{H}^2(\{(\gamma, \delta) \in \mathbb{R}^2 : L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}\}) > 0.$$

It now follows from (2.4), and the fact that F is locally bilipschitz in the set $\mathbb{R}^2 \setminus \{(\gamma,\delta) : \delta = 0\}$ (since $|\det DF(\gamma,\delta)| = 1/\delta^3$), that also

$$\mathcal{H}^2(\{(a,b) \in \mathbb{R}^2 : \ell_{(a,b,c)} \in \Xi(\mathcal{L})\}) \geqslant \mathcal{H}^2(\{F(\gamma,\delta) : L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}\}) > 0.$$

Since the direction of $\ell_{(a,b,c)} = (b,0,c) + \mathrm{span}(a,1,b/2)$ is determined by (a,b), we may now infer that $\mathcal{H}^2(\mathrm{dir}(\Xi(\mathcal{L}))) > 0$, as claimed in (2.3). □

3 | PROOFS CONCERNING HORIZONTAL LINES

We start by proving Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality, we may assume that $\mathcal{L} = \ell(P)$ is a Borel set of lines, that is, $P \subset \mathbb{R}^3$ is a Borel set. For the full details of this reduction, see [7, Section 3] or [1, Theorem 7.9]. The idea is that we can first replace $\cup\mathcal{L}$ by a G_δ-set $G \supset \cup\mathcal{L}$ without affecting $\dim_{\mathrm{H}}(\cup\mathcal{L})$. Then, it is easy to check that the set of parameters $P' := \{p \in \mathbb{R}^3 : \ell(p) \subset G\}$ is a Borel set with $P' \supset P$, in particular $\dim_{\mathrm{H}} P' \geqslant \dim_{\mathrm{H}} P$. Finally, writing $\mathcal{L}' := \ell(P')$, we have

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) = \dim_{\mathrm{H}} G \geqslant \dim_{\mathrm{H}}(\cup\mathcal{L}').$$

So, if the result is known for Borel sets of lines, it follows for $\mathcal{L}$.

Write $\mathcal{L} := \ell(P)$, where $P \subset \mathbb{R}^3$ is Borel. Write also

$$K_y := \left\{ \left(ay + b, \tfrac{b}{2}y + c \right) : (a,b,c) \in P \right\}, \qquad y \in \mathbb{R},$$

and note that K_y is a 'slice' of $\cup\mathcal{L}$ with the plane $\mathbb{W}_y := \{(x,y,t) : x,t \in \mathbb{R}\}$:

$$(\cup\mathcal{L}) \cap \mathbb{W}_y \cong K_y,$$

where '≅' refers to the isometry $\iota_y : \mathbb{R}^2 \to \mathbb{W}_y$, defined by $\iota_y(x,t) = (x,y,t)$. In order to prove that

$$\dim_{\mathrm{H}}(\cup\mathcal{L}) \geqslant \min\{\dim_{\mathrm{H}} \mathcal{L} + 1, 3\}, \tag{3.1}$$

we now claim that

$$\dim_{\mathrm{H}} K_y = \min\{\dim_{\mathrm{H}} P, 2\} \quad \text{for a.e. } y \in \mathbb{R}. \tag{3.2}$$

If $\mathcal{L}$ consisted of line segments of positive length, and not full lines, then we would have to modify (3.2) as follows: for every $\epsilon > 0$, there exists an interval $I \subset \mathbb{R}$ of positive length such that $\dim_{\mathrm{H}} K_y \geqslant \min\{\dim_{\mathrm{H}} P - \epsilon, 2\}$ for a.e. $y \in I$. This interval would (be chosen to) consist of points $y \in \mathbb{R}$ with the property that the plane $\mathbb{W}_y$ intersects a family of segments corresponding to a $(\dim_{\mathrm{H}} P - \epsilon)$-dimensional Borel subset $P' \subset P$. We refer the reader to [7, Section 3] for a very similar argument.

Clearly, (3.1) follows from (3.2) by the 'Fubini inequality' for Hausdorff measures (hence dimension), see [1, Theorem 5.8] or [8, Theorem 7.7]. To prove (3.2), we define

$$v(y) := (y, 1, 0) \quad \text{and} \quad w(y) := (0, y/2, 1), \qquad y \in \mathbb{R}.$$

Then, we note that for $y \in \mathbb{R}$ fixed, K_y can be expressed as

$$\begin{aligned} K_y &= \{(\langle p, v(y)\rangle, \langle p, w(y)\rangle) : p \in P\} \\ &= \{(\langle \pi_{V_y}(p), v(y)\rangle, \langle \pi_{V_y}(p), w(y)\rangle) : p \in P\}, \end{aligned} \tag{3.3}$$

where '$\langle \cdot, \cdot \rangle$' is the Euclidean dot product and π_{V_y} the Euclidean orthogonal projection from $\mathbb{R}^3$ onto the plane

$$V_y := \operatorname{span}(\{v(y), w(y)\}).$$

It is then easy to see that

$$\dim_{\mathrm{H}} K_y = \dim_{\mathrm{H}} \pi_{V_y}(P), \quad y \in \mathbb{R}. \tag{3.4}$$

Indeed, expression (3.3) shows that K_y can be written as the image of $\pi_{V_y}(P)$ under the linear map

$$M_y : V_y \to \mathbb{R}^2, \quad M_y(q) = (\langle q, v(y)\rangle, \langle q, w(y)\rangle),$$

and thus, $\dim_{\mathrm{H}} K_y = \dim_{\mathrm{H}} M_y(\pi_{V_y}(P))$. Moreover, $\dim_{\mathrm{H}} M_y(\pi_{V_y}(P)) = \dim_{\mathrm{H}} \pi_{V_y}(P)$ holds as the linear map M_y is invertible by the linear independence of $v(y)$ and $w(y)$. Hence, (3.4) holds as desired.

To complete the proof, we claim that

$$\dim_{\mathrm{H}} \pi_{V_y}(P) = \min\{\dim_{\mathrm{H}} P, 2\} \quad \text{for a.e. } y \in \mathbb{R}. \tag{3.5}$$

The idea is that $\{\pi_{V_y}\}_{y\in\mathbb{R}}$ is a one-parameter family of orthogonal projections to planes in $\mathbb{R}^3$ which satisfies the hypotheses of [3, Corollary 1].

Which planes are the planes V_y? Note that

$$v(y) \times w(y) = (1, -y, y^2/2) =: e_y.$$

Thus, $V_y = e_y^{\perp}$. Moreover, the lines $\ell_y := \operatorname{span}(e_y)$ are all contained in a 45° rotated copy of the light cone

$$C := \{(x, y, t) \in \mathbb{R}^3 : t^2 = x^2 + y^2\},$$

see [2, Section 2.2] for the details. This implies that the projections $\{\pi_{V_y}\}_{y\in\mathbb{R}}$ satisfy the curvature condition [3, (1)]. In fact, up to the rotation by 45°, this family of projections is precisely the 'model example' mentioned just below [3, (1)]. Therefore, (3.5) follows from [3, Corollary 1], and the proof is complete. □

We conclude the paper by proving Corollary 1.6.

Proof of Corollary 1.6. Firstly, note that $\mathcal{H}^2(\mathrm{dir}(\mathcal{L} \cap \mathcal{L}'(\mathbb{H}))) > 0$. This is because $\mathrm{dir}(\mathcal{L}'(\mathbb{H}))$ contains all the directions on S^2, except for those contained in the null set $\{(x, 0, t) : x, t \in \mathbb{R}\}$. Therefore, we may assume that $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$.

Write $\mathcal{L} = \ell(P)$, where $P \subset \mathbb{R}^3$. Recall that

$$\begin{aligned}\mathcal{L} = \ell(P) &= \left\{ \left(as + b, s, \tfrac{b}{2}s + c \right) : s \in \mathbb{R}, (a, b, c) \in P \right\} \\ &= \left\{ (b, 0, c) + \mathrm{span}\left(a, 1, \tfrac{b}{2} \right) : (a, b, c) \in P \right\}.\end{aligned}$$

Since $\mathcal{H}^2(\mathrm{dir}(\mathcal{L})) > 0$ by assumption, we see that

$$\mathcal{H}^2\left(\left\{ \left(a, \tfrac{b}{2} \right) : (a, b, c) \in P \right\} \right) > 0,$$

and consequently, $\dim_{\mathrm{H}} P \geqslant 2$. The claim now follows from Theorem 1.5. □

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