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INVERSE PROBLEMS FOR SEMILINEAR ELLIPTIC PDE WITH MEASUREMENTS AT A SINGLE POINT

MIKKO SALO AND LEO TZOU

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ABSTRACT. We consider the inverse problem of determining a potential in a semilinear elliptic equation from the knowledge of the Dirichlet-to-Neumann map. For bounded Euclidean domains we prove that the potential is uniquely determined by the Dirichlet-to-Neumann map measured at a single boundary point, or integrated against a fixed measure. This result is valid even when the Dirichlet data is only given on a small subset of the boundary. We also give related uniqueness results on Riemannian manifolds.

1. INTRODUCTION

In this article we study inverse problems for semilinear elliptic equations, with measurements given by the nonlinear Dirichlet-to-Neumann map (DN map) measured at a single point or integrated against a fixed measure. The method is based on higher order linearizations of the DN map. This method was introduced in inverse problems for hyperbolic PDE in [KLU18] where a source-to-solution map was used. It was observed in [LLPMT22] that in the hyperbolic case it may be sufficient to measure a DN map integrated against a suitable fixed function. The work [Tzo21] proved a result showing that measurements of the source-to-solution map at a single point suffice (see [BKT21] for another single point measurement result).

The higher order linearization method in inverse problems for nonlinear elliptic PDE was introduced independently in [FO20] and [LLLS21a]. We note that the first linearization has been used extensively since the work [Isa93], see e.g. [IS94, IN95], and the second linearization had also been used in [Sun96, SU97, KN02, CNV19, AZ21]. The works [LLLS21b, KU20a, KU20b] studied related inverse problems for semilinear elliptic equations with partial data, with [LLST22] addressing fractional power nonlinearities. In [LZ20, KU22, CF21, KKU22, CFK+21] the authors study nonlinear conductivity or magnetic Schrödinger type equations. All these results use the nonlinear DN map with data given on open subsets of the boundary.

In this note we observe that in some of the elliptic results above it is enough to measure the DN map at a single point, or integrated against a fixed measure. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary, and let $m \geq 2$ be an

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integer. Consider the semilinear elliptic equation

$$\begin{cases} \Delta u + q(x)u^m = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q \in C^\alpha(\overline{\Omega})$ is a potential, and C^α with $0 < \alpha < 1$ denotes the space of α-Hölder continuous functions. Let $f \in U_\delta$, where

$$U_\delta := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\}.$$

If $\delta > 0$ is small enough there is a unique small solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ of (1.1), see e.g. [LLST22, Proposition 2.1]. One can then define the corresponding nonlinear DN map Λ_q by

$$\Lambda_q : U_\delta \to C^{1,\alpha}(\partial\Omega), \qquad f \mapsto \partial_\nu u_f|_{\partial\Omega},$$

where ∂_ν denotes the normal derivative on $\partial\Omega$. In [FO20, LLLS21a] it was proved that the full DN map Λ_q uniquely determines q. This was extended in [KU20b, LLLS21b] to the case where one knows $\Lambda_q(f)|_{\Gamma_1}$ for f supported in Γ_2 where $\Gamma_1, \Gamma_2 \subset \partial\Omega$ are open sets.

We show that it is enough to measure $\int_{\partial\Omega} \Lambda_q(f)\, d\mu$ for a fixed measure μ on $\partial\Omega$. When $\mu = \delta_{x_0}$ this corresponds to measurements at a fixed point.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary, let $m \geq 2$ be an integer, and let $\Gamma \subset \partial\Omega$ be a nonempty open set. Suppose that $\mu \not\equiv 0$ is a fixed measure on $\partial\Omega$. If $q_1, q_2 \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$ satisfy*

$$\int_{\partial\Omega} \Lambda_{q_1}(f)\, d\mu = \int_{\partial\Omega} \Lambda_{q_2}(f)\, d\mu \tag{1.2}$$

for all $f \in U_\delta$ with $\mathrm{supp}(f) \subset \Gamma$ where $\delta > 0$ is sufficiently small, then

$$q_1 = q_2 \text{ in } \Omega.$$

In particular, choosing $\mu = \delta_{x_0}$ for some fixed $x_0 \in \partial\Omega$, we see that the condition

$$\Lambda_{q_1}(f)(x_0) = \Lambda_{q_2}(f)(x_0) \qquad \text{for all } f \in U_\delta \text{ with } \mathrm{supp}(f) \subset \Gamma$$

implies that $q_1 = q_2$.

We can give a similar result for semilinear elliptic PDE on manifolds. Let (M, g) be a compact Riemannian manifold with smooth boundary, let $q \in C^\infty(M)$, and let $m \geq 2$. We consider the Dirichlet problem

$$\begin{cases} \Delta_g u + q(x)u^m = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases} \tag{1.3}$$

Again, if $U_\delta := \{f \in C^{2,\alpha}(\partial M) : \|f\|_{C^{2,\alpha}(\partial M)} < \delta\}$, then for any $f \in U_\delta$ with δ small enough the Dirichlet problem has a unique small solution $u \in C^{2,\alpha}(M)$ (see e.g. [LLLS21a, Proposition 2.1]). We may define the DN map

$$\Lambda_q : U_\delta \to C^{1,\alpha}(\partial M), \qquad f \mapsto \partial_\nu u_f|_{\partial M},$$

where ∂_ν denotes the normal derivative with respect to the metric g on ∂M. We have the following result where f can be supported on all of ∂M, but we only measure the DN map at a single point or integrated against a fixed measure.

Theorem 1.2. *Let (M, g) be a compact Riemannian n-manifold with smooth boundary, let $m \geq 2$ be an integer, and let $\mu \not\equiv 0$ be a fixed measure on ∂M. Assume that one of the following conditions is satisfied:*

(1) *(M, g) is transversally anisotropic as in* [LLLS21a, Definition 1.1], *and $m \geq 4$; or*
(2) *(M, g) is a complex manifold satisfying the conditions in* [GST19, Theorem 1.4].

If $q_1, q_2 \in C^\infty(M)$ are such that $q_1 = q_2$ to infinite order on ∂M and

$$\int_{\partial\Omega} \Lambda_{q_1}(f)\, d\mu = \int_{\partial\Omega} \Lambda_{q_2}(f)\, d\mu \tag{1.4}$$

for all $f \in U_\delta$ where $\delta > 0$ is sufficiently small, then $q_1 = q_2$ in M.

The proofs of Theorems 1.1–1.2 are based on the higher order linearization method in [FO20, LLLS21a]. From [LLLS21a, Proposition 2.2] one obtains the identity

$$\begin{aligned}\int_{\partial M} ((D^m\Lambda_{q_1})_0 - (D^m\Lambda_{q_2})_0)(f_1, \ldots, f_m) f_{m+1}\, dS \\ = -(m!) \int_M (q_1 - q_2) v_1 \cdots v_{m+1}\, dV\end{aligned} \tag{1.5}$$

where $(D^m\Lambda_q)_0$ denotes the mth Fréchet derivative on Λ_q at 0 considered as an m-linear form, f_j are Dirichlet data, and v_j are solutions of the linearized equation $\Delta_g v_j = 0$ in M with $v_j|_{\partial M} = f_j$. The single point measurement case formally corresponds to choosing $f_{m+1} = \delta_{x_0}$ with $x_0 \in \partial M$. The corresponding solution v_{m+1} is in $L^1(\Omega)$ but it is not bounded, and this will require some additional arguments.

If one has equality of the DN maps for q_1 and q_2 as in Theorems 1.1–1.2, the identity (1.5) implies that

$$\int_M f v_1 v_2\, dV = 0$$

where $f := (q_1 - q_2) v_3 \cdots v_m v_{m+1}$ and v_j are as above. We choose $v_3, \ldots, v_m$ to be smooth nonvanishing solutions, and v_{m+1} will be the (nonvanishing) $L^1(\Omega)$ solution whose Dirichlet data is a measure. It is then enough to show that $f = 0$, which will imply $q_1 = q_2$. For the partial data result in Theorem 1.1, we need the following extension given in [CGU21, Section 4] of the fundamental result of [DSFKSU09] on the linearized local Calderón problem that was originally proved for $f \in L^\infty(\Omega)$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary, and let $\Gamma \subset \partial\Omega$ be a nonempty open set. Suppose that $f \in L^1(\Omega)$ is such that*

$$\int_\Omega f v_1 v_2\, dx = 0$$

for all $v_j \in C^\infty(\overline{\Omega})$ solving $\Delta v_j = 0$ in Ω with $\mathrm{supp}(v_j|_{\partial\Omega}) \subset \Gamma$. Then $f = 0$ in Ω.

For Theorem 1.2 we will invoke the results in [LLLS21a, GST19] instead.

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2. Proof of Theorem 1.1

For the proof of Theorem 1.1, we give a lemma related to solving the Dirichlet problem when the boundary value is a finite Borel measure μ on $\partial\Omega$. We use the norm given by the total variation,

$$\|\mu\|_{\mathcal{M}(\partial\Omega)} = |\mu|(\partial\Omega) = \sup_{\|\varphi\|_{C(\partial\Omega)=1}} \left| \int_{\partial\Omega} \varphi \, d\mu \right|.$$

We say that $\Psi \in L^1(\Omega)$ solves the Dirichlet problem

$$\begin{cases} \Delta\Psi &= 0 \quad \text{in } \Omega, \\ \Psi &= \mu \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

if for any $w \in C^2(\overline{\Omega})$ with $w|_{\partial\Omega} = 0$ one has

$$\int_{\partial\Omega} \partial_\nu w \, d\mu = \int_\Omega (\Delta w)\Psi \, dx. \tag{2.2}$$

In fact, there is a solution in $L^r(\Omega)$ for $1 \leq r < \frac{n}{n-1}$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, and let μ be a finite complex Borel measure on $\partial\Omega$. Consider the function*

$$\Psi(x) = \int_{\partial\Omega} P(x,y) \, d\mu(y), \qquad x \in \Omega,$$

where $P(x,y)$ is the Poisson kernel for Δ in Ω. Then $\Psi \in L^r(\Omega)$ where $1 \leq r < \frac{n}{n-1}$, and it solves the Dirichlet problem (2.1).

Proof. By applying a partition of unity, boundary flattening transformations and convolution approximation, we can produce a sequence $\psi_j \in C^\infty(\partial\Omega)$ such that $\|\psi_j \, dS - \mu\|_{\mathcal{M}(\partial\Omega)} \to 0$. Let $\Psi_j \in C^\infty(\overline{\Omega})$ solve $\Delta\Psi_j = 0$ in Ω with $\Psi_j|_{\partial\Omega} = \psi_j$. If w is as in the statement of the lemma, integration by parts gives

$$\int_{\partial\Omega} (\partial_\nu w)\psi_j \, dS = \int_\Omega (\Delta w)\Psi_j \, dx.$$

It is thus sufficient to show that $\Psi \in L^r(\Omega)$ and $\Psi_j \to \Psi$ in $L^r(\Omega)$ for $1 \leq r < \frac{n}{n-1}$. We apply the Poisson kernel estimate (see e.g. [Kra05])

$$P(x,y) \leq \frac{C \operatorname{dist}(x,\partial\Omega)}{|x-y|^n} \leq \frac{C}{|x-y|^{n-1}}, \quad x \in \Omega, \ y \in \partial\Omega,$$

for some $C > 0$. If $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \delta\}$, the Minkowski inequality in integral form gives

$$\begin{aligned} \|\Psi(x)\|_{L^r(\Omega_\delta)} &\leq \int_{\partial\Omega} \|P(\,\cdot\,,y)\|_{L^r(\Omega_\delta)} \, d|\mu|(y) \\ &\leq \left[\sup_{y\in\partial\Omega} \left(\int_{\Omega_\delta} \frac{C}{|x-y|^{(n-1)r}} \, dx \right)^{1/r} \right] \|\mu\|_{\mathcal{M}(\partial\Omega)} \,. \end{aligned}$$

The quantity in brackets is finite uniformly over $\delta > 0$ when $r < \frac{n}{n-1}$. Thus we may let $\delta \to 0$ to obtain that $\Psi \in L^r(\Omega)$. Applying the same argument to

$$\Psi_j(x) - \Psi(x) = \int_{\partial\Omega} P(x,y)(\psi_j(y)\,dS(y) - d\mu(y))$$

shows that $\Psi_j \to \Psi$ in $L^r(\Omega)$. □

Proof of Theorem 1.1. Let first $q \in C^\alpha(\overline{\Omega})$ be fixed. Consider Dirichlet data of the form $f_\varepsilon = \varepsilon_1 h_1 + \ldots + \varepsilon_m h_m$ where $h_j \in C^\infty(\partial\Omega)$ satisfy $\mathrm{supp}(h_j) \subset \Gamma$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ where ε_j are sufficiently small. Let u_ε be the solution of (1.1) with Dirichlet data f_ε. By [LLST22, Proposition 2.1] the map $\varepsilon \mapsto u_\varepsilon$ is smooth. By uniqueness of small solutions one has $u_0 = 0$, and by differentiating (1.1) with respect to ε_j one has $\partial_{\varepsilon_j} u_\varepsilon|_{\varepsilon=0} = v_j$ where v_j is the solution of

$$\begin{cases} \Delta v_j &= 0 \quad \text{in } \Omega, \\ v_j &= h_j \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Moreover, applying $\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}$ to (1.1) and evaluating at $\varepsilon = 0$ implies that $w := \partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m} u_\varepsilon|_{\varepsilon=0}$ solves the equation

$$\begin{cases} \Delta w &= -(m!) q v_1 \cdots v_m \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

By elliptic regularity, $v_j \in C^\infty(\overline{\Omega})$ and $w \in C^{2,\alpha}(\overline{\Omega})$. The DN map satisfies

$$\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}(\Lambda_q(f_\varepsilon))|_{\varepsilon=0} = \partial_{\varepsilon_1} \ldots \partial_{\varepsilon_m}(\partial_\nu u_\varepsilon)|_{\varepsilon=0} = \partial_\nu w|_{\partial\Omega}. \tag{2.5}$$

Now assume that $q_1, q_2 \in C^\alpha(\overline{\Omega})$ are such that (1.2) holds. Let w_j be the solution of (2.4) for $q = q_j$. By (1.2) and (2.5), one has

$$\int_{\partial\Omega} \partial_\nu(w_1 - w_2)\,d\mu = 0.$$

Let $\Psi \in L^r(\Omega)$ with $r < \frac{n}{n-1}$ be the solution of $\Delta\Psi = 0$ in Ω with $\Psi|_{\partial\Omega} = \mu$ in the sense of Lemma 2.1. It follows from (2.2) that

$$0 = \int_\Omega \Delta(w_1 - w_2)\Psi\,dx = -(m!)\int_\Omega (q_1 - q_2)v_1 \ldots v_m \Psi\,dx.$$

Now choose $h_3, \ldots, h_m \in C^\infty(\partial\Omega)$ so that $\mathrm{supp}(h_j) \subset \Gamma$, $h_j \geq 0$, and $h_j > 0$ somewhere. By the strong maximum principle $v_j > 0$ in Ω for $3 \leq j \leq m$. We obtain that

$$\int_\Omega [(q_1 - q_2)v_3 \cdots v_m \Psi] v_1 v_2\,dx = 0 \tag{2.6}$$

for any $h_1, h_2 \in C^\infty(\partial\Omega)$ with $\mathrm{supp}(h_j) \subset \Gamma$. Note that the function in brackets is in $L^r(\Omega)$ for $r < \frac{n}{n-1}$. Now we invoke Theorem 1.3, which implies that $(q_1 - q_2)v_3 \cdots v_m \Psi = 0$ in Ω. Since $v_3, \ldots, v_m$ are positive we must have $(q_1 - q_2)\Psi = 0$ in Ω. Finally, since $\mu \not\equiv 0$, the solution Ψ cannot vanish in any open subset of Ω by unique continuation (otherwise one would have $\Psi = 0$ a.e. in Ω by standard unique continuation for solutions of $\Delta\Psi = 0$ in Ω, and (2.2) would imply that $\mu \equiv 0$ by varying w). Thus Ψ is nonzero in a dense set of points in Ω. Since q_j are continuous, this shows that $q_1 = q_2$. □

3. Proof of Theorem 1.2

We now describe how to prove Theorem 1.2. The proof is very similar to that of Theorem 1.1 and we indicate the required modifications. First we note that Lemma 2.1 extends to the case where Ω is replaced by a compact Riemannian manifold (M,g) with smooth boundary and Δ is replaced by Δ_g. This relies on estimates for the Poisson kernel $P(x,y)$ on compact manifolds with boundary:

$$|\nabla_x^k P(x,y)| \leq \frac{C_k}{d_g(x,y)^{n-1+k}}, \qquad x \in M,\ y \in \partial M. \tag{3.1}$$

In fact the case $k=0$ follows e.g. from [HWY09, Lemma 2.2]. The general case follows by writing $\varepsilon = d_g(x,y)$ and by inserting $u(\,\cdot\,) = P(\,\cdot\,,y)$ into the elliptic estimate

$$\|\nabla^k u\|_{L^\infty(B_{\varepsilon/4}(x)\cap M)} \leq C_k \varepsilon^{-k} \|u\|_{L^\infty(B_{\varepsilon/2}(x)\cap M)}.$$

The last estimate is valid by standard elliptic regularity after rescaling into a ball of radius one.

Assuming the conditions in Theorem 1.2, the same argument that leads to (2.6) yields the identity

$$\int_M (q_1 - q_2) v_1 \cdots v_m \Psi \, dV_g = 0 \tag{3.2}$$

where $v_j \in C^\infty(M)$ are arbitrary solutions of the equation $\Delta_g v_j = 0$ in M, and $\Psi \in L^r(M)$ for $1 \leq r < \frac{n}{n-1}$ is the solution of

$$\begin{cases} \Delta_g \Psi &= 0 \quad \text{in } M, \\ \Psi &= \mu \quad \text{on } \partial M. \end{cases}$$

Note that by elliptic regularity, Ψ is smooth in M^{int} and it is also smooth up to the boundary near points $z \in \partial M$ so that $\mu = 0$ near z. To study the situation near $\mathrm{supp}(\mu)$, we observe using (3.1) that for any $x \in M^{\mathrm{int}}$ one has

$$|\Psi(x)| \leq \left| \int_{\partial M} P(x,y)\, d\mu(y) \right| \leq C \int_{\partial M} \frac{1}{d_g(x,y)^{n-1}} \, d|\mu|(y).$$

Write $f := (q_1 - q_2)\Psi$. Using the assumption that $q_1 = q_2$ to infinite order on ∂M, for any $N \geq 0$ there is $C_N > 0$ such that

$$\begin{aligned} |f(x)| &\leq C_N d_g(x,\partial M)^N \int_{\partial M} \frac{1}{d_g(x,y)^{n-1}} \, d|\mu|(y) \\ &\leq C_N d_g(x,\partial M)^{N-(n-1)} |\mu|(\partial M). \end{aligned}$$

Choosing $N \geq n$ gives that f is bounded in M and vanishes on ∂M. Applying similar estimates to derivatives of f in M^{int} proves that f is actually C^∞ up to the boundary in M and it vanishes to infinite order on ∂M.

We rewrite (3.2) in the form

$$\int_M f v_1 \dots v_m \, dV_g = 0$$

where $f = (q_1 - q_2)\Psi$ and $v_j \in C^\infty(M)$ are any solutions of $\Delta_g v_j = 0$ in M. It now follows from [LLLS21a, Proposition 5.1], if (M,g) is transversally anisotropic and $m \geq 4$, or from [GST19, Theorem 1.4], if (M,g) is a complex manifold satisfying the assumptions of that theorem, that $f = 0$. Since $\mu \not\equiv 0$ and M is connected, Ψ cannot vanish in any open set in M^{int} by the unique continuation principle. Indeed, if Ψ would vanish in an open set, then $\Psi = 0$ in M^{int} by unique continuation for solutions of $\Delta_g \Psi = 0$ in M^{int} (see e.g. [Ler19, Theorem 3.8]), and the analogue of (2.2) for Δ_g in M would imply that $\mu \equiv 0$ by varying w. Thus we must also have $q_1 - q_2 = 0$ in M, which concludes the proof of Theorem 1.2.

Remark 3.1. Under assumption (1) in Theorem 1.2, the condition that $q_1 = q_2$ to infinite order on ∂M can be weakened. In fact it would be enough to suppose that $q_1 = q_2$ to suitable finite order near $\mathrm{supp}(\mu)$ on ∂M, since in that case the argument above shows that $(q_1 - q_2)\Psi$ is in $C^1(M)$ and hence [LLLS21a, Proposition 5.1] applies. In a similar vein, under assumption (1) and in the special case $\mu = \delta_{x_0}$, it would be enough to assume that $\nabla^k q_1(x_0) = \nabla^k q_2(x_0)$ for finitely many k.

References

[AZ21] Yernat M. Assylbekov and Ting Zhou, *Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media*, J. Spectr. Theory **11** (2021), no. 1, 1–38. MR 4233204

[BKT21] Amin Boumenir and Vu Kim Tuan, *Reconstructing the wave speed and the source*, Math. Methods Appl. Sci. **44** (2021), no. 18, 14470–14480. MR 4342848

[CF21] Cătălin I. Cârstea and Ali Feizmohammadi, *An inverse boundary value problem for certain anisotropic quasilinear elliptic equations*, J. Differential Equations **284** (2021), 318–349. MR 4227095

[CFK+21] Cătălin I. Cârstea, Ali Feizmohammadi, Yavar Kian, Katya Krupchyk, and Gunther Uhlmann, *The Calderón inverse problem for isotropic quasilinear conductivities*, Adv. Math. **391** (2021), 107956. MR 4300916

[CGU21] Cătălin I. Cârstea, Tuhin Ghosh, and Gunther Uhlmann, *An inverse problem for the porous medium equation with partial data and a possibly singular absorption term*, arXiv:2108.12970 (2021).

[CNV19] Cătălin I. Cârstea, Gen Nakamura, and Manmohan Vashisth, *Reconstruction for the coefficients of a quasilinear elliptic partial differential equation*, Appl. Math. Lett. **98** (2019), 121–127. MR 3964222

[DSFKSU09] David Dos Santos Ferreira, Carlos E. Kenig, Johannes Sjöstrand, and Gunther Uhlmann, *On the linearized local Calderón problem*, Math. Res. Lett. **16** (2009), no. 6, 955–970. MR 2576684

[FO20] Ali Feizmohammadi and Lauri Oksanen, *An inverse problem for a semi-linear elliptic equation in Riemannian geometries*, J. Differential Equations **269** (2020), no. 6, 4683–4719. MR 4104456

[GST19] Colin Guillarmou, Mikko Salo, and Leo Tzou, *The linearized Calderón problem on complex manifolds*, Acta Math. Sin. (Engl. Ser.) **35** (2019), no. 6, 1043–1056. MR 3952702

[HWY09] Fengbo Hang, Xiaodong Wang, and Xiaodong Yan, *An integral equation in conformal geometry*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 1, 1–21. MR 2483810

[IN95] Victor Isakov and Adrian I. Nachman, *Global uniqueness for a two-dimensional semilinear elliptic inverse problem*, Trans. Amer. Math. Soc. **347** (1995), no. 9, 3375–3390. MR 1311909

[IS94] Victor Isakov and John Sylvester, *Global uniqueness for a semilinear elliptic inverse problem*, Comm. Pure Appl. Math. **47** (1994), no. 10, 1403–1410. MR 1295934

[Isa93] V. Isakov, *On uniqueness in inverse problems for semilinear parabolic equations*, Arch. Rational Mech. Anal. **124** (1993), no. 1, 1–12. MR 1233645

[KKU22] Yavar Kian, Katya Krupchyk, and Gunther Uhlmann, *Partial data inverse problems for quasilinear conductivity equations*, Math. Ann. (2022), https://doi.org/10.1007/s00208-022-02367-y.

[KLU18] Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann, *Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations*, Invent. Math. **212** (2018), no. 3, 781–857. MR 3802298

[KN02] Kyeonbae Kang and Gen Nakamura, *Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map*, Inverse Problems **18** (2002), no. 4, 1079–1088. MR 1929283

[Kra05] Steven G. Krantz, *Calculation and estimation of the Poisson kernel*, J. Math. Anal. Appl. **302** (2005), no. 1, 143–148. MR 2107352

[KU20a] Katya Krupchyk and Gunther Uhlmann, *Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities*, Math. Res. Lett. **27** (2020), no. 6, 1801–1824. MR 4216606

[KU20b] ———, *A remark on partial data inverse problems for semilinear elliptic equations*, Proc. Amer. Math. Soc. **148** (2020), no. 2, 681–685. MR 4052205

[KU22] Katya Krupchyk and Gunther Uhlmann, *Inverse problems for nonlinear magnetic Schrödinger equations on conformally transversally anisotropic manifolds*, Anal. PDE (2022), arXiv:2009.05089.

[Ler19] Nicolas Lerner, *Carleman inequalities. an introduction and more*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 353, Springer, Cham, 2019. MR 3932103

[LLLS21a] Matti Lassas, Tony Liimatainen, Yi-Hsuan Lin, and Mikko Salo, *Inverse problems for elliptic equations with power type nonlinearities*, J. Math. Pures Appl. (9) **145** (2021), 44–82. MR 4188325

[LLLS21b] ———, *Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations*, Rev. Mat. Iberoam. **37** (2021), no. 4, 1553–1580. MR 4269409

[LLPMT22] Matti Lassas, Tony Liimatainen, Leyter Potenciano-Machado, and Teemu Tyni, *Uniqueness, reconstruction and stability for an inverse problem of a semi-linear wave equation*, J. Differential Equations **337** (2022), 395–435. MR 4473034

[LLST22] Tony Liimatainen, Yi-Hsuan Lin, Mikko Salo, and Teemu Tyni, *Inverse problems for elliptic equations with fractional power type nonlinearities*, J. Differential Equations **306** (2022), 189–219. MR 4332042

[LZ20] Ru-Yu Lai and Ting Zhou, *Partial data inverse problems for nonlinear magnetic Schrödinger equations*, arXiv:2007.02475 (2020).

[SU97] Ziqi Sun and Gunther Uhlmann, *Inverse problems in quasilinear anisotropic media*, Amer. J. Math. **119** (1997), no. 4, 771–797. MR 1465069

[Sun96] Ziqi Sun, *On a quasilinear inverse boundary value problem*, Math. Z. **221** (1996), no. 2, 293–305. MR 1376299

[Tzo21] Leo Tzou, *Determining Riemannian manifolds from nonlinear wave observations at a single point*, arXiv:2102.01841 (2021).

Department of Mathematics and Statistics, University of Jyvaskyla, Jyvaskyla, Finland
Email address: mikko.j.salo@jyu.fi

Korteweg-de Vries Institute, University of Amsterdam, Amsterdam, Netherlands
Email address: leo.tzou@gmail.com