JYX JYVÄSKYLÄN YLIOPISTO UNIVERSITY OF JYVÄSKYLÄ

This is a self-archived version of an original article. This version may differ from the original in pagination and typographic details.

Author(s): Julia, Antoine; Nicolussi Golo, Sebastiano

Title: Intrinsic rectifiability via flat cones in the Heisenberg group

Year: 2022

Version: Accepted version (Final draft)

Copyright: © Scuola Normale Superiore - Edizioni della Normale

Rights: In Copyright

Rights url: http://rightsstatements.org/page/InC/1.0/?language=en

Please cite the original version:

Julia, A., & Nicolussi Golo, S. (2022). Intrinsic rectifiability via flat cones in the Heisenberg group. Annali della Scuola Normale Superiore di Pisa: Classe di Scienze, 23(2), 829-835. https://doi.org/10.2422/2036-2145.202005_012

INTRINSIC RECTIFIABILITY VIA FLAT CONES IN THE HEISENBERG GROUP

ANTOINE JULIA AND SEBASTIANO NICOLUSSI GOLO

Abstract. We give a geometric criterion for a topological surface in the first Heisenberg group to be an intrinsic Lipschitz graph, using planar cones instead of the usual open cones.

Introduction

We identify the *first Heisenberg group* $\mathbb{H}$ with the manifold $\mathbb{R}^3$ endowed with the group operation

$$(x,y,z)(x',y',z') = \left(x+x', y+y', z+z'+\frac{xy'-x'y}{2}\right).$$

The inverse of (x,y,z) is $(-x,-y,-z)$. If $E \subset \mathbb{H}$ and $p \in \mathbb{H}$, we write pE for the set $\{pe : e \in E\}$. The elements of the one-parameter family of group automorphisms $\delta_\lambda : (x,y,z) \mapsto (\lambda x, \lambda y, \lambda^2 z)$ are called *dilations*.

In recent years, intrinsic Lipschitz graphs in $\mathbb{H}$ have gained more attention, see [7,8,11] and [3–6]. Indeed, these graphs yield a robust notion of rectifiable set in the Heisenberg group, and more generally in Carnot groups. The definition of intrinsic Lipschitz graphs is inspired by the Euclidean characterization of Lipschitz graphs by cones. Here, a *cone* is a set $E \subset \mathbb{H}$ with $\delta_\lambda(E) = E$ for all $\lambda > 0$. A set is an intrinsic Lipschitz graph if it has the following full cone property.

Definition 1 (Full cone property)**.** A set $S \subset \mathbb{H}$ has the *full cone property* if there is an open cone C with $-C = C$, $C \cap \{z = 0\} \neq \emptyset$ and, for all $p \in S$,

$$C \cap p^{-1}S = \emptyset.$$

An example of open cone, for some $\alpha > 0$, is the *α-full cone*:

$$\mathtt{C}(\alpha) := \left\{(x,y,z) : \ |y| < \alpha|x|, \ |z| < \alpha x^2/2\right\}.$$

In this paper, we consider *flat cones*

$$\mathtt{fC}(\alpha) := \{(x,y,0) : |y| < \alpha|x|\}.$$

Our aim is to compare the full cone property with a flat cone property defined as follows.

Definition 2 (Flat cone property)**.** A set $S \subset \mathbb{H}$ has the *α-flat cone property* if for all $p \in S$

$$\mathtt{fC}(\alpha) \cap p^{-1}S = \emptyset.$$

Date: February 4, 2021.

2010 *Mathematics Subject Classification.* 53C17,28A75,22E25.

Key words and phrases. Sub-Riemannian Geometry, Heisenberg Group, Intrinsic Lipschitz graphs.

Both A.J. and S.N.G. have been supported by University of Padova STARS Project "Sub-Riemannian Geometry and Geometric Measure Theory Issues: Old and New"; and by the INdAM – GNAMPA Project 2019 "Rectifiability in Carnot groups".

Note that, if C is an open cone with $-C = C$ and $C \cap \{z = 0\} \neq \emptyset$, then there exists $\alpha > 0$ such that $\mathtt{C}(\alpha) \subset C$, up to a rotation around the z-axis. Recall that rotations around the z-axis are isomorphisms of the homogeneous structure of $\mathbb{H}$ given by left-translations and dilations. Noting that $\mathtt{fC}(\alpha) = \mathtt{C}(\alpha) \cap \{z = 0\}$, it is then clear that intrinsic Lipschitz graphs also have the flat cone property, up to a rotation around the z-axis. We will prove the inverse implication for topological surfaces.

Theorem. *If $S \subset \mathbb{H}$ is a topological surface with the flat cone property, then it is locally a Lipschitz intrinsic graph. Quantitatively, the α-flat cone property implies locally the $\alpha/4$-full cone property.*

Remark 3. If S is a topological surface with the flat cone property, then it is an intrinsic graph and horizontal curves on S are Lipschitz with uniformly bounded slope. The Theorem is thus a geometric version of [2, Theorem 1.2] in the first Heisenberg group.

Remark 4. The topological surface

$$\{(0, y, z), 0 < y < 1, z < 0\} \cup \{(1, y, z) : 0 < y < 1, y < 2z\}$$

has the flat cone property, but is not an intrinsic Lipschitz graph in the direction of the x-axis. It is not hard to modify this set to make it into a connected topological surface without losing the flat cone property. This example shows that the word "locally" in the statement of the Theorem is necessary.

Remark 5. The hypothesis in the Theorem that S is a topological surface is crucial. We register two examples where our strategy fails and that we consider prototypical. The first one has the flat cone property, but not the full cone property at 0 in the direction of the x-axis:

$$\{(x, 0, x^3), x \in [\,0, 1\,]\}.$$

Another example is the following set, which has the flat cone property, but is not an intrinsic Lipschitz graph:

$$\{(0, 0, 0)\} \cup \{(x, y, z) : x = 1\} \backslash \{(1, s, 0) : |s| < 1\}.$$

However, these sets are clearly intrinsic 1-codimensional-rectifiable in the sense of [7, Definition 3.16]. We do not know whether the flat cone property does imply intrinsic rectifiability.

Remark 6. After this paper was submitted, we have learned that a similar result for Carnot groups is contained in a recent work of E. Le Donne and C. Bellettini, see [1, Proposition 2.29].

We present a detailed proof of the Theorem in the subsequent sections. However, the expert reader could be satisfied with the following sketch.
Suppose $p = 0 \in S$: we want to find $\beta > 0$ and an open neighborhood$U \subset \mathbb{H}$ of 0 such that $\mathtt{C}(\beta) \cap (S \cap U) = \emptyset$. We start by considering the intersection $S \cap H$, where $H = \{z = 0\}$ is the so called "horizontal plane": for every y_0 close to 0 there is $x \in [\,-y_0/\alpha, y_0/\alpha\,]$ such that $(x, y_0, 0) \in S$. Now, we notice a crucial fact: all the left-translations $\{(x, y_0, 0)H\}_{x \in \mathbb{R}}$, with y_0 fixed, are affine planes containing a common line, which turns out to be inside the plane $\{y = 0\}$. Therefore,

$$\bigcap_{x \in [\,-y_0/\alpha, y_0/\alpha\,]} (x, y_0, 0)\mathtt{fC}(\alpha)$$

is the union of two half lines in the plane $\{y = 0\}$. By construction, these two half lines do not intersect S.

If we then consider all $y_0 \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$ small enough, we get a family of half lines in $\{y = 0\}$ that do not intersect S: the union of such a family yields a "vertical set" which, in a neighborhood of 0, coincides with $\mathtt{C}(\alpha/2) \cap \{y = 0\}$.
Although promising, this is still not enough to get a full cone. However, the same argument can be carried out in another system of coordinates (x', y', z') for $\mathbb{H}$ in which the x'-axis is slightly tilted with respect to the x-axis; in other words, we consider $x' = x$, $y' = y + tx$, $z' = z$, where $t \in \mathbb{R}$. Indeed, for t small, S will still have the $(\alpha/2)$-flat cone property in the new coordinates and thus we obtain a vertical set in $\{y' = 0\}$ that does not intersect S and coincides in a neighborhood of 0 with $\mathtt{C}(\alpha/4) \cap \{y' = 0\}$.
The union of all these vertical sets coincides with $\mathtt{C}(\alpha/4)$ near 0, as desired. This is proved in Section 1. Of course, the arguments should be carried out in a uniform way for p close to 0 in S, this is the scope of Section 2. The full proof of the Theorem is in Section 3.

Acknowledgements. We thank Davide Vittone and Roberto Monti for inspiring conversations.

1. Flat cones build up full cones

In this section, we show how we get truncated full cones from flat cones. It is worth mentioning [10], where another strategy to pass from flat cones to full open cones is discussed.

For $r, \beta > 0$, define the *truncated full cone*

$$\mathtt{C}(\beta, r) := \left\{(x, y, z) : \ |y| < \beta|x|, \ |z| < \beta x^2/2, \ |x| < r\right\}$$

and the *vertical set*

$$\mathtt{vC}(\beta, r) = \left\{\left(x, 0, ux^2/2\right) : |u| < \beta, |x| < r\right\}.$$

Remark 7. Note that if a set S satisfies $\mathtt{C}(\beta, r) \cap p^{-1}S = \emptyset$, for all $p \in S$, then, for each $p \in S$, the set $S \cap p\{|x| < r/2\}$ has the β-full cone property.

Lemma 8. *Fix $\epsilon, \beta > 0$, then*

$$\mathtt{vC}\left(\frac{\beta}{2}, \frac{2\epsilon}{\beta}\right) \subset \bigcup_{|\eta|<\epsilon} \bigcap_{|s|<\beta^{-1}} (s\eta, \eta, 0)\mathtt{fC}(\beta). \tag{1}$$

Proof. Pick $(x, 0, ux^2/2) \in \mathtt{vC}(\beta/2, 2\epsilon/\beta)$, and notice that $|-ux| < \epsilon$. We will show that $(x, 0, ux^2/2) \in (-sux, -ux, 0)\mathtt{fC}(\beta)$, i.e. that $(sux, ux, 0)(x, 0, ux^2/2) \in \mathtt{fC}(\beta)$, for all $s \in (-\beta^{-1}, \beta^{-1})$. We have

$$(sux, ux, 0)(x, 0, ux^2/2) = (x + sux, ux, 0),$$

so we only need to prove that $|ux| < \beta|x + sux|$, which is clear as $|u| < \beta/2$, $|su| < 1/2$ and thus

$$\beta|x + sux| \geq \beta|x|(1 - |su|) \geq \frac{\beta}{2}|x| \geq |ux|. \qquad \square$$

For $t \in \mathbb{R}$, define the map $M_t(x, y, z) := (x, y + tx, z)$, which is a group automorphism $\mathbb{H} \to \mathbb{H}$, with inverse $M_t^{-1} = M_{-t}$. An immediate consequence of the above definitions is that, for all $r, \beta > 0$,

$$\mathtt{C}(\beta, r) = \bigcup_{|t|<\beta} M_t(\mathtt{vC}(\beta, r)). \tag{2}$$

2. Intrinsic graphs

Given $\Omega \subset \mathbb{R}^2$ and $\phi : \Omega \to \mathbb{R}$, the *intrinsic graph* of ϕ is the subset

$$\begin{aligned}\Gamma_\phi :=&\{(0,\eta,\tau)(\phi(\eta,\tau),0,0) : (\eta,\tau) \in \Omega\}\\ =&\{(\phi(\eta,\tau),\eta,\tau-\eta\phi(\eta,\tau)/2) : (\eta,\tau) \in \Omega\}.\end{aligned}$$

Define the projection $\pi : \mathbb{H} \to \mathbb{R}^2$, $(x,y,z) \mapsto (y, z+xy/2)$. Notice that if $(\eta_0,\tau_0) = \pi(x_0,y_0,z_0)$, then $(x_0,y_0,z_0) = (0,\eta_0,\tau_0)(x_0,0,0)$.

Proposition 9. *If S is a topological surface with the flat cone property, then it is the intrinsic graph of a continuous function defined on an open subset of $\mathbb{R}^2$.*

Proof. Since $\{(\xi,0,0) : \xi \in \mathbb{R}\setminus\{0\}\} \subset \mathtt{fC}(\alpha)$, the map $\pi|_S : S \to \mathbb{R}^2$ is injective. Thus, $S = \Gamma_\phi$ for some $\phi : \pi(S) \to \mathbb{R}$. As S is a topological surface and by the Invariance of Domain Theorem, $\pi(S)$ is open and ϕ is continuous. (An elementary proof of the Invariance of Domain Theorem can be found in [9].) □

As we explained in the introduction, in order to apply Lemma 8, we will use the fact that for every y_0 close to 0 there is $x \in [-y_0/\alpha, y_0/\alpha]$ such that $(x,y_0,0) \in S$.

Proposition 10. *Fix $\eta_0,\tau_0,\beta > 0$ and let $\psi : [-\eta_0,\eta_0] \times [-\tau_0,\tau_0] \to \mathbb{R}$ be continuous, such that $\psi(0) = 0$ and Γ_ψ has the β-flat cone property. Then, letting $\epsilon := \min\{\eta_0, \sqrt{\tau_0\beta}\}$, for all $\eta \in [-\epsilon,\epsilon]$, there exists $s \in [-\beta^{-1},\beta^{-1}]$ with*

$$(s\eta,\eta,0) \in \Gamma_\psi.$$

Proof. Consider the function $\zeta(\eta,\tau) := \tau - \frac{\eta\psi(\eta,\tau)}{2}$ (yielding the third coordinate of the point $(0,\eta,\tau)(\psi(\eta,\tau),0,0)$). Notice that, since $\mathtt{fC}(\beta) \cap \Gamma_\psi = \emptyset$, we have

$$\{(\eta,\tau) : \zeta(\eta,\tau) = 0\} \subset \mathbb{R}^2 \setminus \pi(\mathtt{fC}(\beta)) = \{(\eta,\tau) : \eta^2 \geq 2\beta|\tau|\}.$$

Moreover, by the choice of ϵ, for $\eta \in [-\epsilon,\epsilon]$, there holds $\eta^2 < 2\beta\tau_0$ and therefore $\zeta(\eta,\pm\tau_0) \neq 0$. Since ζ is continuous, and $\zeta(0,\tau_0) = \tau_0 > 0 > -\tau_0 = \zeta(0,-\tau_0)$, we have for all $\eta \in [-\epsilon,\epsilon]$

$$\zeta(\eta,\tau_0) > 0 > \zeta(\eta,-\tau_0). \tag{3}$$

Using once again the continuity of ζ, it follows that for every $\eta \in [-\epsilon,\epsilon]$ there is $\tau \in]-\tau_0,\tau_0[$ with $\zeta(\eta,\tau) = 0$ and thus a point $(s\eta,\eta,0) \in \Gamma_\psi$. By the β-flat cone property of Γ_ψ, either $\eta = 0$ (and we can take $s = 0$) or there holds $|s| \leq \beta^{-1}$. □

We want to apply Proposition 10 and Lemma 8, not only to S but also to $M_t(p^{-1}S)$ for p in a neighborhood of 0 and t in a compact interval. Lemma 12 allows us to do this.

Remark 11. Notice that, for $t \in [-\alpha/2,\alpha/2]$, there holds $\mathtt{fC}(\alpha/2) \subset M_t\mathtt{fC}(\alpha)$. In particular, if S has the α-flat cone property, then for such t, $M_t(p^{-1}S)$ has the $\alpha/2$-flat cone property, for every $t \in [-\alpha/2,\alpha/2]$ and $p \in \mathbb{H}$.

Lemma 12. *Let $S \subset \mathbb{H}$ be a topological surface with the α-flat cone property and $0 \in S$. Then there are $\eta_0,\tau_0 > 0$ such that, defining $V_0 := \pi^{-1}([-\eta_0,\eta_0] \times [-\tau_0,\tau_0])$, for all $t \in [-\alpha/2,\alpha/2]$ and $p \in S \cap V_0$, there exists a continuous function $\phi_{t,p} : [-\eta_0,\eta_0] \times [-\tau_0,\tau_0] \to \mathbb{R}$ such that*

$$\Gamma_{\phi_{t,p}} = M_t(p^{-1}S) \cap V_0.$$

Proof. By Remark 11 and Proposition 9, for all $p \in S$ and $t \in [-\alpha/2,\alpha/2]$, $M_t(p^{-1}S)$ is an intrinsic graph over $\pi(M_t(p^{-1}S))$.

Denote by $\mathbb{D}_r$ the disk of radius r in $\mathbb{R}^2$. Let $\Phi : \mathbb{D}_1 \to S$ be a local chart with $\Phi(0,0) = (0,0,0)$. For $p \in \Phi(\mathbb{D}_1)$ and $t \in [-\alpha/2,\alpha/2]$, define

$$f_{t,p} : \mathbb{D}_1 \to \mathbb{R}^2, \qquad f_{t,p}(v) = \pi(M_t(p^{-1}\Phi(v)).$$

Notice that, since $M_t(p^{-1}S)$ is an intrinsic graph, the $f_{t,p}$ are homeomorphisms onto their images. Define

$$\Omega := \bigcap_{p\in\Phi(\bar{\mathbb{D}}_{1/2})} \bigcap_{|t|\le\alpha/2} f_{t,p}(\mathbb{D}_1). \tag{4}$$

We claim that Ω is a neighborhood of $(0,0)$. We argue by contradiction. Suppose that there exists a sequence $g_n \to 0$ in $\mathbb{R}^2$, along with p_n in $\Phi(\bar{\mathbb{D}}_{1/2})$ and $t_n \in [-\alpha/2, \alpha/2]$, with $g_n \notin f_{t_n,p_n}(\mathbb{D}_1)$. By compactness, we can suppose that $p_n \to p_\infty \in \Phi(\bar{\mathbb{D}}_{1/2})$ and that $t_n \to t_\infty \in [-\alpha/2, \alpha/2]$. Since the $f_{t,p}$ are homeomorphisms onto their images, there is $r > 0$ such that $\mathbb{D}_{2r} \subset f_{t_\infty,p_\infty}(\mathbb{D}_1)$. Since $f_{t_n,p_n} \to f_{t_\infty,p_\infty}$ uniformly, there holds $f_{t_n,p_n}(\partial\mathbb{D}_1) \cap \mathbb{D}_r = \emptyset$ for n large enough. In particular, as $0 \in f_{t_n,p_n}(\mathbb{D}_1)$, we have $\mathbb{D}_r \subset f_{t_n,p_n}(\mathbb{D}_{r_0})$, however, as $g_n \to 0$, for n large enough there also holds $g_n \in \mathbb{D}_r$, a contradiction. This proves the claim.

Finally, if $\eta_0, \tau_0 > 0$ are such that $[-\eta_0, \eta_0] \times [-\tau_0, \tau_0] \subset \Omega$, then we have all the properties stated in the lemma. □

3. Proof of the Theorem

Let $\alpha > 0$ and $S \subset \mathbb{H}$ be a topological surface with the α-flat cone property. We will show that S has locally the $(\alpha/4)$-full cone property, that is, that for every $o \in S$ there is an open neighborhood $U \subset \mathbb{H}$ of o such that, for all $p \in S \cap U$

$$\mathtt{C}(\alpha/4) \cap p^{-1}(S \cap U) = \emptyset.$$

It suffices to work for $o = 0$.

By Lemma 12, there are $\eta_0, \tau_0 > 0$ such that, defining $V_0 := \pi^{-1}([-\eta_0, \eta_0] \times [-\tau_0, \tau_0])$, for all $t \in [-\alpha/2, \alpha/2]$ and $p \in S \cap V_0$, there exists a continuous function $\phi_{t,p} : [-\eta_0, \eta_0] \times [-\tau_0, \tau_0] \to \mathbb{R}$ such that

$$\Gamma_{\phi_{t,p}} = M_t(p^{-1}S) \cap V_0.$$

Fix $p \in S \cap V_0$, $t \in [-\alpha/2, \alpha/2]$ and set $\epsilon := \min\{\eta_0, \sqrt{\tau_0\alpha/2}\}$. Applying Proposition 10 to $\phi_{t,p}$, for each $\eta \in [-\epsilon, \epsilon]$ there exists $s \in [-2/\alpha, 2/\alpha]$ such that $(s\eta, \eta, 0) \in \Gamma_{\phi_{t,p}}$. Therefore, as $\Gamma_{\phi_{t,p}} \subset M_t(p^{-1}S)$ and since $M_t(p^{-1}S)$ has the $(\alpha/2)$-flat cone property by Remark 11, there holds

$$M_t(p^{-1}S) \cap \bigcap_{s\in[-2/\alpha,2/\alpha]} (s\eta, \eta, 0)\mathtt{fC}(\alpha/2) = \emptyset, \qquad \forall \eta \in [-\epsilon, \epsilon].$$

Taking the union over $\eta \in [-\epsilon, \epsilon]$ and applying Lemma 8 yields

$$M_t(p^{-1}S) \cap \mathtt{vC}(\alpha/4, 4\epsilon/\alpha) = \emptyset, \tag{5}$$

which holds for all $p \in S \cap V_0$ and $t \in [-\alpha/2, \alpha/2]$. Apply M_{-t} and the left translation by p to (5), then take a union over $t \in [-\alpha/2, \alpha/2]$ and use (2) to get

$$S \cap p\,\mathtt{C}(\alpha/4, 4\epsilon/\alpha) = \emptyset,$$

which holds for all $p \in S \cap V_0$. Remark 7 implies that $S \cap V_0$ has the $(\alpha/4)$-cone property in a neighborhood of 0. Thus, S has locally the $(\alpha/4)$-full cone property. □

References

[1] Costante Bellettini and Enrico Le Donne, *Sets with constant normal in Carnot groups: properties and examples*, arXiv e-prints (October 2019), arXiv:1910.12117, available at `1910.12117`.

[2] F. Bigolin, L. Caravenna, and F. Serra Cassano, *Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 5, 925–963. MR3400438

[3] Vasileios Chousionis, Katrin Fässler, and Tuomas Orponen, *Boundedness of singular integrals on $C^{1,\alpha}$ intrinsic graphs in the Heisenberg group*, arXiv e-prints (August 2017), arXiv:1708.08444, available at `1708.08444`.

[4] Sebastiano Don, Enrico Le Donne, Terhi Moisala, and Davide Vittone, *A rectifiability result for finite-perimeter sets in Carnot groups*, arXiv e-prints (December 2019), arXiv:1912.00493, available at `1912.00493`.

[5] Sebastiano Don, Annalisa Massaccesi, and Davide Vittone, *Rank-one theorem and subgraphs of BV functions in Carnot groups.*, J. Funct. Anal. **276** (2019), no. 3, 687–715 (English).

[6] Katrin Fässler and Tuomas Orponen, *Riesz transform and vertical oscillation in the Heisenberg group*, arXiv e-prints (October 2018), arXiv:1810.13122, available at `1810.13122`.

[7] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano, *Intrinsic Lipschitz graphs in Heisenberg groups*, J. Nonlinear Convex Anal. **7** (2006), no. 3, 423–441. MR2287539 (2008b:58014)

[8] Bruno Franchi and Raul Paolo Serapioni, *Intrinsic Lipschitz graphs within Carnot groups*, J. Geom. Anal. **26** (2016), no. 3, 1946–1994. MR3511465

[9] Wladyslaw Kulpa, *Poincaré and domain invariance theorem*, Acta Univ. Carolin. Math. Phys. **39** (1998), no. 1-2, 127–136. MR1696596

[10] Roberto Monti and Davide Vittone, *Sets with finite $\mathbb{H}$-perimeter and controlled normal*, Math. Z. **270** (2012), no. 1-2, 351–367. MR2875838

[11] Raul Paolo Serapioni, *Intrinsic difference quotients*, Harmonic analysis, partial differential equations and applications, 2017, pp. 165–192. MR3642743

(A. Julia) Institut de Mathématiques d'Orsay, Université Paris-Saclay, 91405 Orsay, France.

Email address: `antoine.julia@universite-paris-saclay.fr`

(S. Nicolussi Golo) Department of Mathematics and Statistics, 40014 University of Jyväskylä, Finland.

Email address: `sebastiano2.72@gmail.com`