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Sobolev extensions via reflections

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Abstract

We show that certain extension results obtained by Maz'ya and Poborchi for domains with an outward peak can be realized via composition operators generated by reflections. We also study the case of the complementary domains.

1 Introduction

A domain $\Omega \subset \mathbb{R}^n$ is called a (p, q)-extension domain, $1 \leq q \leq p \leq \infty$, if every $u \in W^{1,p}(\Omega)$ has an extension $E(u) \in W^{1,q}_{\rm loc}(\mathbb{R}^n)$ with

$$||E(u)||_{W^{1,q}(\mathbb{R}^n \setminus \Omega)} \leq C||u||_{W^{1,p}(\Omega)}.$$

A Lipschitz domain Ω is a (p, p)-extension domain for all $1 \leq p \leq \infty$ by results due to Calderón and Stein [27]. Jones generalized this result to a much larger class of domains, so-called (ϵ, δ)-domains, but general domains are not necessarily (p, p)-extension domains for any p. For example, in [20, 21, 22], Maz'ya and Poborchi investigated in detail a typical case where the above extension property fails: the case of a domain with an outward peak, also see [19, 25] for related results. Once the sharpness of the peak was fixed, they found the optimal p, q for the (p, q)-extendability.

The idea of using reflections to construct extension operators is implicit in the results for Lipschitz domains. Gol'dshtein, Latfullin and Vodop'yanov initiated the systematic use of reflections for constructing extension operators in the Euclidean plane $\mathbb{R}^2$ in [5, 8]. In [6], Gol'dshtein and Sitnikov showed that the Sobolev extendability for planar outward and inward cuspidal domains of polynomial order can be achieved by a bounded linear extension operator induced by reflections.

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Very recently, Koskela, Pankka and Zhang [18] proved that for every planar Jordan (p,p)-extension domain with $1 < p < \infty$, there exists a reflection over the boundary $\partial\Omega$ which induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^2)$.

In this paper, we study the Sobolev extension via reflections on outward cuspidal domains in the Euclidean space $\mathbb{R}^n$ with $n \geq 3$. From now on, we always assume $n \geq 3$.

We distinguish a horizontal coordinate axis in $\mathbb{R}^n$,

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} = \{z := (t,x) : t \in \mathbb{R} \text{ and } x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

A strictly increasing function $\psi : [0,\infty) \to [0,\infty)$ is said to be a cuspidal function if $\psi \in C^1(0,\infty) \cap C[0,\infty)$ is doubling on $(0,1)$ in the sense that $\psi(2t) \leq C\psi(t)$ when $t \in (0, \frac{1}{2})$, $\psi(0) = 0$, ψ' is increasing on $(0,\infty)$ and

$$\lim_{\rho \to 0^+} \psi'(\rho) = 0.$$

We normalize the function ψ by requiring $\psi(1) = 1$. The corresponding outward cuspidal domain Ω_ψ is defined by setting

uter cuspidal (1.1) $$\Omega_\psi := \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < t \leq 1, |x| < \psi(t)\right\} \cup B((2,0), \sqrt{2}).$$

See Figure 1.

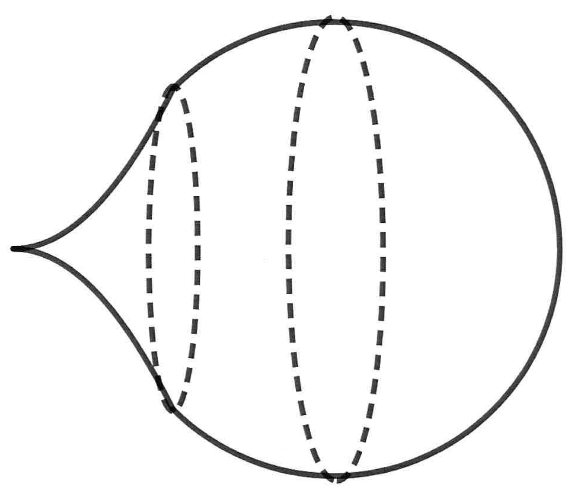

Figure 1: Ω_ψ fig:3

We are interested in those cuspidal functions that satisfy the integrability condition

eq:integral (1.2) $$\int_0^1 \left(\frac{t^s}{\psi(t)}\right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty$$

for some $1 < s < \infty$. The typical examples of desired ψ are

$$\psi(t) = t^{\beta} \text{ with } 1 \leq \beta < s$$

and

$$\psi(t) = t^s \log^{\alpha}\left(\frac{e}{t}\right),\ t^s \log^{\alpha}\log\left(\frac{e}{t}\right), \cdots, t^s \log^{\alpha}\underbrace{\log\cdots\log}_{k}\left(\frac{e}{t}\right), \cdots$$

with $\alpha > \frac{s-1}{n}$. For the case of this model domain Ω_ψ with a cuspidal function ψ satisfying (1.2) for some $1 < s < \infty$, the results due to Maz'ya and Poborchi state that there exists a bounded linear extension operator E_1 from $W^{1,p}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{n} < p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$, and there exists another bounded linear extension operator E_2 from $W^{1,p}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{2+(n-2)s} < p < \infty$ and $1 \leq q \leq \frac{p+(n-1)sp}{1+(n-1)s+(s-1)p}$. For $p = \frac{(n-1)+(n-1)^2 s}{n}$, one has

$$\frac{np}{1+(n-1)s} = \frac{p+(n-1)sp}{1+(n-1)s+(s-1)p} = n-1.$$

For a detailed exposition of these results, see [19]. Interestingly, the given extension operators for the domain Ω_ψ above are linear and the formulas defining the operators do not depend on p once s and the range of p are fixed. Our main result explains this phenomenon.

main resu **Theorem 1.1.** *Let ψ be a cuspidal function satisfying (1.2) for some constant $1 < s < \infty$ and Ω_ψ be the corresponding outward cuspidal domain. Then*
(1) : *There exists a reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ which induces a bounded linear extension operator $E_{\mathcal{R}_1}$ from $W^{1,p}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$.*
(2) : *There exists another reflection $\mathcal{R}_2 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ which induces a bounded linear extension operator $E_{\mathcal{R}_2}$ from $W^{1,p}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$ and $1 \leq q \leq \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$.*

Theorem 1.1 implies that both reflections $\mathcal{R}_1$ and $\mathcal{R}_2$ induce a bounded linear extension operator from $W^{1,\frac{(n-1)+(n-1)^2 s}{n}}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $1 \leq q \leq n-1$. Coming back to the original results due to Maz'ya and Poborchi, their extension operators E_1 and E_2 do not extend all functions in $W^{1,\frac{(n-1)+(n-1)^2 s}{n}}(\Omega_\psi)$ into $W^{1,n-1}(\mathbb{R}^n)$. For this, they constructed a further more complicated extension operator E_3 from $W^{1,\frac{(n-1)+(n-1)^2 s}{n}}(\Omega_\psi)$ to $W^{1,n-1}(\mathbb{R}^n)$. Comparing with the extension operators from the work of Maz'ya and Poborchi, our extension operators induced by reflections have another obvious advantage. If $u_1, u_2 \in W^{1,p}(\Omega_\psi)$ satisfy

$$\operatorname{supp}(u_1) \cap \operatorname{supp}(u_2) = \emptyset,$$

then

$$\mathrm{supp}(E_{\mathcal{R}_1}(u_1)) \cap \mathrm{supp}(E_{\mathcal{R}_1}(u_2)) = \emptyset \text{ and } \mathrm{supp}(E_{\mathcal{R}_2}(u_1)) \cap \mathrm{supp}(E_{\mathcal{R}_2}(u_2)) = \emptyset.$$

One can check from [19] and the references therein that the extension operators constructed by Maz'ya and Poborchi do not have this property.

In general, we say that a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega$, for a bounded domain Ω (whose boundary has volume zero) induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ if there is an open set U containing $\partial\Omega$ so that, for every $u \in W^{1,p}(\Omega)$, the function v defined by setting $v = u$ on $\Omega \cap U$ and $v = u \circ \mathcal{R}$ on $U \setminus \overline{\Omega}$ has a representative which belongs to $W^{1,q}(U)$ with

uu1 (1.3)
$$\|v\|_{W^{1,q}(U)} \le C\|u\|_{W^{1,p}(U\cap\Omega)},$$

for some positive constant C independent of u. Similarly, we say that the reflection $\mathcal{R}$ induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ to $W^{1,q}(\mathbb{R}^n)$, if for every $u \in W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ the function $\tilde{v}$ defined by setting $\tilde{v} = u$ on $U \setminus \overline{\Omega}$ and $\tilde{v} = u \circ \mathcal{R}$ on $U \cap \Omega$ has a representative which belongs to $W^{1,q}(U)$ with

uu2 (1.4)
$$\|\tilde{v}\|_{W^{1,q}(U)} \le C\|u\|_{W^{1,p}(U\setminus\overline{\Omega})}.$$

Here the introduction of the open set U is a convenient way to overcome the non-essential difficulty that functions in $W^{1,p}(G)$ do not necessarily belong to $W^{1,q}(G)$ when $1 \le q < p < \infty$ and G has infinite volume. It follows from the assumption (1.3) (or (1.4)) via the use of a suitable cut-off function that Ω (or $\mathbb{R}^n \setminus \overline{\Omega}$, respectively) is a $(p,q)-$extension domain with a bounded linear extension operator. For this, see Section 2.

The crucial point behind Theorem 1.1 is that we obtain Sobolev estimates on $u \circ \mathcal{R}$ in terms of the data on u. There is a rather long history of such results, for example see [7, 9, 12, 28] and references therein. In the setting of our problem, the most relevant reference is the paper [28] by Ukhlov. What we find surprising in our situation is that a single $\mathcal{R}_1$ induces the best bounded linear extension operator for all values $\frac{(n-1)+(n-1)^2 s}{n} \le p < \infty$ and another single $\mathcal{R}_2$ induces the best bounded linear extension operator for all values $\frac{1+(n-1)s}{2+(n-2)s} \le p \le \frac{(n-1)+(n-1)^2 s}{n}$. In the case of compositions from $W^{1,p}$ to $W^{1,p}$, the relevant estimate is

maar (1.5)
$$|D\mathcal{R}(z)|^p \le C|J_{\mathcal{R}}(z)|$$

almost everywhere, which for $p = n$ is the pointwise condition of quasiconformality. Mappings satisfying (1.5) with $p \ne n$ apparently appeared for the first time in the works of Gehring [4] and of Maz'ya [23], independently. With some work one can show that (1.5) implies the corresponding inequality with p replaced by q when either $q > p > n$ or $1 \le q < p < n$, but not in other cases. On the other hand, for

$n-1<p<\infty$, a result in [6] shows that (1.5) together with $W^{1,p}$-regularity of $\mathcal{R}$ implies the dual estimate

maar1 (1.6)
$$|D\mathcal{R}^{-1}(z)|^{\frac{p}{p+1-n}} \leq C'|J_{\mathcal{R}^{-1}}(z)|.$$

This kind of duality actually also holds for compositions from $W^{1,p}$ to $W^{1,q}$ with $q<p$, see [28]. Also see [13, 14, 28, 31] for general results on the regularity of $\mathcal{R}^{-1}$.

From the argument above, one could also expect that the reflections $\mathcal{R}_1$ and $\mathcal{R}_2$ induce bounded linear extension operators from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ to $W^{1,q}(\mathbb{R}^n)$, for some $1\leq q\leq p<\infty$. As one can easily check, for every cuspidal function ψ, $\mathbb{R}^n\setminus\overline{\Omega_\psi}$ is a so-called (ϵ,δ)-domain and hence a (p,p)-extension domain for every $1\leq p<\infty$ due to Jones [16]. Our next theorem relates this to our reflections.

thm:comp **Theorem 1.2.** *For arbitrary cuspidal function ψ, $\mathbb{R}^n\setminus\overline{\Omega_\psi}$ is a (p,p)-extension domain, for every $1\leq p<\infty$. The reflection $\mathcal{R}_1$ over $\partial\Omega_\psi$ in Theorem 1.1 induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$, whenever $1\leq p\leq n-1$. Moreover, for each $n-1<p<\infty$, no reflection over $\partial\Omega_\psi$ can induce a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$.*

What then about the case $p=\infty$? We say that a domain $\Omega\subset\mathbb{R}^n$ is uniformly locally quasiconvex if there exist constants $C>0$ and $R>0$ such that, for every pair of points $x,y\in\Omega$ with $d(x,y)<R$, there is a rectifiable curve γ connecting x and y in Ω such that the length of γ is bounded from above by $Cd(x,y)$. If the above holds without the distance restriction, Ω is said to be quasiconvex. Recall that Ω is an (∞,∞)-extension domain if and only if it is uniformly locally quasiconvex, see [10] by Hajłasz, Koskela and Tuominen. One can easily check that both Ω_ψ and $\mathbb{R}^n\setminus\overline{\Omega_\psi}$ are uniformly locally quasiconvex, equivalently, they are (∞,∞)-extension domains. We close this introduction with the following analog of Theorem 1.2.

thm:infty **Theorem 1.3.** *For arbitrary cuspidal function ψ, both Ω_ψ and $\mathbb{R}^n\setminus\overline{\Omega_\psi}$ are (∞,∞)-extension domains. The reflection $\mathcal{R}_1$ over $\partial\Omega_\psi$ in Theorem 1.1 induces a bounded linear extension operator from $W^{1,\infty}(\Omega_\psi)$ to $W^{1,\infty}(\mathbb{R}^n)$. On the other hand, no reflection over $\partial\Omega_\psi$ can induce a bounded linear extension operator from $W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ to $W^{1,\infty}(\mathbb{R}^n)$.*

2 Preliminaries

In this paper, $\widehat{\mathbb{R}^n}:=\mathbb{R}^n\cup\{\infty\}$ is the one-point compactification of $\mathbb{R}^n$. Next, $z=(t,x)\in\mathbb{R}\times\mathbb{R}^{n-1}=\mathbb{R}^n$ means a point in the n-dimensional Euclidean space $\mathbb{R}^n$. We write $C=C(a_1,a_2,...,a_n)$ to indicate a constant C that depends only on the parameters $a_1,a_2,...,a_n$; the notation $A\lesssim B$ means there exists a finite constant c with $A\leq cB$, and $A\sim_c B$ means $\frac{1}{c}A\leq B\leq cA$ for a constant $c>1$. Typically $c,C,...$ will be constants that depend on various parameters and may differ even on the same line of inequalities. The Euclidean distance between given points z_1, z_2 in the Euclidean space $\mathbb{R}^n$ is denoted by $d(z_1, z_2)$ or $|z_1 - z_2|$. Then the distance between two sets $A, B \subset \mathbb{R}^n$ is denoted by

$$d(A, B) := \inf\{d(z_1, z_2) : z_1 \in A, z_2 \in B\}.$$

The open ball of radius r, centered at the point z, is denoted by $B(z, r)$. In what follows, $\Omega \subset \mathbb{R}^n$ is always a domain, and $\partial\Omega$ is the boundary of Ω. The r-neighborhood of Ω is

$$B(\Omega, r) := \{z \in \mathbb{R}^n : d(z, \Omega) < r\}.$$

Given a Lebesgue-measurable set $A \subset \mathbb{R}^n$, $|A|$ refers to the n-dimensional Lebesgue measure. The interior of a set $A \subset \mathbb{R}^n$ is denoted by $\mathring{A}$. For a locally integrable function u and a measurable set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$, we define the integral average of u over A by setting

$$\fint_A u(z)dz := \frac{1}{|A|} \int_A u(z)dz.$$

The Sobolev space $W^{1,p}(\Omega)$ for $p \in [1, \infty]$ is the collection of all functions $u \in L^p(\Omega)$ whose norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \||Du|\|_{L^p(\Omega)}$$

is finite. Here $Du = (g_1, g_2, ..., g_n)$ is the *distributional gradient* of u, where g_i is the weak partial derivative of u with respect to x_i. A mapping $f = (f_1, f_2, \cdots, f_m) : \Omega \to \Omega'$ is said to be in the class $W^{1,p}(\Omega, \Omega')$, if every component f_i is in the Sobolev space $W^{1,p}(\Omega)$.

The outward cuspidal domain Ω_ψ has a boundary singularity but it is still rather nice. For example, both the outward cuspidal domain Ω_ψ and its complement $\mathbb{R}^n \setminus \overline{\Omega_\psi}$ satisfy the segment condition.

defn:segment **Definition 2.1.** *We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition if every $x \in \partial\Omega$ has a neighborhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$.*

Smooth functions are dense in our Sobolev space for domains satisfying the segment condition. See [1, Theorem 3.22], [24].

lem:density **Lemma 2.1.** *If the domain $\Omega \subset \mathbb{R}^n$ satisfies the segment condition, then the set of restrictions to Ω of functions in $C_o^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$. In short, $C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p < \infty$.*

Let us give the definition of Sobolev extension domains.

,q)-extension **Definition 2.2.** *Let $1 \le q \le p \le \infty$. We say that a domain $\Omega \subset \mathbb{R}^n$ is a (p,q)-extension domain, if for every $u \in W^{1,p}(\Omega)$, there exists a function $E(u) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ with $E(u)\big|_\Omega \equiv u$ and*

$$\|E(u)\|_{W^{1,q}(\mathbb{R}^n\setminus\overline{\Omega})} \le C\|u\|_{W^{1,p}(\Omega)}$$

with a constant C independent of u.

Lipschitz domains are typical examples of Sobolev extension domains. By the results due to Calderón and Stein [27], Lipschitz domains are (p,p)-extension domains for $1 \le p \le \infty$. For the definition of Lipschitz domains, see [3, Definition 4.4]. As a generalization of the extension result for Lipschitz domains, Jones [16] proved that (ϵ,δ)-domains are also (p,p)-extension domains.

defn:ED **Definition 2.3.** *We say $\Omega \subset \mathbb{R}^n$ is an (ϵ,δ)-domain for some positive constants $0 < \epsilon < 1$ and $\delta > 0$ if, whenever $z_1, z_2 \in \Omega$ with $|z_1 - z_2| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ joining x to y and satisfying*

$$l(\gamma) \le \frac{1}{\epsilon}|z_1 - z_2|$$

and

$$d(z, \Omega^c) \ge \frac{\epsilon|z_1 - z||z_2 - z|}{|z_1 - z_2|} \text{ for all } z \text{ on } \gamma.$$

defn:ref **Definition 2.4.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. A self-homeomorphism $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ is called a reflection over $\partial\Omega$, if $\mathcal{R}(\widehat{\mathbb{R}^n} \setminus \overline{\Omega}) = \Omega$, $\mathcal{R}(\Omega) = \widehat{\mathbb{R}^n} \setminus \overline{\Omega}$ and for every $z \in \partial\Omega$, $\mathcal{R}(z) = z$.*

The following technical lemma justifies our terminology.

cut-off **Proposition 2.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $|\partial\Omega| = 0$ and $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be a reflection over $\partial\Omega$. If $\mathcal{R}$ induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ in the sense of (1.3) (from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ to $W^{1,q}(\mathbb{R}^n)$, respectively) for $1 \le q \le p < \infty$, then Ω ($\mathbb{R}^n \setminus \overline{\Omega}$, respectively) is a (p,q)-extension domain with a linear extension operator.*

Proof. We only consider the case of Ω, since the case of $\mathbb{R}^n \setminus \overline{\Omega}$ is analogous. Let $U \subset \mathbb{R}^n$ be the corresponding open set which contains $\partial\Omega$. For a given function $u \in W^{1,p}(\Omega)$, we define a function $E_{\mathcal{R}}(u)$ by setting

equa:E_r(u)
$$E_{\mathcal{R}}(u)(z) := \begin{cases} u(\mathcal{R}(z)), & \text{for } z \in U \setminus \overline{\Omega}, \\ 0, & \text{for } z \in \partial\Omega, \\ u(z), & \text{for } z \in \Omega. \end{cases} \tag{2.1}$$

Then $E_{\mathcal{R}}(u)$ has a representative that belongs to $W^{1,q}(U)$ with

$$\|E_{\mathcal{R}}(u)\|_{W^{1,q}(U)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

Let $\mathcal{L}:\mathbb{R}^n\to\mathbb{R}$ be a Lipschitz function such that $\mathcal{L}\big|_{\overline{\Omega}}\equiv 1$, $\mathcal{L}\big|_{\mathbb{R}^n\setminus U}\equiv 0$ and $0\le\mathcal{L}(z)\le 1$ for every $z\in\mathbb{R}^n$. We define a function on $\mathbb{R}^n$ by setting

equa:exglo
$$\tilde{E}_{\mathcal{R}}(u):=\mathcal{L}\cdot E_{\mathcal{R}}(u). \tag{2.2}$$

Since $\mathcal{L}$ is Lipschitz with $0\le\mathcal{L}\le 1$, $\tilde{E}_{\mathcal{R}}(u)$ has a representative that belongs to $W^{1,q}(\mathbb{R}^n)$. Now

$$\begin{aligned}\int_{\mathbb{R}^n}|\tilde{E}_{\mathcal{R}}(u)(z)|^q dz &\le \int_\Omega |u(z)|^q dz+\int_U |E_{\mathcal{R}}(u)(z)|^q dz\\ &\le\left(\int_\Omega |u(z)|^p dz+\int_\Omega |Du(z)|^p dz\right)^{\frac{q}{p}},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}|D\tilde{E}_{\mathcal{R}}(u)(z)|^q dz &\le C\int_U |E\mathcal{R}(u)(z)D\mathcal{L}(z)|^q dz+C\int_U|\mathcal{L}(z)\nabla E_{\mathcal{R}}(u)(z)|^q dz\\ &\quad +C\int_\Omega |Du(z)|^q dz\\ &\le C\left(\int_\Omega |u(z)|^p dz+\int_\Omega|\nabla u(z)|^p dz\right)^{\frac{q}{p}}.\end{aligned}$$

By combining these two inequalities, we obtain that $\tilde{E}_{\mathcal{R}}(u)\in W^{1,q}(\mathbb{R}^n)$ with $\tilde{E}_{\mathcal{R}}(u)\big|_\Omega\equiv u$ and

$$\|\tilde{E}_{\mathcal{R}}(u)\|_{W^{1,q}(\mathbb{R}^n)}\le C\|u\|_{W^{1,p}(\Omega)}.$$

Hence, (2.2) defines a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$. □

By Proposition 2.1, in order to prove that a reflection $\mathcal{R}$ over $\partial\Omega$ induces a bounded linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ for some $1\le q\le p\le\infty$, it suffices to prove that, for every $u\in W^{1,p}(\Omega)$, the function $E_{\mathcal{R}}(u)$ defined in (2.1) satisfies the inequality

$$\|E_{\mathcal{R}}(u)\|_{W^{1,q}(U)}\le C\|u\|_{W^{1,p}(\Omega)}$$

with a constant C independent of u.

Let $f:\Omega\to\Omega'$ be a homeomorphism. If for every $z\in U$ there is an open set containing z and a constant $C>1$ such that for all $x,y\in U$, we have

$$\frac{1}{C}|x-y|\le|f(x)-f(y)|\le C|x-y|,$$

we call f a locally bi-Lipschitz homeomorphism.

By combining results in [28, 29, 30, 32], we obtain following two lemmas.

`QCcompo` **Lemma 2.2.** *Suppose that $f:\Omega\to\Omega'$ is a homeomorphism in the class $W^{1,1}_{\rm loc}(\Omega,\Omega')$. Fix $1\le p<\infty$. Then the following assertions are equivalent:*
(1) : *for every locally Lipschitz function u, defined on Ω', the inequality*

$$\int_\Omega |D(u\circ f)(z)|^p dz \le C\int_{\Omega'} |Du(z)|^p dz$$

holds for a positive constant C independent of u;
(2) : *the inequality*

$$|Df(z)|^p \le C(p)|J_f(z)|$$

holds almost everywhere in Ω.

`lem:pQc` **Lemma 2.3.** *Let $1\le q<p<\infty$. Suppose that $f:\Omega\to\Omega'$ is a homeomorphism in the class $W^{1,1}_{\rm loc}(\Omega,\Omega')$. Then the following assertions are equivalent:*
(1) : *for every locally Lipschitz function u, the inequality*

$$\left(\int_\Omega |D(u\circ f)(z)|^q dz\right)^{\frac{1}{q}} \le C\left(\int_{\Omega'} |Du(z)|^p dz\right)^{\frac{1}{p}}$$

holds for a positive constant C independent of u;
(2) :

$$\int_\Omega \frac{|Df(z)|^{\frac{pq}{p-q}}}{|J_f(z)|^{\frac{q}{p-q}}} dz<\infty.$$

The following lemma is a special case of [31, Theorem 3].

`reduinverse` **Lemma 2.4.** *Let $\Omega,\Omega'\subset\mathbb{R}^n$ be domains, and let $f:\Omega\to\Omega'$ be a homeomorphism in the class $W^{1,p}_{\rm loc}(\Omega,\Omega')$ for a fixed $n-1<p<\infty$. If*

`pdisin` (2.3)
$$|Df(z)|^p \le C(p)|J_f(z)|$$

holds for almost every $z\in\Omega$, then the inverse homeomorphism $f^{-1}:\Omega'\to\Omega$ belongs to the class $W^{1,\frac{p}{p+1-n}}_{\rm loc}(\Omega',\Omega)$ with

`inverdisin` (2.4)
$$|Df^{-1}(z)|^{\frac{p}{p+1-n}} \le C(p)|J_{f^{-1}}(z)|$$

for almost every $z\in\Omega'$.

3 Main Results

In this section, we show that the Sobolev extension results for outward cuspidal domain $\Omega_\psi\subset\mathbb{R}^n$ from [20, 21, 22] can be achieved via bounded linear extension operators induced by reflections. Let ψ be a cuspidal function. It follows from the definition that there exists a positive constant C such that we have

eq:condition1 (3.1)
$$\lim_{t\to 0^+}\frac{\psi(t)}{t}=0 \text{ and } \frac{\psi(t)}{t}\le \psi'(t)\le C\frac{\psi(t)}{t} \text{ for every } t\in(0,1),$$

eq:condition2 (3.2)
$$\frac{1}{C}(\psi^{-1})'(t)\le \frac{\psi^{-1}(t)}{t}\le C(\psi^{-1})'(t) \text{ for every } t\in(0,1)$$

and

eq:condition3 (3.3)
$$\lim_{t\to 0^+}\frac{\psi^{-1}(t)}{t}=\infty \text{ and } t\le C\psi^{-1}(t) \text{ for every } t\in(0,1).$$

3.1 Reflection $\mathcal{R}_1$ over $\partial\Omega_\psi$

sec:ref1

In oder to introduce the reflection $\mathcal{R}_1:\widehat{\mathbb{R}^n}\to\widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$, we define a domain $\Delta\subset\mathbb{R}^n$ by setting

equa:delta (3.4)
$$\Delta:=\left\{(t,x)\in\mathbb{R}\times\mathbb{R}^{n-1}:\frac{-1}{2}<t<\frac{1}{2},|x|<\frac{1}{2}\right\}\cup\Omega_\psi.$$

See Figure 2. To begin, we divide $\Delta\setminus\overline{\Omega_\psi}$ into three parts A,B,C by setting

$$A:=\left\{(t,x)\in\mathbb{R}\times\mathbb{R}^{n-1}:\tfrac{-1}{2}<t\le 0,|x|\le|t|\right\},$$

$$B:=\left\{(t,x)\in\mathbb{R}\times\mathbb{R}^{n-1}:\tfrac{-1}{2}<t<\tfrac{1}{2},|t|\le|x|<\tfrac{1}{2}\right\}$$

and

$$C:=\left\{(t,x)\in\mathbb{R}\times\mathbb{R}^{n-1}:0\le t<\tfrac{1}{2},\psi(t)\le|x|\le t\right\}.$$

We define a subdomain $\Omega_1^\psi\subset\Omega_\psi$ around the tip by setting

subdomain (3.5)
$$\Omega_1^\psi:=\left\{(t,x)\in\Omega_\psi;0<t<\frac{1}{2},|x|<\psi(t)\right\}.$$

We will construct a reflection $\mathcal{R}_1$ which maps $\Delta\setminus\overline{\Omega_\psi}$ onto Ω_1^s. We define $\mathcal{R}_1$ on $\Delta\setminus\overline{\Omega_\psi}$ by setting

HOMEOMOR' (3.6)
$$\mathcal{R}_1(t,x):=\begin{cases}\left(-t,\frac{\psi(t)}{6t}x\right), & \text{if } (t,x)\in A,\\ \left(|x|,\left(\frac{\psi(|x|)}{12|x|}t+\frac{\psi(|x|)}{4}\right)\frac{x}{|x|}\right), & \text{if } (t,x)\in B,\\ \left(t,\frac{2\psi(t)}{3(\psi(t)-t)}x+\frac{\psi^2(t)-3t\psi(t)}{3(\psi(t)-t)}\frac{x}{|x|}\right), & \text{if } (t,x)\in C.\end{cases}$$

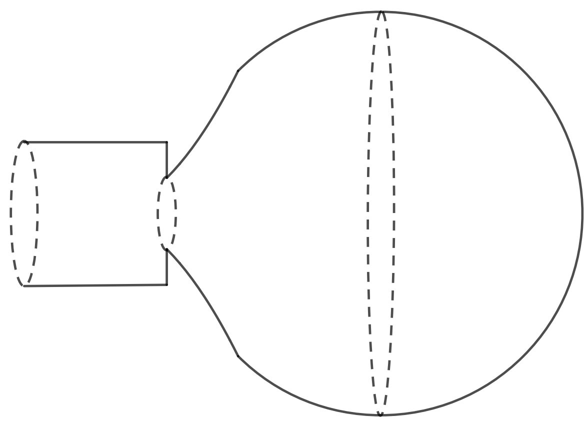

Figure 2: The domain Δ fig:5

We extend $\mathcal{R}_1$ to $\partial\Omega_\psi$ as the identity. Since both $\partial\Delta$ and $\partial(\Omega_\psi \setminus \Omega_1^\psi)$ are bi-Lipschitz equivalent to the unit sphere, it is easy to check that we can construct a reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ such that $\mathcal{R}_1$ is defined as above on $\Delta \setminus \Omega_\psi$, and $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \Delta$.

For $(t, x) \in \mathring{A}$, the resulting differential matrix of $\mathcal{R}_1$ is

differeninA (3.7)
$$D\mathcal{R}_1(t,x) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ \left(\frac{\psi'(t)}{6t} - \frac{\psi(t)}{6t^2}\right)x_1 & \frac{\psi(t)}{6t} & 0 & \cdots & 0 \\ \left(\frac{\psi'(t)}{6t} - \frac{\psi(t)}{6t^2}\right)x_2 & 0 & \frac{\psi(t)}{6t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \left(\frac{\psi'(t)}{6t} - \frac{\psi(t)}{6t^2}\right)x_{n-1} & 0 & \cdots & 0 & \frac{\psi(t)}{6t} \end{pmatrix}. \tag{3.7}$$

By (3.1) and the fact that $|x| < |t|$ for every $(t, x) \in \mathring{A}$, we have

equa:ref11 (3.8)
$$|D\mathcal{R}_1(t,x)| \lesssim 1 \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c \left(\frac{\psi(t)}{t}\right)^{n-1}. \tag{3.8}$$

For $(t,x) \in \mathring{B}$, the resulting differential matrix $\mathcal{R}_1$ is

differeninB

$$D\mathcal{R}_1(t,x) = \begin{pmatrix} 0 & \frac{x_1}{|x|} & \frac{x_2}{|x|} & \cdots & \frac{x_{n-1}}{|x|} \\ \frac{x_1\psi(|x|)}{12|x|^2} & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ \frac{x_2\psi(|x|)}{12|x|^2} & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{x_{n-1}\psi(|x|)}{12|x|^2} & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix}, \tag{3.9}$$

where, for every $i, j \in \{1, 2, \cdots, n-1\}$, we set

$$A_j^i(t,x) = \begin{cases} \frac{x_i^2}{|x|^2}\left(\left(\frac{t\psi'(|x|)}{12|x|} - \frac{t\psi(|x|)}{6|x|^2}\right) + \left(\frac{\psi'(|x|)}{4} - \frac{\psi(|x|)}{4|x|}\right)\right) + \left(\frac{t\psi(|x|)}{12|x|^2} + \frac{\psi(|x|)}{4|x|}\right), & \text{if } \ i = j, \\ \frac{x_i x_j}{|x|^2}\left(\left(\frac{t\psi'(|x|)}{12|x|} - \frac{t\psi(|x|)}{6|x|^2}\right) + \left(\frac{\psi'(|x|)}{4} - \frac{\psi(|x|)}{4|x|}\right)\right), & \text{if } \ i \neq j. \end{cases}$$

Since $|t| \le |x| < \frac{1}{2}$, (3.1) and some computations give

equa:ref12

$$|D\mathcal{R}_1(t,x)| \lesssim 1 \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c \sum_{k=1}^{n-1} \frac{x_k^2\psi(|x|)}{12|x|^3} \prod_{i\neq k} A_i^i \sim_c \left(\frac{\psi(|x|)}{|x|}\right)^{n-1}. \tag{3.10}$$

For $(t,x) \in \mathring{C}$, the resulting differential matrix of $\mathcal{R}_1$ is

differeninC

$$D\mathcal{R}_1(t,x) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ A_t^1(t,x) & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ A_t^2(t,x) & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_t^{n-1}(t,x) & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix}, \tag{3.11}$$

where, for every $i, j \in \{1, 2, \cdots, n-1\}$, we have

$$A_j^i(t,x) := \begin{cases} \frac{2\psi(t)}{3(\psi(t)-t)} + \left(\frac{\psi^2(t)-3t\psi(t)}{3(\psi(t)-t)}\right)\left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right), & \text{if } \ i = j, \\ \left(\frac{3t\psi(t)-\psi^2(t)}{3(\psi(t)-t)}\right)\frac{x_i x_j}{|x|^3}, & \text{if } \ i \neq j. \end{cases}$$

and

$$A_t^i(t,x) := x_i\left(\frac{2(\psi(t) - t\psi'(t))}{3(\psi(t)-t)^2}\right) + \frac{x_i}{|x|}\left(\frac{\psi^2(t)\psi'(t) - 2t\psi(t)\psi'(t) - 2\psi^2(t) + 3t^2\psi'(t)}{3(\psi(t)-t)^2}\right).$$

Since $\psi(t) < |x| < t$, (3.1) and simple computations give

equa:ref13

$$|D\mathcal{R}_1(t,x)| \lesssim 1 \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c \frac{\psi^{n-1}(t)}{|x|^{n-1}}. \tag{3.12}$$

Finally, since $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \overline{\Delta}$, there exists a positive constant $C > 1$ such that for almost every $(t, x) \in B(\Omega_\psi, 1) \setminus \overline{\Delta}$, we have

equa:upper1
$$\frac{1}{C} \leq |D\mathcal{R}_1(t, x)| \leq C \text{ and } \frac{1}{C} \leq |J_{\mathcal{R}_1}(t, x)| \leq C. \tag{3.13}$$

It is easy to see that the restriction of $\mathcal{R}_1$ to $B(\Omega_\psi, 1) \setminus (\Omega_\psi \cup \{0\})$ is locally bi-Lipschitz.

3.2 Reflection $\mathcal{R}_2$ over $\partial\Omega_\psi$

In order to introduce the reflection $\mathcal{R}_2 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$, we define a domain $\Delta' \subset \mathbb{R}^n$ by setting

cone
$$\Delta' := \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n : \frac{-1}{2} < t < \frac{1}{2}, |x| < \psi\left(\frac{1}{2}\right)\right\} \cup \Omega_\psi. \tag{3.14}$$

See Figure 3. We divide $\Delta' \setminus \overline{\Omega_\psi}$ into two parts D, E by setting

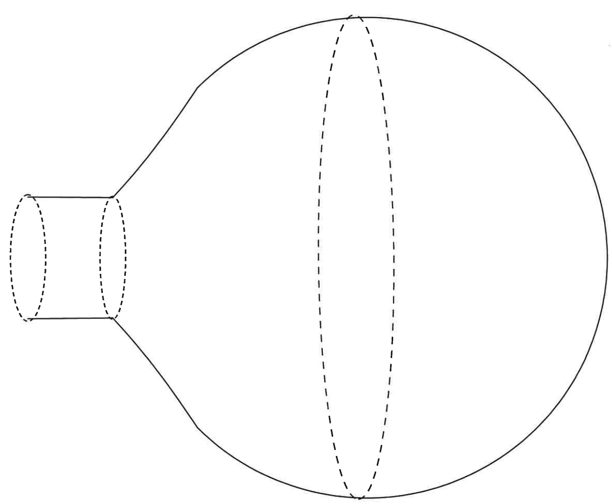

Figure 3: The domain Δ' fig:6

$$D := \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n : \frac{-1}{2} < t \leq 0, |x| \leq \psi(|t|)\right\},$$

and

$$E := \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n : \frac{-1}{2} < t < \frac{1}{2}, \psi(|t|) < |x| < \psi\left(\frac{1}{2}\right)\right\}.$$

We will construct a reflection $\mathcal{R}_2$ over $\partial\Omega_\psi$ that maps $\Delta' \setminus \overline{\Omega_\psi}$ onto Ω_1^ψ. We define the reflection $\mathcal{R}_2$ on $\Delta' \setminus \overline{\Omega_\psi}$ by setting

HOMEOMOR1
$$\mathcal{R}_2(t,x) := \begin{cases} \left(-t, \frac{1}{2}x\right), & \text{if } (t,x) \in D, \\ \left(\psi^{-1}(|x|), \frac{t}{4}\frac{x}{\psi^{-1}(|x|)} + \frac{3}{4}x\right), & \text{if } (t,x) \in E. \end{cases} \tag{3.15}$$

We extend $\mathcal{R}_2$ on $\partial\Omega_\psi$ as the identity. Since both $\partial\Delta'$ and $\partial(\Omega_\psi \setminus \Omega_1^\psi)$ are bi-Lipschitz equivalent to the unit sphere, we can construct a reflection $\mathcal{R}_2$ which is defined on $\Delta' \setminus \overline{\Omega_\psi}$ as in (3.15) and is bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \Delta'$.

For $z = (t,x) \in \mathring{D}$, the resulting differential matrix of $\mathcal{R}_2$ is

differeninA1
$$D\mathcal{R}_2(t,x) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} \end{pmatrix}. \tag{3.16}$$

Hence,

equa:ref21
$$|D\mathcal{R}_2(t,x)| = 1 \text{ and } |J_{\mathcal{R}_2}(t,x)| = \frac{1}{2^{n-1}}. \tag{3.17}$$

For $z = (t,x) \in \mathring{E}$, the resulting matrix of $\mathcal{R}_2$ is

$$D\mathcal{R}_2(t,x) = \begin{pmatrix} 0 & (\psi^{-1})'(|x|)\frac{x_1}{|x|} & (\psi^{-1})'(|x|)\frac{x_2}{|x|} & \cdots & (\psi^{-1})'(|x|)\frac{x_{n-1}}{|x|} \\ \frac{x_1}{4\psi^{-1}(|x|)} & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x) \\ \frac{x_2}{4\psi^{-1}(|x|)} & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{x_{n-1}}{4\psi^{-1}(|x|)} & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x) \end{pmatrix},$$

where, for every $i, j \in \{1, 2, \cdots, n-1\}$, we have

$$A_j^i(t,x) := \begin{cases} \frac{t}{4}\left(\frac{1}{\psi^{-1}(|x|)} - \frac{x_i^2}{|x|(\psi^{-1}(|x|))^2}(\psi^{-1})'(|x|)\right) + \frac{3}{4}, & \text{if } i = j, \\ \frac{-tx_ix_j(\psi^{-1})'(|x|)}{4|x|(\psi^{-1}(|x|))^2}, & \text{if } i \neq j. \end{cases} \tag{3.18}$$

By (3.2) and (3.3), after some simple computations, for every $(t,x) \in \mathring{E}$ we have equa:ref22

$$|D\mathcal{R}_2(t,x)| \lesssim (\psi^{-1})'(|x|) \sim_c \frac{\psi^{-1}(|x|)}{|x|} \quad \text{and} \qquad (3.19)$$

$$|J_{\mathcal{R}_2}(t,x)| \sim_c \sum_{k=1}^{n-1} \frac{(\psi^{-1})'(|x|)x_k^2}{4\psi^{-1}(|x|)|x|} \prod_{i\neq k} A_i^i \sim_c 1.$$

Finally, since $\mathcal{R}_2$ is bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \overline{\Delta'}$, there exists a positive constant $C > 1$ such that for almost every $(t,x) \in B(\Omega_\psi, 1) \setminus \overline{\Delta'}$, we have

equa:lipschitz

$$\frac{1}{C} \leq |D\mathcal{R}_2(t,x)| \leq C \text{ and } \frac{1}{C} \leq |J_{\mathcal{R}_2}(t,x)| \leq C. \qquad (3.20)$$

It is easy to see that the restriction of $\mathcal{R}_2$ to $B(\Omega_\psi, 1) \setminus (\Omega_\psi \cup \{0\})$ is locally bi-Lipschitz.

3.3 Proof of Theorem 1.1

We prove Theorem 1.1 in two parts, considering our two reflections separately.

thm5

Theorem 3.1. *Fix $1 < s < \infty$. Let ψ be a cuspidal function which satisfies inequality (1.2). Then the reflection $\mathcal{R}_1 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ induces a bounded linear extension operator from $W^{1,p}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$.*

Proof. Since Ω_ψ satisfies the segment condition, by Lemma 2.1, $C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$ is dense in $W^{1,p}(\Omega_\psi)$. Let $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$ be arbitrary. We define a function $E_{\mathcal{R}_1}(u)$ as in (2.1) and another function w by setting

defn:funcw

$$w(z) := \begin{cases} u \circ \mathcal{R}_1(z), & \text{if } z \in B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi}, \\ u(z), & \text{if } z \in \overline{\Omega_\psi}. \end{cases} \qquad (3.21)$$

Since $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$ and $\mathcal{R}_1$ is locally Lipschitz on $B(\Omega_\psi, 1) \setminus (\Omega_\psi \cup \{0\})$, the function w is locally Lipchitz on $B(\Omega_\psi, 1) \setminus \{0\}$. We claim that $w \in W^{1,q}(B(\Omega_\psi, 1))$ with

$$\|w\|_{W^{1,q}(B(\Omega_\psi,1))} \leq C\|u\|_{W^{1,p}(\Omega_\psi)}$$

for a constant $C > 1$ independent of u. These claims follow if we prove the above norm estimate with $B(\Omega_\psi, 1)$ replaced by $B(\Omega_\psi, 1) \setminus \{0\}$. Next, since w is locally Lipschitz and $|\partial\Omega_\psi| = 0$, it suffices to estimate the norm over the union of Ω_ψ and $B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi}$. Since $w = u \in W^{1,p}(\Omega_\psi)$ on Ω_ψ, our domain Ω_ψ has finite measure and $q < p$, we are reduced to estimating the norm over the second set in question. On this set, $w = u \circ \mathcal{R}_1$ almost everywhere and hence it suffices to prove the inequality

equa:norm1

$$\left(\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |u \circ \mathcal{R}_1(z)|^q dz \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_\psi} |u(z)|^p dz \right)^{\frac{1}{p}} \qquad (3.22)$$

and the inequality

equa:norm2 (3.23)
$$\left(\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |D(u\circ\mathcal{R}_1)(z)|^q dz\right)^{\frac{1}{q}} \le C\left(\int_{\Omega_\psi} |Du(z)|^p dz\right)^{\frac{1}{p}}.$$

It is easy to see that

$$B(\Omega_\psi,1)\setminus\overline{\Omega_\psi} = (B(\Omega_\psi,1)\setminus\Delta)\cup(\Delta\setminus\overline{\Omega_\psi})$$

and $\Delta\setminus\overline{\Omega_\psi} = A\cup B\cup C$. Since

$$|\partial\Delta| = |\partial A| = |\partial B| = |\partial C| = 0,$$

we have

equa:sum1 (3.24)
$$\begin{aligned}\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |u\circ\mathcal{R}_1(z)|^q dz = &\int_{B(\Omega_\psi,1)\setminus\overline{\Delta}} |u\circ\mathcal{R}_1(z)|^q dz \\ &+\left(\int_{\mathring{A}} + \int_{\mathring{B}} + \int_{\mathring{C}}\right)|u\circ\mathcal{R}_1(z)|^q dz.\end{aligned}$$

Since $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega_\psi,1)\setminus\overline{\Delta}$ and $|\Omega_\psi|<\infty$, by the Hölder inequality we have

equa:sum2 (3.25)
$$\int_{B(\Omega_\psi,1)\setminus\overline{\Delta}} |u\circ\mathcal{R}_1(z)|^q dz \le C\left(\int_{\Omega_\psi} |u(z)|^p dz\right)^{\frac{q}{p}}.$$

By the Hölder inequality and a change of variable, we have

akobi (3.26)
$$\begin{aligned}\int_{\mathring{A}} |u\circ\mathcal{R}_1(z)|^q dz &\le \left(\int_{\mathring{A}} |u\circ\mathcal{R}_1(z)|^p |J_{\mathcal{R}_1}(z)| dz\right)^{\frac{1}{p}} \cdot \left(\int_{\mathring{A}} \frac{1}{|J_{\mathcal{R}_1}^{\frac{q}{p-q}}(z)|} dz\right)^{\frac{p-q}{p}} \\ &\le \left(\int_{\Omega_\psi} |u(z)|^p dz\right)^{\frac{q}{p}} \cdot \left(\int_{\mathring{A}} \frac{1}{|J_{\mathcal{R}_1}^{\frac{q}{p-q}}(z)|} dz\right)^{\frac{p-q}{p}}.\end{aligned}$$

By (3.8), we have

$$\int_{\mathring{A}} \frac{1}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \le C\int_0^{\frac{1}{2}} \left(\frac{t^s}{\psi(t)}\right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty,$$

whenever $\frac{1+(n-1)s}{n} \le p < \infty$ and $1\le q\le \frac{np}{1+(n-1)s}$. Hence, we have

equa:sum3 (3.27)
$$\int_{\mathring{A}} |u\circ\mathcal{R}_1(z)|^q dz \le C\left(\int_{\Omega_\psi} |u(z)|^p dz\right)^{\frac{q}{p}}.$$

Next, via (3.10) and (3.12), we obtain the estimates

$$\int_{\mathring{B}} \frac{1}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \leq C \int_0^{\frac{1}{2}} \left(\frac{|x|^s}{\psi(|x|)} \right)^{\frac{n}{s-1}} \frac{d|x|}{|x|} < \infty$$

and

$$\int_{\mathring{C}} \frac{1}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \leq C \int_0^{\frac{1}{2}} \left(\frac{t^s}{\psi(t)} \right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty,$$

whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$. By repeating the argument leading to (3.27), we obtain the following desired analogs of (3.27):

equa:sum4 (3.28)
$$\int_{\mathring{B}} |u \circ \mathcal{R}_1(z)|^q dz \leq C \left(\int_{\Omega_\psi} |u(z)|^p dz \right)^{\frac{q}{p}}$$

and

equa:sum5 (3.29)
$$\int_{\mathring{C}} |u \circ \mathcal{R}_1(z)|^q dz \leq C \left(\int_{\Omega_\psi} |u(z)|^p dz \right)^{\frac{q}{p}},$$

whenever $\frac{1+(n-1)s}{n} \leq p < \infty$ and $1 \leq q \leq \frac{np}{1+(n-1)s}$. Hence (3.22) follows.

To prove inequality (3.23), by Lemma 2.3, it suffices to show that

$$\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz < \infty.$$

Clearly

$$\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz = \int_{B(\Omega_\psi,1)\setminus\overline{\Delta}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz + \int_{\Delta\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz.$$

First, by inequality (3.13), we have

$$\int_{B(\Omega_\psi,1)\setminus\overline{\Delta}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz < \infty.$$

Since $\Delta \setminus \overline{\Omega_\psi} = A \cup B \cup C$ and $|\partial A| = |\partial B| = |\partial C| = 0$, we have

$$\int_{\Delta\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz = \left(\int_{\mathring{A}} + \int_{\mathring{B}} + \int_{\mathring{C}} \right) \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz.$$

By (3.8), (3.10) and (3.12), we obtain

$$\int_{\mathring{A}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \leq C \int_0^{\frac{1}{2}} \left(\frac{t^s}{\psi(t)} \right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty,$$

$$\int_{\mathring{B}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dz \le C \int_0^{\frac{1}{2}} \left(\frac{|x|^s}{\psi(|x|)} \right)^{\frac{n}{s-1}} \frac{d|x|}{|x|} < \infty,$$

and

$$\int_{\mathring{C}} \frac{|D\mathcal{R}_1(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_1}(z)|^{\frac{q}{p-q}}} dx \le C \int_0^{\frac{1}{2}} \left(\frac{t^s}{\psi(t)} \right)^{\frac{n}{s-1}} \frac{dt}{t} < \infty,$$

whenever $\frac{1+(n-1)s}{n} \le p < \infty$ and $1 \le q \le \frac{np}{1+(n-1)s}$. In conclusion, we have proved that $w \in W^{1,q}(B(\Omega_\psi, 1))$ with the bound

$$\|w\|_{W^{1,q}(B(\Omega_\psi,1))} \le C\|u\|_{W^{1,p}(\Omega_\psi)}$$

whenever $\frac{1+(n-1)s}{n} \le p < \infty$ and $1 \le q \le \frac{np}{1+(n-1)s}$. Since $E_{\mathcal{R}_1}(u) = w$ almost everywhere, the above also holds with w replaced by $E_{\mathcal{R}_1}(u)$.

For an arbitrary $u \in W^{1,p}(\Omega_\psi)$, by the density of $C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$, we can find a sequence of functions $\{u_i\}_{i=1}^\infty \subset C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$ and a subset $N \subset \Omega_\psi$ with $|N| = 0$ such that

equa:limit0 (3.30)
$$\lim_{i\to\infty} \|u_i - u\|_{W^{1,p}(\Omega_\psi)} = 0,$$

and for every $z \in \Omega_\psi \setminus N$,

equa;limit (3.31)
$$\lim_{i\to\infty} |u_i(z) - u(z)| = 0.$$

By the argument above, for every $u_i \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$, we have $E_{\mathcal{R}_1}(u_i) \in W^{1,q}(B(\Omega_\psi, 1))$ and

equa:normc (3.32)
$$\|E_{\mathcal{R}_1}(u_i)\|_{W^{1,q}(B(\Omega_\psi,1))} \le C\|u_i\|_{W^{1,p}(\Omega_\psi)}$$

with a constant C independent of u_i. Since $\mathcal{R}_1$ is locally bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi}$, we have $\mathcal{R}_1(N) \subset B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi}$ with $|\mathcal{R}_1(N)| = 0$. By the definition of $E_{\mathcal{R}_1}(u_i)$ in (2.1), the sequence $\{E_{\mathcal{R}_1}(u_i)\}_{i=1}^\infty$ has a limit at every point $z \in B(\Omega_\psi, 1) \setminus (N \cup \mathcal{R}_1(N))$. Define

equa:definev (3.33)
$$v(z) := \begin{cases} \lim_{i\to\infty} E_{\mathcal{R}_1}(u_i)(z) & \text{if } \ z \in B(\Omega_\psi, 1) \setminus (N \cup \mathcal{R}_1(N)), \\ 0, & \text{if } \ z \in N \cup \mathcal{R}_1(N). \end{cases}$$

Since $\{u_i\}_{i=1}^\infty$ is a Cauchy sequence in $W^{1,p}(\Omega_\psi)$, the inequalities (3.30) and (3.32) yields that $\{E_{\mathcal{R}_1}(u_i)\}_{i=1}^\infty$ is also a Cauchy sequence in $W^{1,q}(B(\Omega_\psi, 1))$. Hence $v \in W^{1,q}(B(\Omega_\psi, 1))$ with

$$\|v\|_{W^{1,q}(B(\Omega_\psi,1))} \le C\|u\|_{W^{1,p}(\Omega_\psi)}.$$

By definition, we conclude that $E_{\mathcal{R}_1}(u)(z) = v(z)$ for every $z \in B(\Omega_\psi, 1) \setminus (N \cup \mathcal{R}_1(N))$. Since $|N \cup \mathcal{R}_1(N)| = 0$, we have $E_{\mathcal{R}_1}(u) \in W^{1,q}(B(\Omega_\psi, 1))$ with

$$\|E_{\mathcal{R}_1}(u)\|_{W^{1,q}(B(\Omega_\psi,1))} = \|v\|_{W^{1,q}(B(\Omega_\psi,1))} \le C\|u\|_{W^{1,p}(\Omega_\psi)}.$$

□

thm6 **Theorem 3.2.** *Fix $1 < s < \infty$. Let ψ be a cuspidal function which satisfies inequality (1.2). Then the reflection $\mathcal{R}_2 : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ induces a bounded linear extension operator from $W^{1,p}(\Omega_\psi)$ to $W^{1,q}(\mathbb{R}^n)$, whenever $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$ and $1 \leq q \leq \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$.*

Proof. Let $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$ be arbitrary. We define a function $E_{\mathcal{R}_2}(u)$ as in (2.1) and another function w by setting

defn:funcw'
$$w(z) := \begin{cases} u \circ \mathcal{R}_2(z), & \text{if } \ z \in B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi}, \\ u(z), & \text{if } \ z \in \overline{\Omega_\psi}. \end{cases} \tag{3.34}$$

We claim that $w \in W^{1,q}(B(\Omega_\psi, 1))$ with

$$\|w\|_{W^{1,q}(B(\Omega_\psi,1))} \leq C\|u\|_{W^{1,p}(\Omega_\psi)}$$

for a constant $C > 1$ independent of u. As in the proof of Theorem 3.1, it suffices to estimate the norm over the union of Ω_ψ and $B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi}$ and we are again reduced to estimating the norm over the second set in question. On this set, $w = u \circ \mathcal{R}_2$ almost everywhere and hence it suffices to prove the inequality

equa:Norm1
$$\left(\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |u \circ \mathcal{R}_2(z)|^q dz \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_\psi} |u(z)|^p dz \right)^{\frac{1}{p}} \tag{3.35}$$

and the inequality

equa:Norm2
$$\left(\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |D(u \circ \mathcal{R}_2)(z)|^q dz \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_\psi} |Du(z)|^p dz \right)^{\frac{1}{p}}. \tag{3.36}$$

Now

$$B(\Omega_\psi, 1) \setminus \overline{\Omega_\psi} = (B(\Omega_\psi, 1) \setminus \Delta') \cup (\Delta' \setminus \overline{\Omega_\psi})$$

and $\Delta' \setminus \overline{\Omega_\psi} = D \cup E$. Since

$$|\partial\Delta'| = |\partial D| = |\partial E| = 0,$$

we have

equa:Sum1
$$\begin{aligned} \int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |u \circ \mathcal{R}_2(z)|^q dz = & \int_{B(\Omega_\psi,1)\setminus\overline{\Delta'}} |u \circ \mathcal{R}_2(z)|^q dz \\ & + \left(\int_{\mathring{D}} + \int_{\mathring{E}} \right) |u \circ \mathcal{R}_2(z)|^q dz. \end{aligned} \tag{3.37}$$

Since $\mathcal{R}_2$ is bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \overline{\Delta'}$ and $|\Omega_\psi| < \infty$, by the Hölder inequality, we have

equa:Sum2
$$\int_{B(\Omega_\psi,1)\setminus\overline{\Delta'}} |u \circ \mathcal{R}_2(z)|^q dz \le C \left(\int_{\Omega_\psi} |u(z)|^p dz \right)^{\frac{q}{p}}. \tag{3.38}$$

Since $|J_{\mathcal{R}_2}(t,x)| \sim 1$ on $\mathring{E} \cup \mathring{D}$, by (3.17) and (3.19), we conclude by computing as in (3.26) that

equa:Sum3
$$\int_{\mathring{E}\cup\mathring{D}} |u \circ \mathcal{R}_2(z)|^q dz \le C \left(\int_{\Omega_\psi} |u(z)|^p dz \right)^{\frac{q}{p}}, \tag{3.39}$$

whenever $\frac{1+(n-1)s}{2+(n-2)s} \le p < \infty$ and $1 \le q \le \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. By combining inequalities (3.37)-(3.39), we obtain inequality (3.35).

To prove inequality (3.36), by Lemma 2.2, it suffices to show that

$$\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz < \infty.$$

Trivially,

$$\int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz = \int_{B(\Omega_\psi,1)\setminus\overline{\Delta'}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz + \int_{\Delta'\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz.$$

Since $\mathcal{R}_2$ is bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \overline{\Delta'}$, we have

$$\int_{B(\Omega_\psi,1)\setminus\overline{\Delta'}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz < \infty.$$

Since $\Delta' \setminus \overline{\Omega_\psi} = D \cup E$, $|\partial D| = |\partial E| = 0$, inequalities (3.17), (3.19) give

Summa3'
$$\begin{aligned} \int_{\Delta'\setminus\overline{\Omega_\psi}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz &\le \int_{\mathring{D}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz + \int_{\mathring{E}} \frac{|D\mathcal{R}_2(z)|^{\frac{pq}{p-q}}}{|J_{\mathcal{R}_2}(z)|^{\frac{q}{p-q}}} dz \\ &\le C \int_0^{\frac{1}{2}} \int_{\psi(t)}^{\psi\left(\frac{1}{2}\right)} |x|^{n-2} ((\psi^{-1})'(|x|))^{\frac{pq}{p-q}} d|x| dt + C \\ &\le C \int_0^{\frac{1}{2}} \int_{\psi(t)}^{\psi\left(\frac{1}{2}\right)} |x|^{n-2-\frac{pq}{p-q}} (\psi^{-1}(|x|))^{\frac{pq}{p-q}} d|x| dt + C \\ &\le C \int_0^{\frac{1}{2}} \left(\frac{t^s}{\psi(t)} \right)^{\frac{n}{s-1}} \frac{dt}{t} + C < \infty, \end{aligned} \tag{3.40}$$

whenever $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$ and $1 \leq q \leq \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. In conclusion, we have proved that $w \in W^{1,q}(B(\Omega_\psi, 1))$ with the bound

$$\|w\|_{W^{1,q}(B(\Omega_\psi,1))} \leq C\|u\|_{W^{1,p}(\Omega_\psi)}$$

whenever $\frac{1+(n-1)s}{2+(n-2)s} \leq p < \infty$ and $1 \leq q \leq \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p}$. Since $E_{\mathcal{R}_2}(u) = w$ almost everywhere, the above also holds with w replaced by $E_{\mathcal{R}_2}(u)$. Hence, we may complete the proof by following the argument of the proof of Theorem 3.1.

□

3.4 Proof of Theorem 1.2

We begin with a useful observation.

lem:reftwo **Lemma 3.1.** *Let ψ be an arbitrary cuspidal function. If there is a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ which induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$, then $\mathcal{R} \in W^{1,p}_{\text{loc}}(G \cap \Omega_\psi, \mathbb{R}^n)$ and*

$$|D\mathcal{R}(z)|^p \leq C|J_{\mathcal{R}}(z)|$$

for almost every $z \in G \cap \Omega_\psi$, where G is a bounded open set containing $\partial\Omega_\psi$.

Proof. Let $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ be a reflection over $\partial\Omega_\psi$ which induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$. Then there exists a bounded open set U containing $\partial\Omega_\psi$ so that the function

equa:T_r(u) (3.41)
$$E_{\mathcal{R}}(u)(z) := \begin{cases} u \circ \mathcal{R}(z), & \text{for } z \in U \cap \Omega_\psi, \\ 0, & \text{for } z \in \partial\Omega_\psi, \\ u(z), & \text{for } z \in U \setminus \overline{\Omega_\psi} \end{cases}$$

belongs to $W^{1,p}(U)$ whenever $u \in W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ and satisfies

$$\|E_{\mathcal{R}}(u)\|_{W^{1,p}(U)} \leq C\|u\|_{W^{1,p}(U\setminus\Omega_\psi)}$$

for a positive constant C independent of u. We employ an idea from [15] and pick a Lipschitz domain G so that $\partial\Omega_\psi \subset G$, $G \subset U$ and $\partial G \subset U$. Since G is Lipschitz and contains the boundary of Ω_ψ, the geometry of Ω_ψ easily yields that $G \setminus \overline{\Omega_\psi}$ is an (ϵ, δ)-domain for some positive ϵ, δ. Since $u - u_{G\setminus\overline{\Omega_\psi}} \in W^{1,p}(G \setminus \overline{\Omega_\psi})$ and (ϵ, δ)-domains are (p, p)-extension domains, we find a function $v \in W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ such that $v = u - u_{G\setminus\overline{\Omega_\psi}}$ on $G \setminus \overline{\Omega_\psi}$ and

palautus0 (3.42)
$$\|v\|_{W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega_\psi})} \leq C\|u - u_{G\setminus\overline{\Omega_\psi}}\|_{W^{1,p}(G\setminus\overline{\Omega_\psi})}.$$

Next, since $G\setminus\overline{\Omega_\psi}$ is a bounded (ϵ,δ)-domain, we have

palautus2
$$\int_{G\setminus\overline{\Omega_\psi}}|u(z)-u_{G\setminus\overline{\Omega_\psi}}|^p dz\leq C\int_{G\setminus\overline{\Omega_\psi}}|Du(z)|^p dz, \tag{3.43}$$

see [2, 26]. By our assumption, (3.42) and (3.43), we have

$$\begin{aligned}\|v\circ\mathcal{R}\|_{W^{1,p}(G\cap\Omega_\psi)}&\leq C\|v\|_{W^{1,p}(U\setminus\overline{\Omega_\psi})}\\ &\leq C\|u-u_{G\setminus\overline{\Omega_\psi}}\|_{W^{1,p}(G\setminus\overline{\Omega_\psi})}\leq C\|Du\|_{L^p(G\setminus\overline{\Omega_\psi})}.\end{aligned}$$

Clearly $v\circ\mathcal{R}=u\circ\mathcal{R}-u_{G\setminus\overline{\Omega_\psi}}$ on $G\cap\Omega_\psi$. Hence, we have proven that

equa:Lpex
$$\int_{G\cap\Omega_\psi}|DE_{\mathcal{R}}(v)(z)|^p dz\leq C\int_{G\setminus\overline{\Omega_\psi}}|Dv(z)|^p dz. \tag{3.44}$$

Let now u be locally Lipschitz on $G\setminus\overline{\Omega_\psi}$ with

$$\int_{G\setminus\overline{\Omega_\psi}}|Du(z)|^p dz<\infty.$$

It follows from (3.43) applied to truncations of u that $u\in W^{1,p}(G\setminus\overline{\Omega_\psi})$, see [15]. Recalling that functions in $W^{1,p}(G\setminus\overline{\Omega_\psi})$ can be extended to $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$, we may apply (3.44) also to our locally Lipschitz function u. Lemma 2.2 now gives the asserted inequality. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let ψ be an arbitrary cuspidal function. It is easy to check that $\mathbb{R}^n\setminus\overline{\Omega_\psi}$ is an (ϵ,δ)-domain, for some positive constants ϵ and δ. Hence, by [16], $\mathbb{R}^n\setminus\overline{\Omega_\psi}$ is a (p,p)-extension domain, for every $p\in[1,\infty)$.

We begin by showing that the reflection $\mathcal{R}_1$ induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$, whenever $1\leq p\leq n-1$. Define the domain Δ as in (3.4) and the domain Ω_1^ψ as in (3.5). By (3.6), the formula of the reflection $\mathcal{R}_1$ on Ω_1^ψ is

REFLEC
$$\mathcal{R}_1(t,x)=\begin{cases}\left(-t,\frac{6tx}{\psi(t)}\right), & \text{if } 0\leq|x|<\frac{1}{6}\psi(t),\\ \left(\frac{12t|x|}{\psi(t)}-3t,t\frac{x}{|x|}\right), & \text{if } \frac{1}{6}\psi(t)\leq|x|<\frac{1}{3}\psi(t),\\ \left(t,\frac{3(\psi(t)-t)}{2\psi(t)}x+\left(\frac{3t}{2}-\frac{\psi(t)}{2}\right)\frac{x}{|x|}\right), & \text{if } \frac{1}{3}\psi(t)\leq|x|<\psi(t).\end{cases} \tag{3.45}$$

For every $(t,x)\in\Omega_1^\psi$ with $0<|x|<\frac{1}{6}\psi(t)$, the resulting differential matrix of $\mathcal{R}_1$ is

$$D_{\mathcal{R}_1}(t,x)=\begin{pmatrix}-1 & 0 & 0 & \cdots & 0\\ x_1\frac{6\psi(t)-6t\psi'(t)}{\psi^2(t)} & \frac{6t}{\psi(t)} & 0 & \cdots & 0\\ x_2\frac{6\psi(t)-6t\psi'(t)}{\psi^2(t)} & 0 & \frac{6t}{\psi(t)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ x_{n-1}\frac{6\psi(t)-6t\psi'(t)}{\psi^2(t)} & 0 & 0 & \cdots & \frac{6t}{\psi(t)}\end{pmatrix}.$$

By (3.1), after a simple computation, for every $(t,x)\in\Omega_1^\psi$ with $0<|x|<\frac{1}{6}\psi(t)$, we have

Distor1 (3.46)
$$|D\mathcal{R}_1(t,x)|\le\frac{Ct}{\psi(t)}\text{ and }|J_{\mathcal{R}_1}(t,x)|=\left(\frac{6t}{\psi(t)}\right)^{n-1}.$$

For every $(t,x)\in\Omega_1^\psi$ with $\frac{1}{6}\psi(t)<|x|<\frac{1}{3}\psi(t)$, the resulting differential matrix is

$$D\mathcal{R}_1(t,x)=\begin{pmatrix} 12|x|\frac{\psi(t)-t\psi'(t)}{\psi^2(t)}-3 & \frac{12tx_1}{|x|\psi(t)} & \frac{12tx_2}{|x|\psi(t)} & \cdots & \frac{12tx_{n-1}}{|x|\psi(t)}\\ \frac{x_1}{|x|} & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x)\\ \frac{x_2}{|x|} & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{x_{n-1}}{|x|} & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x)\end{pmatrix},$$

where, for every $i,j\in\{1,2,\cdots,n-1\}$, we have

$$A_j^i(t,x):=\begin{cases}\frac{t}{|x|}-\frac{tx_i^2}{|x|^3}, & \text{if } i=j,\\ \frac{-tx_ix_j}{|x|^3}, & \text{if } i\neq j.\end{cases}$$

After a simple computation, for every $(t,x)\in\Omega_1^\psi$ with $\frac{1}{6}\psi(t)<|x|<\frac{1}{3}\psi(t)$, we have

Distor2 (3.47)
$$|D\mathcal{R}_1(t,x)|\le\frac{Ct}{\psi(t)}\text{ and }|J_{\mathcal{R}_1}(t,x)|\sim_c\left(\frac{t}{\psi(t)}\right)^{n-1}.$$

For every $(t,x)\in\Omega_1^\psi$ with $\frac{1}{3}\psi(t)<|x|<\psi(t)$, the resulting differential matrix is

$$D\mathcal{R}_1(t,x)=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ A_t^1(t,x) & A_1^1(t,x) & A_2^1(t,x) & \cdots & A_{n-1}^1(t,x)\\ A_t^2(t,x) & A_1^2(t,x) & A_2^2(t,x) & \cdots & A_{n-1}^2(t,x)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ A_t^{n-1}(t,x) & A_1^{n-1}(t,x) & A_2^{n-1}(t,x) & \cdots & A_{n-1}^{n-1}(t,x)\end{pmatrix},$$

where, for every $i,j\in\{1,2,\cdots,n-1\}$, we have

$$A_j^i(t,x):=\begin{cases}\left(\frac{3}{2}-\frac{3t}{2\psi(t)}\right)+\left(\frac{3t}{2}-\frac{\psi(t)}{2}\right)\left(\frac{1}{|x|}-\frac{x_i^2}{|x|^3}\right), & \text{if } i=j,\\ -\left(\frac{3t}{2}-\frac{\psi(t)}{2}\right)\frac{x_ix_j}{|x|^3}, & \text{if } i\neq j.\end{cases}$$

and

$$A_t^i(t,x):=\frac{3x_i}{2}\left(\frac{t\psi'(t)-\psi(t)}{\psi^2(t)}\right)+\left(\frac{3}{2}-\frac{\psi'(t)}{2}\right)\frac{x_i}{|x|}.$$

By (3.1), after a simple computation, for every $(t,x) \in \Omega_1^{\psi}$ with $\frac{1}{3}\psi(t) < |x| < \psi(t)$, we have

Distor3
$$|D\mathcal{R}_1(t,x)| \le \frac{Ct}{\psi(t)} \text{ and } |J_{\mathcal{R}_1}(t,x)| \sim_c \left(\frac{t}{\psi(t)}\right)^{n-1}. \tag{3.48}$$

By combining (3.46), (3.47) and (3.48), we conclude that for $1 \le p \le n-1$,

equa:pdist
$$|D\mathcal{R}_1(z)|^p \le C|J_{\mathcal{R}_1}(z)| \tag{3.49}$$

for almost every $z \in \Delta \cap \Omega_\psi$. By the same inequalities, since $\mathcal{R}_1$ is locally bi-Lipschitz on $B(\Omega_\psi, 1) \setminus \overline{\Delta}$, for every $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$, we have

equa:LpF
$$\begin{aligned}\int_{\mathcal{R}_1(B(\Omega_\psi,1)\setminus\overline{\Omega_\psi})} |u \circ \mathcal{R}_1(z)|^p dz &\le C \int_{\mathcal{R}_1(B(\Omega_\psi,1)\setminus\overline{\Omega_\psi})} |u \circ \mathcal{R}_1(z)|^p |J_{\mathcal{R}_1}(z)| dz \\ &\le \int_{B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}} |u(z)|^p dz.\end{aligned} \tag{3.50}$$

Moreover, by Lemma 2.2 and (3.49), we have

equa:LpD
$$\int_{\mathcal{R}_1(B(\Omega_\psi,1)\setminus\overline{\Omega_\psi})} |D(u \circ \mathcal{R}_1)(z)|^p dz \le \int_{\mathbb{R}^n\setminus\overline{\Omega_\psi}} |Du(z)|^p dz. \tag{3.51}$$

Since $\mathbb{R}^n \setminus \overline{\Omega_\psi}$ satisfies the segment condition, (3.50) and (3.51) allow us to repeat the argument in the proof of Theorem 3.1 so as to conclude that $\mathcal{R}_1$ induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$, whenever $1 \le p \le n-1$.

Next, we show that there is no reflection over $\partial\Omega_\psi$ which can induce a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$, for any $n-1 < p < \infty$. Let $n-1 < p < \infty$ be fixed. Suppose that there exists a reflection $\mathcal{R} : \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ over $\partial\Omega_\psi$, which induces a bounded linear extension operator from $W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega_\psi})$ to $W^{1,p}(\mathbb{R}^n)$. By Lemma 3.1, there exists an open set G which contains $\partial\Omega_\psi$ such that for almost every $z \in G \cap \Omega_\psi$, we have

$$|D\mathcal{R}(z)|^p \le C|J_{\mathcal{R}}(z)|.$$

Then, by Lemma 2.4, for almost every $(t,x) \in \mathcal{R}(G \cap \Omega_\psi)$, we have

ineq:inver
$$|D\mathcal{R}(z)|^{\frac{p}{p+1-n}} \le C|J_{\mathcal{R}}(z)|. \tag{3.52}$$

Let $u \in C_o^\infty(\mathbb{R}^n) \cap W^{1,p}(\Omega_\psi)$ be arbitrary. By definition, $E_{\mathcal{R}}(u)$ is bounded and continuous on G. Pick a Lipschitz domain $\widetilde{G}$ so that $\overline{\Omega_\psi} \subset \widetilde{G}$ and $\partial\widetilde{G} \subset G$. By Lemma 2.2, we have

equa:HNC
$$\|DE_{\mathcal{R}}(u)\|_{L^{\frac{p}{p+1-n}}(\widetilde{G})} \le C\|Du\|_{L^{\frac{p}{p+1-n}}(\Omega_\psi)}. \tag{3.53}$$

We conclude that $E_{\mathcal{R}}(u) \in W^{1,\frac{p}{p+1-n}}(\widetilde{G})$. Since $\widetilde{G}$ is a Lipschitz domain, [15, Lemma 4.1] implies

equa:POIN

$$\int_{\widetilde{G}} |E_{\mathcal{R}}(u)(z) - u_{\Omega_\psi}|^{\frac{p}{p+1-n}} dz \leq C(\widetilde{G}, \Omega_\psi) \int_{\widetilde{G}} |DE_{\mathcal{R}}(u)(z)|^{\frac{p}{p+1-n}} dz. \tag{3.54}$$

Hence, we have

equa:HNC1

$$\|E_{\mathcal{R}}(u)\|_{L^{\frac{p}{p+1-n}}(\widetilde{G})} \leq C \left(\|DE_{\mathcal{R}}(u)\|_{L^{\frac{p}{p+1-n}}(\widetilde{G})} + \|u\|_{L^{\frac{p}{p+1-n}}(\Omega_\psi)} \right). \tag{3.55}$$

By combining inequalities (3.53) and (3.55), we obtain

$$\|E_{\mathcal{R}}(u)\|_{W^{1,\frac{p}{p+1-n}}(\widetilde{G})} \leq \|u\|_{W^{1,\frac{p}{p+1-n}}(\Omega_\psi)}. \tag{3.56}$$

Since $C_o^\infty(\mathbb{R}^n) \cap W^{1,\frac{p}{p+1-n}}(\Omega_\psi)$ is dense in $W^{1,\frac{p}{p+1-n}}(\Omega_\psi)$, for every function $u \in W^{1,\frac{p}{p+1-n}}(\Omega_\psi)$, there exists a sequence of functions $u_i \in C_o^\infty(\mathbb{R}^n) \cap W^{1,\frac{p}{p+1-n}}(\Omega_\psi)$ such that

equa:appro

$$\lim_{i\to\infty} \|u_i - u\|_{W^{1,\frac{p}{p+1-n}}(\Omega_\psi)} = 0, \tag{3.57}$$

and for almost every $z \in \Omega_\psi$,

$$\lim_{i\to\infty} |u_i(z) - u(z)| = 0.$$

By (3.53) and (3.57), $\{E_{\mathcal{R}}(u_i)\}_{i=1}^\infty$ is a Cauchy sequence in $W^{1,\frac{p}{p+1-n}}(\widetilde{G})$. By the completeness of $W^{1,\frac{p}{p+1-n}}(\widetilde{G})$, there exits a function $\omega \in W^{1,\frac{p}{p+1-n}}(\widetilde{G})$ with

$$\lim_{i\to\infty} \|w - E_{\mathcal{R}}(u_i)\|_{W^{1,\frac{p}{p+1-n}}(\widetilde{G})} = 0$$

and $\omega(z) = u(z)$ for almost every $z \in \Omega_\psi$. We define $E_{\mathcal{R}}(u)(z) := \omega(z)$ on $\widetilde{G}$. By (3.53) and (3.57) again, we have

$$\|E_{\mathcal{R}}(u)\|_{W^{1,\frac{p}{p+1-n}}(\widetilde{G})} \leq C\|u\|_{W^{1,\frac{p}{p+1-n}}(\Omega_\psi)}.$$

Hence, Ω_ψ is a Sobolev $\left(\frac{p}{p+1-n}, \frac{p}{p+1-n}\right)$-extension domain. This contradicts the classical result that Ω_ψ is not a (q,q)-extension domain, for any $1 \leq q < \infty$, see [19] and references therein. $\square$

3.5 Proof of Theorem 1.3

Proof of Theorem 1.3. Let ψ be an arbitrary cuspidal function. It is easy to see both Ω_ψ and $\mathbb{R}^n \setminus \overline{\Omega_\psi}$ are uniformly locally quasiconvex. By [10], they are (∞,∞)-extension domains.

To begin, we show that the reflection $\mathcal{R}_1$ induces a bounded linear extension operator from $W^{1,\infty}(\Omega_\psi)$ to $W^{1,\infty}(\mathbb{R}^n)$. Since Ω_ψ is uniformly quasiconvex, every function in $W^{1,\infty}(\Omega_\psi)$ has a Lipschitz representative. Without loss of generality, we assume every function in $W^{1,\infty}(\Omega_\psi)$ is Lipschitz. Let $u \in W^{1,\infty}(\Omega_\psi)$ be arbitrary. Define the extension $E_{\mathcal{R}_1}(u)$ on $B(\Omega_\psi, 1)$ as in (2.1). Since $u \in W^{1,\infty}(\Omega_\psi)$ is Lipschitz and $\mathcal{R}_1$ is locally Lipschitz on $B(\Omega_\psi,1)\setminus(\Omega_\psi \cup \{0\})$, we have $E_{\mathcal{R}_1}(u) \in W^{1,1}_{\mathrm{loc}}(B(\Omega_\psi,1)\setminus\overline{\Omega_\psi})$. By (3.8), (3.10), (3.12) and the fact that $\mathcal{R}_1$ is bi-Lipschitz on $B(\Omega_\psi,1)\setminus\overline{\Delta}$, for almost every $z \in B(\Omega_\psi,1)\setminus\overline{\Omega_\psi}$, we have

$$|DE_{\mathcal{R}_1}(u)(z)| \le C|Du(\mathcal{R}_1(z))|.$$

This implies that

$$\|E_{\mathcal{R}_1}(u)\|_{W^{1,\infty}(B(\Omega_\psi,1))} \le C\|u\|_{W^{1,\infty}(\Omega_\psi)}$$

as desired.

Next, we show that there does not exist a reflection over $\partial\Omega_\psi$ which can induce a bounded linear extension operator from $W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ to $W^{1,\infty}(\mathbb{R}^n)$. Define a function $u \in W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ by setting

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$$u(t,x) = \begin{cases} 1, & \text{if } (t,x) \in \mathbb{R}^n\setminus\overline{\Omega_\psi} \text{ and } t \ge 1, \\ t, & \text{if } (t,x) \in \mathbb{R}^n\setminus\overline{\Omega_\psi} \text{ and } 0 < t < 1, \\ 0, & \text{if } (t,x) \in \mathbb{R}^n\setminus\overline{\Omega_\psi} \text{ and } t \le 0. \end{cases} \tag{3.58}$$

For every $t \in (0,1)$ fixed, we define a 2-dimensional disk $D_t \subset \Omega_\psi$ by setting

$$D_t := \{(t,x) \in \mathbb{R}^n; |x| < \psi(t)\}$$

and define

$$S_t := \{(t,x) \in \mathbb{R}^n; |x| = 2\psi(t)\}.$$

Suppose for the contrary that there exists a reflection $\mathcal{R} : \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ over $\partial\Omega_\psi$ which induces a bounded linear extension operator from $W^{1,\infty}(\mathbb{R}^n\setminus\overline{\Omega_\psi})$ to $W^{1,\infty}(\mathbb{R}^n)$. Define the function $E_{\mathcal{R}}(u)$ on $B(\Omega_\psi,1)$ as in (2.1). By the geometry of Ω_ψ and the fact that $\mathcal{R}$ is continuous and $\mathcal{R}(z) = z$ whenever $z \in \partial\Omega_\psi$, there exists a small enough $t_o \in (0,1)$ such that for every $t \in (0,t_o)$, there exists $(t,x_t) \in D_t$ with $E_{\mathcal{R}}(u)((t,x_t)) = 0$ and there exists $(t,x'_t) \in S_t$ with $E_{\mathcal{R}}(u)((t,x'_t)) = t$ and $d((t,x_t),(t,x'_t)) \le 2\psi(t)$. Hence for every $0 < t < t_o$, we have

$$|E_{\mathcal{R}}(u)((t,x_t)) - E_{\mathcal{R}}(u)((t,x'_t))| \ge t \ge C\psi^{-1}(d((t,x_t),(t,x'_t))).$$

This contradicts the assumption that $E_{\mathcal{R}}(u) \in W^{1,\infty}(B(\Omega_\psi,1))$: since $B(\Omega_\psi,1)$ is uniformly locally quasiconvex, a function in $W^{1,\infty}(B(\Omega_\psi,1))$ must have a Lipschitz representative. □

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