Integral binary Hamiltonian forms and their waterworlds

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Abstract

We give a graphical theory of integral indefinite binary Hamiltonian forms f analogous to the one of Conway for binary quadratic forms and the one of Bestvina-Savin for binary Hermitian forms. Given a maximal order $\mathscr{O}$ in a definite quaternion algebra over $\mathbb{Q}$, we define the *waterworld* of f, analogous to Conway's *river* and Bestvina-Savin's *ocean*, and use it to give a combinatorial description of the values of f on $\mathscr{O}\times\mathscr{O}$. We use an appropriate normalisation of Busemann distances to the cusps (with an algebraic description given in an independent appendix), the $\mathrm{SL}_2(\mathscr{O})$-equivariant Ford-Voronoi cellulation of the real hyperbolic 5-space, and the conformal action of $\mathrm{SL}_2(\mathscr{O})$ on the Hamilton quaternions. [1]

1 Introduction

In the beautiful little book [Con] (see also [Wei, Hat]), Conway uses Serre's tree $X_\mathbb{Z}$ of the modular lattice $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$ (see [Ser2]), considered as an equivariant deformation retract of the upper halfplane model of the hyperbolic plane $\mathbb{H}^2_\mathbb{R}$, in order to give a graphical theory of binary quadratic forms f. The components C of $\mathbb{H}^2_\mathbb{R}-X_\mathbb{Z}$ consist of points closer to a given cusp $p/q\in\mathbb{P}^1(\mathbb{Q})$ of $\mathrm{SL}_2(\mathbb{Z})$ than to all the other ones. When f is indefinite, anisotropic and integral over $\mathbb{Z}$, Conway constructs a line $R(f)$ in $X_\mathbb{Z}$, called the *river* of f, separating the components C of $\mathbb{H}^2_\mathbb{R}-X_\mathbb{Z}$ such that $f(p,q)>0$ from the ones with $f(p,q)<0$. This allows a combinatorial description of the values taken by f on integral points.

Bestvina and Savin in [BeS] have given an analogous construction when $\mathbb{R}$ is replaced by $\mathbb{C}$, $\mathbb{Z}$ by the ring of integers $\mathscr{O}_K$ of a quadratic imaginary extension K of $\mathbb{Q}$, $\mathbb{H}^2_\mathbb{R}$ by $\mathbb{H}^3_\mathbb{R}$ and $X_\mathbb{Z}$ by Mendoza's spine $X_{\mathscr{O}_K}$ in $\mathbb{H}^3_\mathbb{R}$ for the Bianchi lattice $\mathrm{SL}_2(\mathscr{O}_K)$ in $\mathrm{SL}_2(\mathbb{C})$ (see [Men], and also [Ash, §4]). They construct a subcomplex $O(f)$ of $X_{\mathscr{O}_K}$, called the *ocean* of f, for any indefinite anisotropic integral binary Hermitian form f over $\mathscr{O}_K$, separating the components of $\mathbb{H}^3_\mathbb{R}-X_{\mathscr{O}_K}$ on whose point at infinity f is positive from the negative ones, and prove that it is homeomorphic to a 2-plane.

In this paper, we give analogs of these constructions and results for Hamilton's quaternions and maximal orders in definite quaternion algebras over $\mathbb{Q}$.

Let $\mathbb{H}$ be the standard Hamilton quaternion algebra over $\mathbb{R}$, with conjugation $x\mapsto\overline{x}$, reduced norm $\mathtt{n}$ and reduced trace $\mathtt{tr}$. Let $\mathscr{O}$ be a maximal order in a quaternion algebra A over $\mathbb{Q}$, which is definite (that is, $A\otimes_\mathbb{Q}\mathbb{R}=\mathbb{H}$), with class number h_A and discriminant

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D_A. An example is given by the *Hurwitz order* $\mathscr{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+k}{2}$, in which case $h_A = 1$ and $D_A = 2$. We refer for more information to [Vig] and Subsection 2.1. The *Hamilton-Bianchi group* $\mathrm{SL}_2(\mathscr{O})$, which is defined using Dieudonné determinant, is a lattice in $\mathrm{SL}_2(\mathbb{H})$. It acts discretely on the real hyperbolic 5-space $\mathbb{H}^5_{\mathbb{R}}$ with finite volume quotient, and conformally on its space at infinity $\partial_\infty \mathbb{H}^5_{\mathbb{R}} = \mathbb{H} \cup \{\infty\}$. The number of cusps of the hyperbolic orbifold $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$ is ${h_A}^2$ by [KO, Satz 2.1, 2.2], see also [PP2, §3].

Analogously to [Men] in the complex case, we give in Section 3 an appropriate normalisation of the Busemann distance to the cusps, and we construct the Ford-Voronoi cell decomposition of $\mathbb{H}^5_{\mathbb{R}}$ for $\mathrm{SL}_2(\mathscr{O})$, so that the interior of the *Ford-Voronoi cell* $\mathscr{H}_\alpha$ consists of the points in $\mathbb{H}^5_{\mathbb{R}}$ closer to a given cusp $\alpha \in \mathbb{P}^1_r(A)$ of $\mathrm{SL}_2(\mathscr{O})$ than to all the others. If $X_{\mathscr{O}}$ is the codimension 1 skeleton of the Ford-Voronoi cellulation, called the *spine* of $\mathrm{SL}_2(\mathscr{O})$, then the hyperbolic 5-orbifold $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$ retracts by strong deformations onto the finite 4-dimensional orbihedron $\mathrm{SL}_2(\mathscr{O})\backslash X_{\mathscr{O}}$. As explained in Appendix B, this orbihedron coincides (up to a natural cellular isomorphism) with one of the "well-rounded" retracts of arithmetic locally symmetric spaces of general linear groups constructed in [Ash] (extended to retracts of their Borel-Serre compactifications in [AshC]), but our construction is different and much more geometric. Actually, as in [Ash], we construct in Appendix B a $(h_A^2 - 1)$-dimensional family of spines of $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$.

Using uniform 3-, 4- and 5-polytopes, we give in Example 4.4 when $D_A = 2$ and in Example 4.5 when $D_A = 3$, a complete description of the quotient $\mathrm{SL}_2(\mathscr{O})\backslash X_{\mathscr{O}}$ and of the link of its vertex. For instance, if $\mathscr{O}$ is the Hurwitz order, then $\mathrm{SL}_2(\mathscr{O})\backslash X_{\mathscr{O}}$ is obtained by identifying opposite faces and taking the quotient of any 4-dimensional cell of $X_{\mathscr{O}}$ by its stabilizer. In this case, a 4-dimensional cell of $X_{\mathscr{O}}$ identifies with the 24-*cell* (the self-dual convex regular Euclidean 4-polytope with Schläfli symbol $\{3, 4, 3\}$), and its stabilizer is isomorphic with an index 2 subgroup of the Coxeter group $[3, 4, 3]$.

Following H. Weyl [Wey], we will call *Hamiltonian form* a Hermitian form over $\mathbb{H}$ with anti-involution the conjugation. We refer to Subsection 2.3 and for instance to [PP2] for background. See [PP2] also for a sharp asymptotic result on the average number of their integral representations. Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a binary Hamiltonian form, with

$$f(u, v) = a\, \mathtt{n}(u) + \mathtt{tr}(\overline{u}\, b\, v) + c\, \mathtt{n}(v)\ ,$$

which is *integral* over $\mathscr{O}$ (its *coefficients* a, b, c satisfy $a, c \in \mathbb{Z}$ and $b \in \mathscr{O}$) and indefinite (its *discriminant* $\Delta(f) = \mathtt{n}(b) - ac$ is positive). We choose this definition of integrality for simplicity as in [PP2], in order to avoid half-integral coefficients in the matrix of the form. The *group of automorphs* of f is the arithmetic lattice

$$\mathrm{SU}_f(\mathscr{O}) = \{g \in \mathrm{SL}_2(\mathscr{O})\ :\ f \circ g = f\}\ .$$

If C is a Ford-Voronoi cell for $\mathrm{SL}_2(\mathscr{O})$, let $F(C) = \frac{f(a,b)}{\mathtt{n}(\mathscr{O}a+\mathscr{O}b)}$ where $ab^{-1} \in \mathbb{P}^1_r(A)$ is the cusp of C. We will say that C is respectively *positive, negative or flooded* if $F(C) > 0$, $F(C) < 0$ or $F(C) = 0$. Contrarily to the real and complex cases, there are always flooded Ford-Voronoi cells, since by taking a $\mathbb{Z}$-basis of $\mathscr{O}$, the Hamiltonian form f becomes an indefinite integral quadratic form over $\mathbb{Z}$ with $8 \geq 5$ variables, hence always represents 0 by Meyer's theorem. Our countably many flooded Ford-Voronoi cells are thus the analogues of Conway's two *lakes* for an indefinite isotropic integral binary quadratic form over $\mathbb{Z}$. On the components of $\mathbb{H}^2_{\mathbb{R}} - X_{\mathbb{Z}}$ along the lakes, Conway proved that the values of such a form consist of an infinite arithmetic progression. An analogous result holds in our case, that we only state when the class number is one in this introduction in order to simplify the statement (see Proposition 5.3 for the general result.)

Proposition 1.1. *If $h_A = 1$, given a flooded Ford-Voronoi cell C, there exists a finite set of nonconstant affine maps $\{\varphi_i : \mathbb{H} \to \mathbb{R} \ : \ i \in F\}$ defined over $\mathbb{Q}$ such that the set of values of F on the Ford-Voronoi cells meeting C is $\bigcup_{i\in F} \varphi_i(\mathscr{O})$.*

In order to simplify the next statement, assume from now on in this introduction that the flooded Ford-Voronoi cells are pairwise disjoint. We define the *waterworld* $\mathscr{W}(f)$ of f as the subcomplex of the spine $X_{\mathscr{O}}$ separating positive Ford-Voronoi cells from negative ones, that is, $\mathscr{W}(f)$ is the union of the cells of $X_{\mathscr{O}}$ contained in (the boundary of) both a positive and a negative Ford-Voronoi cell. The *coned-off waterworld* $\mathscr{C}\mathscr{W}(f)$ is the union of $\mathscr{W}(f)$ and, for all cells σ of $\mathscr{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathscr{H}_\alpha$, of the cone with base σ and vertex at infinity α. The following result (see Section 5) in particular says that $\mathscr{C}\mathscr{W}(f)$ is a piecewise hyperbolic polyhedral 4-plane contained in the spine of $\mathrm{SL}_2(\mathscr{O})$ except for its ideal cells.

Let $\mathscr{C}(f)$ be the hyperbolic hyperplane of $\mathbb{H}^5_{\mathbb{R}}$ whose boundary is the projective set of zeros $\{[u:v] \in \mathbb{P}^1_r(\mathbb{H}) \ : \ f(u,v) = 0\}$ of f.

Theorem 1.2. *The closest point mapping from the coned-off waterworld $\mathscr{C}\mathscr{W}(f)$ to $\mathscr{C}(f)$ is an $\mathrm{SU}_f(\mathscr{O})$-equivariant homeomorphism.*

Section 2 recalls the necessary information on the definite quaternion algebras over $\mathbb{Q}$, the Hamilton-Bianchi groups, and the binary Hamiltonian forms. Section 3 gives the construction of the normalized Busemann distance to the cusp, and uses it in order to give a quantitative reduction theory à la Hermite (see for instance [Bor2]) for the arithmetic group $\mathrm{SL}_2(\mathscr{O})$. We describe the Ford-Voronoi cellulation for $\mathrm{SL}_2(\mathscr{O})$ and its spine $X_{\mathscr{O}}$ in Section 4. We define the waterworlds and prove their main properties in Section 5. The noncommutativity of $\mathbb{H}$ and the isotropic property of f require at various point of this text a different approach than the one in [BeS].

Recall (see for instance [PP2, §7] and Section 3) that there is a correspondence between positive definite binary Hamiltonian forms with discriminant -1 and the upper halfspace model of the real hyperbolic 5-space. In the independent Appendix A, we give an algebraic formula for the Busemann distance of a point $x \in \mathbb{H}^5_{\mathbb{R}}$ to a cusp $\alpha \in \mathbb{P}^1_r(A)$ in terms of the covolume of the $\mathscr{O}$-flag associated with α, with respect to the volume of the positive definite binary Hamiltonian form associated with x, analogous to the one of Mendoza in the complex case. Furthermore, in the proof of Theorem 3.5, we use the upper bound on the minima of positive definite binary Hamiltonian forms given in [ChP]: If $\gamma_2(\mathscr{O})$ is the upper bound, on all such forms f with discriminant -1, of the lower bound of $f(u,v)$ on all nonzero $(u,v) \in \mathscr{O} \times \mathscr{O}$, then

$$\gamma_2(\mathscr{O}) \leq \sqrt{D_A}\,. \tag{1}$$

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2 Backgrounds

We refer to [PP2] for more informations on the objects considered in this paper, and we only recall what is strictly needed.

2.1 Background on definite quaternion algebras over $\mathbb{Q}$

A *quaternion algebra* over a field F is a four-dimensional central simple algebra over F. We refer to [Vig] for generalities on quaternion algebras. A real quaternion algebra is isomorphic either to $\mathscr{M}_2(\mathbb{R})$ or to Hamilton's quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, with basis elements $1, i, j, k$ as a $\mathbb{R}$-vector space, with unit element 1 and $i^2 = j^2 = -1$, $ij = -ji = k$. We define the *conjugate* of $x = x_0 + x_1 i + x_2 j + x_3 k$ in $\mathbb{H}$ by $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$, its *reduced trace* by $\mathtt{tr}(x) = x + \overline{x}$, and its *reduced norm* by $\mathtt{n}(x) = x\,\overline{x} = \overline{x}\,x$. Note that $\mathtt{n}(xy) = \mathtt{n}(x)\,\mathtt{n}(y)$, $\mathtt{tr}(\overline{x}) = \mathtt{tr}(x)$ and $\mathtt{tr}(xy) = \mathtt{tr}(yx)$ for all $x, y \in \mathbb{H}$. For every matrix $X = (x_{i,j})_{1\le i\le p,\,1\le j\le q} \in \mathscr{M}_{p,q}(\mathbb{H})$, we denote by $X^* = (\overline{x_{j,i}})_{1\le i\le q,\,1\le j\le p} \in \mathscr{M}_{q,p}(\mathbb{H})$ its adjoint matrix. We endow $\mathbb{H}$ with the Euclidean norm $x \mapsto \sqrt{\mathtt{n}(x)}$, making the $\mathbb{R}$-basis $1, i, j, k$ orthonormal.

Let A be a quaternion algebra over $\mathbb{Q}$. We say that A is *definite* (or ramified over $\mathbb{R}$) if the real quaternion algebra $A \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathbb{H}$, and we then fix an identification between A and a $\mathbb{Q}$-subalgebra of $\mathbb{H}$. The *reduced discriminant* D_A of A is the product of the primes $p \in \mathbb{N}$ such that the quaternion algebra $A \otimes_{\mathbb{Q}} \mathbb{Q}_p$ over $\mathbb{Q}_p$ is a division algebra. Two definite quaternion algebras over $\mathbb{Q}$ are isomorphic if and only if they have the same reduced discriminant, which can be any product of an odd number of primes (see [Vig, page 74]).

A $\mathbb{Z}$-*lattice* I in A is a finitely generated $\mathbb{Z}$-module generating A as a $\mathbb{Q}$-vector space. An *order* in A is a unitary subring $\mathscr{O}$ of A which is a $\mathbb{Z}$-lattice. In particular, $A = \mathbb{Q}\mathscr{O} = \mathscr{O}\mathbb{Q}$. Each order of A is contained in a maximal order. For instance $\mathscr{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+k}{2}$ is a maximal order, called the *Hurwitz order*, in $A = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ with $D_A = 2$. Let $\mathscr{O}$ be an order in A. The reduced norm $\mathtt{n}$ and the reduced trace $\mathtt{tr}$ take integral values on $\mathscr{O}$. The invertible elements of $\mathscr{O}$ are its elements of reduced norm 1. Since $\overline{x} = \mathtt{tr}(x) - x$, any order is invariant under conjugation.

The *left order* $\mathscr{O}_\ell(I)$ of a $\mathbb{Z}$-lattice I is $\{x \in A \;:\; xI \subset I\}$. A *left fractional ideal* of $\mathscr{O}$ is a $\mathbb{Z}$-lattice of A whose left order is $\mathscr{O}$. A *left ideal* of $\mathscr{O}$ is a left fractional ideal of $\mathscr{O}$ contained in $\mathscr{O}$. A (left) *ideal class* of $\mathscr{O}$ is an equivalence class of nonzero left fractional ideals of $\mathscr{O}$ for the equivalence relation $\mathfrak{m} \sim \mathfrak{m}'$ if $\mathfrak{m}' = \mathfrak{m}c$ for some $c \in A^\times$. The *class number* h_A of A is the number of ideal classes of a maximal order $\mathscr{O}$ of A. It is finite and independent of the maximal order $\mathscr{O}$, and we have $h_A = 1$ if and only if $D_A = 2, 3, 5, 7, 13$ (see for instance [Vig]).

The *reduced norm* $\mathtt{n}(\mathfrak{m})$ of a nonzero left ideal $\mathfrak{m}$ of $\mathscr{O}$ is the greatest common divisor of the norms of the nonzero elements of $\mathfrak{m}$. In particular, $\mathtt{n}(\mathscr{O}) = 1$. By [Rei, p. 59], we have

$$\mathtt{n}(\mathfrak{m}) = [\mathscr{O} : \mathfrak{m}]^{\frac{1}{2}}\ . \tag{2}$$

The *reduced norm* of a nonzero left fractional ideal $\mathfrak{m}$ of $\mathscr{O}$ is $\frac{\mathtt{n}(c\mathfrak{m})}{\mathtt{n}(c)}$ for any $c \in \mathbb{N} - \{0\}$ such that $c\mathfrak{m} \subset \mathscr{O}$. By Equation (2), if $\mathfrak{m}, \mathfrak{m}'$ are nonzero left fractional ideals of $\mathscr{O}$ with $\mathfrak{m}' \subset \mathfrak{m}$, we have

$$\frac{\mathtt{n}(\mathfrak{m}')}{\mathtt{n}(\mathfrak{m})} = [\mathfrak{m} : \mathfrak{m}']^{\frac{1}{2}}\ . \tag{3}$$

For $K=\mathbb{H}$ or $K=A$, we consider $K\times K$ as a right module over K and we denote by $\mathbb{P}^1_r(K)=(K\times K-\{0\})/K^\times$ the right projective line of K, identified as usual with the Alexandrov compactification $K\cup\{\infty\}$ where $[1:0]=\infty$ and $[x:y]=xy^{-1}$ if $y\neq 0$.

2.2 Background on Hamilton-Bianchi groups

Refering to [Fue, Die, Asl], the *Dieudonné determinant* is the group morphism Det from the group $\mathrm{GL}_2(\mathbb{H})$ of invertible 2×2 matrices with coefficients in $\mathbb{H}$ to $\mathbb{R}_+^*$, defined by, for every $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{H})$,

$$(\mathrm{Det}\, g)^2 \;=\mathtt{n}(a\,d)+\mathtt{n}(b\,c)-\mathtt{tr}(a\,\overline{c}\,d\,\overline{b})\;. \tag{4}$$

An easy computation allows to check that

$$g^{-1}=\frac{1}{(\mathrm{Det}\, g)^2}\begin{pmatrix}\overline{a}\;\mathtt{n}(d)-\overline{c}\,d\,\overline{b} & \overline{c}\;\mathtt{n}(b)-\overline{a}\,b\,\overline{d}\\ \overline{b}\;\mathtt{n}(c)-\overline{d}\,c\,\overline{a} & \overline{d}\;\mathtt{n}(a)-\overline{b}\,a\,\overline{c}\end{pmatrix}\;. \tag{5}$$

If $c\neq 0$, we have (see loc. cit.)

$$(\mathrm{Det}\, g)^2=\mathtt{n}(ac^{-1}dc-bc)\;. \tag{6}$$

The Dieudonné determinant is invariant under the adjoint map $g\mapsto g^*$. Let $\mathrm{SL}_2(\mathbb{H})$ be the group of 2×2 matrices with coefficients in $\mathbb{H}$ and Dieudonné determinant 1. We refer for instance to [Kel] for more information on $\mathrm{SL}_2(\mathbb{H})$.

The group $\mathrm{SL}_2(\mathbb{H})$ acts linearly on the left on the right $\mathbb{H}$-module $\mathbb{H}\times\mathbb{H}$. The projective action of $\mathrm{SL}_2(\mathbb{H})$ on $\mathbb{P}^1_r(\mathbb{H})$, induced by its linear action on $\mathbb{H}\times\mathbb{H}$, is the action by homographies on $\mathbb{H}\cup\{\infty\}$ defined by

$$\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\cdot z=\left\{\begin{array}{ll}(az+b)(cz+d)^{-1} & \text{if } z\neq\infty, -c^{-1}d\\ ac^{-1} & \text{if } z=\infty, c\neq 0\\ \infty & \text{otherwise}\;.\end{array}\right.$$

We use the upper halfspace model $\{(z,r)\;:\;z\in\mathbb{H}, r>0\}$ with Riemannian metric $ds^2(z,r)=\frac{ds^2_{\mathbb{H}}(z)+dr^2}{r^2}$ for the real hyperbolic space $\mathbb{H}^5_{\mathbb{R}}$ with dimension 5. Its space at infinity $\partial_\infty\mathbb{H}^5_{\mathbb{R}}$ is hence $\mathbb{H}\cup\{\infty\}$. The action of $\mathrm{SL}_2(\mathbb{H})$ by homographies on $\partial_\infty\mathbb{H}^5_{\mathbb{R}}$ extends to a left action on $\mathbb{H}^5_{\mathbb{R}}$ by

$$\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\cdot(z,r)=\left(\frac{(az+b)\,\overline{(cz+d)}+a\,\overline{c}\,r^2}{\mathtt{n}(cz+d)+r^2\,\mathtt{n}(c)},\frac{r}{\mathtt{n}(cz+d)+r^2\,\mathtt{n}(c)}\right)\;. \tag{7}$$

In this way, the group $\mathrm{PSL}_2(\mathbb{H})=\mathrm{SL}_2(\mathbb{H})/\{\pm\,\mathrm{id}\}$ is identified with the group of orientation preserving isometries of $\mathbb{H}^5_{\mathbb{R}}$.

For any order $\mathscr{O}$ in a definite quaternion algebra A over $\mathbb{Q}$, we define the *Hamilton-Bianchi group* by

$$\Gamma_{\mathscr{O}}=\mathrm{SL}_2(\mathscr{O})=\mathrm{SL}_2(\mathbb{H})\cap\mathscr{M}_2(\mathscr{O})\;.$$

We have

$$\mathrm{GL}_2(\mathscr{O})=\mathrm{SL}_2(\mathscr{O}) \tag{8}$$

that is, $\mathscr{M}_2(\mathscr{O})^\times = \mathrm{SL}_2(\mathbb{H}) \cap \mathscr{M}_2(\mathscr{O})$, which in particular proves that $\mathrm{SL}_2(\mathscr{O})$ is a subgroup of $\mathrm{SL}_2(\mathbb{H})$. Indeed, if $x \in \mathscr{M}_2(\mathscr{O})^\times$, then $x^{-1} \in \mathscr{M}_2(\mathscr{O})$, and by Equation (4), since the ring $\mathscr{O}$ is stable by conjugation, and as $\mathtt{n}$ and $\mathtt{tr}$ take integral values on $\mathscr{O}$, the numbers $(\mathrm{Det}\, x)^2$ and $(\mathrm{Det}\, x^{-1})^2$ are positive integers, and inverses one of the other. This proves the inclusion of the set on the left-hand side into the one on the right-hand side in Equation (8). Conversely, if $x \in \mathrm{SL}_2(\mathbb{H}) \cap \mathscr{M}_2(\mathscr{O})$, then by Equation (5), we have $x^{-1} \in \mathscr{M}_2(\mathscr{O})$, thus proving the opposite inclusion.

Note that $\Gamma_{\mathscr{O}}$ is a nonuniform arithmetic lattice in the connected real Lie group $\mathrm{SL}_2(\mathbb{H})$ (see for instance [PP1, page 382] for details). In particular, the quotient real hyperbolic orbifold $\Gamma_{\mathscr{O}} \backslash \mathbb{H}^5_{\mathbb{R}}$ has finite volume.

Remark. It would be very interesting to know if the image in $\mathrm{PSL}_2(\mathbb{H})$ of $\mathrm{SL}_2(\mathscr{O})$ is commensurable (up to conjugation) to one of the lattices in $\mathrm{SO}_0(1,5) \simeq \mathrm{PSL}_2(\mathbb{H})$ studied by Vinberg [Vin], Allcock [All], Everitt [Eve], Ratcliffe-Tschantz [RaS] and others.

Recall that the maximal order $\mathscr{O}$ is *left-Euclidean* if for all $a, b \in \mathscr{O}$ with $b \neq 0$, there exists $c, d \in \mathscr{O}$ with $a = cb + d$ and $\mathtt{n}(d) < \mathtt{n}(b)$, or, equivalently, if for every $\alpha \in A$, there exists $c \in \mathscr{O}$ such that $\mathtt{n}(\alpha - c) < 1$. By for instance [Vig, p. 156], $\mathscr{O}$ is left-Euclidean if and only if $D_A \in \{2, 3, 5\}$. The following elementary lemma gives a nice set of generators for $\mathrm{SL}_2(\mathscr{O})$ when is left-Euclidean. For us, it will be useful in Section 4. See also [Spe, §4] and [JW, §8] for the first claim for the Hurwitz order.

Lemma 2.1. *If $\mathscr{O}$ is left-Euclidean, then the group $\mathrm{SL}_2(\mathscr{O})$ is generated by $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T_w = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for $w \in \mathscr{O}$ and $C_{u,v} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ for $u, v \in \mathscr{O}^\times$. In particular, the anti-homography $z \mapsto \overline{z}$ normalizes the action by homographies of $\mathrm{SL}_2(\mathscr{O})$ on $\mathbb{H}$.*

Proof. The last claim follows from the first one, since we have $J^{-1} = J$, $T_w^{-1} = T_{-w}$, $C_{u,v}^{-1} = C_{u^{-1},v^{-1}}$ and for all $z \in \mathbb{H}$, we have

$$\overline{J \cdot \overline{z}} = J \cdot z, \quad \overline{T_w \cdot \overline{z}} = T_{\overline{w}} \cdot z, \quad \overline{C_{u,v} \cdot \overline{z}} = C_{\overline{v}^{-1}, \overline{u}^{-1}} \cdot z\ .$$

Let G be the subgroup of $\mathrm{SL}_2(\mathscr{O})$ generated by the matrices $J, T_w, C_{u,v}$ for $w \in \mathscr{O}$ and $u, v \in \mathscr{O}^\times$ (their Dieudonné determinant is indeed 1). Let us prove that any element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathscr{O})$ belongs to G, by induction on the integer $\mathtt{n}(c)$. If $c = 0$, then $M = C_{a,d}\, T_{a^{-1}b}$ belongs to G. Otherwise, since $\mathscr{O}$ is left-Euclidean, there exists $w, c' \in \mathscr{O}$ such that $a = wc + c'$ and $\mathtt{n}(c') < \mathtt{n}(c)$. Hence

$$M = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ c' & b - w\, d \end{pmatrix}$$

belongs to G by induction. □

Corollary 2.2. *If $\mathscr{O}$ is left-Euclidean, if $\{w_1, w_2, w_3, w_4\}$ is a $\mathbb{Z}$-basis of $\mathscr{O}$ and if S is a generating set of the group of units $\mathscr{O}^\times$, then the set*

$$\{J, T_{w_1}, T_{w_2}, T_{w_3}, T_{w_4}\} \cup \{C_{u,v} \;:\; u, v \in S\}$$

is a generating set for $\mathrm{SL}_2(\mathscr{O})$. □

The action by homographies of the group $\Gamma_{\mathscr{O}} = \mathrm{SL}_2(\mathscr{O})$ preserves the right projective space $\mathbb{P}^1_r(A) = A \cup \{\infty\}$, which is the set of fixed points of the parabolic elements of $\Gamma_{\mathscr{O}}$ acting on $\mathbb{H}^5_{\mathbb{R}} \cup \partial_\infty \mathbb{H}^5_{\mathbb{R}}$. In particular, the topological quotient space $\Gamma_{\mathscr{O}} \backslash (\mathbb{H}^5_{\mathbb{R}} \cup \mathbb{P}^1_r(A))$ is the compactification of the finite volume hyperbolic orbifold $\Gamma_{\mathscr{O}} \backslash \mathbb{H}^5_{\mathbb{R}}$ by its (finite) space of ends.

2.3 Background on binary Hamiltonian forms

A binary Hamiltonian form f is a map $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$ with

$$f(u,v) = a\,\mathtt{n}(u) + \mathtt{tr}(\overline{u}\,b\,v) + c\,\mathtt{n}(v)$$

whose *coefficients* $a = a(f)$, $b = b(f)$ and $c = c(f)$ satisfy $a, c \in \mathbb{R}$, $b \in \mathbb{H}$. Note that $f((u,v)\lambda) = \mathtt{n}(\lambda) f(u,v)$ for all $u, v, \lambda \in \mathbb{H}$.

The *matrix* $M(f)$ of f is the Hermitian matrix $\begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix}$, so that

$$f(u,v) = \begin{pmatrix} u \\ v \end{pmatrix}^* \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The *discriminant* of f is

$$\Delta(f) = \mathtt{n}(b) - ac.$$

An easy computation shows that the Dieudonné determinant of $M(f)$ is equal to $|\Delta(f)|$. A binary Hamiltonian form is *indefinite* if takes both positive and negative values. It is easy to check that a form f is indefinite if and only if $\Delta(f)$ is positive, see [PP2, §4].

The linear action on the left on $\mathbb{H} \times \mathbb{H}$ of the group $\mathrm{SL}_2(\mathbb{H})$ induces an action on the right on the set of binary Hamiltonian forms f by precomposition. The matrix of $f \circ g$ is $M(f \circ g) = g^* M(f)\, g$. For every $g \in \mathrm{SL}_2(\mathbb{H})$, we have

$$\Delta(f \circ g) = \Delta(f)\ . \tag{9}$$

For every indefinite binary Hamiltonian form f, with $a = a(f)$, $b = b(f)$ and $\Delta = \Delta(f)$, let

$$\mathscr{C}_\infty(f) = \{[u:v] \in \mathbb{P}^1_r(\mathbb{H})\ :\ f(u,v) = 0\}\ .$$

In $\mathbb{P}^1_r(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$, the set $\mathscr{C}_\infty(f)$ is the 3-sphere of center $-\frac{b}{a}$ and radius $\frac{\sqrt{\Delta}}{|a|}$ if $a \neq 0$, and it is the union of $\{\infty\}$ with the real affine hyperplane $\{z \in \mathbb{H}\ :\ \mathtt{tr}(\overline{z}b) + c = 0\}$ of $\mathbb{H}$ otherwise. The values of f are positive on (the representatives in $\mathbb{H} \times \mathbb{H}$ in) one of the two components of $\mathbb{P}^1_r(\mathbb{H}) - \mathscr{C}_\infty(f)$ and negative on the other one. The set

$$\mathscr{C}(f) = \{(z,r) \in \mathbb{H} \times\,]0, +\infty[\ :\ f(z,1) + a\,r^2 = 0\}$$

is the (4-dimensional) hyperbolic hyperplane in $\mathbb{H}^5_{\mathbb{R}}$ with boundary at infinity $\mathscr{C}_\infty(f)$. For every $g \in \mathrm{SL}_2(\mathbb{H})$, we have

$$\mathscr{C}_\infty(f \circ g) = g^{-1}\,\mathscr{C}_\infty(f) \quad \text{and} \quad \mathscr{C}(f \circ g) = g^{-1}\,\mathscr{C}(f)\ . \tag{10}$$

Given an order $\mathscr{O}$ in a definite quaternion algebra over $\mathbb{Q}$, a binary Hamiltonian form f is *integral* over $\mathscr{O}$ if its coefficients belong to $\mathscr{O}$. Note that such a form f takes integral values on $\mathscr{O} \times \mathscr{O}$, but the converse might not be true. The lattice $\Gamma_{\mathscr{O}} = \mathrm{SL}_2(\mathscr{O})$ of $\mathrm{SL}_2(\mathbb{H})$

preserves the set of indefinite binary Hamiltonian forms f that are integral over $\mathscr{O}$. The stabilizer in $\Gamma_{\mathscr{O}}$ of such a form f is its *group of automorphs*

$$\mathrm{SU}_f(\mathscr{O}) = \{g \in \Gamma_{\mathscr{O}} \;:\; f \circ g = f\} \;.$$

If f is integral over $\mathscr{O}$, then $\mathrm{SU}_f(\mathscr{O})\backslash\mathscr{C}(f)$ is a finite volume hyperbolic 4-orbifold, since $\mathrm{SU}_f(\mathscr{O})$ is arithmetic and by Borel-Harish-Chandra's theorem (though it might have been known before this theorem).

3 On the reduction theory of binary Hamiltonian forms and Hamilton-Bianchi lattices

In this section, we study the geometric reduction theory of positive definite binary Hamiltonian forms, as in Mendoza [Men] for the Hermitian case. The results will be useful in Section 5. We start by recalling the correspondence between $\mathbb{H}^5_{\mathbb{R}}$ and positive definite binary Hamiltonian forms with discriminant -1.

Let $\mathscr{Q}$ be the 6-dimensional real vector space of binary Hamiltonian forms, and $\mathscr{Q}^+$ its open cone of positive definite ones. The multiplicative group $\mathbb{R}_+^\times$ of positive real numbers acts on $\mathscr{Q}^+$ by multiplication. We will denote by $\mathbb{R}_+^\times f$ the orbit of $f \in \mathscr{Q}^+$ and by $\mathbb{P}_+\mathscr{Q}^+$ the quotient space $\mathscr{Q}^+/\mathbb{R}_+^\times$. It identifies with the image of $\mathscr{Q}^+$ in the projective space $\mathbb{P}(\mathscr{Q})$ of $\mathscr{Q}$.

Let $\langle\cdot,\cdot\rangle_{\mathscr{Q}}$ be the symmetric $\mathbb{R}$-bilinear form (with signature $(4,2)$) on $\mathscr{Q}$ such that for every $f \in \mathscr{Q}$,

$$\langle f, f\rangle_{\mathscr{Q}} = -2\Delta(f) \;.$$

That is, for all $f, f' \in \mathscr{Q}$, we have

$$\langle f, f'\rangle_{\mathscr{Q}} = a(f)\; c(f') + c(f)\; a(f') - \mathtt{tr}(\,\overline{b(f)}\; b(f')\,) \;. \tag{11}$$

By Equation (9), we have, for all $f, f' \in \mathscr{Q}$ and $g \in \mathrm{SL}_2(\mathbb{H})$

$$\langle f \circ g, f' \circ g\rangle_{\mathscr{Q}} = \langle f, f'\rangle_{\mathscr{Q}} \;. \tag{12}$$

Let $\mathscr{Q}_1^+$ the submanifold of $\mathscr{Q}^+$ consisting of the forms with discriminant -1, and let $\Theta : \mathbb{H}^5_{\mathbb{R}} \to \mathscr{Q}_1^+$ be the homeomorphism such that, for every $(z,r) \in \mathbb{H}^5_{\mathbb{R}}$,

$$M(\Theta(z,r)) = \frac{1}{r}\begin{pmatrix} 1 & -z \\ -\overline{z} & \mathtt{n}(z) + r^2 \end{pmatrix} \;.$$

The fact that this map is well defined and is a homeomorphism follows by checking that its composition by the canonical projection $\mathscr{Q}^+ \to \mathbb{P}_+\mathscr{Q}^+$ is the inverse of the homeomorphism denoted by

$$\Phi : \mathbb{R}_+^\times f \mapsto \Big(-\frac{b(f)}{a(f)}, \frac{\sqrt{-\Delta(f)}}{a(f)}\Big)$$

in [PP2, Prop. 22]. By loc. cit., the map Θ is hence (anti-)equivariant under the actions of $\mathrm{SL}_2(\mathbb{H})$: For all $x \in \mathbb{H}^5_{\mathbb{R}}$ and $g \in \mathrm{SL}_2(\mathbb{H})$, we have

$$\Theta(gx) = \Theta(x) \circ g^{-1} \;. \tag{13}$$

Let $\mathscr{O}$ be a maximal order in a definite quaternion algebra A over $\mathbb{Q}$. For every $\alpha \in A$, let
$$I_\alpha = \mathscr{O}\alpha + \mathscr{O}\ ,$$
which is a left fractional ideal of $\mathscr{O}$. Let f_α be the binary Hamiltonian form with matrix
$$M(f_\alpha) = \frac{1}{\mathtt{n}(I_\alpha)} \begin{pmatrix} 1 & -\alpha \\ -\,\overline{\alpha} & \mathtt{n}(\alpha) \end{pmatrix} .$$
Note that f_α is a positive scalar multiple of the *norm form* associated with α: for all $z \in \mathbb{H}$,
$$f_\alpha(u,v) = \begin{pmatrix} \overline{u} & \overline{v} \end{pmatrix} M(f_\alpha) \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\mathtt{n}(I_\alpha)}\ \mathtt{n}(u - \alpha v)\ .$$
Besides depending on α, the form f_α does depend on the choice of the maximal order $\mathscr{O}$. But its homothety class $\mathbb{R}^\times f_\alpha$ depends only on α.

Let f_∞ be the binary Hamiltonian form whose matrix is $M(f_\infty) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, that is, $f_\infty : (u,v) \mapsto \mathtt{n}(v)$. Note that for every $\alpha \in \mathbb{P}^1_r(A) = A \cup \{\infty\}$, the form f_α is nonzero and degenerate (its discriminant is equal to 0), and $\mathbb{R}^\times f_\alpha$ belongs to the boundary of $\mathbb{P}_+\mathscr{Q}^+$ in $\mathbb{P}(\mathscr{Q})$. The map $\Phi^{-1} : \mathbb{H}^5_\mathbb{R} \to \mathbb{P}(\mathscr{Q})$ given by $x \mapsto \mathbb{R}^\times_+\Theta(x)$ extends continuously to a $\mathrm{SL}_2(A)$-(anti-)equivariant homeomorphism between $\mathbb{H}^5_\mathbb{R} \cup \mathbb{P}^1_r(A)$ and its image in $\mathbb{P}(\mathscr{Q})$ by sending α to $\mathbb{R}^\times f_\alpha$ for every $\alpha \in \mathbb{P}^1_r(A)$. Proposition 3.2 below makes precise the scaling factor for the action of $\mathrm{SL}_2(A)$ on the forms f_α for $\alpha \in \mathbb{P}^1_r(A)$. Its proof will use the following beautiful (and probably well-known) formula.

Lemma 3.1. *For all* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{H})$ *and* $z, w \in \mathbb{H}$ *such that* $g \cdot z, g \cdot w \neq \infty$*, we have*
$$\mathtt{n}(g\cdot z - g \cdot w) = \frac{\mathtt{n}(z-w)}{\mathtt{n}(cz+d)\,\mathtt{n}(cw+d)}\ .$$

Proof. Since
$$\begin{pmatrix} az+b & aw+b \\ cz+d & cw+d \end{pmatrix} = g \begin{pmatrix} z & w \\ 1 & 1 \end{pmatrix}$$
and by taking the square of the Dieudonné determinant (see Equation (6)), we have
$$\begin{aligned} &\mathtt{n}(g\cdot z - g\cdot w) = \mathtt{n}((az+b)(cz+d)^{-1} - (aw+b)(cw+d)^{-1}) \\ &= \frac{1}{\mathtt{n}(cw+d)}\ \mathtt{n}((az+b)(cz+d)^{-1}(cw+d) - (aw+b)) \\ &= \frac{1}{\mathtt{n}(cz+d)\,\mathtt{n}(cw+d)}\ \mathtt{n}((az+b)(cz+d)^{-1}(cw+d)(cz+d) - (aw+b)(cz+d)) \\ &= \frac{1}{\mathtt{n}(cz+d)\,\mathtt{n}(cw+d)}\ \mathtt{n}(z-w)\ . \quad \square \end{aligned}$$

Proposition 3.2. *For all* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$ *and* $\alpha = [x:y] \in \mathbb{P}^1_r(A)$*, we have*
$$f_{g\cdot\alpha} \circ g = \frac{\mathtt{n}(\mathscr{O}x + \mathscr{O}y)}{\mathtt{n}(\mathscr{O}(ax+by) + \mathscr{O}(cx+dy))}\ f_\alpha\ .$$

Note that this implies that $f_{g\cdot\alpha}\circ g = f_\alpha$ if $g \in \mathrm{SL}_2(\mathscr{O})$.

Proof. The result is left to the reader when $\alpha = \infty$ or $g\cdot\alpha = \infty$, hence we assume that $\alpha, g\cdot\alpha \neq \infty$. By Lemma 3.1, for all $z \in \mathbb{H}$ such that $g\cdot z \neq \infty$, we have

$$\begin{aligned} f_{g\cdot\alpha}\circ g(z,1) &= \mathtt{n}(cz+d)\ f_{g\cdot\alpha}(g\cdot z,1) = \frac{\mathtt{n}(cz+d)}{\mathtt{n}(I_{g\cdot\alpha})}\ \mathtt{n}(g\cdot z - g\cdot\alpha) \\ &= \frac{1}{\mathtt{n}(I_{g\cdot\alpha})\,\mathtt{n}(c\alpha+d)}\,\mathtt{n}(z-\alpha) = \frac{\mathtt{n}(I_\alpha)}{\mathtt{n}(I_{g\cdot\alpha})\,\mathtt{n}(c\alpha+d)}\ f_\alpha(z,1)\ . \end{aligned}$$

The result easily follows. $\square$

For all $\alpha \in \mathbb{P}^1_r(A) = A \cup \{\infty\}$ and $x \in \mathbb{H}^5_{\mathbb{R}}$, let us define the *distance from x to the point at infinity α* by

$$d_\alpha(x) = \langle f_\alpha, \Theta(x)\rangle_{\mathscr{Q}}\ .$$

See Appendix A for an alternate description of the map $d_\alpha : \mathbb{H}^5_{\mathbb{R}} \to \mathbb{R}$.

The next result gives a few computations and properties of these maps d_α (which depend on the choice of the maximal order $\mathscr{O}$). We will see afterwards that $\ln d_\alpha$ is an appropriately normalised Busemann function for the point at infinity α.

Proposition 3.3. *(1) For all $(z,r) \in \mathbb{H}^5_{\mathbb{R}}$ and $\alpha \in A$, we have*

$$d_\alpha(z,r) = \frac{1}{r\,\mathtt{n}(I_\alpha)}\big(\mathtt{n}(z-\alpha)+r^2\big)\ ,$$

and $d_\infty(z,r) = \frac{1}{r}$.
(2) For all $x \in \mathbb{H}^5_{\mathbb{R}}$ and $\alpha = [u:v] \in \mathbb{P}^1_r(A)$, we have

$$d_\alpha(x) = \frac{\Theta(x)(u,v)}{\mathtt{n}(\mathscr{O}u+\mathscr{O}v)}\ .$$

(3) For all $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(A)$ and $\alpha = [x:y] \in \mathbb{P}^1_r(A)$, we have

$$d_{g\cdot\alpha}\circ g = \frac{\mathtt{n}(\mathscr{O}x+\mathscr{O}y)}{\mathtt{n}(\mathscr{O}(ax+by)+\mathscr{O}(cx+dy))}\ d_\alpha\ .$$

In particular, if $g \in \mathrm{SL}_2(\mathscr{O})$ and $\alpha \in \mathbb{P}^1_r(A)$, then $d_{g\cdot\alpha}\circ g = d_\alpha$.

Proof. (1) Since $M(f_\alpha) = \frac{1}{\mathtt{n}(I_\alpha)}\begin{pmatrix} 1 & -\alpha\\ -\overline{\alpha} & \mathtt{n}(\alpha)\end{pmatrix}$ and $M(\Theta(z,r)) = \frac{1}{r}\begin{pmatrix} 1 & -z\\ -\overline{z} & \mathtt{n}(z)+r^2\end{pmatrix}$, we have, by Equation (11),

$$d_\alpha(z,r) = \langle f_\alpha, \Theta(z,r)\rangle_{\mathscr{Q}} = \frac{1}{r\ \mathtt{n}(I_\alpha)}\big((\mathtt{n}(z)+r^2)+\mathtt{n}(\alpha)-\mathtt{tr}(\overline{\alpha}\,z)\big) = \frac{\mathtt{n}(z-\alpha)+r^2}{r\ \mathtt{n}(I_\alpha)}\ .$$

The computation of d_∞ is similar and easier.
(2) Let $x = (z,r) \in \mathbb{H}^5_{\mathbb{R}}$ and $f = \Theta(x)$. If $v \neq 0$, then $\alpha = uv^{-1}$, and by the definition of Θ and Assertion (1),

$$\begin{aligned} \frac{f(u,v)}{\mathtt{n}(\mathscr{O}u+\mathscr{O}v)} &= \frac{f(\alpha,1)}{\mathtt{n}(I_\alpha)} = \frac{1}{\mathtt{n}(I_\alpha)}\big(\,\overline{\alpha}\ 1\,\big)\ M(f)\begin{pmatrix}\alpha\\ 1\end{pmatrix} \\ &= \frac{\mathtt{n}(\alpha)-\overline{\alpha}\,z-\overline{z}\,\alpha+\mathtt{n}(z)+r^2}{r\ \mathtt{n}(I_\alpha)} = \frac{\mathtt{n}(z-\alpha)+r^2}{r\ \mathtt{n}(I_\alpha)} = d_\alpha(x)\ . \end{aligned}$$

Similarly, if $v=0$, then $\frac{f(u,v)}{\mathtt{n}(\mathscr{O}u+\mathscr{O}v)}=f(1,0)=\frac{1}{r}=d_\alpha(x)$.

(3) For every $w\in\mathbb{H}^5_{\mathbb{R}}$, using the (anti-)equivariance property (13) of Θ, Equation (12) and Proposition 3.2, we have

$$\begin{aligned} d_{g\cdot\alpha}\circ g(w) &= \langle f_{g\cdot\alpha},\Theta(gw)\rangle_{\mathscr{Q}}=\langle f_{g\cdot\alpha},\Theta(w)\circ g^{-1}\rangle_{\mathscr{Q}}=\langle f_{g\cdot\alpha}\circ g,\Theta(w)\rangle_{\mathscr{Q}}\\ &= \frac{\mathtt{n}(\mathscr{O}x+\mathscr{O}y)}{\mathtt{n}(\mathscr{O}(ax+by)+\mathscr{O}(cx+dy))}\,\langle f_\alpha,\Theta(w)\rangle_{\mathscr{Q}}\\ &= \frac{\mathtt{n}(\mathscr{O}x+\mathscr{O}y)}{\mathtt{n}(\mathscr{O}(ax+by)+\mathscr{O}(cx+dy))}\,d_\alpha(w)\ .\quad\Box \end{aligned}$$

Since $\mathrm{SL}_2(\mathscr{O})$ is a noncocompact lattice with cofinite volume in $\mathrm{SL}_2(\mathbb{H})$ and set of parabolic fixed points at infinity $\mathbb{P}^1_r(A)$, there exists (see for instance [Bow]) a Γ-equivariant family of horoballs in $\mathbb{H}^5_{\mathbb{R}}$ centered at the points of $\mathbb{P}^1_r(A)$, with pairwise disjoint interiors. Since $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$ may have several cusps as mentioned in the introduction, there are various choices for such a family, and we now use the normalized distance to the points of $\mathbb{P}^1_r(A)$ in order to define a canonical such family, and we give consequences on the structure of the orbifold $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$.

For all $\alpha\in\mathbb{P}^1_r(A)$ and $s>0$, we define the *normalized horoball centered at α with radius s* as

$$B_\alpha(s)=\{x\in\mathbb{H}^5_{\mathbb{R}}\;:\;d_\alpha(x)\le s\}\ .$$

The terminology is justified by the following result, which proves in particular that $B_\alpha(s)$ is indeed a (closed) horoball. Recall that the Busemann function $\beta:\partial_\infty\mathbb{H}^5_{\mathbb{R}}\times\mathbb{H}^5_{\mathbb{R}}\times\mathbb{H}^5_{\mathbb{R}}\to\mathbb{R}$ is defined, with $t\mapsto\xi_t$ any geodesic ray with point at infinity $\xi\in\partial_\infty\mathbb{H}^5_{\mathbb{R}}$, by

$$(\xi,x,y)\mapsto\beta_\xi(x,y)=\lim_{t\to+\infty}d(x,\xi_t)-d(y,\xi_t)\ .$$

Proposition 3.4. *Let $\alpha\in\mathbb{P}^1_r(A)$ and $s>0$.*
(1) There exists $c_\alpha\in\mathbb{R}$ such that $\ln d_\alpha(x)=\beta_\alpha(x,(0,1))+c_\alpha$ for every $x\in\mathbb{H}^5_{\mathbb{R}}$.
(2) If $\alpha\in A$, then $B_\alpha(s)$ is the Euclidean ball of center $\big(\alpha,\frac{s\,\mathtt{n}(I_\alpha)}{2}\big)$ and radius $\frac{s\,\mathtt{n}(I_\alpha)}{2}$. If $\alpha=\infty$, then $B_\alpha(s)$ is the Euclidean halfspace consisting of all (z,r) with $r\ge\frac{1}{s}$.
(3) For all $g\in\mathrm{SL}_2(\mathscr{O})$, we have $g(B_\alpha(s))=B_{g\cdot\alpha}(s)$.

Proof. (1) If $\alpha=\infty$, then for every $(z,r)\in\mathbb{H}^5_{\mathbb{R}}$, we have $d_\alpha(z,r)=\frac{1}{r}$ and

$$\beta_\infty((z,r),(0,1))=\beta_\infty((0,r),(0,1))=-\ln r\ ,$$

hence the result holds with $c_\infty=0$.

If $\alpha\in A$, since $\mathrm{SL}_2(A)$ acts transitively on $\mathbb{P}^1_r(A)$, let $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(A)$ be such that $\alpha=g\cdot\infty$. Recall that the Busemann function is invariant under the diagonal action of $\mathrm{SL}_2(\mathbb{H})$ on $\partial_\infty\mathbb{H}^5_{\mathbb{R}}\times\mathbb{H}^5_{\mathbb{R}}\times\mathbb{H}^5_{\mathbb{R}}$ and is an additive cocycle in its two variables in $\mathbb{H}^5_{\mathbb{R}}$. By Proposition 3.3 (3) since $\infty=[1:0]$, we hence have, for every $x\in\mathbb{H}^5_{\mathbb{R}}$,

$$\begin{aligned}\ln d_\alpha(x)&=\ln d_{g\cdot\infty}(g(g^{-1}x))=\ln\frac{d_\infty(g^{-1}x)}{\mathtt{n}(\mathscr{O}a+\mathscr{O}c)}\\ &=\beta_\infty(g^{-1}x,(0,1))-\ln\mathtt{n}(\mathscr{O}a+\mathscr{O}c)=\beta_{g\cdot\infty}(x,g(0,1))-\ln\mathtt{n}(\mathscr{O}a+\mathscr{O}c)\\ &=\beta_\alpha(x,(0,1))+\beta_\alpha((0,1),g(0,1))-\ln\mathtt{n}(\mathscr{O}a+\mathscr{O}c)\ .\end{aligned}$$

Hence the result holds, and taking $x=(0,1)$, we have by Proposition 3.3 (1)

$$c_\alpha = \ln \frac{\mathtt{n}(\alpha)+1}{\mathtt{n}(I_\alpha)}\;.$$

(2) If $\alpha\in A$, for every $(z,r)\in\mathbb{H}^5_{\mathbb{R}}$, by Proposition 3.3 (1), we have $d_\alpha(z,r)\leq s$ if and only if $\mathtt{n}(z-\alpha)+r^2\leq s\,r\,\mathtt{n}(I_\alpha)$, that is, if and only if $\mathtt{n}(z-\alpha)+(r-\frac{s\,\mathtt{n}(I_\alpha)}{2})^2\leq\big(\frac{s\,\mathtt{n}(I_\alpha)}{2}\big)^2$. The second claim of Assertion (2) is immediate.

(3) This follows from Proposition 3.3 (3). □

The following result extends and generalizes a result for $D_A=2$ of [Spe, §5].

Theorem 3.5. *Let $\mathscr{O}$ be a maximal order in a definite quaternion algebra A over $\mathbb{Q}$.*

(1) For all distinct $\alpha,\beta\in\mathbb{P}^1_r(A)$, the normalized horoballs $B_\alpha(1)$ and $B_\beta(1)$ have disjoint interior. Furthermore, their intersection is nonempty if and only if $\alpha=\infty$ and $\beta\in\mathscr{O}$, or $\beta=\infty$ and $\alpha\in\mathscr{O}$, or $\alpha,\beta\neq\infty$ and $I_\alpha I_\beta=\mathscr{O}(\alpha-\beta)$, in which case they meet in one and only one point.

(2) We have

$$\mathbb{H}^5_{\mathbb{R}}=\bigcup_{\alpha\in\mathbb{P}^1_r(A)}B_\alpha\big(\sqrt{D_A}\,\big)\;.$$

Before proving this result, let us make two remarks.

(i) Note that $B_0(1)$ and $B_\infty(1)$ intersect (exactly at their common boundary point $(0,1)$) whatever the definite quaternion algebra A over $\mathbb{Q}$ is. Thus the constant $s=1$ in Assertion (1) is optimal. The family $(B_\alpha(1))_{\alpha\in\mathbb{P}^1_r(A)}$ is a (canonical) family of maximal (closed) horoballs centered at the parabolic fixed points of $\mathrm{SL}_2(\mathscr{O})$ with pairwise disjoint interiors. Since $\mathrm{SL}_2(\mathscr{O})$ is a lattice (hence is geometrically finite with convex hull of its limit set equal to the whole $\mathbb{H}^5_{\mathbb{R}}$), the quotient $\mathrm{SL}_2(\mathscr{O})\backslash\big(\mathbb{H}^5_{\mathbb{R}}-\bigcup_{\alpha\in\mathbb{P}^1_r(A)}B_\alpha(1)\big)$ is compact (see for instance [Bow]).

(ii) Assertion (2) is a quantitative version of the standard geometric reduction theory (see for instance [GR, Bor1, Leu]) for the structure of the arithmetic orbifold $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$. It indeed implies that if $\mathscr{R}$ is a finite subset of $\mathrm{SL}_2(A)$ such that $\mathscr{R}\cdot\infty$ is a set of representatives of $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{P}^1_r(A)$, and if $\mathscr{D}_\gamma$ is a fundamental domain for the action on $\mathbb{H}$ of the stabilizer of ∞ in $\gamma^{-1}\,\mathrm{SL}_2(\mathscr{O})\gamma$ for every $\gamma\in\mathscr{R}$, then a weak fundamental domain for the action of $\mathrm{SL}_2(\mathscr{O})$ on $\mathbb{H}^5_{\mathbb{R}}$ is the finite union $\bigcup_{\gamma\in\mathscr{R}}\gamma\mathscr{S}_\gamma$ where $\mathscr{S}_\gamma$ is the Siegel set

$$\mathscr{S}_\gamma=(\mathscr{D}_\gamma\times\,]0,+\infty[\,)\cap\gamma^{-1}B_{\gamma\cdot\infty}(\sqrt{D_A})\;.$$

Proof. (1) Note that two horoballs centered at distinct points at infinity, which are not disjoint but have disjoint interior, meet at one and only one common boundary point. Hence the last claim of Assertion (1) follows from the first two ones.

First assume that $\alpha=\infty$, so that $\beta\in A$. By Proposition 3.4 (2), we have $B_\infty(1)=\{(z,r)\in\mathbb{H}^5_{\mathbb{R}}\;:\;r\geq1\}$ and $B_\beta(1)$ is the horoball centered at β with Euclidean diameter $\mathtt{n}(I_\beta)$ (see Figure 1). They hence meet if and only if $\mathtt{n}(I_\beta)\geq1$, and their interiors meet if and only if $\mathtt{n}(I_\beta)>1$. But since $\mathscr{O}\subset I_\beta$, by Equation (3), we have $\mathtt{n}(I_\beta)\leq\mathtt{n}(\mathscr{O})=1$ with equality if and only if $I_\beta=\mathscr{O}$, that is, $\beta\in\mathscr{O}$. The result follows.

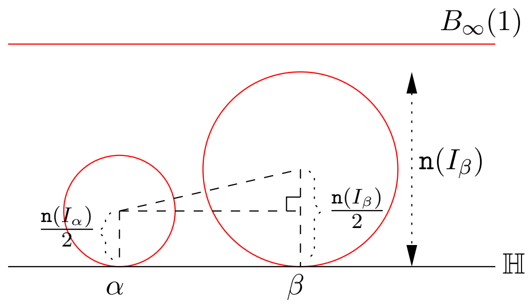

Figure 1: Disjointness of normalized horoballs $B_{\alpha'}(1)$ for $\alpha' \in \mathbb{P}^1_r(A)$.

Up to permuting α and β and applying the above argument, we may now assume that $\alpha, \beta \neq \infty$. The Euclidean balls $B_\alpha(1)$ and $B_\beta(1)$ meet if and only if the distance $d_{\alpha\beta}$ between their Euclidean center is less than or equal to the sum of their radii r_α and r_β, and their interior meet if and only if $d_{\alpha\beta} < r_\alpha + r_\beta$. By Proposition 3.4 (2) and by the multiplicativity of the reduced norms (see [Rei, Thm. 24.11 and p. 181]), we have (see the above picture)

$$\begin{aligned} {d_{\alpha\beta}}^2 - (r_\alpha + r_\beta)^2 &= \Big(\mathtt{n}(\alpha-\beta) + \big(\frac{\mathtt{n}(I_\alpha)}{2} - \frac{\mathtt{n}(I_\beta)}{2}\big)^2\Big) - \big(\frac{\mathtt{n}(I_\alpha)}{2} + \frac{\mathtt{n}(I_\beta)}{2}\big)^2 \\ &= \mathtt{n}(\alpha-\beta) - \mathtt{n}(I_\alpha)\,\mathtt{n}(I_\beta) = \mathtt{n}(\alpha-\beta) - \mathtt{n}(I_\alpha I_\beta)\ . \end{aligned}$$

Since $\alpha - \beta \in I_\alpha I_\beta$ and again by Equation (3), we have $\mathtt{n}(\alpha-\beta) \geq \mathtt{n}(I_\alpha I_\beta)$, with equality if and only if $I_\alpha I_\beta = \mathscr{O}(\alpha - \beta)$. The result follows.

(2) For every $x \in \mathbb{H}^5_\mathbb{R}$, let (u, v) in $\mathscr{O}\times\mathscr{O} - \{0\}$ realizing the minimum on $\mathscr{O}\times\mathscr{O}-\{0\}$ of the positive definite binary Hamiltonian form $\Theta(x)$, whose discriminant is -1. Let $\alpha = [u : v]$. Then by Proposition 3.3 (2) and by Equation (1), we have, since the norm of an integral left ideal is at least 1,

$$d_\alpha(x) = \frac{\Theta(x)(u, v)}{\mathtt{n}(\mathscr{O}u + \mathscr{O}v)} \leq \sqrt{D_A}\ .$$

This proves the result. □

The following observation, which is closely related with the proof of Assertion (1) of Theorem 3.5, will be useful later on.

Lemma 3.6. *For all $\alpha \neq \beta$ in A, the hyperbolic distance between $B_\alpha(1)$ and $B_\beta(1)$ is*

$$d(B_\alpha(1), B_\beta(1)) = \ln \frac{\mathtt{n}(\alpha-\beta)}{\mathtt{n}(I_\alpha I_\beta)}\ .$$

Proof. This follows from the easy exercise in real hyperbolic geometry saying that the distance in the upper halfspace model of the real hyperbolic n-space between two horospheres $\mathscr{H}, \mathscr{H}'$ with Euclidean radius r, r', and with Euclidean distance between their points at infinity equal to λ, is $d(\mathscr{H}, \mathscr{H}') = \ln \frac{\lambda^2}{4rr'}$, if the interiors of $\mathscr{H}$ and $\mathscr{H}'$ are disjoint.

This exercice uses the facts that the common perpendicular between two disjoint horoballs is the geodesic line through their points at infinity and that the (signed) hyperbolic length of an arc of Euclidean circle centered at a point at infinity with angles with the horizontal hyperplane between θ and $\pi/2$ is $-\ln\tan\frac{\theta}{2}$. $\square$

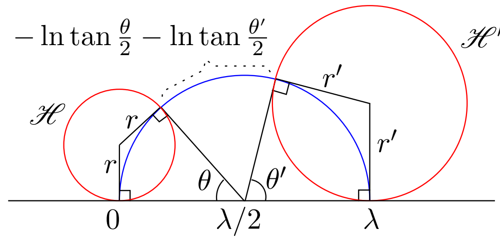

Figure 2

4 The spine of $\mathrm{SL}_2(\mathscr{O})$

Let A be a definite quaternion algebra over $\mathbb{Q}$ and let $\mathscr{O}$ be a maximal order in A. In this section, we describe a canonical $\mathrm{SL}_2(\mathscr{O})$-invariant cell decomposition of the 5-dimensional real hyperbolic space $\mathbb{H}^5_{\mathbb{R}}$. We follow [Men, BeS] when the field $\mathbb{H}$ is replaced by $\mathbb{C}$, the order $\mathscr{O}$ by the ring of integers of a quadratic imaginary extension of $\mathbb{Q}$, and $\mathbb{H}^5_{\mathbb{R}}$ by $\mathbb{H}^3_{\mathbb{R}}$.

For every $\alpha\in\mathbb{P}^1_r(A)$, the *Ford-Voronoi cell* of α for the action of $\mathrm{SL}_2(\mathscr{O})$ on $\mathbb{H}^5_{\mathbb{R}}$ is the set $\mathscr{H}_\alpha$ of points not farther from α than from any other element of $\mathbb{P}^1_r(A)$:

$$\mathscr{H}_\alpha=\{x\in\mathbb{H}^5_{\mathbb{R}}\;:\;\forall\,\beta\in\mathbb{P}^1_r(A),\;\;d_\alpha(x)\le d_\beta(x)\}\,.$$

In the complex case, this set is called the *minimal set* of α, see [Men, Def. 1.3.1].

Proposition 4.1. *Let $\alpha\in\mathbb{P}^1_r(A)$.*
(1) For all $g\in\mathrm{SL}_2(\mathscr{O})$, we have $g(\mathscr{H}_\alpha)=\mathscr{H}_{g\cdot\alpha}$.
(2) We have $B_\alpha(1)\subset\mathscr{H}_\alpha\subset B_\alpha(\sqrt{D_A})$.
(3) The Ford-Voronoi cell $\mathscr{H}_\alpha$ is a noncompact 5-dimensional convex hyperbolic polytope, whose proper cells are compact, and the stabilizer of α in $\mathrm{SL}_2(\mathscr{O})$ acts cocompactly on its boundary $\partial\mathscr{H}_\alpha$.
(4) For every $\beta\in\mathbb{P}^1_r(A)-\{\alpha\}$, let $\mathscr{S}_{\alpha,\beta}=\{x\in\mathbb{H}^5_{\mathbb{R}}\;:\;d_\alpha(x)=d_\beta(x)\}$. Then $\mathscr{S}_{\alpha,\beta}$ is a hyperbolic hyperplane, that intersects perpendicularly the geodesic line with points at infinity α and β. Furthermore, the Ford-Voronoi cells $\mathscr{H}_\alpha$ and $\mathscr{H}_\beta$ have disjoint interior and their (possibly empty) intersection is contained in $\mathscr{S}_{\alpha,\beta}$.

Thus

$$\mathbb{H}^5_{\mathbb{R}}=\bigcup_{\alpha\in\mathbb{P}^1_r(A)}\mathscr{H}_\alpha$$

is a $\mathrm{SL}_2(\mathscr{O})$-invariant cell decomposition of $\mathbb{H}^5_{\mathbb{R}}$, whose codimension 1 skeleton will be studied in the remainder of this section. We will see in Examples 4.4 and 4.5 that the inclusions in Assertion (2) of this proposition, as well as the value $s=\sqrt{D_A}$ such that $\mathbb{H}^5_{\mathbb{R}}=\bigcup_{\alpha\in\mathbb{P}^1_r(A)}B_\alpha(s)$ in Theorem 3.5 (2), are sharp when $D_A=2,3$.

Proof. (1) This follows from Proposition 3.3 (3).

(2) The inclusion on the left-hand side follows from Theorem 3.5 (1): If $x\in B_\alpha(1)$ and $x\notin\mathscr{H}_\alpha$, then there exists $\beta\in\mathbb{P}^1_r(A)-\{\alpha\}$ such that $d_\beta(x)<d_\alpha(x)\le 1$, thus the interiors of $B_\alpha(1)$ and $B_\beta(1)$ have nonempty intersection, a contradiction. If $x\notin B_\alpha(\sqrt{D_A})$, then

by Theorem 3.5 (2), there exists $\beta \in \mathbb{P}^1_r(A) - \{\alpha\}$ such that $x \in B_\beta(\sqrt{D_A})$. Hence $d_\beta(x) \leq \sqrt{D_A} < d_\alpha(x)$, so that $x \notin \mathscr{H}_\alpha$.

(3) and (4) Since $\ln d_\alpha$ is a Busemann function with respect to the point at infinity α by Proposition 3.4 (1), for every $\beta \in \mathbb{P}^1_r(A) - \{\alpha\}$, the set $\mathscr{H}_{\alpha,\beta} = \{x \in \mathbb{H}^5_\mathbb{R} \;:\; d_\alpha(x) \leq d_\beta(x)\}$ is a (closed) hyperbolic halfspace. Its boundary is $\mathscr{S}_{\alpha,\,\beta}$, which is hence a hyperbolic hyperplane that intersects perpendicularly the geodesic line with points at infinity α and β. Being the intersection of the family of hyperbolic halfspaces $(\mathscr{H}_{\alpha,\beta})_{\beta\in\mathbb{P}^1_r(A)-\{\alpha\}}$ with locally finite family of boundaries $(\mathscr{S}_{\alpha,\beta})_{\beta\in\mathbb{P}^1_r(A)-\{\alpha\}}$, and containing the horoball $B_\alpha(1)$, the Ford-Voronoi cell $\mathscr{H}_\alpha$ is a noncompact 5-dimensional convex hyperbolic polytope. Since α is a bounded parabolic fixed point of the lattice $\mathrm{SL}_2(\mathscr{O})$ and by Assertion (2), the stabilizer of α in $\mathrm{SL}_2(\mathscr{O})$ acts cocompactly on $\partial\mathscr{H}_\alpha$, and hence the boundary cells of $\mathscr{H}_\alpha$ are compact. □

The horoballs $B_0(1)$ and $B_\infty(1)$ with disjoint interiors meet at $(0,1) \in \mathbb{H}^5_\mathbb{R}$, and at most two horoballs with disjoint interior can meet at a given point of $\mathbb{H}^5_\mathbb{R}$. Thus, the Ford-Voronoi cells at 0 and at ∞ have nonempty intersection, which is a compact 4-dimensional hyperbolic polytope. This intersection

$$\Sigma_\mathscr{O} = \mathscr{H}_0 \cap \mathscr{H}_\infty \tag{14}$$

is called the *fundamental cell* of the spine of $\mathrm{SL}_2(\mathscr{O})$. It is contained in the hyperbolic hyperplane $\mathscr{S}_{0,\infty} = \{(z,r) \in \mathbb{H}^5_\mathbb{R} : \mathtt{n}(z) + r^2 = 1\}$. We will describe it in Example 4.4 when $D_A = 2$ and in Example 4.5 when $D_A = 3$.

Lemma 4.2. *Let $\alpha \in \mathbb{P}^1_r(A)$ be such that $e = \mathscr{H}_\infty \cap \mathscr{H}_0 \cap \mathscr{H}_\alpha$ is a 3-dimensional cell in the boundary of $\Sigma_\mathscr{O}$. Then*

$$\min\{\mathtt{n}(I_\alpha), \mathtt{n}(I_{\alpha^{-1}})\} \geq \frac{1}{D_A}\;,$$

and the horizontal projection of e to $\mathbb{H}$ is contained in the Euclidean hyperplane

$$\{z \in \mathbb{H} \;:\; \mathtt{tr}(\overline{\alpha}\, z) = 1 + \mathtt{n}(\alpha) - \mathtt{n}(I_\alpha)\}\,.$$

Proof. Note that we have $\alpha \neq 0, \infty$. By Proposition 4.1 (2), the triple intersection $B_\infty(\sqrt{D_A}) \cap B_0(\sqrt{D_A}) \cap B_\alpha(\sqrt{D_A})$ contains the 3-cell e, hence both intersections $B_\infty(\sqrt{D_A}) \cap B_\alpha(\sqrt{D_A})$ and $B_0(\sqrt{D_A}) \cap B_\alpha(\sqrt{D_A})$ are nonempty. Since $B_\infty(\sqrt{D_A})$ is the Euclidean halfspace of points (z,r) with $r \geq \frac{1}{\sqrt{D_A}}$ and $B_\alpha(\sqrt{D_A})$ is a Euclidean ball tangent to the horizontal plane with diameter $\sqrt{D_A}\,\mathtt{n}(I_\alpha)$ by Proposition 3.4 (2), this implies that $\sqrt{D_A}\,\mathtt{n}(I_\alpha) \geq \frac{1}{\sqrt{D_A}}$, so that $D_A\,\mathtt{n}(I_\alpha) \geq 1$. Since $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ belongs to $\mathrm{SL}_2(\mathscr{O})$ and maps 0 to ∞ and α to α^{-1}, and by Proposition 3.4 (3), the intersection $B_\infty(\sqrt{D_A}) \cap B_{\alpha^{-1}}(\sqrt{D_A})$ is nonempty, hence similarly $D_A\,\mathtt{n}(I_{\alpha^{-1}}) \geq 1$.

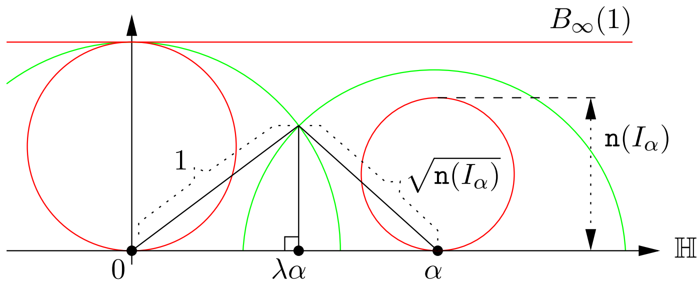

Figure 3

The set of points equidistant to 0 and ∞ is the open Euclidean upper hemisphere of radius 1 centered at 0, and the set of points equidistant to α and ∞ is the open Euclidean upper hemisphere of radius $\sqrt{\mathtt{n}(I_\alpha)}$ centered at α. The projection to $\mathbb{H}$ of the intersection of these hemispheres is contained in the affine Euclidean hyperplane of $\mathbb{H}$ perpendicular to the real vector line containing α that passes through the projection, that we denote by $\lambda\alpha$ with $\lambda > 0$, to that line of any point at Euclidean distance 1 from 0 and at Euclidean distance $\sqrt{\mathtt{n}(I_\alpha)}$ from α. An easy computation (considering the two cases when $\mathtt{n}(\alpha) > 1$ as in Figure 3 or when $\mathtt{n}(\alpha) \leq 1$) using right-angled triangles gives that $\lambda = \frac{1+\mathtt{n}(\alpha)-\mathtt{n}(I_\alpha)}{2\,\mathtt{n}(\alpha)}$. Since $(u, v) \mapsto \frac{1}{2}\,\mathtt{tr}(\overline{u}\, v)$ is the standard Euclidean scalar product on $\mathbb{H}$, this gives the result. □

The *spine* $X_\mathscr{O}$ of $\mathrm{SL}_2(\mathscr{O})$ is the codimension 1 skeleton of the cell decomposition into Ford-Voronoi cells of $\mathbb{H}^5_\mathbb{R}$, that is,

$$X_\mathscr{O} = \bigcup_{\alpha\neq\beta\in\mathbb{P}^1_r(A)} \mathscr{H}_\alpha \cap \mathscr{H}_\beta = \bigcup_{\alpha\in\mathbb{P}^1_r(A)} \partial\mathscr{H}_\alpha \,.$$

It is an $\mathrm{SL}_2(\mathscr{O})$-invariant piecewise hyperbolic polyhedral complex of dimension 4. We refer for instance to [BrH] for the definitions related to polyhedral complexes, CAT(0) spaces and orbihedra. Note that the stabilizers in $\mathrm{SL}_2(\mathscr{O})$ of the cells of $X_\mathscr{O}$ may be nontrivial. The spine is called the *minimal incidence set* in the complex case in [Men] and [ScV], and the *cut locus of the cusp* in [HP, §5] when the class number is one.

For every hyperbolic cell C of $X_\mathscr{O}$ and every $\alpha \in \mathbb{P}^1_r(A)$ such that $C \subset \partial\mathscr{H}_\alpha$, the radial projection along geodesic rays with point at infinity α from C to the horosphere $\partial B_\alpha(1)$ is a homeomorphism onto its image, and the pull-back of the flat induced length metric on this horosphere endows C with a structure of a compact Euclidean polytope. This Euclidean structure does not depend on the choice of α, since the (possibly empty) intersection $\mathscr{H}_\alpha \cap \mathscr{H}_\beta$ is equidistant to $B_\alpha(1)$ and $B_\beta(1)$ for all distinct α, β in $\mathbb{P}^1_r(A)$. It is well known (see for instance [Ait]) that these Euclidean structures on the cells of $X_\mathscr{O}$ endow $X_\mathscr{O}$ with the structure of a CAT(0) piecewise Euclidean polyhedral complex.

Furthermore, $X_\mathscr{O}$ is an $\mathrm{SL}_2(\mathscr{O})$-invariant deformation retract of $\mathbb{H}^5_\mathbb{R}$ along the geodesic rays with points at infinity the points in $\mathbb{P}^1_r(A)$. Since the quotient orbifold with boundary $\mathrm{SL}_2(\mathscr{O})\backslash\big(\mathbb{H}^5_\mathbb{R} - \bigcup_{\alpha\in\mathbb{P}^1_r(A)} B_\alpha(1)\big)$ is compact, the quotient space $\mathrm{SL}_2(\mathscr{O})\backslash X_\mathscr{O}$ is a finite locally CAT(0) piecewise Euclidean orbihedral complex.

The following result gives a description of the cell structure of $\mathrm{SL}_2(\mathscr{O})\backslash X_\mathscr{O}$ when $\mathscr{O}$ is left-Euclidean. See Examples 4.4 and 4.5 for a more detailed study when $D_A = 2, 3$.

Proposition 4.3. *The Hamilton-Bianchi group* $\mathrm{SL}_2(\mathscr{O})$ *acts transitively on the set of* 4*-dimensional cells of its spine* $X_{\mathscr{O}}$ *if and only if* $D_A \in \{2,3,5\}$*. In these cases, the horizontal projection of the fundamental cell* $\Sigma_{\mathscr{O}}$ *to* $\mathbb{H}$ *is the Euclidean Voronoi cell of* 0 *for the* $\mathbb{Z}$*-lattice* $\mathscr{O}$ *in the Euclidean space* $\mathbb{H}$*.*

Proof. If $\mathrm{SL}_2(\mathscr{O})$ acts transitively on the 4-dimensional cells of $X_{\mathscr{O}}$, then $X_{\mathscr{O}} = \mathrm{SL}_2(\mathscr{O})\,\Sigma_{\mathscr{O}}$, and the stabilizer of ∞ in $\mathrm{SL}_2(\mathscr{O})$ acts transitively on the set of 4-dimensional cells in $\partial\mathscr{H}_\infty$, since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathscr{O})$ preserves $\Sigma_{\mathscr{O}} = \mathscr{H}_\infty \cap \mathscr{H}_0$ and exchanges $\mathscr{H}_\infty$ and $\mathscr{H}_0$. This stabilizer consists of the upper triangular matrices with coefficients in $\mathscr{O}$, hence with diagonal coefficients in $\mathscr{O}^\times$. The orbit of $0 \in \mathbb{H}$ under this stabilizer is exactly $\mathscr{O}$. Since $\Sigma_{\mathscr{O}}$ is compact and contained in the open Euclidean upper hemisphere centered at 0 with radius 1, by horizontal projection on $\mathbb{H}$, this proves that $\mathbb{H}$ is covered by the open balls of radius 1 centered at the points of $\mathscr{O}$. Hence $\mathscr{O}$ is left-Euclidean.

Conversely, if $\mathscr{O}$ is left-Euclidean, then the class number of A is 1, and $\mathrm{SL}_2(\mathscr{O})$ acts transitively on the Ford-Voronoi cells. In order to prove that $\mathrm{SL}_2(\mathscr{O})$ acts transitively on the 4-dimensional cells of $X_{\mathscr{O}}$, we hence only have to prove that the stabilizer of ∞ in $\mathrm{SL}_2(\mathscr{O})$ acts transitively on the 4-dimensional cells of $\partial\mathscr{H}_\infty$. For this, let $\alpha \in A$ be such that $\mathscr{H}_\infty \cap \mathscr{H}_\alpha$ is a 4-dimensional cell in $\partial\mathscr{H}_\infty$. Let us prove that $\alpha \in \mathscr{O}$, which gives the result. Due to problems caused by the noncommutativity of $\mathbb{H}$, the proof of [BeS, Prop. 4.3] does not seem to extend exactly. We will use instead Lemma 2.1.

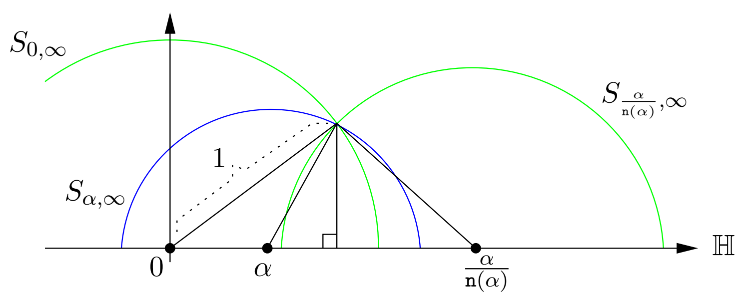

Figure 4

Assume for a contradiction that $\alpha \notin \mathscr{O}$. Since $\mathscr{O}$ is left-Euclidean, there exists $c \in \mathscr{O}$ such that $\mathtt{n}(\alpha - c) < 1$. Up to replacing α by $\alpha - c$, since translations by $\mathscr{O}$ preserve $\mathscr{H}_\infty$, we may assume that $0 < \mathtt{n}(\alpha) < 1$. For every $\beta \in A$ and $\beta' \in \mathbb{P}^1_r(A) - \{\beta\}$, let us denote by $S_{\beta,\beta'}$ the Euclidean upper hemisphere centered at β equidistant from the points at infinity β and β'. In particular, $S_{0,\infty}$ has radius 1. The inversion with respect to the sphere containing $S_{0,\infty}$ acts by an orientation-reversing isometry on $\mathbb{H}^5_{\mathbb{R}}$, and acts on the boundary at infinity $\mathbb{P}^1_r(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$ by $z \mapsto \frac{z}{\mathtt{n}(z)} = \frac{1}{\overline{z}}$. By Lemma 2.1, it hence normalizes $\mathrm{SL}_2(\mathscr{O})$ and, in particular, sends $S_{\alpha,\infty}$ to $S_{\frac{\alpha}{\mathtt{n}(\alpha)},0}$, and fixes $S_{0,\infty}$ (see Figure 4). Since $\mathtt{n}(\alpha) < 1$, the hemisphere $S_{\alpha,\infty}$ is therefore below the union of $S_{0,\infty}$ and $S_{\frac{\alpha}{\mathtt{n}(\alpha)},0}$, which contradicts the fact that $\mathscr{H}_\infty \cap \mathscr{H}_\alpha$, which is contained in $S_{\alpha,\infty}$, is a 4-dimensional cell in $\partial\mathscr{H}_\infty$.

In order to prove the last claim of Proposition 4.3, note that $\mathtt{n}(I_\alpha) = 1$ if $\alpha \in \mathscr{O}$, and that the above proof shows that the 4-dimensional cells contained in $\partial\mathscr{H}_\infty$ and meeting the fundamental cell along a 3-dimensional cell are contained in spheres centered at points in $\mathscr{O}$. Therefore, by Lemma 4.2, the horizontal projection of $\Sigma_{\mathscr{O}}$ is the intersection of the halfspaces containing 0 and bounded by the Euclidean hyperplanes with equation $\mathtt{tr}(\overline{\alpha}\, z) = \mathtt{n}(\alpha)$ for all $\alpha \in \mathscr{O}$. Since this hyperplane is the set of points z in the Euclidean space $\mathbb{H}$ equidistant to 0 and α, this proves that the horizontal projection of $\Sigma_{\mathscr{O}}$ is indeed the Voronoi cell at 0 of the $\mathbb{Z}$-lattice $\mathscr{O}$. □

Example 4.4. Let $A = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k \subset \mathbb{H}$ be the definite quaternion algebra over $\mathbb{Q}$ with $D_A = 2$, and let $\mathscr{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+k}{2}$ be the (maximal) Hurwitz order in A. The Hurwitz order $\mathscr{O}$ is the lattice of type $F_4 = D_4^*$. The group of unit Hurwitz quaternions, called the binary tetrahedral group, has 24 elements

$$\mathscr{O}^\times = \Big\{ \pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2} \Big\}.$$

The Voronoi cell $\Sigma_{\mathscr{O}}^{\mathbb{H}}$ of 0 for the lattice $\mathscr{O}$ in $\mathbb{H}$ is (up to homothety) the 24-*cell*, which is the (unique up to homothety) self-dual, regular, convex Euclidean 4-polytope, whose Schläfli symbol is $\{3, 4, 3\}$. The vertices of $\Sigma_{\mathscr{O}}^{\mathbb{H}}$ are the 24 quaternions

$$\frac{1+i}{2}\mathscr{O}^\times = \Big\{ \frac{\pm 1 \pm i}{2}, \frac{\pm 1 \pm j}{2}, \frac{\pm 1 \pm k}{2}, \frac{\pm i \pm j}{2}, \frac{\pm i \pm k}{2}, \frac{\pm j \pm k}{2} \Big\}.$$

See for instance [CoS, p. 119] for more details and references.

Let $\mathbb{H}_1^\times$ be the subgroup of $\mathbb{H}^\times$ that consists of the quaternions of norm 1. The group morphism $\mathbb{H}_1^\times \times \mathbb{H}_1^\times \to \mathrm{SO}(4)$ that associates to $(u, v) \in \mathbb{H}_1^\times \times \mathbb{H}_1^\times$ the orthogonal transformation $z \mapsto uzv^{-1}$ of the Euclidean space $\mathbb{H}$ endowed with the basis $\{1, i, j, k\}$ is surjective with kernel $\{\pm(1, 1)\}$, see for instance [Ber2, Thm. 8.9.8]. The group of Euclidean symmetries of the 24-cell is the exceptional Coxeter group $F_4 = [3, 4, 3]$. It consists of the 1152 elements $z \mapsto uzv^{-1}$, $z \mapsto u\,\overline{z}\,v^{-1}$ of $\mathrm{O}(4)$, where either both u and v are unit Hurwitz integers or both $u/\sqrt{2}$ and $v/\sqrt{2}$ are in $\frac{1+i}{2}\mathscr{O}^\times$.

By Equation (14) and Proposition 4.3, the fundamental cell of $\mathrm{SL}_2(\mathscr{O})$ is

$$\Sigma_{\mathscr{O}} = \{(z, t) \in \mathbb{H}^5_{\mathbb{R}} : z \in \Sigma_{\mathscr{O}}^{\mathbb{H}},\ \mathtt{n}(z) + t^2 = 1\}.$$

With the notation of Lemma 2.1, the stabilizer of $\Sigma_{\mathscr{O}}$ in $\mathrm{SL}_2(\mathscr{O})$ consists of the 1152 matrices $C_{a,d} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $JC_{a,d} = \begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix}$ with $a, d \in \mathscr{O}^\times$. When $\Sigma_{\mathscr{O}}$ is identified with $\Sigma_{\mathscr{O}}^{\mathbb{H}}$ by the horizontal projection, the diagonal matrices induce by Equation (7) 288 rotational symmetries of $\Sigma_{\mathscr{O}}^{\mathbb{H}}$, and the antidiagonal ones induce another 288 orientation-reversing symmetries, together forming a subgroup of index 2 in the Coxeter group $[3, 4, 3]$.

The quotient $\mathrm{SL}_2(\mathscr{O})\backslash X_{\mathscr{O}}$ is obtained by identifying the opposite 3-dimensional cells of $\Sigma_{\mathscr{O}}$ (which are 24 regular octahedra) by translations by elements of $\mathscr{O}$, and by forming the quotient by the stabilizer of $\Sigma_{\mathscr{O}}$. In particular, all the vertices of $X_{\mathscr{O}}$ are in the same orbit under $\mathrm{SL}_2(\mathscr{O})$.

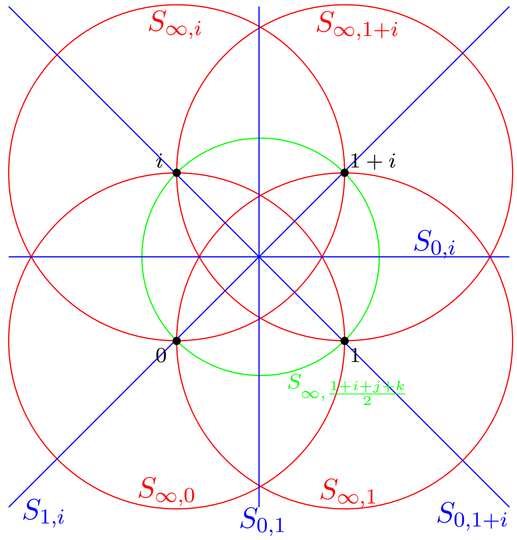

Figure 5: Boundary of equidistant hemispheres and halfplanes in $\mathbb{C} \subset \mathbb{H}$.

Speiser [Spe, §5] observed that the estimate of Proposition 4.1 (2) is sharp in this example: $\mathbb{H}^5_{\mathbb{R}}$ is indeed contained in $\bigcup_{\alpha \in \mathbb{P}^1_r(A)} B_\alpha(\sqrt{2})$, but the $\mathrm{SL}_2(\mathscr{O})$-orbit that contains all the vertices of $\Sigma_{\mathscr{O}}$ is not contained in the union of the interiors of the horoballs $B_\alpha(\sqrt{2})$. Furthermore, Speiser proved that the point

$$v_0 = \Big(\frac{1+i}{2}, \frac{1}{\sqrt{2}}\Big)$$

belongs to the boundary of exactly 10 horoballs $B_\alpha(\sqrt{2})$, the ones with α in

$$E = \Big\{\infty, 0, 1, i, 1+i, \frac{1+i \pm j \pm k}{2}, \frac{1}{1-i} = \frac{1+i}{2}\Big\}.$$

In particular, v_0 is a vertex of the spine $X_{\mathscr{O}}$, contained in the boundary of exactly 10 Ford-Voronoi cells $\mathscr{H}_\alpha$ for α in this set.

The set E contains exactly 5 pairs $\{\alpha, \beta\}$ of distinct elements such that the interiors of the horoballs $B_\alpha(\sqrt{2})$ and $B_\beta(\sqrt{2})$ are disjoint, these pairs being $\{\infty, \frac{1}{1-i}\}$, $\{0, 1+i\}$, $\{1, i\}$, $\{\frac{1+i+j+k}{2}, \frac{1+i-j-k}{2}\}$ and $\{\frac{1+i+j-k}{2}, \frac{1+i-j+k}{2}\}$. If $\{\alpha, \beta\}$ is one of these pairs, the Ford-Voronoi cells $\mathscr{H}_\alpha$ and $\mathscr{H}_\beta$ intersect only at v_0. For all other pairs in E, the intersection is a higher-dimensional cell.

As $0, 1, i, 1+i, \frac{1+i\pm j\pm k}{2}$ are in $\mathscr{O}$ and $\frac{1+i}{2}$ is not in $\mathscr{O}$, there are 8 Ford-Voronoi cells incident to v_0 that intersect $\mathscr{H}_\infty$ in a 4-dimensional 24-cell (see Figure 5, which represents the intersection with the plane in $\mathbb{H}$ containing $0, 1, i$ of the closures of the equidistant spheres and planes between some pairs of elements in $\{\infty, 0, 1, i, 1+i, \frac{1+i+j+k}{2}\}$, so that the horizontal projection of v_0 is the common intersection points of the straight lines). A similar property holds for all the other Ford-Voronoi cells incident to v_0: For example, $\mathscr{H}_0$ intersects in a 4-dimensional cell the Ford-Voronoi cells $\mathscr{H}_\infty, \mathscr{H}_1, \mathscr{H}_i, \mathscr{H}_{\frac{1+i\pm j\pm k}{2}}, \mathscr{H}_{\frac{1+i}{2}}$, but not $\mathscr{H}_{i+1}$ by Theorem 3.5 (1), since $I_0 I_{i+1} = \mathscr{O} \neq \mathscr{O}(1+i)$. Thus the pattern of pairwise intersections into 4-dimensional cells of these 10 Ford-Voronoi cells is given by Figure 6 and the number of 24-cells containing v_0 is exactly $40 = (10 \times 8)/2$, one for each edge of this intersection pattern.

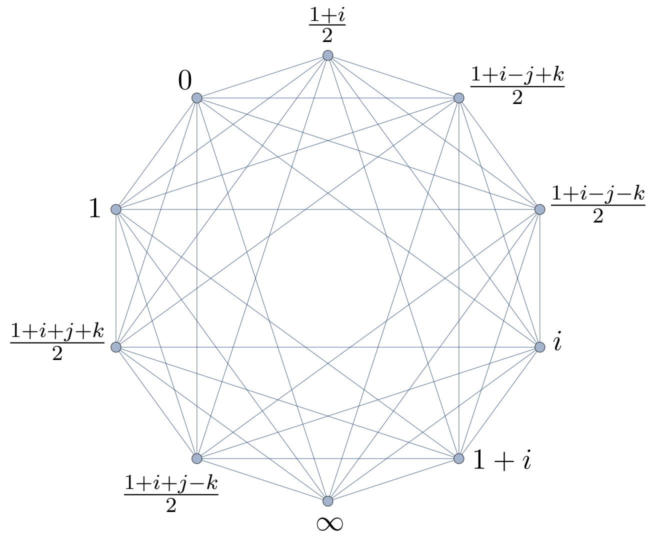

Figure 6: Pattern of intersections into 4-dimensional cells of Ford-Voronoi cells centered at $\{\infty, 0, 1, i, i+1, \frac{1+i\pm j\pm k}{2}, \frac{1+i}{2}\}$.

The boundary of each $\mathscr{H}_\alpha$ is tiled by 24-cells, combinatorially forming the 24-cell honeycomb. The dual of this honeycomb is the 16-cell honeycomb. Therefore, the link of the vertex v_0 in the tessellation of $\partial\mathscr{H}_\alpha$ for all $\alpha \in E$ is the dual of the boundary of the 16-cell, which is the boundary of the 4-cube, such that the intersection of the link with each of the eight 24-cells is a 3-cube.

Gluing together the ten boundaries of 4-cubes (that have been subdivided in eight 3-cubes each) according to the above intersection pattern proves that the link of v_0 in the spine $X_{\mathscr{O}}$ is the 3-skeleton of the 5-cube (which is the 5-dimensional regular polytope with Schläfli symbol $\{4,3,3,3\}$).

Example 4.5. The (unique up to conjugation) maximal order of the definite quaternion algebra $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ of discriminant $D_A = 3$ is $\mathbb{Z}[1, i, \frac{i+j}{2}, \frac{1+k}{2}]$, see [Vig, p. 98]. Using the unique $\mathbb{Q}$-linear map from $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ to $\mathbb{H}$ sending 1 to 1, i to j, j to $k\sqrt{3}$ and k to $-i\sqrt{3}$, we identify $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ with the $\mathbb{Q}$-subalgebra A of $\mathbb{H}$ generated by 1, $i\sqrt{3}$, j and $k\sqrt{3}$, and the maximal order is then identified with $\mathscr{O} = \mathbb{Z}[1, \rho, j, \rho j]$, where

$$\rho = \frac{1+i\sqrt{3}}{2}\ .$$

The group of units of $\mathscr{O}$ is the binary dihedral group of order 12

$$\mathscr{O}^\times = \{\pm 1,\ \pm j,\ \pm\rho,\ \pm\rho^2,\ \pm\rho j,\ \pm\rho^2 j\}\ .$$

The elements of the maximal order $\mathscr{O} = \mathbb{Z}[1,\rho] + \mathbb{Z}[1,\rho]j$ of A are the vertices of the 3-3 duoprism honeycomb in the 4-dimensional Euclidean space $\mathbb{H}$, seen as the orthogonal product $\mathbb{C} \overset{\perp}{\oplus} \mathbb{C}\,j$ of its Euclidean subspaces $\mathbb{C} = \mathbb{R} + \mathbb{R}\,i$ and $\mathbb{C}\,j = \mathbb{R}\,j + \mathbb{R}\,k$. The 9 elements of the set

$$V_{3,3} = \{0, 1, j, 1+j, \rho, \rho j, 1+\rho j, j+\rho, \rho(1+j)\}\ ,$$

contained in $\mathscr{O}$, are the vertices of its fundamental 3-3 *duoprism*, which is a uniform 4-polytope with Schläfli symbol $\{3\}\times\{3\}$ (the Cartesian product of two equilateral triangles, whose 1-skeleton is given in Figure 7). We refer to Coxeter's three papers [Cox1, Cox2, Cox3] for notation and references about uniform polytopes and their Coxeter groups, with the help of the numerous and beautiful articles in Wikipedia and Polytope Wiki.

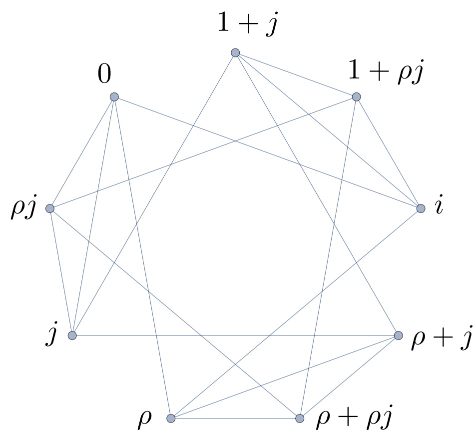

Figure 7: The 1-skeleton of the 3-3 duoprism.

The Voronoi cell $\Sigma^{\mathbb{H}}_{\mathscr{O}}$ of 0 for the lattice $\mathscr{O}$ in $\mathbb{H}$ is the 6-6 *duoprism* whose Schläfli symbol is $\{6\}\times\{6\}$. It is the Cartesian product of two copies of the Voronoi cell of 0 for the hexagonal lattice of the Eisenstein integers in $\mathbb{C}$ whose set of vertices is $V_6 = \{\pm\frac{i}{\sqrt{3}}, \pm\frac{1}{2}\pm\frac{i}{2\sqrt{3}}\}$. Thus, the set of vertices of $\Sigma^{\mathbb{H}}_{\mathscr{O}}$ is $V_6 + V_6\, j$. These 36 vertices, including

$$z_0 = \frac{1}{2} + \frac{i}{2\sqrt{3}} + \frac{j}{2} + \frac{k}{2\sqrt{3}} = (j+\rho)(1+\rho)^{-1}\ ,$$

belong to A and all have reduced norm $\frac{2}{3}$.

By Proposition 4.3, the fundamental cell $\Sigma_{\mathscr{O}}$ of $\mathrm{SL}_2(\mathscr{O})$ is the subset of the Euclidean unit sphere in $\mathbb{H}^n_{\mathbb{C}}$ whose horizontal projection is $\Sigma^{\mathbb{H}}_{\mathscr{O}}$. In particular, all the vertices of $\Sigma_{\mathscr{O}}$ have Euclidean height $\frac{1}{\sqrt{3}}$. Let u and v be either both in $\mathscr{O}^\times$ or both in $\rho^{\frac{1}{2}}\mathscr{O}^\times$. The 288 mappings $z\mapsto uzv^{-1}$ and $z\mapsto u\,\overline{z}\,v^{-1}$ are Euclidean symmetries of $\Sigma^{\mathbb{H}}_{\mathscr{O}}$, and they form the Coxeter group $[[6,2,6]]$ of the symmetries of the 6-6 duoprism $\Sigma^{\mathbb{H}}_{\mathscr{O}}$.

With the notation of Lemma 2.1, the stabilizer of $\Sigma_{\mathscr{O}}$ in $\mathrm{SL}_2(\mathscr{O})$ consists of the 288 matrices $C_{a,d} = \begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}$ and $JC_{a,d} = \begin{pmatrix} 0 & d\\ a & 0\end{pmatrix}$ with $a,d\in\mathscr{O}^\times$. When $\Sigma_{\mathscr{O}}$ is identified with $\Sigma^{\mathbb{H}}_{\mathscr{O}}$ by the horizontal projection, the diagonal matrices induce by Equation (7) 72 rotational symmetries of $\Sigma^{\mathbb{H}}_{\mathscr{O}}$, and the antidiagonal ones induce another 72 orientation-reversing symmetries, together forming a subgroup of index 2 in $[[6,2,6]]$.

A straightforward computation using Mathematica gives that the subgroup of $[[6,2,6]]$ that arises from the diagonal matrices in $\mathrm{SL}_2(\mathscr{O})$ acts transitively on the vertices of $\Sigma^{\mathbb{H}}_{\mathscr{O}}$. Thus, this subgroup acts transitively on the vertices of the fundamental cell $\Sigma_{\mathscr{O}}$, which implies that all vertices of the spine $X_{\mathscr{O}}$ are in the same orbit.

We will now turn to a study of the link of a vertex in $X_{\mathscr{O}}$. Let

$$v_0 = \Big(z_0, \frac{1}{\sqrt{3}}\Big)\ ,$$

which is the vertex of $\Sigma_{\mathscr{O}}$ whose projection to $\mathbb{H}$ is z_0. Let

$$g = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & -3z_0 \end{pmatrix} = \begin{pmatrix} 3z_0 & 1-3z_0^2 \\ 3 & -3z_0 \end{pmatrix} \in \mathrm{GL}_2(A)\,,$$

inducing the the homography $z \mapsto \frac{1}{3}\,(z-z_0)^{-1} + z_0$.

Lemma 4.6. *The element g belongs to the normalizer of $\mathrm{SL}_2(\mathscr{O})$ in $\mathrm{SL}_2(\mathbb{H})$.*

Proof. Computations (using Mathematica and SAGE) show that g conjugates all the generators of $\mathrm{SL}_2(\mathscr{O})$ given in Corollary 2.2 to elements of $\mathrm{SL}_2(\mathscr{O})$, as follows. We have

$$gJg^{-1} = \begin{pmatrix} 3+\rho+j+\rho j & 1-2\rho-2j-2\rho j \\ 4-j-\rho-\rho j & -3-\rho-j-\rho j \end{pmatrix},$$

$$gT_1g^{-1} = \begin{pmatrix} 2+\rho+j+\rho j & 1-\rho-j-\rho j \\ 3 & -\rho-j-\rho j \end{pmatrix},$$

$$gT_jg^{-1} = \begin{pmatrix} -\rho+j+\rho j & 1+\rho-j+\rho j \\ 3j & 3-\rho-2j+\rho j \end{pmatrix},$$

$$gT_\rho g^{-1} = \begin{pmatrix} 2\rho+2j-\rho j & 2-2\rho \\ 3\rho & 2-2\rho+j-2\rho j \end{pmatrix},$$

$$gT_{\rho j}g^{-1} = \begin{pmatrix} -1+\rho-j+2\rho j & 1+\rho+j-\rho j \\ 3\rho j & 2+\rho-j-\rho j \end{pmatrix}.$$

Since $C_{u,v}C_{u',v'} = C_{uu',vv'}$ and $JC_{u,v}J = C_{v,u}$ for all units u, v, u', v' of $\mathscr{O}$, it suffices to check the following elements:

$$gC_{1,-1}g^{-1} = \begin{pmatrix} -3+2\rho+2j+2\rho j & 4 \\ 2+2\rho+2j+2\rho j & 3-2\rho-2j-2\rho j \end{pmatrix},$$

$$gC_{1,j}g^{-1} = \begin{pmatrix} 2\rho+3j & 3-2\rho-j \\ 2+2\rho & 1-2\rho-2\rho j \end{pmatrix},$$

and

$$gC_{1,\rho}g^{-1} = \begin{pmatrix} 2\rho-j+2\rho j & 1+j-2\rho j \\ 2-\rho-j+2\rho j & \rho-j-\rho j \end{pmatrix}.$$

Thus, g belongs to the normalizer of $\mathrm{SL}_2(\mathscr{O})$ in $\mathrm{SL}_2(\mathbb{H})$. □

Proposition 4.7. *If $D_A = 3$, then the set V of $\alpha \in A$ such that v_0 belongs to the boundary of $B_\alpha(\sqrt{3})$ is*

$$V = V_{3,3} \cup g(V_{3,3}) \cup \{\infty, z_0\}\,.$$

For every $\alpha \in A$, the point v_0 of $\mathbb{H}^5_{\mathbb{R}}$ does not belong to the interior of $B_\alpha(\sqrt{3})$.

The second claim implies that when $r < \sqrt{3}$, the family $\big(B_\alpha(r)\big)_{\alpha\in A}$ does not cover $\mathbb{H}^5_{\mathbb{R}}$. In particular, the inclusions in Proposition 4.1 (2) are also sharp when $D_A = 3$.

Proof. First observe that v_0 as well as all the vertices of $\Sigma_{\mathscr{O}}$ are in the horizontal plane $\{(z,t) \in \mathbb{H}^5_{\mathbb{R}} : t = \frac{1}{\sqrt{3}}\}$, which is the boundary of $B_\infty(\sqrt{3})$.

For every $\alpha \in A$, recall from Proposition 3.4 (2) that the horoball $B_\alpha(\sqrt{3})$ is the Euclidean ball tangent to $\mathbb{H}$ at α with Euclidean radius $\frac{\sqrt{3}\,\mathtt{n}(I_\alpha)}{2}$. Writing $\alpha = pq^{-1}$ with $p, q \in \mathscr{O}$ relatively prime, we have $\mathtt{n}(I_\alpha) = \mathtt{n}(q)^{-1}$. Thus if $v_0 \in B_\alpha(\sqrt{3})$, then the Euclidean diameter $\sqrt{3}\,\mathtt{n}(I_\alpha)$ of $B_\alpha(\sqrt{3})$ is at least the Euclidean height $\frac{1}{\sqrt{3}}$ of v_0, that is $\mathtt{n}(q) \leq 3$. Equality is only possible if α is the vertical projection to $\mathbb{H}$ of v_0, that is $\alpha = z_0$. Since $z_0 = (j+\rho)(1+\rho)^{-1}$ and $j+\rho$, $1+\rho$ are relatively prime (their norms are 2 and 3), we have $z_0 \in A$ and $\mathtt{n}(I_{z_0}) = \frac{1}{3}$. Hence the point v_0 does belong to the boundary of $B_{z_0}(\sqrt{3})$, and if $\alpha \neq z_0$, then $\mathtt{n}(q) = 1$ or $\mathtt{n}(q) = 2$.

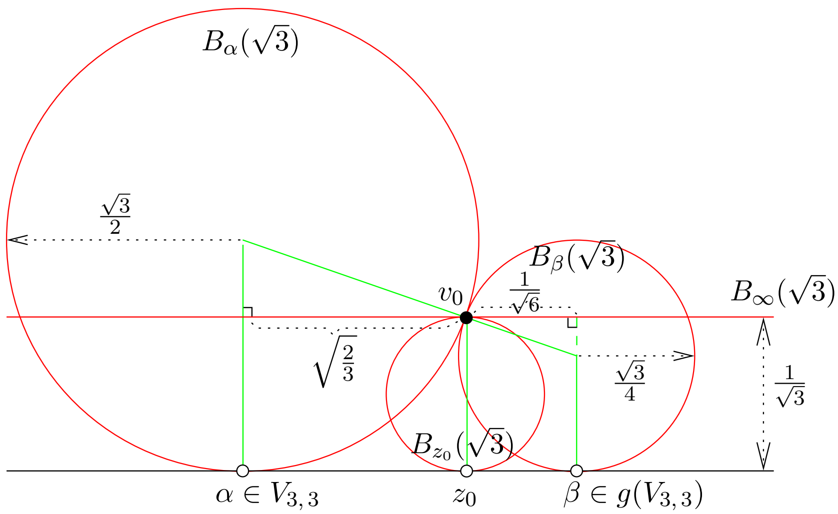

Figure 8: Intersection pattern at v_0 of the covering family of horoballs $\big(B_\alpha(\sqrt{3})\big)_{\alpha\in A}$

First assume that $\mathtt{n}(q) = 1$, or equivalently that $\alpha \in \mathscr{O}$. Then $\mathtt{n}(I_\alpha) = 1$, hence $B_\alpha(\sqrt{3})$ is the Euclidean ball of center $(\alpha, \frac{\sqrt{3}}{2})$ and radius $\frac{\sqrt{3}}{2}$, that intersects the horizontal plane at height $\frac{1}{\sqrt{3}}$ in a horizontal ball centered at $(\alpha, \frac{1}{\sqrt{3}})$ and of radius $\sqrt{\frac{2}{3}}$. The 9 vertices of the fundamental 3-3 duoprism of $\mathscr{O}$ are exactly at this distance from z_0, and all other elements of $\mathscr{O}$ are at greater distance from z_0. Hence (see Figure 8 on its left), v_0 belongs to the boundary of $B_\alpha(\sqrt{3})$ for every $\alpha \in V_{3,3}$ and $v_0 \notin B_\alpha(\sqrt{3})$ if $\alpha \in \mathscr{O} - V_{3,3}$.

We begin the treatment of the remaining case $\mathtt{n}(q) = 2$ by geometric observations. The homography g defined before Lemma 4.6 maps ∞ to z_0, z_0 to ∞, and the sphere in $\mathbb{H}$ of center z_0 and radius r to the sphere in $\mathbb{H}$ of center z_0 and radius $\frac{1}{3r}$, for every $r > 0$. In particular, g maps the sphere in $\mathbb{H}$ of center z_0 and radius $\frac{1}{\sqrt{3}}$ to itself and the Poincaré extension of g to $\mathbb{H}^5_{\mathbb{R}}$ (again denoted by g) fixes v_0. Thus, $g(B_\infty(\sqrt{3})) = B_{z_0}(\sqrt{3})$ and Lemma 4.6 implies that g preserves the $\mathrm{SL}_2(\mathscr{O})$-equivariant family $(B_\alpha(\sqrt{3}))_{\alpha\in A}$ of horoballs.

Now let $\beta = pq^{-1} \in A$ be such that $v_0 \in B_\beta(\sqrt{3})$ and $\mathtt{n}(q) = 2$. Note that the Euclidean perpendicular projection from $\mathbb{H}^5_{\mathbb{R}}$ to $\mathbb{H}$ does not increase the Euclidean distances, and that the projection of the Euclidean center of $B_\beta(\sqrt{3})$ is β and the projection of v_0 is z_0 (see Figure 8). Since the radius of $B_\beta(\sqrt{3})$ is $\frac{\sqrt{3}}{4}$, we hence have $d(z_0, \beta) \leq \frac{\sqrt{3}}{4} < \frac{1}{\sqrt{3}}$. Since g fixes v_0 and $gB_\beta(\sqrt{3}) = B_{g(\beta)}(\sqrt{3})$ by Lemma 4.6, the element $\alpha = g^{-1}(\beta)$, which satisfies $v_0 \in B_\alpha(\sqrt{3})$ and is outside the ball of center z_0 and radius $\frac{1}{\sqrt{3}}$, hence cannot have a denominator of norm 2. Therefore α has denominator (of norm) 1 and by the previous case, it belongs to $V_{3,3}$ and v_0 lies in the boundary of $B_\alpha(\sqrt{3})$. So that $\beta = g(\alpha)$ belongs to $g(V_{3,3})$ and v_0 lies in the boundary of $B_\beta(\sqrt{3})$. $\square$

An easy computation gives

$$g(V_{3,3}) = \Big\{\frac{1+j}{2} = \frac{1}{1-j}\,,\ \frac{1+\rho j}{2} = \frac{1}{1-j\bar\rho}\,,\ \frac{\rho+j}{2} = \frac{1}{\bar\rho - j}\,,\ \frac{\rho(1+j)}{2} = \frac{1}{(1-j)\bar\rho}\,,$$
$$\frac{1+j+\rho j}{2} = \frac{1}{1-j(\bar\rho-1)} + j\,,\ \frac{\rho+j+\rho j}{2}\,,\ \frac{1+\rho+j}{2}\,,\ \frac{1+\rho+\rho j}{2}\,,\ \frac{1+\rho+j+\rho j}{2}\Big\}\,.$$

As any element β in $g(V_{3,3})$ is the sum of an element of $\mathscr{O}$ with the inverse of an element of $\mathscr{O}$ with reduced norm 2, we have $\mathtt{n}(I_\beta) = \frac{1}{2}$ and the horoball $B_\beta(\sqrt{3})$ has Euclidean radius $\frac{\sqrt{3}}{4}$. This horoball intersects the horizontal plane $\{(z,t) \in \mathbb{H}^5_\mathbb{R} : t = \frac{1}{\sqrt{3}}\}$ in a horizontal ball of Euclidean radius $\frac{1}{\sqrt{6}}$. In particular, the points in $g(V_{3,3})$ are at Euclidean distance $\frac{1}{\sqrt{6}}$ of z_0 and the horoballs tangent to v_0 are positioned as in Figure 8.

By Proposition 4.7, the link of v_0 in the cellulation of $\mathbb{H}^5_\mathbb{R}$ by the Ford-Voronoi cells of $\mathscr{O}$ has 20 4-dimensional cells, which are the intersections of a small sphere centered at v_0 with the Ford-Voronoi cells $\mathscr{H}_\alpha$ for α in $V = V_{3,3} \cup g(V_{3,3}) \cup \{\infty, z_0\}$. Furthermore, for all $\alpha \neq \beta$ in V, the horoballs $B_\alpha(\sqrt{3})$ and $B_\beta(\sqrt{3})$ are tangent at v_0 if and only if $\{\alpha, \beta\}$ is one of the 10 pairs

$$\{\infty, z_0\}\,,\ \Big\{0, \frac{1+\rho+j+\rho j}{2}\Big\}\,,\ \Big\{1, \frac{\rho+j+\rho j}{2}\Big\}\,,\ \Big\{\rho, \frac{1+j+\rho j}{2}\Big\}\,,\ \Big\{j, \frac{1+\rho+\rho j}{2}\Big\}\,,$$
$$\Big\{1+j, \frac{\rho+\rho j}{2}\Big\}\,,\ \Big\{1+\rho j, \frac{\rho+j}{2}\Big\}\,,\ \Big\{\rho j, \frac{1+\rho+j}{2}\Big\}\,,\ \Big\{j+\rho, \frac{1+\rho j}{2}\Big\}\,,\ \Big\{\rho+\rho j, \frac{1+j}{2}\Big\}\,.$$

By computing (using Mathematica) the intersections of the horoballs $B_\alpha(1)$ contained in the Ford-Voronoi cells incident to v_0, we find that each Ford-Voronoi cell containing v_0 intersects 9 others in 4-dimensional cells, that are images under $\mathrm{SL}_2(\mathscr{O})$ of the fundamental cell $\Sigma_\mathscr{O}$, combinatorially equal to the 6-6 duoprism $\{6\} \times \{6\}$. The graph in Figure 9 shows the intersection pattern of the $\mathscr{H}_\alpha$ for $\alpha \in V$.

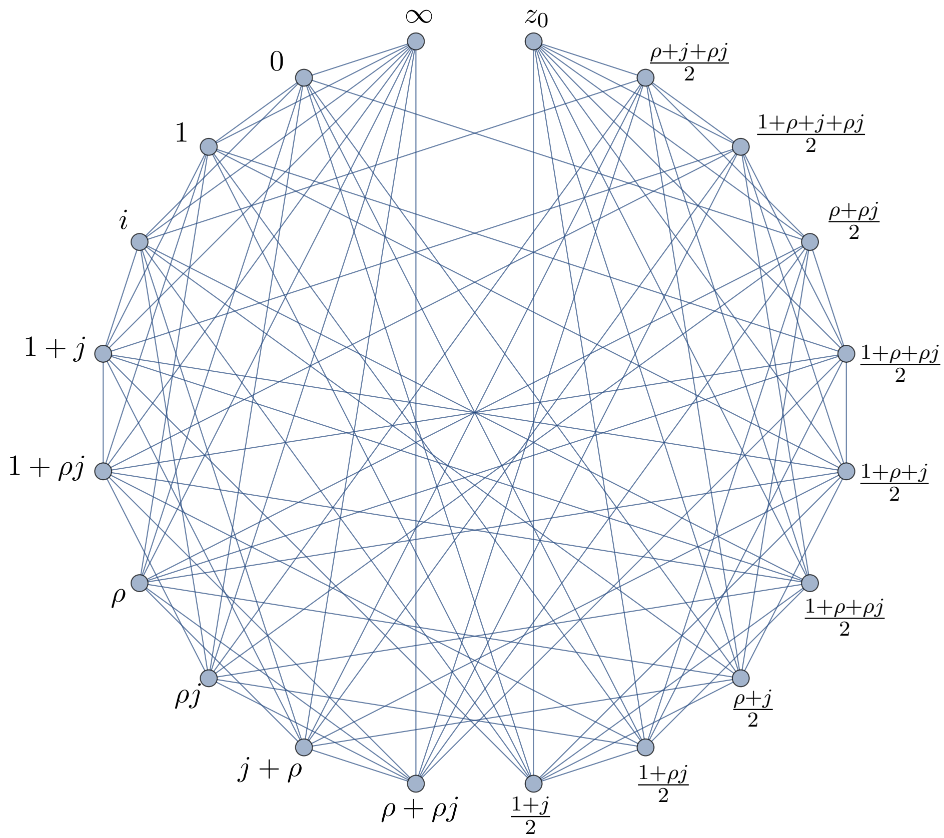

Figure 9: Intersection pattern into 4-dimensional cells of Ford-Voronoi cells $\mathscr{H}_\alpha$ for $\alpha \in V$.

Thus the number of (6-6 duoprismatic) 4-dimensional cells of $X_{\mathscr{O}}$ containing v_0 is exactly $90 = (20 \times 9)/2$, one for each edge of this diagram.

The dual tiling of the 6-6-duoprismatic tiling of $\mathbb{H}$ is the 3-3-duoprismatic tiling. Therefore, the link of v_0 in $\partial\mathscr{H}_\infty$ (hence in all $\partial\mathscr{H}_\alpha$ containing v_0) is the 3-skeleton of the dual of the 3-3 duoprism, namely the 3-3 duopyramid (also known as the triangular duotegum), whose Schläfli symbol is $\{3\}+\{3\}$ and whose symmetry group has order $8 \times 3^3 = 72$. The link of v_0 in $\mathbb{H}^5_{\mathbb{R}}$ is constructed of 20 copies of the 3-3 duopyramid, that are glued together according to the intersection pattern described in Figure 9. Let us describe the symmetries of this link.

Let $G_{v_0,\infty}$ be the stabilizer of v_0 and ∞ in $\mathrm{SL}_2(\mathscr{O})$. The stabilizer of ∞ in $\mathrm{SL}_2(\mathscr{O})$ consists of the upper triangular matrices. An element $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathscr{O})$ fixes z_0 if and only if $az_0 + b = z_0 d$. It is easy to check (using Mathematica) that this equation has 36 solutions. The elements of the intersection of the stabilizers of ∞ and z_0 preserve the geodesic line between z_0 and ∞ and the horospheres centered at ∞, hence fix v_0. Thus

$$G_{v_0,\infty} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathscr{O}) : az_0 + b = z_0 d \right\}.$$

Using the facts that $z_0 = \frac{1+j+\rho+\rho j}{3}$ and $\rho j = j\rho^{-1}$, it is easy to check that the three matrices $g_{\infty,1} = \begin{pmatrix} \rho & \rho^{-1} \\ 0 & \rho^{-1} \end{pmatrix}$, $g_{\infty,2} = \begin{pmatrix} \rho & j\rho \\ 0 & \rho \end{pmatrix}$ and $h_\infty = \begin{pmatrix} \rho j & 0 \\ 0 & j \end{pmatrix}$ are elements of $G_{v_0,\infty}$. As $g_{\infty,1}$ and $g_{\infty,2}$ are elements of order 6 that commute and the intersection of the cyclic groups generated by their squares is trivial, the group generated by $g_{\infty,1}^2$ and $g_{\infty,2}^2$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The element h_∞ has order 4 and conjugates $g_{\infty,1}$ to its inverse, as well as for $g_{\infty,2}$. Hence h_∞ conjugates each element of the abelian group generated by $g_{\infty,1}^2$ and $g_{\infty,2}^2$ to its inverse. Thus these three elements actually generate a group isomorphic to the semi-direct product $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/4\mathbb{Z}$, where the generator of $\mathbb{Z}/4\mathbb{Z}$ acts by the opposite on the abelian group $(\mathbb{Z}/3\mathbb{Z})^2$. This group has 36 elements that therefore is all of $G_{v_0,\infty}$. For $i = 1, 2$, note that $g_{\infty,i}^3 = h_\infty^2 = -\operatorname{id}$, hence $g_{\infty,i} = h_\infty^2 (g_{\infty,i}^2)^{-1}$ does belong to the above semi-direct product. The group $G_{v_0,\infty}$ is a subgroup of index 4 in the stabilizer in $\mathrm{SL}_2(\mathscr{O})$ of the 3-3 duopyramid corresponding to ∞ in the link of v_0.

The subgroup $G_{v_0,\infty}$ acts transitively on $V_{3,3}$: The graph in Figure 10 shows how the points of $V_{3,3}$ are mapped by $g_{\infty,1}$ (in continuous green) and $g_{\infty,2}$ (in dotted red).

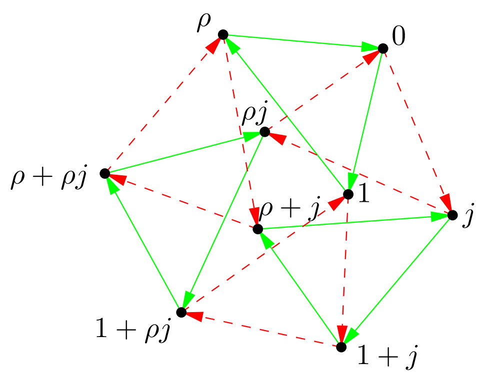

Figure 10: Transitive action of $G_{v_0,\infty}$ on $V_{3,3}$.

Since the inversion g conjugates $g_{\infty,1}$ and $g_{\infty,2}$ to $-g_{\infty,1}$ and $-g_{\infty,2}$ respectively, the group $G_{v_0,\infty}$ also acts transitively on $g(V_{3,3})$.

The element $g_\rho = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, inducing the homography $z \mapsto (1-z)^{-1}$, is an element of the stabilizer of v_0 in $\mathrm{SL}_2(\mathscr{O})$; it fixes $\rho \in V_{3,3}$ and $\frac{1+j+\rho j}{2} \in g(V_{3,3})$, maps ∞ to $0 \in V_{3,3}$ and $\rho j \in V_{3,3}$ to $\frac{1+\rho j}{2} \in g(V_{3,3})$, and does not fix z_0. Since $G_{v_0,\infty}$ acts transitively on $V_{3,3}$ and on $g(V_{3,3})$, it follows that the stabilizer of v_0 acts transitively on $V = V_{3,3} \cup g(V_{3,3}) \cup \{\infty, z_0\}$. One can check (using Mathematica) that the group generated by $G_{v_0,\infty}$ and g_ρ has 720 elements. This induces a group $\overline{G}_{v_0}$ of 360 isometries in the stabilizer of v_0 in the group of isometries of $\mathbb{H}^n_{\mathbb{R}}$.

The intersection pattern of Figure 9 and the type of the 4-dimensional cells coincide with those of the boundary of the bidodecateron, dual to the dodecateron (also called the birectified 5-simplex), see [Wik]. The full group of symmetries of the bidodecateron, whose Coxeter notation is $[[3^4]]$, has $1440 = 4 \times 360$ elements. Assuming that the link of v_0 is the bidodecateron (we have not checked this), the group $\overline{G}_{v_0}$ would be a subgroup of index 4 in $[[3^4]]$, and the stabilizer of v_0 in $\mathrm{SL}_2(\mathscr{O})$ would coincide with the group generated by $g_{\infty,1}$, $g_{\infty,2}$, h_∞ and g_ρ. This concludes the study of Example 4.5.

Remark. When $D_A \in \{2, 3, 5\}$, let $P_{\mathscr{O}}$ be the hyperbolic 5-polytope that consists of the points in the halfspace $\mathscr{H}_{\infty 0}$ whose horizontal projection to $\mathbb{H}$ is $\Sigma_{\mathscr{O}}^{\mathbb{H}}$. The quotient orbifold $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$ is obtained from $P_{\mathscr{O}}$ by gluing the vertical sides of $P_{\mathscr{O}}$ by the translations in the stabilizer of ∞ in $\mathrm{SL}_2(\mathscr{O})$, and then folding by the action of the stabilizer of $\Sigma_{\mathscr{O}}$.

The quotient space $\mathrm{SL}_2(\mathscr{O})\backslash X_{\mathscr{O}}$ obtained by making the above identifications in $\Sigma_{\mathscr{O}}$ is a 4-dimensional cellular retract of $\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{H}^5_{\mathbb{R}}$ that could be used to study the homology of $\mathrm{SL}_2(\mathscr{O})$ and $\mathrm{PSL}_2(\mathscr{O})$ analogously to the study of the Bianchi groups in [Men] and [ScV].

5 Waterworlds

Let A be a definite quaternion algebra over $\mathbb{Q}$ and let $\mathscr{O}$ be a maximal order in A. Let f be an indefinite integral binary Hamiltonian form over $\mathscr{O}$.

The form f defines a function $F = F_f : \mathbb{P}^1_r(A) \to \mathbb{Q}$ by

$$F([x:y]) = \frac{f(x,y)}{\mathtt{n}(\mathscr{O}x + \mathscr{O}y)}\ .$$

This definition does not depend on the representative $(x,y) \in A\times A$ of $[x:y]\in\mathbb{P}^1_r(A)$, and f is uniquely determined by its associated function F. In particular, we may take $x,y\in\mathscr{O}$ in order to compute $F([x:y])$, so that the numerator of the fraction defining $F([x:y])$ belongs to $\mathbb{Z}$. Note that $\mathrm{SL}_2(\mathscr{O})$ acts with finitely many orbits on $\mathbb{P}^1_r(A)$, since the number of cusps is finite, and that the denominator defining $F([x:y])$ is invariant under $\mathrm{SL}_2(\mathscr{O})$. Therefore there exists $N\in\mathbb{N}-\{0\}$ such that F has values in $\frac{1}{N}\mathbb{Z}$, hence the set of values of F is discrete.

Note that for every $g\in\mathrm{SL}_2(\mathscr{O})$, the function $F_{f\circ g}$ associated with the form $f\circ g$ is $F\circ g$ (where we again denote by g the projective transformation of $\mathbb{P}^1_r(A)$ induced by g). In particular, $F\circ g = F$ if $g\in\mathrm{SU}_f(\mathscr{O})$.

As in [Con] for integral indefinite binary quadratic forms, we will think of F as a map which associates a rational number to (the interior of) any Ford-Voronoi cell. For instance, if $D_A = 2$ and $\mathscr{O}$ is the Hurwitz order, then the values of F on the two Ford-Voronoi cells $\mathscr{H}_\infty, \mathscr{H}_0$ containing the fundamental cell $\Sigma_{\mathscr{O}}$ are $f(1,0), f(0,1)$ and the values of F on the 24 Ford-Voronoi cells meeting $\Sigma_{\mathscr{O}}$ in a 3-dimensional cell are $f(u,1)$ for $u\in\mathscr{O}^\times$ (see Figure 11).

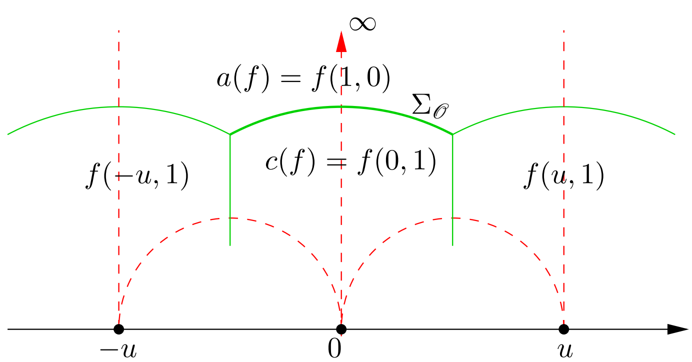

Figure 11: Values of F on Ford-Voronoi cells meeting $\Sigma_{\mathscr{O}}$.

Let $\mathfrak{m}$ be a left fractional ideal of $\mathscr{O}$. For every $s\geq 0$, let

$$\psi_{F,\mathfrak{m}}(s) = \mathrm{Card}\ {}_{\mathrm{SU}_f(\mathscr{O})}\backslash\big\{(u,v)\in\mathfrak{m}\times\mathfrak{m}\ :\ |F(u,v)|\leq s,\quad \mathscr{O}u+\mathscr{O}v=\mathfrak{m}\big\}\ ,$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by F of rational numbers in $\frac{1}{N}\mathbb{Z}$ with absolute value at most s. We showed in [PP2, Theo. 1] and [PP3, Cor. 5.6] that there exists $\kappa > 0$ such that, as s tends to $+\infty$,

$$\psi_{F,\mathtt{m}}(s) = \frac{45\ D_A\ \operatorname{Covol}(\mathrm{SU}_f(\mathscr{O}))}{2\,\pi^2\ \zeta(3)\ \Delta(f)^2\ \prod_{p|D_A}(p^3-1)}\ \ s^4(1+\mathrm{O}(s^{-\kappa}))\ .$$

Lemma 5.1. *The function F takes all signs $0, +, -$.*

Proof. It takes positive and negative values since f is indefinite. The values of F are actually positive at the points in $\mathbb{P}^1_r(A)$ in one of the two components of $\mathbb{P}^1_r(\mathbb{H}) - \mathscr{C}_\infty(f)$ and negative at the ones in the other component. But contrarily to the cases of binary quadratic and Hermitian forms, all indefinite integral binary Hamiltonian forms f over $\mathscr{O}$ represent 0, since by taking a $\mathbb{Z}$-basis of $\mathscr{O}$, the form f becomes an indefinite integral quadratic form over $\mathbb{Z}$ with 8 variables and all indefinite integral quadratic forms over $\mathbb{Z}$ with at least 5 variables represent 0 by Meyer's theorem, see for instance [Ser1, p. 77] or [Cas, p. 75]. □

A Ford-Voronoi cell will be called *flooded* for f if the value of F on its point at infinity is 0. Lemma 5.1 says that there are always flooded Ford-Voronoi cells. See also [Vul, Cor. 4.8]. The flooded Ford-Voronoi cells for f correspond to Conway's *lakes* for an isotropic integral indefinite binary quadratic form over $\mathbb{Z}$, see [Con, page 23]. There were only two lakes, whereas there are now countably infinitely many flooded Ford-Voronoi cells for f, one for each parabolic fixed point of the group of automorphs of f.

Example 5.2. Consider the definite quaternion algebra A with $D_A = 2$, $\mathscr{O}$ the Hurwitz order and a Hamiltonian form f with $a(f) = 0$, $b = b(f), c = c(f) \in \mathbb{Z} - \{0\}$ such that b does not divide c nor $2c$. Then $\mathscr{H}_\infty$ is flooded. Let $\alpha = xy^{-1}$ with $x \in \mathscr{O}$ and $y \in \mathscr{O} - \{0\}$ relatively prime. If $\mathtt{n}(y) \leq 2$, then the Ford-Voronoi cell $\mathscr{H}_\alpha$ is not flooded, since otherwise the equation $b\,\mathtt{tr}(\bar{x}\,y) + c\,\mathtt{n}(y) = 0$ would imply that b divides c or $2c$. If $\mathtt{n}(y) > 2$, then $\mathtt{n}(I_\alpha) = \frac{\mathtt{n}(\mathscr{O}x+\mathscr{O}y)}{\mathtt{n}(y)} = \frac{1}{\mathtt{n}(y)} < \frac{1}{2}$. Hence by Proposition 3.4 (2), we have $B_\alpha(\sqrt{2}) \cap B_\infty(\sqrt{2}) = \emptyset$. Therefore $\mathscr{H}_\alpha \cap \mathscr{H}_\infty = \emptyset$ by Proposition 4.1 (2). This proves that $\mathscr{H}_\infty$ does not meet any other flooded Ford-Voronoi cell. Thus if the hyperbolic 4-orbifold $\mathrm{SU}_f(\mathscr{O})\backslash\mathscr{C}(f)$ has only one cusp, then the flooded Ford-Voronoi cells are pairwise disjoint. We actually do not know when $\mathrm{SU}_f(\mathscr{O})\backslash\mathscr{C}(f)$ has only one cusp.

We have the following analog of the statement of Conway (loc. cit.) that the values of the binary quadratic form along a lake are in an infinite arithmetic progression.

Proposition 5.3. *Let $\alpha_0 \in \mathbb{P}^1_r(A)$ be such that the Ford-Voronoi cell $\mathscr{H}_{\alpha_0}$ is flooded for f. If α_0 belongs to the $\mathrm{SL}_2(\mathscr{O})$-orbit of ∞, let $\Lambda_{\alpha_0} = \mathscr{O}$. Otherwise, let*

$$\Lambda_{\alpha_0} = \mathscr{O} \cap \alpha_0^{-1}\mathscr{O} \cap \mathscr{O}\alpha_0^{-1} \cap \alpha_0^{-1}\mathscr{O}\alpha_0^{-1}\ .$$

Then there exists a finite set of nonconstant affine maps $\{\varphi_j : \mathbb{H} \to \mathbb{R}\ :\ j \in J'\}$ defined over $\mathbb{Q}$ such that the set of values of F on the Ford-Voronoi cells meeting $\mathscr{H}_{\alpha_0}$ is $\bigcup_{j\in J'} \varphi_j(\Lambda_{\alpha_0})$.

Proof. For every $\alpha \in \mathbb{P}^1_r(A)$, let $E_\alpha = \{\beta \in \mathbb{P}^1_r(A) - \{\alpha\}\ :\ \mathscr{H}_\alpha \cap \mathscr{H}_\beta \neq \emptyset\}$. Note that $E_{g\cdot\alpha} = g \cdot E_\alpha$ for every $g \in \mathrm{SL}_2(\mathscr{O})$, by Proposition 4.1 (1).

First assume that α_0 belongs to the $\mathrm{SL}_2(\mathscr{O})$-orbit of ∞. Then up to replacing f by $f \circ g$ for some $g \in \mathrm{SL}_2(\mathscr{O})$ such that $g \cdot \infty = \alpha_0$, we may assume that $\alpha_0 = \infty$.

Let $a = a(f)$, $b = b(f)$ and $c = c(f)$. Note that $\mathscr{H}_\infty$ is flooded for f if and only if $f(0,1) = 0$, that is, if and only if $a = 0$. We then have $b \neq 0$ since f is indefinite. Hence $F(E_\infty) = \big\{\frac{\mathtt{tr}(\overline{u}\,b)+c}{\mathtt{n}(I_u)} \;:\; u \in E_\infty\big\}$. Since the stabilizer of ∞ in $\mathrm{SL}_2(\mathscr{O})$ acts with finitely many orbits on the cells of $\partial\mathscr{H}_\infty$, its finite index subgroup $\mathscr{O}$ acts by translations with finitely many orbits on E_∞. Hence there exists a finite subset J' of A such that $E_\infty = J' + \mathscr{O}$. Since $I_{\alpha+o} = I_\alpha$ for all $\alpha \in A$ and $o \in \mathscr{O}$, the result follows with $\varphi_j : u \mapsto \frac{\mathtt{tr}(\bar{b}(j+u))+c}{\mathtt{n}(I_j)}$ for all $j \in J'$.

Assume now that α_0 does not belong to the $\mathrm{SL}_2(\mathscr{O})$-orbit of ∞, so that in particular $\alpha_0 \in A - \{0\}$. Let Γ_{α_0} be the stabilizer of α_0 in $\mathrm{SL}_2(\mathscr{O})$, which acts with finitely many orbits on E_{α_0}. Let $g = \begin{pmatrix} \alpha_0 & -1 \\ 1 & 0 \end{pmatrix}$, which belongs to $\mathrm{SL}_2(A)$ and whose inverse projectively maps α_0 to ∞. Then (see for instance [PP2, §5]), Λ_{α_0} is a $\mathbb{Z}$-lattice in $\mathbb{H}$, such that the group of unipotent upper triangular matrices with coefficient 1-2 in Λ_{α_0} is a finite index subgroup of $g^{-1}\Gamma_{\alpha_0} g$. A similar argument concludes. $\square$

By a *projective real hyperplane* in $\partial_\infty \mathbb{H}^5_\mathbb{R} = \mathbb{P}^1_r(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$, we mean in what follows the boundary at infinity of a hyperbolic hyperplane in $\mathbb{H}^5_\mathbb{R}$. The ones containing $\infty = [1:0]$ are the union of $\{\infty\}$ with the affine real hyperplanes in $\mathbb{H}$. The ones not containing ∞ are the Euclidean spheres in the affine Euclidean space $\mathbb{H}$.

Lemma 5.4. *The form f is uniquely determined by the values of its associated function F at six points in $\mathbb{P}^1_r(A)$ that do not lie in a projective real hyperplane.*

Proof. Let $a = a(f)$, $b = b(f)$ and $c = c(f)$. Let us first prove that we may assume that the six points in $A \cup \{\infty\}$ are $\infty = [1:0]$, 0, $\alpha_0 = 1$ and $\alpha_1, \alpha_2, \alpha_3 \in A - \{0\}$.

Note that for all $x, y \in A$ and $g \in \mathrm{GL}_2(A)$, if g_1, g_2 are the components of the linear selfmap g of $A \times A$, then

$$F_{f\circ g}([x:y]) = F_f \circ g([x:y])\, \frac{\mathtt{n}(\mathscr{O} g_1(x,y) + \mathscr{O} g_2(x,y))}{\mathtt{n}(\mathscr{O} x + \mathscr{O} y)}\;. \tag{15}$$

Given six points in $\mathbb{P}^1_r(A)$ not in a projective real hyperplane of $\mathbb{P}^1_r(\mathbb{H})$, the first three of them constitute a projective frame of the projective line $\mathbb{P}^1_r(A)$. Hence by the existence part of the fundamental theorem of projective geometry (see [Ber1, Prop. 4.5.10]), there exists an element $g \in \mathrm{GL}_2(A)$ mapping them to $\infty, 0, 1$. Note that this existence part does hold in the noncommutative setting, though the uniqueness part does not. The initial claim follows by Equation (15).

Now, the values of F at the points ∞, 0, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ give a system of six equations on the unknown a, b, c, of the form $a = A_1$, $c = A_2$, $a + \mathtt{tr}\, b + c = A_3$, $\mathtt{tr}(\overline{\alpha_i}\, b) = A_{i+3}$ for $i \in \{1,2,3\}$. Thus a and c are uniquely determined, and b belongs to the intersection of four affine real hyperplanes in $\mathbb{H}$ orthogonal to $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ with equations $\mathtt{tr}(\overline{\alpha_i}\, b) = A'_i$ for $i \in \{0,1,2,3\}$. The result follows since if $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are linearly independent over $\mathbb{R}$, then for all $A'_0, A'_1, A'_2, A'_3 \in \mathbb{R}$, such an intersection contains one and only one point of $\mathbb{H}$. $\square$

Proposition 5.5. *Let v be a vertex of the spine $X_\mathscr{O}$. The form f is uniquely determined by the values of its associated function F on the Ford-Voronoi cells containing v, that is, on the points $\alpha \in \mathbb{P}^1_r(A)$ such that $v \in \mathscr{H}_\alpha$.*

Proof. A dimension count shows that there are at least six Ford-Voronoi cells meeting at each vertex v of the spine. Their points at infinity cannot all be on the same projective real hyperplane P. Otherwise, the intersection of the equidistant hyperbolic hyperplanes between the pair of them yielding a 4-dimensional cell containing v would have dimension at least 1: It would contain a germ of the orthogonal through v to the convex hull of P in $\mathbb{H}^5_{\mathbb{R}}$. The result follows by Lemma 5.4. $\square$

The *waterworld* of f is

$$\mathscr{W}(f) = \bigcup_{\alpha \neq \beta \in \mathbb{P}^1_r(A),\ F(\alpha)F(\beta)<0} \mathscr{H}_\alpha \cap \mathscr{H}_\beta \,.$$

As $F \circ g = F$ if $g \in \mathrm{SU}_f(\mathscr{O})$, the waterworld $\mathscr{W}(f)$ is invariant under the group of automorphs $\mathrm{SU}_f(\mathscr{O})$ of f.

Since f is always isotropic over A, the arguments of Conway and Bestvina-Savin for the anisotropic case no longer apply, and the waterworld of f could be empty. We do not know precisely when the waterworlds are nonempty, and we now study some examples.

Example 5.6. The binary Hamiltonian form $f(u,v) = \mathtt{tr}(\overline{u}\,v)$ is indefinite with discriminant 1. The coefficients of f are rational integers so it is integral over any maximal order $\mathscr{O}$ of any definite quaternion algebra A over $\mathbb{Q}$. Let us prove that the waterworld $\mathscr{W}(f)$ is not empty.

It is easy to check that $\mathscr{C}_\infty(f) = \{z \in \mathbb{H} : \mathtt{tr}\, z = 0\} \cup \{\infty\}$. Let $a \in \mathscr{O}$ be such that $\mathtt{tr}(a) = 1$ (which does exist since $\mathscr{O}$ is maximal, hence $\mathtt{tr} : \mathscr{O} \to \mathbb{Z}$ is onto, see for instance the proof of Proposition 16 in [ChP]). In particular $a \neq 0$, $a \neq -\bar{a}$, and $a, -\bar{a}$ are in two different components of $\partial_\infty \mathbb{H}^5_{\mathbb{R}} - \mathscr{C}_\infty(f)$, so that $F(a)F(-\bar{a}) < 0$. Let us prove that $\mathscr{H}_a$ and $\mathscr{H}_{-\bar{a}}$ intersect in a 4-dimensional cell of $X_{\mathscr{O}}$, which thus belongs to $\mathscr{W}(f)$. By Proposition 4.1 (2), it is sufficient to prove that $B_a(1)$ and $B_{-\bar{a}}(1)$ meet. By Theorem 3.5 (1), this is equivalent to proving that $I_a I_{\bar{a}} = \mathscr{O}(\mathtt{tr}\, a)$. But this holds since $\mathtt{tr}\, a = 1$ and $I_b = \mathscr{O}$ when $b \in \mathscr{O}$.

Figure 12 illustrates the analogous case of the ocean in $\mathbb{H}^3_{\mathbb{R}}$ of the isotropic binary Hermitian form $f(u,v) = \mathtt{tr}(\overline{u}\,v)$ considered as an integral form over the Eisenstein integers $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$. The blue hexagons are the components of the ocean of f in the hyperplane $\mathscr{C}(f) = \{(z,t) \in \mathbb{H}^3_{\mathbb{R}} : \mathrm{Im}\ z = 0\}$ which is a copy of the (upper halfplane model of the) real hyperbolic plane.

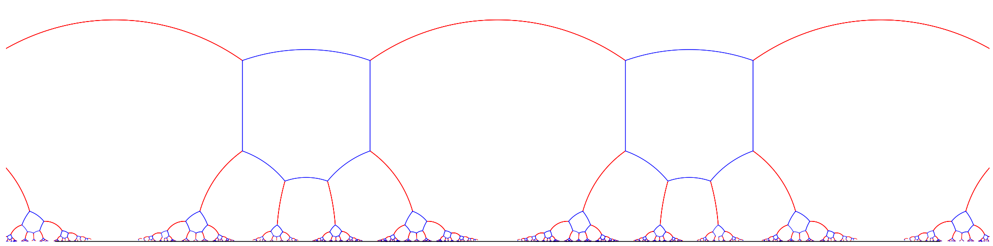

Figure 12: Ocean of the Hermitian form $f(u,v) = \mathtt{tr}(\overline{u}\,v)$ over $\mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$.

We do not have an example of an empty waterworld and, in fact, it may be that no such example exists. However, the ocean of the isotropic binary Hamiltonian form $f(u,v) = \mathtt{tr}(\overline{u}\,v)$ considered over the Gaussian integers $\mathbb{Z}[i]$ is empty (see Figure 13). In order to prove this, let $\alpha \in \mathbb{Q}(i)$ with $\mathtt{tr}\,\alpha \neq 0$. Note that in the commutative case, $\mathtt{n}(I_\alpha) = \mathtt{n}(I_{-\overline{\alpha}})$, so that the Euclidean balls $B_\alpha(1)$ and $B_{-\overline{\alpha}}(1)$ have the same radius. By symmetry, $\mathscr{C}(f)$ is the equidistant hyperbolic hyperplane of $B_\alpha(1)$ and $B_{-\overline{\alpha}}(1)$. Since $\mathbb{Z}[i]$ is Euclidean, the spine of $\mathrm{SL}_2(\mathbb{Z}[i])$ has only one orbit of 2-cells (see [BeS]). Hence all the intersections of the Ford-Voronoi cells are in the orbit of the fundamental cell, and therefore, $\mathscr{H}_\alpha$ and $\mathscr{H}_{-\overline{\alpha}}$ intersect if and only if $B_\alpha(1)$ and $B_{-\overline{\alpha}}(1)$ are tangent, that is, if and only if $B_\alpha(1)$ intersects $\mathscr{C}(f)$.

Since the hyperbolic 3-orbifold $\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathbb{H}^3_\mathbb{R}$ has only one cusp, there exists an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[i])$ such that $\alpha = g\cdot\infty = ac^{-1}$. Since $g\cdot(-c^{-1}d) = \infty$, the point $g\cdot(-c^{-1}d, 1) = (\alpha, \frac{1}{\mathtt{n}(c)})$ is the highest point in $B_\alpha(1) = gB_\infty(1)$. Thus the Euclidean radius of $B_\alpha(1)$ is $\frac{1}{2\,\mathtt{n}(c)}$. As the Euclidean distance of α from $\mathscr{C}_\infty(f) = \{z \in \mathbb{C} : \mathrm{Re}\, z = 0\} \cup \{\infty\}$ is $|\frac{\mathtt{tr}\,\alpha}{2}|$, this implies that $B_\alpha(1)$ intersects $\mathscr{C}(f)$ if and only if $\left|\frac{\mathtt{tr}\,\alpha}{2}\right| \leq \frac{1}{2\,\mathtt{n}(c)}$, that is, if and only if $\mathtt{tr}\,a\,\overline{c} = \pm 1$. This is impossible since the trace of any Gaussian integer is even.

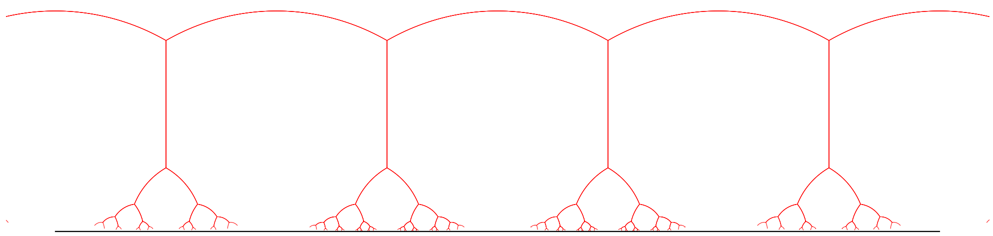

Figure 13: Empty ocean of the Hermitian form $f(u,v) = \mathtt{tr}(\overline{u}\,v)$ over $\mathbb{Z}[i]$.

Proposition 5.7. *If the union of the flooded Ford-Voronoi cells does not separate $\mathbb{H}^5_\mathbb{R}$, and in particular if the flooded Ford-Voronoi cells are pairwise disjoint, then the waterworld of f is nonempty.*

Proof. The assumption says that the topological space

$$X = \mathbb{H}^5_\mathbb{R} - \bigcup_{\alpha \in \mathbb{P}^1_r(A),\ F(\alpha)=0} \mathscr{H}_\alpha$$

is connected. If $\mathscr{W}(f) = \emptyset$, then

$$X = \Big(\bigcup_{\alpha \in \mathbb{P}^1_r(A),\ F(\alpha)<0} \mathscr{H}_\alpha\Big) \cup \Big(\bigcup_{\alpha \in \mathbb{P}^1_r(A),\ F(\alpha)>0} \mathscr{H}_\alpha\Big)$$

would be a partition into two nonempty (since f is indefinite) locally finite, hence closed, unions of closed polyhedra, contradicting the connectedness of X. □

Proposition 5.8. *The quotient $\mathrm{SU}_f(\mathscr{O})\backslash\mathscr{W}(f)$ is compact, and the set of flooded Ford-Voronoi cells consists of finitely many $\mathrm{SU}_f(\mathscr{O})$-orbits.*

Proof. The points at infinity of the flooded Ford-Voronoi cells are the parabolic fixed points of $\mathrm{SL}_2(\mathscr{O})$ contained in $\mathscr{C}_\infty(f)$, hence are the parabolic fixed points of the group of automorphs $\mathrm{SU}_f(\mathscr{O})$. Since $\mathrm{SU}_f(\mathscr{O})$ is a lattice in the real hyperbolic 4-space $\mathscr{C}(f)$, the quotient $\mathrm{SU}_f(\mathscr{O})\backslash\mathscr{C}(f)$ has only finitely many cusps. This proves the second claim.

Let $\alpha, \beta \in \mathbb{P}^1_r(A)$ be such that $F(\alpha)F(\beta) < 0$ and the intersection $\mathscr{H}_\alpha \cap \mathscr{H}_\beta$ is nonempty. Then the intersection $B_\alpha(\sqrt{D_A}) \cap B_\beta(\sqrt{D_A})$ is nonempty by Proposition 4.1 (2), hence the hyperbolic distance between the horoballs $B_\alpha(1)$ and $B_\beta(1)$ is at most $\ln D_A$. By Lemma 3.6, we hence have $\frac{\mathtt{n}(\alpha-\beta)}{\mathtt{n}(I_\alpha I_\beta)} \leq D_A$.

Let $a = a(f)$, $b = b(f)$, $c = c(f)$ and $\Delta = \Delta(f)$. Write $\alpha = [x : y]$ and $\beta = [u : v]$ with $x, y, u, v \in \mathscr{O}$ and $y, v \in \mathbb{Z}$. Note that

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix}^* \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} f(x,y) & z \\ \overline{z} & f(u,v) \end{pmatrix} ,$$

for some $z \in \mathscr{O}$. Since $y, v \in \mathbb{R}$, an easy computation of Dieudonné determinants thus gives

$$\big|\,\mathtt{n}(z) - f(x,y)f(u,v)\big| = \mathtt{n}(xv - uy)\,\Delta \,.$$

Hence $0 \leq -f(x,y)f(u,v) \leq \mathtt{n}(z) - f(x,y)f(u,v) = \mathtt{n}(xv-uy)\,\Delta$ and

$$0 \leq -F(\alpha)F(\beta) = \frac{-f(x,y)f(u,v)}{\mathtt{n}(\mathscr{O}x + \mathscr{O}y)\,\mathtt{n}(\mathscr{O}u + \mathscr{O}v)} \leq \frac{\mathtt{n}(\alpha-\beta)}{\mathtt{n}(I_\alpha)\,\mathtt{n}(I_\beta)}\,\Delta \leq D_A\,\Delta \,.$$

Since the set of values of F is discrete in $\mathbb{R}$, this implies that F takes only finitely many values on the Ford-Voronoi cells that intersect $\mathscr{W}(f)$.

Given any vertex $v \in \mathscr{W}(f)$, for every $g \in \mathrm{SL}_2(\mathscr{O})$, if $F(\alpha) = F(g \cdot \alpha)$ for all $\alpha \in A$ such that the Ford-Voronoi cell $\mathscr{H}_\alpha$ contains v, then $f = f \circ g$ by Proposition 5.5. Since there are only finitely many orbits of $\mathrm{SL}_2(\mathscr{O})$ on the vertices of the spine $X_\mathscr{O}$ and since F takes only finitely many values on the Ford-Voronoi cells meeting the waterworld $\mathscr{W}(f)$, this implies that $\mathrm{SU}_f(\mathscr{O})$ has only finitely many orbits of vertices in $\mathscr{W}(f)$. The result follows. □

Remark. We claim there exist a positive constant and finitely many pairs $\{\alpha, \beta\}$ in A such that, for all indefinite integral binary Hamiltonian forms f over $\mathscr{O}$ up to the action of $\mathrm{SL}_2(\mathscr{O})$, the distance between $\mathscr{H}_\alpha$ and $\mathscr{H}_\beta$ is at most this constant and $F(\alpha)F(\beta) < 0$. This follows, even if the waterworld $\mathscr{W}(f)$ could be empty, from the fact that the flooded Ford-Voronoi cells only have their points at infinity on the 3-sphere $\mathscr{C}_\infty(f)$ in $\mathbb{P}^1_r(\mathbb{H})$, and by the cocompactness of the action of $\mathrm{SL}_2(\mathscr{O})$ on its spine $X_\mathscr{O}$. The above arguments hence allow to give another proof of Corollary 25 in [PP2], saying that the number of $\mathrm{SL}_2(\mathscr{O})$-orbits in the set of indefinite integral binary Hamiltonian forms over $\mathscr{O}$ with given discriminant is finite.

We introduce two variants of $\mathscr{W}(f)$. The *sourced waterworld* $\mathscr{W}_+(f)$ of f is the union of its waterworld and of its flooded Ford-Voronoi cells

$$\mathscr{W}_+(f) = \mathscr{W}(f) \;\cup \bigcup_{\alpha \in \mathbb{P}^1_r(A),\ F(\alpha)=0} \mathscr{H}_\alpha \,.$$

The *coned-off waterworld* $\mathscr{C}\mathscr{W}(f)$ of f is obtained from $\mathscr{W}(f)$ by adding geodesic rays from its boundary points to the points at infinity of the corresponding flooded Ford-Voronoi cells

$$\mathscr{C}\mathscr{W}(f) = \mathscr{W}(f) \;\cup \bigcup_{\alpha \in \mathbb{P}^1_r(A),\, x \in \mathscr{W}(f) \cap \mathscr{H}_\alpha \,:\, F(\alpha)=0} [x, \alpha[\,.$$

Both the sourced waterworld $\mathscr{W}_+(f)$ and the coned-off waterworld $\mathscr{C}\mathscr{W}(f)$ of f are invariant under the group of automorphs $\mathrm{SU}_f(\mathscr{O})$ of f.

Before stating the main result of this paper, we give two lemmas and refer to Section 6 of [BeS] for the proofs

Lemma 5.9. *Let P, P' be hyperbolic hyperplanes in $\mathbb{H}^n_\mathbb{R}$ that do not intersect perpendicularly. Then the closest point mapping from P to P' is a homeomorphism onto a convex open subset of P', which maps any hyperbolic polyhedron of P to a hyperbolic polyhedron of P'.* □

Lemma 5.10. *Let f be an indefinite integral binary Hamiltonian form over $\mathscr{O}$. If ℓ is a geodesic line in $\mathbb{H}^5_\mathbb{R}$ that is perpendicular to the hyperbolic hyperplane $\mathscr{C}(f)$, oriented such that $\ell(\pm\infty) \in \{[x:y] \in \mathbb{P}^1_r(\mathbb{H}) \;:\; \pm f(x,y) > 0\}$, if ℓ meets transversally at a point z the interior of a 4-dimensional cell $\mathscr{H}_{\alpha_-} \cap \mathscr{H}_{\alpha_+}$ of $X_\mathscr{O}$ with $F(\alpha_-) \leq 0$ and $F(\alpha_+) \geq 0$ and $(F(\alpha_-), F(\alpha_+)) \neq (0,0)$, then a germ of ℓ at z pointing towards $\ell(\pm\infty)$ is contained in $\mathscr{H}_{\alpha_\pm}$.*

Proof. The proof of Claim 2 page 12 of [BeS] applies. □

The following result implies Theorem 1.2 in the Introduction.

Theorem 5.11. *Let A be a definite quaternion algebra over $\mathbb{Q}$ and let $\mathscr{O}$ be a maximal order in A. For every indefinite integral binary Hamiltonian form f over $\mathscr{O}$, the closest point mapping $\pi : \mathscr{W}_+(f) \to \mathscr{C}(f)$ is a proper $\mathrm{SU}_f(\mathscr{O})$-equivariant homotopy equivalence. If the flooded Ford-Voronoi cells for f are pairwise disjoint, then the closest point mapping $\pi : \mathscr{C}\mathscr{W}(f) \to \mathscr{C}(f)$ is a $\mathrm{SU}_f(\mathscr{O})$-equivariant homeomorphism and its restriction to the waterworld $\mathscr{W}(f)$ is a $\mathrm{SU}_f(\mathscr{O})$-equivariant homeomorphism onto a contractible 4-manifold with a polyhedral boundary component homeomorphic to $\mathbb{R}^3$ contained in every flooded Ford-Voronoi cell.*

Proof. The $\mathrm{SU}_f(\mathscr{O})$-equivariance properties are immediate. We will subdivide this proof into several steps. Unless otherwise stated, polyhedra are compact and convex.

Claim 1. The closest point mapping $\pi : \mathscr{W}_+(f) \to \mathscr{C}(f)$ has the following properties.

(1) The restriction of π to any cell of $\mathscr{W}(f)$ is a homeomorphism onto its image, which is a hyperbolic polyhedron in the hyperbolic hyperplane $\mathscr{C}(f)$.

(2) The restriction of π to any flooded Ford-Voronoi cell $\mathscr{H}_\alpha$ of f is a proper map onto a noncompact convex hyperbolic polyhedron in $\mathscr{C}(f)$ containing $B_\alpha(1) \cap \mathscr{C}(f)$ and contained in $B_\alpha(\sqrt{D_A}) \cap \mathscr{C}(f)$.

(3) If the flooded Ford-Voronoi cells for f are pairwise disjoint, then the restriction of π to any cell in the boundary of a flooded Ford-Voronoi cells for f is a homeomorphism onto its image, which is a hyperbolic polyhedron in the hyperbolic hyperplane $\mathscr{C}(f)$.

Proof. (1) Any 4-dimensional cell, hence any cell, of $\mathscr{W}(f)$ is a hyperbolic polyhedron in the equidistant hyperbolic hyperplane

$$\mathscr{S}_{\alpha,\beta} = \{x \in \mathbb{H}^5_\mathbb{R} \;:\; d_\alpha(x) = d_\beta(x)\}$$

for some $\alpha \neq \beta$ in $\mathbb{P}^1_r(A)$ with $F(\alpha)F(\beta) < 0$. Note that $\mathscr{S}_{\alpha,\beta}$ is not perpendicular to $\mathscr{C}(f)$, otherwise α and β, which are the points at infinity of a geodesic line perpendicular to $\mathscr{S}_{\alpha,\beta}$, would belong to the closure of the same component of $\partial_\infty\mathbb{H}^5_\mathbb{R} - \mathscr{C}_\infty(f)$, which contradicts the fact that $F(\alpha)F(\beta) < 0$. Hence Assertion (1) of Claim 1 follows from Lemma 5.9.

(2) The closest point mapping from a horoball H to a hyperbolic hyperplane P passing through the point at infinity of H is a proper map (since the intersection of H with any geodesic line not passing through its point at infinity is compact), whose image is $H \cap P$, and which maps the geodesic segment between two points to the geodesic segment between their images. Assertion (2) of Claim 1 hence follows from Proposition 4.1 (2).

(3) If the flooded Ford-Voronoi cells for f are pairwise disjoint, any 4-dimensional cell, hence any cell, in the boundary of a flooded Ford-Voronoi cell for f is a hyperbolic polyhedron in the hyperbolic hyperplane $\mathscr{S}_{\alpha,\beta}$ for some $\alpha \neq \beta$ in $\mathbb{P}^1_r(A)$ with $F(\alpha) = 0$ and $F(\beta) \neq 0$. Note that $\mathscr{S}_{\alpha,\beta}$ is again not perpendicular to $\mathscr{C}(f)$, otherwise α and β would both belong to $\mathscr{C}_\infty(f)$, and the Ford-Voronoi cells $\mathscr{H}_\alpha$ and $\mathscr{H}_\beta$ would both be flooded for f and not disjoint. The last assertion of Claim 1 follows. □

Claim 2. We have the following parity properties.

(1) Any 3-dimensional cell σ of $\mathscr{W}(f)$ not contained in a flooded Ford-Voronoi cell for f belongs to an even number of 4-dimensional cells of $\mathscr{W}(f)$.

(2) If the flooded Ford-Voronoi cells for f are pairwise disjoint, then any 3-dimensional cell σ' of $\mathscr{W}(f)$ contained in a flooded Ford-Voronoi cell for f belongs to an odd number of 4-dimensional cells of $\mathscr{W}(f)$.

Proof. (1) Since σ has codimension 2, the link of σ in the Ford-Voronoi cellulation of the manifold $\mathbb{H}^5_{\mathbb{R}}$ is a circle. Considering its intersection with the 4-dimensional cells, this circle subdivides into closed intervals with disjoint interiors, each one of them contained in some Ford-Voronoi cell. By the assumption on σ, these Ford-Voronoi cells are nonflooded. Hence the sign of F on each one of them is either $+$ or $-$. In such a cyclic arrangement of signs, the number of sign changes is even. Assertion (1) follows.

(2) Similarly, the link of σ' is subdivided into at least 3 closed intervals with disjoint interiors carrying a sign $+, 0, -$. By the assumptions, exactly one of them, denoted by I_0, belongs to a flooded Ford-Voronoi cell $\mathscr{H}_{\alpha_0}$ for some $\alpha_0 \in \mathbb{P}^1_r(A)$, that is, carries the sign 0. Assume for a contradiction that the two intervals adjacent to I_0 carry the same sign. Let $\beta_1, \beta_2 \in \mathbb{P}^1_r(A)$ be such that $\mathscr{H}_{\alpha_0} \cap \mathscr{H}_{\beta_1}$ and $\mathscr{H}_{\alpha_0} \cap \mathscr{H}_{\beta_2}$ are the 4-dimensional cells corresponding to the endpoints of I_0. Note that the points at $+\infty$ of the geodesic lines starting from a given point α_0 of $\mathscr{C}_\infty(f)$, passing through a geodesic line both of whose endpoints β_1, β_2 are contained in the same component C of $\partial_\infty \mathbb{H}^5_{\mathbb{R}} - \mathscr{C}_\infty(f)$ also belong to C. Hence all intervals except I_0 in the link of σ' carry the same sign, which contradicts the fact that σ' belongs to $\mathscr{W}(f)$. As for σ, this proves that the number of sign changes between $+$ and $-$ in the link of σ' is odd. □

Claim 3. If σ and τ are distinct 4-dimensional cells of $\mathscr{W}(f)$ or flooded Ford-Voronoi cells for f, then $\pi(\sigma)$ and $\pi(\tau)$ have disjoint interiors.

Proof. Note that no 4-dimensional cell of $\mathscr{W}(f)$ is contained in a flooded Ford-Voronoi cell for f.

For a contradiction, assume that a point $p \in \mathscr{C}(f)$ is contained in the interior of both $\pi(\sigma)$ and $\pi(\tau)$ and, up to moving it a little bit, is not in the (measure 0) image by π of the codimension 1 skeleton of $X_{\mathscr{O}}$. Let ℓ be the geodesic line through p perpendicular to $\mathscr{C}(f)$, meeting σ and τ at interior points x and y respectively. Since the cell complex $X_{\mathscr{O}}$ is locally finite, we may assume that the geodesic segment $[x,y]$ does not meet any 4-dimensional cell of $\mathscr{W}(f)$ or flooded Ford-Voronoi cell for f other than σ and τ.

Assume for a contradiction that $[x,y]$ is contained in $\sigma \cup \tau$. Then σ and τ are flooded Ford-Voronoi cells, meeting in a 4-dimensional cell C, which is crossed transversally by $[x,y]$ since ℓ does not meet the 3-skeleton of $X_{\mathscr{O}}$. Since σ, τ are flooded, their points at infinity $\alpha, \beta \in \mathbb{P}^1_r(A)$ belong to $\mathscr{C}_\infty(f)$. Hence the hyperbolic hyperplane $\mathscr{S}_{\alpha,\beta}$ equidistant to α and β, which contains C, is perpendicular to $\mathscr{C}(f)$. In particular, ℓ, which is perpendicular to $\mathscr{C}(f)$, is contained in the closure of one of the two connected component of $\mathbb{H}^5_{\mathbb{R}} - \mathscr{S}_{\alpha,\beta}$. This contradicts the fact that ℓ meets transversally C.

Hence $[x,y]$ is not contained in $\sigma \cup \tau$. Let $]x',y'[\;=\; [x,y] - (\sigma \cup \tau) \cap [x,y]$ with x, x', y', y in this order on $[x,y]$, so that $[x',y']$ is contained in a Ford-Voronoi cell $\mathscr{H}_\alpha$ for some $\alpha \in \mathbb{P}^1_r(A)$. Let σ' and τ' be the 4-dimensional cells of $X_{\mathscr{O}}$ containing x' and y' respectively (note that for instance $x = x'$ and $\sigma = \sigma'$ if σ is a 4-dimensional cell of $\mathscr{W}(f)$, but $x \neq x'$ if σ is a flooded Ford-Voronoi cell).

Now Lemma 5.10 implies that, since the two germs of the segment $[x',y']$ at its end-points have opposite direction, the sign of $F(\alpha)$ should be both positive and negative, a contradiction. $\square$

Claim 4. The 3-dimensional cells of the waterworld satisfy the following properties.

(1) No 3-dimensional cell of $\mathscr{W}(f)$ is contained in two distinct flooded Ford-Voronoi cells.

(2) Any 3-dimensional cell σ of $\mathscr{W}(f)$ not contained in a flooded Ford-Voronoi cell for f belongs to exactly two 4-dimensional cells τ and τ' of $\mathscr{W}(f)$, and π embeds their union.

(3) Any 3-dimensional cell σ of $\mathscr{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathscr{H}_\alpha$ for f belongs to exactly one 4-dimensional cell τ of $\mathscr{W}(f)$, and π embeds the union of τ and $\tau' = \bigcup_{x\in\sigma} [x, \alpha[$.

Proof. (1) Assume for a contradiction that σ is a 3-dimensional cell of $\mathscr{W}(f)$ contained in the flooded Ford-Voronoi cells $\mathscr{H}_\alpha$ and $\mathscr{H}_\beta$ with $\alpha \neq \beta$ in $\mathbb{P}^1_r(A)$. Let τ be a 4-dimensional cell of $\mathscr{W}(f)$ containing σ. Then the interiors of the images by π of τ and either $\mathscr{H}_\alpha$ or $\mathscr{H}_\beta$ are not disjoint, which contradicts Claim 3.

Let us prove Assertions (2) and (3). Three n-dimensional polytopes in $\mathbb{H}^n_{\mathbb{R}}$ having a common codimension 1 face cannot have pairwise disjoint interiors, so that the claims on the number of 4-dimensional cells of $\mathscr{W}(f)$ containing σ follows from Claim 3. Since the polyhedra $\pi(\tau)$ and $\pi(\tau')$ are convex, the result follows. $\square$

Claim 5. The 2-dimensional cells of the waterworld satisfy the following properties.

(1) For every 2-dimensional cell σ of $\mathscr{W}(f)$ not contained in a flooded Ford-Voronoi cell for f, the link of σ in $\mathscr{W}(f)$ is a circle and the union of the 4-dimensional cells of $\mathscr{W}(f)$ containing σ embeds in $\mathscr{C}(f)$ by π.

(2) If the flooded Ford-Voronoi cells for f are pairwise disjoint, for every 2-dimensional cell σ' of $\mathscr{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathscr{H}_\alpha$, the link of σ' in $\mathscr{W}(f)$ is an interval and the union of the 4-dimensional cells of $\mathscr{W}(f)$ containing σ' and of the geodesic rays $[x, \alpha[$ for x in the two 3-dimensional cells of $\mathscr{W}(f) \cap \partial \mathscr{H}_\alpha$ containing σ' embeds in $\mathscr{C}(f)$ by π.

Proof. (1) By Claim 4, the link $Lk(\sigma)$ of σ in $\mathscr{W}(f)$ is a disjoint union of circles. Each component of $Lk(\sigma)$ corresponds to a finite set of 4-dimensional cells cyclically arranged around σ. By Claim 4 again, their images by π are not folded, hence are cyclically arranged around $\pi(\sigma)$. If $Lk(\sigma)$ was not connected, the image of two 4-dimensional cells of $\mathscr{W}(f)$ by π would have intersecting interiors, contradicting Claim 3.

(2) An analogous proof gives that the link of σ' in $\mathscr{C}\mathscr{W}(f)$ is a circle. □

Claim 6. The 1-dimensional cells of the waterworld satisfy the following properties.

(1) For every 1-dimensional cell σ of $\mathscr{W}(f)$ not contained in a flooded Ford-Voronoi cell for f, the link of σ in $\mathscr{W}(f)$ is a 2-sphere and the union of the 4-dimensional cells of $\mathscr{W}(f)$ containing σ embeds in $\mathscr{C}(f)$ by π.

(2) If the flooded Ford-Voronoi cells for f are pairwise disjoint, for every 1-dimensional cell σ' of $\mathscr{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathscr{H}_\alpha$, the link of σ' in $\mathscr{W}(f)$ is a 2-disc and the union of the 4-dimensional cells of $\mathscr{W}(f)$ containing σ' and of the geodesic rays $[x, \alpha[$ for x in any 3-cell of $\mathscr{W}(f) \cap \partial\mathscr{H}_\alpha$ containing σ' embeds in $\mathscr{C}(f)$ by π.

Proof. (1) By Claim 5, the links of the vertices of the link $Lk(\sigma)$ of σ in $\mathscr{W}(f)$ are circles, hence $Lk(\sigma)$ is a compact surface, mapping locally homeomorphically to $Lk(\pi(\sigma))$ by π, which is a 2-sphere. Hence $Lk(\pi(\sigma))$ is a union of 2-spheres, again with only one of them by Claim 3.

(2) The proof that the link of σ' in $\mathscr{C}\mathscr{W}(f)$ is a 2-sphere is similar. □

Claim 7. The vertices of the waterworld satisfy the following properties.

(1) For every vertex v of $\mathscr{W}(f)$ not contained in a flooded Ford-Voronoi cell for f, the link of v in $\mathscr{W}(f)$ is a 3-sphere and the union of the 4-dimensional cells of $\mathscr{W}(f)$ containing v embeds in $\mathscr{C}(f)$ by π.

(2) If the flooded Ford-Voronoi cells for f are pairwise disjoint, for every vertex v' of $\mathscr{W}(f)$ contained in a flooded Ford-Voronoi cell $\mathscr{H}_\alpha$, the link of v' in $\mathscr{W}(f)$ is a 3-disc and the union of the 4-dimensional cells of $\mathscr{W}(f)$ containing v' and of the geodesic rays $[x, \alpha[$ for x in any 3-cell of $\mathscr{W}(f) \cap \mathscr{H}_\alpha$ containing v' embeds in $\mathscr{C}(f)$ by π.

Proof. The proof is similar to the previous one. □

Now, the properness of $\pi : \mathscr{W}_+(f) \to \mathscr{C}(f)$ follows from the fact that π is $\mathrm{SU}_f(\mathscr{O})$-equivariant, that $\mathrm{SU}_f(\mathscr{O})$ acts cocompactly on $\mathscr{W}(f)$ and with finitely many orbits on the set of flooded Ford-Voronoi cells by Proposition 5.8, and from its properness when restricted to each flooded Ford-Voronoi cell (see Claim 1).

Claim 7 proves that when the flooded Ford-Voronoi cells for f are pairwise disjoint, the map $\pi : \mathscr{C}\mathscr{W}(f) \to \mathscr{C}(f)$ is a proper local homeomorphism betwen locally compact spaces, hence is a covering map. Since $\mathscr{C}(f)$ is simply connected, π is hence a homeomorphism on each of the connected components of $\mathscr{C}\mathscr{W}(f)$. But since π is injective outside the codimension 1 skeleton by Claim 3, it follows that $\mathscr{C}\mathscr{W}(f)$ is connected and π is a homeomorphism. This concludes the proof of Theorem 5.11. □

A An algebraic description of the distance to the cusps

Let A be a definite quaternion algebra over $\mathbb{Q}$ and let $\mathscr{O}$ be a maximal order in A. In this independent appendix, following Mendoza [Men] in the Hermitian case, we give an algebraic description of the distance functions d_α to the rational points at infinity $\alpha \in \mathbb{P}^1_r(A)$, defined just before Proposition 3.3.

An $\mathscr{O}$*-flag* is a right $\mathscr{O}$-submodule L of the right $\mathscr{O}$-module $\mathscr{O} \times \mathscr{O}$, with rank one (that is, LA is a line in the A-vector space $A \times A$), such that the quotient $(\mathscr{O} \times \mathscr{O})/L$ has no torsion. We denote by $\mathscr{F}_{\mathscr{O}}$ the set of $\mathscr{O}$-flags.

For all right $\mathscr{O}$-submodules M of $A \times A$ and $v \in A \times A - \{0\}$, let us define

$$M_v = \{x \in A \;:\; vx \in M\}\;.$$

Note that for every $\lambda \in A - \{0\}$, we immediately have

$$\lambda M_{v\lambda} = M_v\;. \tag{16}$$

Example A.1. Recall that the *inverse* I^{-1} of a left fractional ideal I of $\mathscr{O}$ is the right fractional ideal of $\mathscr{O}$

$$I^{-1} = \{x \in A \;:\; IxI \subset I\}\;.$$

It is well known and easy to check that for every $a, b \in \mathscr{O}$, if $ab \neq 0$, then

$$(\mathscr{O}a + \mathscr{O}b)^{-1} = a^{-1}\mathscr{O} \cap b^{-1}\mathscr{O}\;. \tag{17}$$

We claim that if $v = (a, b)$, then

$$(\mathscr{O} \times \mathscr{O})_v = (\mathscr{O}a + \mathscr{O}b)^{-1}\;. \tag{18}$$

Indeed, if $ab \neq 0$, then by Equation (17)

$$(\mathscr{O} \times \mathscr{O})_v = \{x \in A \;:\; (ax, bx) \in \mathscr{O} \times \mathscr{O}\} = a^{-1}\mathscr{O} \cap b^{-1}\mathscr{O} = (\mathscr{O}a + \mathscr{O}b)^{-1}\;.$$

The result is immediate if $a = 0$ or $b = 0$.

Proposition A.2. *(1) For every right $\mathscr{O}$-submodule M of $A \times A$ and $v \in A \times A - \{0\}$, the subset M_v of A is a right fractional ideal of $\mathscr{O}$.*
(2) For every $v \in A \times A - \{0\}$, the subset $v(\mathscr{O} \times \mathscr{O})_v$ of $\mathscr{O} \times \mathscr{O}$ is an $\mathscr{O}$-flag.
(3) For all $\mathscr{O}$-flags L and all $v \in L - \{0\}$, we have

$$L = v(\mathscr{O} \times \mathscr{O})_v\;.$$

(4) The map $\mathrm{SL}_2(A) \times \mathscr{F}_{\mathscr{O}} \to \mathscr{F}_{\mathscr{O}}$ defined by

$$(g, L) \mapsto (gv)(\mathscr{O} \times \mathscr{O})_{gv}$$

for any $v \in L - \{0\}$ is an action on the set $\mathscr{F}_{\mathscr{O}}$ of $\mathscr{O}$-flags of the group $\mathrm{SL}_2(A)$.
(5) The map $\Theta' : \mathbb{P}^1_r(A) \to \mathscr{F}_{\mathscr{O}}$ defined by $[a : b] \mapsto (a, b)(\mathscr{O} \times \mathscr{O})_{(a,b)}$ is a $\mathrm{SL}_2(A)$-equivariant bijection.

Proof. (1) This follows immediately from the fact that M is stable by addition and by multiplications on the right by the elements of $\mathscr{O}$.

(2) Let $L = v(\mathscr{O}\times\mathscr{O})_v \subset vA$. Then L is contained in $\mathscr{O}\times\mathscr{O}$ by the definition of $(\mathscr{O}\times\mathscr{O})_v$ and is a right $\mathscr{O}$-submodule of $\mathscr{O}\times\mathscr{O}$ by Assertion (1). Since $v\neq 0$, note that $(\mathscr{O}\times\mathscr{O})_v$ is a nonzero right fractional ideal, so that $L\neq 0$ and L has rank one.

Assume that $w\in\mathscr{O}\times\mathscr{O}$ has its image in $(\mathscr{O}\times\mathscr{O})/L$ which is torsion. Then there exists $y\in\mathscr{O}-\{0\}$ and $x\in A$ such that $wy = vx$. Hence $w = vxy^{-1}$. Since $w\in\mathscr{O}\times\mathscr{O}$, this implies that $xy^{-1}\in(\mathscr{O}\times\mathscr{O})_v$, so that $w\in L$, and the image of w in $(\mathscr{O}\times\mathscr{O})/L$ is zero.

(3) As L has rank one and $v\in L-\{0\}$, we have $L\subset vA\cap(\mathscr{O}\times\mathscr{O}) = v(\mathscr{O}\times\mathscr{O})_v$.

Conversely, for every $x\in(\mathscr{O}\times\mathscr{O})_v$ so that $vx\in\mathscr{O}\times\mathscr{O}$, let us prove that $vx\in L$. Since x belongs to A which is the field of fractions of $\mathscr{O}$, there exists $y\in\mathscr{O}$ such that $xy\in\mathscr{O}$. Hence $(vx)y = v(xy)$ belongs to L, since $v\in L$ and L is a right $\mathscr{O}$-module. In particular, the image of vx in $(\mathscr{O}\times\mathscr{O})/L$ is torsion. Since L is an $\mathscr{O}$-flag, this implies that this image is zero, as wanted. This proves that $v(\mathscr{O}\times\mathscr{O})_v$ is contained in L, hence is equal to L by the previous inclusion.

(4) Let us prove that this map is well defined. If $v,w\in L-\{0\}$, since L has rank one, there exists $x\in A-\{0\}$ such that $w = vx$. Thus, for every $g\in\mathrm{SL}_2(A)$, by the linearity on the right of g and by Equation (16), we have

$$(gw)(\mathscr{O}\times\mathscr{O})_{gw} = (gv)x(\mathscr{O}\times\mathscr{O})_{(gv)x} = (gv)(\mathscr{O}\times\mathscr{O})_{gv}\ .$$

The fact that this map is an action is then immediate: for all $g,g'\in\mathrm{SL}_2(A)$ and $L\in\mathscr{F}_{\mathscr{O}}$, let $v\in L-\{0\}$ and $\lambda\in A$ be such that $gv\lambda\in(gv)(\mathscr{O}\times\mathscr{O})_{gv}-\{0\}$; then using twice Equation (16) and the linearity, we have

$$\begin{aligned} g'(gL) &= g'\big(gv(\mathscr{O}\times\mathscr{O})_{gv}\big) = g'\big(gv\lambda(\mathscr{O}\times\mathscr{O})_{gv\lambda}\big) = g'(gv\lambda)(\mathscr{O}\times\mathscr{O})_{g'(gv\lambda)}\\ &= (g'g)v\lambda(\mathscr{O}\times\mathscr{O})_{(g'g)v\lambda} = (g'g)v(\mathscr{O}\times\mathscr{O})_{(g'g)v} = (g'g)L\ . \end{aligned}$$

(5) For every $\alpha = [a:b]\in\mathbb{P}^1_r(A)$, the subset $(a,b)(\mathscr{O}\times\mathscr{O})_{(a,b)}$, which is an $\mathscr{O}$-flag by Assertion (2), does not depend on the choice of homogeneous coordinates of α by Equation (16). Hence the map Θ' is well defined, and equivariant by the definition in Assertion (4) of the action of $\mathrm{SL}_2(A)$ on $\mathscr{F}_{\mathscr{O}}$.

The fact that Θ' is onto follows from Assertion (3). Clearly, it is one-to-one since if $(a,b)(\mathscr{O}\times\mathscr{O})_{(a,b)} = (c,d)(\mathscr{O}\times\mathscr{O})_{(c,d)}$, then there is $\lambda\in A-\{0\}$ such that $(a,b) = (c,d)\lambda$. □

Let $f:\mathbb{H}\times\mathbb{H}\to\mathbb{R}$ be a positive definite binary Hamiltonian form and let L be a rank one right $\mathscr{O}$-submodule of $\mathscr{O}\times\mathscr{O}$. Then L is a rank 4 free $\mathbb{Z}$-submodule of $\mathbb{H}\times\mathbb{H}$, and we denote by $\langle L\rangle_{\mathbb{R}}$ the 4-dimensional real vector subspace of $\mathbb{H}\times\mathbb{H}$ generated by L, endowed with the restriction of the scalar product $\langle\cdot,\cdot\rangle_f$ on $\mathbb{H}\times\mathbb{H}$ defined by f, hence with the induced volume form. Recall that for all $z,z'\in\mathbb{H}\times\mathbb{H}$, we have

$$\langle z,z'\rangle_f = \frac{1}{2}\big(f(z+z')-f(z)-f(z')\big)\ . \qquad (19)$$

We define the covolume of L for f as

$$\mathrm{Covol}_f\, L = \mathrm{Vol}(\langle L\rangle_{\mathbb{R}}/L)\ .$$

Recall that if $G = (\langle e_i, e_j\rangle_f)_{1\le i,j\le 4}$ is the Gram matrix of a $\mathbb{Z}$-basis (e_1, e_2, e_3, e_4) of L for the scalar product $\langle\cdot,\cdot\rangle_f$, then

$$\operatorname{Covol}_f L = (\det G)^{\frac{1}{2}} \ . \tag{20}$$

See for instance [Ber2, Vol 2, prop. 8.11.6].

Theorem A.3. *For all $x \in \mathbb{H}^5_{\mathbb{R}}$ and $\alpha \in \mathbb{P}^1_r(A)$, we have*

$$d_\alpha(x) = \frac{2}{\sqrt{D_A}} \left(\operatorname{Covol}_{\Theta(x)} \Theta'(\alpha)\right)^{\frac{1}{2}} .$$

Proof. Fix $a, b \in \mathscr{O}$ such that $\alpha = [a : b]$. Let $f = \Theta(x)$, $L = \Theta'(\alpha) = (a, b)(\mathscr{O} \times \mathscr{O})_{(a,b)}$ and $L' = (a, b)\mathscr{O}$. Since $a, b \in \mathscr{O}$, we have $\mathscr{O} \subset (\mathscr{O} \times \mathscr{O})_{(a,b)}$, hence L' is a finite index $\mathbb{Z}$-submodule in L. Furthermore, by Equation (18) and the relation (see Equation (2)) between the norm and reduced norm of a left integral ideal of $\mathscr{O}$, we have

$$\begin{aligned}[][L : L'] &= [(\mathscr{O} \times \mathscr{O})_{(a,b)} : \mathscr{O}] = [(\mathscr{O}a + \mathscr{O}b)^{-1} : \mathscr{O}] = [\mathscr{O} : \mathscr{O}a + \mathscr{O}b] \\ &= \mathtt{n}(\mathscr{O}a + \mathscr{O}b)^2 \ . \end{aligned} \tag{21}$$

Let (x_1, x_2, x_3, x_4) be a $\mathbb{Z}$-basis of $\mathscr{O}$, so that $(\,(a, b)x_i)_{1\le i\le 4}$ is a $\mathbb{Z}$-basis of L'. Using Equation (19) and the fact that $f((u, v)\lambda) = \mathtt{n}(\lambda)f(u, v)$ for all $u, v, \lambda \in \mathbb{H}$, we have for $1 \le i, j \le 4$,

$$\begin{aligned}\langle (a, b)x_i, (a, b)x_j\rangle_f &= \frac{1}{2}\big(f\big((a, b)(x_i + x_j)\big) - f((a, b)\,x_i) - f((a, b)\,x_j)\big) \\ &= \frac{f(a, b)}{2}\,(\mathtt{n}(x_i + x_j) - \mathtt{n}(x_i) - \mathtt{n}(x_j)) = \frac{f(a, b)}{2}\ \mathtt{tr}(\,\overline{x_i}\,x_j) \ .\end{aligned}$$

Note that $(u, v) \mapsto \frac{1}{2}\,\mathtt{tr}(\,\overline{u}\,v)$ is the standard Euclidean scalar product on $\mathbb{H}$ (making the standard basis $(1, i, j, k)$ orthonormal), hence $\big(\frac{1}{2}\,\mathtt{tr}(\,\overline{x_i}\,x_j)\big)_{1\le i,j\le 4}$ is the Gram matrix of the $\mathbb{Z}$-lattice $\mathscr{O}$ in the Euclidean space $\mathbb{H}$. Therefore, by Equation (20) and by [KO, Lem. 5.5], we have

$$\big(\det\big(\,\mathtt{tr}(\,\overline{x_i}\,x_j)\big)_{1\le i,j\le 4}\big)^{\frac{1}{2}} = (2^4)^{\frac{1}{2}}\ \operatorname{Vol}(\mathbb{H}/\mathscr{O}) = 4\frac{D_A}{4} = D_A \ . \tag{22}$$

Thus using Equations (20), (21) and (22), we have

$$\begin{aligned}\operatorname{Covol}_f(L) &= \frac{1}{[L : L']}\ \operatorname{Covol}_f(L') = \frac{1}{[L : L']}\big(\det\big(\langle (a, b)x_i, (a, b)x_j\rangle_f\big)_{1\le i,j\le 4}\big)^{\frac{1}{2}} \\ &= \frac{1}{[L : L']}\Big(\frac{f(a, b)}{2}\Big)^2\big(\det\big(\,\mathtt{tr}(\,\overline{x_i}\,x_j)\big)_{1\le i,j\le 4}\big)^{\frac{1}{2}} = \frac{D_A}{4}\ \frac{f(a, b)^2}{\mathtt{n}(\mathscr{O}a + \mathscr{O}b)^2} \ .\end{aligned}$$

By Proposition 3.3 (2), this proves Theorem A.3. □

B Relation with Ash's classifying spaces for the arithmetic group $\mathrm{SL}_2(\mathscr{O})$

Let A be a definite quaternion algebra over $\mathbb{Q}$ and let $\mathscr{O}$ be a maximal order in A. In this appendix, as suggested by the referee, we relate the spine $X_{\mathscr{O}}$ constructed in Section 4 with one of the retracts defined in the paper [Ash] for the arithmetic subgroups of general linear groups.

We apply the construction in loc. cit. in the special case when its notation n, D, A, S, G and Γ is respectively our notation 2, A, $\mathscr{O}$, $\mathbb{H}$, $\mathrm{GL}_2(\mathbb{H})$ and $\mathrm{GL}_2(\mathscr{O}) = \mathrm{SL}_2(\mathscr{O})$, using Equation (8). We will follow [Ash, §4], which relates one of the retracts of the paper [Ash] with Mendoza's minimal incidence set I_K in the paper [Men], which is a $\mathrm{PGL}_2(\mathscr{O})$-equivariant retract of $\mathbb{H}^3_{\mathbb{R}}$. We identify $\mathbb{H}^5_{\mathbb{R}}$ with $\mathscr{Q}_1^+$ by the $\mathrm{SL}_2(\mathbb{H})$-equivariant homeomorphism Θ of Section 3. We denote by K a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{H})$, say the stabilizer of the point $(0,1) \in \mathbb{H}^5_{\mathbb{R}}$ for the action by homographies of $\mathrm{SL}_2(\mathbb{H})$ on $\mathbb{H}^5_{\mathbb{R}}$, so that the orbital map $g \mapsto g \cdot (0,1)$ induces an $\mathrm{SL}_2(\mathbb{H})$-equivariant homeomorphism $\mathrm{SL}_2(\mathbb{H})/K \to \mathbb{H}^5_{\mathbb{R}}$. Note that K is also a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{H})$. We use the standard homeomorphism $\mathrm{SL}_2(\mathbb{H})/K \to K\backslash \mathrm{SL}_2(\mathbb{H})$ induced by the inverse map, and the homeomorphism $K\backslash \mathrm{SL}_2(\mathbb{H}) \times \mathbb{R}_+^* \to K\backslash \mathrm{GL}_2(\mathbb{H})$ defined by $Kg \mapsto Kg \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$.

Let $\varphi : \mathbb{P}^1_r(A) \to \mathbb{R}_+^*$ be any $\mathrm{SL}_2(\mathscr{O})$-invariant map, called a *set of weights* in [Ash, Def. 2.9]. Then the $\mathrm{GL}_2(\mathscr{O})$-equivariant retract $\widetilde{W}$ in $K\backslash \mathrm{GL}_2(\mathbb{H})$ constructed in [Ash, page 466] is, through the above isomorphisms, the graph in $\mathscr{Q}_1^+ \times \mathbb{R}_+^*$ of a continuous map $\widetilde{\widetilde{W}} \to \mathbb{R}_+^*$ where $\widetilde{\widetilde{W}}$ is the set of $f \in \mathscr{Q}_1^+$ such that the map from $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$ to $\mathbb{R}_+^*$ defined by

$$(a,b) \mapsto \varphi([a:b])\, f(a,b)$$

attains its minimum at least in two elements that generate $A \times A$ as a right A-vector space.

Note that $\widetilde{\widetilde{W}}$ depends on the choice of φ, and actually only on φ modulo a positive multiplicative constant. This gives, since $\mathrm{Card}\big(\mathrm{SL}_2(\mathscr{O})\backslash\mathbb{P}^1_r(A)\big) = h_A^2$, a real $(h_A^2 - 1)$-dimensional family of $\mathrm{GL}_2(\mathscr{O})$-equivariant retracts $\widetilde{W}$ of $K\backslash \mathrm{GL}_2(\mathbb{H})$. We start by introducing an analogous real (h_A^2-1)-dimensional family of deformations of our spine $X_{\mathscr{O}}$.

The smallest discriminant for which our construction gives different cellulations of $\mathbb{H}^5_{\mathbb{R}}$ for the same arithmetic group is $D_A = 11$. Up to isomorphism, A is then the $\mathbb{Q}$-algebra generated by elements i and j satisfying $i^2 = -1$, $j^2 = -11$ and $ij = -ji$ (see [Vig, p. 98]). Note that A has class number $h_A = 2$ and contains exactly two conjugation classes of maximal orders (see the table [Vig, p. 154]). For instance, if $t = \frac{1+j}{2}$, then the order $\mathscr{O} = \mathbb{Z}[t] + i\,\mathbb{Z}[t]$ in A has discriminant 11, hence is maximal.

We fix an $\mathrm{SL}_2(\mathscr{O})$-invariant map $\chi : \mathbb{P}^1_r(A) \to \mathbb{R}_+^*$, and denote it $\alpha \mapsto \chi_\alpha$. For every $\alpha \in \mathbb{P}^1_r(A)$, we can define a new *distance to the cusp* α function

$$d'_\alpha = \chi_\alpha\, d_\alpha\ ,$$

a new family of horoballs for $s > 0$

$$B'_\alpha(s) = \{x \in \mathbb{H}^5_{\mathbb{R}} : d'_\alpha(x) \leq s\} = B_\alpha\big(\frac{s}{\chi_\alpha}\big)\ ,$$

a new family of *Ford-Voronoi cells*

$$\mathscr{H}'_\alpha = \{x \in \mathbb{H}^5_{\mathbb{R}} : \forall\, \beta \in \mathbb{P}^1_r(A),\ \ d'_\alpha(x) \leq d'_\beta(x)\}\ ,$$

and a new *spine*

$$X'_{\mathscr{O}} = \bigcup_{\alpha \neq \beta \in \mathbb{P}^1_r(A)} \mathscr{H}'_\alpha \cap \mathscr{H}'_\beta \ = \bigcup_{\alpha \in \mathbb{P}^1_r(A)} \partial \mathscr{H}'_\alpha\ .$$

Using the crucial argument that χ is $\mathrm{SL}_2(\mathscr{O})$-invariant, an elementary consequence of Proposition 3.3, Proposition 3.4, Theorem 3.5 and Proposition 4.1 gives the following properties, for all $x \in \mathbb{H}^5_{\mathbb{R}}$, $\alpha = [a : b] \in \mathbb{P}^1_r(A)$, $g \in \mathrm{SL}_2(\mathscr{O})$, and $s > 0$.

(i) We have

$$d'_\alpha(x) = \frac{\chi_\alpha\, \Theta(x)(a,b)}{\mathtt{n}(\mathscr{O}a + \mathscr{O}b)}\ . \tag{23}$$

(ii) We have $d'_{g\cdot\alpha} \circ g = d'_\alpha$, $g(B'_\alpha(s)) = B'_{g\cdot\alpha}(s)$, and $g(\mathscr{H}'_\alpha) = \mathscr{H}'_{g\cdot\alpha}$.

(iii) The map $\ln d'_\alpha$, which differs from $\ln d_\alpha$ only by an additive constant, is also a Busemann function for the point at infinity α : there exists $c'_\alpha \in \mathbb{R}$ such that $\ln d'_\alpha(y) = \beta_\alpha(y, (0,1)) + c'_\alpha$ for every $y \in \mathbb{H}^5_{\mathbb{R}}$.

(iv) If $\alpha \in A$, then $B'_\alpha(s)$ is the Euclidean ball of center $\big(\alpha, \frac{s\,\mathtt{n}(I_\alpha)}{2\,\chi_\alpha}\big)$ and radius $\frac{s\,\mathtt{n}(I_\alpha)}{2\,\chi_\alpha}$. If $\alpha = \infty$, then $B'_\alpha(s)$ is the Euclidean halfspace consisting of all (z,r) with $r \geq \frac{\chi_\infty}{s}$.

(v) For all distinct $\alpha, \beta \in \mathbb{P}^1_r(A)$, the horoballs $B'_\alpha(\min\chi)$ and $B'_\beta(\min\chi)$ have disjoint interior.

(vi) We have

$$\mathbb{H}^5_{\mathbb{R}} = \bigcup_{\alpha \in \mathbb{P}^1_r(A)} B'_\alpha\big((\max\chi)\sqrt{D_A}\,\big)\ .$$

(vii) We have $B'_\alpha(\min\chi) \subset \mathscr{H}'_\alpha \subset B'_\alpha((\max\chi)\sqrt{D_A})$.

(viii) The Ford-Voronoi cell $\mathscr{H}'_\alpha$ is a noncompact 5-dimensional convex hyperbolic polytope, whose proper cells are compact. For every $\beta \in \mathbb{P}^1_r(A) - \{\alpha\}$, the Ford-Voronoi cells $\mathscr{H}'_\alpha$ and $\mathscr{H}'_\beta$ have disjoint interior. The spine $X'_{\mathscr{O}}$ is an $\mathrm{SL}_2(\mathscr{O})$-invariant piecewise hyperbolic polyhedral complex of dimension 4, which is a $\mathrm{SL}_2(\mathscr{O})$-equivariant retract of $\mathbb{H}^5_{\mathbb{R}}$.

Now, the relation between our spine $X_{\mathscr{O}}$ and the retracts $\widetilde{W}$ of [Ash] is given by the following result.

Proposition B.1. *For every $\mathrm{SL}_2(\mathscr{O})$-invariant map $\varphi : \mathbb{P}^1_r(A) \to \mathbb{R}^*_+$, let $\chi : \mathbb{P}^1_r(A) \to \mathbb{R}^*_+$ be defined by, for every $\alpha \in \mathbb{P}^1_r(A)$,*

$$\chi_\alpha = \varphi(\alpha)\, \min\{\mathtt{n}(\mathscr{O}a + \mathscr{O}b)\ :\ a, b \in \mathscr{O}\ \text{ and }\ \alpha = [a : b]\}\ .$$

Then, with the above notation, we have

$$\widetilde{\widetilde{W}} = \Theta(X'_{\mathscr{O}})\ .$$

In particular, the virtual cohomological dimension of $\mathrm{SL}_2(\mathscr{O})$ is equal to 4.

Proof. For every $\alpha \in \mathbb{P}^1_r(A)$, let $\mathtt{n}_\alpha = \min\{\mathtt{n}(\mathscr{O}a' + \mathscr{O}b')\ :\ a', b' \in \mathscr{O}\ \text{ and }\ \alpha = [a' : b']\}$, so that the map $\alpha \mapsto \mathtt{n}_\alpha$ is invariant under the action of $\mathrm{SL}_2(\mathscr{O})$ on $\mathbb{P}^1_r(A)$ and

$$\varphi(\alpha) = \frac{\chi_\alpha}{\mathtt{n}_\alpha}\ . \tag{24}$$

Let us denote by $p : \mathscr{O} \times \mathscr{O} - \{(0,0)\} \to \mathbb{P}^1_r(A)$ the surjective map $(a,b) \mapsto [a : b]$.

Lemma B.2. *For every $f \in \mathscr{Q}^+$, if $(a,b) \in \mathscr{O} \times \mathscr{O} - \{(0,0)\}$ minimizes the product map $(\varphi \circ p)f$ on $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$, then $\mathtt{n}(\mathscr{O}a + \mathscr{O}b) = \mathtt{n}_{[a:b]}$.*

Proof. Let $\alpha = [a : b]$. Assume for a contradiction that $\mathtt{n}(\mathscr{O}a+\mathscr{O}b) > \mathtt{n}_\alpha$. Then there exist $a', b' \in \mathscr{O}$ such that $\alpha = [a' : b']$ and $\mathtt{n}(\mathscr{O}a + \mathscr{O}b) > \mathtt{n}(\mathscr{O}a' + \mathscr{O}b')$. Since $[a : b] = [a' : b']$, there exists $\lambda \in A^\times$ such that $a = a'\lambda$ and $b = b'\lambda$. Hence $\mathtt{n}(\mathscr{O}a+\mathscr{O}b) = \mathtt{n}(\mathscr{O}a'+\mathscr{O}b')\,\mathtt{n}(\lambda)$, so that $\mathtt{n}(\lambda) > 1$ and

$$((\varphi \circ p)f)(a,b) = \varphi \circ p(a'\lambda, b'\lambda)\, f(a'\lambda, b'\lambda) = \varphi \circ p(a',b') f(a',b')\,\mathtt{n}(\lambda) > ((\varphi \circ p)f)(a',b')\ ,$$

a contradiction. □

Let $x \in \mathbb{H}^5_{\mathbb{R}}$ and $f = \Theta(x)$, which belongs to $\mathscr{Q}^+_1$. If $(a,b) \in \mathscr{O} \times \mathscr{O} - \{(0,0)\}$ minimizes the map $(\varphi \circ p)f$ on $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$, let us prove that $x \in \mathscr{H}'_\alpha$, where $\alpha = [a : b]$. Otherwise, there exists $\beta \in \mathbb{P}^1_r(A)$ such that $d'_\beta(x) < d'_\alpha(x)$. Let us write $\beta = [c : d]$ with $(c,d) \in \mathscr{O} \times \mathscr{O} - \{(0,0)\}$ such that $\mathtt{n}(\mathscr{O}c + \mathscr{O}d) = \mathtt{n}_\beta$. Then by Lemma B.2 for the penultimate equality, and using twice Equations (23) and (24), we have

$$\begin{aligned}((\varphi \circ p)f)(c,d) = \varphi(\beta)\, f(c,d) = \frac{\chi_\beta}{\mathtt{n}_\beta}\theta(x)(c,d) = \frac{\chi_\beta\, \theta(x)(c,d)}{\mathtt{n}(\mathscr{O}c + \mathscr{O}d)} = d'_\beta(x)\\ < d'_\alpha(x) = \frac{\chi_\alpha\, \theta(x)(a,b)}{\mathtt{n}(\mathscr{O}a + \mathscr{O}b)} = \frac{\chi_\alpha}{\mathtt{n}_\alpha}\theta(x)(a,b) = ((\varphi \circ p)f)(a,b)\ ,\end{aligned}$$

a contradiction to the minimizing property of (a,b).

Conversely, let $x \in \mathscr{H}'_\alpha$ and $f = \Theta(x)$, let $(a,b) \in \mathscr{O} \times \mathscr{O} - \{(0,0)\}$ be such that $\mathtt{n}(\mathscr{O}a+\mathscr{O}b) = \mathtt{n}_\alpha$ where $\alpha = [a : b]$, and let us prove that (a,b) minimizes the map $(\varphi \circ p)f$ on $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$. Otherwise, let $(c,d) \in \mathscr{O} \times \mathscr{O} - \{(0,0)\}$ be such that $((\varphi \circ p)f)(c,d)$ minimizes $(\varphi \circ p)f$ and $((\varphi \circ p)f)(c,d) < ((\varphi \circ p)f)(a,b)$. Let $\beta = [c : d]$. Then using Lemma B.2 for the second equality, we have

$$\begin{aligned}d'_\beta(x) = \frac{\chi_\beta\, \theta(x)(c,d)}{\mathtt{n}(\mathscr{O}c + \mathscr{O}d)} = \frac{\chi_\beta}{\mathtt{n}_\beta}\theta(x)(c,d) = ((\varphi \circ p)f)(c,d)\\ < ((\varphi \circ p)f)(a,b) = \frac{\chi_\alpha}{\mathtt{n}_\alpha}\theta(x)(a,b) = \frac{\chi_\alpha\, \theta(x)(a,b)}{\mathtt{n}(\mathscr{O}a + \mathscr{O}b)} = d'_\alpha(x)\ ,\end{aligned}$$

a contradiction to the fact that x belongs to $\mathscr{H}'_\alpha$.

Since two elements in $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$ generate $A \times A$ as a right A-vector space if and only if their images in $\mathbb{P}^1_r(A)$ are distinct, this implies that an element $x \in \mathbb{H}^5_{\mathbb{R}}$ belongs to $\mathscr{H}'_\alpha$ and $\mathscr{H}'_\beta$ with $\alpha \neq \beta$ if and only if the map $((\varphi \circ p)\theta(x)$ is minimized by two elements in $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$ that generate $A \times A$ as a right A-vector space. This proves the first claim of Proposition B.1.

The second claim follows from [Ash, Theo. (ii), p. 462]. □

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