Weighted Hardy Spaces of Quasiconformal Mappings

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Abstract

We study integral characterizations of weighted Hardy spaces of quasiconformal mappings on the n-dimensional unit ball using the weight $(1-r)^{n-2+\alpha}$. We extend known results for univalent functions on the unit disk. Some of our results are new even in the unweighted setting for quasiconformal mappings.

1 Introduction

Analogously to the definition in the setting of analytical functions on the unit disk, a quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{R}^n$ belongs to the Hardy space $H^p, 0 < p < \infty$, if

$$\sup_{0<r<1} \int_{\mathbb{S}^{n-1}} |f(r\omega)|^p d\sigma(\omega) < \infty.$$

The theory for Hardy spaces of quasiconformal mappings was initiated in [10] and expanded on in [3]. Many of the characterizations of the classical Hardy spaces by way of various maximal functions have already been shown to also hold in the quasiconformal setting. In particular, consider the maximum modulus function $M(r, f) = \sup_{\omega \in \mathbb{S}^{n-1}} |f(r\omega)|, 0 < r < 1$. It was shown in [3] that membership in H^p is equivalent with

$$\int_0^1 M^p(r, f)(1-r)^{n-2} dr < \infty, \tag{1.1}$$

for quasiconformal f, thus extending the well-known characterization of univalent functions of the unit disk that belong to H^p.

In this paper, we consider integral characterizations of weighted Hardy spaces, using (1.1) as the starting point and the weight $(1-r)^{n-2+\alpha}$ for a suitable range of α. Thus our Hardy space H^p_α consists of those f for which

$$\int_0^1 (1-r)^{n-2+\alpha} M^p(r, f) dr < \infty.$$

For univalent functions on the unit disk, the Hardy spaces H^p_α have been studied in [5] and [8]. Our main result is as follows:

Theorem 1.1. *Let $f : \mathbb{B}^n \to \mathbb{R}^n$ be a quasiconformal mapping, $0 < p < \infty$, and $1 - n < \alpha < \infty$. Then the following are equivalent:*

$$\int_0^1 (1-r)^{n-2+\alpha} M^p(r, f) dr < \infty, \tag{1.2}$$

$$\int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |f(x)|^{p-n} |Df(x)|^n dx dr < \infty, \tag{1.3}$$

$$\int_0^1 (1-r)^{n-2+\alpha} \left(\int_{B(0,r)} |Df(x)|^n dx \right)^{p/n} dr < \infty. \tag{1.4}$$

If $\alpha \geq 0$ or $p \geq n$, the above conditions are further equivalent to

$$\int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p-1+\alpha} dx < \infty. \tag{1.5}$$

Here $a_f(x)$ denotes an average derivative, which is equivalent to $|f'(x)|$ for conformal maps. The precise definition is given in the next section.

Our proof of Theorem 1.1 relies on techniques from [3] and [10], but our weighted setting requires some new ideas. For example, the equivalence of (1.3) and (1.4) with the membership in quasiconformal H^p is new even in the unweighted setting. Definitions and background results are given in Section 2, and the proof of Theorem 1.1 is given in Section 3.

2 Background and Preliminaries

To begin with, let us recall some basic notions.

2.1 Cones and Shadows

Given $x \in \mathbb{B}^n$, the open unit ball in $\mathbb{R}^n$, we define

$$B_x = B(x, (1-|x|)/2)$$

and for $\omega \in \mathbb{S}^{n-1}$, we let

$$\Gamma(\omega) = \bigcup_{0<r<1} B_{r\omega}.$$

This is a cone with a tip at ω. Finally, the shadow of B_x is

$$S_x = \left\{ \frac{z}{|z|} : 0 \neq z \in B_x \right\}.$$

Remark 1. *Let $\omega \in S_x$ with $x \neq 0$. Then $d(x, \overline{0w}) < (1-|x|)/2$, where $\overline{0w}$ is the line segment between w and 0. This distance is realized by $z \in \overline{0w}$ for which $\overline{xz}$ is perpendicular to $\overline{0w}$. It follows that $|z-x| < (1-|x|)/2 \leq (1-|z|)/2$. Hence $x \in B_z$ and $x \in \Gamma(w)$.*

2.2 Quasiconformal Mappings

Let $G \subset \mathbb{R}^n$ be a domain and $K \geq 1$. We say that $f : G \to \mathbb{R}^n$ is a K-quasiconformal mapping if f is continuous and one-to-one (hence a homeomorphism onto $f(G)$), $f \in W^{1,n}_{loc}(G, \mathbb{R}^n)$ and $|Df(x)|^n \leq K J_f(x)$ for almost every $x \in G$. For convenience, we write $f : \mathbb{B}^n \to \Omega$ below to specify that f is defined on $\mathbb{B}^n$ with $f(\mathbb{B}^n) = \Omega$.

We continue with properties of quasiconformal mappings. The following estimates can be deduced from [3, Lemma 2.1], also see [9].

Lemma 2.1. *Let $f : \mathbb{B}^n \to \Omega$ be a K-quasiconformal mapping. There exists a constant $C = C(n, K)$ such that, for all $x \in \mathbb{B}^n$, we have*

$$diam(f(B_x))/C \leq d(f(x), \partial\Omega) \leq C\, diam(f(B_x)) \leq C^2 d(f(B_x), \partial\Omega),$$

and $d(f(x), \partial\Omega))/C \leq |f(y) - f(x)| \leq Cd(f(x), \partial\Omega))$ for every $y \in \partial B_x$.

A quasiconformal mapping is only almost everywhere differentiable and hence we will employ the concept of averaged derivative a_f, defined by setting

$$a_f(x) = \exp[\int_{B_x} \log J_f(y) dy/(n|B_x|)].$$

If f is a conformal mapping, then $|Df|^n = J_f$ and especially $a_f(z) = |Df(z)| = |f'(z)|$, see [2] for details and [1] for the origins of the definition.

The following lemma is from [2].

Lemma 2.2. *Let $f : \mathbb{B}^n \to \Omega$ be a K-quasiconformal mapping. There exists a constant $C(n, K)$ so that*

$$d(f(x), \partial\Omega)/C \leq a_f(x)(1 - |x|) \leq Cd(f(x), \partial\Omega)$$

and

$$\frac{1}{C} \fint_{B_x} |Df(y)|^n dy \leq Ca_f^n(x) \leq \fint_{B_x} |Df(y)|^n dy$$

for all $x \in \mathbb{B}^n$.

The following result is [3, Lemma 2.5].

Lemma 2.3. *Let $f : \mathbb{B}^n \to \Omega$ be a K-quasiconformal mapping, and suppose that $u > 0$ satisfies*

$$u(y)/C \leq u(x) \leq Cu(y)$$

for each $x \in \mathbb{B}^n$ and every $y \in B_x$. Let $0 < q \leq n$ and $p \geq q$. Then

$$\int_{\mathbb{B}^n} a_f^p u\, dx \approx \int_{\mathbb{B}^n} a_f^{p-q} |Df(x)|^q u\, dx.$$

with constants only depending on p, q, n, C, K.

We continue with a useful estimate.

Lemma 2.4. *Let $f : \mathbb{B}^n \to \Omega$ be a K-quasiconformal mapping. Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. Then*

$$\int_{\mathbb{S}^{n-1}} \sup_{x \in \Gamma(\omega)} d(f(x), \partial\Omega)^p (1-|x|)^\alpha d\sigma \leq C_1 \int_{\mathbb{S}^{n-1}} \sup_{x \in \Gamma(\omega)} a_f^p(x)(1-|x|)^{p+\alpha} d\sigma \leq C_2 \int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p+\alpha} dx$$

with constants C_1, C_2 that only depend on n, K, p, α.

Proof. By the first chain of inequalities in Lemma 2.1 we conclude that, for all $x \in \mathbb{B}^n$ and each $y \in B_x$,

$$d(f(y), \partial\Omega)/C \leq d(f(x), \partial\Omega) \leq Cd(f(y), \partial\Omega),$$

where $C = C(K,n)$. This together with the first chain of inequalities in Lemma 2.2 yields

$$a_f(y)/C \le a_f(x) \le C a_f(y) \tag{2.1}$$

with $C = C(K,n)$. By Lemma 2.2 and (2.1) applied to $x \in \Gamma(\omega)$ we obtain the estimate

$$d(f(x),\partial\Omega)(1-|x|)^{\alpha/p} \le C_1 a_f(x)(1-|x|)^{1+\alpha/p} \le C_2 \Big(\int_{B_x} a_f^p(y)dy\Big)^{1/p}(1-|x|)^{1-n/p+\alpha/p}$$
$$\le C_3 \Big(\int_{\Gamma(\omega)} a_f(y)^p (1-|y|)^{p-n+\alpha} dy\Big)^{1/p} \tag{2.2}$$

with constants that only depend on n, K, p, α.

On the other hand, by the Fubini theorem

$$\int_{\mathbb{B}^n} |h(x)|dx \approx \int_{\mathbb{S}^{n-1}} \int_{\Gamma(\omega)} |h(y)|(1-|y|)^{1-n} dy d\sigma$$

for any integrable function on $\mathbb{B}^n$. Especially, this holds for $h(x) = a_f^p(x)(1-|x|)^{p-1+\alpha}$ and hence the claim follows from (2.2). □

A measure μ on $\mathbb{B}^n$ is called a Carleson measure if there is a constant C_μ such that

$$\mu(\mathbb{B}^n \cap B(\omega, r)) \le C_\mu r^{n-1}$$

for all $\omega \in \mathbb{S}^{n-1}$, $r > 0$. The following lemma, see [6] and [3, Lemma 5.6], gives us a family of Carleson measures.

Lemma 2.5. *If f is quasiconformal on $\mathbb{B}^n$, $0 < p < n$, and $f(x) \neq 0$ for all $x \in \mathbb{B}^n$, then the measure μ induced by $d\mu = |Df(x)|^p |f(x)|^{-p}(1-|x|)^{p-1}dx$ is a Carleson measure on $\mathbb{B}^n$.*

2.3 Modulus

Given a collection of locally rectifiable curves Γ in $\mathbb{R}^n$, the modulus $\mathrm{Mod}(\Gamma)$ of Γ is defined as:

$$\mathrm{Mod}(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^n dx,$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho ds \ge 1$ when $\gamma \in \Gamma$. Given a K-quasiconformal mapping $f : \Omega \to \mathbb{R}^n$, one has $\mathrm{Mod}(\Gamma)/C \le \mathrm{Mod}(f\Gamma) \le C\mathrm{Mod}(\Gamma)$, where $C = C(K,n)$. See e.g. [9] for a proof.

We recall two useful estimates, see [9]. Given $E \subset \mathbb{S}^{n-1}$, $0 < r < 1$, and the family Γ of radial segments joining $rE := \{rx : x \in E\}$ and E, we have

$$\mathrm{Mod}(\Gamma) = \sigma(E)(\log(1/r))^{1-n},$$

where $\sigma(E)$ is the surface area of E. Moreover, we always have the upper bound

$$\mathrm{Mod}(\Gamma) \le \frac{\omega_{n-1}}{\log(R/r)^{n-1}},$$

if each $\gamma \in \Gamma$ joins $S^{n-1}(x,r)$ to $S^{n-1}(x,R)$, $0 < r < R$.

According to Beurling's theorem, for a given quasiconformal mapping f, the radial limit

$$f(\omega) := \lim_{r \to 1} f(r\omega)$$

exists for almost every $\omega \in \mathbb{S}^{n-1}$. One of our key tools is the following modulus estimate that can be found in [3] and in [10].

Lemma 2.6. *[3, Lemma 4.2, Remark 4.3] There exists a constant $C = C(n, K)$ such that if f is K-quasiconformal on $\mathbb{B}^n$, $x \in \mathbb{B}^n$, $M > 1$, and $\alpha \geq 0$, then*

$$\sigma(\{\omega \in S_x : d(f(\omega), f(x))(1-|x|)^\alpha > Md(f(x), \partial\Omega)(1-|x|)^\alpha\}) \leq \frac{C\sigma(S_x)}{(\log M)^{n-1}}\,.$$

2.4 Nontangential and Radial Maximal Functions

Given $p > 0, \alpha \geq 0$, we define the weighted radial maximal and nontangential maximal functions by setting

$$\begin{aligned} &Mf_{p,\alpha}(\omega) = \sup\nolimits_{0<r<1} |f(r\omega)|(1-r)^{\alpha/p}, \omega \in \mathbb{S}^{n-1} \\ \text{and}\quad &M^* f_{p,\alpha}(\omega) = \sup\nolimits_{x\in\Gamma(\omega)} |f(x)|(1-|x|)^{\alpha/p}, \omega \in \mathbb{S}^{n-1} \end{aligned}$$

Even though the nontangential maximal function can be larger than the radial one, we have the following estimate.

Lemma 2.7. *Let $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ be a K-quasiconformal mapping and let $0 < p < \infty$ and $\alpha \geq 0$. There exists a constant $C = C(n, K, p, \alpha)$ such that*

$$\int_{\mathbb{S}^{n-1}} (M^*_{p,\alpha}(\omega))^p d\sigma(\omega) \leq C(n, K) \int_{\mathbb{S}^{n-1}} (M_{p,\alpha} f(\omega))^p d\sigma(\omega). \tag{2.3}$$

Proof. Given $\omega \in \mathbb{S}^{n-1}$ and $x_0 \in \Gamma(\omega)$, there exists $0 < r_0 < 1$ such that $x_0 \in B_{r_0\omega}$. By the definition of $B_{r_0\omega}$, we have $\frac{1}{2}(1-r_0) \leq 1-|x_0| \leq 2(1-r_0)$ and $|r_0\omega - x_0| \leq \frac{1}{2}(1-r_0)$. Set $r_1 = (1+r_0)/2$. Then $r_1\omega \in \partial B_{r_0\omega}$. Hence Lemma 2.1 yields that

$$|f(r_0\omega) - f(x_0)| \leq C(n, K)|f(r_0\omega) - f(r_1\omega)|. \tag{2.4}$$

By the triangle inequality

$$|f(x_0)| \leq |f(r_0\omega)| + |f(r_0\omega) - f(x_0)| \tag{2.5}$$

and

$$|f(r_0\omega) - f(r_1\omega)| \leq |f(r_0\omega)| + |f(r_1\omega)|. \tag{2.6}$$

By combining (2.4),(2.5),(2.6) we obtain

$$|f(x_0)| \leq C(n, K)\left(|f(r_0\omega)| + |f(r_1\omega)|\right).$$

Since $1 - r_1 = (1-r_0)/2$, we conclude that

$$(M^*_{p,\alpha} f(\omega))^p \leq C(n, K, \alpha)(M_{p,\alpha} f(\omega))^p,$$

and (2.3) follows. □

3 Proof of Theorem 1.1

We begin with the following lemma. We will employ it in the proof of Lemma 3.3 to fix the problem that $M(r,f)(1-r)^{\alpha/p}$ for $\alpha > 0$ need not be nondecreasing even though $M(r,f)$ is.

Lemma 3.1. *Let $M : [0,1) \to [0,\infty)$ be increasing and continuous with $M(0) = 0$. Let $p > 0$, $\alpha \geq 0$ and define $N(r) = \sup_{0\leq t\leq r} M(t)(1-t)^{\alpha/p}$. Then*

$$\int_0^1 (1-r)^{n-2+\alpha} M^p(r)dr < \infty \tag{3.1}$$

if and only if

$$\int_0^1 (1-r)^{n-2} N^p(r)dr < \infty. \tag{3.2}$$

Proof. Since

$$\begin{aligned} M^p(r)(1-r)^{n-2+\alpha} &= M^p(r)(1-r)^{\alpha}(1-r)^{n-2} \\ &\leq \left(\sup_{0\leq t\leq r} M^p(t)(1-t)^{\alpha}\right)(1-r)^{n-2} = N^p(r)(1-r)^{n-2}, \end{aligned}$$

we have that (3.2) implies (3.1) for any p.

Towards the other direction, we may assume that $N(r)$ is unbounded. Moreover, if the desired conclusion is true for the case $p = 1$ and all M as in our formulation, then by applying it to $\widehat{M}(r) := M^p(r)$ we obtain our claim for all $p > 0$. Thus it suffices prove that (3.1) implies (3.2) for $p = 1$.

We define a sequence of points $r_k \in [0,1)$ as follows. Let $r_0 = 0$ and set $r_k = \inf\{r : N(r) = 2^{k-1}\}$. Then the continuity and monotonicity of $N(r)$ gives that $2^{k-1} = N(r_k) = M(r_k)(1-r_k)^{\alpha}$. Hence

$$\begin{aligned} \int_0^1 N(r)(1-r)^{n-2}dr &\leq \sum_{k=0}^{\infty} N(r_{k+1}) \int_{r_k}^{r_{k+1}} (1-r)^{n-2}dr \\ &= \sum_{k=0}^{\infty} \frac{N(r_{k+1})}{n-1} \left[(1-r_k)^{n-1} - (1-r_{k+1})^{n-1}\right] \\ &= \sum_{k=0}^{\infty} \frac{M(r_{k+1})(1-r_{k+1})^{\alpha}}{n-1} \left[(1-r_k)^{n-1} - (1-r_{k+1})^{n-1}\right] \\ &= \frac{2}{n-1} \sum_{k=0}^{\infty} M(r_k)(1-r_k)^{\alpha} \left[(1-r_k)^{n-1} - (1-r_{k+1})^{n-1}\right] \\ &= \frac{2}{n-1} \sum_{k=0}^{\infty} M(r_k) \left[(1-r_k)^{n-1+\alpha} - (1-r_k)^{\alpha}(1-r_{k+1})^{n-1}\right] \\ &\leq \frac{2}{n-1} \sum_{k=0}^{\infty} M(r_k) \left[(1-r_k)^{n-1+\alpha} - (1-r_{k+1})^{n-1+\alpha}\right] \end{aligned}$$

We also have that

$$\begin{aligned} \int_0^1 (1-r)^{n-2+\alpha} M(r)dr &\geq \sum_{k=0}^{\infty} M(r_k) \int_{r_k}^{r_{k+1}} (1-r)^{n-2+\alpha}dr \\ &= \sum_{k=0}^{\infty} \frac{M(r_k)}{n-1+\alpha} \left[(1-r_k)^{n-1+\alpha} - (1-r_{k+1})^{n-1+\alpha}\right]. \end{aligned}$$

The desired implication follows. ☐

We continue with a result on Carleson measures.

Lemma 3.2. *Let $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ be a quasiconformal mapping, $0 < p < \infty$, $\alpha \geq 0$, and let μ be a Carleson measure on $\mathbb{B}^n$. There is a constant $C = C(n, K, C_\mu)$ such that*

$$\int_{\mathbb{B}^n} |f(x)|^p (1-|x|)^\alpha d\mu \leq C \int_{\mathbb{S}^{n-1}} (Mf_{p,\alpha})^p(w) d\sigma(\omega).$$

Proof. By Lemma 2.7 it suffices to show that there exists a constant $C(n, K)$ such that

$$\int_{\mathbb{B}^n} |f(x)|^p (1-|x|)^\alpha d\mu \leq C \int_{\mathbb{S}^{n-1}} (M^* f_{p,\alpha})^p d\sigma(\omega).$$

For each $\lambda > 0$, set

$$E_\lambda = \{x \in \mathbb{B}^n : |f(x)|(1-|x|)^{\alpha/p} > \lambda\}$$

and

$$U_\lambda = \left\{ \omega \in \mathbb{S}^{n-1} : \sup_{x \in \Gamma(\omega)} |f(x)|(1-|x|)^{\alpha/p} > \lambda \right\}.$$

Then U_λ is an open subset of $\mathbb{S}^{n-1}$. Recall the definion of the shadows S_x from Subsection 2.1. They are spherical caps. We can decompose U_λ into a Whitney-type decomposition of spherical caps. That is, we can write,

$$U_\lambda = \bigcup_{k=1}^{\infty} S_{x_k},$$

where no $\omega \in U_\lambda$ belongs to more than $N(n)$ spherical caps S_{x_k} and

$$d(S_{x_k}, \partial U_\lambda) \approx \operatorname{diam}(S_{x_k}) \approx (1 - |x_k|),$$

with universal constants. If $x \in E_\lambda$ and $x \neq 0$, then $M^* f_{p,\alpha}(\omega) > \lambda$ whenever $x \in \Gamma(\omega)$. Moreover, $\frac{x}{|x|} \in S_{x_k}$ for some k. Thus, by the definition of S_k and the properties of the Whitney-type decomposition, there exists a universal constant C such that $1 - |x| \leq C(1 - |x_k|)$. Hence $E_\lambda \subset \cup_{k=1}^{\infty} B(x_k/|x_k|, C(1 - |x_k|))$. Therefore

$$\begin{aligned}
\mu(E_\lambda) &\leq \sum_{k=1}^{\infty} \mu(B(x_k/|x_k|, C(1-|x_k|)) \cap \mathbb{B}^n) \\
&\leq C(n, C_\mu) \sum_{k=1}^{\infty} (1 - |x_k|)^{n-1} \\
&\leq C(n, C_\mu) \sum_{k=1}^{\infty} \sigma(S_{x_k}) \leq C(n, C_\mu) \sigma(U_\lambda).
\end{aligned}$$

This together with the Cavalieri formula gives

$$\begin{aligned}
\int_{\mathbb{B}^n} |f(x)|^p (1-|x|)^\alpha d\mu &= \int_0^\infty p\lambda^{p-1} \mu(E_\lambda) d\lambda \\
&\leq C(n, C_\mu) \int_0^\infty p\lambda^{p-1} \sigma(U_\lambda) d\lambda \\
&= C(n, C_\mu) \int_{\mathbb{S}^{n-1}} \sup_{x \in \Gamma(\omega)} |f(x)|^p (1-|x|)^\alpha d\sigma(\omega) \\
&= C \int_{\mathbb{S}^{n-1}} (M^* f_{p,\alpha}(\omega))^p d\sigma(\omega).
\end{aligned}$$

□

We are now ready to establish a maximal characterization for H^p_α. By Lemma 2.7 we could also replace the radial maximal function by the nontangential one below.

Lemma 3.3. *Let* $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ *be a quasiconformal mapping,* $0 < p < \infty$ *and* $\alpha \geq 0$*. Then*

$$\int_{\mathbb{S}^{n-1}} \sup_{0<r<1} |f(r\omega)|^p (1-r)^\alpha d\sigma(\omega) < \infty \tag{3.3}$$

if and only if

$$\int_0^1 (1-r)^{n-2+\alpha} M^p(r,f) dr < \infty. \tag{3.4}$$

Proof. Assume first that $f(0) = 0$ and suppose that (3.4) holds. Set

$$N(r,f) = \sup_{0 \leq t \leq r} M(t,f)(1-t)^{\frac{\alpha}{p}}.$$

Then, by Lemma 3.1, we have that

$$\int_0^1 N^p(r,f)(1-r)^{n-2} dr < \infty.$$

Recall our notation $Mf_{p,\alpha}(\omega) = \sup_{0<r<1} |f(r\omega)|(1-r)^{\frac{\alpha}{p}}$. Now

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \sup_{0<r<1} |f(r\omega)|^p (1-r)^\alpha d\sigma(\omega) &= \int_{\mathbb{S}^{n-1}} (Mf_{p,\alpha}(\omega))^p d\sigma(\omega) \\ &= \int_0^\infty p\lambda^{p-1} \sigma(\{\omega \in \mathbb{S}^{n-1} : Mf_{p,\alpha}(\omega) > \lambda\}) d\lambda. \end{aligned} \tag{3.5}$$

Fix $\lambda > 0$ and let $E = \{\omega \in \mathbb{S}^{n-1} : Mf_{p,\alpha}(\omega) > \lambda\}$. Suppose that E is nonempty. Then there is $\omega \in \mathbb{S}^{n-1}$ and $r \in (0,1)$ such that $N(r,f) = \frac{\lambda}{2}$, since $N(r,f)$ is continuous. Our function $N(r,f)$ is also nondecreasing and we let

$$r_\lambda = \max\{r : N(r,f) = \lambda/2\}. \tag{3.6}$$

We may assume that λ is large enough so that $1/2 < r_\lambda < 1$. Let Γ_E be the family of radial line segments connecting $B(0, r_\lambda)$ and $E \subset \mathbb{S}^{n-1}$. Then

$$\mathrm{Mod}(\Gamma_E) = \sigma(E)(\log(1/r_\lambda))^{1-n} \geq \sigma(E) 2^{1-n} (1-r_\lambda)^{1-n}.$$

By the definitions of E and r_λ, for any $\gamma \in \Gamma_E$, the image curve $f(\gamma)$ connects $B(0, (\lambda/2)(1-r_\lambda)^{-\alpha/p})$ to $\mathbb{R}^n \setminus B(0, \lambda(1-r_\lambda)^{-\alpha/p})$, and therefore the modulus of the image family $f\Gamma_E$ satisfies

$$\mathrm{Mod}(f\Gamma_E) \leq \omega_{n-1}(\log 2)^{1-n}.$$

By combining the above two estimates and using the quasi-invariance of the modulus, we arrive at the upper bound

$$\sigma(E) \leq C(n,K)(1-r_\lambda)^{n-1}.$$

In order to prove (3.3) we may assume that $Mf_{p,\alpha}$ is unbounded on $\mathbb{S}^{n-1}$. Define a measure ν on $[0,1]$ by setting $d\nu = (1-r)^{n-2} dr$ and recall the definition of r_λ from (3.6). Now

$$\nu(\{r : N(r,f) > \lambda/2\}) = \int_{r_\lambda}^1 (1-r)^{n-2} dr = \frac{(1-r_\lambda)^{n-1}}{n-1}.$$

Thus

$$\begin{aligned}
&\int_0^\infty p\lambda^{p-1}\sigma(\{\omega\in\mathbb{S}^{n-1}: Mf_{p,\alpha}(\omega)>\lambda\})d\lambda\\
&\le \sigma(\mathbb{S}^{n-1})2^pN^p(1/2,f)+\int_{2N(1/2,f)}^\infty p\lambda^{p-1}\sigma(\{\omega\in\mathbb{S}^{n-1}: Mf_{p,\alpha}(\omega)>\lambda\})d\lambda\\
&\le \sigma(\mathbb{S}^{n-1})2^pN^p(1/2,f)+C(n,K,p)\int_0^\infty \lambda^{p-1}(1-r_\lambda)^{n-1}d\lambda\\
&\le \sigma(\mathbb{S}^{n-1})2^pN^p(1/2,f)+C(n,K,p)\int_0^\infty \lambda^{p-1}\int_{\{r:N(r,f)>\lambda/2\}}(1-r)^{n-2}drd\lambda\\
&\le \sigma(\mathbb{S}^{n-1})2^pN^p(1/2,f)+C(n,K,p)\int_0^1 (1-r)^{n-2}N^p(r,f)dr<\infty,
\end{aligned}$$

and hence (3.3) follows by (3.5).

In the case $f(0)\neq 0$, we consider the quasiconformal mapping g defined by setting $g(x)=f(x)-f(0)$. Then (3.4) also holds with f replaced by g, and by the first part of our proof (3.3) follows with f replaced by g. We conclude with (3.3) via the triangle inequality.

Towards the other direction, suppose that (3.3) holds. Set $r_k:=1-2^{-k}$ and choose $x_k\in\mathbb{B}^n$ so that $|x_k|=r_k$ and $|f(x_k)|=M(r_k,f)$. Then

$$\int_0^1(1-r)^{n-2+\alpha}M^p(r,f)dr\le 2^n\sum_{k=1}^\infty(2^{-k})^{n-1+\alpha}M^p(r_k,f)=2^n\int_{\mathbb{B}^n}|f(x)|^p(1-|x|)^\alpha d\mu,$$

where $d\mu=\sum_{k=1}^\infty(1-|x|)^{n-1}\delta_{x_k}$. Notice that μ is a Carleson measure. Hence Lemma 3.2 gives us the estimate

$$\int_0^1(1-r)^{n-2+\alpha}M^p(r,f)dr\le C(n,K,C_\mu)\int_{\mathbb{S}^{n-1}}\sup_{0<r<1}|f(r\omega)|^p(1-r)^\alpha d\sigma(\omega).$$

□

We continue with the following estimate whose proof is based on a good-λ inequality.

Lemma 3.4. *Let $f:\mathbb{B}^n\to\Omega\subset\mathbb{R}^n$ be a K-quasiconformal mapping, $0<p<\infty$, and $\alpha\ge 0$. Let*

$$v(\omega)=\sup_{x\in\Gamma(\omega)}d(f(x),\partial\Omega)(1-|x|)^{\alpha/p}\in L^p(\mathbb{S}^{n-1}).$$

There exists $C=C(n,K,p,\alpha)$ such that

$$\int_{\mathbb{S}^{n-1}}(Mf_{p,\alpha}(\omega))^pd\sigma(\omega)\le C\int_{\mathbb{S}^{n-1}}v^p(\omega)d\sigma(\omega).$$

Proof. Recall that

$$M^*f_{p,\alpha}(\omega)=\sup_{x\in\Gamma(\omega)}|f(x)|(1-|x|)^{\alpha/p}$$
$$\text{and}\quad Mf_{p,\alpha}(\omega)=\sup_{0<r<1}|f(r\omega)|(1-r)^{\alpha/p}.$$

Let $L>2$. By the Cavalieri formula

$$\int_{\mathbb{S}^{n-1}}(Mf_{p,\alpha}(\omega))^pd\sigma(\omega)=L^p\int_0^\infty p\lambda^{p-1}\sigma(\{\omega\in\mathbb{S}^{n-1}: Mf_{p,\alpha}(\omega)>L\lambda\})d\lambda. \tag{3.7}$$

Set $\Sigma_\lambda = \sigma(\{\omega \in \mathbb{S}^{n-1} : Mf_{p,\alpha}(\omega) > L\lambda\})$. Then, for any $\gamma > 0$, we have

$$\Sigma_\lambda \leq \sigma(\{\omega \in \mathbb{S}^{n-1} : Mf_{p,\alpha}(\omega) > L\lambda, v(\omega) \leq \gamma\}) + \sigma(\{\omega \in \mathbb{S}^{n-1} : v(\omega) > \gamma\}).$$

If γ is a fixed multiple of λ, then the latter term is what we want, but we need to obtain a suitable estimate for the first term.

Towards this end, set

$$E_{L\lambda,\gamma} = \{\omega \in \mathbb{S}^{n-1} : Mf_{p,\alpha}(\omega) > L\lambda, v(\omega) \leq \gamma\}$$

and define

$$U_\lambda = \{\omega \in \mathbb{S}^{n-1} : M^* f_{p,\alpha}(\omega) > \lambda\}.$$

Since Clearly $E_{L\lambda,\gamma} \subset U_\lambda$. We utilize a generalized Whitney decomposition of the open set U_λ as in the proof of Lemma 3.2:

$$U_\lambda = \bigcup_k S_{x_k}.$$

where the caps S_{x_k} have uniformly bounded overlaps and

$$d(S_{x_k}, \partial U(\lambda)) \approx \mathrm{diam}(S_{x_k}) \approx (1 - |x_k|). \tag{3.8}$$

Suppose $\omega \in S_{x_k}$ is such that $v(\omega) \leq \gamma$ and $Mf_{p,\alpha}(\omega) > L\lambda$. According to (3.8), we can choose $\bar{\omega} \in \partial U(\lambda)$ with

$$d(\omega, \bar{\omega}) \leq C\mathrm{diam}(S(x_k)). \tag{3.9}$$

Let $\bar{x}_k \in \Gamma(\bar{\omega})$ satisfy $|\bar{x}_k| = |x_k|$. By (3.8), we conclude that $d(x_k, \bar{x}_k) \leq C(1 - |x_k|)$. Hence Lemma 2.1 allows us to conclude that

$$d(f(x_k), f(\bar{x}_k))(1 - |x_k|)^{\alpha/p} \leq Cd(f(x_k), \partial\Omega)(1 - |x_k|)^{\alpha/p} \leq Cv(\omega) \leq C\gamma. \tag{3.10}$$

Since $\bar{\omega} \notin U(\lambda)$, we may deduce from (3.10) that

$$|f(x_k)|(1 - |x_k|)^{\alpha/p} \leq (|f(\bar{x}_k)| + d(f(x_k, f(\bar{x}_k)))(1 - |x_k|)^{\alpha/p} \leq \lambda + C\gamma. \tag{3.11}$$

Next, the assumption that $Mf_{p,\alpha}(\omega) > L\lambda$, allows us to choose choose $r_\omega \in (0, 1)$ such that

$$|f(r_\omega \omega)|(1 - r_\omega)^{\alpha/p} \geq \frac{1}{2} Mf_{p,\alpha}(\omega) \geq \frac{1}{2} L\lambda. \tag{3.12}$$

We proceed to show that

$$1 - r_\omega \leq C_0(1 - |x_k|) \tag{3.13}$$

for an absolute constant C_0. If (3.13) fails, then $1 - |x_k| \leq \frac{1}{C_0}(1 - r_\omega)$, which implies by (3.9) that

$$d(w, \bar{w}) \leq C\mathrm{diam}(S(x_k)) \leq C(1 - |x_k|) \leq \frac{C}{C_0}(1 - r_\omega).$$

This shows that $r_\omega \omega \in \Gamma_{\bar{\omega}}$ when $C_0 > 2C$. Since $\frac{L}{2} > 1$, we conclude that

$$M^* f_{p,\alpha}(\bar{\omega}) > \lambda,$$

which contradicts the assumption that $\bar{\omega} \notin U_\lambda$.

We may assume that $C_0 \geq 1$. By (3.12) together with (3.13) we obtain

$$L\lambda \leq 2|f(r_\omega \omega)|(1-r_\omega)^{\alpha/p} \leq 2C_0^{\alpha/p}|f(r_\omega \omega)|(1-|x_k|)^{\alpha/p}. \tag{3.14}$$

Let us fix the value of L by choosing $L = 4C_0^{\alpha/p}$. Then (3.14) yields

$$2\lambda \leq |f(r_\omega \omega)|(1-|x_j|)^{\alpha/p}. \tag{3.15}$$

We proceed to estimate $\sigma(S_{x_k} \cap E_{L\lambda,\gamma})$. Let $\omega \in S_{x_k} \cap E_{L\lambda,\gamma}$. Then there is $r_\omega \in (0,1)$ so that both (3.13) and (3.15) hold. Consider the collection of all the corresponding caps $S_{r_\omega \omega}$. By the Besicovitch covering theorem we find a countable subcollection of these caps, say $S_{r_1\omega_1}, S_{r_2\omega_2}, \ldots$, so that

$$S_{x_k} \cap E_{L\lambda,\gamma} \subset \bigcup_j S_{r_j \omega_j} \tag{3.16}$$

and $\sum_j \chi_{S_{r_j\omega_j}}(w) \leq C_n$ for all $\omega \in \mathbb{S}^{n-1}$. By (3.13) we further have

$$\sum_j \sigma(S_{r_j\omega_j}) \leq C_1 \sigma(S_{x_k}) \tag{3.17}$$

for an absolute constant C_1.

Fix one of the caps $S_{r_j\omega_j} =: S_j$ and let $A \geq 1$. Write

$$E_1^j(A) = \{w \in S_j \cap S_{x_k} \cap E_{L\lambda,\gamma} : \ |f(w) - f(r_j\omega_j)| \geq Ad(f(r_j\omega_j), \partial\Omega)\}$$

and

$$E_2^j(A) = \{w \in S_j \cap S_{x_k} \cap E_{L\lambda,\gamma} : \ |f(w) - f(x_k)| \geq Ad(f(x_k), \partial\Omega)\}.$$

We claim that we can find a constant C_2 only depending on C_0, p, α so that the choice $\lambda = C_2 A\gamma$ guarantees that

$$S_j \cap S_{x_k} \cap E_{L\lambda,\gamma} = E_1^j(A) \cup E_2^j(A). \tag{3.18}$$

Let $\omega \in S_j \cap S_{x_k} \cap E_{L\lambda,\gamma}$. Suppose first that

$$|f(\omega) - f(r_j\omega_j)|(1-r_j)^{\alpha/p} \geq A\gamma. \tag{3.19}$$

Since $\omega \in E_{L\lambda,\gamma} \cap S_j$, we have

$$\gamma \geq d(f(r_j\omega_j), \partial\Omega)(1-r_j)^{\alpha/p},$$

and we deduce from (3.19) that $\omega \in E_1^j(A)$. We are left to consider the case

$$|f(\omega) - f(r_j\omega_j)|(1-r_j)^{\alpha/p} < A\gamma. \tag{3.20}$$

Under this condition, the triangle inequality together with (3.13), (3.14) and (3.11) give us

$$\begin{aligned} &|f(\omega) - f(x_k)|(1-|x_k|)^{\alpha/p} \geq |f(\omega)|(1-|x_k|)^{\alpha/p} - |f(x_k)|(1-|x_k|)^{\alpha/p} \\ \geq&(|f(r_j\omega_j)| - |f(\omega) - f(r_j\omega_j)|)\frac{(1-r_j)^{\alpha/p}}{C_0^{\alpha/p}} - |f(x_k)|(1-x_k)^{\alpha/p} \\ \geq&\frac{L\lambda}{2C_0^{\alpha/p}} - \frac{A\gamma}{C_0^{\alpha/p}} - (\lambda + C\gamma) \geq 2\lambda - (\lambda + C\gamma) - \frac{A\gamma}{C_0^{\alpha/p}}. \end{aligned} \tag{3.21}$$

We now fix the relation between λ and γ by setting $\lambda = (C + \frac{A}{C_0^{\alpha/p}} + 1)\gamma$. Then (3.21) reduces to

$$|f(\omega) - f(x_k)|(1-|x_k|)^{\alpha/p} \geq A\gamma \geq Ad(f(x_k), \partial\Omega)(1-|x_k|)^{\alpha/p}$$

and we conclude that $\omega \in E_2^j(A)$.

According to Lemma 2.6,

$$\sigma(E_1^j(A)) \leq \frac{C_2\sigma(S_j)}{(\log A)^{n-1}}, \tag{3.22}$$

where C_2 depends only on K, n. Thus (3.22) together with (3.17) gives

$$\sum_j \sigma(E_1^j(A)) \leq \frac{C_1C_2\sigma(S_{x_k})}{(\log A)^{n-1}}. \tag{3.23}$$

We also deduce via Lemma 2.6 that

$$\sigma(\cup_j E_2^j(A)) \leq \sigma(\{\omega \in S_{x_k} : \ |f(w) - f(x_k)| \geq Ad(f(x), \partial\Omega)\} \leq \frac{C_2\sigma(S_{x_k})}{(\log A)^{n-1}}. \tag{3.24}$$

Now (3.18) together with (3.23) and (3.24) gives

$$\sigma(S_{x_k} \cap E_{L\lambda,\gamma}) \leq \frac{C_3\sigma(S_{x_k})}{(\log A)^{n-1}}, \tag{3.25}$$

where C_3 depends only on K, n.

By the choice of the caps S_{x_k}, the definition of $E_{L\lambda,\gamma}$ and (3.25) give via summing over k the estimate

$$\begin{aligned}\Sigma_\lambda \leq &\sigma(E_{L\lambda,\gamma}) + \sigma(\{\omega \in \mathbb{S}^{n-1} : v(\omega > \gamma)\}) \\ \leq &\frac{C_3\sigma(U_\lambda)}{(\log A)^{n-1}} + \sigma(\{\omega \in \mathbb{S}^{n-1} : v(\omega > \gamma)\}).\end{aligned} \tag{3.26}$$

We insert (3.26) into (3.7) and conclude that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} (Mf_{p,\alpha}(\omega))^p d\sigma(\omega) &= L^p \int_0^\infty p\lambda^{p-1}\Sigma_\lambda d\lambda \\ &\leq L^p \int_0^\infty p\lambda^{p-1}\frac{C_3\sigma(U_\lambda)}{(\log A)^{n-1}} d\lambda + L^p \int_0^\infty p\lambda^{p-1}\sigma(\{\omega \in \mathbb{S}^{n-1} : v(\omega > \gamma)\})d\lambda \\ &\leq \frac{C_3L^p}{(\log A)^{n-1}} \int_{\mathbb{S}^{n-1}} (M^*_{p,\alpha}f(\omega))^p d\omega + L^p \int_0^\infty p\lambda^{p-1}\sigma(\{\omega \in \mathbb{S}^{n-1} : v(\omega > \gamma)\})d\lambda.\end{aligned} \tag{3.27}$$

Suppose that the integral on the left-hand side of (3.27) is finite. Then Lemma 2.7 allows us to choose A only depending on K, n, p, α, L, C_3 so that the integral of $(M^*_{p,\alpha}f)^p$ can be embedded into the left-hand side. In this case our claim follows via the Cavalieri formula, recalling that $\lambda = (C + \frac{A}{C_0^{\alpha/p}} + 1)\gamma$. We are left with the case where the integral on the left-hand-side of (3.27) is infinite. In this case, we replace f by the K-quasiconformal map f^j defined by setting $f^j(x) = f((1-1/j)x)$. Since the corresponding integral is now finite, we obtain a uniform estimate for the integral of $Mf^j_{p,\alpha}$ in terms of the integral of v^j, defined analogously. The desired estimate follows via the Fatou lemma by letting j tend to infinity since it easily follows that $v_j(\omega) \leq v(\omega)$ for all ω and that $Mf^j_{p,\alpha}(\omega) \to Mf_{p,\alpha}(\omega)$ for a.e. ω. □

Lemma 3.5. *Let $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ be a quasiconformal mapping, $0 < p < \infty$ and $\alpha \geq 0$. Then the following are equivalent:*

1. $\int_{\mathbb{S}^{n-1}} (Mf_{p,\alpha})^p d\sigma < \infty$
2. $\int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p-1+\alpha} dx < \infty$
3. $\int_{\mathbb{S}^{n-1}} \sup_{x\in\Gamma(\omega)} d(f(x), \partial\Omega)^p (1-|x|)^\alpha d\sigma < \infty$

Proof. ($\mathbf{1} \implies \mathbf{2}$) Suppose first that $0 < p \leq 1$. We may assume that $f \neq 0$ in $\mathbb{B}^n$. Then the measure given by $d\mu = |Df|^p |f|^{-p}(1-|x|)^{p-1}dx$ is a Carleson measure by Lemma 2.5 and hence Lemma 2.3 and Lemma 3.2 give

$$\begin{aligned}\int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p-1+\alpha}dx &\leq C\int_{\mathbb{B}^n} |Df|^p(1-|x|)^{p-1+\alpha}dx \\ \leq C\int_{\mathbb{B}} |f(x)|^p(1-|x|)^\alpha d\mu(x) &\leq C\int_{\mathbb{S}^{n-1}} (Mf_{p,\alpha})^p(\omega)d\sigma.\end{aligned}$$

We are left to deal with the case $p > 1$. Pick $y \in \partial\Omega$. By Lemma 2.3 and Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p-1+\alpha}dx &\leq C\int_{\mathbb{B}^n} |Df(x)|a_f^{p-1}(x)(1-|x|)^{p-1+\alpha}dx \\ \leq C\int_{\mathbb{B}^n} |Df(x)|d(f(x),\partial\Omega)^{p-1}(1-|x|)^\alpha dx &\leq C\int_{\mathbb{B}^n} |Df(x)||f(x)-y|^{p-1}(1-|x|)^\alpha dx.\end{aligned}$$

Since $f(x)-y \neq 0$ in $\mathbb{B}^n$, the measure induced by $d\mu = |Df(x)||f(x)-y|^{-1}dx$ is a Carleson measure by Lemma 2.5. Hence we can apply Lemma 3.2 to conclude that $\int_{\mathbb{B}^n} a_f^p(x)(1-|x|)^{p-1+\alpha}dx < \infty$.

($\mathbf{2} \implies \mathbf{3}$) This follows from Lemma 2.4.

($\mathbf{3} \implies \mathbf{1}$) By Lemma 2.2 we have that $d(f(x),\partial\Omega) \leq Ca_f(x)(1-|x|)$. Hence ($\mathbf{1}$) follows from ($\mathbf{3}$) by Lemma 3.4. □

Lemma 3.6. *Let $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ be a quasiconformal mapping with $f(0) = 0$. Let $p \geq n$ and let $\alpha > 1-n$. Then*

$$\int_0^1 (1-r)^{n-2+\alpha} M^p(r,f)dr \leq C\int_{\mathbb{B}^n} a_f(x)^p(1-|x|)^{p-1+\alpha}dx.$$

Proof. Define $v(\omega) = \sup_{x\in\Gamma(\omega)} d(f(x),\partial\Omega)(1-|x|)^{\alpha/p}$. By Lemma 2.4 we only need to show that

$$\int_0^1 (1-r)^{n-2+\alpha} M^p(r,f)dr \leq C\int_{\mathbb{S}^{n-1}} v^p d\sigma. \tag{3.28}$$

For each $i \geq 1$, let $r_i = 1-2^{-i}$ and pick $x_i \in \mathbb{S}^{n-1}(r_i)$ with $|f(x_i)| = M(r_i, f)$. Then

$$\begin{aligned}\int_0^1 (1-r)^{n-2+\alpha}M^p(r,f)dr &= \sum_{i=1}^\infty \int_{r_{i-1}}^{r_i} (1-r)^{n-2+\alpha}M^p(r,f)dr \\ &\leq C\sum_{i=1}^\infty |f(x_i)|^p(1-|x_i|)^{n-1+\alpha}.\end{aligned} \tag{3.29}$$

Let $\tilde{C}$ be a constant, to be determined later, and let $G(f) = \{i \in \mathbb{N} : |f(x_i)| \leq \tilde{C}d(f(x_i),\partial\Omega)\}$ and $B(f) = \mathbb{N}\setminus G(f)$. For $i \in G(f)$ and $\omega \in S_{x_i}$, we have

$$|f(x_i)|^p(1-|x_i|)^{n-1+\alpha} \leq \tilde{C}^p d(f(x_i),\partial\Omega)^p(1-|x_i|)^{n-1+\alpha} \leq \tilde{C}^p v_f(\omega)^p(1-|x_i|)^{n-1}. \tag{3.30}$$

Define $\delta = 1/\tilde{C}$. Then, for $i \in B(f)$, we have $d(f(x_i), \partial\Omega) \leq \delta|f(x_i)|$. Set $\omega_i = \frac{x_i}{|x_i|}$, and let $y_{i-1} = r_{i-1}\omega_i$. Then we have $x_i \in B_{y_{i-1}}$. Hence Lemma 2.1 gives

$$|f(x_i) - f(y_{i-1})| \leq \mathrm{diam} f(B_{y_{i-1}}) \leq Cd(f(B_{y_{i-1}}), \partial\Omega) \leq C^2 d(f(x_i), \partial\Omega).$$

Therefore, by the choice of x_{i-1}, we obtain

$$|f(x_i)| \leq |f(y_{i-1})| + C^2\delta|f(x_i)| \leq |f(x_{i-1})| + C^2\delta|f(x_i)|.$$

If $\tilde{C}$ is sufficiently large, then $C^2\delta < 1$ and we have

$$|f(x_i)| \leq \lambda|f(x_{i-1})|, \tag{3.31}$$

where $\lambda = 1/(1 - C^2\delta)$. After multiplying both sides of (3.31) by $(1 - |x_i|)^{(n-1+\alpha)/p}$ and raising to power p, we conclude that

$$\begin{aligned} |f(x_i)|^p(1 - |x_i|)^{n-1+\alpha} &\leq \lambda^p|f(x_{i-1})|^p(1 - |x_i|)^{n-1+\alpha} \\ &= \lambda^p|f(x_{i-1})|^p 2^{-(n-1+\alpha)}(1 - |x_{i-1}|)^{n-1+\alpha}. \end{aligned} \tag{3.32}$$

Now, notice that $n - 1 + \alpha > 0$ when $\alpha > 1 - n$. By recalling that $\delta = 1/\tilde{C}$ and $\lambda = 1/(1 - C^2\delta)$, we find $\tilde{C}= \tilde{C}(p, C)$ such that $\lambda^p 2^{-(n-1+\alpha)} < 1$ and $\tilde{C} \geq C_0$. Then there exists $c < 1$, such that

$$|f(x_i)|^p(1 - |x_i|)^{n-1+\alpha} \leq c|f(x_{i-1})|^p(1 - |x_{i-1}|)^{n-1+\alpha}.$$

Since $x_0 = 0$ and $f(0) = 0$, we have $0 \in G(f)$. If $i - 1 \in B(f)$, we repeat the above argument with i replaced by $i - 1$ and arrive at

$$|f(x_i)|^p(1 - |x_i|)^{n-1+\alpha} \leq c^2|f(x_{i-2})|^p(1 - |x_{i-2}|)^{n-1+\alpha}.$$

We repeat inductively until $i - k \in G(f)$. In conclusion, there exists k such that $l \in B(f)$, for all $i - k < l \leq i$, $i - k \in G(f)$ and

$$|f(x_i)|^p(1 - |x_i|)^{n-1+\alpha} \leq c^k|f(x_{i-k})|^p(1 - |x_{i-k}|)^{n-1+\alpha}. \tag{3.33}$$

Since $c < 1$, inequality (3.33) yields the estimate

$$\begin{aligned} \sum_{i=0}^{\infty} |f(x_i)|^p(1 - |x_i|)^{n-1+\alpha} &\leq C \sum_{i \in G(f)} |f(x_i)|^p(1 - |x_i|)^{n-1+\alpha} \\ &\leq C\tilde{C} \sum_{i \in G(f)} d(f(x_i), \partial\Omega)^p(1 - |x_i|)^{\alpha}(1 - |x_i|)^{n-1}. \end{aligned} \tag{3.34}$$

Set $S_i = S_{x_i}$ for $i \in G(f)$. Since $|x_i| = 1 - 2^{-i}$, the definition of the shadows S_i yields that

$$\sum_{j>i} \sigma(S_j) \leq \sigma(S_i).$$

Thus, we also have

$$\sigma(\{\omega \in S_i : \sum_{j>i} \chi_{S_j}(\omega) \geq 2\}) \leq \frac{1}{2}\sigma(S_i).$$

Hence there exist sets $\hat{S}_i \subset S_i$ with $\sigma(\hat{S}_i) \geq \frac{1}{2}\sigma(S_i) = C_n(1-|x_i|)^{n-1}$ and so that no point of $\hat{S}_i$ belongs to more than one S_j. Then

$$\sum_{i\in G(f)} \chi_{\hat{S}_i}(\omega) \leq 2.$$

By combining (3.30) and (3.34), we arrive at

$$\sum_{i=0}^{\infty} |f(x_i)|^p (1-|x_i|)^{n-1+\alpha} \leq \frac{C\tilde{C}}{C_n} \sum_{i\in G(f)} \int_{\hat{S}_i} v^p(\omega) d\sigma \leq \frac{C\tilde{C}}{C_n} \int_{\mathbb{S}^{n-1}} v^p(\omega) d\sigma.$$

This together with (3.29) gives (3.28) and hence our claim follows. □

Proof of Theorem 1.1. We begin with the equivalence of (1.2), (1.3) and (1.4).

We assume first that $f(0) = 0$ and handle the cases $0 < p \leq n$ and $p > n$ separately.

Case 1 Suppose that $0 < p \leq n$. Then

$$\begin{aligned} &\int_{B(0,r)} |f|^{p-n} |Df|^n dx \leq K \int_{B(0,r)} |f|^{p-n} J_f(x) dx = K \int_{f(B(0,r))} |y|^{p-n} dy \\ &\overset{(*)}{\leq} K \int_{B(0,\sqrt[n]{|f(B(0,r))|})} |y|^{p-n} dy = KC(n) \int_{\mathbb{S}^{n-1}} \int_0^{\sqrt[n]{(|f(B(0,r))}} t^{p-1} dt d\sigma = C(K,n,p) |f(B(0,r))|^{p/n} \\ &= C \left(\int_{B(0,r)} J_f(x) dx \right)^{p/n} \leq C \left(\int_{B(0,r)} |Df(x)|^n dx \right)^{p/n}, \end{aligned}$$

where $(*)$ holds since the weight function $|y|^{p-n}$ is radially decreasing when $0 < p \leq n$. We have proved that (1.4) yields (1.3).

Now let $g = |f|^{(p-n)/n} f$. Since quasiconformal mappings are differentiable almost everywhere, a calculation gives

$$Dg = |f|^{(p-n)/n} \left(I + \frac{p-n}{n} \frac{f^T f}{|f|^2} \right) Df \quad (\text{for a.e. } x \in \mathbb{B}^n),$$

and so $|Dg| \lesssim |f|^{(p-n)/n} |Df|$. Then the Fubini theorem and Lemma 2.3 yield

$$\begin{aligned} \int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |f|^{p-n} |Df|^n &\gtrsim \int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |Dg|^n dx dr \\ \approx \int_{\mathbb{B}^n} |Dg|^n (1-|x|)^{n-1+\alpha} dx &\approx \int_{\mathbb{B}^n} a_g^n(x) (1-|x|)^{n-1+\alpha} dx. \end{aligned}$$

Hence, assuming that (1.3) in the statement of the theorem holds, we can apply Lemma 3.6 to the quasiconformal mapping g so as to conclude that

$$\begin{aligned} &\int_0^1 (1-r)^{n-2+\alpha} M^p(r,f) dr \approx \int_0^1 (1-r)^{n-2+\alpha} M^n(r,g) dr \\ &\lesssim \int_{\mathbb{B}^n} a_g(x)^n (1-|x|)^{n-1+\alpha} dx \lesssim \int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |f|^{p-n} |Df|^n dx dr. \end{aligned} \tag{3.35}$$

We have shown that (1.3) implies (1.2).

The implication that (1.2) $\Longrightarrow$ (1.4) is a straightforward application of Hölder's inequality and the definition of $M(r,f)$. Indeed,

$$
\begin{aligned}
&\int_0^1 (1-r)^{n-2+\alpha}\left(\int_{B(0,r)} |Df(x)|^n dx\right)^{p/n} dr\\
\leq &\int_0^1 (1-r)^{n-2+\alpha}\left(\int_{B(0,r)} |Df(x)|^n |f(x)|^{p-n} M^{n-p}(r,f) dx\right)^{p/n} dr\\
= &\int_0^1 (1-r)^{(n-2+\alpha)(n-p)/n} M^{(n-p)p/n}(r,f)\left((1-r)^{n-2+\alpha}\int_{B(0,r)} |Df(x)|^n |f(x)|^{p-n} dx\right)^{p/n} dr\\
\leq &\left(\int_0^1 (1-r)^{n-2+\alpha} M^p(r,f) p dr\right)^{(n-p)/n}\left(\int_0^1 (1-r)^{n-2+\alpha}\int_{B(0,r)} |Df|^n |f|^{p-n} dx dr\right)^{p/n}.
\end{aligned}
\tag{3.36}
$$

Since

$$\int_{B(0,r)} |Df|^n |f|^{p-n} dx dr \leq K\int_{f(B(0,r))} |y|^{p-n} dy \leq K\int_{B(0,M(r,f))} |y|^{p-n} dy \leq C(n,K,p) M^p(r,f),$$

the desired result follows.

Case 2 We assume that $p > n$. Now

$$\int_{B(0,r)} |Df|^n dx \leq K\int_{B(0,r)} J_f(x) dx \leq K|f(B(0,r))| \leq KM^n(r,f)$$

Therefore, (1.2) implies (1.4).

Set $g = |f|^{(p-n)/n} f$. Then $|g|^n = |f|^p$, and $M^p(r,f) = M^n(r,g)$. Analogously to (3.35), Lemma 3.6 gives

$$
\begin{aligned}
&\int_0^1 (1-r)^{n-2+\alpha} M^p(r,f) dr \leq \int_0^1 (1-r)^{n-2+\alpha} M^n(r,g) dr\\
&\lesssim \int_0^1 (1-r)^{n-2+\alpha}\int_{B(0,r)} |f|^{p-n} |Df|^n dx dr.
\end{aligned}
\tag{3.37}
$$

Hence (1.3) implies (1.2).

We need to show that (1.4) yields (1.3). First of all,

$$
\begin{aligned}
&\int_0^1 (1-r)^{n-2+\alpha}\int_{B(0,r)} |f|^{p-n} |Df|^n dx dr \leq \int_0^1 (1-r)^{n-2+\alpha} M^{p-n}(r,f)\int_{B(0,r)} |Df|^n dx dr\\
\leq &\left(\int_0^1 \left((1-r)^{n-2+\alpha} M(r,f)\right)^p dr\right)^{(p-n)/p}\left(\int_0^1 (1-r)^{n-2+\alpha}\left(\int_{B(0,r)} |Df|^n dx\right)^{p/n} dr\right)^{n/p}.
\end{aligned}
$$

If the latter term on the right-hand side is finite, then we obtain via (3.37) that

$$\int_0^1 (1-r)^{n-2+\alpha}\int_{\mathbb{B}^n} |f|^{p-n} |Df|^n dx dr \leq C\int_0^1 (1-r)^{n-2+\alpha}\left(\int_{\mathbb{B}^n} |Df|^n dx\right)^{p/n} dr.$$

Since the constant in this inequality only depends on n, K, p, α the general case easily follows by applying this estimate with f replaced by f_j, defined by setting $f_j(x) = f((1-1/j)x)$, and by passing to the limit.

We have shown the equivalence of (1.2),(1.3) and (1.4) under the additional assumption that $f(0) = 0$. The general case follows since each of them holds for a quasiconformal mapping f if and only if it holds for g, defined by setting $g(x) = f(x) - f(0)$.

We are left with the equivalence of (1.2) and (1.5), when $\alpha \geq 0$ and when $1 - n < \alpha < 0$ with $p \geq n$.

For $\alpha \geq 0$, by Lemma 3.3 and Lemma 3.5, we know that (1.2) is equivalent to (1.5).

Suppose that $1 - n < \alpha < 0$ and $p \geq n$. By Lemma 3.6 we already know that (1.5) implies (1.2). Hence we only need to show that (1.3) implies (1.5). Fix $y \in \partial f(\mathbb{B}^n)$. Then, Lemma 2.2 and 2.3 ensure that

$$\begin{aligned}
\int_{\mathbb{B}^n} a_f(x)^p (1-|x|)^{p-1+\alpha} dx &\approx \int_{\mathbb{B}^n} a_f(x)^{p-n} (1-|x|)^{p-n} |Df(x)|^n (1-|x|)^{n-1+\alpha} dx \\
&\leq C \int_{\mathbb{B}^n} d(f(x), \partial\Omega)^{p-n} |Df(x)|^n (1-|x|)^{n-1+\alpha} dx \\
&\leq C \int_{\mathbb{B}^n} |f(x) - y|^{p-n} |Df(x)|^n (1-|x|)^{n-1+\alpha} dx \\
&\approx \int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |f(x) - y|^{p-n} |Df(x)|^n dx dr.
\end{aligned} \tag{3.38}$$

Notice that $|f(x) - y| \leq |f(x)| + |y|$ and $|f(x) - y|^{p-n} \leq C(p,n)(|f(x)|^{p-n} + |y|^{p-n})$. By (3.38), we only need to show that

$$\begin{aligned}
&\int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} (|f(x)|^{p-n} + |y|^{p-n}) |Df(x)|^n dx dr = \\
=&\int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |f(x)|^{p-n} |Df(x)|^n dx dr + |y|^{p-n} \int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |Df(x)|^n dx dr \\
=&(I) + (II) < \infty.
\end{aligned}$$

By the equivalance of (1.2) and (1.3), we know that $(I) < \infty$. On the other hand, we have

$$\begin{aligned}
&\int_0^1 (1-r)^{n-2+\alpha} \int_{B(0,r)} |Df(x)|^n dx dr \\
\leq\ &\left(\int_0^1 (1-r)^{n-2+\alpha} \left(\int_{B(0,r)} |Df(x)|^n \right)^{p/n} dx dr \right)^{n/p} \left(\int_0^1 (1-r)^{n-2+\alpha} dr \right)^{(p-n)/p}.
\end{aligned}$$

The equivalence of (1.2) and (1.4) ensures that the first term is finite and the second term is also finite since $\int_0^1 (1-r)^{n-2+\alpha} dr < \infty$ when $-1 < \alpha < 0$. We have shown that $(II) < \infty$. This completes the proof of Theorem 1.1. $\square$

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