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ON THE DIMENSION OF KAKEYA SETS IN THE FIRST HEISENBERG GROUP

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(Communicated by)

Abstract. We define Kakeya sets in the Heisenberg group and show that the Heisenberg Hausdorff dimension of Kakeya sets in the first Heisenberg group is at least 3. This lower bound is sharp since, under our definition, the $\{xoy\}$-plane is a Kakeya set with Heisenberg Hausdorff dimension 3.

1. Introduction

The study of Kakeya sets in Euclidean space is one of the central topics in geometric measure theory. A set $E \subset \mathbb{R}^n$ is a Kakeya set if for every $e \in S^{n-1}$ there exists a unit line segment I_e parallel to e such that $I_e \subset E$.

A natural question is to determine the least Hausdorff dimension of Kakeya sets.

The answer is known in 2-dimensional Euclidean space. Indeed, Kakeya sets in $\mathbb{R}^2$ turn out to be of Hausdorff dimension equal to 2 which can be shown by multiple ways. See [7, 9, 15]. However, for Kakeya sets in higher dimensional Euclidean space, the sharp lower bound is not known. We would like to remark some progress: Bourgain used two different methods to provide lower bounds [3, 4], which were further improved by Wolff [18] and Katz-Tao [11] respectively. Recently, Katz-Zahl [13] and Guth-Zahl [10] enhanced the results of [18] in $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively, which show that the best known lower bound is $5/2 + \epsilon_0$ in $\mathbb{R}^3$ with ϵ_0 an absolute constant and $3 + 1/40$ in $\mathbb{R}^4$. In $\mathbb{R}^n$ with $n \geq 5$, the best known lower bound $3 + (2 - \sqrt{2})(n - 4)$ was established in [12] by Katz-Tao.

As an analogy to Euclidean Kakeya sets, we can define Kakeya sets in the Heisenberg group. In this paper, we denote by $\mathbb{H}^n$ the n-th Heisenberg group. When $n = 1$, we write $\mathbb{H}$ instead of $\mathbb{H}^1$ for simplicity.

Definition 1.1. *A set $E \subset \mathbb{H}^n$ is a Kakeya set if for every unit line segment $I \subset \mathbb{R}^{2n} \times \{0\}$ centred at the origin, there exists $q \in \mathbb{H}^n$ such that $qI \subset E$.*

Here and in what follows, by a unit line segment we mean an isometric copy of the unit open interval $(0, 1)$. Moreover, by Heisenberg Hausdorff measure we mean the

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one induced by the Korányi metric on the first Heisenberg group. For the definition of the Korányi metric, we refer the readers to Section 2.

Remark 1.2. The Kakeya sets we consider here are often also called Besicovitch sets in some other literature. And some authors use Kakeya sets to denote those inside of which a unit line segment can be continuously rotated through all directions. However it is not clear how to well define the latter concept on sub-Riemannian setting. Hence it is quite interesting to consider the latter concept as a further study.

The Heisenberg Hausdorff dimension of Euclidean Kakeya sets has been studied in [16] where the author showed a lower bound on the dimension of Kakeya sets. In [17], the author studied Kakeya sets for general metric spaces in axiomatic sense.

According to Definition 1.1, it is not hard to show that the Euclidean Hausdorff dimension of Kakeya sets in $\mathbb{H}$ is at least 2. This can be done as follows. Under orthogonal projection to the $\{xoy\}$-plane, every Kakeya set E in $\mathbb{H}$ becomes a Kakeya set E' in $\mathbb{R}^2$. Hence the known lower bound of the Euclidean Hausdorff dimension of E' is also the one of E since orthogonal projection in $\mathbb{R}^3$ is Lipschitz. Moreover, 2 is sharp since the $\{xoy\}$-plane is a Kakeya set in $\mathbb{H}$ with Euclidean Hausdorff dimension 2. However, the Heisenberg Hausdorff dimension of the $\{xoy\}$-plane is 3 and the orthogonal projection from $\mathbb{H}$ to the $\{xoy\}$-plane is no longer Lipschitz with respect to the Korányi metric. Hence to calculate a lower bound of Heisenberg Hausdorff dimension of Kakeya sets seems to be a nontrivial problem.

In this note, we will show the following

Theorem 1.3. *In the first Heisenberg group $\mathbb{H}$ equipped with the Korányi metric, every Kakeya set has Heisenberg Hausdorff dimension at least 3 and this lower bound is sharp.*

In the following, E will denote a Kakeya set in the first Heisenberg group.

Our method to show Theorem 1.3 is based on the idea of [2, 9]. We first encode each horizontal line segment in E by a quadruple in $\mathbb{R}^4$ forming a subset $L(E) \subset \mathbb{R}^4$. Then we transfer the computation for dimensions of each intersection of E and a plane belonging to a one parameter family to that of a subset in $\mathbb{R}^3$ obtained by certain projections acting on $L(E)$. This can be seen as a duality principle. Finally we use a recent Marstrand-type projection theorem in $\mathbb{R}^3$ by Käenmäki-Orponen-Venieri [14] and a co-area inequality by Eilenberg-Harrold, Jr. [8] to conclude the proof.

Since every Kakeya set in the first Heisenberg group has Heisenberg Hausdorff dimension at least 3, a further question that may be asked is to find a lower bound of 3-dimensional Heisenberg Hausdorff measure among all Kakeya sets in the first Heisenberg group. Unlike the Euclidean Kakeya set, which may have n-dimensional Lebesque measure zero in every $\mathbb{R}^n$ (for example, see [1]), it is not easy to show a counterpart for Kakeya sets in Heisenberg group. Hence we would like to ask the following question

Problem 1.4. *Does there exist a Kakeya set in the first Heisenberg group with zero 3-dimensional Heisenberg Hausdorff measure?*

The paper is organised as follows. In section 2, we recall some background in Heisenberg groups, the Marstrand-type projection theorem in $\mathbb{R}^3$ and the co-area inequality. In section 3, we prove Theorem 1.3.

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2. Preliminaries

The first part of this section is dedicated to a brief introduction to the first Heisenberg group $\mathbb{H}$. For a detailed one, we refer the readers to [5].

The first Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$, equipped with the group multiplication, for any $w = (x, y, t)$ and $w' = (x', y', t')$, as follows

$$ww' = \left(x + x', y + y', t + t' + \frac{1}{2}[xy' - x'y]\right). \tag{2.1}$$

We introduce the Korányi metric on the first Heisenberg group. This is the left invariant metric given by

$$d_{\mathbb{H}}(w, w') := \|(w')^{-1} \cdot w\|_{\mathbb{H}} \tag{2.2}$$

where $\|\cdot\|_{\mathbb{H}}$ is defined as

$$\|(x, y, t)\|_{\mathbb{H}} = ((x^2 + y^2)^2 + 16t^2)^{1/4}.$$

We define horizontal lines in the first Heisenberg group $\mathbb{H}$ as lines which can be obtained by a left translation of some line passing through the origin and lying in the $\{xoy\}$-plane.

By the definition of horizontal lines, we know that for any $b \in \mathbb{R}$ and $q \in \mathbb{H}$, qI_b and qJ_b are horizontal line and horizontal line segment respectively where

$$I_b(\tau) = (\tau, b\tau, 0),\ \tau \in \mathbb{R} \tag{2.3}$$

and

$$J_b(\tau) = (\tau, b\tau, 0),\ \tau \in (-\frac{1}{2\sqrt{b^2+1}}, \frac{1}{2\sqrt{b^2+1}}).$$

The following observation is needed in the proof of Theorem 1.3.

Lemma 2.1. *For any $b \in \mathbb{R}$ and $q = (q_1, q_2, q_3) \in \mathbb{H}$,*

(1) *qI_b and qJ_b can be parameterised as*

$$qI_b(s) = (s, bs + a, -\frac{as}{2} + d),\ s \in \mathbb{R} \tag{2.4}$$

and

$$qJ_b(s) = (s, bs + a, -\frac{as}{2} + d),\ s \in (\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}}) \tag{2.5}$$

where $a = q_2 - bq_1$, $d = q_3 + \frac{1}{2}aq_1$ and $\epsilon = q_1 - \frac{1}{2\sqrt{b^2+1}}$.

(2) *If we denote*

$$l_{(a,b,d)} := \{(s, bs + a, -\frac{as}{2} + d) \in \mathbb{H} \mid s \in \mathbb{R}\} \tag{2.6}$$

and

$$l^{\epsilon}_{(a,b,d)} := \left\{(s, bs + a, -\frac{as}{2} + d) \in \mathbb{H} \mid s \in (\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}})\right\}.$$

Then $l^{\epsilon}_{(a,b,d)}$ has length 1 with respect to $d_{\mathbb{H}}$ for every a, b, d, ϵ.

(3) *If* $b \in (-\sqrt{3}, \sqrt{3})$*, then the orthogonal projection of* $l^{\epsilon}_{(a,b,d)}$ *to the* x*-axis has Euclidean length greater than* $\frac{1}{2}$*.*

Proof. (1) Let I_b be parameterised as in (2.3). Then by the Heisenberg multiplication law (2.1), we have

$$qI_b = \{(q_1 + \tau, q_2 + b\tau, q_3 + \frac{1}{2}(q_1 b\tau - q_2\tau)) \mid \tau \in \mathbb{R}\ \}.$$

Letting $s = q_1 + \tau$, $a = q_2 - bq_1$ and $d = q_3 + \frac{1}{2}aq_1$, we arrive at (2.4). In addition, letting $\epsilon = q_1 - \frac{1}{2\sqrt{b^2+1}}$, we verify that (2.5) holds.

(2) Using the definition of $d_{\mathbb{H}}$ and the fact that left translation is an isometry with respect to $d_{\mathbb{H}}$, we deduce the result.

(3) From (2.5), the orthogonal projection of $l^{\epsilon}_{(a,b,d)}$ to the x-axis is the interval $(\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}}) \subset x$-axis. Hence when $b \in (-\sqrt{3}, \sqrt{3})$, the length of the interval is greater than $\frac{1}{2}$. □

Remark 2.2. In the sense of sub-Riemannian geometry, there exists more general horizontal curves in $\mathbb{H}$ besides horizontal lines. Indeed, associated to the group operation (2.1), we can define the left invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t}.$$

A Lipschitz curve $\gamma = (\gamma_1, \gamma_2, \gamma_3) : [a, b] \to \mathbb{H}$ is said to be horizontal if $\dot{\gamma}(s) \in \mathrm{Span}\{X(\gamma(s)), Y(\gamma(s))\}$, i.e. $\dot{\gamma}(s) = a(s)X(\gamma(s)) + b(s)Y(\gamma(s))$, for almost every $s \in [a, b]$. One can check that horizontal lines are indeed horizontal curves under the this definition. For more information from the sub-Riemannian point of view, we refer readers to [5, Chapter 2].

In this paper, we denote by $\mathcal{H}^s_{\mathbb{H}}$ (resp. $\mathcal{H}^s_{\mathbb{R}}$) the s-dimensional Hausdorff measure induced by the Korányi metric (resp. Euclidean metric) and by $\dim^{\mathbb{H}}_{\mathcal{H}}$ (resp. $\dim^{\mathbb{R}}_{\mathcal{H}}$) the Hausdorff dimension of sets induced by Korányi metric (resp. Euclidean metric). In addition, given any set $A \subset \mathbb{H}$, we denote

$$L(A) := \{(a, b, d, \epsilon) \in \mathbb{R} \times (-\sqrt{3}, \sqrt{3}) \times \mathbb{R} \times \mathbb{R} \mid l^{\epsilon}_{(a,b,d)} \subset A\} \tag{2.7}$$

and

$$L(A, c) := \{(a, b, d, \epsilon) \in L(A) \mid l^{\epsilon}_{(a,b,d)} \cap \{x = c\} \neq \emptyset\}. \tag{2.8}$$

Next, we recall a version of a Marstrand-type projection theorem [14, Theorem 1.2]:

Theorem 2.3. *Suppose that* $\gamma : [0, 2\pi) \to \mathbb{R}^3, \theta \mapsto \gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$*. If* $K \subset \mathbb{R}^3$ *is Borel set, then* $\dim^{\mathbb{R}}_{\mathcal{H}} \rho_{\gamma(\theta)}(K) = \min\{\dim^{\mathbb{R}}_{\mathcal{H}} K, 1\}$ *for almost every* $\theta \in [0, 2\pi)$*.*

Here, and in what follows, for any $x \in \mathbb{R}^3 \setminus \{0\}$, $\rho_x : \mathbb{R}^3 \to \mathrm{Span}(x)$ denotes the Euclidean orthogonal projection to the straight line passing through the origin and x.

We also need the following co-area inequality [8, Theorem 1]:

Theorem 2.4. *Let X be an arbitrary metric space, $0 \le \alpha < \infty$ be real numbers and $F \subset X$ be any subset. Then, for any 1-Lipschitz map $f : X \to \mathbb{R}$ we have*

$$\int_{\mathbb{R}}^{*} \mathcal{H}_X^{\alpha}(F \cap f^{-1}(y))\, dy \le \mathcal{H}_X^{\alpha+1}(F). \tag{2.9}$$

Here, $\int_{\mathbb{R}}^{*} g\, dy$ is the upper integral of $g : \mathbb{R} \to [0, +\infty)$. That is

$$\int_{\mathbb{R}}^{*} g(y)\, dy = \inf \int_{\mathbb{R}} h(y)\, dy$$

where the infimum is taken over all measurable functions $h : \mathbb{R} \to [0, +\infty)$ satisfying $0 \le g(y) \le h(y)$ for a.e. $y \in \mathbb{R}$.

3. Proof of Theorem 1.2

Proof of Theorem 1.3. Since every set is contained in a G_δ-set of the same dimension, we may assume E to be G_δ.

Step 1. Properties of $L(E, c)$
First, we need

Claim I: *For every $c \in \mathbb{R}$, $L(E, c)$ is a G_δ set in $\mathbb{R}^4$.*

Proof of Claim I. Recalling (2.7),

$$L(E, c) = \{(a, b, d, \epsilon) \in L(E) \mid l^{\epsilon}_{(a,b,d)} \cap \{x = c\} \neq \emptyset\}.$$

Since E is a G_δ set, we can find a sequence of open sets $\{E_i\}_{i\in\mathbb{N}}$ such that $E_i \supset E_{i+1}$ for each $i \in \mathbb{N}$ and

$$E = \bigcap_{i\in\mathbb{N}} E_i. \tag{3.1}$$

Consider the sets

$$L(E_i, c) = \{(a, b, d, \epsilon) \in L(E_i) \mid (l^{\epsilon}_{(a,b,d)} \cap \{x = c\}) \neq \emptyset\}.$$

By (3.1) it is not hard to see

$$L(E, c) = \bigcap_{i\ge 1} L(E_i, c).$$

We just need to show that $L(E_i, c)$ is open for any $c \in \mathbb{R}$ and $i \in \mathbb{N}$.

Consider an arbitrary quadruple $(a, b, d, \epsilon) \in L(E_i, c)$, i.e.

$$l^{\epsilon}_{(a,b,d)} \cap \{x = c\} \neq \emptyset \quad \text{and} \quad l^{\epsilon}_{(a,b,d)} \subset E_i, \tag{3.2}$$

Thanks to the openness of E_i and the interval $(\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}})$, we deduce that for (a', b', d', ϵ') close enough to (a, b, d, ϵ), we have

$$l^{\epsilon'}_{(a',b',d')} \cap \{x = c\} \neq \emptyset \quad \text{and} \quad l^{\epsilon'}_{(a',b',d')} \subset E_i$$

and hence $(a', b', d', \epsilon') \in L(E_i, c)$, which implies $L(E_i, c)$ is open. □

Furthermore, we have the following

Claim II: *There exists at least one $c_0 \in \mathbb{R}$ satisfying*

$$\mathcal{H}^1_{\mathbb{R}}(\pi_{123}(L(E, c_0))) > 0 \tag{3.3}$$

where π_{123} is the orthogonal projection from $\mathbb{R}^4$ to the subspace spanned by the first three coordinates.

Proof of Claim II. Since E is a Kakeya set in $\mathbb{H}$, by Definition 1.1 and recalling (2.7), we have

$$\pi_2(L(E)) \supset (-\sqrt{3}, \sqrt{3})$$

where π_2 is the orthogonal projection from $\mathbb{R}^4$ to the subspace spanned by the second coordinate, which implies

$$\mathcal{H}^1_{\mathbb{R}}(\pi_2(L(E))) > 0. \tag{3.4}$$

By observing that

$$L(E) = \bigcup_{c \in \mathbb{Q}} L(E, c)$$

and using (3.4), we infer that there exists c_0 such that

$$\mathcal{H}^1_{\mathbb{R}}(\pi_2(L(E, c_0))) > 0. \tag{3.5}$$

Noting that

$$\mathcal{H}^1_{\mathbb{R}}(\pi_{123}(L(E, c))) \geq \mathcal{H}^1_{\mathbb{R}}(\pi_2(L(E, c))), \ \forall c \in \mathbb{R},$$

we conclude the proof. □

We end step 1 with the following

Claim III: *We can find a Borel set $B \subset \pi_{123}(L(E, c_0))$ with*

$$\mathcal{H}^1_{\mathbb{R}}(B) > 0 \tag{3.6}$$

and at least one of the following holds:

$$B \subset \pi_{123}(L(E, c)), \quad \forall c \in [c_0 - \frac{1}{4}, c_0] \tag{3.7}$$

or

$$B \subset \pi_{123}(L(E, c)), \quad \forall c \in [c_0, c_0 + \frac{1}{4}]. \tag{3.8}$$

Proof of Claim III. Using Lemma 2.1(3), we observe that if $l^{\epsilon}_{(a,b,d)} \cap \{x = c_0\} \neq \emptyset$, then either

$$l^{\epsilon}_{(a,b,d)} \cap \{x = c_0 - \frac{1}{4}\} \neq \emptyset,$$

or

$$l^{\epsilon}_{(a,b,d)} \cap \{x = c_0 + \frac{1}{4}\} \neq \emptyset.$$

Hence we have

$$L(E, c_0) \subset [L(E, c_0) \cap L(E, c_0 - \frac{1}{4})] \cup [L(E, c_0) \cap L(E, c_0 + \frac{1}{4})].$$

We conclude

$$\pi_{123}(L(E, c_0)) \subset \pi_{123}(L(E, c_0) \cap L(E, c_0 - \frac{1}{4})) \cup \pi_{123}(L(E, c_0) \cap L(E, c_0 + \frac{1}{4})). \tag{3.9}$$

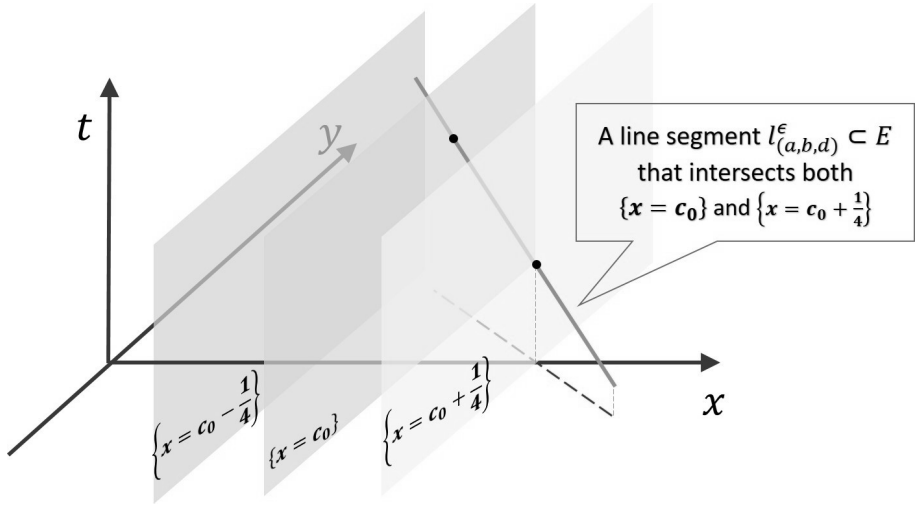

FIGURE 1. A line segment that intersects both $\{x = c_0\}$ and $\{x = c_0 + \frac{1}{4}\}$.

From the above inclusion, we can assume, without loss of generality, that

$$\mathcal{H}^1_{\mathbb{R}}\left(\pi_{123}(L(E,c_0) \cap L(E,c_0+\frac{1}{4}))\right) \geq \frac{1}{2}\mathcal{H}^1_{\mathbb{R}}(L(E,c_0)) > 0$$

where the last inequality results from CLAIM II.

On the other hand, if $(a,b,d,\epsilon) \in L(E,c_0) \cap L(E,c_0+\frac{1}{4})$, then for any $c \in [c_0, c_0+\frac{1}{4}]$, we have $(a,b,d,\epsilon) \in L(E,c)$, which indicates

$$L(E,c_0) \cap L(E,c_0+\frac{1}{4}) \subset L(E,c) \text{ for any } c \in [c_0, c_0+\frac{1}{4}].$$

By CLAIM I, for any $c \in \mathbb{R}$, we know that $L(E,c)$ is a G_δ set and hence $\pi_{123}(L(E,c_0) \cap L(E,c_0+\frac{1}{4}))$ is an analytic set. Hence we can apply Corollary 2 in [6] to choose B to be a closed subset of $\pi_{123}(L(E,c_0) \cap L(E,c_0+\frac{1}{4}))$ with $\mathcal{H}^1_{\mathbb{R}}(B) > 0$. Therefore B satisfies the assumption of the claim. □

CLAIM III enables us to choose $c_0 \in \mathbb{R}$ such that, without loss of generality, there exists a Borel set $B \subset \pi_{123}(L(E,c_0))$ satisfying (3.6), i.e.

$$\mathcal{H}^1_{\mathbb{R}}(B) > 0.$$

and (3.8). Therefore, we infer that

$$\dim^{\mathbb{R}}_{\mathcal{H}}(B) \geq 1. \tag{3.10}$$

Step 2. Establish a duality principle.
Recall the definition of $l_{(a,b,d)}$ in (2.6). For every $c \in [c_0, c_0+1/4]$, we consider $E_c \subset \{x = c\} \cap E \subset \mathbb{H}$ defined by

$$\begin{aligned} E_c :&= \{l_{a,b,d} \cap \{x = c\} \mid (a,b,d) \in B\} \\ &= \{(c, bc+a, -\frac{ac}{2}+d) \mid (a,b,d) \in B\}. \end{aligned} \tag{3.11}$$

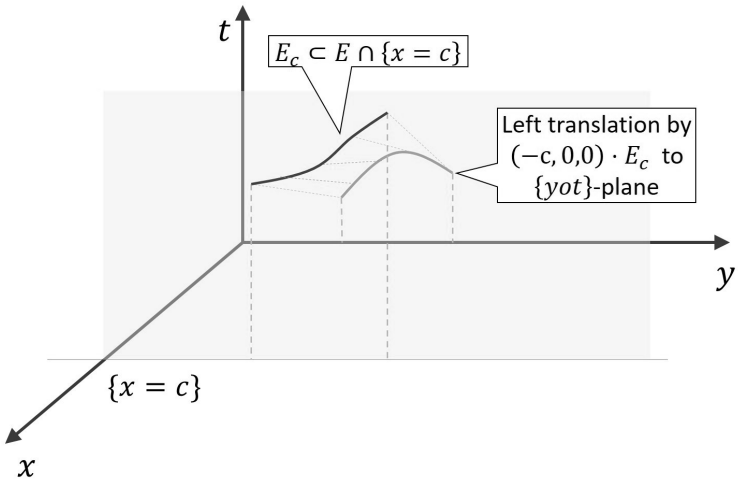

FIGURE 2. Translating E_c to $\{yot\}$-plane

Use left translation $T_{(-c,0,0)}$ to translate E_c to $\{yot\}$-plane, which means $E_c^1 := T_{(-c,0,0)}(E_c)$ lies in $\{yot\}$-plane and has same dimension as E_c. See the above Figure 2.

Recalling (3.11) and the Heisenberg multiplication law (2.1), we deduce that

$$\begin{aligned} E_c^1 &= T_{(-c,0,0)}(E_c) \\ &= \{(-c,0,0)\cdot(c, bc+a, -\frac{ac}{2}+d) \mid (a,b,d)\in B\} \\ &= \{(0, bc+a, -ac-\frac{bc^2}{2}+d) \mid (a,b,d)\in B\}. \end{aligned} \tag{3.12}$$

Notice that the third coordinate of points in E_c^1 expressed in (3.12) takes the form

$$-ac-\frac{bc^2}{2}+d = \left\langle (-c, -\frac{c^2}{2}, 1), (a,b,d)\right\rangle$$

where $\langle , \rangle$ is the Euclidean inner product in $\mathbb{R}^3$.

By considering the t-axis as $\mathbb{R}$ and letting

$$\varphi : \{yot\} \to \mathbb{H},\ (0,y,t)\mapsto (0,0,t), \tag{3.13}$$

we can write

$$\begin{aligned} \varphi(E_c^1) &= \left\{ \left\langle (-c, -\frac{c^2}{2}, 1), (a,b,d)\right\rangle \mid (a,b,d)\in B \right\} \\ &= \left(1+\frac{c^2}{2}\right)\rho_{(-c,-\frac{c^2}{2},1)}(B). \end{aligned} \tag{3.14}$$

Equation (3.14) implies that $\varphi(E_c^1)$ can be viewed as a Euclidean projection of B to the one parameter family of lines $\Gamma = \{\gamma_c : \mathbb{R}\to\mathbb{R}^3 \mid t\mapsto (-ct, -\frac{c^2}{2}t, t), c\in [c_0, c_0+1/4]\}$ up to scalings. Letting $t=1$, we observe that $c\mapsto(-c, -\frac{c^2}{2}, 1)$ forms a part of parabola $\mathcal{H}i$ in $\mathbb{R}^3$. Hence this one parameter family of lines forms part

of a cone C_1 in $\mathbb{R}^3$, i.e.

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = -2yz\}.$$

Moreover, the intersection of Γ and the unit sphere in $\mathbb{R}^3$ is contained in a circle and can be parameterised as

$$\widetilde{\gamma}(c) = \left\{ \frac{2}{2+c^2}(-c, -\frac{c^2}{2}, 1) \mid c \in [c_0, c_0 + 1/4] \right\}.$$

We see the arc $\widetilde{\gamma}$ and the parabola $\mathcal{H}i$ are both conical curves, they can be included in one same cone as Figure 3 depicts.

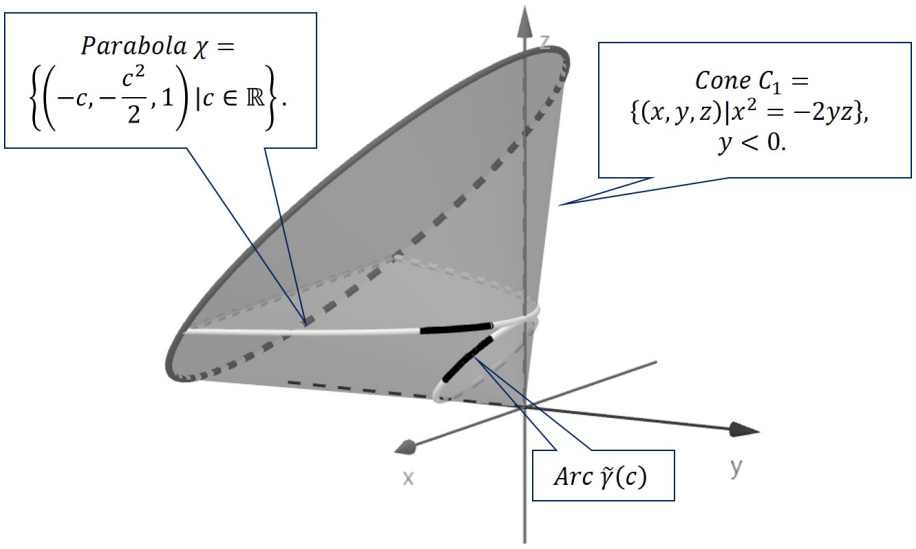

FIGURE 3. Parabola $\mathcal{H}i$ and Arc $\widetilde{\gamma}$ can be included in one same cone

Step 3. Conclusion.
In Theorem 2.3, the family of lines passing through the origin and

$$\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$$

also spans a cone $C_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ in $\mathbb{R}^3$.

We observe that the cone C_2 can be obtained by a rotation $\mathcal{R} : (x, y, z) \mapsto (x, \frac{\sqrt{2}}{2}(y+z), \frac{\sqrt{2}}{2}(z-y))$ acting on C_1 and $\widetilde{\gamma}$ is mapped to an arc of γ, i.e.

$$\begin{aligned}\gamma(\theta(c)) = \mathcal{R} \circ \widetilde{\gamma}(c) &= \frac{2}{2+c^2}\left(-c, \frac{\sqrt{2}}{4}(2-c^2), \frac{\sqrt{2}}{4}(2+c^2)\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{-2\sqrt{2}c}{2+c^2}, \frac{2-c^2}{2+c^2}, 1\right),\end{aligned}$$

for $c \in [c_0, c_0 + \frac{1}{4}]$, where $\theta(c)$ is determined from

$$(\cos(\theta(c)), \sin(\theta(c))) = \left(\frac{-2\sqrt{2}c}{2+c^2}, \frac{2-c^2}{2+c^2} \right).$$

This implies

$$\rho_{(-c,-\frac{c^2}{2},1)}((x,y,z)) = \rho_{\gamma(\theta(c))}(\mathcal{R}(x,y,z)), \quad \forall (x,y,z) \in \mathbb{R}^3. \tag{3.15}$$

By Claim III and (3.10), we know B is Borel and $\dim_{\mathcal{H}}^{\mathbb{R}}(B) \geq 1$. We use (3.15) and apply Theorem 2.3 to the family of lines passing through the origin and $\{\gamma(\theta(c))\}_{c\in[c_0,c_0+1/4]}$ to deduce that

$$\dim_{\mathcal{H}}^{\mathbb{R}}[\rho_{(-c,-\frac{c^2}{2},1)}(B)] = \dim_{\mathcal{H}}^{\mathbb{R}}[\rho_{\gamma(\theta(c))}(\mathcal{R}(B))] = 1 \quad a.e.\ c \in [c_0, c_0 + 1/4]. \tag{3.16}$$

Recalling the definition of φ in (3.13) and according to (2.2), we know φ is 1-Lipschitz with respect to $d_{\mathbb{H}}$ and for any set $A \subset t$-axis,

$$\dim_{\mathcal{H}}^{\mathbb{H}}(A) = 2\dim_{\mathcal{H}}^{\mathbb{R}}(A).$$

Hence for any $c \in [c_0, c_0 + 1/4]$ such that (3.16) holds, combining (3.11), (3.14), (3.16) and the above equality, we conclude

$$\begin{aligned}\dim_{\mathcal{H}}^{\mathbb{H}}(\{x = c\} \cap E) &\geq \dim_{\mathcal{H}}^{\mathbb{H}}(E_c) = \dim_{\mathcal{H}}^{\mathbb{H}}(E_c^1) \geq \dim_{\mathcal{H}}^{\mathbb{H}}(\phi(E_c^1)) \\ &= 2\dim_{\mathcal{H}}^{\mathbb{R}}(\rho_{(-1,\frac{c}{2},c^2)}(B)) \\ &= 2\end{aligned}$$

and for any $0 < \alpha < 2$, we deduce

$$\mathcal{H}_{\mathbb{H}}^{\alpha}(\{x = c\} \cap E) = \infty.$$

By definition of $d_{\mathbb{H}}$, the map $f : (\mathbb{H}, d_{\mathbb{H}}) \to \mathbb{R}, (x,y,t) \to (x,0,0)$ is 1-Lipschitz. Now letting $X = \mathbb{H}$, $Y = [c_0, c_0 + 1/4]$ and $F = E \cap \{(x,y,t) \in \mathbb{H} \mid x \in [c_0, c_0 + 1/4]\}$ in Theorem 2.4, for any $0 < \alpha < 2$, we derive

$$\mathcal{H}_{\mathbb{H}}^{\alpha+1}(F) \geq \int_{[c_0,c_0+1/4]}^{*} \mathcal{H}_{\mathbb{H}}^{\alpha}(F \cap f^{-1}(y))\, dy = \infty,$$

which implies

$$\dim_{\mathcal{H}}^{\mathbb{H}}(E) \geq \dim_{\mathcal{H}}^{\mathbb{H}}(E \cap \{(x,y,t) \in \mathbb{H} \mid x \in [c_0, c_0 + 1/4]\}) = \dim_{\mathcal{H}}^{\mathbb{H}}(F) \geq 3.$$

We finish the proof. □

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