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On the statistics of pairs of logarithms of integers

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With an appendix by Étienne Fouvry

June 14, 2022

Abstract

We study the statistics of pairs of logarithms of positive integers at various scalings, either with trivial weights or with weights given by the Euler function, proving the existence of pair correlation functions. We prove that at the linear scaling, which is not the usual scaling by the inverse of the average gap, the pair correlations exhibit a level repulsion similar to radial distribution functions of fluids. We prove total loss of mass phenomena at superlinear scalings, and constant nonzero asymptotic behaviour at sublinear scalings. The case of Euler weights has applications to the pair correlation of the lengths of common perpendicular geodesic arcs from the maximal Margulis cusp neighborhood to itself in the modular curve $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$. [1]

1 Introduction

When studying the asymptotic distribution of a sequence of finite subsets of $\mathbb{R}$, finer information is sometimes given by the statistics of the spacings between pairs or k-tuples of elements, seen at an appropriate scaling. These problems often arise in quantum chaos, including energy level spacings or clusterings, and in statistical physics, including molecular repulsion or interstitial distribution. A general setting for such a study may be described as follows. Let $\mathscr{F} = ((F_N)_{N\in\mathbb{N}},\ \omega)$ be a nondecreasing sequence of finite subsets F_N of a finite dimensional Euclidean space E, endowed with a *multiplicity function* $\omega : \bigcup_{N\in\mathbb{N}} F_N \to\ ]\,0,+\infty\,[$ (or *weight* function). Note that the standard unfolding technique (see for instance the comments after Theorem 2.1) might not work in order to study the statistics of pairs when the weights are not constant equal to 1. Let $\psi : \mathbb{N} \to\ ]\,0,+\infty\,[$ be a nondecreasing *scaling function.* We define the *pair correlation measure of $\mathscr{F}$ at time N with scaling $\psi(N)$* as the measure on E with finite support

$$\mathscr{R}_N^{\mathscr{F},\psi} = \sum_{x,y\in F_N\,:\,x\neq y} \omega(x)\,\omega(y)\,\Delta_{\psi(N)(y-x)}\ ,$$

where Δ_z denotes the unit Dirac mass at z. Standard pair correlation studies use a specific scaling, that we will introduce later on. When the sequence of measures $(\mathscr{R}_N^{\mathscr{F},\psi})_{N\in\mathbb{N}}$, appropriately renormalized, weak-star converges to a measure $g\,\mathrm{Leb}_E$ absolutely continuous with respect to the Lebesgue measure Leb_E of E, the Radon-Nikodym derivative $g = g_{\mathscr{F},\psi}$

[1] **Keywords:** pair correlation, logarithms of integers, level repulsion, Euler function. **AMS codes:** 11K38, 11J83, 11N37, 53C22.

is called the asymptotic *pair correlation function* of $\mathscr{F}$ *for the scaling* ψ. When $g_{\mathscr{F},\psi}$ vanishes on a neighbourhood of 0 in E, we say that the pair $(\mathscr{F},\psi)$ *exhibits a strong level repulsion*, the standard level repulsion requiring only $g_{\mathscr{F},\psi}$ to vanish at 0.

If the family $\mathscr{F}$ consists of subsets of the unit interval $[0,1]$, then it is customary to use the cardinality of the finite set F_N as the scaling function. See for example [BocZ], where $F_N = \{\frac{p}{q} : p,q \in \mathbb{N}, p \leqslant q, (p,q) = 1, 0 < q \leqslant N\}$ is the set of Farey fractions of order N in $[0,1]$ (without multiplicities, hence $\omega \equiv 1$), so that $\psi(N) = \frac{3N^2}{\pi} + \mathrm{O}(N \ln N)$. Montgomery studied (under the Riemann hypothesis) the pair correlations of the imaginary parts of the zeros (with their multiplicity as zeros) of the Riemann zeta function ζ in the seminal paper [Mon]. The number of zeros $\frac{1}{2} + it$ of ζ with imaginary part t in the interval $[0,N]$ is asymptotic to $\frac{N \ln N}{2\pi}$ as $N \to +\infty$ and the scaling used in [Mon] is, analogously to the unit interval case, the standard one by the inverse of the *average gap* : $\psi(N) = (\frac{N \ln N}{2\pi})/N = \frac{\ln N}{2\pi}$. In this paper, in contrast to the above references as well as for instance [RS], [LS] and [HK], we insist that we will consider pair correlations with arbitrary scaling functions, as it has for instance been done when studying the number variance for the Riemann zeros, see [Ber].

In Sections 2 and 3, we study the pair correlations of the family of the logarithms of positive integers

$$\mathscr{L}_{\mathbb{N}} = \big((L_N = \{\ln n \ : \ 0 < n \leqslant N\})_{N \in \mathbb{N}}, \omega \equiv 1 \big)$$

without multiplicities. In order to simplify the statements in this introduction, we only consider power scalings $\psi : N \mapsto N^\alpha$ for $\alpha \geqslant 0$, and we denote these scaling functions by id^α.

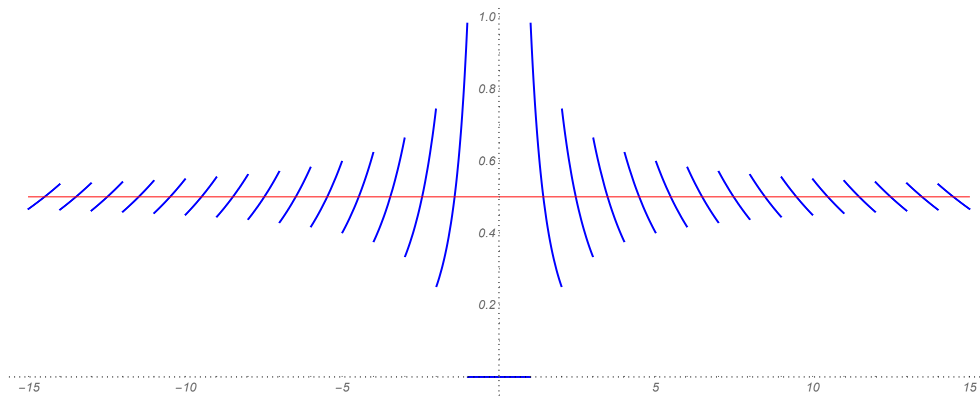
Theorem 1.1 *Let* $\alpha \geqslant 0$. *As* $N \to +\infty$, *the normalized pair correlation measures* $\frac{1}{N^{\max\{2-\alpha,0\}}} \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}},\,\mathrm{id}^\alpha}$ *on* $\mathbb{R}$ *weak-star converge to a measure* $g_{\mathscr{L}_{\mathbb{N}},\,\mathrm{id}^\alpha}\ \mathrm{Leb}_{\mathbb{R}}$ *with pair correlation function given by*

$$g_{\mathscr{L}_{\mathbb{N}},\,\mathrm{id}^\alpha} : t \mapsto \begin{cases} \frac{1}{2}\, e^{-|t|} & \text{if } \alpha = 0 \\ \frac{1}{2} & \text{if } 0 < \alpha < 1 \\ \frac{1}{2\,t^2}\, \lfloor |t| \rfloor \big(\lfloor |t| \rfloor + 1 \big) & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \,. \end{cases}$$

We refer to Theorems 2.1 and 3.1, for more complete versions of Theorem 1.1, with congruence restrictions and with more general scaling functions, as well as for error terms. These error terms, as well as the ones in Theorems 4.1 and 5.1, constitute the main technical parts of this paper.

When $\alpha \leqslant 2$, the renormalisation by $\frac{1}{N^{2-\alpha}}$ in Theorem 1.1 is naturally chosen in order for the pair correlation function to be finite. As the finite set L_N, whose order is N, is contained in the minimal interval $[0, \ln N]$, the average gap in L_N is $\frac{\ln N}{N}$. Scaling by the inverse $\psi(N) = \frac{N}{\ln N}$ of the average gap (as in a particular case of Theorem 3.1), as well as by id^α for $0 < \alpha < 1$ (as in the above statement) gives a nonzero constant pair correlation function (as expected by the standard unfolding technique). As in the above result for $\alpha > 1$ and more generally by Theorem 3.1, if the scaling function ψ grows faster than linearly, then the pair correlation function vanishes : the empirical measures $\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}},\,\psi}$ have a total loss of mass at infinity, regardless of what the renormalisation is (the support of the measure itself converges to infinity). The transition from nonzero constant to zero correlation occurs at linear scalings, where a more exotic pair correlation function appears. Since $g_{\mathscr{L}_{\mathbb{N}},\,\mathrm{id}}$ vanishes on $]-1,1[$, the pair $(\mathscr{L}_{\mathbb{N}},\,\mathrm{id})$ exhibits a strong level repulsion.

The figure below gives the graph of the pair correlation function $g_{\mathscr{L}_{\mathbb{N}},\,\mathrm{id}}$ of $\mathscr{L}_{\mathbb{N}}$ at the linear scaling $\psi = \mathrm{id} : N \mapsto N$ in the interval $[-15, 15]$ compared with the graph of the constant function $\frac{1}{2}$. The graph is similar to certain radial distribution functions in statistical physics, see for example [ZP, Sect. II], [SdH, Fig. 7], [Cha, page 199] or [Boh, page 18].



Instead of the pair correlations, one can study the gaps between consecutive elements in the subsets F_N of the real line or, most often, of the unit interval. Marklof and Strömbergsson [MaS] have computed the gap distribution of the fractional parts of the family $\mathscr{L}_{\mathbb{N}}$ (with a linear scaling, which corresponds to the average gap, and linear renormalisation) and showed that the limiting gap distribution has two jump discontinuities.

In Section 6, we prove that the pair correlation measures of the lengths of the common perpendiculars between the maximal Margulis cusp neighbourhood and itself in the modular curve $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ are (up to a factor 2) the pair correlation measures of the weighted family

$$\mathscr{L}^{\varphi}_{\mathbb{N}} = \big((L_N = \{\ln n \;:\; 0 < n \leqslant N\})_{N\in\mathbb{N}},\; \omega = \varphi\circ\exp\big)$$

of logarithms of integers, with weights given by the Euler function $\varphi : n \mapsto \mathrm{Card}(\mathbb{Z}/n\mathbb{Z})^{\times}$, see Proposition 6.1. See [PS1, PS2] for results on the pair correlation of the lengths of closed geodesics in negatively curved manifolds.

We study the pair correlations of the arithmetically defined family $\mathscr{L}^{\varphi}_{\mathbb{N}}$ in Sections 4 and 5, where we find the pair correlation function without scaling and with linear scaling.

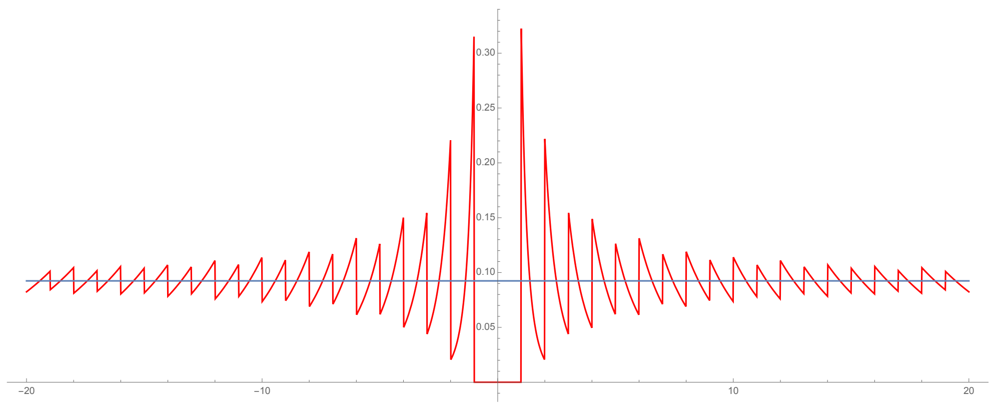
Theorem 1.2 *(1) As $N \to +\infty$, the pair correlation measures $\mathscr{R}_N^{\mathscr{L}^{\varphi}_{\mathbb{N}},1}$ on $\mathbb{R}$, renormalized to be probability measures, weak-star converge to the probability measure $g_{\mathscr{L}^{\varphi}_{\mathbb{N}},1}\,\mathrm{Leb}_{\mathbb{R}}$, with pair correlation function $g_{\mathscr{L}^{\varphi}_{\mathbb{N}},1} : s \mapsto e^{-2\,|s|}$.*

(2) As $N \to +\infty$, the normalized pair correlation measures $\frac{1}{N^3}\,\mathscr{R}_N^{\mathscr{L}^{\varphi}_{\mathbb{N}},\,\mathrm{id}}$ (with linear scaling) on $\mathbb{R}$ weak-star converge to the measure $g_{\mathscr{L}^{\varphi}_{\mathbb{N}},\,\mathrm{id}}\,\mathrm{Leb}_{\mathbb{R}}$, with pair correlation function

$$g_{\mathscr{L}^{\varphi}_{\mathbb{N}},\,\mathrm{id}} : s \mapsto \frac{1}{s^4}\prod_{p\ \mathrm{prime}}\Big(1-\frac{2}{p^2}\Big)\sum_{k=1}^{\lfloor |s|\rfloor} k^3 \prod_{p\ \mathrm{prime},\ p\,|\,k}\Big(1+\frac{1}{p(p^2-2)}\Big)\,. \tag{1}$$

We refer to Theorems 4.1 and 5.1 for more complete versions of Theorem 1.2 with congruence restrictions, and for error terms. When the congruences are nontrivial, the proof of the second claim of Theorem 1.2 uses a generalization of Mirsky's formula (see [Mir]) that is proved in Appendix A by Étienne Fouvry.

The figure below gives the graph of the pair correlation function $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\,\mathrm{id}}$ compared with the graph of the constant function with value $\frac{1}{4}\prod_{p\ \mathrm{prime}}\big(1-\frac{2}{p^2}\big)\big(1+\frac{1}{p^2(p^2-2)}\big)\simeq 0.09239$, which is the limit of the pair correlation function $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\,\mathrm{id}}$ at $\pm\infty$ by Proposition 5.5.



Theorems 4.1 and 5.1 imply pair correlation results for the lengths of the common perpendiculars of cusps neighborhoods in the modular curve and on quotients of the hyperbolic plane by Hecke congruence subgroups of $\mathrm{PSL}(\mathbb{Z})$, see Corollary 6.2 for precise statements.

Further directions. It would be interesting, given a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$, to study the asymptotic of the pair correlation measures of the complex translation lengths $\ell_{\mathbb{C}}(\gamma)$ with absolute value at most N of the elements $\gamma\in\Gamma$, and given a discrete subgroup Γ of a semi-simple connected real Lie group G with finite center and without compact factor, of the Cartan projections $\mu(\gamma)$ with Killing norm at most N of the elements $\gamma\in\Gamma$. See Section 6 for the problem of the asymptotic of the pair correlation measures of common perpendiculars in negative curvature, which will be studied more completely in subsequent works of the authors. See [PP2] for an abstract pair correlation result under exponential growth assumptions on the family $(F_N)_{N\in\mathbb{N}}$ with scaling functions ψ of moderate growths, and [PP3] for a version of this paper on the pair correlations of complex logarithms of lattice points.

When the finite-dimensional Euclidean space E (where the family of finite sets $(F_N)_{N\in\mathbb{N}}$ sits) is replaced by a locally compact metric space (X,d), we may also consider the positive measure on $]0,+\infty[$ with finite support $\mathscr{R}_N^{\mathscr{F},\psi}=\sum_{x,y\in F_N\,:\,x\neq y}\ \omega(x)\,\omega(y)\,\Delta_{\psi(N)d(x,y)}$.

Acknowledgements: The authors thank a lot Etienne Fouvry for his proofs of Lemma 4.2, Proposition 5.5 and Theorem A.1 and for agreeing to contribute the appendix to this paper. We thank the referee for her/his numerous very helpful comments. This research was supported by the French-Finnish CNRS IEA BARP.

Notation. We introduce here some of the notation used throughout the paper.

The pushforward of a measure μ by a mapping f is denoted by $f_*\mu$, and its total mass by $\|\mu\|$. We denote by $\mathrm{sg}:\mathbb{R}\to\mathbb{R}$ the change of sign map $t\mapsto -t$.

For every interval I in $\mathbb{R}$, we denote by Leb_I the Lebesgue measure on I and by $\mathbb{1}_I$ the characteristic function of I. We denote by $\operatorname{BV}(I)$ the vector space of measurable functions $f : I \to \mathbb{R}$ with finite total variation $\operatorname{Var}(f)$. For every $k \in \mathbb{N}$, we denote by $C^k_{\rm c}(I)$ the real vector space of C^k-smooth functions $f : I \to \mathbb{C}$ with compact support in I. We denote by $\|f\|_\infty = \sup_{x \in I} |f(x)|$ the uniform norm of $f \in C^0_{\rm c}(I)$.

In addition to the above, more or less standard, notation, we will use the following indexing sets in Sections 2, 3, 4 and 5. Let us fix throughout the paper $a, b \in \mathbb{N} - \{0\}$ with $a \leqslant b$. For every $N \in \mathbb{N} - \{0\}$, let

$$\begin{aligned} I_N = I_{N,a,b} &= \{(m,n) \in \mathbb{N}^2 \;:\; 0 < m, n \leqslant N,\ m \neq n,\ \ m, n \equiv a \mod b\}\ , \\ I^-_N &= \{(m,n) \in \mathbb{N}^2 \;:\; 0 < m < n \leqslant N,\ \ m, n \equiv a \mod b\} \\ I^+_N &= \{(m,n) \in \mathbb{N}^2 \;:\; 0 < n < m \leqslant N,\ \ m, n \equiv a \mod b\}\ , \end{aligned}$$

so that $I_N = I^-_N \sqcup I^+_N$ is the disjoint union of I^-_N and I^+_N.

We use Landau's O-notation: For every function g of a variable in $\mathbb{N} - \{0\}$, possibly depending on parameters (including a and b), we will denote by $\operatorname{O}(g)$ any function f on $\mathbb{N} - \{0\}$ such that there exists a constant C' depending only on the parameter b and a constant N_0 possibly depending on the parameters such that for every $N \geqslant N_0$, we have $|f(N)| \leqslant C' \,|g(N)|$. We write explicitly $\operatorname{O}_b(g)$ when we want to insist on the possible dependence on the parameter b. We think that obtaining N_0 independent of the parameters A and/or α in Theorems 3.1, 4.1 and 5.1 is not possible.

In the proofs of Lemma 4.2 and of Proposition 5.5, and in the whole Appendix A, we use a stronger version of this notation that is more uniform on parameters. This variant is described in detail in the proof of Lemma 4.2 and the parts of the paper using this notation are indicated in the text.

2 Pair correlations without weights nor scaling

For every $N \in \mathbb{N} - \{0\}$, the (not normalised) *pair correlation measure* of the logarithms of integers congruent to a modulo b at time N, with trivial multiplicities and with trivial scaling function, is

$$\nu_N = \sum_{(m,n) \in I_{N,a,b}} \Delta_{\ln m - \ln n}\ .$$

If we consider the following nondecreasing sequence of finite subsets of $\mathbb{R}$ with trivial multiplicity

$$\mathscr{L}^{a,b}_{\mathbb{N}} = \big((L^{a,b}_N = \{\ln n : 0 < n \leqslant N,\ \ n \equiv a \mod b\})_{N \in \mathbb{N}},\ \omega \equiv 1\big)\ ,$$

then, with the notation of the introduction, we have $\mathscr{L}^{1,1}_{\mathbb{N}} = \mathscr{L}_{\mathbb{N}}$ and $\nu_N = \mathscr{R}_N^{\mathscr{L}^{a,b}_{\mathbb{N}},1}$.

Theorem 2.1 *As $N \to +\infty$, the measures ν_N on $\mathbb{R}$, renormalized to be probability measures, weak-star converge to the measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with Radon-Nikodym derivative the function $g_{\mathscr{L}^{a,b}_{\mathbb{N}},1} : s \mapsto \frac{1}{2}\, e^{-|s|}$:*

$$\frac{\nu_N}{\|\nu_N\|} \;\overset{*}{\rightharpoonup}\; g_{\mathscr{L}^{a,b}_{\mathbb{N}},1}\ \operatorname{Leb}_{\mathbb{R}}\ .$$

Furthermore, for every $f \in C_c^0(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$, we have

$$\frac{\nu_N}{\|\nu_N\|}(f) = \frac{1}{2}\int_{s\in\mathbb{R}} f(s)\, e^{-|s|}\, ds \;+\; \mathrm{O}_b\Big(\frac{\|f\|_\infty + \mathrm{Var}(f)}{N}\Big)\,.$$

When $a = b = 1$, this result implies the case $\alpha = 0$ of Theorem 1.1 in the introduction, with pair correlation function $g_{\mathscr{L}_{\mathbb{N}},1} = g_{\mathscr{L}_{\mathbb{N}}^{1,1},1}$.

The proof below uses at the very beginning the standard unfolding technique (see [Boh, p. 14] and sections 3 and 5 of [MaS]) in order to use the uniform distribution on the unit interval. Note first that, from the point of view of pair correlations, we can study the behaviour of the finite sequences $(\ln\frac{n}{N})_{1\leqslant n\leqslant N}$ on $]-\infty, 0]$ instead of $(\ln n)_{1\leqslant n\leqslant N}$ on $[0,+\infty[$. If A is a Borel subset of $]-\infty,0]$, then

$$\lim_{N\to+\infty}\frac{1}{N}\Big|\big\{n\in\mathbb{N}-\{0\} : n\leqslant N \;\text{ and }\; \ln\frac{n}{N}\in A\big\}\Big| = \lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\Delta_{\ln\frac{n}{N}}(A)$$

$$= \big(\ln_*\lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\Delta_{\frac{n}{N}}\big)(A) = (\ln_*\mathrm{Leb}_{]0,1]})(A) = \int_A e^s\, ds\,.$$

In particular, we have $\frac{n}{N} = \int_{-\infty}^{\ln\frac{n}{N}} e^s\, ds$, so that, using the notation of [MaS], the unfolded sequence of $(\eta_n = \ln\frac{n}{N})_{1\leqslant n\leqslant N}$ is indeed the sequence $(\widetilde{\eta}_n = \frac{n}{N})_{1\leqslant n\leqslant N}$, whose distribution is regular. This elementary remark does not spare us from a more refined study when dealing with congruences and for the error term, and is not appropriate when various weights and scalings are introduced.

Proof of Theorem 2.1. For every $q\in\mathbb{N}-\{0\}$ with $q\equiv a \mod b$, let $q'\in\mathbb{N}$ be such that $q = a+q'b$ and

$$J_q = \{p\in\mathbb{N} : 0<p<q,\ p\equiv a \mod b\} = \{a+kb : 0\leqslant k<q'\}\,. \tag{2}$$

Let

$$\omega_q = \sum_{p\in J_q}\Delta_{\frac{p}{q}}\ ,$$

which is a finitely supported measure on $[0,1]$, with total mass $\|\omega_q\| = q'$. When $q'\neq 0$, we hence have $\|\omega_q\| = \frac{q}{b} + \mathrm{O}(1)$ and $\frac{1}{\|\omega_q\|} = \frac{b}{q} + \mathrm{O}_b(\frac{1}{q^2})$. When $q'\neq 0$, we denote by $\overline{\omega_q} = \frac{\omega_q}{\|\omega_q\|}$ the renormalisation of ω_q to a probability measure on $[0,1]$. By well known Riemann sum arguments, we have, as $q\to+\infty$,

$$\overline{\omega_q} \;\overset{*}{\rightharpoonup}\; \mathrm{Leb}_{[0,1]}\ .$$

Let $f\in\mathrm{BV}([0,1])$, and note that f is bounded, with $\|f\|_\infty\leqslant|f(0)|+\mathrm{Var}(f)$. Denoting by M_k and m_k the maximum and minimum respectively of f on $[\frac{a+kb}{q},\frac{a+(k+1)b}{q}]$ for $0\leqslant k<q'$, we have

$$\Big|\int_0^1 f(t)\,dt - \frac{b}{q}\,\omega_q(f)\Big| = \Big|\int_0^1 f(t)\,dt - \sum_{p\in J_q}\frac{b}{q}\,f(\frac{p}{q})\Big|$$

$$\leqslant \Big|\int_0^{\frac{a}{q}} f(t)\,dt\Big| + \sum_{k=0}^{q'-1}\Big|\int_{\frac{a+kb}{q}}^{\frac{a+(k+1)b}{q}} f(t)dt - \frac{b}{q}\,f(\frac{a+kb}{q})\Big|$$

$$\leqslant \frac{b}{q}\,\|f\|_\infty + \sum_{k=0}^{q'-1}\frac{b}{q}\,(M_k-m_k) \leqslant (\|f\|_\infty+\mathrm{Var}(f))\,\frac{b}{q}\,.$$

When $q' \neq 0$, since $|\,\omega_q(f)\,| \leqslant \|\omega_q\|\;\|f\|_\infty = \mathrm{O}(q\,\|f\|_\infty)$, we hence have

$$\begin{aligned}\overline{\omega_q}(f) &= \int_0^1 f(t)\,dt - \int_0^1 f(t)\,dt + \frac{b}{q}\,\omega_q(f) + \mathrm{O}\,\big(\frac{1}{q^2}\big)\,\omega_q(f)\\ &= \int_0^1 f(t)\,dt + \mathrm{O}\,\Big(\frac{\|f\|_\infty + \mathrm{Var}(f)}{q}\Big)\;.\end{aligned}$$

For every $N \in \mathbb{N} - \{0\}$, with $N \geqslant a + b$, let us define

$$\mu_N^- = \sum_{(m,\,n)\in I_N^-} \Delta_{\frac{m}{n}} \;\;=\;\; \sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \omega_q\;,$$

which is a finitely supported measure on $[0,1]$. Its total mass is equal to

$$\|\mu_N^-\| = \sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \|\omega_q\| = \sum_{0\leqslant q'\leqslant\lfloor\frac{N-a}{b}\rfloor} (q' + \mathrm{O}(1)) = \frac{N^2}{2\,b^2} + \mathrm{O}(N)\;.$$

Hence $\frac{1}{\|\mu_N^-\|} = \frac{2\,b^2}{N^2} + \mathrm{O}(\frac{1}{N^3})$. For $f \in \mathrm{BV}([0,1])$, we have (taking $\|\omega_q\|\;\overline{\omega_q}(f) = 0$ if $q = a$)

$$\begin{aligned}\frac{\mu_N^-(f)}{\|\mu_N^-\|} &= \frac{1}{\|\mu_N^-\|}\sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \|\omega_q\|\;\overline{\omega_q}(f)\\ &= \int_0^1 f(t)\,dt + \frac{1}{\|\mu_N^-\|}\sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \mathrm{O}\,\big(\|f\|_\infty + \mathrm{Var}(f)\big)\\ &= \int_0^1 f(t)\,dt + \mathrm{O}\,\Big(\frac{\|f\|_\infty + \mathrm{Var}(f)}{N}\Big)\;.\end{aligned}$$

Notice that ln is an increasing homeomorphism from $]0,1]$ to $]-\infty,0]$. For every element $N \in \mathbb{N} - \{0\}$, let us define

$$\nu_N^\pm = \sum_{(m,\,n)\in I_N^\pm} \Delta_{\ln\frac{m}{n}}\;,$$

so that $\nu_N^- = \ln_*\mu_N^- = \nu_N\;|_{]-\infty,0]}$, and $\|\nu_N^-\| = \|\mu_N^-\|$. We have, for every $f \in BV(]-\infty,0])$,

$$\begin{aligned}\frac{\nu_N^-(f)}{\|\nu_N^-\|} = \frac{\mu_N^-(f\circ\ln)}{\|\mu_N^-\|} &= \int_0^1 f\circ\ln(t)\,dt + \mathrm{O}\,\Big(\frac{\|f\circ\ln\|_\infty + \mathrm{Var}(f\circ\ln)}{N}\Big)\\ &= \int_{-\infty}^0 f(s)\,e^s\,ds + \mathrm{O}\,\Big(\frac{\|f\|_\infty + \mathrm{Var}(f)}{N}\Big)\;.\end{aligned}$$

Since $\nu_N = \nu_N^- + \nu_N^+$, since $\nu_N^+ = \mathrm{sg}_*\,\nu_N^-$, we have $\|\nu_N^\pm\| = \frac{1}{2}\;\|\nu_N\|$ and this proves the second assertion of Theorem 2.1. The first assertion follows by the density of $C_c^1(\mathbb{R})$ in $C_c^0(\mathbb{R})$ for the uniform norm. □

Let us give some numerical illustrations of Theorem 2.1 when $a = b = 1$. For every $N \in \mathbb{N} - \{0\}$, let

$$\mathscr{D}_N : s \mapsto \frac{\mathrm{Card}\{(p,q) \in I_N : \ln\frac{p}{q} \leqslant s\}}{\mathrm{Card}\; I_N}\;,$$

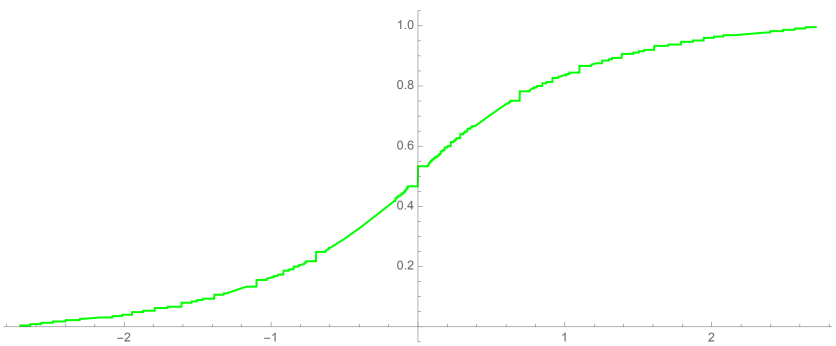
which is the cumulative distribution function at time N of the differences of pairs of logarithms of integers, that is, for all $s, s' \in \mathbb{R}$ with $s < s'$, we have

$$\frac{\nu_N}{\|\nu_N\|}\,(\,]s, s'\,]) = \mathscr{D}_N(s') - \mathscr{D}_N(s)\ .$$

The first assertion of Theorem 2.1 says that as $N \to +\infty$ the function $\mathscr{D}_N$ converges pointwise to the C^1 (but not C^2) function

$$\mathscr{D} : s \mapsto \begin{cases} \frac{1}{2}\, e^{s} & \text{if } s \leqslant 0 \\ 1 - \frac{1}{2}\, e^{-s} & \text{if } s \geqslant 0 \end{cases}$$

(with derivative $\mathscr{D}' = g_{\mathscr{L}_{\mathbb{N}},1}$), which is the asymptotic cumulative distribution function of the differences of pairs of logarithms of integers. This is illustrated by the figure below, which shows $\mathscr{D}_{15}$ in green.



3 Pair correlations without weights and with scaling

In this section, we study the pair correlations of logarithms of integers at various scalings, now assumed to converge to $+\infty$. We fix two positive functions, respectively denoted by $\psi : \mathbb{N} - \{0\} \to\ ]0, +\infty[$ and $\psi' : \mathbb{N} - \{0\} \to\ ]0, +\infty[$, with ψ' assumed to have a positive lower bound, which will give the scaling factors on the difference of pairs of logarithms and the renormalizing factors on their distribution, respectively.

For every $N \in \mathbb{N} - \{0\}$, the (not normalised) *pair correlation measure* of the logarithms of integers congruent to a modulo b at time N *with trivial multiplicities and with scaling* $\psi(N)$ is the (Borel, positive) measure with finite support in $\mathbb{R}$ defined by

$$\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi} = \sum_{(m,n)\in I_{N,a,b}} \Delta_{\psi(N)(\ln m - \ln n)}\ ,$$

and the normalized one is $\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$.

Theorem 3.1 *Assume that the nondecreasing positive function ψ satisfies $\lim_{+\infty} \psi = +\infty$ and $\lim_{N\to+\infty} \frac{\psi(N)}{N} = \lambda_\psi \in [0, +\infty]$. As $N \to +\infty$, the measures $\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$ on $\mathbb{R}$, normalized by $\psi'(N)$ as given below, weak-star converge to a measure $g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\ \mathrm{Leb}_{\mathbb{R}}$ absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$,*

$$\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi} \quad \overset{*}{\rightharpoonup} \quad g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\ \mathrm{Leb}_{\mathbb{R}}\ ,$$

with Radon-Nikodym derivative the function

$$g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi} : t \mapsto \begin{cases} 0 & \text{if } \lambda_\psi = +\infty, \text{ for any } \psi' \text{ as above ,} \\ \frac{1}{2\,b^2} & \text{if } \lambda_\psi = 0 \text{ and } \psi'(N) = \frac{N^2}{\psi(N)} \text{ ,} \\ \frac{1}{2\,t^2}\, \big\lfloor \frac{|t|}{b\,\lambda_\psi} \big\rfloor \Big(\big\lfloor \frac{|t|}{b\,\lambda_\psi} \big\rfloor + 1 \Big) & \text{if } \lambda_\psi \neq 0, +\infty \text{ and } \psi'(N) = \psi(N) \text{ .} \end{cases} \tag{3}$$

Furthermore, if $\lambda_\psi \neq 0, +\infty$ and $\psi'(N) = \psi(N)$, for every $f \in C^1_{\rm c}(I)$ with support contained in $[-A, A]$ where $A \geqslant$, we have

$$\frac{1}{\psi'(N)}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}(f)$$
$$= \int_{s\in\mathbb{R}} f(s)\, g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}(s)\, ds + \; \mathrm{O}_b\Big(\frac{A^3}{\lambda_\psi^5}\, \|f\|_\infty \big(\big|\lambda_\psi - \frac{\psi(N)}{N}\big| + \frac{A}{N} \big) + \|f'\|_\infty (1 + \lambda_\psi^{-3}) \frac{A^3}{N} \Big) \,.$$

The pair correlation function $g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$ depends on b but it is independent of a. The above result shows in particular that renormalizing to probability measures (taking $\psi'(N) = N^2 - N$) is inappropriate, as the limiting measure would always be 0.

When $\alpha > 0$, $a = b = 1$ and $\psi = \mathrm{id}^\alpha : N \to N^\alpha$, the measure $\frac{1}{\psi'(N)}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$ corresponds to the one denoted by $\frac{1}{N^{\max\{2-\alpha,0\}}}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\mathrm{id}^\alpha}$ in the introduction if $\alpha \leqslant 2$. If $\alpha > 2$, then $\lambda_\psi = +\infty$, and the constant renormalizing function $\psi' = 1$ satisfies the hypotheses of the first case of Equation (3), so that the measure $\frac{1}{\psi'(N)}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$ also corresponds to the one denoted by $\frac{1}{N^{\max\{2-\alpha,0\}}}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\mathrm{id}^\alpha}$ in the introduction. The above result thus implies the cases $\alpha > 0$ of Theorem 1.1 in the introduction, as well as the comment about the scaling by the inverse of the average gap $\psi(N) = \frac{N}{\ln N}$, for which $\lambda_\psi = 0$.

The fact that $g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$ vanishes when $\lambda_\psi = +\infty$ means that the sequence of measures $\big(\frac{1}{\psi'(N)}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\big)_{N\in\mathbb{N}}$ on $\mathbb{R}$ has a total loss of mass at infinity. For error terms when $\lambda_\psi = +\infty$ and $\lambda_\psi = 0$, see respectively Equation (7) and Equation (10).

Proof. Note that the change of variables $(m, n) \mapsto (n, m)$ in I_N proves that we have $\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\,|_{]-\infty,0]} = \mathrm{sg}_*\big(\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\,|_{[0,+\infty[}\big)$. We will thus only study the convergence of the measures $\frac{1}{\psi'(N)}\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$ on $[0, +\infty[$, and deduce the global result by the symmetry of $g_{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}$.

For every $N \in \mathbb{N} - \{0\}$ and for every $p \in \mathbb{N}$ with $p \equiv 0 \mod b$ and $0 < p < N$, let $N_p = \lfloor \frac{N-p-a}{b} \rfloor$, let

$$J_{p,\,N} = \{q \in \mathbb{N} : 1 \leqslant q \leqslant N - p,\ q \equiv a \mod b\} = \{a + kb : 0 \leqslant k \leqslant N_p\} \;, \tag{4}$$

and let

$$\omega_{p,\,N} = \sum_{q\in J_{p,\,N}} \Delta_{\psi(N)\frac{p}{q}} \quad \text{and} \quad \mu_N^+ = \sum_{0<p<N,\ p\equiv 0 \bmod b} \omega_{p,\,N} \;.$$

Then $\omega_{p,\,N}$ is a measure on $[0, +\infty[$, with finite support contained in $[\frac{\psi(N)}{N-p} p, \psi(N)p]$. The support of the measure μ_N^+ on $]0, +\infty[$ is contained in $[\frac{\psi(N)}{N}, \psi(N)N]$. The motivation for the definition of the measure μ_N^+ comes from the following lemma.

Lemma 3.2 *For every $A > 0$ and for every $f \in C^1_{\rm c}(\mathbb{R})$ with compact support contained in $[0, A]$, we have, as $N \to +\infty$,*

$$\big|\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}(f) - \mu_N^+(f)\,\big| = {\rm O}\Big(A^3\; \|f'\|_\infty\; \big(\frac{N}{\psi(N)}\big)^2\Big)\;.$$

In particular, if $\frac{1}{\psi'(N)}\big(\frac{N}{\psi(N)}\big)^2$ tends to 0 as $N \to +\infty$, the measures $\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\,|_{[0,+\infty]}$ and $\frac{1}{\psi'(N)}\,\mu_N^+$ on $[0, +\infty]$ are asymptotic for the weak-star convergence of measures on $[0, +\infty]$, and we will study the weak-star convergence of the latter one.

Proof. By the change of variable $(p, q) \mapsto (m = p + q, n = q)$, we have

$$\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\,|_{[0,+\infty]} = \sum_{(m,\,n)\in I_N^+} \Delta_{\psi(N)\,\ln\frac{m}{n}} = \sum_{\substack{0<q\leqslant N-p,\; q\equiv a \mod b\\ 0<p<N,\; p\equiv 0 \mod b}} \Delta_{\psi(N)\,\ln(1+\frac{p}{q})}\;.$$

By definition, we have

$$\mu_N^+ = \sum_{\substack{0<q\leqslant N-p,\; q\equiv a \mod b\\ 0<p<N,\; p\equiv 0 \mod b}} \Delta_{\psi(N)\,\frac{p}{q}}\;.$$

Since the support of f is contained in $[0, A]$, if a pair (p, q) occurs in the index of the sum defining either $\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}(f)$ or $\mu_N^+(f)$ with nonzero summand, then $\psi(N)\ln(1 + \frac{p}{q}) \leqslant A$. This implies that $\frac{p}{q} = {\rm O}\big(\frac{A}{\psi(N)}\big)$ since $\lim\limits_{+\infty} \psi = +\infty$, and that $p = {\rm O}\big(\frac{A\,N}{\psi(N)}\big)$ since $q \leqslant N$. For all $x, y \in [0, +\infty[$, we have

$$|\Delta_x(f) - \Delta_y(f)| = |f(x) - f(y)| \leqslant \|f'\|_\infty |x - y|\;.$$

Recall that $|\ln(1 + t) - t| = {\rm O}(t^2)$ as $t \to 0$. Hence, by a uniform majoration of the terms of the sum below,

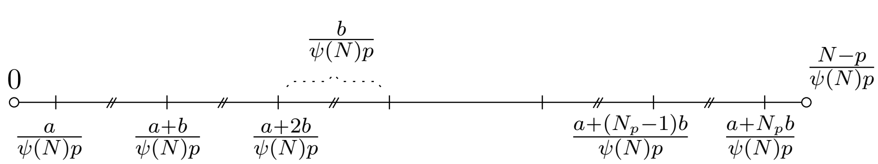
$$\big|\, \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}(f) - \mu_N^+(f)\,\big| \leqslant \sum_{\substack{1\leqslant q\leqslant N\\ 1\leqslant p\leqslant {\rm O}(\frac{A\,N}{\psi(N)})}} \|f'\|_\infty\; \psi(N)\;\; {\rm O}\Big(\big(\frac{p}{q}\big)^2\Big) = {\rm O}\Big(A^3\; \|f'\|_\infty\; \big(\frac{N}{\psi(N)}\big)^2\Big)\;. \quad \square$$

Let us now study the convergence properties of the (renormalized) measures $\omega_{p,\,N}$ and of their sums μ_N^+ as $N \to +\infty$.

Let $\iota : \,]0, +\infty[\; \to\;]0, +\infty[$ be the involutive diffeomorphism $t \mapsto \frac{1}{t}$. We have

$$\iota_*\omega_{p,\,N} = \sum_{q\,\in J_{p,\,N}} \Delta_{\frac{q}{\psi(N)p}}\;.$$

If $N_p \geqslant 0$, as q varies in $J_{p,\,N}$, the above Dirac masses are taken on the distribution of points given by the following picture.



Hence if $N_p \geqslant 0$, as in the proof of Theorem 2.1, for every C^1 function $f :]0, +\infty[\to \mathbb{R}$ with compact support, we have

$$\Big| \int_0^{\frac{N-p}{\psi(N)p}} f(t)\,dt - \frac{b}{\psi(N)p}\,\iota_*\omega_{p,\,N}(f) \Big| = \Big| \int_0^{\frac{N-p}{\psi(N)p}} f(t)\,dt - \sum_{q\,\in J_{p,\,N}} \frac{b}{\psi(N)p}\, f\big(\frac{q}{\psi(N)p}\big) \Big|$$

$$\leqslant \Big| \int_0^{\frac{a}{\psi(N)p}} f(t)\,dt \Big| + \sum_{k=0}^{N_p-1} \Big| \int_{\frac{a+kb}{\psi(N)p}}^{\frac{a+(k+1)b}{\psi(N)p}} f(t)dt - \frac{b}{\psi(N)p}\, f\Big(\frac{a+kb}{\psi(N)p}\Big) \Big| + \Big| \int_{\frac{a+N_pb}{\psi(N)p}}^{\frac{N-p}{\psi(N)p}} f(t)\,dt \Big|$$

$$\leqslant \big(2\,\|f\,|_{]0,\frac{N-p}{\psi(N)p}]}\,\|_\infty + \mathrm{Var}(f\,|_{]0,\frac{N-p}{\psi(N)p}]})\big)\,\frac{b}{\psi(N)p}\;.$$

If $N_p < 0$, then $N - p < a \leqslant b$ and $J_{p,N} = \varnothing$, hence $\omega_{p,N}$. Therefore

$$\Big| \int_0^{\frac{N-p}{\psi(N)p}} f(t)\,dt - \frac{b}{\psi(N)p}\,\iota_*\omega_{p,\,N}(f) \Big| \leqslant \|f\,|_{]0,\frac{N-p}{\psi(N)p}]}\,\|_\infty \frac{N-p}{\psi(N)p} \leqslant \|f\,|_{]0,\frac{N-p}{\psi(N)p}]}\,\|_\infty \frac{b}{\psi(N)p}\;,$$

and the above majoration when $N_p \geqslant 0$ is still valid.

Hence for every C^1 function $f :]0, +\infty[\to \mathbb{R}$ with compact support, since ι is a diffeomorphism, we have

$$\begin{aligned}\omega_{p,\,N}(f) &= \frac{\psi(N)p}{b} \int_0^{\frac{N-p}{\psi(N)p}} f\circ\iota(s)\;ds + \mathrm{O}\left(\big\|f\circ\iota\big|_{]0,\frac{N-p}{\psi(N)p}]}\,\big\|_\infty + \mathrm{Var}(f\circ\iota\big|_{]0,\frac{N-p}{\psi(N)p}]})\right)\\ &= \frac{\psi(N)p}{b} \int_{\frac{\psi(N)p}{N-p}}^{+\infty} f(t)\;\frac{dt}{t^2} + \mathrm{O}\left(\big\|f\,|_{[\frac{\psi(N)p}{N-p},+\infty[}\,\big\|_\infty + \mathrm{Var}(f\,|_{[\frac{\psi(N)p}{N-p},+\infty[})\right)\;.\end{aligned}$$

For every $t > 0$, let

$$\theta_N(t) = \frac{1}{t^2} \sum_{\substack{0<p<N\\ p\equiv 0 \mod b}} \frac{p}{b}\;\mathbb{1}_{[\frac{\psi(N)p}{N-p},+\infty[}(t)\,.$$

Then

$$\begin{aligned}\theta_N(t) &= \frac{1}{t^2} \sum_{0<k<N/b} k\;\mathbb{1}_{[\frac{\psi(N)bk}{N-bk},+\infty[}(t) = \frac{1}{t^2} \sum_{0<k\leqslant \frac{tN}{b(\psi(N)+t)}} k\\ &= \frac{1}{2\,t^2}\Big\lfloor \frac{tN}{b(\psi(N)+t)}\Big\rfloor\Big(\Big\lfloor \frac{tN}{b(\psi(N)+t)}\Big\rfloor + 1\Big)\;.\end{aligned} \tag{5}$$

Let $\theta_N(0) = 0$. In particular, for every $t \geqslant 0$, we have $\theta_N(t) = 0$ if and only if $t \in [0, \frac{b\,\psi(N)}{N-b}[$.

Thus, if the support of f is contained in the interval $[0, A]$, since $\frac{\psi(N)p}{N-p} \leqslant A$ if and only if $p \leqslant \frac{AN}{\psi(N)+A}$, we have,

$$\begin{aligned}\mu_N^+(f) &= \sum_{0<p<N,\;p\equiv 0 \mod b} \omega_{p,\,N}(f)\\ &= \psi(N) \int_0^{+\infty} f(t)\Big(\sum_{\substack{0<p<N\\ p\equiv 0 \mod b}} \frac{p}{b}\;\mathbb{1}_{[\frac{\psi(N)p}{N-p},+\infty[}(t)\Big)\,\frac{dt}{t^2} + \mathrm{O}\left(\big(\|f\|_\infty + \mathrm{Var}(f)\big)\frac{AN}{\psi(N)}\right)\\ &= \psi(N) \int_0^{+\infty} f(t)\;\theta_N(t)\;dt + \mathrm{O}\left(\big(\|f\|_\infty + \mathrm{Var}(f)\big)\frac{AN}{\psi(N)}\right)\;.\end{aligned} \tag{6}$$

Case 1. Assume first that $\lambda_\psi = +\infty$, that is, that $\lim_{N\to+\infty} \frac{N}{\psi(N)} = 0$. Then for every $A \geqslant 1$, if N is large enough, then for every $t \in [0, A]$, we have $\theta_N(t) = 0$. Thus, whatever the normalizing function ψ' is (with a positive lower bound, by the assumption at the beginning of Section 3), we have a total loss of mass at infinity :

$$\frac{1}{\psi'(N)}\,\mu_N^+ \overset{*}{\rightharpoonup} 0\ .$$

More precisely, for every C^1 function $f :]0, +\infty[\to \mathbb{R}$ with compact support contained in $[0, A]$, we have

$$\frac{1}{\psi'(N)}\,\mu_N^+(f) = \mathrm{O}\Big(\big(\|f\|_\infty + \mathrm{Var}(f)\big)\frac{AN}{\psi(N)\psi'(N)}\Big)\ .$$

Note that $\mathrm{Var}(f) = \int_0^A |f'(t)|\,dt \leqslant A\,\|f'\|_\infty$, and that $\frac{N}{\psi(N)} \leqslant \frac{1}{A^2}$ for N large enough. By the comment following Lemma 3.2 and the fact that ψ' has a positive lower bound, we have

$$\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{L}_\mathbb{N},\psi} \overset{*}{\rightharpoonup} 0\ ,$$

thus proving Equation (3) under the assumptions of Case 1, and

$$\frac{1}{\psi'(N)}\,\mathscr{R}_N^{\mathscr{L}_\mathbb{N}^{a,b},\psi}(f) = \mathrm{O}\Big(\big(\|f\|_\infty + \|f'\|_\infty\big)\frac{A\,N}{\psi(N)\psi'(N)}\Big)\ . \tag{7}$$

Case 2. Now assume that $\lambda_\psi = 0$, that is, that $\lim_{N\to+\infty} \frac{\psi(N)}{N} = 0$. By Equation (5), if we have $N > b$ and $t \geqslant \frac{b\psi(N)}{N-b}$, then $\frac{tN}{b(\psi(N)+t)} \geqslant 1$ and

$$\theta_N(t) = \frac{1}{2\,t^2}\Big(\frac{tN}{b(\psi(N)+t)} + \mathrm{O}(1)\Big)^2 = \frac{N^2}{2\,b^2\psi(N)^2}\Big(1 + \mathrm{O}\big(\frac{t}{\psi(N)}\big) + \mathrm{O}\big(\frac{\psi(N)}{tN}\big)\Big)^2\ ,$$

therefore

$$\frac{\psi(N)^2}{N^2}\theta_N(t) = \frac{1}{2\,b^2} + \mathrm{O}\big(\frac{t}{\psi(N)}\big) + \mathrm{O}\big(\frac{\psi(N)}{tN}\big)\ . \tag{8}$$

Since θ_N vanishes on $[0, \frac{b\,\psi(N)}{N-b}[$, this proves that $\frac{\psi(N)^2}{N^2}\ \theta_N$ is bounded on any compact subset of $[0, +\infty[$, and pointwise converges to the constant function $\frac{1}{2\,b^2}$. Hence by the Lebesgue dominated convergence theorem, we have

$$\frac{\psi(N)}{N^2}\,\mu_N^+ \overset{*}{\rightharpoonup} \frac{1}{2\,b^2}\ \mathrm{Leb}_{[0,+\infty[}\ . \tag{9}$$

More precisely, for every $A \geqslant 3$, for every C^1 function $f :]0, +\infty[\to \mathbb{R}$ with compact support contained in $[0, A]$, by Equations (6) and (8) and since $\psi(N) \leqslant N$ for N large enough, we have

$$\begin{aligned}\frac{\psi(N)}{N^2}\;\mu_N^+(f) &= \frac{1}{2\,b^2}\int_{\frac{b\,\psi(N)}{N-b}}^{+\infty} f(t)\;dt + \mathrm{O}\Big(\frac{1}{\psi(N)}\int_{\frac{b\,\psi(N)}{N-b}}^{A} t\,|f(t)|\;dt\Big)\\ &\quad+\mathrm{O}\Big(\frac{\psi(N)}{N}\int_{\frac{b\,\psi(N)}{N-b}}^{A}\frac{1}{t}\,|f(t)|\;dt\Big)+\mathrm{O}\Big(\big(\|f\|_\infty+\mathrm{Var}(f)\big)\frac{A}{N}\Big)\\ &=\frac{1}{2\,b^2}\int_0^{+\infty} f(t)\;dt+\mathrm{O}\Big(\frac{\psi(N)}{N}\|f\|_\infty\Big)+\mathrm{O}\Big(\frac{A^2}{\psi(N)}\|f\|_\infty\Big)\\ &\quad+\mathrm{O}\Big(\frac{\psi(N)}{N}\,\|f\|_\infty\big(\ln A-\ln\frac{b\,\psi(N)}{N-b}\big)\Big)+\mathrm{O}\Big(\big(\|f\|_\infty+\|f'\|_\infty\big)\frac{A^2}{N}\Big)\\ &=\frac{1}{2\,b^2}\int_0^{+\infty} f(t)\;dt+\mathrm{O}\Big(\|f\|_\infty\big(\frac{\psi(N)\ln A}{N}+\frac{A^2}{\psi(N)}-\frac{\psi(N)}{N}\ln\frac{\psi(N)}{N}\big)\Big)\\ &\quad+\mathrm{O}\Big(\|f'\|_\infty\frac{A^2}{N}\Big)\;.\end{aligned}$$

Let $\psi'(N)=\frac{N^2}{\psi(N)}$ and note that $\frac{1}{\psi'(N)}\big(\frac{N}{\psi(N)}\big)^2=\frac{1}{\psi(N)}$ tends to 0 as $N\to+\infty$. By Equation (9) and by the comment following Lemma 3.2, we hence have

$$\frac{1}{\psi'(N)}\;\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}\;\overset{*}{\rightharpoonup}\;\frac{1}{2\,b^2}\;\mathrm{Leb}_{\mathbb{R}}\;,$$

thus proving Equation (3) under the assumptions of Case 2. Furthermore, for every C^1 function $f:\mathbb{R}\to\mathbb{R}$ with compact support contained in $[-A,A]$ where $A\geqslant 1$, we have

$$\begin{aligned}\frac{1}{\psi'(N)}\;\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b},\psi}(f)&=\frac{1}{2\,b^2}\int_{\mathbb{R}} f(t)\;dt+\mathrm{O}\Big(\|f\|_\infty\big(\frac{\psi(N)\ln A}{N}+\frac{A^2}{\psi(N)}-\frac{\psi(N)}{N}\ln\frac{\psi(N)}{N}\big)\Big)\\ &\quad+\mathrm{O}\Big(\|f'\|_\infty\frac{A^3}{\psi(N)}\Big)\;.\end{aligned}\tag{10}$$

Case 3. Let us finally assume that $\lim_{N\to+\infty}\frac{\psi(N)}{N}=\lambda_\psi$ belongs to $]0,+\infty[$. Let us consider the map $\theta_\infty:[0,+\infty[\to\mathbb{R}$ defined by $\theta_\infty(0)=0$ and on $]0,+\infty[$ by

$$t\mapsto\frac{1}{t^2}\sum_{k=1}^{\infty}\;k\;\mathbb{1}_{[b\lambda_\psi k,+\infty[}(t)=\frac{1}{2\,t^2}\,\Big\lfloor\frac{t}{b\,\lambda_\psi}\Big\rfloor\big(\Big\lfloor\frac{t}{b\,\lambda_\psi}\Big\rfloor+1\big)\;.$$

It vanishes on $[0,b\,\lambda_\psi[$, is uniformly bounded, tends to $\frac{1}{2\,b^2\,\lambda_\psi^2}$ as $t\to+\infty$, and is piecewise continous, with discontinuities at $b\,\lambda_\psi\mathbb{N}-\{0\}$. See the first picture in the introduction when $a=b=\lambda_\psi=1$.

By Equation (5), the sequence of uniformly bounded maps $(\theta_N)_{N\in\mathbb{N}}$ converges almost everywhere to θ_∞ (more precisely, it converges at least at every point of $[0,+\infty[\,-\,b\,\lambda_\psi\mathbb{N}$). Hence by Equation (6) and by the Lebesgue dominated convergence theorem, we have

$$\frac{1}{\psi(N)}\;\mu_N^+\;\overset{*}{\rightharpoonup}\;\theta_\infty\;\mathrm{Leb}_{[0,+\infty[}\;.$$

Let $A\geqslant 1$, $k\in\mathbb{N}-\{0\}$ and N large enough so that $\frac{\psi(N)}{N}\geqslant\frac{\lambda_\psi}{2}$. Note that $b\lambda_\psi k\leqslant A$ implies that $k\leqslant\frac{A}{b\lambda_\psi}\leqslant\frac{2A}{b\lambda_\psi}$, and that $\frac{\psi(N)bk}{N-bk}\leqslant A$ implies that $k\leqslant\frac{AN}{b(\psi(N)+A)}\leqslant\frac{2A}{b\lambda_\psi}$. Hence for every $t \in [0, A]$, we have

$$| \, \theta_\infty(t) - \theta_N(t) \, | \leqslant \frac{1}{t^2} \sum_{k=1}^{\frac{2A}{b\lambda_\psi}} k \; \big| \, \mathbb{1}_{[b\,\lambda_\psi k, +\infty[}(t) - \mathbb{1}_{[\frac{\psi(N)bk}{N-bk}, +\infty[}(t) \, \big| \; .$$

Besides, we have $b\,\lambda_\psi k \geqslant \frac{b\lambda\psi}{2}$ and $\frac{\psi(N)bk}{N-bk} \geqslant \frac{b\lambda\psi}{2}$. The function $\big| \, \mathbb{1}_{[b\,\lambda_\psi k, +\infty[} - \mathbb{1}_{[\frac{\psi(N)bk}{N-bk}, +\infty[} \big|$ vanishes outside the closed interval between $b\,\lambda_\psi k$ and $\frac{\psi(N)bk}{N-bk}$, and has value 1 on the interior of this interval. We hence have

$$\int_0^{+\infty} \big| \, \mathbb{1}_{[b\,\lambda_\psi k, +\infty[}(t) - \mathbb{1}_{[\frac{\psi(N)bk}{N-bk}, +\infty[}(t) \, \big| \; \frac{dt}{t^2} \leqslant \Big| \int_{b\,\lambda_\psi k}^{\frac{\psi(N)bk}{N-bk}} \frac{dt}{t^2} \Big| \leqslant \frac{4}{b^2 \lambda_\psi^2} \Big| \, b\,\lambda_\psi k - \frac{\psi(N)bk}{N-bk} \, \Big| \; .$$

For every continuous function $f : [0, +\infty[\, \rightarrow \mathbb{R}$ with compact support in $[0, A]$, we therefore have

$$\begin{aligned}\Big| \int_0^{+\infty} f(t) \; (\theta_\infty(t) - \theta_N(t)) \; dt \Big| &= \mathrm{O}\Big(\frac{\|f\|_\infty}{\lambda_\psi^2} \sum_{k=1}^{\frac{2A}{b\lambda_\psi}} k \; \big| \; b\,\lambda_\psi k - \frac{\psi(N)bk}{N-bk} \; \big| \Big) \\ &= \mathrm{O}\Big(\frac{A^3}{\lambda_\psi^5} \; \|f\|_\infty \; \big| \, \lambda_\psi - \frac{\psi(N)}{N} + \mathrm{O}(\frac{A}{N}) \, \big| \Big) \; .\end{aligned}$$

By Equation (6), for every C^1 function $f : [0, +\infty[\, \rightarrow \mathbb{R}$ with compact support in $[0, A]$, we thus have

$$\begin{aligned}\frac{1}{\psi(N)} \mu_N^+(f) = &\int f \; \theta_\infty \; d\,\mathrm{Leb}_{[0,+\infty[} \\ &+ \mathrm{O}\Big(\frac{A^3}{\lambda_\psi^5} \; \|f\|_\infty \; \big(\big| \, \lambda_\psi - \frac{\psi(N)}{N} \big| + \mathrm{O}(\frac{A}{N}) \big) \Big) + \mathrm{O}\Big(\|f'\|_\infty \frac{A^2}{N} \Big) \; .\end{aligned}$$

With $g_{\mathscr{L}_{\mathbb{N}}^{a,b}, \psi} : \mathbb{R} \rightarrow \mathbb{R}$ given by $t \mapsto \theta_\infty(|t|)$, by Lemma 3.2 and the comment following it since $\frac{1}{\psi(N)} \big(\frac{N}{\psi(N)} \big)^2$ tends to 0 as $N \rightarrow +\infty$, it follows that

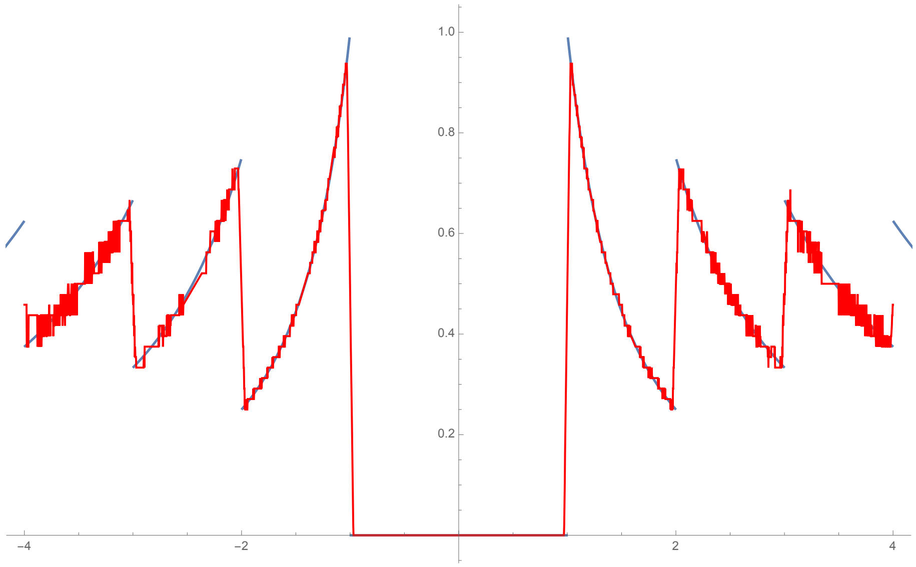
$$\frac{1}{\psi(N)} \; \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b}, \psi} \;\; \overset{*}{\rightharpoonup} \;\; g_{\mathscr{L}_{\mathbb{N}}^{a,b}, \psi} \; \mathrm{Leb}_{\mathbb{R}} \; ,$$

thus proving Equation (3) under the assumptions of Case 3. Furthermore, for every $A \geqslant 1$ and every C^1 function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support contained in $[-A, A]$, we have

$$\begin{aligned}\frac{1}{\psi(N)} \; \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b}, \psi}(f) = &\int_{\mathbb{R}} f(t) \; g_{\mathscr{L}_{\mathbb{N}}^{a,b}, \psi}(t) \; dt \\ &+ \mathrm{O}\Big(\frac{A^3}{\lambda_\psi^5} \; \|f\|_\infty \big(\, \big| \, \lambda_\psi - \frac{\psi(N)}{N} \big| + \frac{A}{N} \, \big) \Big) + \mathrm{O}\Big(\|f'\|_\infty \frac{A^3}{N} \Big) + \mathrm{O}\Big(\frac{A^3}{\lambda_\psi^2 \psi(N)} \|f'\|_\infty \Big) \; . \quad (11)\end{aligned}$$

Since $\psi(N) \sim \lambda_\psi N$ as $N \rightarrow +\infty$, this concludes the proof of Theorem 3.1. □

Let us give a numerical illustration of Theorem 3.1 when $a = b = 1$ and $\psi(N) = N$. The following figure shows in red an approximation of the pair correlation function $g_{\mathscr{L}_{\mathbb{N}}, \psi}$ computed using $\mathscr{R}_{2000}^{\mathscr{L}_{\mathbb{N}}, \psi}$, and in blue the pair correlation function $g_{\mathscr{L}_{\mathbb{N}}, \psi}$, in the interval $[-4, 4]$.



4 Pair correlations with Euler weights without scaling

In this section, we study the weighted family

$$\mathscr{L}_{\mathbb{N}}^{a,b,\varphi} = \big((L_N^{a,b} = \{\ln n : 0 < n \leqslant N, \ \ n \equiv a \mod b\})_{N\in\mathbb{N}}, \ \omega = \varphi \circ \exp \big) .$$

The (not normalised) *pair correlation measure* of the logarithms of integers congruent to a modulo b at time N with multiplicities given by the Euler function φ, for the trivial scaling function, is

$$\widetilde{\nu}_N = \sum_{(m,\,n)\in I_N} \varphi(n)\ \varphi(m)\ \Delta_{\ln m - \ln n} \ .$$

With the notation of the introduction, we have $\mathscr{L}_{\mathbb{N}}^{1,1,\varphi} = \mathscr{L}_{\mathbb{N}}^{\varphi}$ and $\widetilde{\nu}_N = \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},1}$.

Theorem 4.1 *As $N \to +\infty$, the measures $\widetilde{\nu}_N$ on $\mathbb{R}$, renormalized to be probability measures, weak-star converge to the measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with Radon-Nikodym derivative the function $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},1} : s \mapsto e^{-2|s|}$:*

$$\frac{\widetilde{\nu}_N}{\|\,\widetilde{\nu}_N\|} \ \overset{*}{\rightharpoonup} \ g_{\mathscr{L}_{\mathbb{N}}^{\varphi},1}\ \mathrm{Leb}_{\mathbb{R}} \ .$$

Furthermore, for all $f \in C^1_c(\mathbb{R})$ and $\alpha \in [\frac{1}{2}, 1[$, we have

$$\frac{\widetilde{\nu}_N}{\|\,\widetilde{\nu}_N\|}(f) = \int_{s\in\mathbb{R}} f(s)\ e^{-2|s|}\ ds \ + \ \mathrm{O}_b\Big(\frac{\ln N}{N^{1-\alpha}}\,\|f\|_\infty + \frac{1}{N^\alpha}\|e^{|s|} f'(s)\|_\infty\Big) \ .$$

When $a = b = 1$, the measure $\widetilde{\nu}_N$ corresponds to the one denoted by $\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{\varphi},1}$ in the introduction. The above result gives the first assertion of Theorem 1.2 in the introduction. Furthermore, it proves that the pair correlation function $g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},1}$ exists and is independent of a and b.

Proof. The first assertion of Theorem 4.1 follows from the second one, by taking for instance $\alpha = \frac{1}{2}$ and by the density of $C^1_c(\mathbb{R})$ in $C^0_c(\mathbb{R})$ for the uniform norm.

For every $q \in \mathbb{N} - \{0\}$ with $q \equiv a \mod b$, let $q' \in \mathbb{N}$ be such that $q = a + q'b$ and let J_q be given by Equation (2). We now define

$$\widetilde{\omega}_q = \sum_{p \in J_q} \varphi(p)\ \Delta_{\frac{p}{q}}\ ,$$

which is a finitely supported measure on $[0,1]$, and nonzero if and only if $q' \neq 0$. In order to compute its total mass, we will use the following elementary adaptation of Mertens' formula (see for example [HaW, Thm. 330]). We have not found its proof in the literature, hence we provide one, due to Fouvry.

Let $(a,b) \in \mathbb{N} - \{0\}$ be the greatest common divisor of a and b. Let

$$c_{a,b} = \frac{\varphi((a,b))}{b\,(a,b)} \prod_{p \text{ prime},\ p\,|\,b} \Big(1 - \frac{1}{p^2}\Big)^{-1}\ .$$

Note that $0 < c_{a,b} \leqslant \min\big\{\frac{1}{\varphi(b)}, \frac{\pi^2}{6\,b}\big\} \leqslant 1$. In particular, $c_{a,b}$ tends to 0 as $b \to +\infty$ uniformly in a. Furthermore, there exists $C' > 0$ (independent of a, b) such that $c_{a,b} \geqslant \frac{1}{C'\,b\,\ln\ln(3b)}$ by for instance [HaW, Thm. 328]. When $a = b = 1$, we have $c_{a,b} = 1$, and the following result is exactly Mertens' formula.

Lemma 4.2 *There exists $C > 0$ such that for all integers $a, b \geqslant 1$ and real numbers $x \geqslant 1$, we have*

$$\Big|\sum_{1 \leqslant n \leqslant x,\ n \equiv a \mod b} \varphi(n) - \frac{3\,c_{a,b}}{\pi^2}\,x^2\Big| \leqslant C\ x \ln(2x)\ .$$

Let $S(x,a,b)$ be the above sum. This lemma implies an almost optimal uniformity in the parameters a and b on the asymptotic of $S(x,a,b)$: since $c_{a,b} \geqslant \frac{1}{C'\,b\,\ln\ln(3b)}$, we have

$$S(x,a,b) \sim \frac{3\,c_{a,b}}{\pi^2}\,x^2$$

uniformly for $1 \leqslant a \leqslant b \leqslant \frac{x}{\ln(2x)(\ln\ln(3x))^2}$.

Proof. (Fouvry) In this proof, for every function g of a variable in $[1, +\infty[$, possibly depending on parameters, we use the notation $\mathrm{O}(g)$ in order to denote any function f on $[1, +\infty[$ such that there exists a constant C, independent of the variable and of all the parameters, such that $|f| \leqslant C|g|$. We do not need a more precise error term.

We refer for instance to [HaW, Sect. 16.3-16.4] for the definition of the Möbius function $\mu : \mathbb{N} - \{0\} \to \{-1, 0, 1\}$, of the Dirichlet convolution $f * g$ of two maps $f, g : \mathbb{N} - \{0\} \to \mathbb{R}$ and for the Möbius inversion formula, which in particular gives that $\varphi = \mu * \mathrm{id}$. Hence

$$S(x,a,b) = \sum_{\substack{1 \leqslant n \leqslant x \\ n \equiv a \mod b}} \ \sum_{md = n} \mu(d)\,m \ = \sum_{1 \leqslant d \leqslant x} \mu(d) \sum_{\substack{1 \leqslant m \leqslant x/d \\ md \equiv a \mod b}} m\ .$$

Let us fix $d \geqslant 1$. Let us consider the congruence equation $m\,d \equiv a \mod b$ with unknown m. It has no solution if the greatest common divisor (b,d) of b and d does not divide a. If (b,d) does divide a, let $a' = \frac{a}{(b,d)}$, $b' = \frac{b}{(b,d)}$ and $d' = \frac{d}{(b,d)}$, so that the congruence equation is equivalent to $m\,d' \equiv a' \mod b'$. Since d' is coprime with b', it is invertible modulo b', and we denote its inverse by $\overline{d'}$. The congruence equation becomes $m \equiv a'\,\overline{d'} \mod b'$. The classical formula $\displaystyle\sum_{1\leqslant m\leqslant y,\; m\equiv a'\,\overline{d'} \mod b'} 1 = \frac{y}{b'} + \mathrm{O}(1)$ gives, by a summation by parts, the equality

$$\sum_{1\leqslant m\leqslant y,\; m\equiv a'\,\overline{d'} \mod b'} m \;=\; \frac{y^2}{2b'} + \mathrm{O}(y)\;.$$

Therefore

$$\begin{aligned} S(x,a,b) &= \sum_{1\leqslant d\leqslant x,\;(b,d)|a} \mu(d)\Big(\frac{(b,d)}{2\,b}\big(\frac{x}{d}\big)^2 + \mathrm{O}\big(\frac{x}{d}\big)\Big)\\ &= \frac{x^2}{2\,b}\Big(\sum_{1\leqslant d\leqslant x,\;(b,d)|a} \mu(d)\frac{(b,d)}{d^2}\Big) + \mathrm{O}(x\ln(2x))\;. \end{aligned}$$

Using the Eulerian product formula of the zeta function, giving $\displaystyle\prod_{p \text{ prime}} (1-\frac{1}{p^2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, and the expression $\varphi(n) = n \displaystyle\prod_{p \text{ prime},\; p\,|\,n} (1-\frac{1}{p})$ of the Euler function in terms of the prime factors, we have, by decomposing d into prime powers and using the definition of the Möbius function,

$$\begin{aligned} \sum_{1\leqslant d\leqslant x,\;(b,d)|a} \mu(d)\frac{(b,d)}{d^2} &= \sum_{d\geqslant 1,\;(b,d)|a} \mu(d)\frac{(b,d)}{d^2} \;+\mathrm{O}\big(\frac{b}{x}\big)\\ &= \prod_{p \text{ prime},\; p\nmid b} (1-\frac{1}{p^2}) \prod_{p \text{ prime},\; p\,|\,a,\; p\,|\,b} (1-\frac{1}{p}) \;+\mathrm{O}\big(\frac{b}{x}\big)\\ &= \frac{1}{\zeta(2)} \prod_{p \text{ prime},\; p\,|\,b} (1-\frac{1}{p^2})^{-1}\;\frac{\varphi((a,b))}{(a,b)} + \mathrm{O}\big(\frac{b}{x}\big) = \frac{6\;b\;c_{a,b}}{\pi^2} + \mathrm{O}\big(\frac{b}{x}\big)\;. \end{aligned}$$

This proves Lemma 4.2. □

Lemma 4.2 says that if $q' \neq 0$ (that is, when $q > a$), then

$$\|\,\widetilde{\omega}_q\| = \frac{3\,c_{a,b}\,q^2}{\pi^2} + \mathrm{O}(q\ln(2q))\;, \tag{12}$$

and in particular $\frac{1}{\|\,\widetilde{\omega}_q\|} = \frac{\pi^2}{3\,c_{a,b}\,q^2}\big(1+\mathrm{O}_b(\frac{\ln q}{q})\big)$.

Lemma 4.3 *We have as $q \to +\infty$,*

$$\frac{\widetilde{\omega}_q}{\|\,\widetilde{\omega}_q\|} \;\overset{*}{\rightharpoonup}\; 2\,t\;d\,\mathrm{Leb}_{[0,1]}(t)\;.$$

More precisely, for all $f \in C^1([0,1])$ and $\alpha \in [\frac{1}{2},1\,[$, we have

$$\frac{\widetilde{\omega}_q}{\|\,\widetilde{\omega}_q\|}(f) = \int_0^1 2\,t\,f(t)\;\,dt \;+\; \mathrm{O}\Big(\frac{\ln q}{q^{1-\alpha}}\|f\|_\infty + \frac{1}{q^\alpha}\|f'\|_\infty\Big)\;.$$

Proof. The first assertion follows from the second one, by taking for instance $\alpha = \frac{1}{2}$ and by the density of $C_c^1(\mathbb{R})$ in $C_c^0(\mathbb{R})$ for the uniform norm.

Let $Q = \lfloor q^\alpha \rfloor \in \mathbb{N} - \{0\}$. For all $n \in \{0, \dots, Q-1\}$ and $t \in \,]\, n\, q^{-\alpha}, (n+1)q^{-\alpha}]$, we have by the mean value theorem

$$f(t) = f(n\, q^{-\alpha}) + \mathrm{O}(q^{-\alpha}\, \|f'\|_\infty)\ .$$

Since $n+1 \leqslant Q \leqslant q^\alpha$, we hence have

$$\begin{aligned}\int_{n\,q^{-\alpha}}^{(n+1)q^{-\alpha}} 2\,t\,f(t)\;dt &= \big(f(n\,q^{-\alpha}) + \mathrm{O}(q^{-\alpha}\,\|f'\|_\infty)\big)\int_{n\,q^{-\alpha}}^{(n+1)q^{-\alpha}} 2\,t\;dt \\ &= \frac{1}{q^\alpha}\big((2\,n+1)\,q^{-\alpha} f(n\,q^{-\alpha}) + \mathrm{O}(n\,q^{-2\alpha}\|f'\|_\infty)\big)\ . \end{aligned} \tag{13}$$

Using the formula for $\frac{1}{\|\,\widetilde{\omega}_q\|}$ following Equation (12) and twice Lemma 4.2, we have

$$\begin{aligned} &\sum_{p \in \,]\,n\,q^{1-\alpha},\,(n+1)q^{1-\alpha}] \cap J_q} f\big(\frac{p}{q}\big)\;\frac{1}{\|\,\widetilde{\omega}_q\|}\;\varphi(p) \\ &= \big(f(n\,q^{-\alpha}) + \mathrm{O}(q^{-\alpha}\,\|f'\|_\infty)\big)\frac{\pi^2}{3\,c_{a,b}\,q^2}\,\big(1 + \mathrm{O}\,\big(\frac{\ln q}{q}\big)\big) \\ &\quad\times\Big(\frac{3\,c_{a,b}((n+1)^2 - n^2)}{\pi^2}q^{2-2\alpha} + \mathrm{O}\,\big((n+1)q^{1-\alpha}\ln((n+1)q^{1-\alpha})\big)\Big) \\ &= \frac{1}{q^\alpha}\Big((2\,n+1)\,q^{-\alpha}\,f(n\,q^{-\alpha}) + \mathrm{O}\,\big(\frac{n}{q^{2\alpha}}\,\|f'\|_\infty + \frac{n\ln q}{q}\,\|f\|_\infty\big)\Big)\ . \end{aligned} \tag{14}$$

Again using Equation (12) and Lemma 4.2, since $|1 - Q\,q^{-\alpha}| = \mathrm{O}(q^{-\alpha})$, we have

$$\sum_{p \in \,]\,Q\,q^{1-\alpha},\,q[\, \cap J_q} \frac{\varphi(p)}{\|\,\widetilde{\omega}_q\|}\,\big|f\big(\frac{p}{q}\big)\big| = \mathrm{O}\,\big(\frac{q^2 - (Q\,q^{1-\alpha})^2}{q^2}\,\|f\|_\infty\big) = \mathrm{O}\,\big(q^{-\alpha}\,\|f\|_\infty\big)\ . \tag{15}$$

By cutting the sum defining $\widetilde{\omega}_q$ and the integral from 0 to 1, by using Equations (13), (14) and (15), since $Q \leqslant q^\alpha$ and again $|1 - Q\,q^{-\alpha}| = \mathrm{O}(q^{-\alpha})$, we have (using $\alpha \geqslant \frac{1}{2}$ for the last equality)

$$\begin{aligned} &\Big|\;\frac{\widetilde{\omega}_q}{\|\,\widetilde{\omega}_q\|}(f) - \int_0^1 2\,t\,f(t)\;\;dt\;\Big| \\ &= \Big|\;\sum_{n=0}^{Q-1}\Big(\sum_{p \in \,]\,n\,q^{1-\alpha},\,(n+1)q^{1-\alpha}] \cap J_q} \frac{\varphi(p)}{\|\,\widetilde{\omega}_q\|}\;f\big(\frac{p}{q}\big) - \int_{n\,q^{-\alpha}}^{(n+1)q^{-\alpha}} 2\,t\,f(t)\;dt\Big)\;\Big| \\ &\quad+ \sum_{p \in \,]\,Q\,q^{1-\alpha},\,q[\, \cap J_q} \frac{\varphi(p)}{\|\,\widetilde{\omega}_q\|}\;\big|f\big(\frac{p}{q}\big)\big| \;\;+ \int_{Q\,q^{-\alpha}}^{1} 2\,t\,|f(t)|\;dt \\ &= \mathrm{O}(q^{-\alpha}\,\|f\|_\infty) + \frac{1}{q^\alpha}\sum_{n=0}^{Q-1}\mathrm{O}\,\big(\frac{n}{q^{2\alpha}}\|f'\|_\infty + \frac{n\ln q}{q}\|f\|_\infty\big) \\ &= \mathrm{O}(q^{-\alpha}\,\|f\|_\infty) + \frac{1}{q^\alpha}\,\mathrm{O}\,\big(\frac{Q^2}{q^{2\alpha}}\|f'\|_\infty + \frac{Q^2\ln q}{q}\|f\|_\infty\big) = \mathrm{O}\,\big(\frac{\ln q}{q^{1-\alpha}}\,\|f\|_\infty + \frac{\|f'\|_\infty}{q^\alpha}\big)\ . \end{aligned}$$

This proves Lemma 4.3. $\square$

For every $N\in\mathbb{N}-\{0\}$, let us define

$$\widetilde{\mu}_N^- = \sum_{(m,\,n)\in I_N^-} \varphi(m)\;\varphi(n)\;\Delta_{\frac{m}{n}} \;\;=\sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \varphi(q)\;\widetilde{\omega}_q\;,$$

which is a finitely supported measure on $[0,1]$, which is nonzero if N is large enough. By Lemma 4.2, its total mass is

$$\begin{aligned}\|\widetilde{\mu}_N^-\| &= \sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \varphi(q)\;\|\widetilde{\omega}_q\| = \sum_{(m,n)\in I_N^-} \varphi(m)\;\varphi(n)\\ &= \frac{1}{2}\Big(\big(\sum_{\substack{1\leqslant q\leqslant N\\ q\equiv a \mod b}} \varphi(q)\big)^2 - \sum_{\substack{1\leqslant q\leqslant N\\ q\equiv a \mod b}} \varphi(q)^2\Big) = \frac{9\,c_{a,b}^2\,N^4}{2\,\pi^4} + \mathrm{O}(N^3\ln N)\;.\end{aligned}$$

For $f\in C^1([0,1])$, by Lemma 4.3, by Equation (12), since $q^{1+\alpha}(\ln q)\leqslant N^{1+\alpha}(\ln N)$ and $q^{2-\alpha}\leqslant N^{2-\alpha}$ when q occurs in the summations below, and by Lemma 4.2, we have

$$\begin{aligned}\frac{\widetilde{\mu}_N^-(f)}{\|\widetilde{\mu}_N^-\|} &= \frac{1}{\|\widetilde{\mu}_N^-\|}\sum_{1\leqslant q\leqslant N,\; q\equiv a \mod b} \varphi(q)\;\|\widetilde{\omega}_q\|\;\frac{\widetilde{\omega}_q(f)}{\|\widetilde{\omega}_q\|}\\ &= \int_0^1 2\,t\,f(t)\,dt + \frac{1}{\|\widetilde{\mu}_N^-\|}\sum_{\substack{1\leqslant q\leqslant N\\ q\equiv a \mod b}} \varphi(q)\;\|\widetilde{\omega}_q\|\;\mathrm{O}\,\big(\frac{\ln q}{q^{1-\alpha}}\,\|f\|_\infty + \frac{1}{q^\alpha}\|f'\|_\infty\big)\\ &= \int_0^1 2\,t\,f(t)\,dt + \mathrm{O}\,\big(\frac{1}{N^4}\sum_{\substack{1\leqslant q\leqslant N\\ q\equiv a \mod b}} \varphi(q)\big(q^{1+\alpha}(\ln q)\,\|f\|_\infty + q^{2-\alpha}\,\|f'\|_\infty\big)\\ &= \int_0^1 2\,t\,f(t)\,dt + \mathrm{O}\,\big(\frac{\ln N}{N^{1-\alpha}}\,\|f\|_\infty + \frac{1}{N^\alpha}\|f'\|_\infty\big)\;.\end{aligned}$$

For every $N\in\mathbb{N}-\{0\}$, let us define

$$\widetilde{\nu}_N^{\pm} = \sum_{(m,\,n)\in I_N^{\pm}} \varphi(m)\;\varphi(n)\;\Delta_{\ln\frac{m}{n}}\;,$$

so that $\widetilde{\nu}_N^- = \ln_*\widetilde{\mu}_N^- = \widetilde{\nu}_N\,|_{]-\infty,0]}$, and $\|\widetilde{\nu}_N^-\| = \|\widetilde{\mu}_N^-\|$. We have, for every $f\in C^1_{\rm c}(]-\infty,0])$,

$$\begin{aligned}\frac{\widetilde{\nu}_N^-(f)}{\|\widetilde{\nu}_N^-\|} = \frac{\widetilde{\mu}_N^-(f\circ\ln)}{\|\widetilde{\mu}_N^-\|} &= \int_0^1 2\,t\,f\circ\ln(t)\,dt + \mathrm{O}\,\big(\frac{\ln N}{N^{1-\alpha}}\,\|f\circ\ln\|_\infty + \frac{1}{N^\alpha}\|(f\circ\ln)'\|_\infty\big)\\ &= \int_{-\infty}^0 2\,f(s)\,e^{2s}\,ds + \mathrm{O}\,\big(\frac{\ln N}{N^{1-\alpha}}\,\|f\|_\infty + \frac{1}{N^\alpha}\|e^{-s}f'(s)\|_\infty\big)\;.\end{aligned}$$

Since $\widetilde{\nu}_N = \widetilde{\nu}_N^- + \widetilde{\nu}_N^+$, since $\widetilde{\nu}_N^+ = {\rm sg}_*\,\widetilde{\nu}_N^-$, we have $\|\widetilde{\nu}_N^{\pm}\| = \frac{1}{2}\,\|\widetilde{\nu}_N\|$ and Theorem 4.1 follows. $\square$

Let us give some numerical illustrations of Theorem 4.1 with $a=b=1$. For every $N\in\mathbb{N}-\{0\}$, let

$$\widetilde{\mathscr{D}}_N : s\mapsto \frac{\sum_{0<m\neq n\leqslant N\,:\,(\ln m-\ln n)\leqslant s}\varphi(m)\;\varphi(n)}{\sum_{0<m\neq n\leqslant N}\varphi(m)\;\varphi(n)}\;.$$

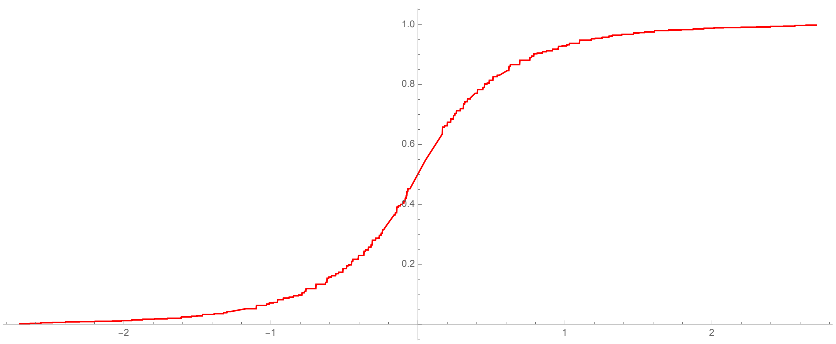
This is the cumulative distribution function at time N of the unscaled differences of the logarithms of natural numbers weighted by the Euler function, that is, for all $s, s' \in \mathbb{R}$ with $s < s'$, we have

$$\frac{\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{\varphi},1}}{\|\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{\varphi},1}\|}(\,]s, s'\,]) = \widetilde{\mathscr{D}}_N(s') - \widetilde{\mathscr{D}}_N(s)\ .$$

Theorem 4.1 with $a = b = 1$ says that the function $\widetilde{\mathscr{D}}_N$ pointwise converges as $N \to +\infty$ to the C^1 (but not C^2) function

$$\widetilde{\mathscr{D}} : s \mapsto \begin{cases} \frac{1}{2}\, e^{2s} \text{ if } s \leqslant 0 \\ 1 - \frac{1}{2}\, e^{-2s} \text{ if } s \geqslant 0 \end{cases}$$

(with derivative $\widetilde{\mathscr{D}}' = g_{\mathscr{L}_{\mathbb{N}}^{\varphi},1}$). This is illustrated by the figure below, which shows $\widetilde{\mathscr{D}}_{15}$ in red.



5 Pair correlations with Euler weights and linear scaling

In this section, we study the pair correlations of the family $\mathscr{L}_{\mathbb{N}}^{a,b,\varphi}$ defined at the beginning of Section 4, now with a linear scaling. We leave to the reader the study of a general scaling ψ, assumed to converge to $+\infty$. For every $N \in \mathbb{N} - \{0\}$, the (not normalised) *pair correlation measure* of the logarithms of integers, congruent to a modulo b, at time N with multiplicities given by the Euler function and with scaling N is the (Borel, positive) measure with finite support in $\mathbb{R}$ defined by

$$\widetilde{\mathscr{R}}_N = \sum_{(m,\,n) \in I_N} \varphi(m)\ \varphi(n)\ \Delta_{N(\ln m - \ln n)}\ .$$

With the notation of the introduction, we have $\widetilde{\mathscr{R}}_N = \mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}$.

For every $k \in \mathbb{N} - \{0\}$, we consider the arithmetic constant $c_{a,b,k}$ defined in Equation (23) of Appendix A. Note that $c_{a,b,k} > 0$ is uniformly bounded from above when a, b, k vary in $\mathbb{N} - \{0\}$, and has a positive lower bound in terms of a and k when b is fixed, by Equation (24) of Appendix A.

Theorem 5.1 *As $N \to +\infty$, the family $\big(\frac{1}{N^3}\ \widetilde{\mathscr{R}}_N\big)_{N \in \mathbb{N}}$ of measures on $\mathbb{R}$ weak-star converges to the measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with Radon-Nikodym derivative the function*

$$g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}} : s \mapsto \frac{1}{s^4} \sum_{1 \leqslant k \leqslant \lfloor\, |s|\, \rfloor,\ k \equiv 0 \mod b} c_{a,b,k}\ k^3\ ,$$

that is, as $N \to +\infty$,

$$\frac{1}{N^3}\,\widetilde{\mathscr{R}}_N \;\overset{*}{\rightharpoonup}\; g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}\ \mathrm{Leb}_{\mathbb{R}}\ .$$

Furthermore, for all $f \in C^1(\mathbb{R})$ with compact support contained in $[-A, A]$, where $A \geqslant 1$, and for any $\alpha \in \left[\frac{1}{2}, 1\right[$, we have

$$\frac{1}{N^3}\,\widetilde{\mathscr{R}}_N(f) = \int_{s\in\mathbb{R}} f(s)\, g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}(s)\, ds \;+\; \mathrm{O}_b\Big(\frac{A^4 \ln^2 N}{N^{1-\alpha}}\,\|f\|_\infty + \frac{A^3}{N^\alpha}\|f'\|_\infty\Big)\ .$$

When $a = b = 1$, the measure $\widetilde{\mathscr{R}}_N$ corresponds to the one denoted by $\mathscr{R}_N^{\mathscr{L}_{\mathbb{N}}^{\varphi},\,\mathrm{id}}$ in the introduction. The above result gives the second assertion of Theorem 1.2 in the introduction, with pair correlation function $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\,\mathrm{id}} = g_{\mathscr{L}_{\mathbb{N}}^{1,1,\varphi},\,\mathrm{id}}$, using Mirsky's value of $c_{1,1,k}$ given by Equation (25), as explained in Remark A.9 of Appendix A.

Note that, as the proof below shows, the total mass of $\widetilde{\mathscr{R}}_N$ is equivalent to $c\,N^4$ as $N \to +\infty$, for some constant $c > 0$, hence renormalising $\widetilde{\mathscr{R}}_N$ to be a probability measure makes it weak-star converge to the zero measure on the noncompact space $\mathbb{R}$ (a total loss of mass phenomenon).

Proof. The first assertion of Theorem 5.1 follows from the second one, by taking for instance $\alpha = \frac{1}{2}$ and by the density of $C^1_c(\mathbb{R})$ in $C^0_c(\mathbb{R})$ for the uniform norm.

The change of variables $(m, n) \mapsto (n, m)$ in I_N gives $\widetilde{\mathscr{R}}_N|_{]-\infty,0]} = \mathrm{sg}_*\big(\widetilde{\mathscr{R}}_N|_{[0,+\infty[}\big)$. We will thus only study the convergence of the measures $\frac{1}{N^3}\,\widetilde{\mathscr{R}}_N$ on $[0, +\infty[$, and deduce the global result by the symmetry of $g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}$.

For every $N \in \mathbb{N} - \{0\}$ and for every $p \in \mathbb{N}$ with $p \equiv 0 \bmod b$ and $0 < p < N$, let $J_{p,\,N}$ be given by Equation (4). We now define the key auxiliary measure by

$$\widetilde{\omega}_{p,\,N} = \sum_{q\in J_{p,\,N}} \varphi(q)\ \varphi(q+p)\ \Delta_{\frac{q}{N\,p}}\ .$$

Then $\widetilde{\omega}_{p,\,N}$ is a measure on $[0, +\infty[$, with finite support contained in $[\frac{1}{N\,p}, \frac{N-p}{N\,p}]$, hence in $[0, 1]$. The measure $\widetilde{\omega}_{p,\,N}$ is nonzero if and only if $N \geqslant a+p$. In order to compute its total mass, we use an adaptation with congruences of a formula by Mirsky (see [Mir, Thm. 9]) proved in the appendix by Fouvry. Theorem A.1 applied with $x = N - p$ and $k = p$ says that if $N \geqslant a + p$, then

$$\|\,\widetilde{\omega}_{p,N}\| = \frac{c_{a,b,p}}{3}\,(N-p)^3 + \mathrm{O}\left(N^2(p + \ln^2 N)\right)\ , \tag{16}$$

and in particular $\frac{1}{\|\,\widetilde{\omega}_{p,N}\|} = \frac{3}{c_{a,b,p}\,(N-p)^3}\big(1 + \mathrm{O}(\frac{N^2(p+\ln^2 N)}{(N-p)^3})\big)$.

The next result implies that the measures $\widetilde{\omega}_{p,N}$, once normalized to be probability measures, weak-star converge to the measure $d\mu(t) = 3\big(\frac{N\,p}{N-p}\big)^3 t^2\ d\,\mathrm{Leb}_{[\frac{1}{N\,p},\,\frac{N-p}{N\,p}]}$, which is absolutely continuous with respect to the Lebesgue measure on the interval $[\frac{1}{N\,p}, \frac{N-p}{N\,p}]$.

Lemma 5.2 *For every $p \in \mathbb{N}$ with $0 < p < N$ and $p \equiv 0 \mod b$, for every $\alpha \in \,]0,1[$ and for every $f \in C^1_{\rm c}(]0,1])$, we have*

$$\frac{\widetilde{\omega}_{p,N}}{\|\widetilde{\omega}_{p,N}\|}(f) = \int_{\frac{1}{N\,p}}^{\frac{N-p}{N\,p}} 3\big(\frac{N\,p}{N-p}\big)^3\; t^2\; f(t)\; dt$$
$$+ \operatorname{O}\Big(\big(\frac{(N\,p)^3}{(N-p)^3(N\,p)^{\alpha}} + \frac{(N\,p)^3(p+\ln(N\,p))^2}{(N-p)^3(N\,p)^{1-\alpha}}\big)\;\|f\|_\infty + \frac{1}{(N\,p)^{\alpha}}\;\|f'\|_\infty\Big)\;.$$

Proof. As in the proof of Lemma 4.3, we will estimate the difference of the main terms in the above centered formula by cutting the sum defining the renormalized measure $\widetilde{\omega}_{p,N}$ and by cutting similarly the integral from $\frac{1}{N\,p}$ to $\frac{N-p}{N\,p}$.

Let $Q = \lfloor (N\,p)^{\alpha}\frac{N-p}{N\,p}\rfloor \in \mathbb{N}$. For all $n \in \{0,\dots,Q-1\}$, we thus define

$$S_n = \sum_{q\,\in\,]\,n\,(N\,p)^{1-\alpha},\,(n+1)(N\,p)^{1-\alpha}]\cap J_{p,N}} f\big(\frac{q}{N\,p}\big)\;\frac{1}{\|\,\widetilde{\omega}_{p,N}\|}\;\varphi(q)\;\varphi(q+p)$$

and

$$I_n = \int_{n\,(N\,p)^{-\alpha}}^{(n+1)(N\,p)^{-\alpha}} 3\big(\frac{N\,p}{N-p}\big)^3\; t^2\; f(t)\; dt\;.$$

Let us also define the following remaining terms

$$S_{\rm end} = \sum_{q\,\in\,]\,Q\,(N\,p)^{1-\alpha},\,N-p]\cap J_{p,N}} f\big(\frac{q}{N\,p}\big)\;\frac{1}{\|\,\widetilde{\omega}_{p,N}\|}\;\varphi(q)\;\varphi(q+p)$$

and

$$I_{\rm end} = \int_{Q\,(N\,p)^{-\alpha}}^{\frac{N-p}{N\,p}} 3\big(\frac{N\,p}{N-p}\big)^3\; t^2\; f(t)\; dt\;.$$

For all $t \in \,]\,n\,(N\,p)^{-\alpha},(n+1)(N\,p)^{-\alpha}]$, we have by the mean value theorem

$$f(t) = f(n\,(N\,p)^{-\alpha}) + \operatorname{O}((N\,p)^{-\alpha}\,\|f'\|_\infty)\;.$$

Using the formula for $\frac{1}{\|\,\widetilde{\omega}_{p,N}\|}$ following Equation (16) and twice Theorem A.1, and using the inequality $(n+1) \leqslant Q \leqslant (N\,p)^{\alpha}\frac{N-p}{N\,p}$, we have

$$\begin{aligned}
S_n =& \Big(f(n\,(N\,p)^{-\alpha}) + \operatorname{O}((N\,p)^{-\alpha}\,\|f'\|_\infty)\Big)\Big(\frac{1}{(N-p)^3}\big(1+\operatorname{O}\big(\frac{N^2(p+\ln^2 N)}{(N-p)^3}\big)\big)\Big)\\
&\times\Big(\big((n+1)(N\,p)^{1-\alpha}\big)^3 - \big(n(N\,p)^{1-\alpha}\big)^3\\
&\qquad + \operatorname{O}\big(\big(p+(n+1)(N\,p)^{1-\alpha}\big)^2\big(p+\ln((n+1)(N\,p)^{1-\alpha})\big)^2\big)\Big)\\
=&\frac{(N\,p)^3}{(N-p)^3(N\,p)^{\alpha}}\Big((3n^2+3n+1)(N\,p)^{-2\alpha} f(n\,(N\,p)^{-\alpha}) + \operatorname{O}\big(\frac{n^2}{(N\,p)^{3\alpha}}\;\|f'\|_\infty\big)\\
&\qquad + \operatorname{O}\big(\big(\frac{n^2N^2(p+\ln^2 N)}{(N-p)^3(N\,p)^{2\alpha}} + \frac{(n+\frac{p}{(Np)^{1-\alpha}})^2(p+\ln(N-p))^2}{(N\,p)^{1+\alpha}}\big)\,\|f\|_\infty\big)\Big)\;. \qquad (17)
\end{aligned}$$

We also have

$$
\begin{aligned}
I_n &= \Big(f(n\,(N\,p)^{-\alpha}) + \mathrm{O}((N\,p)^{-\alpha}\,\|f'\|_\infty)\Big)\int_{n\,(N\,p)^{-\alpha}}^{(n+1)(N\,p)^{-\alpha}} 3\big(\frac{N\,p}{N-p}\big)^3\; t^2\; dt \\
&= \frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha}\Big((3n^2+3n+1)(N\,p)^{-2\alpha}f(n\,(N\,p)^{-\alpha}) + \mathrm{O}\big(\frac{n^2}{(N\,p)^{3\alpha}}\;\|f'\|_\infty\big)\Big)\;. \quad (18)
\end{aligned}
$$

Similarly, since $Q \geqslant (N\,p)^\alpha \frac{N-p}{N\,p} - 1$ and $N-p \leqslant N\,p$, we have

$$
S_{\rm end} = \mathrm{O}\Big(\frac{(N-p)^3 - (Q(N\,p)^{1-\alpha})^3}{(N-p)^3}\;\|f\|_\infty\Big) = \mathrm{O}\Big(\frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha}\;\|f\|_\infty\Big) \qquad (19)
$$

and

$$
I_{\rm end} = \mathrm{O}\Big(\big(\frac{N\,p}{N-p}\big)^3\Big(\big(\frac{N-p}{N\,p}\big)^3 - (Q(N\,p)^{-\alpha})^3\Big)\;\|f\|_\infty\Big) = \mathrm{O}\Big(\frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha}\;\|f\|_\infty\Big)\;. \qquad (20)
$$

Note that $p + \ln^2 N \leqslant (p + \ln(N\,p))^2$, that $\sum_{n=0}^{Q-1} n^2 = \mathrm{O}\big((N\,p)^{3\alpha-3}(N-p)^3\big)$ and that

$$
\sum_{n=0}^{Q-1}\big(n + \frac{p}{(Np)^{1-\alpha}}\big)^2 = \mathrm{O}\Big(\big(Q + \frac{p}{(Np)^{1-\alpha}}\big)^3\Big) = \mathrm{O}\big(\frac{(N\,p)^{3\alpha}}{p^3}\big) = \mathrm{O}\big((N\,p)^{3\alpha}\big)\;. \qquad (21)
$$

Putting together Equations (17), (18), (19), (20) and (21), we have (again using the inequality $N-p \;\leqslant N\,p$)

$$
\begin{aligned}
&\Big|\;\frac{\widetilde{\omega}_{p,N}}{\|\widetilde{\omega}_{p,N}\|}(f) - \int_{\frac{1}{N\,p}}^{\frac{N-p}{N\,p}} 3\big(\frac{N\,p}{N-p}\big)^3\; t^2\; f(t)\; dt\;\Big| = \Big|\;\sum_{n=0}^{Q-1}(S_n - I_n) + S_{\rm end} - I_{\rm end}\;\Big| \\
&\leqslant \sum_{n=0}^{Q-1}|S_n - I_n| + |S_{\rm end}| + |I_{\rm end}| \\
&= \mathrm{O}\Big(\frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha}\;\|f\|_\infty\Big) + \frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha}\sum_{n=0}^{Q-1}\mathrm{O}\Big(\;\frac{n^2}{(N\,p)^{3\alpha}}\;\|f'\|_\infty + \\
&\qquad\big(\;\frac{n^2N^2(p+\ln^2 N)}{(N-p)^3(N\,p)^{2\alpha}} + \frac{\big(n + \frac{p}{(Np)^{1-\alpha}}\big)^2(p+\ln(N-p))^2}{(N\,p)^{1+\alpha}}\;\big)\;\|f\|_\infty\Big) \\
&= \mathrm{O}\Big(\frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha}\;\|f\|_\infty + \frac{1}{(N\,p)^\alpha}\;\|f'\|_\infty + \frac{(N\,p)^3(p+\ln(N\,p))^2}{(N-p)^3(N\,p)^{1-\alpha}}\;\|f\|_\infty\Big)\;.
\end{aligned}
$$

This proves Lemma 5.2. □

Now, let us introduce the sum

$$
\widetilde{\mu}_N^+ = \sum_{0<p<N,\; p\equiv 0 \mod b} \iota_*\widetilde{\omega}_{p,N} \;=\; \sum_{\substack{1\leqslant q\leqslant N-p,\; 0<p<N\\ q\equiv a \mod b,\; p\equiv 0 \mod b}} \varphi(q)\;\varphi(q+p)\;\Delta_{N\frac{p}{q}}\;,
$$

where as previously $\iota : t \mapsto \frac{1}{t}$ (noting that the measures $\widetilde{\omega}_{p,\,N}$ are supported in $]0, +\infty[\,$).

Lemma 5.3 *For every $f \in C^1([0,+\infty[)$ with compact support contained in $[0,A]$, where $A \geqslant 1$, we have*

$$\big|\, \widetilde{\mathscr{R}}_N(f) - \widetilde{\mu}_N^+(f) \,\big| = \mathrm{O}(A^3\, N^2\, \|f'\|_\infty)\ .$$

Proof. Using the change of variables $(p,q) \mapsto (m = p+q, n = q)$, we have

$$\widetilde{\mathscr{R}}_N\,|_{[0,+\infty[} = \sum_{(m,n)\in I_N^+} \varphi(m)\ \varphi(n)\ \Delta_{N \ln \frac{m}{n}} \ = \sum_{\substack{0<p<N,\ 1\leqslant q\leqslant N-p\\ p\equiv 0 \mod b,\ q\equiv a \mod b}} \varphi(q)\ \varphi(q+p)\ \Delta_{N\ln(1+\frac{p}{q})}\ .$$

As in the proof of Lemma 3.2, since the support of f is contained in $[0,A]$, if a pair (p,q) occurs in the index of the sum defining either $\widetilde{\mathscr{R}}_N(f)$ or $\widetilde{\mu}_N^+(f)$ with nonzero corresponding summand, then $\frac{p}{q} = \mathrm{O}\big(\frac{A}{N}\big)$ and $p = \mathrm{O}(A)$. By the mean value theorem, we then have

$$\begin{aligned}\big|\, f\big(N\,\frac{p}{q}\big) - f\big(N\ln(1+\frac{p}{q})\big)\,\big| &\leqslant \|f'\|_\infty \big|\, N\,\frac{p}{q} - N\ln(1+\frac{p}{q})\,\big| \\ &= \|f'\|_\infty\, N\ \mathrm{O}\big(\big(\frac{p}{q}\big)^2\big) = \mathrm{O}\big(\frac{A^2}{N}\|f'\|_\infty\big)\ .\end{aligned}$$

Thus, using Theorem A.1 and Equation (24) in Appendix A, we have

$$\begin{aligned}\big|\,\widetilde{\mathscr{R}}_N(f) - \widetilde{\mu}_N^+(f)\big| &\leqslant \sum_{1\leqslant p\leqslant \mathrm{O}(A),\ 1\leqslant q\leqslant N} \varphi(q)\ \varphi(q+p)\,\mathrm{O}\big(\frac{A^2}{N}\|f'\|_\infty\big) \\ &\leqslant \sum_{1\leqslant p\leqslant \mathrm{O}(A)} \mathrm{O}(N^3)\,\mathrm{O}\big(\frac{A^2}{N}\|f'\|_\infty\big) = \mathrm{O}\big(A^3\,N^2\,\|f'\|_\infty\big)\ .\end{aligned}$$

This proves Lemma 5.3. □

Lemma 5.4 *For all $\alpha \in [\frac{1}{2}, 1[$ and $f \in C^1([0,+\infty[)$ with compact support contained in $[0,A]$, where $A \geqslant 1$, we have, as $N \to +\infty$,*

$$\frac{1}{N^3}\ \widetilde{\mu}_N^+(f) = \int_0^\infty f(s)\ g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}(s)\ ds + \mathrm{O}\Big(\frac{A^4\ln^2 N}{N^{1-\alpha}}\ \|f\|_\infty + \frac{A^3}{N^\alpha}\ \|f'\|_\infty\Big)\ .$$

Proof. Let A and f be as in the statement. Since the support of $\widetilde{\omega}_{p,N}$ is contained in $[\frac{1}{N\,p}, \frac{N-p}{N\,p}]$, the support of $\iota_*\widetilde{\omega}_{p,N}$ is contained in $[\frac{N\,p}{N-p}, N\,p]$. In particular the measures $\widetilde{\mu}_N^+$ and $g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}(s)\ ds$ both vanish on $[0,1]$. Hence we may assume that the support of f is contained in $[1,+\infty[$, so that the support of $f\circ\iota$ is contained in $]0,1]$.

Note that $\|f\circ\iota\|_\infty = \|f\|_\infty$ and $\|(f\circ\iota)'\|_\infty = \|t^2 f'(t)\|_\infty \leqslant A^2\|f'\|_\infty$, since the support of f' is contained in $[0,A]$.

By the definition of $\widetilde{\mu}_N^+$, by Equation (16) and Lemma 5.2, since $N-p \leqslant N$ and by the restriction on p, explained in the proof of Lemma 5.3, in the summation defining $\widetilde{\mu}_N^+(f)$

due to the support of f, we have

$$\begin{aligned}
\widetilde{\mu}_N^+(f) &= \sum_{0<p<N,\; p\equiv 0 \mod b} \iota_*\widetilde{\omega}_{p,\,N}(f) = \sum_{0<p<N,\; p\equiv 0 \mod b} \|\widetilde{\omega}_{p,\,N}\| \;\frac{\widetilde{\omega}_{p,N}}{\|\widetilde{\omega}_{p,\,N}\|}(f\circ\iota)\\
&= \sum_{0<p<N,\; p\equiv 0 \mod b,\; p\leqslant \mathrm{O}(A)} \Big(\frac{c_{a,b,p}}{3}\,(N-p)^3 + \mathrm{O}\big(N^2(p+\ln^2 N)\big)\Big)\\
&\quad\times\Big(\int_{\frac{1}{N\,p}}^{\frac{N-p}{N\,p}} 3\big(\frac{N\,p}{N-p}\big)^3\; t^2\; f\circ\iota(t)\; dt\\
&\quad + \mathrm{O}\Big(\big(\frac{(N\,p)^3}{(N-p)^3(N\,p)^\alpha} + \frac{(N\,p)^3(p+\ln(N\,p))^2}{(N-p)^3(N\,p)^{1-\alpha}}\big)\;\|f\circ\iota\|_\infty + \frac{1}{(N\,p)^\alpha}\;\|(f\circ\iota)'\|_\infty\Big)\Big)\\
&= N^3 \sum_{0<p<N,\; p\equiv 0 \mod b,\; p\leqslant \mathrm{O}(A)} c_{a,b,p}\; p^3 \int_{\frac{N\,p}{N-p}}^{N\,p} \frac{1}{s^4}\; f(s)\; ds\\
&\quad + N^3 \sum_{0<p<N,\; p\equiv 0 \mod b,\; p\leqslant \mathrm{O}(A)} \Big(\mathrm{O}\big(\frac{p^3 N^2(p+\ln^2 N)}{(N-p)^3}\;\|f\|_\infty\big)\\
&\qquad\qquad + \mathrm{O}\Big(\big(\frac{p^{3-\alpha}}{N^\alpha} + \frac{p^{2+\alpha}(p+\ln(N\,p))^2}{N^{1-\alpha}}\big)\;\|f\|_\infty + \frac{A^2}{(N\,p)^\alpha}\;\|f'\|_\infty\Big)\Big)\;.
\end{aligned}$$

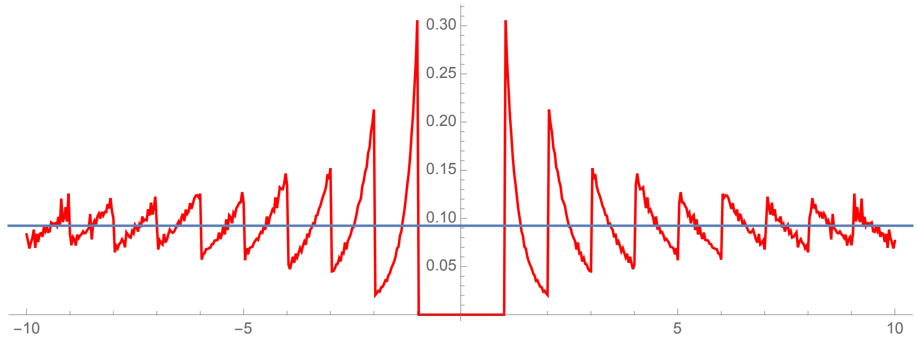
Noting that $\int_p^{\frac{N\,p}{N-p}} \frac{1}{s^4}\; f(s)\,ds = \mathrm{O}\big(\frac{N^2p+N\,p^2+p^3}{(N\,p)^3}\;\;\|f\|_\infty\big)$, we therefore have, for N large compared to A,

$$\begin{aligned}
\frac{1}{N^3}\;\widetilde{\mu}_N^+(f) &= \Big(\sum_{0<p<N,\; p\equiv 0 \mod b} c_{a,b,p}\; p^3 \int_p^{+\infty} \frac{1}{s^4}\; f(s)\; ds\Big) + \mathrm{O}\big(\frac{A^4}{N}\;\|f\|_\infty\big)\\
&\qquad + \mathrm{O}\Big(\frac{A^4\ln^2 N}{N}\;\|f\|_\infty + \big(\frac{A^4}{N^\alpha} + \frac{A^4\ln^2 N}{N^{1-\alpha}}\big)\;\|f\|_\infty + \frac{A^3}{N^\alpha}\;\|f'\|_\infty\Big)\\
&= \int_0^{+\infty} f(s)\; g_{\mathscr{L}_{\mathbb{N}}^{a,b,\varphi},\,\mathrm{id}}(s)\; ds + \mathrm{O}\Big(\;\big(\frac{A^4}{N^\alpha} + \frac{A^4\ln^2 N}{N^{1-\alpha}}\big)\;\|f\|_\infty + \frac{A^3}{N^\alpha}\;\|f'\|_\infty\Big)\;.
\end{aligned}$$

This proves Lemma 5.4, using that $\frac{A^4}{N^\alpha} = \mathrm{O}(\frac{A^4\ln^2 N}{N^{1-\alpha}})$ if $\alpha \geqslant \frac{1}{2}$. □

Theorem 5.1 now follows from Lemmas 5.3 and 5.4, as explained in the beginning of the proof. □

We close this section with a numerical illustration of Theorem 5.1 when $a = b = 1$. The following figure shows in red an approximation of the pair correlation function $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\,\mathrm{id}} = g_{\mathscr{L}_{\mathbb{N}}^{1,1,\varphi},\,\mathrm{id}}$ computed using $\widetilde{\mathscr{R}}_{2000}$ in the interval $[-10, 10]$, to be compared with the graph of $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\,\mathrm{id}}$ in the introduction.



The fact that the graph of $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\mathrm{id}}$ has a horizontal asymptote near $\pm\infty$ follows from the following result.

Proposition 5.5 *We have* $\lim_{s\to\pm\infty} g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\mathrm{id}}(s) = \frac{1}{4}\prod_{p \text{ prime}}\big(1-\frac{2}{p^2}\big)\big(1+\frac{1}{p^2(p^2-2)}\big)$.

Proof. (Fouvry) In this proof, we use the same convention concerning $\mathrm{O}(\cdot)$ as in the beginning of the proof of Lemma 4.2.

We consider the multiplicative[2] function $f : n \mapsto \prod_{p \text{ prime},\, p\,|\,n}\big(1+\dfrac{1}{p(p^2-2)}\big)$ and the constant $C_1 = \prod_{p \text{ prime}}\big(1+\dfrac{1}{p^2(p^2-2)}\big)$. Let us prove that uniformly in $x \geqslant 1$, we have

$$\sum_{1\leqslant n\leqslant x} n^3 f(n) = \frac{C_1}{4}\,x^4 + \mathrm{O}(x^3)\,. \tag{22}$$

By Equation (1) and the symmetry under $s\mapsto -s$ of $g_{\mathscr{L}_{\mathbb{N}}^{\varphi},\mathrm{id}}$, this proves Proposition 5.5.

Let $g = f * \mu$ be the Dirichlet convolution of f with the Möbius function μ. Then g is multiplicative. For every prime p, we have

$$g(p) = f(p)\,\mu(1) + f(1)\,\mu(p) = \frac{1}{p(p^2-2)}$$

and $g(p^k) = f(p^k)\,\mu(1) + f(p^{k-1})\,\mu(p) = 0$ for every $k \geqslant 2$. Therefore, for every $m \geqslant 1$, we have

$$g(m) = \mu(m)^2 \prod_{p \text{ prime},\, p\,|\,m} \frac{1}{p(p^2-2)}\ .$$

Lemma 5.6 *For every* $m \geqslant 1$*, we have* $0 \leqslant g(m) \leqslant m^{-3}\prod_{p \text{ prime}}\big(1-\dfrac{2}{p^2}\big)^{-1}$.

Proof. This is immediate if $\mu(m) = 0$. Otherwise, $m = p_1 \dots p_k$ with $p_1, \dots, p_k$ pairwise distinct primes, and

$$0 \leqslant m^3 g(m) = \prod_{i=1}^{k} \frac{p_i^3}{p_i(p_i^2-2)} = \prod_{i=1}^{k}\Big(1-\frac{2}{p_i^2}\Big)^{-1} \leqslant \prod_{p \text{ prime}}\Big(1-\frac{2}{p^2}\Big)^{-1} < +\infty\ . \quad \square$$

[2]Recall that an arithmetic function f is *multiplicative* if $f(1) = 1$ and for all coprime integers m, n, we have $f(mn) = f(m)f(n)$.

Therefore, using the Möbius inversion formula $f = g * \mathbf{1}$ for the first equality, Lemma 5.6 for the fifth equality and an Eulerian product (since g is multiplicative and vanishes on integers divisible by a nontrivial square) for the sixth equality, we have, with $S(x) = \sum_{1\leqslant k\leqslant x} f(k)$,

$$\begin{aligned} S(x) &= \sum_{\substack{m,n\geqslant 1\\ mn\leqslant x}} g(m) = \sum_{1\leqslant m\leqslant x} g(m) \sum_{1\leqslant n\leqslant x/m} 1 = \sum_{1\leqslant m\leqslant x} g(m)\big(\frac{x}{m} + \mathrm{O}(1)\big)\\ &= x\sum_{m=1}^{\infty}\frac{g(m)}{m} + \mathrm{O}\,\big(x\sum_{m\geqslant x}\frac{g(m)}{m}\big) + \mathrm{O}\,\big(\sum_{1\leqslant m\leqslant x} g(m)\big) = x\sum_{m=1}^{\infty}\frac{g(m)}{m} + \mathrm{O}(1)\\ &= x\prod_{p\ \text{prime}}\big(1 + \frac{1}{p^2(p^2-2)}\big) + \mathrm{O}(1) = C_1\, x + \mathrm{O}(1)\ . \end{aligned}$$

By summation by parts, we hence have

$$\begin{aligned} \sum_{1\leqslant n\leqslant x} n^3 f(n) &= \int_1^x t^3 d[S(t)] = \big[t^3(C_1\,t + \mathrm{O}(1))\big]_1^x - 3\int_1^x t^2(C_1\,t+\mathrm{O}(1))\,dt\\ &= \frac{C_1}{4}\,x^4 + \mathrm{O}(x^3)\ . \end{aligned}$$

This proves Equation (22) and concludes the proof of Proposition 5.5. □

6 Pair correlations of common perpendiculars in the modular curve $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$

In this section, we give a geometric motivation for the introduction of the Euler function as multiplicities in the family $\mathscr{L}^{\varphi}_{\mathbb{N}}$ of logarithms of natural numbers. We refer to [PP1, BPP] for more information.

Let Y be a nonelementary geodesically complete connected proper locally CAT(-1) good orbispace, so that the underlying space of Y is $\Gamma\backslash\widetilde{Y}$ with $\widetilde{Y}$ a geodesically complete proper CAT(-1) space and Γ a discrete group of isometries of $\widetilde{Y}$ preserving no point nor pair of points in $\widetilde{Y} \cap \partial_\infty\widetilde{Y}$. Let D^- and D^+ be connected proper nonempty properly immersed locally convex closed subsets of Y, that is, D^- and D^+ are the locally finite Γ-orbits of proper nonempty closed convex subsets $\widetilde{D}^-$ and $\widetilde{D}^+$ of $\widetilde{Y}$. A *common perpendicular* α between D^- and D^+ is the Γ-orbit of the unique shortest arc $\widetilde{\alpha}$ between $\widetilde{D}^-$ and $\gamma\widetilde{D}^+$ for some $\gamma \in \Gamma$ such that $d(\widetilde{D}^-, \gamma\widetilde{D}^+) > 0$. The *multiplicity* $\mathrm{mult}(\alpha)$ of α is the ratio A/B where

- A is the number of elements $(\gamma_-, \gamma_+) \in (\Gamma/\Gamma_{D^-}) \times (\Gamma/\Gamma_{\gamma D^+})$ such that $\widetilde{\alpha}$ is the unique shortest arc between $\gamma_-\widetilde{D}^-$ and $\gamma_+\gamma\widetilde{D}^+$, and
- B is the cardinality of the pointwise stabilizer of $\widetilde{\alpha}$ in Γ.

The *length* $\lambda(\alpha)$ of the common perpendicular α is the length of the geodesic segment $\widetilde{\alpha}$ in $\widetilde{Y}$. For every ℓ in the set $\mathrm{OL}^{\natural}(D^-, D^+)$ of lengths of common perpendiculars, the *length multiplicity* of ℓ is the sum of the multiplicities of the common perpendiculars between D^-, D^+ having the length ℓ :

$$\omega(\ell) = \sum_{\substack{\alpha \text{ common perpendicular}\\ \text{beween } D^- \text{ and } D^+ \text{ with } \lambda(\alpha)=\ell}} \mathrm{mult}(\alpha)\ .$$

If $\mathrm{Perp}(D^-, D^+)$ is the set of all common perpendiculars from D^- to D^+ with multiplicities, then $(\lambda(\alpha))_{\alpha\in\mathrm{Perp}(D^-,\,D^+)}$ is the *marked ortholength spectrum* from D^- to D^+, and the set $\mathrm{OL}(D^-,\, D^+) = (\mathrm{OL}^\natural(D^-, D^+),\ \omega)$ of the lengths of the common perpendiculars endowed with the length multiplicity ω is the *ortholength spectrum* from D^- to D^+.

The *pair correlation measure of the common perpendiculars from D^- to D^+* is the pair correlation measure of the family

$$\mathscr{F}_{D^-,D^+} = \big((F_N = \mathrm{OL}^\natural(D^-, D^+) \cap [0, 2\ln N])_{N\in\mathbb{N}},\ \omega\big)\,.$$

We study the asymptotic properties of the pair correlation measures $\mathscr{R}_N^{\mathscr{F}_{D^-,D^+},\,\psi}$ for appropriately growing scaling fonctions ψ in [PP2, §4], and we only consider in this paper the following example.

Let

$$\widetilde{Y} = \mathbb{H}^2_\mathbb{R} = \big(\{z\in\mathbb{C} : \mathrm{Im}\ z > 0\}, ds^2 = \frac{d(\mathrm{Re}\ z)d(\mathrm{Im}\ z)}{(\mathrm{Im}\ z)^2}\big)$$

be the upper halfspace model of the real hyperbolic plane with constant curvature -1. For every $b\in\mathbb{N}-\{0\}$, let $\Gamma_0[b]$ be *Hecke's congruence subgroup modulo* b of the *modular group* $\mathrm{PSL}_2(\mathbb{Z})$, which is the preimage of the upper triangular subgroup of $\mathrm{PSL}_2(\mathbb{Z}/b\mathbb{Z})$ under the reduction morphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/b\mathbb{Z})$. It acts faithfully by homographies on $\mathbb{H}^2_\mathbb{R}$, and is a lattice in the isometry group of $\mathbb{H}^2_\mathbb{R}$. Let $Y^b = \Gamma_0[b]\backslash\mathbb{H}^2_\mathbb{R}$, which is a finite (ramified) cover of the *modular curve* $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_\mathbb{R}$. Let $\widetilde{D}^- = \widetilde{D}^+$ be the horoball $\mathscr{H}_\infty = \{z\in\mathbb{C} : \mathrm{Im}\ z \geqslant 1\}$ in $\mathbb{H}^2_\mathbb{R}$, whose image $D^- = D^+$ in Y^b is a *Margulis neighbourhood* of a cusp of Y^b. If $b = 1$, then $D^- = D^+$ is actually the maximal Margulis neighbourhood of the unique cusp of Y^b. In order to emphasize the dependence on the integer b, we will use the notation $\mathscr{F}^b_{D^-,D^+} = \mathscr{F}_{D^-,D^+}$ for the family of lengths of common perpendiculars between D^- and D^+ in Y^b.

The following result says that the pair correlation measures of the common perpendiculars from this Margulis cusp neighbourhood to itself are, up to the homothety of factor 2, the pair correlation measures of the logarithms of the natural numbers congruent to 0 modulo b, with multiplicities given by the Euler function φ.

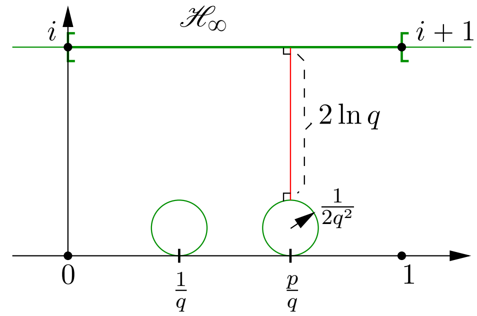
We use in the following result the notation $\mathscr{R}_N^{\mathscr{L}_\mathbb{N}^{b,b,\varphi},\,\psi}$ of the introduction with

$$\mathscr{L}_\mathbb{N}^{b,b,\varphi} = \big(\,(L_N = \{\ln n\ :\ 0 < n \leqslant N,\ n \equiv 0 \mod b\})_{N\in\mathbb{N}},\ \omega = \varphi\circ\exp\big)\,.$$

Proposition 6.1 *For every scaling function ψ and every $N \in \mathbb{N}$, if $f : t \mapsto 2t$, then* $\mathscr{R}_N^{\mathscr{F}^b_{D^-,D^+},\,\psi} = f_*\big(\mathscr{R}_N^{\mathscr{L}_\mathbb{N}^{b,b,\varphi},\,\psi}\big)$.

Proof. The orbit of $\mathscr{H}_\infty$ under $\Gamma_0[b]$ consists, besides $\mathscr{H}_\infty$ itself, of the Euclidean disks $\mathscr{H}_{\frac{p}{q}}$ of Euclidean radius $\frac{1}{2q^2}$ tangent to the horizontal line at the rational numbers $\frac{p}{q}$ with $q > 0$, $q \equiv 0 \mod b$ and $(p, q) = 1$.

Every common perpendicular between D^- and D^+ has a unique representative which starts from the Euclidean segment $[i, i+1[$ on the boundary of $\mathscr{H}_\infty$ and ends on the boundary of $\mathscr{H}_{\frac{p}{q}}$ with $\frac{p}{q} \in \mathbb{Q} \cap [0,1[$ and $q \equiv 0 \mod b$. Its hyperbolic length is $2\ln q$. In particular, we have $\mathrm{OL}^\natural(D^-, D^+) = \{2\ln q : q \geqslant 2,\ q \equiv 0 \mod b\}$.



Since $\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle of $\mathbb{H}^2_\mathbb{R}$, the multiplicities of the common perpendiculars are equal to 1. Hence the length multiplicity of $2\ln q$ is exactly the number of elements $p \in \mathbb{Z}/q\mathbb{Z}$ coprime with q, that is, $\omega(2\ln q) = \varphi(q)$. □

The following results, computing the pair correlation functions at trivial or linear scaling of the lengths of the common perpendiculars from the Margulis cusp neighbourhood at infinity to itself in Hecke's modular curve $\Gamma_0[b]\backslash\mathbb{H}^2_\mathbb{R}$, follow immediately from Theorems 4.1 and 5.1 with $a = b$, which also give an error term, using Proposition 6.1.

Corollary 6.2 *(1) For every $b \in \mathbb{N} - \{0\}$, as $N \to +\infty$, the pair correlation measures $\mathscr{R}_N^{\mathscr{F}^b_{D^-,D^+},1}$ on $\mathbb{R}$, renormalized to be probability measures, weak-star converge to a measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with pair correlation function given by $s \mapsto \frac{1}{2}\, e^{-|s|}$.*

(2) For every $b \in \mathbb{N} - \{0\}$, as $N \to +\infty$, the pair correlation measures $\frac{1}{N^3}\mathscr{R}_N^{\mathscr{F}^b_{D^-,D^+},\mathrm{id}}$ on $\mathbb{R}$ weak-star converge to a measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with pair correlation function given by $s \mapsto \frac{8}{s^4} \sum_{1 \leqslant k \leqslant \lfloor \frac{|s|}{2} \rfloor,\ k \equiv 0 \mod b} c_{b,b,k}\, k^3$, where $c_{b,b,k}$ is defined in Equation (23). □

A Appendix : A Mirsky formula with congruences, by Etienne Fouvry

The aim of this appendix is to give a proof of a version with congruences, and with a uniform estimate on the parameters, of an arithmetic formula due to Mirsky [Mir]. This improved version is used in the proof of Theorem 5.1.

Let $a, b, k \in \mathbb{N}$ be fixed integers satisfying $a, b \geqslant 1$. Denoting by φ the Euler function, we give an asymptotic formula, as $x \geqslant 1$ tends to $+\infty$, for the quantity

$$S(x; a, b, k) = \sum_{\substack{1 \leqslant n \leqslant x \\ n \equiv a \mod b}} \varphi(n)\ \varphi(n+k)\ .$$

Throughout this appendix, the letter p denotes as usual a varying positive prime in $\mathbb{Z}$, and we use the convention on $\mathrm{O}(\cdot)$ of the beginning of Lemma 4.2. For all integers $\alpha, \beta \geqslant 1$, we denote as usual by (α, β) and $[\alpha, \beta]$ their positive gcd et lcm, respectively.

Theorem A.1 *Let*

$$c_{a,b,k} = \frac{1}{b} \prod_{\substack{p \\ (p,b)\,|\,a+k}} \big(1 - \frac{(p,b)}{p^2}\big) \prod_{\substack{p \\ (p,b)\,|\,a}} \big(1 - (p,b)\,\frac{\kappa_{a,b,k}(p)\;\kappa'_k(p)}{p^2}\big)\ , \tag{23}$$

where

$$\kappa_{a,b,k}(p) = \begin{cases} (1 - \frac{(p,b)}{p^2})^{-1} & \text{if} \quad (p,b) \mid a+k \\ 1 \quad \text{otherwise} \end{cases} \qquad \text{and} \qquad \kappa'_k(p) = \begin{cases} 1 - \frac{1}{p} & \text{if} \quad p \mid k \\ 1 \quad \text{otherwise.} \end{cases}$$

There exists an absolute constant $K > 0$ such that for all integers $a, b \geqslant 1$ and $k \geqslant 0$ and real number $x \geqslant 1$, we have

$$\big|\, S(x; a, b, k) - c_{a,b,k} \big(\frac{x^3}{3} + \frac{k\,x^2}{2} \big) \ \big| \leqslant K\big(x(x+k)(\ln 2x)\ln(2x+k)\big)\ .$$

Before proving this theorem, we give some considerations on the constant $c_{a,b,k}$ in Remark A.2 and some considerations on the uniformity properties of the asymptotic on $S(x; a, b, k)$ in Remark A.3.

Remark A.2 We start from the obvious inequalities, for every prime p,

$$1 \leqslant \kappa_{a,b,k}(p) \leqslant 2 \quad \text{and} \quad \frac{1}{2} \leqslant \kappa'{}_k(p) \leqslant 1\ .$$

From these inequalities and the definition (23), we deduce that $c_{a,b,k}$ satisfies the following inequalities

$$\frac{1}{b}\,\Lambda(a,b,k) \prod_{\substack{p \geqslant 3 \\ (p,b)|a+k}} \Big(1 - \frac{(p,b)}{p^2}\Big) \prod_{\substack{p \geqslant 3 \\ (p,b)|a}} \Big(1 - 2 \cdot \frac{(p,b)}{p^2}\Big) \leqslant c_{a,b,k} < \frac{1}{b}\ , \tag{24}$$

where the local factor $\Lambda(a,b,k)$, obtained by separating the case $p = 2$ in the Euler products, is defined by

$$\Lambda(a,b,k) = \begin{cases} 5/8 & \text{if } 2 \nmid b \text{ and } 2 \mid k, \\ 1/2 & \text{if } 2 \nmid b \text{ and } 2 \nmid k, \\ 1 & \text{if } 2 \mid b,\ 2 \nmid a \text{ and } 2 \mid k, \\ 1/2 & \text{if } 2 \mid b,\ 2 \nmid a \text{ and } 2 \nmid k, \\ 1/4 & \text{if } 2 \mid b,\ 2 \mid a \text{ and } 2 \mid k, \\ 1/2 & \text{if } 2 \mid b,\ 2 \mid a \text{ and } 2 \nmid k\ . \end{cases}$$

We have the positive lower bound $\Lambda(a,b,k) \geqslant \frac{1}{4}$ in all cases. This shows in particular that $c_{a,b,k} > 0$. A deeper look also leads to the statement that the positive product $b\,c_{a,b,k}$ can be arbitrarily small : it suffices to consider the case where all the integers a, b and k are all divisible by the t smallest primes, and letting the integer t tend to infinity.

Theorem A.1 (without the uniform control on k) was already known when $a = b = 1$, this result is due to [Mir, Thm. 9, Eq. (30)], with

$$c_{1,1,k} = \prod_{p} \big(1 - \frac{2}{p^2}\big) \prod_{p\,|\,k} \big(1 + \frac{1}{p(p^2-2)}\big)\ . \tag{25}$$

For this computation, see Remark A.9 below.

Remark A.3 Consider the case where $a = b = 1$. Since Equation (25) gives the lower bound $c_{1,1,k} \geqslant \prod_p \left(1 - \frac{2}{p^2}\right) > 0$ for every $k \in \mathbb{N}$, by discussing separately the case $k \geqslant 2x$ so that $\ln(2x + k) \leqslant 2 \ln k$ if $k \geqslant 2$ and the case $k \leqslant 2x$ so that $\ln(2x + k) \leqslant 2 \ln(2x)$ if $x \geqslant 1$, Theorem A.1 gives, for x tending to infinity, the following asymptotic behavior of $S(x; 1, 1, k)$ with a large uniformity over the parameter k, that was not present in Mirsky's result in loc. cit..

Corollary A.4 *As x tends to infinity, we have*

$$S(x; 1, 1, k) \sim c_{1,1,k} \left(\frac{x^3}{3} + \frac{kx^2}{2} \right),$$

uniformly for

$$0 \leqslant k \leqslant \exp\left(x / \ln^2(2x)\right) . \quad \square$$

In the opposite direction, we now fix $k = 1$ and we return to Equation (23), to write, using for instance [HaW, Thm. 328] for the last very classical estimate,

$$c_{a,b,1} \gg \frac{1}{b} \prod_{p|(b,a+1)} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \geqslant 3 \\ p|(b,a)}} \left(1 - \frac{2}{p}\right) \gg \frac{1}{b} \prod_{\substack{p \geqslant 3 \\ p|b}} \left(1 - \frac{2}{p}\right) \gg \frac{1}{b} \prod_{p|b} \left(1 - \frac{1}{p}\right)^2 = \frac{\varphi(b)^2}{b^3}$$
$$\gg \frac{1}{b \, (\ln \ln(3b))^2} .$$

It is now easy to deduce from Theorem A.1 the following corollary, where the uniformity over b is almost optimal.

Corollary A.5 *As x tends to infinity, we have*

$$S(x; a, b, 1) \sim c_{a,b,1} \cdot \frac{x^3}{3},$$

uniformly for

$$1 \leqslant b \leqslant x / (\ln(2x))^2 (\ln \ln(3x))^3 . \quad \square$$

Proof of Theorem A.1. For every $x \geqslant 1$, let us first prove that there exists a constant $c_{a,b,k} \in \,]0, 1]$ such that the sum

$$\widetilde{S}(x) = \sum_{\substack{1 \leqslant n \leqslant x \\ n \equiv a \bmod b}} \frac{\varphi(n)}{n} \cdot \frac{\varphi(n+k)}{n+k}$$

satifies the asymptotic formula, uniformly in $a, b \geqslant 1$, $k \geqslant 0$ and $x \geqslant 1$,

$$\widetilde{S}(x) = c_{a,b,k} \, x + \mathrm{O}((\ln 2x) \ln(2x + k)) . \tag{26}$$

Theorem A.1 follows classically, by applying Abel's summation formula

$$\sum_{1 \leqslant n \leqslant x} a_n f(n) = \Big(\sum_{1 \leqslant n \leqslant x} a_n \Big) \, f(x) - \int_1^x \Big(\sum_{1 \leqslant n \leqslant t} a_n \Big) \, f'(t) \, dt$$

to the numerical sequence $\left(a_n = \frac{\varphi(n)}{n} \cdot \frac{\varphi(n+k)}{n+k}\ \delta_n\right)_{n\geqslant 1}$, where $\delta_n = 1$ if $n \equiv a \bmod b$ and $\delta_n = 0$ otherwise, and to the function $f : [1, +\infty[\to \mathbb{R}$ of class C^1 defined by $x \mapsto x(x+k)$. We indeed have

$$\begin{aligned} S(x; a, b, k) &= \widetilde{S}(x)x(x+k) - \int_1^x \widetilde{S}(t)(2t+k)\, dt \\ &= \frac{c_{a,b,k}}{3}\, x^3 + \frac{c_{a,b,k}\, k}{2}\, x^2 + (\frac{2}{3} + \frac{k}{2})c_{a,b,k} + \mathrm{O}\left(x(x+k)(\ln 2x)\ln(2x+k)\right) , \end{aligned}$$

which gives the result since $c_{a,b,k} \leqslant 1$.

Let us denote by **1** the constant arithmetic function with value 1. The convolution equality $\varphi = \mu \star \mathrm{id}$ implies by division that $\frac{\varphi}{\mathrm{id}} = \frac{\mu}{\mathrm{id}} \star \mathbf{1}$. Applying twice this formula, we have

$$\widetilde{S}(x) = \sum_{1\leqslant d\leqslant x} \frac{\mu(d)}{d} \sum_{1\leqslant \delta\leqslant x+k} \frac{\mu(\delta)}{\delta} \sum_{\substack{1\leqslant n\leqslant x \\ d\,|\,n,\ \delta\,|\,(n+k) \\ n\equiv a \bmod b}} 1 .$$

The system of three congruences $n \equiv \begin{cases} 0 \bmod d \\ -k \bmod \delta \\ a \bmod b \end{cases}$ has a solution $n \leqslant x$ if and only if there exists an integer $m \leqslant x/d$ such that $n = dm$ and

$$\begin{cases} dm \equiv -k \mod \delta \\ dm \equiv a \mod b . \end{cases} \tag{27}$$

When $(d, \delta) \nmid k$ or when $(d, b) \nmid a$, no solution exists. We hence have

$$\widetilde{S}(x) = \sum_{\substack{1\leqslant d\leqslant x,\ 1\leqslant \delta\leqslant x+k \\ (d,\delta)\,|\,k,\ (d,b)\,|\,a}} \frac{\mu(d)}{d}\, \frac{\mu(\delta)}{\delta}\, \mathrm{Card}\Big\{m \leqslant x/d : \begin{array}{l} dm \equiv -k \mod \delta \\ dm \equiv a \mod b \end{array} \Big\} .$$

If $(d, \delta) \mid k$ and $(d, b) \mid a$, let us denote by $\overline{\frac{d}{(d,\delta)}}$ the multiplicative inverse of the integer $\frac{d}{(d,\delta)}$ modulo $\frac{\delta}{(d,\delta)}$ and by $\overline{\frac{d}{(d,b)}}$ the multiplicative inverse of the integer $\frac{d}{(d,b)}$ modulo $\frac{b}{(d,b)}$. The system of two congruences (27), after division of its first equation by (d, δ) and of its second equation by (d, b), is then equivalent to the system

$$\begin{cases} m \equiv -\frac{k}{(d,\delta)}\, \overline{\frac{d}{(d,\delta)}} \mod \frac{\delta}{(d,\delta)} \\ m \equiv \frac{a}{(d,b)}\, \overline{\frac{d}{(d,b)}} \mod \frac{b}{(d,b)} . \end{cases}$$

This system has a solution if and only if the following divisibility condition holds

$$\begin{aligned} &\Big(\frac{\delta}{(d,\delta)}, \frac{b}{(d,b)}\Big) \mid \frac{k}{(d,\delta)}\, \overline{\frac{d}{(d,\delta)}} + \frac{a}{(d,b)}\, \overline{\frac{d}{(d,b)}} \\ \Leftrightarrow\ &\Big(\frac{\delta}{(d,\delta)}, \frac{b}{(d,b)}\Big) \mid \frac{k}{(d,\delta)}\, \frac{d}{(d,b)} + \frac{a}{(d,b)}\, \frac{d}{(d,\delta)} \end{aligned}$$

Since the integers $\frac{d}{(d,b)}$ and $\frac{d}{(d,\delta)}$ are coprime with the gcd $\left(\frac{\delta}{(d,\delta)}, \frac{b}{(d,b)}\right)$, we deduce by multiplication that the above condition holds if and only if we have

$$\big(\delta(d,b), b(d,\delta)\big) \mid d(k+a) .$$

The following lemma, where $\mathrm{O}(1)$ is uniformly bounded in $\alpha, \beta, \alpha_0, \beta_0$, is elementary.

Lemma A.6 *For all integers $\alpha, \beta, \alpha_0, \beta_0 \geqslant 1$ and real number $y \geqslant 1$, we have*

$$\operatorname{Card}\{m \leqslant y : m \equiv \alpha_0 \mod \alpha \text{ and } m \equiv \beta_0 \mod \beta\} = \begin{cases} 0 \text{ if } \alpha_0 \not\equiv \beta_0 \mod (\alpha, \beta) \\ \frac{y}{[\alpha,\beta]} + \mathrm{O}(1) \text{ otherwise} . \end{cases} \square$$

This lemma implies that $\operatorname{Card}\Big\{m \leqslant x/d : \begin{array}{c} dm \equiv -k \bmod \delta \\ dm \equiv a \bmod b \end{array}\Big\} = \dfrac{x}{d\big[\frac{\delta}{(d,\delta)}, \frac{b}{(d,b)}\big]} + \mathrm{O}(1)$ under the assumption that $(d,\delta) \mid k$, $(d,b) \mid a$ and $\big(\delta(d,b), b(d,\delta)\big) \mid d(k+a)$, where $\mathrm{O}(1)$ is uniformly bounded in a, b, k, d, δ. By the classical majoration of the harmonic series, we have

$$\sum_{1 \leqslant d \leqslant x,\ 1 \leqslant \delta \leqslant x+k} \frac{1}{d}\,\frac{1}{\delta} = \mathrm{O}((\ln 2x)\ln(2x+k)) .$$

By the relation between lcm, gcd and product of two positive integers, we hence have

$$\widetilde{S}(x) = x \sum_{\substack{1 \leqslant d \leqslant x,\ 1 \leqslant \delta \leqslant x+k \\ (d,\delta) \mid k,\ (d,b) \mid a \\ (\delta(d,b), b(d,\delta)) \mid d(k+a)}} \frac{\mu(d)}{d}\,\frac{\mu(\delta)}{\delta}\,\frac{(\delta(d,b),\, b(d,\delta))}{d\delta b} + \mathrm{O}((\ln 2x)\ln(2x+k)) ,$$

uniformly in $a, b \geqslant 1$, $k \geqslant 0$ and $x \geqslant 1$.

Completing the sum with the indices $d > x$ and $\delta > x+k$ introduces an error of the form (uniformly in $a, b \geqslant 1$, $k \geqslant 0$ and $x \geqslant 1$)

$$\mathrm{O}\Big(\sum_{d \geqslant x,\ \delta \geqslant 1} \frac{(d,\delta)}{d^2\delta^2}\Big) = \mathrm{O}\Big(\sum_{t \geqslant 1,\ d' \geqslant x/t,\ \delta' \geqslant 1} \frac{t}{(td')^2(t\delta')^2}\Big) = \mathrm{O}\Big(\frac{1}{x}\Big) .$$

This proves Formula (26) by setting

$$c_{a,b,k} = \sum_{\substack{d,\ \delta \geqslant 1 \\ (d,\delta) \mid k,\ (d,b) \mid a \\ (\delta(d,b), b(d,\delta)) \mid d(k+a)}} \frac{\mu(d)}{d}\,\frac{\mu(\delta)}{\delta}\,\frac{(\delta(d,b),\, b(d,\delta))}{d\delta b} . \tag{28}$$

Let us now prove Equation (23). By Remark A.2, this implies that $0 < c_{a,b,k} \leqslant 1$, hence completes the proof of Theorem A.1.

Proof of Equation (23). For every integer $d \geqslant 1$, let χ_d be the characteristic function of the set of integers $\delta \geqslant 1$ such that $(\delta, d) \mid k$. For every integer $d \geqslant 1$, let us define

$$\psi_d : \delta \mapsto \big(\delta, \frac{b}{(d,b)}\,(d,\delta)\big) . \tag{29}$$

Note that the assertion $(\delta(d,b), b(d,\delta)) \mid d(k+a)$ is equivalent to the assertion

$$\psi_d(\delta) \ \big|\ \frac{d}{(d,b)}\,(k+a) .$$

For every integer $d \geqslant 1$, let χ^*_d be the characteristic function of the set of integers $\delta \geqslant 1$ such that the above divisibility assertion is satisfied. Let us define

$$c^* : d \mapsto \sum_{\delta \geqslant 1} \frac{\mu(\delta)}{\delta^2}\,\chi_d(\delta)\,\chi^*_d(\delta)\,\psi_d(\delta) \tag{30}$$

(this arithmetic function c^* depends on the constants a, b, k). Equation (28) then becomes

$$c_{a,b,k} = \frac{1}{b} \sum_{\substack{d \geqslant 1 \\ (d,b) \mid a}} \frac{\mu(d)}{d^2} (d,b)\, c^*(d)\,. \tag{31}$$

In order to transform the series $c^*(d)$ defined by Formula (30) into an Eulerian product and in order to analyse it, we will use the following two lemmas.

Lemma A.7 *For every integer $d \geqslant 1$, the arithmetic functions χ_d, χ_d^* and ψ_d are multiplicative.*

Proof. We have $\chi_d(1) = \chi_d^*(1) = \psi_d(1) = 1$. Let δ_1, δ_2 be two coprime integers.

The equality $(\delta_1\delta_2, d) = (\delta_1, d)(\delta_2, d)$ and the fact that (δ_1, d) and (δ_2, d) are coprime imply the multiplicativity of χ_d.

In order to prove the multiplicativity of the function ψ_d, we write

$$\psi_d(\delta_1\delta_2) = \big(\delta_1\delta_2, \frac{b}{(d,b)}\,(d, \delta_1\delta_2)\big) = \big(\delta_1, \frac{b}{(d,b)}\,(\delta_1, d)(\delta_2, d)\big)\big(\delta_2, \frac{b}{(d,b)}\,(\delta_1, d)(\delta_2, d)\big)\,.$$

Since δ_1 is coprime to (δ_2, d) and since δ_2 is coprime to (δ_1, d), we obtain as wanted the equality $\psi_d(\delta_1\delta_2) = \psi_d(\delta_1)\,\psi_d(\delta_2)$.

Finally, the multiplicativity of the function χ_d^* is a consequence of the multiplicativity of the function ψ_d and of the fact that $\psi_d(\delta_1)$ and $\psi_d(\delta_2)$ are coprime. □

Lemma A.8 *For every prime p and every integer $d \geqslant 1$, we have*

$$\psi_d(p) = \begin{cases} p & \text{if} \quad p \mid d, \\ (p,b) & \text{otherwise,} \end{cases}$$

and

$$\chi_d(p)\ \chi_d^*(p) = 1 \Leftrightarrow \begin{cases} p \mid (d,k) \quad \text{and} \quad p \mid \frac{d}{(d,b)}\,(k+a), \\ \text{or} \\ p \nmid d \quad \text{and} \quad (p,b) \mid k+a\,. \end{cases}$$

Proof. The first formula follows from the definition of $\psi_d(p)$ (see Formula (29)) by considering the three cases ($p \mid d$), ($p \nmid d$ and $p \mid b$), and ($p \nmid d$ and $p \nmid b$).

The second formula follows from the first one, from the definitions of $\chi_d(p)$ and $\chi_d^*(p)$, and from the fact that $\chi_d(p)\ \chi_d^*(p) = 1$ if and only if $\chi_d(p) = \chi_d^*(p) = 1$, by considering the two cases ($p \mid d$) and ($p \nmid d$). □

The arithmetic function $\delta \mapsto \mu(\delta)\chi_d(\delta)\ \chi_d^*(\delta)\ \psi_d(\delta)$ being multiplicative by Lemma A.7, and vanishing on the nontrivial powers of primes, the series defining $c^*(d)$ in Formula (30) may be written as an Eulerian product

$$c^*(d) = \prod_p \big(1 - \frac{\chi_d(p)\ \chi_d^*(p)\ \psi_d(p)}{p^2}\big) = \prod_{\substack{p \\ \chi_d(p)\ \chi_d^*(p)=1}} \big(1 - \frac{\psi_d(p)}{p^2}\big)\,. \tag{32}$$

By Equations (31) and (32), and by Lemma A.8, we have

$$c_{a,b,k} = \frac{1}{b} \sum_{\substack{d \geqslant 1 \\ (d,b) \mid a}} \frac{\mu(d)}{d^2} (d,b) \prod_{\substack{p \nmid d \\ (p,b) \mid k+a}} \big(1 - \frac{(p,b)}{p^2}\big) \prod_{\substack{p \mid (d,k) \\ p \mid \frac{d}{(d,b)}(k+a)}} \big(1 - \frac{1}{p}\big)\,.$$

Let us define $\Gamma_{a,b,k} = \prod_{\substack{p \\ (p,b)\,|\,k+a}} \big(1 - \frac{(p,b)}{p^2}\big)$, so that

$$c_{a,b,k} = \frac{\Gamma_{a,b,k}}{b} \sum_{\substack{d\geqslant 1 \\ (d,b)\,|\,a}} \frac{\mu(d)}{d^2}\,(d,b) \prod_{\substack{p\,|\,d \\ (p,b)\,|\,k+a}} \big(1 - \frac{(p,b)}{p^2}\big)^{-1} \prod_{\substack{p\,|\,(d,k) \\ p\,|\,\frac{d}{(d,b)}(k+a)}} \big(1 - \frac{1}{p}\big)\;. \tag{33}$$

For every integer $d \geqslant 1$ without square factor such that $(d,b) \mid a$, we have

$$\begin{aligned}
\prod_{\substack{p\,|\,(d,k) \\ p\,|\,\frac{d}{(d,b)}(k+a)}} \big(1 - \frac{1}{p}\big) &= \prod_{\substack{p\,|\,(d,k) \\ p\,|\,\frac{d}{(d,b)}}} \big(1 - \frac{1}{p}\big) \prod_{\substack{p\,|\,(d,k) \\ p\,|\,k+a}} \big(1 - \frac{1}{p}\big) \prod_{\substack{p\,|\,(d,k) \\ p\,|\,(\frac{d}{(d,b)},k+a)}} \big(1 - \frac{1}{p}\big)^{-1} \\
&= \prod_{p\,|\,(\frac{d}{(d,b)},k)} \big(1 - \frac{1}{p}\big) \prod_{p\,|\,(d,a,k)} \big(1 - \frac{1}{p}\big) \prod_{p\,|\,(\frac{d}{(d,b)},a,k)} \big(1 - \frac{1}{p}\big)^{-1} \\
&= \prod_{p\,|\,(\frac{d}{(d,b)},k)} \big(1 - \frac{1}{p}\big) \prod_{p\,|\,(d,a,b,k)} \big(1 - \frac{1}{p}\big) \\
&= \prod_{p\,|\,(\frac{d}{(d,b)},k)} \big(1 - \frac{1}{p}\big) \prod_{p\,|\,(d,b,k)} \big(1 - \frac{1}{p}\big) = \prod_{p\,|\,(d,k)} \big(1 - \frac{1}{p}\big)\;.
\end{aligned}$$

Thus, Equation (33) writes $c_{a,b,k}$ as a series $\frac{\Gamma_{a,b,k}}{b} \sum_{\substack{d\geqslant 1 \\ (d,b)\,|\,a}} \frac{f(d)}{d^2}$ where f is a multiplicative function, which vanishes on the nontrivial powers of primes. By Eulerian product, we have therefore proved Equation (23). □

Remark A.9 When $a = b = 1$, we indeed recover Mirsky's result [Mir, Thm. 9, Eq. (30)]. Indeed, by Equation (23), we have

$$\begin{aligned}
c_{1,1,k} &= \prod_{p} \big(1 - \frac{1}{p^2}\big) \prod_{p\,|\,k} \Big(1 - \frac{(1-\frac{1}{p^2})^{-1}(1-\frac{1}{p})}{p^2}\Big) \prod_{p\nmid k} \Big(1 - \frac{(1-\frac{1}{p^2})^{-1}}{p^2}\Big) \\
&= \prod_{p} \big(1 - \frac{1}{p^2}\big) \prod_{p\,|\,k} \Big(1 - \frac{p-1}{p(p^2-1)}\Big) \prod_{p} \big(1 - \frac{1}{p^2-1}\big) \prod_{p\,|\,k} \Big(1 - \frac{1}{p^2-1}\Big)^{-1} \\
&= \prod_{p} \big(1 - \frac{2}{p^2}\big) \prod_{p\,|\,k} \big(1 + \frac{1}{p(p^2-2)}\big)\;.
\end{aligned}$$

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