METRIC RECTIFIABILITY OF $\mathbb{H}$-REGULAR SURFACES WITH HÖLDER CONTINUOUS HORIZONTAL NORMAL

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ABSTRACT. Two definitions for the rectfiability of hypersurfaces in Heisenberg groups $\mathbb{H}^n$ have been proposed: one based on $\mathbb{H}$-regular surfaces, and the other on Lipschitz images of subsets of codimension-1 vertical subgroups. The equivalence between these notions remains an open problem. Recent partial results are due to Cole-Pauls, Bigolin-Vittone, and Antonelli-Le Donne.

This paper makes progress in one direction: the metric Lipschitz rectifiability of $\mathbb{H}$-regular surfaces. We prove that $\mathbb{H}$-regular surfaces in $\mathbb{H}^n$ with α-Hölder continuous horizontal normal, $\alpha > 0$, are metric bilipschitz rectifiable. This improves on the work by Antonelli-Le Donne, where the same conclusion was obtained for C^∞-surfaces.

In $\mathbb{H}^1$, we prove a slightly stronger result: every codimension-1 intrinsic Lipschitz graph with an ϵ of extra regularity in the vertical direction is metric bilipschitz rectifiable. All the proofs in the paper are based on a new general criterion for finding bilipschitz maps between "big pieces" of metric spaces.

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Date: July 9, 2021.

2010 *Mathematics Subject Classification.* 30L05 (primary) 28A78 (secondary).

Key words and phrases. Uniform rectifiability, big pieces of Lipschitz images, intrinsic graphs.

D.D.D. is partially supported by the Academy of Finland (grant 288501 '*Geometry of subRiemannian groups*' and by grant 322898 '*Sub-Riemannian Geometry via Metric-geometry and Lie-group Theory*') and by the European Research Council (ERC Starting Grant 713998 GeoMeG '*Geometry of Metric Groups*'). K.F. is supported by the Academy of Finland trough the grant 321696 'Singular integrals, harmonic functions, and boundary regularity in Heisenberg groups'. T.O. is supported by the Academy of Finland via the project *Quantitative rectifiability in Euclidean and non-Euclidean spaces*, grant Nos. 309365, 314172.

1. Introduction

This paper concerns the relationship between two notions of *codimension-1 rectifiability* in the Heisenberg group $(\mathbb{H}^n, d_{\mathbb{H}}) = (\mathbb{R}^{2n+1}, \cdot, d_{\mathbb{H}})$, where "$\cdot$" is the group product

$$(x_1, \ldots, x_{2n}, t) \cdot (x_1', \ldots, x_{2n}', t') = \left(\sum_{i=1}^{2n} x_i + x_i', t + t' + \tfrac{1}{2} \sum_{i=1}^{n} x_i x_{n+i}' - x_{n+i} x_i' \right) \in \mathbb{R}^{2n} \times \mathbb{R},$$

and $d_{\mathbb{H}}$ is the Korányi distance $d_{\mathbb{H}}(p,q) := \|q^{-1} \cdot p\|$ (with $\|(x,t)\| := \sqrt[4]{|x|^4 + 16t^2}$ for $(x,t) \in \mathbb{R}^{2n} \times \mathbb{R}$). Metric notions in $\mathbb{H}^n$, notably Hausdorff measures, are defined using the metric $d_{\mathbb{H}}$. Metric notions in $\mathbb{R}^n$ are defined using the standard Euclidean distance.

In $\mathbb{R}^n$, the notion of rectifiability can be defined in two equivalent ways. For $0 < m < n$, an $\mathcal{H}^m$ measurable set $E \subset \mathbb{R}^n$ is called *m-rectifiable* if $\mathcal{H}^m$ almost all of E can be covered by either

(1) countably many Lipschitz m-images, or
(2) countably many Lipschitz m-graphs.

Here, a *Lipschitz m-image* means a Lipschitz image of $\mathbb{R}^m$, while a *Lipschitz m-graph* means a set of the form $\{v + A(v) : v \in V\}$, where $V \subset \mathbb{R}^n$ is an m-dimensional subspace, and $A \colon V \to V^{\perp}$ is a Lipschitz map. The equivalence of "Lipschitz image rectifiability" and "Lipschitz graph rectifiability" is well-known. In particular, Lipschitz m-graphs are trivially Lipschitz m-images, since $v \mapsto v + A(v)$ is Lipschitz whenever A is.

We then discuss the analogues of these notions in $\mathbb{H}^n$. Recall first that $(\mathbb{H}^n, d_{\mathbb{H}})$ is a metric space of Hausdorff dimension $2n+2$. A common notion of codimension-1 rectifiability (see [20, Definition 4.33]) is *intrinsic Lipschitz graph (iLG) rectifiability*: an $\mathcal{H}^{2n+1}$ measurable set $E \subset \mathbb{H}^n$ is called iLG rectifiable if $\mathcal{H}^{2n+1}$ almost all of E can be covered by countably many iLGs over *vertical subgroups of codimension* 1. Here, vertical subgroups refer to codimension-1 subspaces of $\mathbb{R}^{2n+1}$ containing the t-axis, cf. Section 3, while iLGs were introduced by Franchi, Serapioni, and Serra Cassano [19] in 2006. They are natural $\mathbb{H}^n$ counterparts of Lipschitz graphs in $\mathbb{R}^n$, see Definition 1.2.

The notion of *Lipschitz image (LI) rectifiability* in $\mathbb{H}^n$ was first studied by Pauls [24] in 2004. An $\mathcal{H}^{2n+1}$ measurable set $E \subset \mathbb{H}^n$ is called LI rectifiable if $\mathcal{H}^{2n+1}$ almost all of E can be covered by countably many Lipschitz images of closed subsets of codimension-1 vertical subgroups. All of these subgroups (for $n \geqslant 1$ fixed) are isometrically isomorphic to each other. If $n = 1$, they are further isometrically isomorphic to the *parabolic plane*

$$\Pi := (\mathbb{R}^2, +, \|\cdot\|), \quad \text{where} \quad \|(y,t)\| := \sqrt[4]{y^4 + 16t^2},$$

and if $n \geqslant 2$, they are isometrically isomorphic to $(\mathbb{H}^{n-1} \times \mathbb{R}, \cdot_{\mathbb{H}^{n-1}\times\mathbb{R}}, \|\cdot\|)$ with

$$((z,t),s) \cdot_{\mathbb{H}^{n-1}\times\mathbb{R}} ((z',t'),s') = \Big(\big(z + z', t + t' + \tfrac{1}{2} \sum_{i=1}^{n-1} (z_i z_{n-1+i}' - z_i' z_{n-1+i})\big), s + s' \Big) \quad (1.1)$$

and

$$\|((z,t),s)\| = \sqrt[4]{|(z,s)|^4 + 16t^2}.$$

So, E is LI rectifiable if and only if $\mathcal{H}^{2n+1}$ almost all of E can be covered by countably many Lipschitz images of closed subsets of Π (if $n = 1$) or $\mathbb{H}^{n-1} \times \mathbb{R}$ (if $n \geqslant 2$). The metric induced by $\| \cdot \|$ in Π is denoted d_Π, and in $\mathbb{H}^{n-1} \times \mathbb{R}$ by $d_{\mathbb{H}^{n-1}\times\mathbb{R}}$.

The connection between iLG and LI rectifiability in $\mathbb{H}^n$ is poorly understood. It is neither known if (a) LIs of vertical subgroups are iLG rectifiable, nor if (b) iLGs are LI rectifiable. It may appear surprising that question (b) is open: after all, to show that Lipschitz m-graphs in $\mathbb{R}^n$ are Lipschitz m-images, one only needed to observe that the graph map $v \mapsto v + A(v)$ is Lipschitz whenever A is. In $\mathbb{H}^n$, this argument fails completely. We will discuss the matter further in a moment.

The purpose of this paper is to make progress in question (b). In brief, we will show intrinsic $C^{1,\alpha}$-graphs in $\mathbb{H}^n$ are LI rectifiable for any $\alpha > 0$. In $\mathbb{H}^1$, we can say something a little better. For precise statements, see Theorems 1.6, 1.7, and 1.11. Before formulating these new results in detail, we define our objects of study more carefully, and describe some previous work on the topic.

Definition 1.2. An *intrinsic graph over the vertical subgroup* $\mathbb{W} = \{x_1 = 0\}$ in $\mathbb{H}^n$ is a set of the form

$$S = \{w \cdot \varphi(w) : w \in \mathbb{W}\}, \tag{1.3}$$

where $\varphi \colon \mathbb{W} \to \mathbb{V} =: \{(x_1, 0, \ldots, 0) : x_1 \in \mathbb{R}\}$ is an arbitrary function. The graph S determines φ uniquely. Further, S is an intrinsic L-**Lipschitz** graph (L-iLG) over $\mathbb{W}$ if it satisfies a *cone condition* of the form

$$S \cap (p \cdot \mathcal{C}(\alpha)) = \{p\}, \qquad p \in S,\ 0 < \alpha < L^{-1}.$$

Here $\mathcal{C}(\alpha) := \{q \in \mathbb{H}^n : \|\pi_{\mathbb{W}}(p)\| \leqslant \alpha \|\pi_{\mathbb{V}}(p)\|\}$, and $\pi_{\mathbb{W}} \colon \mathbb{H}^n \to \mathbb{W}$ and $\pi_{\mathbb{V}} \colon \mathbb{H}^n \to \mathbb{V}$ are the *vertical and horizontal* projections induced by the splitting $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$. If $S \subset \mathbb{H}^n$ is an (L-)iLG, the function φ is called an (L-)*intrinsic Lipschitz function*.

Now, viewing (1.3), it is clear that an iLG $S \subset \mathbb{H}^n$ has a "canonical" parametrisation by the *graph map* $\Phi \colon \mathbb{W} \to S$, defined by $\Phi(w) := w \cdot \varphi(w)$. However:

- $\varphi \colon (\mathbb{W}, d_{\mathbb{H}}) \to (\mathbb{V}, d_{\mathbb{H}})$ is not always a Lipschitz function, and
- the graph map $\Phi \colon (\mathbb{W}, d_{\mathbb{H}}) \to (S, d_{\mathbb{H}})$ is "almost never" Lipschitz.

Regarding the first point, [19, Example 3.3] suggests that $\varphi(0, x_2, t) = (1 + t^{1/2}, 0, 0)$ is an intrinsic Lipschitz function in $\mathbb{H}^1$, which is not a Lipschitz function. For the second point, consider the constant function $\varphi(0, x_2, t) \equiv (1, 0, 0)$. Then the graph map

$$\Phi(0, x_2, t) = (0, x_2, t) \cdot (1, 0, 0) = (1, x_2, t - \tfrac{x_2}{2})$$

parametrises the vertical plane $S = \mathbb{W}' = \{x_1 = 1\} \subset \mathbb{H}^1$, a prototypical iLG. However, Φ is not Lipschitz on any open subset of $\mathbb{W}$, because the only rectifiable curves on $\mathbb{W}, \mathbb{W}'$ are the *horizontal lines* contained on $\mathbb{W}, \mathbb{W}'$, and Φ sends the horizontal lines on $\mathbb{W}$ to non-horizontal lines on $\mathbb{W}'$ (by **right** translations).

In spite of these difficulties, the graph map is sometimes useful for Lipschitz parametrising iLGs: if an iLG $S \subset \mathbb{H}^1$ has enough *a priori* regularity, then Φ can precomposed with something known as the *characteristic straightening map* $\Psi \colon \Pi \to \mathbb{W}$ in such a way that $\Phi \circ \Psi \colon \Pi \to S$ is locally Lipschitz – or even bilipschitz. The following theorem is due to Cole and Pauls [12] from 2006 (the addition of the letters "bi" is due to Bigolin and Vittone [7] from 2010):

Theorem 1.4 (Cole-Pauls, Bigolin-Vittone)**.** *Every non-characteristic point on a Euclidean C^1 surface $S \subset \mathbb{H}^1$ has a neighbourhood which is the bilipschitz image of an open subset of Π.*

Both proofs reduce the problem to intrinsic Lipschitz graphs $S = \Phi(\mathbb{W}) \subset \mathbb{H}^1$, and the Euclidean C^1-smoothness of S then translates to properties of the intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{V}$. The essential hypothesis is that φ is a Euclidean C^1-function, although it might suffice that φ is Euclidean Lipschitz, viewed as a function $\mathbb{R}^2 \to \mathbb{R}$. The characteristic straightening map only plays a small (and rather implicit) role in this paper, see Lemma 4.10. So, we refer to the "Outline of proofs" section in [6], or the proof of [7, Theorem 3.1] for more details. In brief, the regularity of φ is, in Theorem 1.4, required to control the regularity of Ψ, and if φ fails to be Lipschitz $\mathbb{R}^2 \to \mathbb{R}$, the map Ψ does not appear to be useable for the Lipschitz parametrisation problem.

In fact, Bigolin and Vittone in [7] show that Theorem 1.4 can fail without the C^1-regularity assumption. For $\frac{1}{2} < \alpha < 1$, they consider the intrinsic Lipschitz function

$$\varphi(0, x_2, t) = \begin{cases} -\frac{t^\alpha}{1-\alpha}, & \text{if } t \geqslant 0, \\ 0, & \text{if } t < 0, \end{cases} \qquad (0, x_2, t) \in \mathbb{W}, \tag{1.5}$$

and its intrinsic graph $S = \Phi(\mathbb{W})$, which fails to be Euclidean C^1-regular in any neighbourhood of the line $\{(0, x_2, 0) : x_2 \in \mathbb{R}\}$. They show that no Lipschitz map from an open subset of Π to a neighbourhood of $0 \in \Gamma$ can have a Lipschitz inverse.

The example (1.5) is a good prelude to the results of this paper. The main novelties will be to

(a) say something about the LI rectifiability of iLGs below the critical C^1-regularity of φ (in particular, our results apply to the example in (1.5) for $\frac{1}{2} < \alpha < 1$),
(b) consider the problem in higher Heisenberg groups, where the technique via the characteristic straightening map does not seem to be easily available.

Here is the first main result:

Theorem 1.6. *Let $\alpha > 0$, and let $S \subset \mathbb{H}^n$ be the intrinsic graph of a globally defined but compactly supported intrinsic $C^{1,\alpha}$-function. Then S has big pieces of bilipschitz images of the parabolic plane (Π, d_Π) if $n = 1$, or of $(\mathbb{H}^{n-1} \times \mathbb{R}, d_{\mathbb{H}^{n-1} \times \mathbb{R}})$ if $n \geqslant 2$. In particular, S is LI rectifiable.*

The following corollary is easier to read:

Theorem 1.7. *Let $S \subset \mathbb{H}^n$ be a $C^{1,\alpha}_{\mathbb{H}}$-surface. Then S is LI rectifiable.*

Remark 1.8. A first version of the present paper, by the third author, contained Theorems 1.6 and 1.7 in $\mathbb{H}^1$. After that version appeared on the arXiv, Antonelli and Le Donne proved in [2], building on [22], that every C^∞ hypersurface S in $\mathbb{H}^n$, $n \geqslant 2$, is rectifiable by bilipschitz images of subsets of $\mathbb{H}^{n-1} \times \mathbb{R}$.

The result of Antonelli-Le Donne follows from Theorem 1.7 (or rather, the bilipschitz version stated in Theorem 3.28): the C^∞ regularity of S, and a result of Balogh [5], imply that $\mathcal{H}^{2n+1}$ almost every point on S has an open neighbourhood U such that $S \cap U$ is given as the level set of a C^∞ function $f : U \to \mathbb{R}$ with nonvanishing *horizontal gradient* $\nabla_{\mathbb{H}} f$. Using that f is Euclidean C^∞ (or even Euclidean C^2), one concludes that f satisfies condition (3.4) in Definition 3.2, possibly on a slightly smaller open set. Thus, outside an $\mathcal{H}^{2n+1}$ null set, S is locally a $C^{1,\alpha}_{\mathbb{H}}$ surface, and it follows from Theorem 3.28 that S is rectifiable by bilipschitz images of compact subsets of $\mathbb{H}^{n-1} \times \mathbb{R}$.

The notions of regularity appearing in the theorems above will be formally introduced in Section 3. In brief, a $C^{1,\alpha}_{\mathbb{H}}$-surface is an $\mathbb{H}$-regular surface whose *horizontal normal* is α-Hölder continuous (in the metric $d_{\mathbb{H}}$). Then, roughly speaking, an intrinsic $C^{1,\alpha}$-function is a function $\mathbb{W} \to \mathbb{V}$ whose intrinsic graph $\Phi(\mathbb{W})$ is a $C^{1,\alpha}_{\mathbb{H}}$-surface, but this is a little inaccurate; see Definition 3.6 and Remark 3.7 for more precision.

The next example points out that Theorem 1.7 applies to the function in (1.5):

Example 1.9. *The horizontal normal of the intrinsic graph of the function φ from (1.5) is*

$$\nu_{\mathbb{H}}(\Phi(0,x_2,t)) = \left(\frac{-1}{\sqrt{1+(c_\alpha t^{2\alpha-1})^2}}, \frac{c_\alpha t^{2\alpha-1}}{\sqrt{1+(c_\alpha t^{2\alpha-1})^2}} \right), \qquad x_2 \in \mathbb{R},\ t > 0,$$

where $c_\alpha := \alpha/(1-\alpha)^2$, and $\nu_{\mathbb{H}}(\Phi(0,x_2,t)) \equiv (-1,0)$ for $x_2 \in \mathbb{R}$, $t \leqslant 0$. This follows by combining the expression for the intrinsic gradient *of φ on* [7, p. 166] *with a known relationship between the horizontal normal of $\Phi(\mathbb{W})$ and the intrinsic gradient of φ, see the references around (3.13). With the explicit expression in hand, let us show that the horizontal normal map $\nu_{\mathbb{H}} \colon (\Phi(\mathbb{W}), d_{\mathbb{H}}) \to S^1$ is $(2\alpha-1)/\alpha$-Hölder. First, observe by calculating derivatives that the functions $f, g \colon \mathbb{R} \to \mathbb{R}$ defined by $f(r) := -1/\sqrt{1+r^2}$ and $g(r) := r/\sqrt{1+r^2}$ are (globally) Lipschitz. Consequently, for $(0,x_2,s), (0,x_2',t) \in \mathbb{W}$, we have*

$$\begin{aligned}|\nu_{\mathbb{H}}(\Phi(0,x_2,s)) - \nu_{\mathbb{H}}(\Phi(0,x_2',t))| &\lesssim |f(c_\alpha s^{2\alpha-1}) - f(c_\alpha t^{2\alpha-1})| + |g(c_\alpha s^{2\alpha-1}) - g(c_\alpha t^{2\alpha-1})| \\ &\lesssim c_\alpha \cdot |s^{2\alpha-1} - t^{2\alpha-1}|.\end{aligned}$$

Next, noting that $(2\alpha-1)/\alpha \in (0,1]$ for $\frac{1}{2} < \alpha \leqslant 1$, we have

$$|s^{2\alpha-1} - t^{2\alpha-1}| \leqslant |s^\alpha - t^\alpha|^{(2\alpha-1)/\alpha} \lesssim_\alpha d_{\mathbb{H}}(\Phi(0,x_2,s), \Phi(0,x_2',t))^{(2\alpha-1)/\alpha},$$

as claimed. It follows that $\Phi(\mathbb{W})$ satisfies the assumptions of Theorem 1.7, which are made precise in Definition 3.2 (see also Remark 3.5).

The next definition explains the rest of the terminology in Theorem 1.6:

Definition 1.10 (BPGBI). Fix $n \in \mathbb{N}$ and set $(G, d_G) = (\Pi, d_\Pi)$ if $n = 1$, and $(G, d_G) = (\mathbb{H}^{n-1} \times \mathbb{R}, d_{\mathbb{H}^{n-1}\times\mathbb{R}})$ if $n \geqslant 2$. Let $E \subset \mathbb{H}^n$ be closed and $(2n+1)$-regular. Then, E has *big pieces of G bilipschitz images* (BPGBI) if there exist constants $L \geqslant 1$ and $\theta > 0$ such that the following holds: for every $p \in E$ and $0 < r \leqslant \operatorname{diam}_{\mathbb{H}}(E)$, there exists a compact set $K \subset B(0,r) \subset G$ and an L-bilipschitz map $f \colon K \to \mathbb{H}^n$ such that

$$\mathcal{H}^{2n+1}(f(K) \cap [E \cap B(p,r)]) \geqslant \theta r^{2n+1}.$$

So, up to some technical assumptions, Theorem 1.6 states that intrinsic $C^{1,\alpha}$-graphs in $\mathbb{H}^n$ are *uniformly rectifiable* by (G, d_G), in the spirit of David and Semmes [13]. We next formulate a stronger result in $\mathbb{H}^1$. Informally, the point is that we can relax $C^{1,\alpha}$-regularity to Lipschitz regularity in the "horizontal" directions, but we still need to assume an ϵ of additional *a priori* regularity in the vertical direction. To motivate the definition, we note that intrinsic $C^{1,\alpha}$-functions are locally Euclidean $(1+\alpha)/2$ Hölder continuous along vertical lines by [10, Proposition 4.2]. In $\mathbb{H}^1$, it turns out that this property of *extra vertical Hölder regularity* alone implies the conclusion of Theorem 1.6.

Theorem 1.11. *Let $S \subset \mathbb{H}^1$ be the intrinsic graph of a globally defined but compactly supported intrinsic Lipschitz function with extra vertical Hölder regularity, see Definition 4.1. Then S has big pieces of bilipschitz images of the parabolic plane (Π, d_Π). In particular, S is LI rectifiable.*

The "extra vertical Hölder regularity" is often weaker than intrinsic $C^{1,\alpha}$-regularity: for example, intrinsic Lipschitz functions of the form $\varphi(0,y,t) = \tilde{\varphi}(y)$, with $\tilde{\varphi} \colon \mathbb{R} \to \mathbb{R}$ Lipschitz, are not necessarily intrinsic $C^{1,\alpha}$, but they are very smooth along vertical lines.

Theorem 1.11 can be used to give alternative proofs for the LI rectifiability of Euclidean C^1 hypersurfaces in $\mathbb{H}^1$ (originally due to Cole-Pauls [12]) and of the $n = 1$ case of Theorem 1.6 (originally due to the third author). Intrinsic graphs that satisfy the assumptions of Theorem 1.11 are examples of sets on which the 3-dimensional *Heisenberg Riesz transform* is L^2-bounded, see [16]. The essential property of such graphs used here is that they can be well approximated by *Lipschitz flags* (Definition 4.8), and that Lipschitz flags can be bilipschitz parametrised using the characteristic straightening map of Cole-Pauls and Bigolin-Vittone. No counterparts for these properties are known in higher dimensions.

We close this section with a few questions:

Questions. *Are all intrinsic Lipschitz graphs in $\mathbb{H}^n$ LI rectifiable? If so, do they have big pieces of Lipschitz images of G, or BPGBI? In the converse direction: Are (bi-)Lipschitz images of vertical subgroups in $\mathbb{H}^n$ iLG rectifiable?*

1.1. **Bilipschitz maps between big pieces of metric spaces.** As mentioned above Theorem 1.6, adapting the techniques of Cole-Pauls and Bigolin-Vittone seems difficult without something close to C^1-regularity, or if $n \geqslant 2$. Instead, Theorems 1.6 and 1.11 will follow from an application of a general result concerning metric spaces, Theorem 1.14 below. We now formulate the abstract hypotheses of that theorem.

Let (G, d_G) and (M, d_M) be metric spaces. Assume the following "local correspondence" between G and M, for constants $\alpha > 0$, $L \geqslant 1$, $A \geqslant 1$, and for some $x_0 \in G$ and $p_0 \in M$ fixed. For every $x \in B_G(x_0, 1)$, $p \in B_M(p_0, 1)$ and $n \in \mathbb{N} \cup \{0\}$ there exists a map $i^n_{x \to p} \colon G \to M$ with $i^n_{x\to p}(x) = p$ such that

$$L^{-1}d_G(y,z) - A2^{-n(1+\alpha)} \leqslant d_M(i^n_{x\to p}(y), i^n_{x\to p}(z)) \leqslant Ld_G(y,z) + A2^{-n(1+\alpha)}, \quad y,z \in B(x, 2^{-n}). \tag{1.12}$$

Moreover, if $n \geqslant 0$, $x, y \in B_G(x_0, 1)$ and $p, q \in B_M(p_0, 1)$ with $d_G(x,y) \leqslant 2^{-n}$ and $i^n_{x\to p}(y) = q$, then

$$d_M(i^n_{x\to p}(z), i^{n+1}_{y\to q}(z)) \leqslant A2^{-n(1+\alpha)} \qquad z \in B(x, 2^{-n}). \tag{1.13}$$

These assumptions are reminiscent of [14, (1.6)-(1.9)]. See also [8, Appendix 1] for related results. The assumption (1.12) postulates that the maps $i^n_{x\to p}$ are bilipschitz continuous at the scale 2^{-n}, up to an error which is much smaller than 2^{-n}. The assumption (1.13) is "compatibility condition": it states that the maps $i^n_{x\to p}$ and $i^{n+1}_{y\to q}$ nearly coincide at scale 2^{-n}, again up to an error which is much smaller than 2^{-n}. The *a priori* condition "$i^n_{x\to p}(y) = q$" above (1.13) is a technically convenient way of assuming that p and q are close to each other on M: indeed it follows from the hypotheses, including (1.12), that

$$d_M(p,q) = d_M(i^n_{x\to p}(x), i^n_{x\to p}(y)) \leqslant Ld_G(x,y) + A2^{-n(1+\alpha)} \lesssim_{A,L} 2^{-n}.$$

The assumptions (1.12)-(1.13) are designed to enable the construction of bilipschitz maps from subsets of G to M, at least if G is complete and doubling:

Theorem 1.14. *Assume that* (G, d_G) *is a complete metric space with* $\operatorname{diam}_G(G) \geqslant 1$ *and equipped with a nontrivial doubling measure* μ*, and* (M, d_M) *is complete. Assume that the conditions* (1.12)-(1.13) *hold for some* $\alpha > 0$*,* $L \geqslant 1$*, and* $A \geqslant 1$*. Then, there exists a constant* $\delta > 0$*, a compact set* $K \subset B(x_0, 1)$ *with* $\mu(K) \geqslant \delta\mu(B(x_0, 1))$ *and a* $2L$*-bilipschitz embedding* $F \colon K \to M$ *with* $F(K) \subset B(p_0, 1)$*. The constant* $\delta > 0$ *only depends on the doubling constant of* (G, d_G, μ)*, and the constants* α*,* L*, and* A *in* (1.12)-(1.13).

The plan of the paper is to prove Theorem 1.14 in Section 2 and Appendix A, and then apply it to prove Theorem 1.6 for $\mathbb{H}^n$, $n > 1$, in Section 3. In Section 4 we prove Theorem 1.11, which yields as a corollary the case $n = 1$ of Theorem 1.6. Theorem 1.7 is "morally" a direct corollary of Theorem 1.6: by the implicit function theorem of Franchi, Serapioni, and Serra Cassano [18, Theorem 6.5], a $C^{1,\alpha}_{\mathbb{H}}$-surface is locally parametrisable by an intrinsic $C^{1,\alpha}$-function over some vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$. However, the parametrisation may only be defined on a strict subset of $\mathbb{W}$, and fail to literally satisfy the assumptions of Theorem 1.6. As far as we know, there is no extension theorem for intrinsic $C^{1,\alpha}$-functions available. To bypass the issue, Appendix B contains a proposition saying that every point on a $C^{1,\alpha}_{\mathbb{H}}$-surface has a neighbourhood which is contained on the intrinsic graph of a globally defined, compactly supported intrinsic $C^{1,\alpha/3}$-function. With this proposition in hand, Theorem 1.7 is indeed a corollary of Theorem 1.6.

1.2. Notations.

For $A, B > 0$, we write $A \lesssim B$ if there is a constant $C > 0$ such that $A \leqslant CB$. If we want to specify that the value of C is allowed to depend on an auxiliary parameter h, we will write $A \lesssim_h B$.

1.3. Acknowledgements.

K.F. and T.O. would like to thank Enrico Le Donne and Séverine Rigot for many fruitful discussions on the topic of the paper. We are also grateful to Davide Vittone for tips on proving Proposition B.1, which was needed to reduce Theorem 1.7 to Theorem 1.6, and to Damian Dąbrowski for pointing out the reference [4]. D.D.D. would like to thank Enrico Le Donne and Raul Serapioni for important suggestions on the subject. Finally, we thank the anonymous referees for a careful reading of the manuscript, and for making many useful suggestions.

2. Proof of the main theorem for metric spaces

The proof of Theorem 1.14 is inspired by the recent work of Le Donne and Young [22] on the Carnot rectifiability of sub-Riemannian manifolds.

Before starting the proof of Theorem 1.14 in earnest, we need to introduce some terminology. Assume for a moment that $n_0 \geqslant 0$, and there exist families $\{\mathcal{D}_n\}_{n \geqslant n_0}$ of subsets of G, known as *cubes*, with the following properties:

(i) Each $\mathcal{D}_n$ consists of a finite number of disjoint non-empty compact sets, and in particular $\operatorname{card} \mathcal{D}_{n_0} = 1$.

(ii) For $n > n_0$, each cube $Q \in \mathcal{D}_n$ is contained in a unique cube $\hat{Q} \in \mathcal{D}_{n-1}$, called the *parent* of Q. For $Q \in \mathcal{D}_{n-1}$ fixed, write $\operatorname{ch}(Q) := \{Q \in \mathcal{D}_n : \hat{Q} = Q\}$.

(iii) $\operatorname{diam}_G(Q) < 2^{-n}$ for all $Q \in \mathcal{D}_n$.

(iv) There are constants $\epsilon, \tau > 0$ such that if $n \geqslant 0$ and $Q_1, Q_2 \in \mathcal{D}_n$ are distinct, then $d_G(Q_1, Q_2) \geqslant \tau 2^{-(1+\epsilon)n}$.

An (ϵ, n_0, τ)-*Cantor set* is any set of the form

$$K := \bigcap_{n \geqslant n_0} K_n := \bigcap_{n \geqslant n_0} \bigcup_{Q \in \mathcal{D}_n} Q,$$

where the families $\mathcal{D}_n$, $n_0 \geqslant 0$, satisfy properties (i)-(iv).

Definition 2.1. A metric measure space (X, d, μ) *admits fat Cantor sets* if for all $\epsilon > 0$ and $n_0 \geqslant 0$, there exist constants $\delta = \delta(n_0) > 0$, and $\tau = \tau(\epsilon) > 0$ such that the following holds. For all $x \in X$, there exists an (ϵ, n_0, τ)-Cantor set $K \subset B(x, 1)$ with $\mu(K) \geqslant \delta\mu(B(x, 1))$.

We will only use the assumption that (G, d_G, μ) is doubling and complete to ensure that it admits fat Cantor sets.

Proposition 2.2. *Every doubling and complete metric measure space (X, d, μ) of diameter $\geqslant 1$ admits fat Cantor sets. In other words, for every $\epsilon > 0$ and $n_0 \geqslant 0$, the constants $\delta(n_0) > 0$ and $\tau(\epsilon) > 0$ can be found as in Definition 2.1. They are also allowed to depend on the doubling constant of (X, d, μ).*

The proof of the proposition above is essentially contained in [22, Section 4.1], but since the statement is not explicitly given in [22], we repeat the details in Appendix A. What follows next is a proof of Theorem 1.14, assuming Proposition 2.2. Fix $x_0 \in G$ and $p_0 \in M$. Let $\epsilon := \alpha/2$ (where $\alpha > 0$ is the parameter appearing in (1.12)-(1.13)), let $n_0 \geqslant 0$ be a large integer to be determined later, and let $K \subset B(x_0, 1)$ be an (ϵ, n_0, τ)-Cantor set associated to families of cubes $\{\mathcal{D}_n\}_{n \geqslant n_0}$, as in (i)-(iv). For each $Q \in \mathcal{D}_n$, $n \geqslant n_0$, pick a *centre* $c_Q \in Q$. Set $\mathcal{D}_{n_0 - 1} := \{G\}$ and $K_{n_0-1} := G$.

The map $F \colon K \to M$ will be defined as the limit of certain intermediate maps $F_n \colon K_n \to M$ for $n \geqslant n_0 - 1$. Set $F_{n_0 - 1} \equiv p_0$. To proceed, assume that $n \geqslant n_0$, and the map $F_{n-1} \colon K_{n-1} \to M$ has already been defined. Then, fix $Q_{n-1} \in \mathcal{D}_{n-1}$, and set

$$F_n|_Q := i^n_{c_Q \to F_{n-1}(c_Q)}|_Q, \qquad Q \in \mathrm{ch}(Q_{n-1}) \subset \mathcal{D}_n.$$

The next lemma shows, in particular, that if $x \in K$, then the sequence $(F_n(x))_{n \in \mathbb{N}}$ is Cauchy in (M, d_M). Hence $F(x) := \lim_{n \to \infty} F_n(x)$ exists by the completeness of (M, d_M).

Lemma 2.3. *If $n_0 \geqslant 1$ is large enough (depending on A and the constant L in (1.12)), the following holds for all $w \in K$ and $n \geqslant n_0$:*

$$d_M(F_n(w), F_{n+1}(w)) \leqslant A2^{-n(1+\alpha)}. \tag{2.4}$$

Proof. For $w \in K$, let $c_n(w)$ be the centre of $Q_n(w)$, where $Q_n(w)$ is the unique cube in $\mathcal{D}_n$ containing w. One can infer (2.4) from the compatibility condition (1.13) in the following way:

$$d_M(F_n(w), F_{n+1}(w)) = d_M(i^n_{c_n(w) \to F_{n-1}(c_n(w))}(w), i^{n+1}_{c_{n+1}(w) \to F_n(c_{n+1}(w))}(w)) \leqslant A2^{-n(\alpha+1)}.$$

To check that the assumptions of (1.13) are really in force, use the notational substitutions

$$q := F_n(c_{n+1}(w)),\ p := F_{n-1}(c_n(w)),\ x := c_n(w), y := c_{n+1}(w), \quad \text{and} \quad z := w.$$

Then, note that $d_G(x, y) \leqslant \mathrm{diam}_G(Q_n(x)) < 2^{-n}$ by (iii), $z \in B(x, 2^{-n})$ again by (iii), and

$$q = F_n(c_{n+1}(w)) = i^n_{c_n(w) \to F_{n-1}(c_n(w))}(c_{n+1}(w)) = i^n_{x \to p}(y). \tag{2.5}$$

by definition. This shows that the assumptions of (1.13) are valid, except for one small issue: is it clear that $p, q \in B(p_0, 1)$? For $n = n_0$, simply $p = F_{n_0-1}(c_{n_0}(w)) = p_0$. Also, using (2.5) and (1.12), we find that

$$d_M(q,p) = d_M(i^n_{x\to p}(y), i^n_{x\to p}(x)) \leqslant L d_G(x,y) + A2^{-n(1+\alpha)} \lesssim_L A2^{-n}.$$

Recalling the definitions of p, q, and using the estimate above repeatedly shows that $\max\{d_M(p,p_0), d_M(q,p_0)\} \lesssim_L A2^{-n_0}$ for all $n \geqslant n_0$. In particular, $p, q \in B(p_0, 1)$ if $n_0 \geqslant 1$ is large enough, depending on A and the constant L in (1.12). The proof of the lemma is complete. □

As an immediate corollary, one deduces the useful estimate

$$d_M(F_n(x), F(x)) \lesssim A2^{-n(1+\alpha)}, \qquad x \in K,\ n \geqslant n_0. \tag{2.6}$$

It remains to prove that F is $2L$-bilipschitz on K if the index $n_0 \in \mathbb{N}$ was chosen large enough, depending on the parameters $\alpha > 0$, $L \geqslant 1$ and $A \geqslant 1$. Fix $x, y \in K$ arbitrary with $x \neq y$, and let $Q \in \mathcal{D}_n$, $n \geqslant n_0$, be the smallest cube with $x, y \in Q$ (thus x, y lie in distinct cubes in $\mathcal{D}_{n+1}$). Write $c := c_Q \in Q$. Then, using properties (iii)-(iv) of the cubes $\mathcal{D}_n$,

$$\max\{d_G(x,c), d_G(y,c)\} \leqslant 2^{-n} \quad \text{and} \quad \tau 2^{-(n+1)(1+\epsilon)} \leqslant d_G(x,y) \leqslant 2^{-n}. \tag{2.7}$$

Also, by definition,

$$F_n(x) = i^n_{c\to F_{n-1}(c)}(x) \quad \text{and} \quad F_n(y) = i^n_{c\to F_{n-1}(c)}(y),$$

since $x, y \in Q$. We deduce from (1.12) (using (2.7)) that

$$d_M(F_n(x), F_n(y)) = d_M(i^n_{c\to F_{n-1}(c)}(x), i^n_{c\to F_{n-1}(c)}(y)) \geqslant L^{-1} d_G(x,y) - A2^{-n(1+\alpha)}. \tag{2.8}$$

We also have the upper bound analogous to (2.8),

$$d_M(F_n(x), F_n(y)) \leqslant L d_G(x,y) + A2^{-n(1+\alpha)}.$$

Recalling that $\epsilon = \alpha/2$ and $\tau = \tau(\varepsilon)$, the error term $A2^{-n(1+\alpha)}$ is smaller than

$$L^{-1}\tau 2^{-(n+1)(1+\epsilon)}/4 \leqslant L^{-1} d_G(x,y)/4$$

for $n \geqslant n_0$, if $n_0 \in \mathbb{N}$ was chosen large enough, depending on A, L and α. Thus,

$$L^{-1}\frac{3d_G(x,y)}{4} \leqslant d_M(F_n(x), F_n(y)) \leqslant L\frac{5d_G(x,y)}{4}. \tag{2.9}$$

Finally, if $n_0 \geqslant 0$ is large enough, depending again on α, L and A, one sees from (2.6)-(2.7) that

$$\max\{d_M(F(x), F_n(x)), d_M(F(y), F_n(y))\} \leqslant L^{-1}\frac{d_G(x,y)}{8}. \tag{2.10}$$

It then follows by combining (2.9), (2.10), and the triangle inequality that

$$L^{-1}\frac{d_G(x,y)}{2} \leqslant d_M(F(x), F(y)) \leqslant L2d_G(x,y),$$

as desired. The proof of Theorem 1.14 is complete, except for the claim $F(K) \subset B(p_0, 1)$. Pick $x \in K \subset Q_{n_0} \subset B(c_{Q_{n_0}}, 2^{-n_0})$, and write, using (1.12),

$$d_M(p_0, F_{n_0}(x)) = d_M(i^{n_0}_{c_{Q_{n_0}}\to p_0}(c_{Q_{n_0}}), i^{n_0}_{c_{Q_{n_0}}\to p_0}(x)) \leqslant L d_G(c_{Q_{n_0}}, x) + A2^{-n_0} \leqslant (L+A)2^{-n_0}.$$

Further, it follows from (2.6) that $d_M(F(x), F_{n_0}(x)) \lesssim A2^{-n_0}$. So, if $n_0 \geqslant 2$ was chosen large enough, depending on A and L, one has $F(x) \in B(p_0, \frac{1}{2})$. Finally, since $\mathrm{diam}_G(K) \leqslant$

$2^{-n_0} \leqslant \frac{1}{4L}$ for n_0 large enough depending on L, and F is $2L$-Lipschitz, one has $F(K) \subset B(F(x), \frac{1}{2}) \subset B(p_0, 1)$. The proof of Theorem 1.14 is complete.

3. Graphs and surfaces with Hölder-continuous horizontal normals

Throughout this section, we use coordinates $(x_1, \dots, x_{2n}, t)$ on $\mathbb{H}^n$ as defined at the beginning of Section 1. In these coordinates, a frame for the left invariant vector fields on $\mathbb{H}^n$ is given by

$$X_i = \partial_{x_i} - \tfrac{x_{n+i}}{2}\partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \tfrac{x_i}{2}\partial_t, \text{ for } i = 1, \dots, n, \quad \text{and} \quad T = \partial_t, \tag{3.1}$$

which yields the nontrivial commutator relations

$$[X_i, X_{n+i}] = T, \quad i = 1, \dots, n.$$

The *horizontal gradient* of a function $f : U \subset \mathbb{H}^n \to \mathbb{R}$ is

$$\nabla_{\mathbb{H}} f = (X_1 f, \dots, X_{2n} f).$$

A *vertical subgroup* $\mathbb{W}$ *of codimension* 1 in $\mathbb{H}^n$ is, in the above coordinate system, a $2n$-dimensional subspace of $\mathbb{R}^{2n+1}$ containing the t-axis. Given such a subgroup $\mathbb{W}$, we denote by $\mathbb{V}$ its Euclidean orthogonal complement. We recall that $\pi_{\mathbb{W}} \colon \mathbb{H}^n \to \mathbb{W}$ is the *vertical projection* to $\mathbb{W}$, and $\pi_{\mathbb{V}} \colon \mathbb{H}^n \to \mathbb{V}$ is the *horizontal projection* to $\mathbb{V}$, induced by the splitting $\mathbb{H} = \mathbb{W} \cdot \mathbb{V}$, see [19, Proposition 2.2]. In particular,

$$p = \pi_{\mathbb{W}}(p) \cdot \pi_{\mathbb{V}}(p), \quad \text{for all } p \in \mathbb{H}^n.$$

3.1. Definitions and preliminaries.

A $C^{1,\alpha}_{\mathbb{H}}$*-surface* is locally a non-critical level set of a $C^{1,\alpha}_{\mathbb{H}}$-function $f \colon \mathbb{H}^n \to \mathbb{R}$. Here is the precise definition:

Definition 3.2 ($C^{1,\alpha}_{\mathbb{H}}$-surfaces)**.** Let $S \subset \mathbb{H}^n$ be an $\mathbb{H}$-regular surface in the sense of Franchi, Serapioni, and Serra Cassano [18, Definition 6.1]: for every $p \in S$, there exists an open ball $B(p,r)$ and a function $f \in C^1_{\mathbb{H}}(B(p,r))$ such that $\nabla_{\mathbb{H}} f(p) \neq 0$, and

$$S \cap B(p,r) = \{q \in B(p,r) : f(q) = 0\}. \tag{3.3}$$

For $0 < \alpha \leqslant 1$, the set S is called a $C^{1,\alpha}_{\mathbb{H}}$*-surface* if one can choose f so that there exists a constant $H = H_p \geqslant 1$ such that

$$|\nabla_{\mathbb{H}} f(q_1) - \nabla_{\mathbb{H}} f(q_2)| \leqslant H d_{\mathbb{H}}(q_1, q_2)^{\alpha}, \qquad q_1, q_2 \in S \cap B(p,r). \tag{3.4}$$

Remark 3.5. If $S \cap B(p,r) = B(p,r) \cap \{f = 0\}$, as above, then [18, Theorem 6.5] states that, after making $r > 0$ possibly a little smaller, the inward-pointing horizontal normal of $E = \{f < 0\}$ is given by the expression

$$\nu_E(q) = -\frac{\nabla_{\mathbb{H}} f(q)}{|\nabla_{\mathbb{H}} f(q)|}, \qquad q \in S \cap B(p,r).$$

Clearly, if f satisfies (3.4), then this choice of horizontal normal is (locally) α-Hölder continuous as a map $(S, d_{\mathbb{H}}) \to S^{2n-1}$.

Conversely, if $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular surface with α-Hölder continuous ν_E, if p is an arbitrary point in S, and $r > 0$ is small enough, we claim that there exists $f \in C^1_{\mathbb{H}}(B(p,r))$ so that (3.3) and (3.4) hold. Not every function f which satisfies (3.3) necessarily fulfills (3.4), cf. the related Remark 3.7 below. In order to find f which satisfies simultaneously the two conditions, it is convenient to write S locally as an *intrinsic graph*. First, by assumption there exists $i \in \{1, \dots, 2n\}$ so that the component ν_E^i does not vanish on

$S \cap B(p,r)$ for small enough $r > 0$. Without loss of generality, we may assume that $i = 1$ and $\nu_E^i < 0$ on $S \cap B(p,r)$. Then, the implicit function theorem of Franchi, Serapioni, and Serra Cassano [18, Theorem 6.5], combined with [1, Theorem 1.2], implies that for small enough $r > 0$, the set $S \cap B(p,r)$ can be written as intrinsic graph in X_1-direction of a function φ whose *intrinsic gradient* $\nabla^{\varphi}\varphi$ (in the sense of Definition 3.9) exists and is a continuous $\mathbb{R}^{2n-1}$-valued function. This allows us to express locally the horizontal normal ν_E in a convenient form, cf. (3.13). Using this expression, it is easy to see that α-Hölder continuity of ν_E implies α-Hölder continuity of $[\nabla^{\varphi}\varphi] \circ \pi_{\mathbb{W}}|_{S \cap B(p,r)}$ for r small enough, cf. (3.14). Then

$$f(x_1, \ldots, x_{2n}, t) := x_1 - \varphi\left(\pi_{\mathbb{W}}(x_1, \ldots, x_{2n}, t)\right) = x_1 - \varphi\left(0, x_2, \ldots, x_{2n}, t + \tfrac{1}{2}x_1 x_{n+1}\right)$$

has the properties (3.3) and (3.4). To see this, use [1, Proposition 2.22] for the expression of $\nabla_{\mathbb{H}} f$ in terms of $\nabla^{\varphi}\varphi$, and note that $X_1 f = 1$.

A case of particular interest in this paper are intrinsic graphs $S \subset \mathbb{H}^n$ that also happen to be $C^{1,\alpha}_{\mathbb{H}}$-surfaces. Specifically, consider the following definition:

Definition 3.6 (Intrinsic $C^{1,\alpha}$-functions and intrinsic $C^{1,\alpha}$-graphs)**.** Let

$$\mathbb{W} := \{(0, x_2, \ldots, x_{2n}, t) : (x_2, \ldots, x_{2n}, t) \in \mathbb{R}^{2n}\}$$

and $\mathbb{V} := \{(x_1, 0 \ldots, 0) : x_1 \in \mathbb{R}\} \cong \mathbb{R}$. Let $U \subset \mathbb{W}$ be open, and let $\varphi \colon U \to \mathbb{V}$ be continuous. Write $\Phi(w) := w \cdot \varphi(w)$ for the *graph map* of φ. We say that φ is an *intrinsic $C^{1,\alpha}$-function on U*, denoted $\varphi \in C^{1,\alpha}_{\mathbb{H}}(U)$ if

(i) $S := \Phi(U) \subset \mathbb{H}^n$ is a $C^{1,\alpha}_{\mathbb{H}}$-surface in the sense of Definition 3.2, and
(ii) the horizontal normal $\nu_{\mathbb{H}} = (\nu_{\mathbb{H}}^1, \ldots, \nu_{\mathbb{H}}^{2n}) \colon S \to \mathbb{R}^{2n}$ of the subgraph

$$\{w \cdot v : v < \varphi(w)\}$$

satisfies $\nu_{\mathbb{H}}^1(p) < 0$ for all $p \in S$.

The intrinsic graph S of any intrinsic $C^{1,\alpha}$-function is an *intrinsic $C^{1,\alpha}$-graph*.

Remark 3.7. The conditions (i)-(ii) are $C^{1,\alpha}$-versions of the conditions appearing in [1, Theorem 1.2(i)]. Condition (ii) is not as odd as it looks: a similar hypothesis would also be required to characterise $C^{1,1}$-functions $f \colon \mathbb{R} \to \mathbb{R}$ via the properties of $\Gamma(f) := \{(x, f(x)) : x \in \mathbb{R}\}$. To see this, consider $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \mathrm{sgn}(x)\sqrt{|x|}$. Then $\Gamma(f)$ is a $C^{1,1}$-surface as a subset of $\mathbb{R}^2$, because $\Gamma(f)$ can also be written as $\Gamma(f) = \{(\mathrm{sgn}(y)y^2, y) : y \in \mathbb{R}\}$, where $y \mapsto \mathrm{sgn}(y)y^2 \in C^{1,1}(\mathbb{R})$. Nonetheless, $f \notin C^{1,1}(\mathbb{R})$.

A previous notion of *intrinsic $C^{1,\alpha}$-functions* and *graphs* in $\mathbb{H}^1$ already exists, see [10, Definition 2.16] or (3.11) below, and it looks different than Definition 3.6. The connection needs to be clarified immediately, because a result concerning intrinsic $C^{1,\alpha}$-functions in the sense of [10], namely [10, Proposition 4.2], will be used also in this paper. Definition 2.16 in [10] was stated for $\mathbb{H}^1$, but it can be extended in an obvious way to higher dimensional Heisenberg groups, and this version appears in Proposition 3.10. The precise formulation requires the notion of *intrinsic differentiability*.

Definition 3.8. A function $\varphi : \mathbb{W} \to \mathbb{V}$ is *intrinsically differentiable* at the point $w_0 \in \mathbb{W}$ if there exists a map $L : \mathbb{W} \to \mathbb{V}$ whose intrinsic graph $\{w \cdot L(w) : \ w \in \mathbb{W}\}$ is a vertical subgroup and which satisfies

$$\lim_{\|w\|\to 0} \frac{\|L(w)^{-1} \cdot \varphi^{(p^{-1})}(w)\|}{\|w\|} = 0, \quad w \in \mathbb{W}.$$

Here $p = \Phi(w_0)$, and $\varphi^{(p^{-1})} \colon \mathbb{W} \to \mathbb{V}$ is the unique function $\mathbb{W} \to \mathbb{V}$ whose intrinsic graph is $p^{-1} \cdot \Phi(\mathbb{W})$ (see (3.19) for a formula).

For equivalent definitions, see [25, Proposition 4.76]. If φ is intrinsically differentiable at w_0, then there is a unique map $L : \mathbb{W} \to \mathbb{V}$ with the properties stated in Definition 3.8. This map is called the *intrinsic differential* of φ at w_0 and it is denoted by $d^\varphi\varphi(w_0)$. Moreover, there is a unique vector $\nabla^\varphi\varphi(w_0) \in \mathbb{R}^{2n-1}$, such that

$$d^\varphi\varphi(w_0)(w) = \langle \nabla^\varphi\varphi(w_0), \pi(w)\rangle, \quad w \in \mathbb{W},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{2n-1}$, and

$$\pi(0, x_2, \ldots, x_{2n}, t) = (x_2, \ldots, x_{2n}).$$

Definition 3.9. If $\varphi : \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable at $w_0 \in \mathbb{W}$, its *intrinsic gradient at w_0* is the vector $\nabla^\varphi\varphi(w_0)$. The components of $\nabla^\varphi\varphi(w_0)$ are denotes as follows:

$$\nabla^\varphi\varphi(w_0) = (D_2^\varphi\varphi(w_0), \ldots, D_{2n}^\varphi\varphi(w_0)).$$

More information about the functions $D_i^\varphi\varphi$, $i = 2, \ldots, 2n$, will only be required once we arrive at the proof of Proposition 3.29, so we postpone the detailed discussion. At this point we just mention that the component $D_{n+1}^\varphi\varphi$ will play a distinguished role as we consider intrinsic graphs in the X_1 direction, and $[X_1, X_i] = 0$ for all $i = 1, \ldots, 2n$, except for $i = n + 1$.

Proposition 3.10. *Let $0 < \alpha \leqslant 1$ and $\varphi \colon \mathbb{W} \to \mathbb{V}$ be a compactly supported function between the subgroups $\mathbb{W}$ and $\mathbb{V}$ in $\mathbb{H}^n$. Then $\varphi \in C_{\mathbb{H}}^{1,\alpha}(\mathbb{W})$ in the sense of Definition 3.6 if and only if φ is intrinsically differentiable, and the intrinsic gradient $\nabla^\varphi\varphi$ satisfies*

$$|\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(w) - \nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(0)| \leqslant H\|w\|^\alpha, \qquad w \in \mathbb{W},\ p \in \Phi(\mathbb{W}) \tag{3.11}$$

for a constant $H \geqslant 1$.

Remark 3.12. Since [10, Definition 2.16] imposes "global" Hölder continuity for $\nabla^\varphi\varphi$, whereas the assumptions in Definition 3.6 are of local nature, the notions cannot be equivalent without some *a priori* assumptions – as the compact support of φ in Proposition 3.10. We also remark that condition (3.11) implies that $\nabla^\varphi\varphi$ is continuous, as can be deduced for instance from formula (3.15) below.

Proof of Proposition 3.10. Assume that $\varphi \in C_{\mathbb{H}}^{1,\alpha}(\mathbb{W})$ in the sense of Definition 3.6. Then, [1, Theorem 1.2] states in particular that φ is intrinsically differentiable, the intrinsic gradient $\nabla^\varphi\varphi(w)$ exists for all $w \in \mathbb{W}$, and $w \mapsto \nabla^\varphi\varphi(w)$ is continuous (see also [25, Theorem 4.95]).

Additionally, [1, Theorem 1.2] promises that the horizontal normal $\nu_{\mathbb{H}}$ in Definition 3.6(ii) has the representation

$$\nu_{\mathbb{H}}(\Phi(w)) = \left(\frac{-1}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}}, \frac{\nabla^\varphi\varphi(w)}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}} \right), \qquad w \in \mathbb{W}. \tag{3.13}$$

Let us now argue that $\nu_{\mathbb{H}}$, above, is (globally) α-Hölder continuous on $S = \Phi(\mathbb{W})$. Recall that by the assumption that S is a $C_{\mathbb{H}}^{1,\alpha}$-surface, and Remark 3.5, for every $p \in S$

there exists **some** α-Hölder continuous choice of a horizontal normal $\nu_{\mathbb{H}}^{p}$, defined in a neighbourhood $S \cap U$ of p (note that there are two horizontal normals at every point in S). However, it is easy to see that $\nu_{\mathbb{H}}^{p}$ must coincide on $S \cap U$ with either $\nu_{\mathbb{H}}$ or $-\nu_{\mathbb{H}}$ whenever $S \cap U$ is connected (that is, the sign depends only on U). But since S is locally connected, and since $\nu_{\mathbb{H}}(p) \equiv (-1, 0, \ldots, 0)$ outside the compact set $\Phi(\operatorname{spt}\varphi) \subset S$, one infers that the horizontal normal $\nu_{\mathbb{H}}$ in (3.13) is α-Hölder continuous on S.

Another remark: using again that $\operatorname{spt}\varphi$ is compact, $\nabla^{\varphi}\varphi$ is continuous and supported in $\operatorname{spt}\varphi$, we see that $L := \|\nabla^{\varphi}\varphi\|_{L^{\infty}(\mathbb{W})} < \infty$. From (3.13) and the α-Hölder continuity of $\nu_{\mathbb{H}}$ on $S = \Phi(\mathbb{W})$, it can easily be deduced that

$$\ell : (S, d_{\mathbb{H}}) \to \mathbb{R}, \quad p \mapsto \ell(p) := \sqrt{1 + |\nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(p))|^2}$$

is α-Hölder continuous with Hölder constant bounded in terms of L and the α-Hölder constant of $\nu_{\mathbb{H}}$. Then

$$\left|\ell(p)\nu_{\mathbb{H}}(p) - \ell(p')\nu_{\mathbb{H}}(p')\right| \leqslant \ell(p)\left|\nu_{\mathbb{H}}(p) - \nu_{\mathbb{H}}(p')\right| + \left|\ell(p) - \ell(p')\right| \lesssim_L d(p,p')^{\alpha} \tag{3.14}$$

for $p, p' \in S$. Now we are quite well equipped to check that φ satisfies (3.11). To this end, fix $w \in \mathbb{W}$ and $p \in \Phi(\mathbb{W})$. Observe the explicit formula

$$\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(w) = \nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(p \cdot w)), \qquad w \in \mathbb{W},\ p \in \Phi(\mathbb{W}), \tag{3.15}$$

proven in Lemma 3.22 below. Thus, starting from the left hand side of (3.11),

$$\begin{aligned} |\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(w) - \nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(0)| &= |\nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(p \cdot w)) - \nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(p))| \\ &= |\nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(\Phi(\pi_{\mathbb{W}}(p \cdot w)))) - \nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(p))| \\ &\overset{(3.14)}{\lesssim}_L d_{\mathbb{H}}(\Phi(\pi_{\mathbb{W}}(p \cdot w)), p)^{\alpha}. \end{aligned} \tag{3.16}$$

The following formula is needed to make sense of the right hand side:

Lemma 3.17. *For any $p \in \mathbb{H}^n$ and $w \in \mathbb{W}$, the following relation holds:*

$$\Phi(\pi_{\mathbb{W}}(p \cdot w)) = p \cdot \Phi^{(p^{-1})}(w). \tag{3.18}$$

Here $\Phi^{(p^{-1})}$ is the graph map of $\varphi^{(p^{-1})}$.

Proof. The function $\varphi^{(p^{-1})} : \mathbb{W} \to \mathbb{V}$ is explicitly given by

$$\varphi^{(p^{-1})}(w) = \pi_{\mathbb{V}}(p^{-1}) \cdot \varphi(\pi_{\mathbb{W}}(p \cdot w)), \tag{3.19}$$

where $\pi_{\mathbb{V}}(x_1, \ldots, x_{2n}, t) = x_1$ is the horizontal projection induced by the splitting $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, see for instance [9, Lemma 4.7] (for $n = 1$). The map $\pi_{\mathbb{V}}$ is a group homomorphism $\mathbb{H}^n \to \mathbb{V}$ with $\pi_{\mathbb{V}}(w) = 0$ for all $w = (0, x_2, \ldots, x_{2n}, t) \in \mathbb{W}$. Therefore,

$$\begin{aligned} &\Phi(\pi_{\mathbb{W}}(p \cdot w)) = p \cdot \Phi^{(p^{-1})}(w) \\ \overset{(3.19)}{\iff}\ &\pi_{\mathbb{W}}(p \cdot w) \cdot \varphi(\pi_{\mathbb{W}}(p \cdot w)) = p \cdot [w \cdot \pi_{\mathbb{V}}(p^{-1}) \cdot \varphi(\pi_{\mathbb{W}}(p \cdot w))] \\ \iff\ &\pi_{\mathbb{W}}(p \cdot w) = p \cdot w \cdot \pi_{\mathbb{V}}(p^{-1}) = p \cdot w \cdot [\pi_{\mathbb{V}}(p)]^{-1} \\ \iff\ &p \cdot w = \pi_{\mathbb{W}}(p \cdot w) \cdot \pi_{\mathbb{V}}(p) = \pi_{\mathbb{W}}(p \cdot w) \cdot \pi_{\mathbb{V}}(p \cdot w). \end{aligned}$$

The last equation is true by the definition of the projections $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}$, so the proof is complete. ☐

We deduce that the right hand side of (3.16) equals, up to a multiplicative constant, the term $d_{\mathbb{H}}(\Phi^{(p^{-1})}(w), 0)^{\alpha}$, which we will now further bound from above in order to conclude the proof of the first implication in Proposition 3.10. To do so, we observe that [1, Theorem 1.2] implies more than mere intrinsic differentiability of φ: it shows that φ is *uniformly intrinsically differentiable* in the sense of [3, Definition 3.16]. Recalling that φ has compact support, this implies that there exists a function $\varepsilon = \varepsilon_{\mathrm{spt}\varphi} : (0, \infty) \to (0, \infty)$ with $\lim_{s \to 0} \varepsilon(s) = 0$ such that

$$|\varphi^{(p^{-1})}(y,t) - \langle \nabla^{\varphi}\varphi(w_0), y\rangle| \leqslant \varepsilon\left(\|(y,t)\|\right)\|(y,t)\|, \qquad p = \Phi(w_0) \in \Phi(\mathbb{W}),\ (y,t) \in \mathrm{spt}\,\varphi^{(p^{-1})}.$$

Then there is a constant $\delta > 0$ such that

$$|\varphi^{(p^{-1})}(y,t)| \leqslant |\varphi^{(p^{-1})}(y,t) - \langle \nabla^{\varphi}\varphi(w_0), y\rangle| + |\langle \nabla^{\varphi}\varphi(w_0), y\rangle| \leqslant (1+L)\|(y,t)\|$$

for all $p = \Phi(w_0) \in \Phi(\mathbb{W})$ and all $(y,t) \in \mathrm{spt}\,\varphi^{(p^{-1})}$ with $\|(y,t)\| \leqslant \delta$, where $L := \|\nabla^{\varphi}\varphi\|_{L^{\infty}(\mathbb{W})}$. On the other hand, if $\|(y,t)\| \in \mathrm{spt}\,\varphi^{(p^{-1})}$ satisfies $\|(y,t)\| > \delta$, then trivially,

$$|\varphi^{(p^{-1})}(y,t)| = \|\pi_{\mathbb{V}}(\Phi(w_0)^{-1}) \cdot \varphi(\pi_{\mathbb{W}}(\Phi(w_0) \cdot w)\| \leqslant \tfrac{2\|\varphi\|_{L^{\infty}(\mathbb{W})}}{\delta}\|(y,t)\|.$$

In conclusion, there exists a constant $L' \geqslant 1$ such that for all $p \in \Phi(\mathbb{W})$, it holds

$$\|\Phi^{(p^{-1})}(w)\| \leqslant \|w\| + |\varphi^{(p^{-1})}(w)| \leqslant L'\|w\|, \quad w \in \mathbb{W}. \tag{3.20}$$

Finally, plugging formula (3.18) into (3.16) and using the left-invariance of $d_{\mathbb{H}}$,

$$|\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(w) - \nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(0)| \lesssim d_{\mathbb{H}}(\Phi^{(p^{-1})}(w), 0)^{\alpha} \overset{(3.20)}{\lesssim} \|w\|^{\alpha}. \tag{3.21}$$

Now (3.21) proves that φ satisfies (3.11).

The converse implication stated in Proposition 3.10 is not needed in the paper, so we only sketch the argument. Let φ be intrisically differentiable with intrinsic gradient satisfying (3.11). Then φ is again uniformly intrinsically differentiable. This is a consequence of Proposition 3.29, see also [10, Remark 2.24]. Therefore, according to [1, Theorem 1.2], the intrinsic graph $S = \Phi(\mathbb{W})$ is an $\mathbb{H}$-regular surface, and the condition (ii) in Definition 3.6 holds. So, it remains to check that the horizontal normal of S is α-Hölder continuous. This follows from the expression (3.13) (which is available by [1, Theorem 1.2]) using estimates similar to those above (3.60). □

The proof of Proposition 3.10 referred to the following auxiliary lemma.

Lemma 3.22. *Let $\varphi : \mathbb{W} \to \mathbb{V}$ be defined on the codimension-1 vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$. If $p \in \mathbb{H}^n$ and $w \in \mathbb{W}$ are such that φ is intrinsically differentiable at $\pi_{\mathbb{W}}(p \cdot w)$, then*

$$\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(w) = \nabla^{\varphi}\varphi(\pi_{\mathbb{W}}(p \cdot w)). \tag{3.23}$$

Proof. By its very definition, a function $\psi : \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable at a point $w_0 \in \mathbb{W}$ if and only if $\psi^{(p_0^{-1})}$ for $p_0 = w_0 \cdot \psi(w_0)$ is intrinsically differentiable at 0, and in that case

$$\nabla^{\psi}\psi(w_0) = \nabla^{\psi^{(p_0^{-1})}}\psi^{(p_0^{-1})}(0), \tag{3.24}$$

see, for instance, [25, Definition 4.71, Proposition 4.76.] and [1, top of p.192]. Now fix φ, and points $w \in \mathbb{W}$, and $p \in \mathbb{H}^n$ as in the statement of the lemma. We first apply formula (3.24) to $\psi := \varphi$ and $w_0 := \pi_{\mathbb{W}}(p \cdot w)$. Hence,

$$p_0 = w_0 \cdot \varphi(w_0) = \pi_{\mathbb{W}}(p \cdot w) \cdot \varphi(\pi_{\mathbb{W}}(p \cdot w)) = p \cdot w \cdot \pi_{\mathbb{V}}(p^{-1}) \cdot \varphi(\pi_{\mathbb{W}}(p \cdot w)) \overset{(3.19)}{=} p \cdot w \cdot \varphi^{(p^{-1})}(w)$$

and (3.24) reads

$$\nabla^{\psi}\psi(\pi_{\mathbb{W}}(p\cdot w)) = \nabla^{\varphi^{([p\cdot w\cdot\varphi^{(p^{-1})}(w)]^{-1})}}\varphi^{([p\cdot w\cdot\varphi^{(p^{-1})}(w)]^{-1})}(0). \tag{3.25}$$

On the other hand, denoting

$$q_0 = w\cdot\varphi^{(p^{-1})}(w),$$

we observe that the graph of $[\varphi^{(p^{-1})}]^{(q_0^{-1})}$ is $q_0^{-1}\cdot\Gamma'$, where Γ' is the graph of $\varphi^{(p^{-1})}$, so

$$q_0^{-1}\cdot\Gamma' = q_0^{-1}\cdot p^{-1}\cdot\Phi(\mathbb{W}) = [p\cdot q_0]^{-1}\cdot\Phi(\mathbb{W}).$$

This shows that

$$\varphi^{([p\cdot q_0]^{-1})} = [\varphi^{(p^{-1})}]^{(q_0^{-1})}$$

and hence

$$(3.25) = \nabla^{\varphi^{([p\cdot q_0]^{-1})}}\varphi^{([p\cdot q_0]^{-1})}(0) = \nabla^{[\varphi^{(p^{-1})}]^{(q_0^{-1})}}[\varphi^{(p^{-1})}]^{(q_0^{-1})}(0).$$

In particular, $[\varphi^{(p^{-1})}]^{(q_0^{-1})}$ is intrinsically differentiable at 0. Formula (3.24) applied to $\psi := \varphi^{(p^{-1})}$, $p_0 = q_0$ and $w_0 := w$ yields

$$\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(w) = \nabla^{[\varphi^{(p^{-1})}]^{(q_0^{-1})}}[\varphi^{(p^{-1})}]^{(q_0^{-1})}(0). \tag{3.26}$$

The lemma follows by observing that the right hand sides of (3.25) and (3.26) are equal. □

With the main definitions now in place, we repeat Theorem 1.6 below.

Theorem 3.27. *Let $S = \Phi(\mathbb{W}) \subset \mathbb{H}^n$, where $\varphi \in C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ is compactly supported. Then S has BPGBI in the sense of Definition 1.10.*

Before proving this, let us deduce the qualitative corollary, Theorem 1.7:

Theorem 3.28. *Let $S \subset \mathbb{H}^n$ be a $C^{1,\alpha}_{\mathbb{H}}$-surface. Then $\mathcal{H}^{2n+1}$ almost all of S can be covered by bilipschitz images of closed subsets of codimension-1 vertical subgroups. In particular, S is LI rectifiable.*

Proof. There are (at least) two possible approaches. One is to use the implicit function theorem [18, Theorem 6.5] to express the surface S locally as the intrinsic graph of a *locally defined* intrinsic $C^{1,\alpha}$-function. This function does not, literally, satisfy the assumptions of Theorem 3.27, but the proof of Theorem 3.27 could be localised with some effort.

The biggest difficulty in this approach is of expository nature as localising the proof of Theorem 3.27 would lead to a more cumbersome version of Proposition 3.29 below. So we take the following alternative route: in Appendix B, we show that every point on S has a neighbourhood which can be contained on the intrinsic graph Γ of a globally defined, compactly supported intrinsic $C^{1,\alpha/3}$-function. Since Γ is LI rectifiable by Theorem 3.27, the same is also true for S. □

It remains to prove Theorem 3.27. Recall the notation $\mathbb{V}$ and $\mathbb{W}$ from Definition 3.6. We will frequently abbreviate $(0, x_2, \ldots, x_{2n}, t) =: (x_2, \ldots, x_{2n}, t) =: (y, t)$ for points in $\mathbb{W}$, and continue to use the notation

$$\langle\nabla^{\varphi}\varphi(w), y\rangle = \sum_{i=2}^{2n} D_i^{\varphi}\varphi(w)x_i$$

for an intrinsically differentiable φ. The functions $D_i^{\varphi}\varphi$ have been introduced as the components of the intrinsic gradient $\nabla^{\varphi}\varphi$ in Definition 3.9, but under additional regularity assumptions on φ, they can also be obtained as derivatives of φ in the direction of the vector fields

$$D_j^{\varphi} := X_j, \quad j \in \{2, \ldots, 2n\} \backslash \{n+1\} \quad \text{and} \quad D_{n+1}^{\varphi} := \partial_{x_{n+1}} + \varphi \partial_t.$$

A first result of this type is [1, Proposition 3.7], and more specific statements will follow shortly in the proof of the next result, which is a key ingredient in the proof of Theorem 3.27.

Proposition 3.29. *Assume that $\varphi : \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable on $\mathbb{W}$ and it has a continuous intrinsic gradient which satisfies (3.11) with constant $H \geqslant 1$ and $0 < \alpha \leqslant 1$. Suppose further that $L := \|\nabla^{\varphi}\varphi\|_{L^{\infty}(\mathbb{W})} < \infty$, and $p = \Phi(w)$ for some $w \in \mathbb{W}$. Then,*

$$|\varphi^{(p^{-1})}(y,t) - \langle \nabla^{\varphi}\varphi(w), y\rangle| \lesssim \|(y,t)\|^{1+\alpha}, \qquad (y,t) \in \mathbb{W}. \tag{3.30}$$

The implicit constant in (3.30) only depends on H and L.

The proposition says, in a "left-invariant" way, that φ is locally well-approximated by linear functions. The main corollary is Proposition 3.41, which quantifies how well the intrinsic graph of φ around $\Phi(w)$ is approximated by the *vertical tangent plane* determined by $\nabla^{\varphi}\varphi(w)$.

Proof of Proposition 3.29. For $n = 1$ the proposition was established in [10, Proposition 2.23]. The case $n > 1$ can be proven in a simpler way without the arguments that were used in [10, Proposition 4.2]. We include here a self-contained proof for that case. By the definition of the intrinsic differentiability and intrinsic gradients, we have

$$|\varphi^{(p^{-1})}(y,t) - \langle \nabla^{\varphi}\varphi(w), y\rangle| = |\varphi^{(p^{-1})}(y,t) - \langle \nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}(0), y\rangle|$$

and so we just prove that

$$|\varphi(y,t) - \langle \nabla^{\varphi}\varphi(0), y\rangle| \lesssim \|(y,t)\|^{\alpha+1}, \quad \text{for all } (y,t) \in \mathbb{W}, \tag{3.31}$$

under the assumption that $p = 0$ and $\varphi(0) = 0$. Notice that the constants L and H are not changed under left translations.

To explain the idea, let us first consider a $C^{1,\alpha}(\mathbb{R})$ function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$. Then, for $y > 0$,

$$|h(y) - h'(0)y| = \left| \int_0^y h'(s) - h'(0)\, ds \right| \leqslant \int_0^y |h'(s) - h'(0)|\, ds \lesssim |y|^{1+\alpha}.$$

To prove (3.31), we apply the same idea, but we integrate along integral curves of vector fields D_j^{φ}, $j = 2, \ldots, 2n$. We use the assumption $n > 1$ to ensure that the origin can be connected to any point

$$(y,t) = (0, x_2, \ldots, x_{2n}, t) \in \mathbb{W}$$

by a curve $\gamma : I \to \mathbb{W}$ that is defined as a concatenation $\gamma := \gamma_1 \star \cdots \star \gamma_{2n+3}$ of the following curves:

- γ_1 is an integral curve of

  $$D_{n+1}^{\varphi} = \partial_{x_{n+1}} + \varphi \partial_t$$

  that connects 0 to a point of the form

  $$a := (0, \ldots, 0, x_{n+1}, 0, \ldots, 0, \tau(x_{n+1})).$$

- $\gamma_2 \star \cdots \star \gamma_{2n+3}$ is a concatenation of integral curves of
$$D_j^{\varphi} = X_j, \quad j \in \{2, \ldots, 2n\} \backslash \{n+1\}$$
with the property that
$$\text{length}_{\mathbb{H}^n}(\gamma_2 \star \cdots \star \gamma_{2n+3}) \lesssim \|(y,t)\|$$
and $\gamma_2 \star \cdots \star \gamma_{2n+3}$ connects a to (y,t).

A similar construction was used in [2, Proposition 6.10]. We now explain in detail how $\gamma_1, \ldots, \gamma_{2n+3}$ are defined. First, let λ_{n+1} be an integral curve of D_{n+1}^{φ}, given by
$$\lambda_{n+1}(s) := (0, \ldots 0, s, 0, \ldots, \tau(s)), \tag{3.32}$$
where s is the component corresponding to the coordinate x_{n+1}, and $\tau : J \to \mathbb{R}$ is a solution of the Cauchy problem
$$\begin{cases} \tau'(s) = \varphi(0, \ldots, 0, s, 0, \ldots, 0, \tau(s)), \\ \tau(0) = 0. \end{cases}$$
Such a solution exists by Peano's theorem since φ is continuous, and we assume that J is the maximal interval of existence for τ containing the point 0. As moreover $\nabla^{\varphi}\varphi$ is continuous, which follows from the assumption (3.11) by Remark 3.12, we find by [10, Lemma 4.4] that
$$(\varphi \circ \lambda_{n+1})'(s) = D_{n+1}^{\varphi}\varphi(\lambda_{n+1}(s)), \quad s \in J. \tag{3.33}$$
To be precise, [10, Lemma 4.4] is stated in $\mathbb{H}^1$, but since λ_{n+1} is entirely contained in the $x_{n+1}t$-plane, we can apply the result to
$$(x_{n+1}, t) \mapsto \varphi(0, \ldots, 0, x_{n+1}, 0, \ldots, 0, t),$$
interpreted as a function on a vertical subgroup in $\mathbb{H}^1$. Moreover, using the same proof as for [10, (4.4)], one can show that in fact $J = \mathbb{R}$, and
$$|\tau(s)| \lesssim_L |s|^2, \qquad s \in \mathbb{R}. \tag{3.34}$$

The curve γ_1 will be an appropriately parametrized subcurve of λ_{n+1}. If $x_{n+1} > 0$, we naturally define γ_1 to be the restriction of λ_{n+1} to the interval $[0, x_{n+1}]$. If $x_{n+1} < 0$, we have to reverse the order of the parametrization, so we make the general definition: $\gamma_1(s) := \lambda_{n+1}(\text{sgn}(x_{n+1})s)$, $s \in [0, |x_{n+1}|]$, if $x_{n+1} \neq 0$, and we let γ_1 be the constant curve $\gamma_1 \equiv 0$ otherwise. Note that the points $a := \gamma_1(|x_{n+1}|)$ and (y,t) belong to
$$\mathbb{G} := \{(z_1, \ldots, z_{2n+1}) \in \mathbb{R}^{2n+1} : z_1 = 0 \quad \text{and} \quad z_{n+1} = x_{n+1}\},$$
and $\mathbb{G}$ with the group law and metric induced from $\mathbb{H}^n$ is isometrically isomorphic to $\mathbb{H}^{n-1}$ with $d_{\mathbb{H}}$. We next show that there exists a compact interval $I = [|x_{n+1}|, b]$ such that a and (y,t) can be connected by a concatenation $\gamma_2 \star \cdots \star \gamma_{2n+3} : I \to \mathbb{G}$ of integral curves of $D_j^{\varphi} = X_j$, $j \in \{2, \ldots, 2n\} \backslash \{n+1\}$ so that
$$\max_{s \in I} \|(\gamma_2 \star \cdots \star \gamma_{2n+3})(s)\| \lesssim_L \|(y,t)\| \quad \text{and} \quad \text{length}_{\mathbb{H}^n}(\gamma_2 \star \cdots \star \gamma_{2n+3}) \lesssim d_{\mathbb{H}}(a, (y,t)). \tag{3.35}$$
The first inequality in (3.35) follows from the second one since
$$\|a\| \overset{(3.34)}{\lesssim_L} |x_{n+1}| \quad \text{and} \quad |x_{n+1}| \leqslant \|(y,t)\| \tag{3.36}$$
and
$$\|(\gamma_2 \star \cdots \star \gamma_{2n+3})(s)\| \leqslant \text{length}_{\mathbb{H}^n}(\gamma_2 \star \cdots \star \gamma_{2n+3}) + \|a\|, \quad s \in I.$$

Thus it suffices to find curves which satisfy the second condition in (3.35). To achieve this, first concatenate curves $\gamma_2, \ldots, \gamma_{2n-1}$ in the following way: follow the unique integral curve of $D_2^{\varphi} = X_2$ starting at a for time x_2, then follow the integral curve of $D_j^{\varphi} = X_j$ for time x_j, etc. for $j = 3, \ldots, n, n+2, \ldots, 2n$, until you reach $a' = (y, \tau(x_{n+1}) + \frac{1}{2}\sum_{i=2}^{n} x_i x_{n+i})$. Then connect a' to (y,t) by a curve $\gamma_{2n} \star \cdots \star \gamma_{2n+3}$ with

$$\mathrm{length}_{\mathbb{H}^n}(\gamma_{2n} \star \cdots \star \gamma_{2n+3}) \lesssim \left| t - \left(\tau(x_{n+1}) + \tfrac{1}{2} \sum_{i=2}^{n} x_i x_{n+i} \right) \right|^{1/2}.$$

This is possible since $[X_2, X_{n+2}] = \partial_t$. Now $\gamma_2 \star \cdots \star \gamma_{2n+3}$ connects a to (y,t) as desired, and

$$\begin{aligned} \mathrm{length}_{\mathbb{H}^n}(\gamma_2 \star \cdots \star \gamma_{2n+3}) &\lesssim |x_2| + \cdots + |x_{2n}| + \left| t - \left(\tau(x_{n+1}) + \tfrac{1}{2} \sum_{i=2}^{n} x_i x_{n+i} \right) \right|^{1/2} \\ &\lesssim d_{\mathbb{H}}(a, (y,t)) + |\tau(x_{n+1}) - t|^{1/2} \\ &\lesssim d_{\mathbb{H}}(a, (y,t)). \end{aligned} \tag{3.37}$$

Hence we have found curves $\gamma_2, \ldots, \gamma_{2n+3}$ so that the conditions in (3.35) are satisfied.

Now we are ready to implement the idea explained at the beginning of the proposition. Namely, we will write $\varphi(y,t)$ as an integral of $(\varphi \circ \gamma)'$, where γ is the concatenation of the curves $\gamma_1, \ldots, \gamma_{2n+3}$. To relate this to the intrinsic gradient, we apply again the intrinsic differentiability of φ, and observe that

$$(\varphi \circ \lambda_j)'(s) = D_j^{\varphi} \varphi(\lambda_j(s)), \tag{3.38}$$

for all $j = 2, \ldots, 2n$, where λ_{n+1} is as in (3.32), and λ_j for $j \neq n+1$ is an arbitrary integral curve of $D_j^{\varphi} = X_j$. For $j = n+1$, we saw (3.38) already in (3.33). For $j \neq n+1$, the vector field D_j^{φ} is linear and independent of φ, and (3.38) follows directly from the intrinsic differentiability assumption by the argument given at the beginning of the proof of [1, Proposition 3.7]. Hence, if γ is the piecewise C^1 curve given by

$$\gamma(s) = (\gamma_1 \star \cdots \star \gamma_{2n+3})(s) = (y(s), t(s)), \quad s \in [0, b],$$

we get that $\varphi \circ \gamma$ is piecewise C^1 and

$$\begin{aligned} |\varphi(y,t) - \langle \nabla^{\varphi} \varphi(0), y \rangle| &= \left| \int_0^b (\varphi \circ \gamma)'(s) \, \mathrm{d}s - \int_0^b \langle \nabla^{\varphi} \varphi(0), \dot{y}(s) \rangle \, \mathrm{d}s \right| \\ &\overset{(3.38)}{\leqslant} \int_0^b |\nabla^{\varphi} \varphi(\gamma(s)) - \nabla^{\varphi} \varphi(0)| \, \mathrm{d}s, \\ &\lesssim_H \int_0^b \|\gamma(s)\|^{\alpha} \, \mathrm{d}s \end{aligned} \tag{3.39}$$

where in the last inequality we have used the assumption that φ satisfies (3.11). In the first inequality, we used the fact that for almost all $s \in [0, b]$, the tangent vector $\dot{y}(s)$ exists by construction, and is of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ with

$$(\varphi \circ \gamma)'(s) \overset{(3.38)}{=} D_j^{\varphi} \varphi(\gamma(s)) = \langle \nabla^{\varphi} \varphi(\gamma(s)), \dot{y}(s) \rangle$$

if the x_j-component of $\dot{y}(s)$ is $+1$, and

$$(\varphi \circ \gamma)'(s) \overset{(3.38)}{=} -D_j^{\varphi}\varphi(\gamma(s)) = \langle \nabla^{\varphi}\varphi(\gamma(s)), \dot{y}(s)\rangle$$

if the x_j-component of $\dot{y}(s)$ is -1. Having established (3.39), we will estimate from above the expression

$$\int_0^b \|\gamma(s)\|^{\alpha}\,\mathrm{d}s = \int_0^{|x_{n+1}|} \|\gamma_1(s)\|^{\alpha}\,\mathrm{d}s + \int_{|x_{n+1}|}^{b} \|(\gamma_2 \star \cdots \star \gamma_{2n+3})(s)\|^{\alpha}\,\mathrm{d}s.$$

Firstly, since (3.34) holds, we have

$$\int_0^{|x_{n+1}|} \|\gamma_2(s)\|^{\alpha}\,\mathrm{d}s \lesssim \int_0^{|x_{n+1}|} |s|^{\alpha} + \sqrt{|\tau(\operatorname{sgn}(x_{n+1})s)|^{\alpha}}\,\mathrm{d}s \lesssim_L \|(y,t)\|^{\alpha+1}$$

and secondly, by definition of γ_j, $j \in \{2, \ldots, 2n+3\}$, we have

$$|b - |x_{n+1}|| = \text{length}_{\mathbb{H}^n}(\gamma_2 \star \cdots \star \gamma_{2n+3}) \overset{(3.36),(3.37)}{\lesssim}_L \|(y,t)\|.$$

This combined with the first condition in (3.35) yields

$$\int_{|x_{n+1}|}^{b} \|(\gamma_2 \star \cdots \star \gamma_{2n+3})(s)\|^{\alpha}\,ds \lesssim_L \|(y,t)\|^{\alpha+1}.$$

Putting together (3.39) and the last estimates, we can conclude

$$|\varphi(y,t) - \langle \nabla^{\varphi}\varphi(0), y\rangle| \lesssim_H \int_0^b \|\gamma(s)\|^{\alpha}\,\mathrm{d}s \lesssim_{H,L} \|(y,t)\|^{\alpha+1}$$

and the proof of Proposition 3.29 is complete. □

Remark 3.40. Recall from Proposition 3.10 that a compactly supported function $\varphi \in C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ (as in Theorem 3.27) satisfies (3.11) for some constant $H \geqslant 1$. The letter H will refer to this constant for the rest of Section 3. We also remark that φ is intrinsic Lipschitz, recall Definition 1.2. Indeed, a $C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ function is intrinsically differentiable with continuous intrinsic gradient, and the compact support assumption implies that $\nabla^{\varphi}\varphi \in L^{\infty}(\mathbb{W})$. In the case $n = 1$, [10, Lemma 2.22] states that then φ is intrinsic Lipschitz. One could adapt the proof to higher dimensions using Proposition 3.29, or alternatively refer to (3.20) to conclude also for $n > 1$ that φ is intrinsic Lipschitz. We denote by L the maximum of the intrinsic Lipschitz constant of φ, and the sup-norm $\|\nabla^{\varphi}\varphi\|_{L^{\infty}(\mathbb{W})}$. The compact support assumption of $\varphi \in C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ is initially needed to ensure that $\max\{H, L\} < \infty$. However, the constants are then left-invariant: if $p \in \mathbb{H}$, then $\varphi^{(p^{-1})}$ is intrinsically differentiable, its intrinsic gradient is continuous, and satisfies (3.11) with the same constant H (see [10, Lemma 2.25]), $\|\nabla^{\varphi^{(p^{-1})}}\varphi^{(p^{-1})}\|_{L^{\infty}(\mathbb{W})} \leqslant L$ by Lemma 3.22, and $\varphi^{(p^{-1})}$ is intrinsic Lipschitz with constant L, even though the support of $\varphi^{(p^{-1})}$ of course depends on p.

We use Proposition 3.29 to quantify how well the intrinsic graph $S \subset \mathbb{H}^n$ of a compactly supported $C^{1,\alpha}_{\mathbb{H}}$ function is approximated at a point $p \in S$ by a certain vertical plane. For $p \in S$, let $\mathbb{W}_p = W_p \times \mathbb{R}$ be the unique vertical subgroup with the property that W_p is a $(2n-1)$ dimensional subspace of $\mathbb{R}^{2n}$ which is perpendicular to the line spanned by the vector $\nu_{\mathbb{H}}(p)$ using coordinates as in (3.13).

Proposition 3.41. *Fix $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. Let $\varphi \in C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ be compactly supported on the codimension-1 vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$, and write $S := \Phi(\mathbb{W})$. Then, there exists a constant $A = A(H, L) \geqslant 1$ such that for every $p \in S$,*

$$\mathrm{dist}_{\mathbb{H}}(q, S) \leqslant A d_{\mathbb{H}}(p, q)^{1+\alpha}, \qquad q \in p \cdot \mathbb{W}_p. \tag{3.42}$$

Here H and L are defined as in Remark 3.40, that is, φ satisfies (3.11) with constant H, φ is intrinsic L-Lipschitz, and $\|\nabla^{\varphi}\varphi\|_{L^\infty(\mathbb{W})} \leqslant L$.

Proof of Proposition 3.41. The plan is to apply estimate (3.30) from Proposition 3.29. Note that

$$L_p(y, t) := (0, y, t) \cdot (\langle \nabla^{\varphi}\varphi(w), y\rangle, 0, 0)$$

defines a map $\mathbb{W} \to \mathbb{W}_p$ since

$$W_p = \left\{ \left(\sum_{i=2}^{2n} D_i^{\varphi}\varphi(w)x_i, x_2, \ldots, x_{2n} \right) : (x_2, \ldots, x_{2n}) \in \mathbb{R}^{2n-1} \right\} \tag{3.43}$$

is a $(2n-1)$-plane perpendicular to

$$\begin{pmatrix} -1 \\ D_2^{\varphi}\varphi(w) \\ \vdots \\ D_{2n}^{\varphi}\varphi(w) \end{pmatrix}$$

in $\mathbb{R}^{2n}$. In fact, L_p is a bijection $\mathbb{W} \to \mathbb{W}_p$, and for all $(y, t) = (0, x_2, \ldots, x_{2n}, t) \in \mathbb{W}$,

$$\|L_p(y,t)\| = \left\| \left(\langle \nabla^{\varphi}\varphi(w), y\rangle, y, t - \tfrac{1}{2}x_{n+1}\langle \nabla^{\varphi}\varphi(w), y\rangle \right) \right\| \sim_L \|(y,t)\|, \tag{3.44}$$

because $|\nabla^{\varphi}\varphi(w)| \leqslant \|\nabla^{\varphi}\varphi\|_{L^\infty(\mathbb{W})} \leqslant L$. The last observation immediately implies the inequality "$\lesssim_L$" in (3.44). It also gives the converse inequality, since

$$\begin{aligned} \|(y,t)\| &\lesssim |y| + |t|^{\frac{1}{2}} \lesssim |y| + \left|t - \tfrac{1}{2}x_{n+1}\langle \nabla^{\varphi}\varphi(w), y\rangle\right|^{\frac{1}{2}} + \left|\tfrac{1}{2}x_{n+1}\langle \nabla^{\varphi}\varphi(w), y\rangle\right|^{\frac{1}{2}} \\ &\lesssim_L |y| + \left|t - \tfrac{1}{2}x_{n+1}\langle \nabla^{\varphi}\varphi(w), y\rangle\right|^{\frac{1}{2}} \lesssim_L \|L_p(y,t)\|. \end{aligned}$$

Moreover, for arbitrary $w = (y, t) \in \mathbb{W}$, we have $d_{\mathbb{H}}(q, S) \leqslant d_{\mathbb{H}}(q, p \cdot \Phi^{(p^{-1})}(y, t))$, where $\Phi^{(p^{-1})}$ is the graph map of $\varphi^{(p^{-1})}$, simply because $\Phi^{(p^{-1})}(\mathbb{W}) = p^{-1} \cdot S$. Therefore, by the left-invariance of $d_{\mathbb{H}}$, one has for $q = p \cdot L_p(y, t)$,

$$\begin{aligned} \mathrm{dist}_{\mathbb{H}}(q, S) \leqslant d_{\mathbb{H}}(p^{-1} \cdot q, \Phi^{(p^{-1})}(y,t)) &= d_{\mathbb{H}}(L_p(y,t), \Phi^{(p^{-1})}(y,t)) \\ &= |\varphi^{(p^{-1})}(y,t) - \langle \nabla^{\varphi}\varphi(w), y\rangle| \overset{(3.30)}{\lesssim}_{H,L} \|(y,t)\|^{1+\alpha} \\ &\overset{(3.44)}{\sim}_{H,L} \|L_p(y,t)\|^{1+\alpha} = d_{\mathbb{H}}(p,q)^{1+\alpha}. \end{aligned}$$

This proves (3.42), and hence the proposition. □

3.1.1. *Reduction to unit scale.* The rest of Section 3 is devoted to the proof of Theorem 3.27 in the case $n > 1$. With the earlier preparations in place, a proof for the case $n = 1$ could be obtained along the same lines, but some steps would require a separate discussion. Instead, we will deduce the case $n = 1$ later from a more general result, see Theorem 4.62. So we fix $n > 1$ for the rest of this section, and constants will be allowed to depend on n without special mentioning.

Theorem 3.27 for $n > 1$ is essentially a corollary of Theorem 1.14 applied to

$$(G, d_G, \mu) = (\mathbb{H}^{n-1} \times \mathbb{R}, d_{\mathbb{H}^{n-1}\times\mathbb{R}}, \mathcal{L}^{2n}) \quad \text{and} \quad (M, d_M) = (S, d_{\mathbb{H}}), \tag{3.45}$$

where (G, d_G) is defined as in (1.1) and the line below it.

Once the hypotheses of Theorem 1.14 have been verified – a task occupying the next section – the theorem will yield the existence of $2L'$-bilipschitz maps $f \colon K \to S \cap B(p, 1)$, $p \in S$, where $K \subset G$ with $\mathcal{H}^{2n+1}(K) \geqslant \delta > 0$. The constant $\delta > 0$ will only depend on α, the Hölder constant H in (3.11), the constant L that bounds the intrinsic Lipschitz constant of φ and the L^∞-norm of $\nabla^\varphi \varphi$, whereas the constant L' will depend only on L. This is saying, in particular, that the BPGBI condition holds at unit scale. How about other scales? The following easy lemma shows that property (3.11) improves under "zooming in":

Lemma 3.46. *Let $\varphi : \mathbb{W} \to \mathbb{V}$ be intrinsically differentiable with continuous intrinsic gradient $\nabla^\varphi \varphi$ that satisfies (3.11) with constants $\alpha > 0$ and $H \geqslant 1$. For $r > 0$, let*

$$\varphi_r(w) := \tfrac{1}{r}[\varphi \circ \delta_r], \quad w \in \mathbb{W}.$$

Then, $\psi := \varphi_r$ is an intrinsically differentiable function with intrinsic graph $\delta_{1/r}(\Phi(\mathbb{W}))$, its intrinsic gradient is continuous, and satisfies (3.11) with constants α and $r^\alpha H$, that is

$$|\nabla^{\psi^{(p^{-1})}} \psi^{(p^{-1})}(w) - \nabla^{\psi^{(p^{-1})}} \psi^{(p^{-1})}(0)| \leqslant r^\alpha H \|w\|^\alpha, \quad w \in \mathbb{W}, p \in \delta_{1/r}(\Phi(\mathbb{W})).$$

Proof. Fix $0 < r \leqslant 1$ and let w be an arbitrary point in $\mathbb{W}$. Then

$$\delta_{\frac{1}{r}}[w \cdot \varphi(w)] = \delta_{\frac{1}{r}}(w) \cdot \delta_{\frac{1}{r}}(\varphi(w)) = \delta_{\frac{1}{r}}(w) \cdot \delta_{\frac{1}{r}}(\varphi(\delta_r(\delta_{\frac{1}{r}}(w)))),$$

which shows that $\delta_{\frac{1}{r}}(\Phi(\mathbb{W}))$ is the intrinsic graph of $\psi = \varphi_r$ as defined in the lemma. Since the Heisenberg dilations are group isomorphisms which commute with vertical projections, it is easy to see that ψ is intrinsically differentiable with intrinsic gradient

$$\nabla^\psi \psi = \nabla^\varphi \varphi \circ \delta_r.$$

Moreover, since $\psi^{(p^{-1})} = \frac{1}{r}\varphi^{(\delta_r(p)^{-1})} \circ \delta_r$ for $p \in \delta_{1/r}(\Phi(\mathbb{W}))$, we have

$$\nabla^{\psi^{(p^{-1})}} \psi^{(p^{-1})} = \nabla^{\varphi^{(\delta_r(p)^{-1})}} \varphi^{(\delta_r(p)^{-1})} \circ \delta_r,$$

which yields the remaining claims in the lemma. □

Returning to the proof of Theorem 3.27, let $p \in S$ and, first, $0 < r \leqslant C$, where $C := 2\,\mathrm{diam}_{\mathbb{H}}(\Phi(\mathrm{spt}\,\varphi))$. Using the previous lemma, and also recalling that dilations have no effect on intrinsic Lipschitz constants, $S_{1/r} := \delta_{1/r}(S)$ is an intrinsic graph of an intrinsic Lipschitz function with essentially bounded intrinsic gradient satisfying (3.11) with constants depending only on the corresponding constants for S, and C. Therefore, by the BPGBI property at scale $r = 1$, to be established in the next section, every ball $S_{1/r} \cap B(p, 1)$ contains the image of a $2L'$-bilipschitz map g from a compact set $K \subset G$ with $\mathcal{H}^{2n+1}(K) \geqslant \delta = \delta(C) > 0$. Now, one may simply pre- and post-compose g with the natural dilations in G and $\mathbb{H}^n$ to produce a $2L'$-bilipschitz map $g_r \colon \delta_r(K) \to S \cap B(\delta_r(p), r)$ (note also that $\mathcal{H}^{2n+1}(\delta_r(K)) = r^{2n+1}\mathcal{H}^{2n+1}(K) \geqslant \delta r^{2n+1}$).

Next, consider the case $r > C$. Then, if $p \in S$ is arbitrary, the set $S \cap B(p, r)$ satisfies

$$\mathcal{H}^{2n+1}([S \cap B(p,r)] \cap \mathbb{W}) \gtrsim \mathcal{H}^{2n+1}(S \cap B(p,r)).$$

Thus, the restriction of Id to $[S \cap B(p,r)] \cap \mathbb{W}$ (composed with an isometry $G \cong \mathbb{W}$) yields the desired bilipschitz map in this case.

3.2. **Proof for graphs with Hölder continuous normals.** In this section we complete the proof of Theorem 3.27. After the reduction to unit scale in Section 3.1.1, it remains to verify the hypotheses of Theorem 1.14 for $n > 1$, $(G, d_G) := (\mathbb{H}^{n-1} \times \mathbb{R}, d_{\mathbb{H}^{n-1}\times\mathbb{R}})$, and $(M, d_M) := (S, d_{\mathbb{H}})$, where $S = \Phi(\mathbb{W})$, as in Theorem 3.27. To be accurate, also take $x_0 = 0 \in G$, and fix $p_0 \in M$ arbitrary. We start by defining the maps $i_{w\to p} \colon G \to S$. They will not depend on the scale index $k \geqslant 0$, that is, $i^k_{w\to p} = i_{w\to p}$ for all $k \geqslant 0$, and they can also be defined for all points $w \in G$, $p \in M$ (and not only those close to x_0 and p_0).

To construct the maps $i_{w\to p} \colon G \to S$, we will first define certain bilipschitz maps $\Psi_p : \mathbb{W} \to \mathbb{W}_p$. We know that $\mathbb{W}$ is isometric to $\mathbb{W}_p$, so without further restrictions, this would be an easy task, but keeping in mind (1.13), we want to make sure that the mappings Ψ_p change in a controlled way as we let p vary in S. It would be possible to arrange this even for isometric Ψ_p, but the construction is simpler if we allow for bilipschitz distortion, and the main ideas are contained in the following lemma.

Lemma 3.47. *For $n \geqslant 2$ and $D := (a_2, \ldots, a_n, c, b_2, \ldots, b_n) \in \mathbb{R}^{2n-1}$, define*

$$\psi_D(x_2, \ldots, x_{2n}) := cx_{n+1} + \sum_{i=2}^{n}(a_i x_i + b_i x_{n+i}),$$

and consider the vertical subgroups

$$\mathbb{W} := \{(x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n : \; x_1 = 0\} \text{ and } \mathbb{W}' := \{(x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n : \; x_1 = \psi_D(x_2, \ldots, x_{2n})\}.$$

Then the map $\Psi_D(0, x_2, \ldots, x_{2n}, t) :=$

$$(\psi_D(x_2, \ldots, x_{2n}), b_2x_{n+1} + x_2, \ldots, b_nx_{n+1} + x_n, x_{n+1}, -a_2x_{n+1} + x_{n+2}, \ldots, -a_nx_{n+1} + x_{2n}, t)$$

has the following properties:

1. *$\Psi_D : (\mathbb{W}, \cdot) \to (\mathbb{W}', \cdot)$ is a group isomorphism,*
2. *$\Psi_D : (\mathbb{W}, d_{\mathbb{H}}) \to (\mathbb{W}', d_{\mathbb{H}})$ is L_D-bilipschitz with L_D depending continuously on D,*
3. *$d_{\mathbb{H}}(\Psi_D(w), \Psi_{D'}(w)) \lesssim \max\{|D - D'|, |D - D'|^{1/2}\}\, \|w\|$, for all $w \in \mathbb{W}$.*

Proof. We start by noting that

$$\begin{aligned}\psi_D(x_2, \ldots, x_{2n}) &= cx_{n+1} + \sum_{i=2}^{n}(a_i x_i + b_i x_{n+i}) \\ &= \psi_D(b_2x_{n+1} + x_2, \ldots, b_nx_{n+1} + x_n, x_{n+1}, -a_2x_{n+1} + x_{n+2}, \ldots, -a_nx_{n+1} + x_{2n}),\end{aligned}$$

which can be used to show that $\Psi_D(\mathbb{W}) = \mathbb{W}'$. It further follows directly from the definition that Ψ_D is injective, $\Psi_D(0) = 0$, and $\Psi_D(w^{-1}) = (\Psi_D(w))^{-1}$. In order to see that

$$\Psi_D(w \cdot w') = \Psi_D(w) \cdot \Psi_D(w'), \quad w, w' \in \mathbb{W}, \tag{3.48}$$

it suffices to verify the identity for the last components of the points, as the first components agree obviously by linearity of Ψ_D. For $w = (0, x_2, \ldots, x_{2n}, t)$, $w' = (0, x_2', \ldots, x_{2n}', t')$, we find that

$$\begin{aligned}
[\Psi_D(w \cdot w')]_{2n+1} &= t + t' + \tfrac{1}{2}\sum_{i=2}^{n}(x_i x_{n+i}' - x_{n+i}x_i') \\
&= t + t' + \tfrac{1}{2}\left(c x_{n+1} + \sum_{i=2}^{n}(a_i x_i + b_i x_{n+i})\right) x_{n+1}' - \tfrac{1}{2}\left(c x_{n+1}' + \sum_{i=2}^{n}(a_i x_i' + b_i x_{n+i}')\right) x_{n+1} \\
&\quad + \tfrac{1}{2}\sum_{i=2}^{n}\left((b_i x_{n+1} + x_i)(-a_i x_{n+1}' + x_{n+i}') - (b_i x_{n+1}' + x_i')(-a_i x_{n+1} + x_{n+i})\right) \\
&= [\Psi_D(w) \cdot \Psi_D(w')]_{2n+1},
\end{aligned}$$

which shows (3.48) and thus completes the proof of the first claim in the lemma. Using the group isomorphism property and the fact that Ψ_D commutes with Heisenberg dilations, we next observe that

$$d_{\mathbb{H}}(\Psi_D(w), \Psi_D(w')) = \|\Psi_D(w')^{-1} \cdot \Psi_D(w)\| = \|\Psi_D(w'^{-1} \cdot w)\| \in [c_D d_{\mathbb{H}}(w, w'), C_D d_{\mathbb{H}}(w, w')],$$

where

$$c_D := \min\{\|\Psi_D(v)\| : \|v\| = 1\} \quad \text{and} \quad C_D := \max\{\|\Psi_D(v)\| : \|v\| = 1\}.$$

This concludes the second part of the lemma, up to the continuity of $D \mapsto L_D$, which will follow from the third part. To verify the third part, let us fix $D, D' \in \mathbb{R}^{2n-1}$, and an arbitrary point $w = (0, x_2, \ldots, x_{2n}, t)$ in $\mathbb{W}$, and compute $\Psi_{D'}(w)^{-1} \cdot \Psi_D(w) =$

$$(\psi_{(D-D')}(w), (b_2 - b_2')x_{n+1}, \ldots, (b_n - b_n')x_{n+1}, 0, (a_2' - a_2)x_{n+1}, \ldots, (a_n' - a_n)x_{n+1}, \tau),$$

where

$$\tau := \tfrac{1}{2}x_{n+1}\left[\psi_{(D-D')}(w) + \sum_{i=2}^{n}\left((b_i - b_i')x_{n+i} + (a_i - a_i')x_i\right)\right].$$

This shows that

$$d_{\mathbb{H}}(\Psi_D(w), \Psi_{D'}(w)) \lesssim \max\{|D - D'|, |D - D'|^{1/2}\}\|w\|,$$

as claimed. □

The mappings defined in Lemma 3.47 will be used later in the case where the components of D are the entries of an intrinsic gradient $\nabla^{\varphi}\varphi(w)$. For $p = \Phi(w)$, we then denote

$$\Psi_p := \Psi_{(D_2^{\varphi}\varphi(w), \ldots, D_{2n}^{\varphi}\varphi(w))}, \tag{3.49}$$

so that $\Psi_p(\mathbb{W}) = \mathbb{W}_p$ is the vertical plane appearing in Proposition 3.41.

Remark 3.50. It is important to note that Ψ_p is different from the obvious parametrization $L_p : \mathbb{W} \to \mathbb{W}_p$ used in the proof of Proposition 3.41. While L_p is intrinsic Lipschitz, the map Ψ_p is metrically Lipschitz and it is obtained by precomposing L_p with a map that serves as "characteristic straightening map" in this setting.

Remark 3.51. The proof of Theorem 3.27 can be modified so that it yields 2-bilipschitz maps instead of $2L'$-bilipschitz maps for a constant $L' = L'(L) > 1$. In the case $n = 1$ this is due to the third author in an earlier version of this paper, and it is based on replacing the bilipschitz map $\Psi_p : \mathbb{W} \to \mathbb{W}_p$ in (3.49) by the isometry $(0, y, t) \mapsto (ye_p, t)$, where e_p represents a horizontal unit vector vector (in the $\{X_1, X_2\}$-frame) perpendicular to the horizontal normal $\nu_{\mathbb{H}}(p)$.

Returning to the proof of Theorem 3.27, we proceed to construct the mappings $i_{w\to p} : G \to \Phi(\mathbb{W})$ for $w \in G$, $p \in S = \Phi(\mathbb{W})$. Since $G = \mathbb{H}^{n-1} \times \mathbb{R}$ (as in (1.1)) is isometrically isomorphic to $\mathbb{W}$ via the map

$$F : ((z_1, \ldots, z_{2n-2}, t), s) \mapsto (0, z_1, \ldots, z_{n-1}, s, z_n, \ldots, z_{2n-2}, t)$$

the idea for the construction of $i_{w\to p}(v)$ is informally the following: identify $w^{-1} \cdot_G v \in G$ with the point $F(w^{-1} \cdot_G v) \in \mathbb{W}$, then map this point to the vertical plane $\mathbb{W}_p$ by means of the bilipschitz map Ψ_p from (3.49), left translate by the point p, and finally let $i_{w\to p}(v)$ be a point in S of minimal distance from $p \cdot \Psi_p(F(w^{-1} \cdot_G v)) \in p \cdot \mathbb{W}_p$, keeping in mind Proposition 3.41. Such a point may not be unique, but this does not matter as long as the choice is made depending only on the product $w^{-1} \cdot_G v$, and not on the points v and w individually.

We now explain the construction in detail. First, if $u \in G$, let

$$q := q[p, u] \in S$$

be any point satisfying

$$d_{\mathbb{H}}\left(p \cdot \Psi_p(F(u)), q\right) = \mathrm{dist}_{\mathbb{H}}\left(p \cdot \Psi_p(F(u)), S\right). \tag{3.52}$$

Then, if $v, w \in G$ and $p \in S$, let

$$i_{w\to p}(v) := q[p, w^{-1} \cdot_G v]. \tag{3.53}$$

The definition implies that if $w, w', v, v' \in G$ with $w^{-1} \cdot_G v = (w')^{-1} \cdot_G v'$, then

$$i_{w\to p}(v) = i_{w'\to p}(v'). \tag{3.54}$$

To simplify notation in the sequel, we define $\mathrm{Tan}_p^w \colon (G, d_G) \to (\mathbb{W}_p, d_{\mathbb{H}})$ to be the map given by

$$\mathrm{Tan}_p^w(v) = \Psi_p(F(w^{-1} \cdot_G v)), \quad v \in G. \tag{3.55}$$

It follows from Lemma 3.47 that $\mathrm{Tan}_p^w \colon (G, d_G) \to (\mathbb{W}_p, d_{\mathbb{H}})$ is a bilipschitz map with bilipschitz constant bounded in terms of $\|\nabla^{\varphi}\varphi\|_{L^\infty(\mathbb{W})}$. Evidently $i_{w\to p}(w) = p \cdot \mathrm{Tan}_p^w(w) = p$. Also note that the isomorphism property of $\Psi_p \circ F$ implies the following "chain rule":

$$\mathrm{Tan}_p^{w_1}(w_3) = \mathrm{Tan}_p^{w_1}(w_2) \cdot \mathrm{Tan}_p^{w_2}(w_3), \qquad w_1, w_2, w_3 \in G,\ p \in S. \tag{3.56}$$

Since $p \cdot \mathrm{Tan}_p^w(v) \in p \cdot \mathbb{W}_p$, one infers from (3.42) and the definition of $i_{w\to p}(v)$ that

$$\begin{aligned} d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v), i_{w\to p}(v)) &= \mathrm{dist}_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v), S) \\ &\leqslant A d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v), p)^{1+\alpha} \lesssim_L A d_G(w, v)^{1+\alpha}. \end{aligned} \tag{3.57}$$

Using this estimate, one has

$$\begin{aligned} &|d_{\mathbb{H}}(i_{w\to p}(v), i_{w\to p}(v')) - d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v), p \cdot \mathrm{Tan}_p^w(v'))| \\ &\quad \leqslant d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v), i_{w\to p}(v)) + d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v'), i_{w\to p}(v')) \\ &\quad \lesssim_L A \max\{d_G(w, v)^{1+\alpha}, d_G(w, v')^{1+\alpha}\}, \qquad v, v' \in G. \end{aligned} \tag{3.58}$$

Moreover

$$d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^w(v), p \cdot \mathrm{Tan}_p^w(v')) = d_{\mathbb{H}}(\Psi_p(F(w^{-1} \cdot_G v)), \Psi_p(F(w^{-1} \cdot_G v'))),$$

and since $\Psi_p \circ F$ is bilipschitz with a constant that depends only on $\|\nabla^\varphi \varphi\|_{L^\infty(\mathbb{W})}$, condition (1.12) follows with the help of (3.58).

It remains to check condition (1.13). Using the homogeneity of G (see the discussion around (3.63) for further details), it suffices to verify the case "$x = 0$" of condition (1.13): if $w \in G$ with $\|w\| \leqslant 2$, $p \in S$, and $i_{0\to p}(w) = q \in S$, then

$$d_{\mathbb{H}}(i_{0\to p}(v), i_{w\to q}(v)) \lesssim_L AH \max\{\|w\|^{1+\alpha/2}, \|v\|^{1+\alpha/2}, d_G(v,w)^{1+\alpha/2}\} \tag{3.59}$$

for all $v \in G$ with $\|v\| \leqslant 1$. (In particular, it may be interesting to note that the $C^{1,\alpha}_{\mathbb{H}}$-hypothesis only gives the condition (1.13) with exponent $\alpha/2$.) To estimate the left hand side of (3.59), the strategy will be to first obtain a corresponding estimate for

$$d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^0(v), q \cdot \mathrm{Tan}_q^w(v)),$$

and eventually conclude (3.59) from this bound combined with (3.57). Consider $v, w \in G$ with $\|w\| \leqslant 2$. Start by applying the "chain rule" (3.56), and the triangle inequality, as follows:

$$\begin{aligned} d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^0(v), q \cdot \mathrm{Tan}_q^w(v)) &= d_{\mathbb{H}}([p \cdot \mathrm{Tan}_p^0(w)] \cdot \mathrm{Tan}_p^w(v), q \cdot \mathrm{Tan}_q^w(v)) \\ &\leqslant d_{\mathbb{H}}([p \cdot \mathrm{Tan}_p^0(w)] \cdot \mathrm{Tan}_p^w(v), [p \cdot \mathrm{Tan}_p^0(w)] \cdot \mathrm{Tan}_q^w(v)) \\ &\quad + d_{\mathbb{H}}([p \cdot \mathrm{Tan}_p^0(w)] \cdot \mathrm{Tan}_q^w(v), q \cdot \mathrm{Tan}_q^w(v)) =: I_1 + I_2. \end{aligned}$$

To estimate I_1, note that by left-invariance

$$I_1 = d_{\mathbb{H}}(\mathrm{Tan}_p^w(v), \mathrm{Tan}_q^w(v)) = d_{\mathbb{H}}(\Psi_p(F(w^{-1} \cdot_G v)), \Psi_q(F(w^{-1} \cdot_G v))).$$

Then (3.49) and the third part of Lemma 3.47 imply that

$$I_1 = d_{\mathbb{H}}(\Psi_p(F(w^{-1} \cdot_G v)), \Psi_q(F(w^{-1} \cdot_G v))) \lesssim_L |\nabla^\varphi \varphi(\pi_{\mathbb{W}}(p)) - \nabla^\varphi \varphi(\pi_{\mathbb{W}}(q))|^{1/2} d_G(v,w).$$

To proceed, we note that $p = a \cdot \varphi(a)$ and $q = b \cdot \varphi(b)$ satisfy

$$\begin{aligned} |\nabla^\varphi \varphi(b) - \nabla^\varphi \varphi(a)| &= |\nabla^\varphi \varphi(\pi_{\mathbb{W}}(p \cdot \varphi(a)^{-1} \cdot a^{-1} \cdot b \cdot \varphi(a))) - \nabla^\varphi \varphi(\pi_{\mathbb{W}}(p))| \\ &= |\nabla^{\varphi^{(p^{-1})}} \varphi^{(p^{-1})}(\varphi(a)^{-1} \cdot a^{-1} \cdot b \cdot \varphi(a))) - \nabla^{\varphi^{(p^{-1})}} \varphi^{(p^{-1})}(0)| \\ &\overset{(3.11)}{\leqslant} H\|\varphi(a)^{-1} \cdot a^{-1} \cdot b \cdot \varphi(a)\|^\alpha \\ &= H\|\varphi(a)^{-1} \cdot a^{-1} \cdot b \cdot \varphi(b) \cdot \varphi(b)^{-1} \cdot \varphi(a)\|^\alpha \\ &\leqslant 2H d_{\mathbb{H}}(p,q)^\alpha, \end{aligned}$$

using Lemma 3.22 when passing to the second line. It follows for $p \in S$ and $q = i_{0\to p}(w)$ with $\|w\| \leqslant 2$ that

$$\begin{aligned} |\nabla^\varphi \varphi(\pi_{\mathbb{W}}(p)) - \nabla^\varphi \varphi(\pi_{\mathbb{W}}(q))| &\leqslant 2H d_{\mathbb{H}}(p,q)^\alpha = 2H d_{\mathbb{H}}(i_{0\to p}(0), i_{0\to p}(w))^\alpha \\ &\overset{(3.58)}{\lesssim}_L H[\|w\|^\alpha + A\|w\|^\alpha] \lesssim_L AH\|w\|^\alpha. \end{aligned} \tag{3.60}$$

One needed here the assumption $\|w\| \leqslant 2$ since (3.58) initially gives a term of the form $\|w\|^{1+\alpha}$. The estimate above implies that

$$I_1 \lesssim_L |\nabla^\varphi \varphi(\pi_{\mathbb{W}}(p)) - \nabla^\varphi \varphi(\pi_{\mathbb{W}}(q))|^{1/2} d_G(v,w) \lesssim_L AH\|w\|^{\alpha/2} d_G(v,w).$$

Thus, the term I_1 is bounded from above by the right hand side of (3.59).

The term I_2 has the form $I_2 = \|\mathfrak{b}^{-1} \cdot \mathfrak{a} \cdot \mathfrak{b}\|$ with

$$\mathfrak{a} = q^{-1} \cdot p \cdot \mathrm{Tan}_p^0(w) \quad \text{and} \quad \mathfrak{b} = \mathrm{Tan}_q^w(v).$$

Note that $\|\mathfrak{a}\| = d_{\mathbb{H}}(q, p \cdot \mathrm{Tan}_p^0(w)) \lesssim_L A\|w\|^{1+\alpha}$ by (3.57) (this is the place where the relation $q = i_{0\to p}(w)$ is used), whereas $\|\mathfrak{b}\| \sim_L d_G(w,v)$. Now, writing $\mathfrak{a} = (x_a, t_a)$ and $\mathfrak{b} = (x_b, t_b)$, one can easily compute that

$$\mathfrak{b}^{-1} \cdot \mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a} \cdot \left(0, 0, \sum_{i=1}^{n} (x_{a,i}x_{b,n+i} - x_{b,i}x_{a,n+i})\right) =: \mathfrak{a} \cdot (0,0,\omega(x_a,x_b)). \tag{3.61}$$

(This is just the fundamental "commutator relation" in $\mathbb{H}^n$.) Consequently,

$$\begin{aligned} I_2 &\leqslant \|\mathfrak{a}\| + \sqrt{|\omega(x_a,x_b)|} \lesssim_L A\|w\|^{1+\alpha} + \sqrt{\|\mathfrak{a}\|\|\mathfrak{b}\|} \\ &\lesssim_L A\|w\|^{1+\alpha} + A^{1/2}\|w\|^{1/2+\alpha/2} d_G(w,v)^{1/2} \\ &\lesssim_L A \max\{\|w\|^{1+\alpha/2}, d_G(w,v)^{1+\alpha/2}\}. \end{aligned}$$

This shows that also I_2 is bounded by the right hand side of (3.59). Glancing again at the estimates for I_1 and I_2, one sees that

$$d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^0(v), q \cdot \mathrm{Tan}_q^w(v)) \lesssim_L AH \max\{\|w\|^{1+\alpha/2}, d_G(v,w)^{1+\alpha/2}\}, \tag{3.62}$$

which is even a bit better than (3.59). The estimate (3.59) now follows from the triangle inequality:

$$\begin{aligned} d_{\mathbb{H}}(i_{0\to p}(v), i_{w\to q}(v)) \leqslant\ & d_{\mathbb{H}}(i_{0\to p}(v), p \cdot \mathrm{Tan}_p^0(v)) \\ &+ d_{\mathbb{H}}(p \cdot \mathrm{Tan}_p^0(v), q \cdot \mathrm{Tan}_q^w(v)) \\ &+ d_{\mathbb{H}}(i_{w\to q}(v), q \cdot \mathrm{Tan}_q^w(v)). \end{aligned}$$

The middle term here is controlled by (3.62), and the first and third terms are controlled by (3.57), recalling the bounds for $\|v\| \leqslant 1$ and $\|w\| \leqslant 2$, which ensure that we can replace "α" by "$\alpha/2$" in (3.57). This concludes the proof of (3.59).

Finally, we address the point left open above, that (3.59) looks slightly less general than (1.13). To check (1.13) properly, we need to fix $w_1, w_2 \in B_G(0,1)$ and $p, q \in S$ with $i_{w_1\to p}(w_2) = q$, and verify that

$$d_{\mathbb{H}}(i_{w_1\to p}(w_3), i_{w_2\to q}(w_3)) \lesssim \max\{d_G(w_1,w_2), d_G(w_1,w_3)), d_G(w_2,w_3)\}^{1+\alpha/2} \tag{3.63}$$

for all $w_3 \in G$ with $d_G(w_1,w_3) \leqslant 1$. However, set $w := w_1^{-1} \cdot_G w_2$ and $v := w_1^{-1} \cdot_G w_3$, and observe that

$$w_2^{-1} \cdot_G w_3 = w^{-1} \cdot_G v \quad \text{and} \quad w_1^{-1} \cdot_G w_3 = 0^{-1} \cdot_G v.$$

These relations, and (3.54), show that

$$i_{w_1\to p}(w_3) = i_{0\to p}(v) \quad \text{and} \quad i_{w_2\to q}(w_3) = i_{w\to q}(v).$$

Thus, (3.63) follows from (3.59) applied with w and v, as above.

This completes the verification of the hypotheses of Theorem 1.14, and hence the proof of Theorem 3.27.

4. LIPSCHITZ FLAGS AND EXTRA VERTICAL HÖLDER REGULARITY

This section is devoted to the first Heisenberg group, $\mathbb{H}^1$. For convenience we use coordinates (x, y, t), instead of (x_1, x_2, t), to denote points in $\mathbb{H}^1$. As usual, we may identify $\mathbb{W}$ with $\mathbb{R}^2$ by mapping $(0, y, t)$ to (y, t), and we identify $(x, 0, 0) \in \mathbb{V}$ with $x \in \mathbb{R}$.

Definition 4.1. We say that an intrinsic Lipschitz function $\varphi : \mathbb{W} \to \mathbb{V}$ has *extra vertical Hölder regularity with constants* $0 < \alpha \leqslant 1$ *and* $H > 0$ if

$$|\varphi(y, t) - \varphi(y, t')| \leqslant H|t - t'|^{\frac{1+\alpha}{2}}, \tag{4.2}$$

for all $y, t, t' \in \mathbb{R}$.

Intrinsic Lipschitz functions are always $1/2$-Hölder continuous with respect to the Euclidean metric along vertical lines. Condition (4.2) constitutes an amount of extra regularity at small scales which is not implied by the intrinsic Lipschitz condition alone, see for instance [6, Example 1.3].

Remark 4.3. The definition is left invariant: if φ has extra vertical Hölder regularity with constants α and H, then for every $p \in \mathbb{H}^1$, the function $\varphi^{(p^{-1})}$ whose intrinsic graph is $p^{-1} \cdot \Phi(\mathbb{W})$ also has extra vertical Hölder regularity with the same constants. Moreover, for $r > 0$, the function $\delta_{\frac{1}{r}} \circ \varphi \circ \delta_r$, whose intrinsic graph is $\delta_{\frac{1}{r}}(\Phi(\mathbb{W}))$, has extra vertical Hölder regularity with constants α and Hr^{α}. So condition (4.2) improves under "zooming in".

Remark 4.4. An intrinsic Lipschitz function **with compact support** has extra vertical Hölder regularity with constants $0 < \alpha \leqslant 1$ and $H > 0$ if and only if there is $H' > 0$ such that

$$|\varphi(y, t) - \varphi(y, t')| \leqslant \begin{cases} H'|t - t'|^{\frac{1+\alpha}{2}}, & \text{if } |t - t'| \leqslant 1, \\ H'|t - t'|^{\frac{1-\alpha}{2}}, & \text{if } |t - t'| \geqslant 1, \end{cases} \quad , \qquad y, t, t' \in \mathbb{R}, \tag{4.5}$$

that is, φ has extra vertical Hölder regularity in the sense of [16, Theorem 5.1].

Before studying in more detail the intrinsic graphs of functions that satisfy the conditions in Definition 4.1, we give two examples of such functions.

Example 4.6. *Under the identification* $\mathbb{W} \triangleq \mathbb{R}^2$ *and* $\mathbb{V} \triangleq \mathbb{R}$ *described before Definition 4.1, every compactly supported Euclidean Lipschitz function* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *is an intrinsic Lipschitz function that satisfies the extra vertical Hölder regularity condition in Definition 4.1 with* $\alpha = 1$.

Example 4.7. *Let* $0 < \alpha \leqslant 1$, $\mathbb{W}, \mathbb{V} \subset \mathbb{H}^1$ *as above, and let* $\varphi : \mathbb{W} \to \mathbb{V}$ *be a compactly supported* $C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ *function. Since* $\operatorname{spt}\varphi$ *is compact,* $\nabla^{\varphi}\varphi$ *is continuous and compactly supported, hence* $L := \|\nabla^{\varphi}\varphi\|_{L^{\infty}(\mathbb{W})} < \infty$. *According to* [10, Lemma 2.22], *this implies that* φ *is intrinsic Lipschitz. The extra vertical Hölder regularity condition follows from* [10, Proposition 4.2], *keeping in mind the characterization of compactly supported* $C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ *functions stated in Proposition 3.10.*

Intrinsic graphs of intrinsic Lipschitz functions with extra vertical Hölder regularity turn out to be well approximable at all points and small scales by intrinsic Lipschitz graphs of a special form, the *Lipschitz flags*; see Proposition 4.25 for the precise statement. In the proof of Theorem 1.11, Lipschitz flags will play an analogous role as vertical tangent planes did in the proof of Theorem 1.6, so we start with some observations of general nature that serve as a counterpart for Section 3.1.

4.1. **Approximation by Lipschitz flags.**

Definition 4.8. We say that $F \subset \mathbb{H}^1$ is a *Lipschitz flag* if there exists a Euclidean Lipschitz map $\psi : \mathbb{R} \to \mathbb{R}$ such that F is the intrinsic graph $\Phi(\mathbb{W})$ of the map $\varphi : \mathbb{W} \to \mathbb{V}$ defined by

$$\varphi(y,t) = \psi(y). \tag{4.9}$$

Lipschitz flags are bilipschitz equivalent to the parabolic plane. This observation appeared already in [17, Lemma 7.5], but we include the proof for completeness.

Lemma 4.10. *If $F \subset \mathbb{H}^1$ is a Lipschitz flag given by an L-Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$, then there is a $\sim (1+L)$-bilipschitz map*

$$\Psi_F : (\mathbb{W}, d_{\mathbb{H}}) \to (F, d_{\mathbb{H}}).$$

Proof. Let F be a Lipschitz flag, so F is the intrinsic graph $\Phi(\mathbb{W})$ of the map $\varphi : \mathbb{W} \to \mathbb{V}$ defined as in (4.9) for the Euclidean L-Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$. We will show that the map $\Psi_F : (\mathbb{W}, d_{\mathbb{H}}) \to (F, d_{\mathbb{H}})$ given by

$$\Psi_F(y,t) := \left(\psi(y), y, t - \frac{1}{2}y\psi(y) + \int_0^y \psi(\eta)\, d\eta\right) \tag{4.11}$$

is the $\sim (1+L)$-bilipschitz map which we are looking for. Firstly, we observe that

$$\Psi_F(y,t) = \Phi\left(y, t + \int_0^y \psi(\eta)\, d\eta\right),$$

where Φ is the graph map[1] of φ, hence $\Psi_F(\mathbb{W}) = \Phi(\mathbb{W})$. Moreover, since ψ is an L-Lipschitz function, we get

$$\begin{aligned} &d_{\mathbb{H}}(\Psi_F(y,t), \Psi_F(y',t')) \\ &= \left\| \left(\psi(y') - \psi(y), y' - y, t' - t + \tfrac{1}{2}(y - y')(\psi(y') + \psi(y)) + \int_y^{y'} \psi(\eta)\, d\eta\right) \right\| \\ &= \left\| \left(\psi(y') - \psi(y), y' - y, t' - t + \int_y^{y'} \left(\tfrac{\psi(\eta) - \psi(y)}{2}\right) + \left(\tfrac{\psi(\eta) - \psi(y')}{2}\right) d\eta\right) \right\| \\ &\lesssim (1+L)\left(|y' - y| + \sqrt{|t' - t|}\right), \end{aligned} \tag{4.12}$$

for all $(y,t), (y',t') \in \mathbb{W}$. On the other hand, since

$$|t' - t| \leqslant \left| t' - t + \int_y^{y'} \left(\tfrac{\psi(\eta) - \psi(y)}{2}\right) + \left(\tfrac{\psi(\eta) - \psi(y')}{2}\right) d\eta \right| + \left| \int_y^{y'} \left(\tfrac{\psi(\eta) - \psi(y)}{2}\right) + \left(\tfrac{\psi(\eta) - \psi(y')}{2}\right) d\eta \right|,$$

it follows

$$d_{\mathbb{H}}((y,t),(y',t')) \lesssim |y - y'| + |t - t'|^{1/2} \lesssim (1+L) d_{\mathbb{H}}(\Psi_F(y,t), \Psi_F(y',t')),$$

for all $(y,t), (y',t') \in \mathbb{W}$. Hence, putting together (4.12) and the last inequality, we get that Ψ_F is a $\sim (1+L)$-bilipschitz map, as desired. □

[1]The map Ψ_F is the composition of the graph map Φ with the "characteristic straightening map" mentioned in the introduction. First, the characteristic straightening map sends horizontal lines in $\mathbb{W}$ to integral curves of $\nabla^{\varphi} = \partial_y + \psi(y)\partial_t$, and these integral curves are then mapped by Φ to the horizontal curves that foliate the flag F.

Given an intrinsic Lipschitz graph $S \subset \mathbb{H}^1$, we will define for each $p \in S$ a Lipschitz flag that intersects S in a Lipschitz curve passing through p. We start with some general definitions that will be used throughout this section. We state them in terms of the intrinsic Lipschitz function $\varphi^{(p^{-1})}$, whose intrinsic graph is $p^{-1} \cdot S$, recall (3.19) .

Definition 4.13. Let $S = \Phi(\mathbb{W}) = \{w \cdot \varphi(w) \; : \; w \in \mathbb{W}\}$ be the intrinsic graph of an intrinsic L-Lipschitz function $\varphi : \mathbb{W} \to \mathbb{V}$. To each point $p \in S$, we associate the function

$$\begin{aligned} \psi_p : \mathbb{R} &\longrightarrow \mathbb{R} \\ s &\longmapsto \varphi^{(p^{-1})}(s, \tau_p(s)), \end{aligned} \tag{4.14}$$

where $\tau_p : \mathbb{R} \to \mathbb{R}$ is some solution of the Cauchy problem

$$\begin{cases} \dot{\tau}_p(s) = \varphi^{(p^{-1})}(s, \tau_p(s)), & \text{for } s \in \mathbb{R} \\ \tau_p(0) = 0. \end{cases} \tag{4.15}$$

Note that $\varphi^{(p^{-1})}$ is intrinsic Lipschitz with the same constant as φ, and then the global existence of τ_p follows as in [2, (6.27)]. Proposition 6.10 in [2] is only for higher dimensions, but this part of the argument works also for our setting $\mathbb{H}^1$. Moreover, using the same proof as for [2, (6.30)], it follows that τ_p satisfies the inequality

$$|\tau_p(s)| \lesssim_L |s|^2, \qquad s \in \mathbb{R}. \tag{4.16}$$

Notice further that $\gamma_p : s \mapsto (0, s, \tau_p(s))$ is a C^1 curve with

$$\dot{\gamma}_p(s) = \begin{pmatrix} 0 \\ 1 \\ \varphi^{(p^{-1})}(\gamma_p(s)) \end{pmatrix} = D_2^{\varphi^{(p^{-1})}}|_{\gamma_p(s)},$$

where we recall that D_2^{φ} is the vector field $D_2^{\varphi} = \partial_y + \varphi \partial_t$. As a consequence, from [2, Proposition 6.6] it also follows that $s \mapsto \psi_p(s) = \varphi^{(p^{-1})}(\gamma_p(s))$ is Euclidean Lipschitz with Lipschitz constant depending only on the intrinsic Lipschitz constant of φ. The solution τ_p may not be unique, but we never need other properties of it than the ones described above, so any choice will do. For completeness, we also mention that $\Phi^{(p^{-1})} \circ \gamma_p$ is a Lipschitz curve in $(\mathbb{H}^1, d_{\mathbb{H}})$ by [21, Theorem 4.2.16].

Definition 4.17. Let $S = \Phi(\mathbb{W})$ be the intrinsic graph of an intrinsic Lipschitz function $\varphi : \mathbb{W} \to \mathbb{V}$. For each point $p \in S$, we define

$$F_p := \left\{(0, y, t) \cdot (\psi_p(y), 0, 0) \; : \; (y, t) \in \mathbb{R}^2\right\}. \tag{4.18}$$

We note the following properties of F_p defined as in (4.18):

(1) F_p is a Lipschitz flag,
(2) $0 \in F_p$,
(3) $p \cdot F_p$ is also a Lipschitz flag,
(4) $F_p \cap (p^{-1} \cdot S) \supseteq \Phi^{(p^{-1})}(\gamma_p(\mathbb{R}))$.

Item (1) follows immediately from the fact that ψ_p is a Euclidean Lipschitz function. The item (2) follows since $p \in S$, and so $\psi_p(0) = \varphi^{(p^{-1})}(0, 0) = 0$. Next, (3) follows by computing explicitly that

$$p \cdot F_p = \{(0, y, t) \cdot (\psi(y), 0, 0) : (y, t) \in \mathbb{R}^2\},$$

where $\psi(y) := \psi_p(y - y_p) + x_p$ and $p = (x_p, y_p, t_p)$. Since $\psi_p : \mathbb{R} \to \mathbb{R}$ is Lipschitz, so is ψ. Finally, (4) is clear from the definitions since $\psi_p(y) = \varphi^{(p^{-1})}(y, \tau_p(y))$ and F_p is of the form (4.18).

Definition 4.19. Let $S = \Phi(\mathbb{W})$ be the intrinsic graph of an intrinsic Lipschitz function $\varphi : \mathbb{W} \to \mathbb{V}$. For each point $p \in S$, we let $\Psi_p := \Psi_{F_p} : \mathbb{W} \to F_p$ be the map given by the formula (4.11) applied to the Lipschitz flag $F = F_p$ from (4.18), that is,

$$\Psi_p(y,t) = \left(\varphi^{(p^{-1})}(y, \tau_p(y)), y, t - \frac{1}{2} y \varphi^{(p^{-1})}(y, \tau_p(y)) + \int_0^y \varphi^{(p^{-1})}(\eta, \tau_p(\eta))\, d\eta\right). \quad (4.20)$$

The reader may think of Ψ_p as a surrogate for the bilipschitz map defined in (3.49), which sends $\mathbb{W}$ to the vertical plane $\mathbb{W}_p$. In this analogy, the Lipschitz flag F_p plays the role of $\mathbb{W}_p$. To emphasise this conceptual similarity, we decided to use again the symbol "Ψ_p" in Definition 4.19. The analogy is however not perfect: if $\varphi^{(p^{-1})}$ is not intrinsic linear, the map Ψ_p from Definition 4.19 is not a group homomorphism and hence we lack a counterpart for the chain rule (3.56), which we proved for the map defined in (3.55).

The following properties of $\Psi_{F_p} = \Psi_p$ defined as in (4.20) will be used:

(1) $\Psi_p(0) = 0$ since $\varphi^{(p^{-1})}(0,0) = 0$,
(2) $\Psi_p(y,t) = \Phi^{(p^{-1})}(y, \tau_p(y)) \cdot (0,0,t)$, as $\int_0^y \varphi^{(p^{-1})}(\eta, \tau_p(\eta))\, d\eta = \int_0^y \dot{\tau}_p(\eta)\, d\eta = \tau_p(y)$,
(3) Ψ_p is bilipschitz with a constant that depends only on the intrinsic Lipschitz constant of φ (by Lemma 4.10 and the paragraph before Definition 4.19).

Remark 4.21. If $S = \Phi(\mathbb{W})$ is itself a Lipschitz flag, that is, the intrinsic graph of an intrinsic Lipschitz function $\varphi : \mathbb{W} \to \mathbb{V}$ that does not depend on the t-variable, then

$$\Psi_p(y,t) = \Phi^{(p^{-1})}\left(y, t + \int_0^y \varphi^{(p^{-1})}(\eta, \tau_p(\eta))\, d\eta\right), \quad (4.22)$$

and in particular, $\Psi_p(\mathbb{W}) = \Phi^{(p^{-1})}(\mathbb{W})$ and hence $S = p \cdot \Psi_p(\mathbb{W})$ in that case. Also note that here $\varphi^{(p^{-1})}$ does not depend on the t-variable, and so the integral in (4.22) can be written without the dependence on τ_p.

As we noted below Definition 4.17, the surfaces $F_p = \Psi_p(\mathbb{W})$ and $p^{-1} \cdot S = \Phi^{(p^{-1})}(\mathbb{W})$ intersect at least along a curve. The next lemma shows that they approximate each other well also in a neighborhood of that curve if φ has extra vertical Hölder regularity.

Lemma 4.23. *Let $\varphi : \mathbb{W} \to \mathbb{V}$ be an intrinsic Lipschitz function with extra vertical Hölder regularity with constants $0 < \alpha \leqslant 1$ and $H > 0$. Then*

$$d_{\mathbb{H}}(\Psi_p(y,t), \Phi^{(p^{-1})}(y, \tau_p(y) + t)) \leqslant H|t|^{(1+\alpha)/2} \leqslant H\|(y,t)\|^{1+\alpha}, \qquad (y,t) \in \mathbb{R}^2.$$

Remark 4.24. From the proof of Lemma 4.23 one can infer that if φ has extra vertical Hölder regularity in the stronger sense of (4.5), then

$$d_{\mathbb{H}}(\Psi_p(y,t), \Phi^{(p^{-1})}(y, \tau_p(y) + t)) \leqslant H' \min\{|t|^{(1+\alpha)/2}, |t|^{(1-\alpha)/2}\}, \quad \text{for } (y,t) \in \mathbb{R}^2.$$

Proof of Lemma 4.23. Recall that τ_p is a solution of the Cauchy problem (4.15), and we have that

$$\Psi_p(y,t) = \Phi^{(p^{-1})}(y, \tau_p(y)) \cdot (0,0,t)$$

for all $(y,t) \in \mathbb{R}^2$. As a consequence,

$$
\begin{aligned}
&d_{\mathbb{H}}(\Psi_p(y,t), \Phi^{(p^{-1})}(y, \tau_p(y)+t)) = d_{\mathbb{H}}(\Phi^{(p^{-1})}(y,\tau_p(y)) \cdot (0,0,t), \Phi^{(p^{-1})}(y,\tau_p(y)+t)) \\
&= \|\varphi^{(p^{-1})}(y,\tau_p(y)+t)^{-1} \cdot (0,y,\tau_p(y)+t)^{-1} \cdot (0,y,\tau_p(y)) \cdot \varphi^{(p^{-1})}(y,\tau_p(y)) \cdot (0,0,t)\| \\
&= |\varphi^{(p^{-1})}(y,\tau_p(y)+t) - \varphi^{(p^{-1})}(y,\tau_p(y))| \leqslant H|t|^{\frac{1+\alpha}{2}},
\end{aligned}
$$

as claimed. □

In the following we denote by $[A]_\delta^{\mathbb{H}^1}$ the δ-neighbourhood of a set $A \subset \mathbb{H}^1$ with respect to $d_{\mathbb{H}}$, and $[B]_\delta^{\mathbb{R}^2}$ stands for the Euclidean δ-neighbourhood of a set $B \subset \mathbb{R}^2$. A direct corollary of Lemma 4.23 is the following result:

Proposition 4.25. *Let $\varphi : \mathbb{W} \to \mathbb{V}$ be an intrinsic L-Lipschitz function with extra vertical Hölder regularity with constants $0 < \alpha \leqslant 1$ and $H > 0$. Then, there is $c = c(L) \geqslant 1$ such that for all $r > 0$ and all $p \in S = \Phi(\mathbb{W})$ it follows*

(1) $S \cap B(p,r) \subset [p \cdot \Psi_p(\mathbb{W})]_{c\delta}^{\mathbb{H}^1}$,
(2) $(p \cdot \Psi_p(\mathbb{W})) \cap B(p,r) \subset [S]_{c\delta}^{\mathbb{H}^1}$,

where $\delta := \delta(H,r) := Hr^{1+\alpha}$.

Remark 4.26. Continuing Remark 4.24, we note that if an intrinsic L-Lipschitz function φ has extra vertical Hölder regularity with constants $0 < \alpha \leqslant 1$ and $H' > 0$ in the stronger sense (4.5), then the conclusion of Proposition (4.25) can be improved by replacing "δ" with $H' \min\{r^{1+\alpha}, r^{1-\alpha}\}$. In other words, the intrinsic graph of φ is well approximated by Lipschitz flags also at large scales.

Proof of Proposition 4.25. Fix $r > 0$ and $p \in S$ as in the assumptions of the proposition. Since the metric $d_{\mathbb{H}}$ is left invariant, it is sufficient to show that there is $c = c(L) \geqslant 1$ such that

(i) $\left(p^{-1} \cdot S\right) \cap B(0,r) \subset [\Psi_p(\mathbb{W})]_{c\delta}^{\mathbb{H}^1}$,
(ii) $\Psi_p(\mathbb{W}) \cap B(0,r) \subset [p^{-1} \cdot S]_{c\delta}^{\mathbb{H}^1}$.

We consider (i). Let $q \in (p^{-1} \cdot S) \cap B(0,r)$. We will prove that $q \in [\Psi_p(\mathbb{W})]_{c\delta}^{\mathbb{H}^1}$ for a constant c depending only on L. Firstly, since $q \in \Phi^{(p^{-1})}(\mathbb{W}) \cap B(0,r)$, we have that $q = \Phi^{(p^{-1})}(0,y,\tau_p(y)+t)$ for some $(y,t) \in \mathbb{R}^2$ and $\|q\| = \|\Phi^{(p^{-1})}(y,\tau_p(y)+t)\| \leqslant r$. More precisely, by the definition of $\Phi^{(p^{-1})}$, we find

$$
|\varphi^{(p^{-1})}(y,\tau_p(y)+t)| \leqslant r, \quad |y| \leqslant r, \quad \text{and} \quad \left|\tau_p(y) + t - \tfrac{1}{2} y \varphi^{(p^{-1})}(y,\tau_p(y)+t)\right| \leqslant \tfrac{r^2}{4},
$$

$$
|t| \leqslant \left|\tau_p(y) + t - \tfrac{1}{2} y \varphi^{(p^{-1})}(y,\tau_p(y)+t)\right| + |\tau_p(y)| + \left|\tfrac{1}{2} y \varphi^{(p^{-1})}(y,\tau_p(y)+t)\right| \overset{(4.16)}{\leqslant} C_L r^2 + \tfrac{3r^2}{4}. \tag{4.27}
$$

Now applying Lemma 4.23 to the point (y,t), we obtain

$$
d_{\mathbb{H}}(\Psi_p(y,t), q) = d_{\mathbb{H}}(\Psi_p(y,t), \Phi^{(p^{-1})}(y,\tau_p(y)+t)) \leqslant H|t|^{(1+\alpha)/2} \overset{(4.27)}{\lesssim}_L \delta,
$$

so $q \in [\Psi_p(\mathbb{W})]_{c\delta}^{\mathbb{H}^1}$, as desired.

Next we consider (ii). Let $q \in \Psi_p(\mathbb{W}) \cap B(0,r)$. We want to prove that $q \in [p^{-1} \cdot S]^{\mathbb{H}^1}_{c\delta}$ for a constant c that depends only on L. Since $q = \Psi_p(y,t) = \Phi^{(p^{-1})}(y,\tau_p(y)) \cdot (0,0,t)$ for some $(y,t) \in \mathbb{R}^2$ and $\|q\| \leqslant r$, we have that

$$|\varphi^{(p^{-1})}(y,\tau_p(y))| \leqslant r, \quad |y| \leqslant r, \quad \text{and} \quad \left|t + \tau_p(y) - \tfrac{1}{2}y\varphi^{(p^{-1})}(y,\tau_p(y))\right| \leqslant \tfrac{r^2}{4}.$$

and so

$$|t| \leqslant \left|t + \tau_p(y) - \tfrac{y}{2}\varphi^{(p^{-1})}(y,\tau_p(y))\right| + |\tau_p(y)| + \left|\tfrac{y}{2}\varphi^{(p^{-1})}(y,\tau_p(y))\right| \overset{(4.16)}{\leqslant} \tfrac{3r^2}{4} + C_L r^2. \quad (4.28)$$

Now we apply Lemma 4.23 to the point (y,t), hence

$$d_{\mathbb{H}}(q, \Phi^{(p^{-1})}(y,\tau_p(y)+t)) = d_{\mathbb{H}}(\Psi_p(y,t), \Phi^{(p^{-1})}(y,\tau_p(y)+t)) \leqslant H|t|^{(1+\alpha)/2} \overset{(4.28)}{\lesssim_L} \delta,$$

so $q \in [p^{-1} \cdot S]^{\mathbb{H}^1}_{c\delta}$ for c depending only on L, as desired. This completes the proof. □

Let $\pi \colon \mathbb{H}^1 \to \mathbb{R}^2$ be the projection $\pi(x,y,t) = (x,y)$. Then π is 1-Lipschitz $(\mathbb{H}^1, d_{\mathbb{H}}) \to (\mathbb{R}^2, |\cdot|)$, which easily implies the following statement:

Lemma 4.29. *Assume that $A_1, A_2 \subset \mathbb{H}^1$, $p \in \mathbb{H}^1$, $r > 0$, and $\delta > 0$ are such that*

$$A_1 \cap B(p,r) \subset [A_2]^{\mathbb{H}^1}_{\delta},$$

then

$$\pi(A_1 \cap B(p,r)) \subset [\pi(A_2 \cap B(p, r+\delta))]^{\mathbb{R}^2}_{\delta}.$$

The lemma will applied with $A_2 = F$, a Lipschitz flag. Then $\pi(F) \subset \mathbb{R}^2$ is a Lipschitz graph, and the lemma says that $\pi(A_1 \cap B(p,r))$ is contained in the Euclidean δ-neighbourhood of the Lipschitz graph $\pi(F)$ whenever if $A_1 \cap B(p,r) \subset [F]^{\mathbb{H}^1}_{\delta}$.

4.2. **Proof for graphs with extra vertical Hölder regularity.** In this section, we prove Theorem 1.11 from the introduction, which we restate here for the reader's convenience.

Theorem 4.30. *Let $S \subset \mathbb{H}^1$ be the intrinsic graph of a globally defined but compactly supported intrinsic Lipschitz function with extra vertical regularity. Then S has big pieces of bilipschitz images of the parabolic plane (Π, d_Π). In particular, S is LI rectifiable.*

The theorem will be proven as an application of Theorem 1.14 and a reduction to unit scale analogous to the one after Lemma 3.46. We will verify the hypotheses of Theorem 1.14 for $(G, d_G) = (\Pi, d_\Pi)$ and $(M, d_M) = (S, d_{\mathbb{H}})$, with $x_0 = 0 \in G$ and an arbitrary point $p_0 \in S$. Since the map $(y,t) \mapsto (0,y,t)$ is an isometric isomorphism between $(\Pi, +, d_\Pi)$ and $(\mathbb{W}, \cdot, d_{\mathbb{H}})$, it suffices to construct maps $i_{w\to p} : \mathbb{W} \to S$ with the desired properties. As in Section 3.2, the maps $i_{w\to p}$ will be independent of the "scale" parameter $k \in \mathbb{N}$, and can be defined for all $p \in S$ and all $w \in \mathbb{W}$.

Definition 4.31. Let $S = \Phi(\mathbb{W})$ be the intrinsic graph of an intrinsic Lipschitz function $\varphi : \mathbb{W} \to \mathbb{V}$. For each point $p \in S$ and $u \in \mathbb{W}$, let $q := q[p,u] \in S$ be any point satisfying

$$d_{\mathbb{H}}(p \cdot \Psi_p(u), q) = \mathrm{dist}_{\mathbb{H}}(p \cdot \Psi_p(u), S). \quad (4.32)$$

Then, define $i_{w\to p} : \mathbb{W} \to S$ as

$$i_{w\to p}(v) := q[p, w^{-1} \cdot v]. \quad (4.33)$$

Remark 4.34. Remark 4.21 implies that if $S = \Phi(\mathbb{W})$ is itself a Lipschitz flag, then $i_{w\to p}(v) = p \cdot \Psi_p(w^{-1} \cdot v)$ for all $v, w \in \mathbb{W}$. In general, the extra vertical Hölder regularity allows to control the distance between $i_{w\to p}(v)$ and $p \cdot \Psi_p(w^{-1} \cdot v)$.

If φ has extra vertical Hölder regularity with constants $0 < \alpha \leqslant 1$ and $H > 0$, then Lemma 4.23 immediately implies that

$$\begin{aligned} d_{\mathbb{H}}(p \cdot \Psi_p(w^{-1} \cdot w'), i_{w\to p}(w')) &= \mathrm{dist}_{\mathbb{H}}\left(\Psi_p(w^{-1} \cdot w'), p^{-1} \cdot S\right) \\ &= \mathrm{dist}_{\mathbb{H}}\left(\Psi_p\left(w^{-1} \cdot w'\right), \Phi^{(p^{-1})}(\mathbb{W})\right) \\ &\leqslant H d_{\mathbb{H}}(w, w')^{1+\alpha}, \end{aligned} \tag{4.35}$$

for all $p \in S = \Phi(\mathbb{W})$ and $w, w' \in \mathbb{W}$.

Once again, $i_{w\to p}(v)$ does not depend on the points v and w individually, but only on the product $w^{-1} \cdot v$, and by definition $i_{w\to p}(w) = p$. To apply Theorem 1.14, we need to verify the two hypotheses (1.12) and (1.13). We start by showing that (1.12) holds for a constant which depends only on the bilipschitz constant of Ψ_p, which in turn depends only on the intrinsic Lipschitz constant of φ, recall the comment below Definition 4.19. Now (1.12) follows immediately from (4.35) and the triangle inequality:

$$|d_{\mathbb{H}}(i_{w\to p}(w'), i_{w\to p}(w'')) - d_{\mathbb{H}}(\Psi_p(w^{-1} \cdot w'), \Psi_p(w^{-1} \cdot w'))| \lesssim H \max\{d_{\mathbb{H}}(w, w'), d_{\mathbb{H}}(w, w'')\}^{1+\alpha}, \tag{4.36}$$

for all $p \in S$ and $w, w', w'' \in \mathbb{W}$. We proceed to verify condition (1.13) in our situation:

Proposition 4.37. *Let $\varphi : \mathbb{W} \to \mathbb{V}$ be an intrinsic L-Lipschitz function that has extra vertical Hölder regularity with constants $0 < \alpha \leqslant 1$ and $H > 0$. If $w_1, w_2 \in \mathbb{W}$ satisfy $\|w_1\|, \|w_2\| \leqslant 1$, and $p, q \in \Phi(\mathbb{W})$ satisfy $i_{w_1\to p}(w_2) = q$, then*

$$d_{\mathbb{H}}\left(i_{w_1\to p}(w_3), i_{w_2\to q}(w_3)\right) \lesssim_{H,L} \max\left\{d_{\mathbb{H}}(w_1, w_2), d_{\mathbb{H}}(w_1, w_3), d_{\mathbb{H}}(w_2, w_3)\right\}^{1+\frac{\alpha}{2}} \tag{4.38}$$

for all $w_3 \in \mathbb{W}$ with $d_{\mathbb{H}}(w_1, w_3) \leqslant 1$.

Remark 4.39. If φ does not depend on the t-variable, that is, $\Phi(\mathbb{W})$ is itself a Lipschitz flag and the extra Hölder regularity holds with constant $H = 0$, then the left hand side of (4.38) vanishes for all $w_1, w_2, w_3 \in \mathbb{W}$ with $i_{w_1\to p}(w_2) = q$. Indeed, Remark 4.34 implies in this case that

$$i_{w_1\to p}(w_3) = p \cdot \Psi_p(w_1^{-1} \cdot w_3), \quad i_{w_2\to q}(w_3) = q \cdot \Psi_q(w_2^{-1} \cdot w_3),$$

and

$$q = i_{w_1\to p}(w_2) = p \cdot \Psi_p(w_1^{-1} \cdot w_2). \tag{4.40}$$

Hence, the left hand side of (4.38) can be written as

$$d_{\mathbb{H}}(i_{w_1\to p}(w_3), i_{w_2\to q}(w_3)) = d_{\mathbb{H}}(\Psi_p(w_1^{-1} \cdot w_3), \Psi_p(w_1^{-1} \cdot w_2) \cdot \Psi_q(w_2^{-1} \cdot w_3)).$$

We recall from Remark 4.21 that

$$\Psi_q(y,t) = \Phi^{(q^{-1})}\left(y, t + \int_0^y \varphi^{(q^{-1})}(\eta, \tau_q(\eta))\, d\eta\right). \tag{4.41}$$

Let us spell out the formula for $\Phi^{(q^{-1})}$:

$$\Phi^{(q^{-1})} \overset{(4.40)}{=} [\Phi^{(p^{-1})}]^{(\Psi_p(w_1^{-1} \cdot w_2)^{-1})} = \Psi_p(w_1^{-1} \cdot w_2)^{-1} \cdot \Phi^{(p^{-1})}(\pi_{\mathbb{W}}(\Psi_p(w_1^{-1} \cdot w_2) \cdot [\cdot])), \tag{4.42}$$

where we have applied the formula $\Phi(\pi_{\mathbb{W}}(p \cdot v)) = p \cdot \Phi^{(p^{-1})}(v)$ from Lemma 3.17 in the last equality. Using (4.41)-(4.42), and writing $w_i = (0, y_i, t_i)$, it is not difficult to show that

$$\Psi_p(w_1^{-1} \cdot w_2) \cdot \Psi_q(w_2^{-1} \cdot w_3)$$
$$= \Psi_p(w_1^{-1} \cdot w_2) \cdot \Phi^{(q^{-1})}\left(y_3 - y_2, t_3 - t_2 + \int_0^{y_3-y_2} \varphi^{(q^{-1})}(\eta, \tau_q(\eta))\, d\eta\right) \overset{(4.42)}{=} \Psi_p(w_1^{-1} \cdot w_3).$$

We omit some computations, as the remark only serves to motivate Proposition 4.37.

Proof of Proposition 4.37. We first apply the triangle inequality:

$$\begin{aligned} d_{\mathbb{H}}\left(i_{w_1 \to p}(w_3), i_{w_2 \to q}(w_3)\right) \leqslant\ & d_{\mathbb{H}}\left(p \cdot \Psi_p(w_1^{-1} \cdot w_3), i_{w_1 \to p}(w_3)\right) \\ & + d_{\mathbb{H}}(p \cdot \Psi_p(w_1^{-1} \cdot w_3), q \cdot \Psi_q(w_2^{-1} \cdot w_3)) \\ & + d_{\mathbb{H}}\left(q \cdot \Psi_q(w_2^{-1} \cdot w_3), i_{w_2 \to q}(w_3)\right). \end{aligned}$$

The estimate (4.35) shows that the first and third terms are bounded from above by $Hd_{\mathbb{H}}(w_1, w_3)^{1+\alpha}$ and $Hd_{\mathbb{H}}(w_2, w_3)^{1+\alpha}$, respectively, which is better than claimed. So, the heart of the matter is to prove an upper bound for the second term. This is the content of the next lemma. □

Lemma 4.43. *Under the same assumptions as in Proposition 4.37, we have*

$$d_{\mathbb{H}}(p \cdot \Psi_p(w_1^{-1} \cdot w_3), q \cdot \Psi_q(w_2^{-1} \cdot w_3)) \lesssim_{H,L} \max\left\{d_{\mathbb{H}}(w_1, w_2), d_{\mathbb{H}}(w_1, w_3), d_{\mathbb{H}}(w_2, w_3)\right\}^{1+\frac{\alpha}{2}}.$$

Proof. We fix points p and $q = i_{w_1 \to p}(w_2)$ as in the assumptions of Proposition 4.37. We may assume with no loss of generality that $w_1 \neq w_2$, since otherwise $q = p$ and the claimed estimate is clear. By left invariance of $d_{\mathbb{H}}$, we have

$$d_{\mathbb{H}}(p \cdot \Psi_p(w_1^{-1} \cdot w_3), q \cdot \Psi_q(w_2^{-1} \cdot w_3)) = d_{\mathbb{H}}(\Psi_p(w_1^{-1} \cdot w_3), p^{-1} \cdot q \cdot \Psi_q(w_2^{-1} \cdot w_3)). \quad (4.44)$$

Let us denote $p^{-1} \cdot q =: (x, y, t)$. We then define the curves

$$\gamma_p : \mathbb{R} \to \mathbb{R}^2, \quad \gamma_p(s) := (\varphi^{(p^{-1})}(s, \tau_p(s)), s)$$

and

$$\gamma_q : \mathbb{R} \to \mathbb{R}^2, \quad \gamma_q(s) := (\varphi^{(q^{-1})}(s, \tau_q(s)) + x, s + y),$$

whose traces are the π-projections of the corresponding Lipschitz flags:

$$\gamma_p(\mathbb{R}) = \pi\left(\Psi_p(\mathbb{W})\right) \quad \text{and} \quad \gamma_q(\mathbb{R}) = \pi\left(p^{-1} \cdot q \cdot \Psi_q(\mathbb{W})\right)$$

We observe that the curves γ_p and γ_q come close in at least one point. Writing

$$w_1 = (0, y_1, t_1), \quad w_2 = (0, y_2, t_2), \quad w_3 = (0, y_3, t_3), \quad (4.45)$$

and recalling that $q = i_{w_1 \to p}(w_2)$ by the assumption in the lemma, we have

$$\begin{aligned} |\gamma_p(y_2 - y_1) - \gamma_q(0)| = |\gamma_p(y_2 - y_1) - (x, y)| &\leqslant d_{\mathbb{H}}(\Psi_p(w_1^{-1} \cdot w_2), p^{-1} \cdot q) \\ &= d_{\mathbb{H}}(p \cdot \Psi_p(w_1^{-1} \cdot w_2), i_{w_1 \to p}(w_2)) \\ &\overset{(4.35)}{\leqslant} Hd_{\mathbb{H}}(w_1, w_2)^{1+\alpha}. \end{aligned} \quad (4.46)$$

We next show, *a fortiori*, that the curves γ_p and γ_q also stay close to each other for some time. Precisely, we claim that

$$\operatorname{dist}_{\mathbb{R}^2}(\gamma_p(s + y_2 - y_1), \gamma_q(\mathbb{R})) \lesssim_{H,L} \max\left\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\right\}^{1+\alpha}, \quad s \in \mathbb{R}. \quad (4.47)$$

Note that for $s = 0$, the right hand side of (4.47) equals $d_{\mathbb{H}}(w_1, w_2)^{1+\alpha}$, as expected. To prove the claim for arbitrary $s \in \mathbb{R}$, we first observe that

$$\gamma_p(s + y_2 - y_1) = \pi(\Psi_p(s + y_2 - y_1, 0)).$$

Since Ψ_p is Lipschitz according to the remark below Definition 4.19,

$$\|\Psi_p(s + y_2 - y_1, 0)\| \lesssim_L \|(s + y_2 - y_1, 0)\|,$$

and we find

$$\gamma_p(s + y_2 - y_1) \in \pi\left(\Psi_p(\mathbb{W}) \cap B(0, r)\right)$$

for some $0 < r \lesssim_L \max\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}$. It follows from Proposition 4.25 and Lemma 4.29 that

$$\gamma_p(s + y_2 - y_1) \in \left[\pi\left(\left(p^{-1} \cdot \Phi(\mathbb{W})\right) \cap B(0, r + c\delta)\right)\right]^{\mathbb{R}^2}_{c\delta}, \tag{4.48}$$

where the constant c depends only on L, and $\delta := Hr^{1+\alpha}$. Since

$$d_{\mathbb{H}}(p, q) \leqslant d_{\mathbb{H}}(p^{-1} \cdot q, \Psi_p(w_1^{-1} \cdot w_2)) + \|\Psi_p(w_1^{-1} \cdot w_2)\| \overset{(4.35)}{\lesssim_{H,L}} d_{\mathbb{H}}(w_1, w_2)^{1+\alpha} + d_{\mathbb{H}}(w_1, w_2),$$

and $d_{\mathbb{H}}(w_1, w_2) \leqslant 1$, we have

$$B\left(0, r + c\delta\right) \subset B(p^{-1} \cdot q, R) \tag{4.49}$$

for some $R \geqslant r$ with $R \lesssim_{H,L} \max\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}$. By another instance of Proposition 4.25 (applied to the point q), Lemma 4.29, and left translation by p^{-1}, we find that

$$\left[\pi\left(\left(p^{-1} \cdot \Phi(\mathbb{W})\right) \cap B(p^{-1} \cdot q, R)\right)\right]^{\mathbb{R}^2}_{c\delta} \subseteq \left[\pi\left(p^{-1} \cdot q \cdot \Psi_q(\mathbb{W})\right)\right]^{\mathbb{R}^2}_{2cHR^{1+\alpha}} = \left[\gamma_q(\mathbb{R})\right]^{\mathbb{R}^2}_{c_{L,H}R^{1+\alpha}}. \tag{4.50}$$

for a constant $0 < c_{L,H} < \infty$ that depends only on L and H. Then the claim (4.47) follows by combining the inclusions (4.48), (4.49), and (4.50).

We now fix $s \in \mathbb{R}$, and let $s' \in \mathbb{R}$ be any point such that $|\gamma_p(s + y_2 - y_1) - \gamma_q(s')| \lesssim_{H,L} \max\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha}$. The existence of s' is guaranteed by (4.47). We next show that s' cannot be too far from s. Indeed, considering the first component of $\gamma_p(s + y_2 - y_1) - \gamma_q(s')$, we see immediately from (4.47) that

$$|\varphi^{(p^{-1})}(s+y_2-y_1, \tau_p(s+y_2-y_1)) - \varphi^{(q^{-1})}(s', \tau_q(s')) - x| \lesssim_{H,L} \max\{|s+y_2-y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha}. \tag{4.51}$$

Considering the second component, we find the estimate

$$|s + y_2 - y_1 - s' - y| \lesssim_{H,L} \max\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha}. \tag{4.52}$$

By the initial estimate (4.46), we know that

$$|(y_2 - y_1) - y| \leqslant H d_{\mathbb{H}}(w_1, w_2)^{1+\alpha}, \tag{4.53}$$

so (4.52) yields

$$|s - s'| \lesssim_{H,L} \max\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha}. \tag{4.54}$$

This last estimate allows us to deduce a version of (4.51) with "s'" replaced by "s". Indeed, recalling that $s \mapsto \psi_q(s) = \varphi^{(q^{-1})}(s, \tau_q(s))$ is a Lipschitz function $\mathbb{R} \to \mathbb{R}$ whose Lipschitz constant depends only on L (see below Definition 4.13), we find

$$\begin{aligned}
&|\varphi^{(p^{-1})}(s+y_2-y_1,\tau_p(s+y_2-y_1))-\varphi^{(q^{-1})}(s,\tau_q(s))-x| \\
&\quad\leqslant |\varphi^{(q^{-1})}(s,\tau_q(s))-\varphi^{(q^{-1})}(s',\tau_q(s'))| \qquad (4.55)\\
&\quad+|\varphi^{(p^{-1})}(s+y_2-y_1,\tau_p(s+y_2-y_1))-\varphi^{(q^{-1})}(s',\tau_q(s'))-x| \\
&\quad\overset{(4.51)}{\lesssim}_{L,H} |s-s'|+\max\{|s+y_2-y_1|,d_{\mathbb{H}}(w_1,w_2)\}^{1+\alpha} \\
&\quad\overset{(4.54)}{\lesssim}_{L,H} \max\{|s+y_2-y_1|,d_{\mathbb{H}}(w_1,w_2)\}^{1+\alpha}. \qquad (4.56)
\end{aligned}$$

After these preparations, we are ready to deduce the desired upper bound for (4.44) by considering $s:=y_3-y_2$. We will show that

$$d_{\mathbb{H}}(\Psi_p(w_1^{-1}\cdot w_3),p^{-1}\cdot q\cdot\Psi_q(w_2^{-1}\cdot w_3))\lesssim_{L,H}\max\{d_{\mathbb{H}}(w_1,w_2),d_{\mathbb{H}}(w_1,w_3),d_{\mathbb{H}}(w_2,w_3)\}^{1+\frac{\alpha}{2}}. \tag{4.57}$$

It is convenient to estimate the expression on the left hand side as follows

$$\begin{aligned}
&d_{\mathbb{H}}(\Psi_p(w_1^{-1}\cdot w_3),p^{-1}\cdot q\cdot\Psi_q(w_2^{-1}\cdot w_3)) \\
&\quad\leqslant d_{\mathbb{H}}(\Psi_p(w_1^{-1}\cdot w_3),\Psi_p(w_1^{-1}\cdot w_2)\cdot\Psi_q(w_2^{-1}\cdot w_3)) \qquad (4.58)\\
&\quad+d_{\mathbb{H}}(\Psi_q(w_2^{-1}\cdot w_3),\Psi_p(w_1^{-1}\cdot w_2)^{-1}\cdot p^{-1}\cdot q\cdot\Psi_q(w_2^{-1}\cdot w_3)) \qquad (4.59)
\end{aligned}$$

First, the term (4.59) can be bounded using the fundamental commutator relation as in (3.61) with

$$\mathfrak{a}:=[p\cdot\Psi_p(w_1^{-1}\cdot w_2)]^{-1}\cdot q \quad\text{and}\quad \mathfrak{b}:=\Psi_q(w_2^{-1}\cdot w_3).$$

This yields

$$\begin{aligned}
(4.59) &\lesssim \|[p\cdot\Psi_p(w_1^{-1}\cdot w_2)]^{-1}\cdot q\|+\|[p\cdot\Psi_p(w_1^{-1}\cdot w_2)]^{-1}\cdot q\|^{\frac{1}{2}}\|\Psi_q(w_2^{-1}\cdot w_3)\|^{\frac{1}{2}}\\
&\lesssim_{L,H} d_{\mathbb{H}}(w_1,w_2)^{1+\alpha}+d_{\mathbb{H}}(w_1,w_2)^{\frac{1+\alpha}{2}}d_{\mathbb{H}}(w_2,w_3)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality follows from $q=i_{w_1\to p}(w_2)$, the estimate (4.35), and the Lipschitz continuity of Ψ_p. Hence, recalling that $d_{\mathbb{H}}(w_1,w_2)\leqslant 2$, the expression (4.59) can be bounded from above by the right hand side of (4.57).

Next, we handle the term (4.58). Since points on the t-axis commute with all other elements in $\mathbb{H}^1$, it follows from the definition of Ψ_p and Ψ_q that (4.58) is independent of the vertical components of w_1,w_2,w_3. Writing these points in coordinates, as in (4.45), and recalling that $s=y_3-y_2$, we thus find

$$(4.58)=\|\Psi_p(s+y_2-y_1,0)^{-1}\cdot\Psi_p(y_2-y_1,0)\cdot\Psi_q(s,0)\|. \tag{4.60}$$

While Ψ_p and Ψ_q are in general not group homomorphisms, their second components are linear:

$$[\Psi_p]_2(y,t)=[\Psi_q]_2(y,t)=y,\quad (y,t)\in\mathbb{W}.$$

Thus, the second coordinate of the product in (4.60) vanishes by linearity, and it suffices to consider the first and and third coordinate, which we denote by I_1 and I_2, respectively, so that

$$(4.58)\lesssim|I_1|+|I_2|^{\frac{1}{2}}. \tag{4.61}$$

Using that $\varphi^{(q^{-1})}(0,0) = 0$, we may write

$$I_1 = \varphi^{(p^{-1})}(y_2 - y_1, \tau_p(y_2 - y_1)) - \varphi^{(q^{-1})}(0,0) - x$$
$$+ x + \varphi^{(q^{-1})}(s, \tau_q(s)) - \varphi^{(p^{-1})}(s + y_2 - y_1, \tau_p(s + y_2 - y_1)).$$

The term I_1 is the sum of two expressions of the same form as in the estimate (4.56), and we thus deduce that

$$|I_1| \lesssim \max\{|y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha} + \max\{|s + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha}.$$

Clearly, $|y_2 - y_1| \leqslant d_{\mathbb{H}}(w_1, w_2)$ and by the choice of $s = y_3 - y_2$, we have

$$|s + y_2 - y_1| = |y_3 - y_1| \leqslant d_{\mathbb{H}}(w_1, w_3).$$

Thus we see that $|I_1|$ is bounded from above by the right hand side of (4.57), using again that $d_{\mathbb{H}}(w_1, w_3), d_{\mathbb{H}}(w_1, w_2) \leqslant 2$. It remains to bound $|I_2|$, where I_2 denotes the third component of the product in (4.60). A direct computation yields

$$I_2 = -\tau_p(s + y_2 - y_1) + \tau_p(y_2 - y_1) + \tau_q(s) + s\varphi^{(p^{-1})}(y_2 - y_1, \tau_p(y_2 - y_1)),$$

and we continue as follows:

$$\begin{aligned}
|I_2| &= |\tau_p(s + y_2 - y_1) - \tau_p(y_2 - y_1) - \tau_q(s) - s\varphi^{(p^{-1})}(y_2 - y_1, \tau_p(y_2 - y_1))| \\
&= \left| \int_{y_2 - y_1}^{s + y_2 - y_1} \dot{\tau}_p(\sigma)\, d\sigma - \int_0^s \dot{\tau}_q(\sigma) + \varphi^{(p^{-1})}(y_2 - y_1, \tau_p(y_2 - y_1))\, d\sigma \right| \\
&= \left| \int_0^s \dot{\tau}_p(\sigma + y_2 - y_1) - \dot{\tau}_q(\sigma) - \varphi^{(p^{-1})}(y_2 - y_1, \tau_p(y_2 - y_1))\, d\sigma \right| \\
&= \left| \int_0^s \left[\varphi^{(p^{-1})}(\sigma + y_2 - y_1, \tau_p(\sigma + y_2 - y_1)) - \varphi^{(q^{-1})}(\sigma, \tau_q(\sigma)) - x \right] \right. \\
&\qquad \left. + \left[x + \varphi^{(q^{-1})}(0, \tau_q(0)) - \varphi^{(p^{-1})}(y_2 - y_1, \tau_p(y_2 - y_1)) \right] d\sigma \right| \\
&\overset{(4.56)}{\lesssim}_{L,H} \int_{J_s} \max\{|\sigma + y_2 - y_1|, d_{\mathbb{H}}(w_1, w_2)\}^{1+\alpha}\, d\sigma \\
&\lesssim_{H,L} |s| \max\{d_{\mathbb{H}}(w_1, w_2), d_{\mathbb{H}}(w_1, w_3)\}^{1+\alpha} \\
&\lesssim_{H,L} \max\{d_{\mathbb{H}}(w_1, w_2), d_{\mathbb{H}}(w_1, w_3), d_{\mathbb{H}}(w_2, w_3)\}^{2+\alpha},
\end{aligned}$$

where $J_s := [s, 0]$ if $s \leqslant 0$ and $J_s := [0, s]$ if $s \geqslant 0$. To justify the application of (4.56) above, we have applied inside the integral an analogous argument as we did to bound the term I_1.

Finally inserting the bounds for $|I_1|$ and $|I_2|$ in (4.61), we conclude that (4.58) is bounded from above by the right hand side of (4.57). Combined with the bound for (4.59), this concludes the proof of the lemma. □

Proof of Theorem 4.30. The BPGBI condition "at unit scale" follows from Theorem 1.14, whose hypotheses (1.12) and (1.13) we have verified in (4.36) and Proposition 4.37, respectively. Here

$$(G, d_G, \mu) = (\mathbb{R}^2, d_\Pi, \mathcal{L}^2), \quad \text{and} \quad (M, d_M) = (S, d_{\mathbb{H}}),$$

with $x_0 = 0 \in G$, and $p_0 \in S$ arbitrary, and we recall that (G, d_G) is isometric to $(\mathbb{W}, d_{\mathbb{H}})$. More precisely, Theorem 1.14 yields the existence of $2L$-bilipschitz maps $f\colon K \to S \cap$

$B(p,1)$, $p \in S$, where $K \subset G$ with $\mathcal{H}^3(K) \geqslant \delta > 0$. The constant L only depends on the intrinsic Lipschitz constant of φ, and $\delta > 0$ depends in addition on α and the constant H in (4.2). Since property (4.2) improves under "zooming in", see Remark 4.3, we can argue analogously as in Section 3.1.1. Let $p \in S$ and, first, $0 < r \leqslant C$, where $C := 2\,\mathrm{diam}_{\mathbb{H}}(\Phi(\mathrm{spt}\,\varphi))$. Using Remark 4.3 and the support assumption on φ, we see that $S_{1/r} := \delta_{1/r}(S)$ is an intrinsic graph of an intrinsic Lipschitz function (with the same constant) satisfying (4.2) with constants α and $H' = H'(H,C)$.

Therefore, by the BPGBI property at scale $r = 1$, every ball $S_{1/r} \cap B(p,1)$ contains the image of a $2L$-bilipschitz map g from a compact set $K \subset G$ with $\mathcal{H}^3(K) \geqslant \delta = \delta(C) > 0$. Now, one may simply pre- and post-compose g with the natural dilations in G and $\mathbb{H}^1$ to produce a $2L$-bilipschitz map $g_r \colon \delta_r(K) \to S \cap B(\delta_r(p), r)$ (note also that $\mathcal{H}^3(\delta_r(K)) = r^3\mathcal{H}^3(K) \geqslant \delta r^3$).

Next, consider the case $r > C$. Then, if $p \in S$ is arbitrary, the set $S \cap B(p,r)$ satisfies

$$\mathcal{H}^3([S \cap B(p,r)] \cap \mathbb{W}) \gtrsim \mathcal{H}^3(S \cap B(p,r)).$$

Thus, the restriction of Id to $[S \cap B(p,r)] \cap \mathbb{W}$ yields the desired bilipschitz map. The proof of Theorem 4.30 is thus complete. □

4.3. **Application to C^1 and intrinsic $C^{1,\alpha}$ surfaces.** As a first application of Theorem 4.30, we deduce the case $n = 1$ of Theorem 3.27, recalling from Example 4.7 that a compactly supported $C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ function is intrinsic Lipschitz and satisfies the extra vertical Hölder regularity condition.

Theorem 4.62. *Let $S = \Phi(\mathbb{W}) \subset \mathbb{H}^1$, where $\varphi \in C^{1,\alpha}_{\mathbb{H}}(\mathbb{W})$ is compactly supported. Then S has big pieces of bilipschitz images of the parabolic plane (Π, d_Π).*

Remark 4.63. As a corollary of Theorem 4.30, we also obtain that every Euclidean C^1 surface in $\mathbb{H}^1$ is rectifiable by bilipschitz images of subsets of the parabolic plane. As written in the introduction, this was known before by the work of Cole-Pauls and Bigolin-Vittone, cf. Theorem 1.4, but we briefly explain how to deduce it from Theorem 4.30. The reduction uses again the result by Balogh [5], which says that the set $\Sigma(S)$ of *characteristic points* of a Euclidean C^1 surface in $\mathbb{H}^1$ has vanishing 3-dimensional Hausdorff measure with respect to $d_{\mathbb{H}}$.

We will argue that outside $\Sigma(s)$, the surface S can be written locally as intrinsic graph of a compactly supported Euclidean C^1 function, and hence as intrinsic Lipschitz graph with extra vertical Hölder regularity. This will show that S is rectifiable by bilipschitz images of subsets of the parabolic plane.

We now turn to the details. Let $p \in S \backslash \Sigma(S)$. For $r > 0$ small enough, $S \cap B(p,r)$ is contained in the level set $\{f = 0\}$ of a Euclidean C^1 function $f : \mathbb{R}^3 \to \mathbb{R}$ with non-vanishing gradient in $B(p,r)$. Without loss of generality, we may assume that $f(0) = 0$, $Xf(0) > 0$ and

$$S \cap B(p,r) = \{q \in B(p,r) : \; f(q) = 0\}$$

for $r > 0$ with the property that $Xf(q) > 0$ for all $q \in B(p,r)$. Since $Xf(0) = \partial_x f(0)$, we may further assume, by making r smaller if necessary, that $\partial_x f(q) > 0$ for all $q \in B(p,r)$. In order to write $S \cap B(p,r)$, for small enough r, as intrinsic graph of a Euclidean C^1 function, we first consider the diffeomorphism

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \quad F(x,y,t) = \left(x, y, t + \tfrac{xy}{2}\right).$$

Then $F(S \cap B(p,r))$ is contained in the level set of $f \circ F^{-1}$, and hence it is again a Euclidean C^1 surface. Since the derivative of F at the origin is the identity, and $\partial_x f(q) > 0$ for all $q \in B(p,r)$, we can apply the usual implicit function theorem to deduce that, if $r > 0$ is small enough, there is an open set $U \subset \mathbb{R}^2$, and a Euclidean C^1 function $\psi : U \to \mathbb{R}$ such that $F(S \cap B(p,r))$ is the Euclidean graph of ψ over the set U in the yt-plane:

$$F(S \cap B(p,r)) = \{(\psi(y,t), y, t) : \ (y,t) \in U\}.$$

It is easy to see that the preimage of this set under F is then given by the **intrinsic** graph of ψ,

$$S \cap B(p,r) = \left\{(\psi(y,t), y, t - \tfrac{1}{2} y\psi(y,t)) : \ (y,t) \in U\right\}.$$

We will next modify ψ to obtain a Euclidean C^1 function φ that is defined on the entire plane, but compactly supported. To this end, let $B', B \subset U$ be concentric balls, relatively open in the yt-plane $\mathbb{W}$ (identified with $\mathbb{R}^2$), such that

$$[B(p,r') \cap S] \subseteq \{w \cdot \psi(w) : \ w \in B'\} \subseteq \{w \cdot \psi(w) : \ w \in B\} \subseteq [B(p,r) \cap S]$$

for some $0 < r' < r$. We define

$$\varphi(w) := \begin{cases} \psi(w), & \text{if } w \in B', \\ \xi(w), & \text{if } w \in B \backslash B', \\ 0, & \text{otherwise,} \end{cases}$$

with a suitable C^1 function ξ in order that φ is also C^1. More precisely, φ is a compactly supported C^1 function defined in $\mathbb{W}$ such that $S \cap B(p,r') = \Phi(\mathbb{W}) \cap B(p,r')$. By Example 4.6, we know that φ is also an intrinsic Lipschitz function with extra vertical regularity. Finally, it follows from Theorem 4.30 that $\Phi(\mathbb{W})$ is rectifiable by bilipschitz images, and hence the same holds for $S \cap B(p,r')$. Repeating the argument for every noncharacteristic point in S proves that S is rectifiable by bilipschitz images of subsets of the parabolic plane, and in particular LI rectifiable.

Appendix A. Fat Cantor sets in metric measure spaces

Here is again the statement of Proposition 2.2:

Proposition A.1. *Every doubling and complete metric measure space (X,d,μ) of diameter $\geqslant 1$ admits fat Cantor sets. In other words, for every $\epsilon > 0$ and $n_0 \geqslant 0$, the constants $\delta(n_0) > 0$ and $\tau(\epsilon) > 0$ can be found as in Definition 2.1. They are also allowed to depend on the doubling constant of (X,d,μ).*

Proof. Let $\mathcal{Q} = \cup\{\mathcal{Q}_n : z \in \mathbb{Z}\}$ be a family of **closed** (and hence compact) Christ cubes on (X,d,μ), see [11, Theorem 11]. Thus, the cubes here are closures of the cubes defined in [11, Theorem 11]. By changing the indexing of the families $\mathcal{Q}_n$ slightly, one may assume that $2^{-n} \lesssim_X \operatorname{diam}_X(Q) < 2^{-n}$ for all $Q \in \mathcal{Q}_n$. According to [11, (3.6)], the cubes in $\mathcal{Q}$ can be chosen so that they have *small boundary regions* in the following sense: there are constants $C \geqslant 1$ and $\eta > 0$ such that $\mu(\partial_\rho Q) \leqslant C\rho^\eta \mu(Q)$ for all $Q \in \mathcal{Q}$, where

$$\partial_\rho Q := \{x \in Q : \operatorname{dist}(x, Q^c) < \rho 2^{-n}\}, \qquad Q \in \mathcal{Q}_n. \tag{A.2}$$

Fix $x \in X$. To begin the construction of a fat Cantor set inside $B(x,1)$, fix also the parameters $\epsilon > 0$ and $n_0 \in \mathbb{N}$, and let Q_0 be a cube in $\mathcal{Q}_0$ containing x (there may be several options, since the cubes in $\mathcal{Q}_0$ are closures of "dyadic" cubes, but any choice will do). Since $\operatorname{diam}_X(Q_0) < 2^{-n_0} \leqslant 1$, one has $Q_0 \subset B(x,1)$. Set $\mathcal{D}_{n_0} := \{Q_0\}$. Now, if one simply declared that $\mathcal{D}_n := \{Q \in \mathcal{Q}_n : Q \subset Q_0\}$, then one would already have the conditions (i)-(iii) listed at the beginning of Section 2. Then, the Cantor set K defined by

$$K := \bigcap_{n \geqslant n_0} \bigcup_{Q \in \mathcal{D}_n} Q$$

would satisfy $\mu(K) = \mu(Q_0) \sim_{n_0} \mu(B(x,1))$ by the doubling hypothesis.

To secure, in addition, the separation condition (iv), one need to remove some boundary regions, and apply (A.2). Namely, fix a constant $\tau = \tau(\epsilon) > 0$, to be determined a little later, and define

$$Q_0' := Q_0 \setminus \partial_{\tau 2^{-n_0 \epsilon}} Q_0 \quad \text{and} \quad \mathcal{D}_{n_0}' := \{Q_0'\}.$$

Then, assume that $\mathcal{D}_n'$ has already been defined for some $n \geqslant n_0$. Assume also that the sets in $\mathcal{D}_n'$ are obtained as compact subsets of sets in $\mathcal{D}_n$: for every $Q \in \mathcal{D}_n$, there corresponds a compact set $Q' \in \mathcal{D}_n'$ with $Q' \subset Q$ (but it may, and will, sometimes happen that $Q' = \emptyset$). To define $\mathcal{D}_{n+1}'$, fix $Q \in \mathcal{D}_{n+1}$, and let $\widehat{Q}' \in \mathcal{D}_n'$ be the compact set contained in the $\mathcal{D}_n$-parent $\widehat{Q} \supset Q$. Define

$$Q' := [Q \setminus \partial_{\tau 2^{-n\epsilon}} Q] \cap \widehat{Q}'.$$

Then evidently $Q' \subset \widehat{Q}'$, and the conditions (i)-(iii) from the beginning of Section 2 remain valid for the modified collections $\mathcal{D}_n'$, $n \geqslant n_0$. But now also condition (iv) is valid. Indeed, if $Q_1', Q_2' \in \mathcal{D}_n$ are distinct, and there still existed a point $x \in Q_1' \subset Q_1$ with $\operatorname{dist}(x, Q_2') < \tau 2^{-(1+\epsilon)n}$, then clearly $x \in \partial_{\tau 2^{-n\epsilon}} Q_1$, and hence in fact $x \notin Q_1'$.

So, the only remaining concern is the μ-measure of the new Cantor set

$$K' := \bigcap_{n \geqslant n_0} \bigcup_{Q' \in \mathcal{D}_n'} Q'.$$

Evidently, if $x \in Q_0 \setminus K'$, then $x \in \partial_{\tau 2^{-n\epsilon}} Q$ for some $Q \in \mathcal{D}_n$ with $Q \subset Q_0$ (hence $n \geqslant n_0$). Recalling the estimate for the μ-measure of boundary regions above (A.2), one infers that

$$\begin{aligned}
\mu(Q_0 \setminus K') &\leqslant \sum_{n \geqslant n_0} \sum_{\substack{Q \in \mathcal{D}_n \\ Q \subset Q_0}} \mu(\partial_{\tau 2^{-\epsilon n}} Q) \\
&\leqslant C \sum_{n \geqslant n_0} (\tau 2^{-n\epsilon})^{\eta} \sum_{\substack{Q \in \mathcal{D}_n \\ Q \subset Q_0}} \mu(Q) \\
&= C\tau^{\eta} \sum_{n \geqslant 0} 2^{-n\epsilon\eta} \mu(Q_0) \lesssim_{\epsilon,\eta} C\tau^{\eta}\mu(Q_0).
\end{aligned}$$

Therefore, choosing $\tau = \tau(C, \epsilon, \eta) > 0$ sufficiently small, one has $\mu(Q_0 \setminus K') \leqslant \mu(Q_0)/2$, hence $\mu(K') \gtrsim \mu(Q_0) \gtrsim_{n_0} \mu(B(x,1))$. The proof is complete. ☐

Appendix B. Containing pieces of $C^{1,\alpha}_{\mathbb{H}}$-surfaces on intrinsic graphs

This appendix contains the proof of the following proposition which was needed in the proof of Theorem 1.7 (or Theorem 3.28).

Proposition B.1. *Let $S \subset \mathbb{H}^n$ be a $C^{1,\alpha}_{\mathbb{H}}$-surface, $0 < \alpha \leqslant 1$. Then, for every $p_0 \in S$, there exists $r_0 > 0$, a vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$ with complementary horizontal subgroup $\mathbb{V}$, and a* *compactly supported intrinsic $C^{1,\alpha/3}$-function $\varphi\colon \mathbb{W} \to \mathbb{V}$ such that $S \cap B(p_0, r_0)$ is contained on the intrinsic graph of φ.*

Proof. Fix $p_0 \in S$, and let $r_0 > 0$ and $B := B(p_0, r_0)$ first be so small that $S \cap B$ can be written as

$$S \cap \bar{B} = \{p \in B : f(p) = 0\}$$

for some $f \in C^1_{\mathbb{H}}(B(p_0, 10r_0))$ satisfying $\nabla_{\mathbb{H}} f(p_0) \neq 0$, and

$$|\nabla_{\mathbb{H}} f(p_1) - \nabla_{\mathbb{H}} f(p_2)| \leqslant H d_{\mathbb{H}}(p_1, p_2)^{\alpha}, \qquad p_1, p_2 \in B(p_0, 10r_0). \tag{B.2}$$

It follows from (B.2) that $f, \nabla_{\mathbb{H}} f \in L^{\infty}(B(p_0, 10r_0))$. By making r_0 smaller, one may further improve (B.2) to

$$|\nabla_{\mathbb{H}} f(p_1) - \nabla_{\mathbb{H}} f(p_2)| \leqslant \min\{H d_{\mathbb{H}}(p_1, p_2)^{\alpha}, \epsilon\}, \qquad p_1, p_2 \in B(p_0, 10r_0), \tag{B.3}$$

where $\epsilon > 0$ is a small absolute constant to be chosen later. For notational convenience, we will also assume that $\nabla_{\mathbb{H}} f(p_0) = (X_1 f(p_0), \ldots, X_{2n} f(p_0)) = (1, 0, \ldots, 0)$, but any other non-zero constant vector would work equally well: it is only crucial to choose $\mathbb{W}$ (as in the statement of the proposition) so that $\nabla_{\mathbb{H}} f(p_0)$ is the horizontal normal of $\mathbb{W}$. Under the present assumption, set $\mathbb{W} := \{(0, y, t) : y \in \mathbb{R}^{2n-1}, t \in \mathbb{R}\}$ and $\mathbb{V} := \{(x_1, 0, 0) : x_1 \in \mathbb{R}\}$.

Let $C > 20$ be another constant to be determined later, which may depend on the *data*

$$\|f\|_{L^{\infty}(B(p_0, r_0))},\ \|\nabla_{\mathbb{H}} f\|_{L^{\infty}(B)},\ \|p_0\|,\ r_0,\ H \quad \text{and} \quad \epsilon. \tag{B.4}$$

Then, initially extend f by setting $f(x_1, y, t) := x_1$ for $(x_1, y, t) \in \mathbb{H}^n \setminus B(p_0, Cr_0)$. The main task is now to extend f to a function $f_1 \in C^{1,\alpha/3}_{\mathbb{H}}(\mathbb{H}^n)$ in such a manner that $X_1 f_1 \geqslant \frac{1}{2}$; then $\{f_1 = 0\}$ will be an intrinsic $C^{1,\alpha/3}$-graph containing $S \cap B$. The extension of f to f_1 can be accomplished, up to a few additional details, by using the standard proof of the Whitney extension theorem [18, Theorem 6.8] in $\mathbb{H}^n$. An underlying observation is that $\nabla_{\mathbb{H}} f \approx (1, 0, \ldots, 0)$ on $B \cup [\mathbb{H}^n \setminus B(p_0, Cr_0)]$, and if C is chosen large enough, depending on the data in (B.4), the extension f_1 can be arranged to have the same property.

Define

$$k(p) := \nabla_{\mathbb{H}} f(p), \qquad p \in \bar{B},$$

recalling that f was initially defined on $B(p_0, 10r_0)$. Also define $k(p) := (1, 0, \ldots, 0) \equiv \nabla_{\mathbb{H}} f(p)$ for $p \in \mathbb{H}^n \setminus B(p_0, Cr_0)$, so both k and f are now defined on the closed set

$$F := \bar{B} \cup [\mathbb{H}^n \setminus B(p_0, Cr_0)].$$

Recalling (B.3), and that $\nabla_{\mathbb{H}} f(p) = (1, 0, \ldots, 0)$, we note that

$$|k(p_1) - k(p_2)| \leqslant \min\{H d_{\mathbb{H}}(p_1, p_2)^{\alpha}, \epsilon\}, \qquad p_1, p_2 \in F. \tag{B.5}$$

Towards applying the Whitney extension theorem [18, Theorem 6.8], consider the following quantity $R(q, p)$ appearing in its statement:

$$R(q, p) := \frac{f(q) - f(p) - \langle k(p), \pi(p^{-1} \cdot q) \rangle}{d_{\mathbb{H}}(p, q)}, \qquad p, q \in F.$$

Here π is the projection $\pi(x_1, \ldots, x_{2n}, t) = (x_1, \ldots, x_{2n})$, and $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^{2n}$. Recall that π is a Lipschitz map $\mathbb{H}^n \to \mathbb{R}^{2n}$, and also a group homomorphism, that is, $\pi(p \cdot q) = \pi(p) + \pi(q)$ for $p, q \in \mathbb{H}^n$. The following estimate for $|R(p, q)|$ will be needed, and next verified:

$$|R(q, p)| \lesssim \min\{H d_{\mathbb{H}}(p, q)^{\alpha}, \epsilon\}, \qquad p, q \in F \cap B(p_0, 2Cr_0). \tag{B.6}$$

For $p, q \in \bar{B}$, the estimate follows immediately from (B.3) and [1, Lemma 4.2], which further cites [23, Theorem 2.3.3]. The case $p, q \in \mathbb{H}^n \setminus B(p_0, Cr_0)$ is clear, as $R(p,q) = 0$ (recalling that $k(p) = (1, 0, \ldots, 0)$ and $f(x_1, \ldots, x_{2n}, t) = x_1$). Finally, consider points $q \in B(p_0, 2Cr_0) \setminus B(p_0, Cr_0)$ and $p \in \bar{B}$ (the case where the roles of p and q are reversed is similar, and even slightly easier). Then $|k(p) - (1, 0, \ldots, 0)| \leqslant \epsilon$, $f(q) = (1, 0, \ldots, 0) \cdot \pi(q)$, and $d_{\mathbb{H}}(p,q) \gtrsim Cr_0$. Consequently,

$$
\begin{aligned}
|R(q,p)| &= \left| \frac{f(q) - f(p) - \langle k(p), \pi(p^{-1} \cdot q) \rangle}{d_{\mathbb{H}}(p,q)} \right| \\
&\lesssim \frac{|\langle ((1, 0, \ldots, 0) - k(p)), \pi(q) \rangle|}{Cr_0} + \frac{|f(p) - \langle k(p), \pi(p) \rangle|}{Cr_0} \\
&\lesssim \epsilon \leqslant \min\{d_{\mathbb{H}}(p,q)^{\alpha}, \epsilon\},
\end{aligned}
$$

noting that $|\pi(q)| \lesssim \|p_0\| + Cr_0$, and choosing $C \geqslant 1$ eventually so large that $(\|p_0\| + Cr_0)/(Cr_0) \leqslant 2$ and $(Cr_0)^{\alpha} \geqslant \epsilon$, and

$$
|f(p) - \langle k(p), \pi(p) \rangle| \lesssim \|f\|_{L^\infty(\bar{B})} + \|\nabla_{\mathbb{H}} f\|_{L^\infty(\bar{B})}(\|p_0\| + r_0) \leqslant \epsilon C r_0.
$$

This completes the proof of (B.6).

Next, we claim that f can be extended to a function $f_1 \in C^{1,\alpha/3}_{\mathbb{H}}(\mathbb{H}^n)$ with the additional property that

$$
|\nabla_{\mathbb{H}} f_1(p) - (1, 0, \ldots, 0)| \leqslant \tfrac{1}{2}, \qquad p \in \mathbb{H}^n. \tag{B.7}
$$

The proof follows the usual argument for the Whitney extension theorem, see [18, Theorem 6.8] or [15, §6.5], and one just needs to check that the resulting extension is in $C^{1,\alpha/3}_{\mathbb{H}}(\mathbb{H}^n)$, and that (B.7) is satisfied. We start by setting up some notation. For any $p \in \mathbb{H}^n$, let

$$
r(p) := \mathrm{dist}_{\mathbb{H}}(p, F)/20.
$$

Since $U := F^c = B(p_0, Cr_0) \setminus \bar{B}$ is bounded in our scenario, the numbers $r(p)$ above are uniformly bounded to begin with (in [18] and [15], one needs to take instead $r(p) = \min\{1, \mathrm{dist}_{\mathbb{H}}(p, F)\}/20$ to fix this). Thus, by the $5r$ covering theorem, there exists a countable set $S \subset \mathbb{H}^n \setminus F$ such that

$$
U = \bigcup_{s \in S} B(s, 5r(s)),
$$

and the balls $B(s, r(s))$, $s \in S$, are disjoint. One may then proceed to define the (smooth) partition of unity $\{\nu_s\}_{s \in S}$ of U, subordinate to the cover $\{B(s, 10r(s))\}_{s \in S}$, as in the proof of either [15, §6.5] or [18, Theorem 6.8]. The key properties are that

$$
\sum_{s \in S} \nu_s = \mathbf{1}_U \quad \text{and} \quad \sum_{s \in S} \nabla_{\mathbb{H}} \nu_s(p) \equiv 0, \tag{B.8}
$$

and

$$
|\nabla^j_{\mathbb{H}} \nu_s(p)| \lesssim \frac{1}{r(p)^j}, \qquad p \in U,\ s \in S,\ j \in \{1, 2\}. \tag{B.9}
$$

Here $\nabla^2_{\mathbb{H}}$ simply refers to any second order horizontal derivative. Moreover, the supports of the functions ν_s have bounded overlap, that is, for $p \in U$ fixed, there are only $\lesssim 1$ indices $s \in S$ with $v_s(p) \neq 0$ or $\nabla_{\mathbb{H}} v_s(p) \neq 0$. To be precise, in the proof of [18, Theorem 6.8], condition (B.9) is only stated for first-order horizontal derivatives, that is, for $j = 1$. However, the bound for the second order derivatives easily follows from formula [18, (57)], observing that the functions defined there are all obtained from a fixed smooth function by rescaling with a factor proportional to $1/r(p)$ and using properties of $q \mapsto d_{\mathbb{H}}(p,q)$.

Now, the extension f_1 is defined as follows:

$$f_1(p) := \begin{cases} f(p), & \text{if } p \in F, \\ \sum_{s \in S} \nu_s(p)[f(\hat{s}) + \langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p)\rangle], & \text{if } p \in U. \end{cases}$$

Here, for $p \in U$ given, $\hat{p} \in F$ is any point satisfying $\mathrm{dist}_{\mathbb{H}}(p,F) = d_{\mathbb{H}}(p,\hat{p})$. Since the definition is precisely the same as the one in [18, Theorem 6.8], the function f_1 is readily a $C^1_{\mathbb{H}}(\mathbb{H}^n)$-extension of f, and moreover

$$\nabla_{\mathbb{H}} f_1(q) = k(q), \qquad q \in F. \tag{B.10}$$

To prove, further, that $f_1 \in C^{1,\alpha/3}_{\mathbb{H}}(\mathbb{H}^n)$, and that (B.7) holds, one needs to look closer at the differences $|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)|$. The following estimates are copied from [15, p. 250] (and completely omitted in [18], as there is virtually no difference between $\mathbb{H}^n$ and $\mathbb{R}^n$ in this argument). First, the horizontal gradient of f_1 on U is evidently

$$\nabla_{\mathbb{H}} f_1(p) = \sum_{s \in S} \{[f(\hat{s}) + \langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p)\rangle] \nabla_{\mathbb{H}} \nu_s(p) + \nu_s(p) k(\hat{s})\}, \qquad p \in U. \tag{B.11}$$

By (B.10), $\nabla_{\mathbb{H}} f_1$ and $\nabla_{\mathbb{H}} f$ coincide on F, hence satisfy the same estimates, and in particular (B.5). To understand the behaviour of $\nabla_{\mathbb{H}} f_1$ outside F, consider first the case $p \in U$ and $q \in F$. First,

$$|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)| \leqslant |\nabla_{\mathbb{H}} f_1(p) - k(\hat{p})| + |k(\hat{p}) - k(q)|. \tag{B.12}$$

Since $d_{\mathbb{H}}(\hat{p},q) \leqslant d_{\mathbb{H}}(\hat{p},p) + d_{\mathbb{H}}(p,q) \leqslant 2d_{\mathbb{H}}(p,q)$, the second term in (B.12) can be estimated by

$$|k(\hat{p}) - k(q)| \overset{\text{(B.5)}}{\lesssim} \min\{H d_{\mathbb{H}}(p,q)^{\alpha}, \epsilon\}. \tag{B.13}$$

The first term in (B.12) is estimated as follows, recalling (B.8) and (B.11):

$$\begin{aligned} |\nabla_{\mathbb{H}} f_1(p) - k(\hat{p})| &\overset{\text{(B.8)\&(B.11)}}{=} \left| \sum_{s \in S} [f(\hat{s}) + \langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p)\rangle] \nabla_{\mathbb{H}} \nu_s(p) + \nu_s(p)[k(\hat{s}) - k(\hat{p})] \right| \\ &\overset{\text{(B.8)}}{\leqslant} \left| \sum_{s \in S} [f(\hat{s}) - f(\hat{p}) + \langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot \hat{p})\rangle] \nabla_{\mathbb{H}} \nu_s(p) \right| \\ &\quad + \left| \sum_{s \in S} [\langle (k(\hat{s}) - k(\hat{p})), \pi(\hat{p}^{-1} \cdot p)\rangle] \nabla_{\mathbb{H}} \nu_s(p) \right| \\ &\quad + \left| \sum_{s \in S} \nu_s(p)[k(\hat{s}) - k(\hat{p})] \right| =: \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \tag{B.14}$$

To arrive at the expression for Σ_2, we added here the term

$$\sum_{s \in S} \langle k(\hat{p})), \pi(\hat{p}^{-1} \cdot p)\rangle \nabla_{\mathbb{H}} \nu_s(p) = \langle k(\hat{p})), \pi(\hat{p}^{-1} \cdot p)\rangle \sum_{s \in S} \nabla_{\mathbb{H}} \nu_s(p) \overset{\text{(B.8)}}{=} 0.$$

With the expressions for Σ_1, Σ_2, and Σ_3 in hand, we can now continue to bound the right hand side of (B.12). First, the term Σ_1 essentially contains $R(\hat{p},\hat{s})$, and can be bounded using (B.6) and (B.9), and the bounded overlap of the supports of the functions ν_s (in applying (B.6), note that easily $\hat{p}, \hat{s} \in F \cap B(p_0, 2Cr_0)$, as $\hat{p}, \hat{s}$ are among the points in F closest to $p, s \in B(p_0, Cr_0)$):

$$\Sigma_1 \lesssim \frac{d_{\mathbb{H}}(\hat{s}, \hat{p})}{r(p)} \cdot \min\{H d_{\mathbb{H}}(\hat{s}, \hat{p})^\alpha, \epsilon\}.$$

Repeating verbatim the estimate on [15, p. 251], we moreover find that $d_{\mathbb{H}}(\hat{s}, \hat{p}) \lesssim d_{\mathbb{H}}(p, \hat{p}) = \mathrm{dist}_{\mathbb{H}}(p, F) = r(p)$ for all $s \in S$ relevant in the summation above, that is, for those $s \in S$ where $\nu_s(p) \neq 0$ or $\nabla_{\mathbb{H}} \nu_s(p) \neq 0$. Consequently,

$$\Sigma_1 \lesssim \min\{H d_{\mathbb{H}}(p, \hat{p})^\alpha, \epsilon\} \leqslant \min\{H d_{\mathbb{H}}(p, q)^\alpha, \epsilon\}.$$

Next, to estimate Σ_2, one uses the same ingredients as above, except that the appeal to (B.6) is replaced by (B.5):

$$\Sigma_2 \lesssim \frac{d_{\mathbb{H}}(\hat{p}, p)}{r(p)} \cdot \min\{H d_{\mathbb{H}}(\hat{s}, \hat{p})^\alpha, \epsilon\} \lesssim \min\{H d_{\mathbb{H}}(p, q)^\alpha, \epsilon\}.$$

Virtually the same argument gives the same upper bound for Σ_3. Starting from (B.12), and recalling (B.13), one finally infers that

$$|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)| \lesssim \min\{H d_{\mathbb{H}}(p, q)^\alpha, \epsilon\}, \qquad p \in U, \ q \in F. \tag{B.15}$$

By symmetry, (B.15) also holds if $p \in F$ and $q \in U$. Combining this with (B.5), one concludes that (B.15) holds for all pairs $p, q \in \mathbb{H}^n$ with (a) both $p, q \in F$, or (b) one point in F and the other one in U. How about the the case (c) $p, q \in U$? The estimate

$$|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)| \lesssim \epsilon \tag{B.16}$$

follows by recalling that $|k(\hat{p}) - (1, 0, \ldots, 0)| \leqslant \epsilon$ and $|k(\hat{q}) - (1, 0, \ldots, 0)| \leqslant \epsilon$ by (B.5), then repeating the estimate from (B.14) and using the triangle inequality. So, it remains to show that $|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)| \lesssim d_{\mathbb{H}}(p, q)^{\alpha/3}$. The implicit constants here may depend on all the data in (B.4). One may assume that $d_{\mathbb{H}}(p, q) \leqslant 1$, since otherwise this is implied by (B.16). Consider first the case where

$$d_{\mathbb{H}}(p, q) \leqslant r(p)^3. \tag{B.17}$$

Recall, once again, the formulae for $\nabla_{\mathbb{H}} f_1(p), \nabla_{\mathbb{H}} f_1(q)$ from (B.11); the plan is to make crude term-by-term estimates. Note that if $s \in S$ is fixed, then

$$|\nu_s(p) k(\hat{s}) - \nu_s(q) k(\hat{s})| \lesssim \|\nabla_{\mathbb{H}} \nu_s\|_{L^\infty} d_{\mathbb{H}}(p, q) \lesssim \frac{d_{\mathbb{H}}(p, q)}{r(p)} \leqslant d_{\mathbb{H}}(p, q)^{2/3},$$

using (B.9) for $j = 1$, and the assumption (B.17). Similarly, using (B.9) for $j = 2$,

$$\begin{aligned} |f(\hat{s}) \nabla_{\mathbb{H}} \nu_s(p) - f(\hat{s}) \nabla_{\mathbb{H}} \nu_s(q)| &\lesssim \|f\|_{L^\infty(B(p_0, 2Cr_0))} \|\nabla^2_{\mathbb{H}} \nu_s\|_{L^\infty} d(p, q) \\ &\lesssim \frac{\|f\|_{L^\infty(B(p_0, 2Cr_0))}}{r(p)^2} d_{\mathbb{H}}(p, q) \lesssim \|f\|_{L^\infty(B(p_0, 2Cr_0))} d_{\mathbb{H}}(p, q)^{1/3}. \end{aligned}$$

Finally, to deal with the last term

$$\triangle := |\langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p) \rangle \nabla_{\mathbb{H}} \nu_s(p) - \langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot q) \rangle \nabla_{\mathbb{H}} \nu_s(q)| \tag{B.18}$$

that arises from $|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)|$, we assume without loss of generality that $\nabla_{\mathbb{H}} \nu_s(p) \neq 0$; if $\nabla_{\mathbb{H}} \nu_s(p) = \nabla_{\mathbb{H}} \nu_s(q) = 0$, the estimate is trivial. As explained between (B.14) and (B.15), the assumption $\nabla_{\mathbb{H}} \nu_s(p) \neq 0$ ensures that $d_{\mathbb{H}}(\hat{s}, \hat{p}) \lesssim r(p)$. Hence we have

$$|\langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p) \rangle| \lesssim d_{\mathbb{H}}(\hat{s}, p) \lesssim d_{\mathbb{H}}(\hat{s}, \hat{p}) + d_{\mathbb{H}}(\hat{p}, p) \lesssim r(p). \tag{B.19}$$

This allows us to bound the term $\triangle$ in (B.18) as follows

$$\triangle \lesssim |\langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p)\rangle| |\nabla_{\mathbb{H}} \nu_s(p) - \nabla_{\mathbb{H}} \nu_s(q)| + |\langle k(\hat{s}), \pi(\hat{s}^{-1} \cdot p) - \pi(\hat{s}^{-1} \cdot q)| |\nabla_{\mathbb{H}} \nu_s(q)|$$
$$\overset{(B.19),(B.9)}{\lesssim} r(p) \frac{1}{r(p)^2} d_{\mathbb{H}}(p,q) + d_{\mathbb{H}}(p,q) \frac{1}{r(p)} \overset{(B.17)}{\lesssim} d_{\mathbb{H}}(p,q)^{2/3},$$

recalling also that the implicit constants in "$\lesssim$" are allowed to depend on the data in (B.4). These bounds combined with the bounded overlap of the supports of the functions ν_s show that

$$|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)| \overset{(B.11)}{\lesssim} d_{\mathbb{H}}(p,q)^{1/3}$$

under the assumption (B.17). Finally, assume that

$$d_{\mathbb{H}}(p,q) \geqslant r(p)^3. \tag{B.20}$$

In this remaining case, one may apply (B.15) as follows:

$$|\nabla_{\mathbb{H}} f_1(p) - \nabla_{\mathbb{H}} f_1(q)| \leqslant |\nabla_{\mathbb{H}} f_1(p) - k(\hat{p})| + |\nabla_{\mathbb{H}} f_1(q) - k(\hat{p})|$$
$$\lesssim d_{\mathbb{H}}(p,\hat{p})^{\alpha} + d_{\mathbb{H}}(q,\hat{p})^{\alpha} \lesssim r(p)^{\alpha} + d_{\mathbb{H}}(p,q)^{\alpha} \lesssim d_{\mathbb{H}}(p,q)^{\alpha/3},$$

using the assumption (B.20) in the final estimate. Recalling also the cases (a)-(b) discussed after (B.15), it has now been established that $f_1 \in C^{1,\alpha/3}_{\mathbb{H}}(\mathbb{H}^n)$, and (B.16) holds for all $p, q \in \mathbb{H}^n$. Consequently, (B.7) holds if $\epsilon > 0$ was chosen small enough to begin with, and then

$$X_1 f_1(p) \geqslant \tfrac{1}{2}, \qquad p \in \mathbb{H}^n. \tag{B.21}$$

It follows from (B.21) that for every $p \in \mathbb{H}^n$, the map $s \mapsto f_1(p \cdot (s, 0, \ldots, 0))$ is strictly increasing with derivative $\partial_s[s \mapsto f_1(p \cdot (s, 0, \ldots, 0))] = X_1 f_1(p \cdot (s, 0, \ldots, 0)) \geqslant \frac{1}{2}$. Consequently, for every $p \in \mathbb{H}^n$, the line $p \cdot \mathbb{V} = \{p \cdot (s, 0, \ldots, 0) : s \in \mathbb{R}\}$ intersects $\{f_1 = 0\}$ in exactly one point, so the set $\{f_1 = 0\}$ is the intrinsic graph of a certain function $\varphi \colon \mathbb{W} \to \mathbb{V}$. Recalling that $f_1 \in C^{1,\alpha/3}_{\mathbb{H}}(\mathbb{H}^n)$, and noting (B.21), the conclusion is that $\{f_1 = 0\}$ is an intrinsic $C^{1,\alpha/3}$-graph. Moreover, since $f_1(p) = f(p)$ for all $p \in \bar{B}$, the set $S \cap \bar{B} \subset \{f = 0\} \cap \bar{B}$ is contained on the graph. Finally, the function φ is compactly supported, because $f_1(x_1, \ldots, x_{2n}, t) = x_1$ for all $(x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n \setminus B(p_0, Cr_0)$. Consequently,

$$\{f_1 = 0\} \cap [\mathbb{H}^n \setminus B(p_0, Cr_0)] \subset \{(x_1, \ldots, x_{2n}, t) : x_1 = 0\} = \mathbb{W}.$$

This implies that $\varphi \equiv 0$ outside a sufficiently large ball centred at the origin. The proof of the proposition is complete. □

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