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AN INVERSE PROBLEM FOR THE RELATIVISTIC BOLTZMANN EQUATION

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ABSTRACT. We consider an inverse problem for the Boltzmann equation on a globally hyperbolic Lorentzian spacetime (M, g) with an unknown metric g. We consider measurements done in a neighbourhood $V \subset M$ of a timelike path μ that connects a point x^- to a point x^+. The measurements are modelled by a source-to-solution map, which maps a source supported in V to the restriction of the solution to the Boltzmann equation to the set V. We show that the source-to-solution map uniquely determines the Lorentzian spacetime, up to an isometry, in the set $I^+(x^-) \cap I^-(x^+) \subset M$. The set $I^+(x^-) \cap I^-(x^+)$ is the intersection of the future of the point x^- and the past of the point x^+, and hence is the maximal set to where causal signals sent from x^- can propagate and return to the point x^+. The proof of the result is based on using the nonlinearity of the Boltzmann equation as a beneficial feature for solving the inverse problem.

CONTENTS

1. Introduction

In this paper we study what information can be recovered from indirect measurements of a system governed by the Boltzmann equation. The Boltzmann equation describes nonlinear particle dynamics which arise in many areas of physics, such as atmospheric chemistry, cosmology and condensed matter physics. For example, in cosmology, the Boltzmann equation describes how radiation is scattered by matter such as dust, stars and plasma on an Einstein spacetime. In condensed matter physics, the Boltzmann equation may describe the transportation of electrons or electron-phonon excitations in a media, which can be a metal or a semiconductor. In such situations, the geometry of the spacetime or resistivity of the media may be described by a Lorentzian manifold. In particular, we investigate the inverse problem of recovering the corresponding Lorentzian manifold of a system behaving according to the Boltzmann equation by making measurements in a confined, possibly small, area in space and time.

In the kinetic theory we adopt, particles travel on a Lorentzian manifold (M, g) along trajectories defined by either future-directed timelike geodesics (for positive mass particles) or future-directed lightlike geodesics (in the case of zero mass particles). In the absence of collisions and external forces, the kinematics of a particle density distribution $u \in C^\infty(TM)$ is captured by the *Vlasov equation* [41], [40] (or Liouville-Vlasov equation [9])

$$\mathcal{X}u(x, p) = 0 \quad \text{for } (x, p) \in \overline{\mathcal{P}}^+ M.$$

Here $\overline{\mathcal{P}}^+ M$ is the subset of TM of the future directed causal (i.e. not spacelike) vectors and $\mathcal{X} : C^\infty(TM) \to C^\infty(TM)$ is the geodesic vector field. In terms of the Christoffel symbols $\Gamma^\alpha_{\lambda\mu}$, the latter is given as

$$\mathcal{X} = \sum_{\alpha, \lambda, \mu} p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu \frac{\partial}{\partial p^\alpha}.$$

The behaviour of binary collisions is characterized by a *collision operator*

$$(1.1) \quad \mathcal{Q}[u, v](x, p) = \int_{\Sigma_{x,p}} \left[u(x, p)v(x, q) - u(x, p')v(x, q') \right] A(x, p, q, p', q') dV(x, p; q, p', q'),$$

where $u, v \in C^\infty(\overline{\mathcal{P}}^+ M)$. Here $dV(x, p; q, p', q')$ is the induced volume form on the submanifold

$$\Sigma_{x,p} := \{p\} \times \{(q, p', q') \in (\overline{\mathcal{P}}_x M)^3 \, : \, p + q = p' + q'\} \subset (T_x M)^4$$

and for each $(x, p) \in \overline{\mathcal{P}}^+ M$ the function

$$A(x, p, \cdot, \cdot, \cdot) \in C^\infty(\Sigma_{x,p})$$

is called a *collision kernel* (or a shock cross-section).

We assume that (M, g) is a globally hyperbolic C^∞ smooth Lorentzian manifold (see Section 2). Global hyperbolicity allows us to impose an initial state for the particle density function u by using a Cauchy surface $\mathcal{C} \subset M$. Given an initial state of no particles in the causal past $\mathcal{C}^-$ of $\mathcal{C}$, and a particle source f, supported in the causal future $\mathcal{C}^+$ of $\mathcal{C}$, the kinematics of a distribution of particles u is given by the *relativistic Boltzmann equation* [9], [41], [40],

$$(1.2) \quad \begin{aligned} \mathcal{X}u(x, p) - \mathcal{Q}[u, u](x, p) &= f(x, p), \quad (x, p) \in \overline{\mathcal{P}}^+ M \\ u(x, p) &= 0, \quad\quad\quad (x, p) \in \overline{\mathcal{P}}^+ \mathcal{C}^-. \end{aligned}$$

Though we do not consider it in this paper, Boltzmann's H-Theorem (which states that the entropy flux is nonincreasing in time) can be shown to hold for the relativistic Boltzmann kinematic model (1.2) when the collisions are reversible ($A(x, p, q, p', q') = A(x, p', q', p, q)$). We refer the reader to the works of [9], [41], or [40] for more details on this matter.

We study an inverse problem where we make observations in an open neighbourhood V of a timelike geodesic in M. We assume that V has compact closure without further notice. The observations are captured by the *source-to-solution map for light observations*,

$$(1.3) \qquad \Phi_{L+V} : B \to C_b(L^+V), \quad \Phi_{L+V}(f) := \Phi(f)|_{L^+V}.$$

Here Φ is the *source-to-solution map* for the relativistic Boltzmann equation

$$(1.4) \qquad \Phi : B \to C_b(\overline{\mathcal{P}}^+ M), \quad \Phi(f) = u,$$

where u solves (1.2) with a source f. The set B is a neighbourhood of the origin in $C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+) = \{f \in C(\overline{\mathcal{P}}^+ \mathcal{C}^+) : \operatorname{supp}(f) \subset K\}$ for a fixed compact set $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ and $C_b(\overline{\mathcal{P}}^+ M) = \{f \in C(\overline{\mathcal{P}}^+ M) : f \text{ is bounded}\}$. We also denote by $L^+V \subset TM$ the bundle of light-like future directed vectors whose base points are in V. Loosely speaking, the operator Φ_{L+V} corresponds to measuring the photons received in $V \subset M$ from particle interactions governed by the Boltzmann equation (1.2).

Known uniqueness and existence results to (1.2) depend inextricably on the properties of the collision kernel A. For general collision kernels, existence of solutions to (1.2) is not known. Complicating the analysis is the fact that it is not completely known what are the physical restrictions on the form of the collision kernel [41, p. 155]. Further, in the case of Israel molecules, where one has a reasonable description of what the collision kernel should be, the collision operator can not be seen as a continuous map between weighted L^p spaces [41, Appendix F]. It is not always clear what is the relationship between conditions on the collision operator $\mathcal{Q}$ and the induced conditions on its collision kernel A.

To address the well-posedness of (1.2), we consider collision kernels of the following type. Here we denote

$$\Sigma := \bigcup_{(x,p) \in \overline{\mathcal{P}}M} \Sigma_{x,p} \subset TM^4$$

and L^+M and $\mathcal{P}M$ are the bundles of light-like and timelike vectors over M respectively.

Definition 1.1 (Admissible Kernels)**.** *We say that $A : \Sigma \subset TM^4 \to \mathbb{R}$ is an* admissible collision kernel *with respect to a relatively compact open set $W \subset M$ if A satisfies*

(1) $A \in C^\infty(\Sigma)$.

(2) *The set $\pi \operatorname{supp} A$ is compact and contains W as a subset.*

(3) $A(x, p, q, p', q') > 0$, *for all* $(x, p, q, p', q') \in \Sigma \cap (L^+W \times \overline{\mathcal{P}}W \times \mathcal{P}W \times \mathcal{P}W)$.

(4) *There is a constant $C > 0$ such that for all $(x, p) \in \overline{\mathcal{P}}^+ M$,*

$$\|A(x, p, \cdot, \cdot, \cdot)\|_{L^1(\Sigma_{x,p})} := \int_{\Sigma_{x,p}} |A(x, p, q, p', q')| dV(x, p; q, p', q') \le C.$$

(5) *For every $(x, p) \in \overline{\mathcal{P}}^+ M$ the function $F_{x,p}(\lambda) := \|A(x, \lambda p, \cdot, \cdot, \cdot)\|_{L^1(\Sigma_{x,\lambda p})}$, $\lambda \in \mathbb{R}$, is continuously differentiable at $\lambda = 0$ and satisfies $F_{x,p}(0) = 0$.*

If there is no reason to emphasize the set W we just use the term *admissible collision kernel*. The most important case is when W is a time-like diamond as in the main theorem, see Figure 1.

The reader may take this as an assumption, although many of the steps in our main proof can be carried through for more general W.

We note that the condition (3) ensures that colliding particles traveling at less than the speed of light scatters lightlike particles. The conditions (4) and (5) are imposed to have control of collisions of relatively high and low momenta particles. These conditions are required in our proof of the well-posedness of the Cauchy problem for the Boltzmann equation (1.2):

Theorem 1.2. *Let (M, g) be a globally hyperbolic C^∞-smooth Lorentzian manifold. Let also $\mathcal{C}$ be a Cauchy surface of M and $K \subset \overline{\mathcal{P}}^+\mathcal{C}^+$ be compact. Assume that $A : \Sigma \to \mathbb{R}$ is an admissible collision kernel in the sense of Definition 1.1.*

Then, there are open neighbourhoods $B_1 \subset C_K(\overline{\mathcal{P}}^+\mathcal{C}^+)$ and $B_2 \subset C_b(\overline{\mathcal{P}}^+M)$ of the respective origins such that if $f \in B_1$, the relativistic Boltzmann problem (1.2) with source f has a unique solution $u \in B_2$. Further, there is a constant $c_{A,K} > 0$ such that

$$\|u\|_{C(\overline{\mathcal{P}}^+M)} \le c_{A,K}\|f\|_{C(\overline{\mathcal{P}}^+M)}.$$

The well-posedness of (1.2) has also been addressed in the following works. For exponentially bounded data, and an L^1-type bound on the cross-section of the collision kernel, it was shown in [8] that a short-time unique solution to (1.2) exists in the setting of a 4-dimensional, globally hyperbolic spacetime. Also for globally hyperbolic geometries, under conditions which require the collision operator to be a continuous map between certain weighted Sobolev spaces, it was proven in [3] that a unique short-time solution exists to (1.2) in arbitrary dimension. If the geometry of (M, g) is close to Minkowski in a precise sense, the collision kernel satisfies certain growth bounds and the initial data satisfied a particular form of exponential decay, then unique global solutions exist to the Cauchy problem for the Boltzmann equation [18]. We refer to [9], [41] and [40] for more information about the well-posedness of (1.2). Also, the recent paper [26], related to our work, contains a well-posedness result in Euclidean spaces.

We are now ready to present our main theorem. In our theorem, measurements of solutions to the Boltzmann equation are made on a neighborhood V of a timelike geodesic $\hat{\mu}$ in (M, g). We prove that measurements made on V both determines a subset W of M and the restriction of the metric to W (up to isometry). The subset W is naturally limited by the finite propagation speed of light. The situation is illustrated in Figure 1 below.

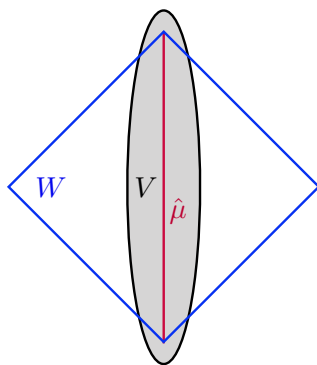FIGURE 1. Illustration of the timelike geodesic $\hat{\mu}$, known set V, and the yet unknown set W.

Before we continue, we introduce our notation. We consider source-to-solutions maps of the Bolzmann equation (1.2) defined on sources, which are supported on different compact sets $K \subset \overline{\mathcal{P}}^+M$. As follows from Theorem 1.2, for each such K the source-to-solution map $B_1 \to$

$B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$ of the Boltzmann equation is well defined, where $B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. In this case, we denote $B_{1,K} = B_K$. In the next theorem we consider source-to-solution maps of Boltzmann equations defined on

$$(1.5) \qquad \mathcal{B} := \bigcup_{\{K:\pi(K)\subset Z\}} B_{1,K} \subset C_c(\overline{\mathcal{P}}^+ \mathcal{C}^+),$$

where Z is a fixed compact subset of M.

Theorem 1.3. *Let (M_1, g_1) and (M_2, g_2) be globally hyperbolic C^∞-smooth Lorentzian manifolds of dimension ≥ 3. Assume that V is a mutual open subset of M_1 and M_2 and that $g_1|_V = g_2|_V$. Let $\mu : [-1,1] \to V$ be a smooth timelike curve. Let also $-1 < s^- < s^+ < 1$, and define $x^\pm := \mu(s^\pm)$ and*

$$W_j := I_j^-(x^+) \cap I_j^+(x^-), \quad j = 1, 2.$$

Let $Z \subset V \subset M$ be a compact set such that $\mu[-1,1] \subset\subset Z$.

Suppose that A_1 and A_2 are admissible kernels in the sense of Definition 1.1. Assume that the source-to-solution maps of the Boltzmann equation on (M_1, g_1) and (M_2, g_2) with kernels A_1 and A_2 respectively agree,

$$\Phi_{2, L^+V} = \Phi_{1, L^+V},$$

on $\mathcal{B}$. Here $\mathcal{B}$ is as in (1.5). Then there is an isometric C^∞-smooth diffeomorphism $F : W_1 \to W_2$,

$$F^* g_2 = g_1 \text{ on } W_1.$$

Note that alternative to V being a mutual open set of M_1 and M_2, we could instead assume that there are open sets $V_1 \subset M_1$ and $V_2 \subset M_2$ and an isometry $\mathcal{I} : V_1 \to V_2$ so that the corresponding source-to-solution maps Φ_{j,L^+V_j}, $j = 1, 2$, satisfy $\Phi_{1,L^+V_1}\mathcal{I}^* = \mathcal{I}^*\Phi_{2,L^+V_2}$. In this case there would be an isometry $W_1 \to W_2$. We work with the assumption that V is a mutual open set for simplicity.

Inverse problems have been studied for equations with various nonlinearity types. Many of the earlier works rely on the fact that a solution to a related linear inverse problem exists. However, it was shown in Kurylev-Lassas-Uhlmann [24] that the nonlinearity can be used as a beneficial tool to solve inverse problems for nonlinear equations. They proved that local measurements of the scalar wave equation with a quadratic nonlinearity on a Lorentzian manifold determines topological, differentiable, and conformal structure of the Lorentzian manifold. Our proof of Theorem 1.3, which we explain shortly in the next section, builds upon this work [24].

Recently, Lai, Uhlmann and Yang [26] studied an inverse problem for the Boltzmann equation in the Euclidean setting. With an L^1 bound and symmetry constraint on the collision kernel, they show that one may reconstruct the collision kernel from boundary measurements. Linear equations such as Vlasov, radiative transfer (also called linear Boltzmann), or generalized transport equations model kinematics of particles which do not undergo collisions. We list a very modest selection of the literature on inverse problems for these equations next. In Euclidean space, Choulli and Stefanov [10] showed that one can recover the absorption and production parameters of a radiative transfer equation from measurements of the particle scattering. They also showed that an associated boundary operator, called the Albedo operator, determines the absorption and production parameters for radiative transfer equations in [11] and [12]. Other results for the recovery of the coefficients of a radiative transfer equation in Euclidean space from the Albedo operator have been proven by Tamasan [44], Tamasan and Stefanov [42], Stefanov and Uhlmann [43], Bellassoued and Boughanja [6], and Lai and Li [25]. Under certain curvature constraints on the metric, an inverse problem for a radiative transfer equation was studied in

the Riemannian setting by McDowall [35], [36]. A review of some of these and other inverse problems for radiative transfer and linear transportation equations is given by Bal [2].

The mentioned work [24] invented a *higher order linearization method* for inverse problems of nonlinear equations in the Lorentzian setting for the wave equation. Other works studying inverse problems for nonlinear hyperbolic equations by using the higher order linearization method include: Lassas, Oksanen, Stefanov and Uhlmann [30]; Lassas, Uhlmann and Wang [32], [31]; and Wang and Zhou [45]; Lassas, Liimatainen, Potenciano-Machado and Tyni [29]. The higher order linearization method in inverse problems for nonlinear elliptic equations was used recently in Lassas, Liimatainen, Lin and Salo [28] and Feizmohammadi and Oksanen [16], and in the partial data case in Krupchyk and Uhlmann [23] and Lassas, Liimatainen, Lin and Salo [27].

1.1. **Theorem 1.3 proof summary.** Now we will explain the key ideas in our proof of Theorem 1.3. To begin let $\hat{\mu} : [-1, 1] \to M$ be a future directed, timelike geodesic and let V be an open neighbourhood of the graph of $\hat{\mu}$ where we do measurements. For some $-1 < s^- < s^+ < 1$, let $x^\pm = \mu(s^\pm)$, $W = I^-(x^+) \cap I^+(x^-)$ and $w \in W$. We adapt the method of higher order linearization introduced in [24] for the nonlinear wave equation to our Boltzmann setting (1.2). Using this approach, we use the nonlinearity to produce a point source at w for the linearized Boltzmann equation.

The point source at w is generated formally as follows. Let $\mathcal{C}$ be a Cauchy surface in M and $f_1, f_2 \in C_c(\overline{\mathcal{P}}^+ V)$ be two sources of particles. For sufficiently small parameters ϵ_1 and ϵ_2, let $u_{\epsilon_1 f_1 + \epsilon_2 f_2}$ be the solution to

$$(1.6) \qquad \begin{aligned} \mathcal{X} u_{\epsilon_1 f_1 + \epsilon_2 f_2} - \mathcal{Q}[u_{\epsilon_1 f_1 + \epsilon_2 f_2}, u_{\epsilon_1 f_1 + \epsilon_2 f_2}] &= \epsilon_1 f_1 + \epsilon_2 f_2 & \text{in} \quad \overline{\mathcal{P}}^+ M \\ u_{\epsilon_1, \epsilon_2}(x, p) &= 0 & \text{in} \quad \overline{\mathcal{P}}^+ \mathcal{C}^- \end{aligned}$$

By computing the mixed derivative

$$\Phi^{2L}(f_1, f_2) := \frac{\partial}{\partial \epsilon_1} \frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_1 = \epsilon_2 = 0} u_{\epsilon_1 f_1 + \epsilon_2 f_2},$$

we show that $\Phi^{2L}(f_1, f_2)$ solves the equation

$$(1.7) \qquad \mathcal{X}\big(\Phi^{2L}(f_1, f_2)\big) = \mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)] + \mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)]$$

on $\overline{\mathcal{P}}^+ M$. Here the functions $\Phi^L(f_l)$, $l = 1, 2$, are solutions to the linearization of the Boltzmann equation at $u = 0$, which solve

$$(1.8) \qquad \begin{aligned} \mathcal{X}\big(\Phi^L(f_l)\big) &= f_l & \text{in} \quad \overline{\mathcal{P}}^+ M. \\ \Phi^L(f_l) &= 0 & \text{in} \quad \overline{\mathcal{P}}^+ \mathcal{C}^- \end{aligned}$$

The transport equation (1.8) is also called the *Vlasov equation*.

Next, we build particular functions f_1 and f_2 such that the right hand side of (1.7),

$$\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)] + \mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)],$$

is a point source at w. To do this, we note that since $w \in W$, there exists lightlike geodesics η and $\tilde{\eta}$, initialized in our measurement set V, which have their first intersection at w. We choose timelike geodesics $\gamma_{(\hat{x}, \hat{p})}$ and $\gamma_{(\hat{y}, \hat{q})}$ with initial data $(\hat{x}, \hat{p})$ and $(\hat{y}, \hat{q})$ in the bundle $\mathcal{P}^+ V$ of future-directed timelike vectors and which approximate the geodesics η and $\tilde{\eta}$. Then, in Lemma 4.6, we create $\dim(M) - 2$ Jacobi fields on the geodesic $\gamma_{(\hat{y}, \hat{q})}$ such that the variation of $\dot{\gamma}_{(\hat{y}, \hat{q})}$ generated by the Jacobi fields is a submanifold $Y_2 \subset \mathcal{P}^+ W \subset TM$. In particular, since Y_2 and $Y_1 := \text{graph}(\gamma_{(\hat{x}, \hat{p})}) \subset TM$ are geodesic flowouts, they can be considered as distributional solutions to the linear transport equation (1.8). The Jacobi fields are also constructed so that the projection of $Y_2 \subset TM$ to M near the intersection points of $\gamma_{(\hat{x}, \hat{p})}$ and $\gamma_{(\hat{y}, \hat{q})}$ is a $\dim(M) - 1$

dimensional submanifold of M which intersects the other geodesic $\gamma_{(\hat{x}, \hat{p})}$ transversally in M. This transversality condition enables us to employ microlocal techniques to show in Theorem 4.3 and Corollary 4.4 that $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)] + \mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)]$ represents a point source.

To differentiate $u_{\epsilon_1 f_1 + \epsilon_2 f_2}$ with respect to both ϵ_1 and ϵ_2, we also prove that the source to solution map of the Boltzmann equation (1.2) is two times Frechét differentiable in Lemma 3.4. To analyze the nonlinear term $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)] + \mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)]$ when $\Phi^L(f_1)$ and $\Phi^L(f_2)$ are distributions in $\mathcal{D}'(TM)$, namely the delta distributions of the submanifolds Y_1 and Y_2 of TM, we view them as conormal distributions in $I^{k_l}(N^*Y_l)$ for some $k_l \in \mathbb{R}$ and $l = 1, 2$. This allows us to compute the terms $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)]$ and $\mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)]$ by using the calculus of Fourier integral operators. Slight complications to this analysis are due the fact that the distributions $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)]$ and $\mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)]$ have relations (in the sense of the theory of Fourier integral operators) in the bundle of causal vectors, which is a manifold with boundary.

We circumvent these complications by only measuring lightlike signals. This means that we compose the source to solution map with a lightlike section $P : V_e \subset V \to L^+V$, where V_e open subset of V. This reduces our analysis of the term $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)]$ in (1.7), $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)]$, to an analysis of the term

$$- \int_{\Sigma_{x,p}} \Phi^L(f_1)(x, p') \Phi^L(f_2)(x, q') A(x, p, q, p', q') dV_{x,p}(q, p', q'),$$

where x lies in the open subset $V_e \subset M$ and $p = P(x) \in L^+V$, since the other part of the collision operator (1.1) in this case yields zero. We analyze similarly the second right hand side term in (1.7).

By the above construction, we construct a singularity at $w \in W \subset M$, which we may observe the singularity by using the source to solution map for light observations Φ_{L^+V}. Similar to [24], from our analysis we may recover the so-called *earliest observation time functions* from the knowledge of Φ_{L^+V}. Time separation function at w compute the optimal travel time of light signal received from w. The collection of all earliest observation time functions from w determines the *earliest light observation set* that is the set of points $y \in \mathcal{U}$ that can be connected to w by lightlike geodesics which have no interior cut points. This set is denoted by $\mathcal{E}_{\mathcal{U}}(w)$. Here $\mathcal{U}$ is an open subset of V and we consider $\mathcal{E}_{\mathcal{U}}(w)$ as the map $w \in W \mapsto \mathcal{E}_{\mathcal{U}}(w)$. We refer to Section 5.3 or [24] for explicit definitions.

We apply the above to construct and measure point sources on two different Lorentzian manifolds (M_1, g_1) and (M_2, g_2) to prove:

Proposition 1.4. *Let Φ_{1, L^+V} and Φ_{2, L^+V} be the source-to-solution maps for light observation satisfying the conditions in Theorem 1.3. Assume that the conditions of Theorem 1.3 are satisfied for the Lorentzian manifolds (M_1, g_1) and (M_2, g_2). Then $\Phi_{2, L^+V} = \Phi_{1, L^+V}$ implies*

$$\{\mathcal{E}_{\mathcal{U}}^1(w_1) : \ w_1 \in W_1\} = \{\mathcal{E}_{\mathcal{U}}^2(w_2) : \ w_2 \in W_2\}.$$

As shown in [24, Theorem 1.2], the sets $\mathcal{E}_{\mathcal{U}}^j(w)$, $j = 1, 2$, determine the unknown regions W_j and the conformal classes of the Lorentzian metrics g_j on them. That is, there is a diffeomorphism $F : (W_1, g_1|_{W_1}) \to (W_2, g_1|_{W_2})$ such that $F^*g_2 = cg_1$ on W_1. After the reconstruction of the conformal class of the metric $(W, g|_W)$, we conclude the proof of Theorem 1.3 by showing that the source-to-solution map in V uniquely determines the conformal factor $c(x)$ in W. This implies that the source-to-solution map in the set V determines uniquely the isometry type of the Lorentzian manifold $(W, g|_W)$.

Paper outline. Section 2 contains all the preliminary information. There we introduce our notation (2.1) and recall Lagrangian distributions (2.2). We provide some expository material

on the collisionless (Vlasov) and Boltzmann models of particle kinematics in Sections 3.1 and 3.2 respectively. In Section 4, we analyze the operator $\mathcal{Q}_{gain}$ from a microlocal perspective. Viewing particles as conormal distributions, we describe fully the wavefront set which arises from particle collisions. Additionally, in Section 4.1 we construct a submanifold $S_2 \subset TM$, whose geodesic flowout is $Y_2 \subset TM$, such that the graph Y_1 of a geodesic and Y_2 satisfies the properties discussed in Section 1.1. In Section 5, we provide the proof of Theorem 1.3, broken down into several key steps:

(1) We show that from our source to solution map data, we can determine if our particle sources interact in W.

- In section 5.1, given choices of future-directed timelike vectors $(\hat{x}, \hat{p})$ and $(\hat{y}, \hat{q})$, as in Section 1.1, we construct the sources f_1 and f_2 described in Section 1.1.

- In this section we additionally construct smooth C_c^∞ approximations h_1^ϵ and h_2^ϵ, $\epsilon > 0$, of the sources f_1 and f_2 respectively. This is done mainly for technical reasons, since we only consider the source-to-solution operator of the Boltzmann equation for smooth sources.

- In Section 5.2 we describe a particular future-directed lightlike vector field $P_e : V_e \to L^+ V_e$, $V_e \subset V$ that admits as an integral curve a fixed optimal geodesic γ. We compose it with the collision operator (1.1) and prove

$$\text{singsupp}(\mathcal{S}) = \gamma \cap V_e,$$

where $\mathcal{S}$ is known from our measurements, namely $\mathcal{S} = \lim_{\epsilon \to 0} \Phi^{2L}(h_1^\epsilon, h_2^\epsilon) \circ P_e$.

(2) In section 5.4 we prove Proposition 1.4, which says that we may determine earliest light observation sets from our measurements of $\mathcal{S}$.

(3) From the work in Section 5.4 which connects source-to-solution map data to the earliest light observation sets, we show we may determine the conformal class of the metric.

(4) Lastly, in the section 5.6 we finish the proof by showing that we can identify the conformal factor of the metric.

Auxiliary lemmas, which include lemmas on the existence of solutions to the Cauchy problem of the Boltzmann equation with small data and the linearization of the source-to-solution map, among others, are contained in the Appendices.

2. PRELIMINARIES

2.1. **Notation.** Throughout this paper, (M, g) will be an n-dimensional Lorentzian spacetime with $n \geq 3$. We additionally assume that (M, g) is globally hyperbolic. Globally hyperbolicity (see e.g. [7]) implies that $M = \mathbb{R} \times N$, for some smooth $n-1$ dimensional submanifold $N \subset M$, and that the metric takes the form

$$g(x) = -g_{00}(x)dx^0 \otimes dx^0 + g_N(x), \quad x = (x^0, \overline{x}), \ x^0 \in \mathbb{R}, \ \overline{x} \in N,$$

where $g_{00} \in C^\infty(M)$ is a positive function and $g_N(x^0, \cdot)$ is a smooth Riemannian metric on N, for each $x^0 \in \mathbb{R}$. Global hyperbolicity implies that the manifold (M, g) has a global smooth timelike vector field τ. This vector field defines the causal structure for (M, g). Further, a globally hyperbolic manifold is both causally disprisoning and causally pseudoconvex (see for example [5]). Causally disprisoning means that for each inextendible causal geodesic $\gamma : (a, b) \to M$, $-\infty \leq a < b \leq \infty$, and any $t_0 \in (a, b)$, the closures in M of the sets $\gamma(a, t_0]$ and $\gamma[t_0, b)$ are

not compact in M. The manifold M is pseudoconvex if for every compact set of M, there is a geodesically convex compact set which properly contains it.

We use standard notation for the causal structure of (M, g). For $x, y \in M$ we write $x \ll y$ (respectively $x \gg y$) if $x \neq y$ and there is a future directed timelike geodesic from x to y (respectively from y to x). We write $x < y$ (respectively $x > y$) if $x \neq y$ and there is a future directed causal geodesic from x to y (respectively from y to x).

Given $x \in M$, let $J^+(x) := \{y : x < y \text{ or } x = y\} \subset M$ and $I^+(x) := \{y : x \ll y\} \subset M$ denote respectively the causal and chronological future of x. Similarly, let $J^-(x)$ and $I^-(x)$ denote the causal and chronological past of x. The set of points in M which may be reached by lightlike geodesics emanating from a point $x \subset M$ is

$$\mathcal{L}^{\pm}(x) := \{y \in TM : y = \exp_x(sp), \ p \in L_x^{\pm}M, \ s \in [0, \infty)\},$$

where $L_x^{\pm}M \subset TM$ is the set of future $(+)$ or past $(-)$ directed lightlike vectors in T_xM. If $U \subset M$, we also write

$$(2.9) \qquad J^{\pm}(U) = \bigcup_{x \in U} J^{\pm}(x), \quad I^{\pm}(U) = \bigcup_{x \in U} I^{\pm}(x), \quad \mathcal{L}^{\pm}(U) = \bigcup_{x \in U} \mathcal{L}^{\pm}(x).$$

We express the elements of TM as (x, p) where $x \in M$ and $p \in T_xM$. Since $TM = T\mathbb{R} \times TN$, each $(x, p) \in TM$ can be written in the form

$$x = (x^0, \bar{x}), \quad \text{and} \quad p = (p^0, \bar{p}),$$

for $x^0 \in \mathbb{R}$, $\bar{x} \in N$, $p^0 \in T_{x^0}\mathbb{R}$, and $\bar{p} \in T_{\bar{x}}N$. Given a local coordinate frame $\partial_\alpha : U \subset M \to TM$, $\alpha = 1, \ldots, n$, we identify $TU \approx U \times \mathbb{R}^n$. We use (x, p^α) to denote this local expression.

For $(x, p) \in TM$, we denote by $\gamma_{(x,p)}$ the geodesic with initial position x and initial velocity p. The velocity of $\gamma_{(x,p)}$ at $s \in \mathbb{R}$ is denoted by $\dot{\gamma}_{(x,p)}(s) \in T_{\gamma_{(x,p)}(s)}M$, $s \in \mathbb{R}$. To simplify the notation we occasionally refer also to the curve $s \mapsto (\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) \in TM$ by $\dot{\gamma}_{(x,p)}(s)$.

The appropriate phase spaces for our particles will be comprised of the following subbundles of TM. Let $U \subset M$ be open and let $m \geq 0$. The mass shell of mass m on U is

$$P^mU := \{(x, p) \in TM \setminus \{0\} : \sqrt{-g(p, p)} = m, \ \pi(x, p) = x \in U\} \subset TM,$$

where $\pi : TM \to M$ is the canonical projection. The time-like bundle on U is

$$\mathcal{P}U := \bigcup_{m > 0} P^mU.$$

We denote the inclusion of the light-like bundle $LU := P^0U$ to this union as

$$\overline{\mathcal{P}}U := \bigcup_{m \geq 0} P^mU$$

which is the bundle of causal vectors on U. Notice that $\overline{\mathcal{P}}U$ excludes the zero section; that is, a zero vector is not causal in our conventions. We also write for $t \in \mathbb{R}$, $x \in \mathbb{R} \times N$

$$P_t^rU := (T_t\mathbb{R} \times TN) \cap P^rU, \quad P_x^rU := T_xM \cap P^rU$$
$$\mathcal{P}_tU := (T_t\mathbb{R} \times TN) \cap \mathcal{P}U, \quad \mathcal{P}_xU := (T_xM) \cap \mathcal{P}U,$$

and write similarly for $\overline{\mathcal{P}}_xU$ and $\overline{\mathcal{P}}_tU$.

Each one of the bundles above consists of future-directed and past-directed components which are distinguished by adding "+" or "−" to the superscript. For example, we denote the manifold of future-directed causal vectors on an open set $U \subset M$ by

$$\overline{\mathcal{P}}^+U := \{(x, p) \in TM \setminus \{0\} : x \in U, \ -g(p, p)|_x \geq 0, \ g(\tau, p) < 0\} \subset \overline{\mathcal{P}}U.$$

Let $S \subset \overline{\mathcal{P}}_t M$, $t \in \mathbb{R}$ be a submanifold of TM. The *geodesic flowout* of S in TM is the set

$$K_S := \{(x,p) \in TM : (x,p) = \dot{\gamma}_{(y,q)}(s), \ s \in (-T(y,-q), T(y,q)), \ (y,q) \in S\} \subset \overline{\mathcal{P}}M.$$

Here the interval $(-T(y,-q), T(y,q))$ is inextendible, i.e. the maximal interval where the geodesic $\gamma_{(y,q)}$ is defined.

Remark 2.1. The geodesic flowout K_S is a smooth manifold. To see this, note that K_S is the image of the map $(s,(y,q)) \mapsto \varphi_s(y,q)$, where $\varphi_s(y,q)$ is the integral curve of the geodesic vector field $\mathcal{X}$ at parameter time s starting from $(y,q) \in S$. Since $\mathcal{X}$ is transversal to $\overline{\mathcal{P}}_t M$ (see the proof of Lemma 3.1) we have that this map is an immersion, see e.g. [33, Theorem 9.20]. Given that M is globally hyperbolic, there are no closed causal geodesics. From this it follows that this map is also injective and thus an embedding. Consequently, its image K_S is a smooth submanifold of TM.

We denote $G_S := \pi(K_S)$, which is the set

$$G_S = \{x \in M : x = \gamma_{(y,q)}(s), \ s \in (-T(y,-q), T(y,q)), \ (y,q) \in S\} \subset M.$$

Unlike K_S, the set G_S might not be a manifold since the geodesics describing this set might have conjugate points.

If X is a smooth manifold and Y is a submanifold of X, the conormal bundle of Y is defined as

$$N^*Y = \{(x,\xi) \in T^*X \setminus \{0\} : x \in Y, \xi \perp T_x Y\}.$$

Here $\perp$ is understood with respect to the canonical pairing of vectors and covectors.

2.2. Lagrangian distributions and Fourier integral operators.
Here we define the classes of distributions and operators we work with in this paper. We will follow the notation in Duistermaat's book [13]. See also the original sources [22, 14].

Let X be a smooth manifold of dimension $n \in \mathbb{N}$ and $\Lambda \subset T^*X \setminus \{0\}$ be a conic Lagrangian manifold [13, Section 3.7]. We denote the space of symbols [13, Definition 2.1.2] of order $\mu \in \mathbb{R}$ (and type $1,0$) on the conic manifold $X \times \mathbb{R}^k \setminus \{0\}$ by $S^\mu(X \times \mathbb{R}^k \setminus \{0\})$. The Hörmander space of Lagrangian distributions of order $m \in \mathbb{R}$ over Λ is denoted by $I^m(X; \Lambda)$, and consists of locally finite sums $u = \sum_j u_j \in \mathcal{D}'(X)$ of oscillatory integrals

$$u_j(x) = \int_{\mathbb{R}^{k_j}} e^{i\varphi_j(x,\xi)} a_j(x,\xi) d\xi, \quad x \in X.$$

Here $a_j \in S^{m-k_j/2+n/4}(X \times \mathbb{R}^{k_j} \setminus \{0\})$. The phase function φ_j is defined on an open cone $\Gamma_j \subset X \times \mathbb{R}^{k_j}$ and satisfies the following two conditions: (a) it is nondegenerate, $d\varphi_j \neq 0$ on Γ_j and (b) the mapping

$$\Gamma_j \to \Lambda, \ (x,\xi) \mapsto (x, d_x\varphi_j(x,\xi))$$

defines a diffeomorphism between the set $\{(x,\xi) \in \Gamma_j : d_\xi \varphi_j(x,\xi) = 0\}$ and some open cone in Λ. When the base manifold X is clear, we abbreviate $I^m(\Lambda) = I^m(X; \Lambda)$.

Below we let X, Y, Z be C^∞-smooth manifolds. Let Λ be a conic Lagrangian manifold in $T^*(X \times Y) \setminus \{0\}$. The manifold corresponds to a canonical relation Λ' defined by

(2.10) $$\Lambda' = \{(x,y; \xi_x, \xi_y) \in T^*(X \times Y) : (x,y; \xi_x, -\xi_y) \in \Lambda\}.$$

Equivalently, one may start with a canonical relation and obtain a Lagrangian manifold. In this paper we chose to represent canonical relations as twisted manifolds Λ' of Lagrangian manifolds. Considering an element in $I^m(X \times Y; \Lambda)$ as a Schwartz kernel defines an operator

$$F : C_c^\infty(Y) \to \mathcal{D}'(X).$$

The class of operators of this form are called Fourier integral operators of order $m \in \mathbb{R}$ associated with the relation Λ'. We denote this class of operators by

$$I^m(X, Y; \Lambda').$$

We may also identify the space $I^m(X; \Lambda)$, $\Lambda \subset T^*X$, with $I^m(X, \{0\}; \Lambda \times \{(0,0)\})$.

The wavefront set of F is denoted by $WF'(F)$ and it is the set

$$(2.11) \qquad WF'(F) = \{(x, \xi^x, y, \xi^y) \in (T^*X \times T^*Y) \setminus \{0\} \,:\, (x, y, \xi^x, -\xi^y) \in WF(G)\},$$

where $WF(G)$ is the wavefront set of the distribution kernel G of F. We also define

$$(2.12) \qquad WF'_X(F) = \{(x, \xi^x) \in (T^*X) \setminus \{0\} \,:\, (x, y, \xi^x, 0) \in WF(G)\},$$

$$(2.13) \qquad WF'_Y(F) = \{(y, \xi^y) \in (T^*Y) \setminus \{0\} \,:\, (x, y, 0, \xi^y) \in WF(G)\}.$$

Consider two Fourier integral operators $u_1 \in I^{m_1}(X, Y; \Lambda'_1)$ and $u_2 \in I^{m_2}(Y, Z; \Lambda'_2)$ (i.e. Schwartz kernels in $I^{m_1}(X \times Y; \Lambda_1)$ and $I^{m_2}(Y \times Z; \Lambda_2)$, respectively) with respective orders m_1 and m_2 and relations Λ'_1 and Λ'_2. Sufficient conditions for u_1 and u_2 to form a well defined composition $u_1 \circ u_2 \in I^{m_1 + m_2}(X, Z; \Lambda'_1 \circ \Lambda'_2)$ are described in theorems [13, Theorem 2.4.1, Theorem 4.2.2] which provide the rules of basic microlocal operator calculus, often referred to as transversal intersection calculus. The relation $\Lambda'_1 \circ \Lambda'_2$ is defined as

$$\Lambda'_1 \circ \Lambda'_2 := \{(x, z\,;\, \xi^x, \xi^z) \in T^*(X \times Z) :$$
$$(2.14) \qquad\qquad (x, y\,;\, \xi^x, \xi^y) \in \Lambda'_1, \quad (y, z\,;\, \xi^y, \xi^z) \in \Lambda'_2, \text{ for some } (y\,;\, \xi^y) \in T^*Y\}.$$

Lastly, we remark that products of Lagrangian distributions are naturally defined as distributions over an interesting pair of conic Lagrangian manifolds $\Lambda_0, \Lambda_1 \in T^*X \setminus \{0\}$, which are described next. We refer to [15, 21, 37, 19, 20] for a thorough presentation of such distributions.

To begin, a pair (Λ_0, Λ_1) of conic Lagrangian manifolds $\Lambda_0, \Lambda_1 \in T^*X \setminus \{0\}$, is called an intersecting pair if their intersection is clean: $\Lambda_0 \cap \Lambda_1$ is a smooth manifold and

$$T^*_\lambda(\Lambda_0 \cap \Lambda_1) = T^*_\lambda \Lambda_0 \cap T^*_\lambda \Lambda_1 \text{ for all } \lambda \in \Lambda_0 \cap \Lambda_1.$$

Let (Λ_0, Λ_1) be an intersecting pair of conic Lagrangians with $\mathrm{codim}(\Lambda_0 \cap \Lambda_1) = k$, and $\mu, \nu \in \mathbb{R}$. For $\ell \in \mathbb{Z}_+$, we denote by $S^{\mu,\nu}(X \times (\mathbb{R}^\ell \setminus \{0\}) \times \mathbb{R}^k)$ the space of symbol-valued symbols on $X \times (\mathbb{R}^\ell \setminus \{0\}) \times \mathbb{R}^k$ (see [15, 19]). We say that $u \in \mathcal{D}'(X)$ is a paired Lagrangian distribution of order (μ, ν) associated to (Λ_0, Λ_1) if u can be expressed as a locally finite sum of the form

$$u = u_0 + u_1 + v,$$

where $u_0 \in I^{\mu+\nu}(\Lambda_0)$, $u_1 \in I^\nu(\Lambda_1)$, and

$$v(x) = \int_{\mathbb{R}^\ell} \int_{\mathbb{R}^k} e^{i\varphi(x;\theta;\sigma)} a(x; \theta; \sigma) \, d\theta d\sigma$$

for $a(x; \theta; \sigma) \in S^{\tilde{\mu},\tilde{\nu}}(X \times (\mathbb{R}^\ell \setminus \{0\}) \times \mathbb{R}^k)$, some $\ell \in \mathbb{Z}_+$, $\mu = \tilde{\mu} + \tilde{\nu} + \frac{\ell+k}{2} - \frac{n}{4}$, and $\nu = -\tilde{\nu} - \frac{n}{2}$. Above, the multiphase function $\varphi(x; \theta; \sigma)$ satisfies the following three conditions: for any $\lambda_0 \in \Lambda_0 \cap \Lambda_1$, (a) there is an open conic set $\Gamma \subset X \times (\mathbb{R}^\ell \setminus \{0\}) \times \mathbb{R}^k$ such that $\varphi(x; \theta, \sigma) \in C^\infty(\Gamma)$, (b) $\varphi(x; \theta, 0)$ is a phase function parametrizing Λ_0 in a conic neighbourhood of λ_0, and (c) $\varphi(x; 0, \sigma)$ is a phase function parametrizing Λ_1 in a conic neighbourhood of λ_0. In particular,

$$(2.15) \qquad\qquad u \in I^{\mu,\nu}(\Lambda_0, \Lambda_1) \implies WF(u) \subset \Lambda_0 \cup \Lambda_1.$$

We denote the set of paired Lagrangian distributions of order (μ, ν), $\mu, \nu \in \mathbb{R}$, associated to (Λ_0, Λ_1) by $I^{\mu,\nu}(\Lambda_0, \Lambda_1)$.

3. Vlasov and Boltzmann Kinetic Models

3.1. **The Vlasov model.** We consider a system of particles on a Lorentzian manifold (M, g) having positions $x \in M$ and momenta $p \in \mathcal{P}_x M$ in a statistical fashion as a density distribution $u \in \mathcal{D}'(\mathcal{P}M)$. We assume that the support of u is contained in some compact and proper subset $K \subset \overline{\mathcal{P}M}$. The Vlasov model describes the trajectories of a system of particles in a relativistic setting where there are no external forces and where collisions between particles are negligible. In this situation, the individual particles travel along geodesics determined by initial positions and velocities. On the level of density distributions, the behaviour of the system of particles is captured by the *Vlasov equation*

$$(3.16) \qquad \mathcal{X}u = f,$$

where $f \in \mathcal{D}'(\mathcal{P}M)$ is a source of particles and $\mathcal{X} : TM \to TTM$ on TM is the geodesic vector field. Locally, this vector field has the expression

$$\mathcal{X} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\lambda\mu}(x)p^\lambda p^\mu \frac{\partial}{\partial p^\alpha}.$$

Above the functions $\Gamma^\alpha_{\lambda\mu}$, $\lambda, \mu, \alpha = 0, \ldots, n = \dim(M) - 1$, are the Christoffel symbols of the Lorentz metric g.

The geodesic vector field $-i\mathcal{X} : C^\infty(\mathcal{P}M) \to C^\infty(\mathcal{P}M)$ may be viewed as Fourier Integral Operator (FIO) with principal symbol

$$(3.17) \qquad \sigma_\mathcal{X} : T^*TM \to \mathbb{R}, \quad \sigma_\mathcal{X}(x, p^\alpha\,; \xi^x_\alpha, \xi^p_\alpha) = p^\alpha \xi^x_\alpha - \Gamma^\alpha_{\lambda\mu}(x)p^\lambda p^\mu \xi^p_\alpha.$$

Writing $\langle\,\cdot\,,\,\cdot\,\rangle$ for the dual paring between TTM and T^*TM, $\sigma_\mathcal{X}$ takes the form

$$\sigma_\mathcal{X}(x, p\,; \xi) = \langle \xi, \mathcal{X} \rangle|_{(x,p)} = \langle \xi, \partial_s \dot{\gamma}_{(x,p)}(s) \rangle\big|_{s=0}.$$

Therefore, a function $\phi \in C^\infty(TM)$ satisfies $\sigma_\mathcal{X}(x, p, d\phi|_{(x,p)}) = 0$ at each $(x, p) \in TM$ if and only if ϕ is constant along geodesic velocity curves $s \mapsto \dot{\gamma}_{(x,p)}(s)$. Thus the bicharacteristic strip of $-i\mathcal{X}$ with initial data $(\tilde{x}, \tilde{p}\,; \tilde{\xi}) \in \sigma_\mathcal{X}^{-1}(0)$ is the parametrized curve

$$s \mapsto (x(s), p(s)) = \dot{\gamma}_{(\tilde{x},\tilde{p})}(s), \quad \xi(s) = d\phi|_{\dot{\gamma}_{(\tilde{x},\tilde{p})}(s)},$$

where $\phi \in C^\infty(TM)$ is any smooth function which is constant along geodesic velocity curves and satisfies $d\phi|_{(\tilde{x},\tilde{p})} = \tilde{\xi}$. We denote by $\Lambda_\mathcal{X}$ the collection of pairs

$$(3.18) \qquad ((x, p\,; \xi), (y, q\,; \eta)) \in T^*TM \times T^*TM$$

that lie on the same bicharacteristic strip of $\mathcal{X}$.

Lemma 3.1. *Let (M, g) be a globally hyperbolic C^∞-smooth Lorentzian manifold and write it in the standard form $M = \mathbb{R} \times N$ of global time and space. Let $t \in \mathbb{R}$. The geodesic vector field $-i\mathcal{X}$ on $\mathcal{P}M$ is a strictly hyperbolic operator of multiplicity 1 with respect to the submanifold $\mathcal{P}_t M := (T_t\mathbb{R} \times TN) \cap \mathcal{P}M$.*

We omit the proof of the lemma, which is a straightforward verification of the conditionsin [13, Definition 5.1.1].

In this section, the space (M, g) is assumed to be globally hyperbolic C^∞-smooth Lorentzian manifold, but not necessarily geodesically complete. Given the initial data that u vanishes in the past of a Cauchy surface and a source f, we will show that the Vlasov equation $\mathcal{X}u = f$ has a unique solution. There is a substantial amount of literature on this topic (see for example [9], [1], [41], and [40]). For example, Lemma 3.1 together with standard results for hyperbolic Cauchy problems (see e.g. [13, Theorem 5.1.6]) demonstrates uniqueness of solutions to the Vlasov equation.

Let us denote by $\gamma_{(x,p)} : (-T_1, T_2) \to M$ the inextendible geodesic which satisfies

$$\gamma_{(x,p)}(0) = x \text{ and } \dot{\gamma}_{(x,p)}(0) = p. \tag{3.19}$$

Since (M, g) is not necessarily geodesically complete, we might have that $T_1 < \infty$ or $T_2 < \infty$. Existence of solutions to $\mathcal{X}u = f$, with initial data $u = 0$, in the case where $f(x,p)$ is a smooth function on $\overline{\mathcal{P}}^+ M$ with compact support in the base variable $x \in M$, can be shown by checking that

$$u(x,p) := \int_{-\infty}^{0} f(\gamma_{(x,p)}(t), \dot{\gamma}_{(x,p)}(t))dt \quad \text{on} \quad (x,p) \in \overline{\mathcal{P}}^+ M \tag{3.20}$$

is well-defined and satisfies $\mathcal{X}u = f$. Indeed, on a globally hyperbolic Lorentzian manifold for a given compact set $K_\pi \subset M$ and a causal geodesic γ, there are $t_1, t_2 \in \mathbb{R}$ such that for parameter times $t \notin [t_1, t_2]$ we have $\gamma(t) \notin K_\pi$. Thus the integral above is actually over a finite interval. Because f and the geodesic flow on (M, g) are smooth, the function $u(x,p)$ is smooth. If (M, g) is not geodesically complete, and if $\gamma_{(x,p)} : (-T_1, T_2) \to M$, we interpret the integral above to be over $(-T_1, 0]$. We interpret similarly for all similar integrals without further notice.

If $\mathcal{C}$ is a Cauchy surface of M, we write $\mathcal{C}^\pm$ for the causal future $(+)$ or the causal past $(-)$ of $\mathcal{C}$:

$$\mathcal{C}^\pm = J^\pm(\mathcal{C}).$$

In particular, the notation $\overline{\mathcal{P}}^+ \mathcal{C}^\pm$ refers to the subset of TM of future directed causal vectors on the causal future $(+)$ or past $(-)$ of the Cauchy surface $\mathcal{C}$, i.e. $\overline{\mathcal{P}}^+ \mathcal{C}^\pm = \overline{\mathcal{P}}^+ [J^\pm(\mathcal{C})]$ (see (2.9) for more details). If $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ is compact and $k \geq 0$ is an integer, we define the Banach space

$$C_K^k(\overline{\mathcal{P}}^+ \mathcal{C}^+) := \{f \in C^k(\overline{\mathcal{P}}^+ \mathcal{C}^+) : \text{supp}(f) \subset K\},$$

equipped with the norm of C^k.

For our purposes it is convenient to have explicit formulas for solutions to the Vlasov equation and therefore we give a proof using the solution formula (B.109).

Theorem 3.2. *Assume that (M, g) is a globally hyperbolic C^∞-smooth Lorentzian manifold. Let $\mathcal{C}$ be a Cauchy surface of (M, g), $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ be compact and $k \geq 0$. Let also $f \in C_K^k(\overline{\mathcal{P}}^+ M)$. Then, the problem*

$$\mathcal{X}u(x,p) = f(x,p) \quad on \quad \overline{\mathcal{P}}^+ M$$
$$u(x,p) = 0 \qquad on \quad \overline{\mathcal{P}}^+ \mathcal{C}^- \tag{3.21}$$

has a unique solution u in $C^k(\overline{\mathcal{P}}^+ M)$. We write $u = \mathcal{X}^{-1}(f)$ and call $\mathcal{X}^{-1} : C_K^k(\overline{\mathcal{P}}^+ M) \to C^k(\overline{\mathcal{P}}^+ M)$ the solution operator to (3.21). In particular, if $Z \subset \overline{\mathcal{P}}^+ M$ is compact, there is a constant $c_{k,K,Z} > 0$ such that

$$\|u|_Z\|_{C^k(Z)} \leq c_{k,K,Z} \|f\|_{C^k(\overline{\mathcal{P}}^+ M)}. \tag{3.22}$$

If $k = 0$, the estimate above is independent of Z:

$$\|u\|_{C(\overline{\mathcal{P}}^+ M)} \leq c_K \|f\|_{C(\overline{\mathcal{P}}^+ M)}.$$

We have placed the proof of the theorem in Appendix B. The proof follows from the explicit formula (B.109) for the solution. We call $\mathcal{X}^{-1}$ the solution operator to (3.21). The source-to-solution map Φ^L of the Vlasov equation (3.21) is defined as

$$\Phi^L : C_K^k(\overline{\mathcal{P}}^+ M) \to C^k(\overline{\mathcal{P}}^+ M), \quad \Phi^L(f) = u,$$

where u is the unique solution to (3.21) with the source $f \in C_K^k(\overline{\mathcal{P}}^+ M)$ and $k \geq 0$. As we will soon see, the Vlasov equation is the linearization of the Boltzmann equation with a source-to-solution map Φ. We also consider the setting where we relax the condition that the source f above is smooth and show that the solution operator to (3.21) (when considering separately timelike and lightlike vectors) has a unique continuous extension to the class of distribution f, which satisfy $WF(f) \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset$. For the precise statement and proof thereof, please see Appendix B.

3.2. **The Boltzmann model.** The Boltzmann model of particle kinetics in (M, g) modifies the Vlasov model to take into account collisions between particles. This modification is characterized by the (relativistic) *Boltzmann equation*

$$\mathcal{X}u(x,p) - \mathcal{Q}[u,u](x,p) = 0.$$

Here $\mathcal{Q}[\,\cdot\,, \cdot\,]$ is called the *collision operator*. It is explicitly given by

$$(3.23) \qquad \mathcal{Q} : C_c^\infty(\overline{\mathcal{P}}M) \times C_c^\infty(\overline{\mathcal{P}}M) \to C^\infty(\overline{\mathcal{P}}M)$$

$$\mathcal{Q}[u_1, u_2](x,p) = \int_{\Sigma_{x,p}} \left[u_1(x,p)u_2(x,q) - u_1(x,p')u_2(x,q') \right] A(x,p,q,p',q') dV_{x,p}(q,p',q'),$$

where $dV_{x,p}(q,p',q')$ for fixed (x,p) is a volume form defined on

$$(3.24) \qquad \Sigma_{x,p} = \{(p,q,p',q') \in (\overline{\mathcal{P}}_x M)^4 : p + q = p' + q'\} \subset T_x M^4,$$

which is induced by a volume form $dV(x,p,q,p',q')$ on the manifold

$$(3.25) \qquad \Sigma = \bigcup_{(x,p) \in \overline{\mathcal{P}}^+} \Sigma_{x,p}.$$

We call $A = A(x,p,q,p',q')$ the *collision kernel* and assume that it is admissible in the sense of Definition 1.1.

Heuristically, $\mathcal{Q}$ describes the average density of particles with position and momentum (x,p), which are gained and lost from the collision of two particles u_1, u_2. The contribution to the average density from the gained particles is

$$(3.26) \qquad \mathcal{Q}_{gain}[u_1, u_2](x,p) := -\int_{\Sigma_{x,p}} u_1(x,p')u_2(x,q')A(x,p,q,p',q')dV_{x,p}(q,p',q'),$$

and the contribution from the lost particles is

$$(3.27) \qquad \mathcal{Q}_{loss}[u_1, u_2](x,p) := u_1(x,p) \int_{\Sigma_{x,p}} u_2(x,q)A(x,p,q,p',q')dV_{x,p}(q,p',q').$$

The existence and uniqueness to the initial value problem for the Boltzmann equation has been studied in the literature under various assumptions on the geometry of the Lorentzian manifold, properties of the collision kernel and assumptions on the data, see for example [8, 3, 18]. These references consider the Boltzmann equation on L^2-based function spaces.

We consider the following initial value problem for the Boltzmann equation on a globally hyperbolic manifold (M, g) with a source f

$$\mathcal{X}u - \mathcal{Q}[u,u] = f, \quad \text{on } \overline{\mathcal{P}}^+ M$$

$$u = 0, \quad \text{on } \overline{\mathcal{P}}^+ \mathcal{C}^-,$$

where $\mathcal{X}$ is the geodesic vector field on TM, $\mathcal{C}$ is a Cauchy surface of M and $\mathcal{C}^\pm$ denotes the causal future (+)/past (-) of $\mathcal{C}$. We assume that (M, g) is globally hyperbolic and that the collision kernel of $\mathcal{Q}$ is admissible in the sense of Definition 1.1. Our conditions on the collision kernel allow us to consider the Boltzmann equation in the space of continuous functions. We also assume that the sources f are supported in a fixed compact set $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$. Note that

(since $0 \notin \overline{\mathcal{P}}^+ M$) this especially means that f is supported outside a neighbourhood of the zero section of TM. We work with the following function spaces, each equipped with the supremum norm:

$$C_b(\overline{\mathcal{P}}^+ \mathcal{C}^+) := \{h \in C(\overline{\mathcal{P}}^+ \mathcal{C}^+) \, : \, h \text{ is bounded}\},$$

and

$$C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+) := \{h \in C(\overline{\mathcal{P}}^+ \mathcal{C}^+) \, : \, \operatorname{supp}(h) \subset K\}.$$

Theorem 1.2. *Let (M, g) be a globally hyperbolic C^∞-smooth Lorentzian manifold. Let also $\mathcal{C}$ be a Cauchy surface of M and $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ be compact. Assume that $A : \Sigma \to \mathbb{R}$ is an admissible collision kernel in the sense of Definition 1.1.*

There are open neighbourhoods $B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ and $B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$ of the respective origins such that if $f \in B_1$, the Cauchy problem

$$\mathcal{X} u(x, p) - \mathcal{Q}[u, u](x, p) = f(x, p) \quad \text{on } \overline{\mathcal{P}}^+ M$$

(3.28)
$$u(x, p) = 0 \quad \text{on } \overline{\mathcal{P}}^+ \mathcal{C}^-$$

has a unique solution $u \in B_2$. There is a constant $c_{A,K} > 0$ such that

$$\|u\|_{C(\overline{\mathcal{P}}^+ M)} \le c_{A,K} \|f\|_{C(\overline{\mathcal{P}}^+ M)}.$$

We give a proof of Theorem 1.2 in Appendix B. As a direct consequence, we find:

Corollary 3.3. *Assume as in Theorem 1.2 and adopt its notation. The source-to-solution map*

$$\Phi : B_1 \to B_2, \quad \Phi(f) = u$$

is well-defined. Here $u \in B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$ is the unique solution to the Boltzmann equation (3.28) with the source $f \in B_1 \subset C_K(\overline{\mathcal{P}}^+ M)$.

The proof of Corollary 3.3 appears in Appendix B.

Given the existence of the source-to-solution map Φ associated to the Boltzmann equation, we now formally calculate the first and second Frechét differentials of Φ, which will correspond to the first and second linearizations of the Boltzmann equation. Let $\mathcal{C}$ be a Cauchy surface in M, and $f_1, f_2 \in C_c^\infty(\overline{\mathcal{P}}^+ \mathcal{C}^+)$, and consider the 2-parameter family of functions

$$(\epsilon_1, \epsilon_2) \mapsto \Phi(\epsilon_1 f_1 + \epsilon_2 f_2)$$

where ϵ_1, ϵ_2 are small enough so that $\epsilon_1 f_1 + \epsilon_2 f_2 \in B_1$. Formally expanding in ϵ_1 and ϵ_2, we obtain

$$\Phi(\epsilon_1 f_1 + \epsilon_2 f_2) = \epsilon_1 \Phi^L(f_1) + \epsilon_2 \Phi^L(f_2) + \epsilon_1 \epsilon_2 \Phi^{2L}(f_1, f_2) + \text{higher order terms},$$

where the higher order terms tend to zero as $(\epsilon_1, \epsilon_2) \to (0, 0)$ in $C_b(\overline{\mathcal{P}}^+ M)$. Substituting this expansion of $\Phi(\epsilon_1 f_1 + \epsilon_2 f_2)$ into the Boltzmann equation and differentiating in the parameters ϵ_1 and ϵ_2 at $\epsilon_1 = \epsilon_2 = 0$ yields the equations

$$\mathcal{X} \Phi^L(f_j) = f_j, \quad j = 1, 2,$$

and

$$\mathcal{X} \Phi^{2L}(f_1, f_2) = \mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)] + \mathcal{Q}[\Phi^L(f_2), \Phi^L(f_1)].$$

We call these equations the first and second linearizations of the Boltzmann equation. Notice that the first and second linearizations are a Vlasov-type equation (3.21) with a source term.

The next lemma makes the above formal calculation precise. We have placed the proof of the lemma in Appendix B.

Lemma 3.4. *Assume as in Theorem 1.2 and adopt its notation. Let $\Phi : B_1 \to B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$, $B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$, be the source-to-solution map of the Boltzmann equation.*

The map Φ is twice Frechét differentiable at the origin of $C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. If $f, h \in B_1$, then:

(1) *The first Frechét derivative Φ' of the source-to-solution map Φ at the origin satisfies*

$$\Phi'(0; f) = \Phi^L(f),$$

where Φ^L is the source-to-solution map of the Vlasov equation (3.21).

(2) *The second Frechét derivative Φ'' of the source-to-solution map Φ at the origin satisfies*

$$\Phi''(0; f, h) = \Phi^{2L}(f, h),$$

where $\Phi^{2L}(f, h) \in C(\overline{\mathcal{P}}^+ M)$ is the unique solution to the equation

$$
\begin{aligned}
(3.29) \qquad \mathcal{X}\Phi^{2L}(f, h) &= \mathcal{Q}[\Phi^L(f), \Phi^L(h)] + \mathcal{Q}[\Phi^L(h), \Phi^L(f)], \quad &&\text{on } \overline{\mathcal{P}}^+ M, \\
\Phi^{2L}(f, h) &= 0, \quad &&\text{on } \overline{\mathcal{P}}^+ \mathcal{C}^-.
\end{aligned}
$$

We remark that the terms $\mathcal{Q}[\Phi^L(f), \Phi^L(h)] + \mathcal{Q}[\Phi^L(h), \Phi^L(f)]$ in (3.29) might not have compact support in $\overline{\mathcal{P}}^+ M$, and thus unique solvability of (3.29) does not follow directly from Theorem 3.2. However, a unique solution to (3.29) is shown to exist in the proof of the above lemma.

In our main theorem, Theorem 1.3, the measurement data consists of solutions to the Boltzmann equation restricted to our measurement set V. By Lemma 3.4 above, we obtain that the measurement data also determines the solutions to the first and second linearization of the Boltzmann equation restricted to V. We will see from (3.29) that the second linearization Φ^{2L} captures information about the (singular) behaviour of the collision term $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)]$. In the next section we will analyze the microlocal behaviour of $\mathcal{Q}[\Phi^L(f_1), \Phi^L(f_2)]$. Then, in Section 5, we use this analysis to recover information about when particles collide in the unknown region W. From such particle interactions in W, we will parametrize points in the unknown set W by light signals measured in V, which are obtained by restricting Φ^{2L} to lightlike vectors.

4. Microlocal analysis of particle interactions

In this section we consider the gain term $\mathcal{Q}_{gain}$ of the collision operator and prove that we can extend $\mathcal{Q}_{gain}$ to conormal distributions over a certain class of submanifolds in $\mathcal{P}M$.

Definition 4.1. *[Admissible intersection property] We say that submanifolds $Y_1 \subset \mathcal{P}M$ and $Y_2 \subset \mathcal{P}M$ have an admissible intersection property at $x_0 \in \pi(Y_1) \cap \pi(Y_2)$ if there exists an open neighbourhood $U_{x_0} = U \subset M$ of x_0 and coordinates*

$$(4.30) \qquad x = (x', x'') : U \to \mathbb{R}^n, \quad x' = (x^1, \ldots, x^d), \quad x'' = (x^{d+1}, \ldots, x^n)$$

centered at x_0 such that

$$
\begin{aligned}
(4.31) \qquad Y_1 \cap TU &= \{(x, p) \in TU : x' = 0, \ p' = 0, \ p'' = (1, 0, \ldots, 0)\} \\
Y_2 \cap TU &= \{(x, p) \in TU : x'' = 0, \ p'' = 0, \ p' = (1, 0, \ldots, 0)\}
\end{aligned}
$$

in the associated canonical coordinates $(x, p) = (x', x'', p', p'')$ of TU. Moreover, for $X \subset M$ we say that the pair Y_1 and Y_2 has the admissible intersection property in X if either the property holds at every $x_0 \in \pi(Y_1) \cap \pi(Y_2) \cap X$ or $\pi(Y_1) \cap \pi(Y_2) \cap X = \emptyset$.

Given a neighbourhood U as above we define the following conic Lagrangian submanifolds:

$$\Lambda_0 := N^*\{(0, 0)\} = T^*_{x_0}\mathbb{R}^n, \quad \Lambda_1 := N^*\{x \in U : x' = 0\} \quad \text{and} \quad \Lambda_2 := N^*\{x \in U : x'' = 0\}.$$

We remark that our statement of Definition 4.1 is motivated by the following case which arises in our proof of Theorem 1.3. Let $V \subset M$ be as in Theorem 1.3. In the proof of the theorem we will consider two delta-type distributions supported over a submanifold $S_1 \subset \mathcal{P}^+V$ of dimension $(n-2)$ and a single point space $S_2 = \{(x,p)\} \subset \mathcal{P}^+V$. The submanifold S_1 is constructed in Corollary 4.6 using Lemma 4.5 so that the geodesic flowouts $Y_1 := K_{S_1}$ and $Y_2 := K_{S_2}$ have admissible intersection property in $\pi \mathrm{supp} A$ and hence transverse intersection when projected to the base manifold M in both settings $(M,g) = (M_j, g_j)$, $j = 1,2$ simultaneously. This intersection is a finite number of discrete points. Moreover, due to admissible intersection property, the fibre component $T_x M \cap Y_j$ of Y_j at each point x sufficiently near an intersection point $x_0 \in \pi Y_1 \cap \pi Y_2 \cap (\pi \mathrm{supp} A)$ is a single vector (the vector $(1,0,\ldots,0)$ in coordinates) and hence the map $\pi : TM \to M$ defines diffeomorphism from Y_j to the projection $G_{S_j} := \pi K_{S_j}$ in that neighbourhood for both $j = 1,2$. Globally the set G_{S_1} (also G_{S_2} if S_2 is not required to be a single point space) may fail to be manifold due to caustic effects.

Our goal is to extend the operator $\mathcal{Q}_{gain}[\,\cdot\,,\,\cdot\,] : C_c^\infty(\mathcal{P}M) \times C_c^\infty(\mathcal{P}M) \to C^\infty(\overline{\mathcal{P}}M)$ to conormal distributions over submanifolds of $\mathcal{P}M$, such as those described above. The analysis of such an extension requires microlocal analysis on manifolds with boundary, since $\overline{\mathcal{P}}M$ has a boundary given by the collection of lightlike vectors. The analysis of distributions over manifolds with boundary can be involved and technical. We are able to avoid difficulties related to manifolds with boundary by introducing an auxiliary vector field and composing it with $\mathcal{Q}_{gain}$. This is explained next.

Fix an open set $U \subset M$ and a smooth vector field $P : U \to \overline{\mathcal{P}}U$. Let $u, v \in C_c^\infty(\mathcal{P}M)$, and let $x \in U$. We define the operator $\mathcal{Q}_{gain}^P : C_c^\infty(\mathcal{P}M) \times C_c^\infty(\mathcal{P}M) \to C^\infty(U)$ as

$$(4.32) \qquad \mathcal{Q}_{gain}^P[u,v](x) = \mathcal{Q}_{gain}[u,v](P(x)).$$

We will see in this section that we are able to analyze $\mathcal{Q}_{gain}^P$ operator by using standard techniques such as those in [13]. The analysis of $\mathcal{Q}_{gain}^P$ presented in this section will be used to study the singular structure of solutions $\Phi_{L+V} f$ to (1.2) for given sources f which are constructed in Section 5. We note now that in Section 5, we we will choose a specific P, which we will denote by P_e.

We first record a couple of auxiliary lemmas. The first lemma considers the conormal bundle of a submanifold of $U \times \mathcal{P}U \times \mathcal{P}U$. The points of $U \times \mathcal{P}U \times \mathcal{P}U$ are denoted by $(x, y, z, p, q) = (x, (y,p), (z,q))$.

Lemma 4.2. *Let Y_1 and Y_2 and U be as in Definition 4.1 and adopt also the associated notation. Define*

$$\Lambda_R = N^*\Big(\bigcup_{x \in U} \{x\} \times \mathcal{P}_x U \times \mathcal{P}_x U \Big).$$

The submanifold Λ_R of $T^*(U \times \mathcal{P}U \times \mathcal{P}U) \setminus \{0\}$ equals the set

$$\Lambda_R = \big\{ \big(x, y, z, p, q \,;\, \xi^x, \xi^y, \xi^z, \xi^p, \xi^q \big) \in T^*(U \times \mathcal{P}U \times \mathcal{P}U) \setminus \{0\} :$$
$$\xi^x + \xi^y + \xi^z = 0, \ \xi^p = \xi^q = 0, \ x = y = z \}.$$

We have

$$\Lambda_R' = \{ \big((x; \xi^x), (y, z, p, q \,;\, \xi^y, \xi^z, \xi^p, \xi^q)\big) \in \ T^*U \times T^*(\mathcal{P}U \times \mathcal{P}U) \setminus \{0\} :$$
$$(4.33) \qquad \xi^x = \xi^y + \xi^z \neq 0, \ \xi^p = \xi^q = 0, \ x = y = z \}.$$

The spaces $\Lambda_R' \times (N^*[Y_1 \times Y_2])$ and $T^*U \times \mathrm{diag}\, T^*(\mathcal{P}M \times \mathcal{P}M)$ intersect transversally in $T^*U \times T^*(\mathcal{P}M \times \mathcal{P}M) \times T^*(\mathcal{P}M \times \mathcal{P}M)$.

Please see Appendix A.1 for a proof.

For the next theorem let $Y_1, Y_2 \subset \mathcal{P}M$ be smooth manifolds which satisfy the properties in Definition 4.1. Fix $x_0 \in \pi(Y_1) \cap \pi(Y_2) \cap X$ (below we choose $X = \pi\mathrm{supp}A$) and let Λ_j, $j = 1, 2$, be the Lagrangian manifolds specified in the definition. In canonical coordinates $(x, \xi) = (x', x'', \xi', \xi'')$ on T^*U we obtain the expressions

$$\Lambda_0 = \{(x, \xi) : x' = 0, \ x'' = 0\}, \tag{4.34}$$

$$\Lambda_1 = \{(x, \xi) : x' = 0, \ \xi'' = 0\}, \tag{4.35}$$

$$\Lambda_2 = \{(x, \xi) : x'' = 0, \ \xi' = 0\}. \tag{4.36}$$

Thus, the elements of the manifolds $\Lambda_0, \Lambda_1, \Lambda_2$ can be parametrized by the free coordinates (ξ', ξ''), (x'', ξ'), and (x', ξ''), respectively. In the canonical coordinates
$(x, p; \xi^x, \xi^p) = (x', x'', p', p''; (\xi^x)', (\xi^x)'', (\xi^p)', (\xi^p)'')$ in T^*TU we have that

$$N^*Y_1 = \{(x, p; \xi^x, \xi^p) : x' = 0, \ p' = 0, \ p'' = (1, 0, \ldots, 0), \ (\xi^x)'' = 0\}, \tag{4.37}$$

$$N^*Y_2 = \{(x, p; \xi^x, \xi^p) : x'' = 0, \ p'' = 0, \ p' = (1, 0, \ldots, 0), \ (\xi^x)' = 0\}, \tag{4.38}$$

so the manifolds N^*Y_1 and N^*Y_2 are locally parametrized by the coordinates $(x'', (\xi^x)', \xi^p)$ and $(x', (\xi^x)'', \xi^p)$ respectively. The identity (4.40) in the next theorem is written in terms of the coordinates $(x'', (\xi^x)', \xi^p)$, $(x', (\xi^x)'', \xi^p)$ and $((\xi^x)', (\xi^x)'')$ for the symbols $\sigma(f_1)$, $\sigma(f_2)$ and $\sigma(\mathcal{Q}_{gain}^P[f_1, f_2])$ respectively.

We extend $\mathcal{Q}_{gain}^P$ to conormal distributions as follows.

Theorem 4.3. *Let (M, g) be a globally hyperbolic Lorentzian manifold. Let $Y_1, Y_2 \subset \mathcal{P}M$, be smooth manifolds that have the admissible intersection property (Definition 4.1) at some $x_0 \in \pi Y_1 \cap \pi Y_2$ and let Λ_l, $l = 0, 1, 2$, and $U \subset M$ be as in Definition 4.1. Let P be a smooth section of the bundle $\pi : \overline{\mathcal{P}}U \to U$ and $\mathcal{Q}$ be the collision operator with an admissible collision kernel.*

Then the operator $\mathcal{Q}_{gain}^P : C_c^\infty(\mathcal{P}M) \times C_c^\infty(\mathcal{P}M) \to C^\infty(U)$ defined in (4.32) extends into a continuous map

$$I_{comp}^{m_1}(\mathcal{P}M; N^*Y_1) \times I_{comp}^{m_2}(\mathcal{P}M; N^*Y_2) \to \mathcal{D}'(U), \quad m_1, m_2 \in \mathbb{R}.$$

For $(f_1, f_2) \in I_{comp}^{m_1}(\mathcal{P}M; N^*Y_1) \times I_{comp}^{m_2}(\mathcal{P}M; N^*Y_2)$ we have that

$$WF(\mathcal{Q}_{gain}^P[f_1, f_2]) \subset \Lambda_0 \cup \Lambda_1 \cup \Lambda_2.$$

Moreover, microlocally away from both Λ_1 and Λ_2, we have that

$$\mathcal{Q}_{gain}^P[f_1, f_2] \in I^{m_1+m_2+3n/4}(U; \Lambda_0 \setminus (\Lambda_1 \cup \Lambda_2)) \tag{4.39}$$

together with the symbol

$$\sigma(\mathcal{Q}_{gain}^P[f_1, f_2])((\xi^x)', (\xi^x)'') = C\sigma(f_1)(0; (\xi^x)', 0)\sigma(f_2)(0; (\xi^x)'', 0), \tag{4.40}$$

where C is given in the canonical coordinates $p = (p', p'')$ and $q = (q', q'')$ by

$$C = cA(x_0, \hat{p}(x_0), p + q - \hat{p}(x_0), p, q)\big|_{p=(1,0,\ldots,0), \ q=(1,0,\ldots,0)},$$

and $\hat{p}(x_0) \in \overline{\mathcal{P}}_{x_0}U$ is such that $P(x_0) = (x_0, \hat{p}(x_0))$. Here c is some non-zero constant and the manifolds $\Lambda_0 \setminus (\Lambda_1 \cup \Lambda_2)$, $T^*U \cap N^*Y_1$, and $T^*U \cap N^*Y_2$ are parametrised by the canonical coordinates $((\xi)', (\xi^x)'')$, $(x'', (\xi^x)', \xi^p)$ and $(x', (\xi^x)'', \xi^p)$ respectively.

Proof of Theorem 4.3. We write $I^m(\mathcal{P}M; \Lambda) = I^m(\Lambda)$ for a Lagrangian manifold $\Lambda \subset T^*(\mathcal{P}M)$ and $m \in \mathbb{R}$. To prove the claims of the proposition, we first represent $\mathcal{Q}^P_{gain}[f_1, f_2]$ as a composition of a Fourier integral operator and a tensor product of f_1 and f_2. Then we appeal to results for Fourier integral operators in [13] to conclude the proof. The majority of the proof consists of demonstrating that the Fourier integral operators we use to decompose $\mathcal{Q}^P_{gain}[f_1, f_2]$ satisfy the conditions required by [13, Theorem 2.4.1] and [13, Corollary 1.3.8].

Let $(x, p) = (x', x'', p', p'')$ be local coordinates in TU, where U is an open neighbourhood of x_0, as described in Definition 4.1. Let us define an integral operator

$$R : C^\infty_c(\mathcal{P}M \times \mathcal{P}M) \to \mathcal{D}'(U)$$

by the formula

$$\langle R[\phi], \psi \rangle = \int_{\mathcal{P}U \oplus \mathcal{P}U} \phi(x, p, x, q) \psi(x) \, dv(x, p, q)$$

where dv is the induced volume form on the direct sum bundle $\mathcal{P}U \oplus \mathcal{P}U := \{((x, p), (y, q)) \in \mathcal{P}U \times \mathcal{P}U : x = y\}$ and where $\phi \in C^\infty_c(\mathcal{P}M \times \mathcal{P}M)$ and $\psi \in C^\infty_c(U)$. The induced volume form dv is given by considering $\mathcal{P}U \oplus \mathcal{P}U$ as a submanifold of $\mathcal{P}U \times \mathcal{P}U$ equipped with product Liouville volume form.

Let P be a smooth local section of the bundle $\pi : \overline{\mathcal{P}U} \to U$. We represent P as $P(x) = (x, \hat{p}(x))$ by using the coordinates x. Next, consider the operator

$$\langle R_A[\phi], \psi \rangle = \int_{\mathcal{P}U \oplus \mathcal{P}U} A(x, \hat{p}(x), p + q - \hat{p}(x), p, q) \phi(x, p, x, q) \psi(x) \, dv(x, p, q)$$

Notice that

(4.41) $$R_A[h_1 \otimes h_2] = \mathcal{Q}^P_{gain}[h_1, h_2], \quad h_1, h_2 \in C^\infty_c(\mathcal{P}M).$$

Here $h_1 \otimes h_2$ is the tensor product of h_1 and h_2, i.e. $(h_1 \otimes h_2)((x, p), (y, q)) = h_1(x, p)h_2(y, q)$. The distribution kernel associated to the operator R is

$$k_R \in \mathcal{D}'(U \times \mathcal{P}M \times \mathcal{P}M).$$

Since k_R is a delta distribution over the submanifold

$$\bigcup_{x \in U} \{x\} \times \mathcal{P}_x M \times \mathcal{P}_x M \subset U \times \mathcal{P}M \times \mathcal{P}M,$$

it can be viewed as a conormal distribution

$$k_R \in I^{-n/4}(U \times \mathcal{P}M \times \mathcal{P}M; \Lambda_R),$$
$$\Lambda_R := N^* \Big(\bigcup_{x \in U} \{x\} \times \mathcal{P}_x M \times \mathcal{P}_x M \Big).$$

Hence R is a Fourier integral operator of class $I^{-n/4}(U, \mathcal{P}M \times \mathcal{P}M, \Lambda'_R)$. Since the collision kernel A is smooth, R_A is also of class $I^{-n/4}(U, \mathcal{P}M \times \mathcal{P}M, \Lambda'_R)$. By Lemma 4.2, the Lagrangian manifold Λ_R is given by

$$\Lambda_R = \{ (x, y, z, p, q; \xi^x, \xi^y, \xi^z, \xi^p, \xi^q) \in T^*(U \times \mathcal{P}U \times \mathcal{P}U) \setminus \{0\} :$$
$$\xi^x + \xi^y + \xi^z = 0, \ \xi^p = \xi^q = 0, \ x = y = z \}.$$

Further, we have that $WF'(R_A) \subset \Lambda'_R$, where Λ'_R equals by its definition (2.10) the set

$$\Lambda'_R = \{ ((x; \xi^x), (y, z, p, q; \xi^y, \xi^z, \xi^p, \xi^q)) \in T^*U \times T^*(\mathcal{P}U \times \mathcal{P}U) \setminus \{0\} :$$
(4.42) $$\xi^x = \xi^y + \xi^z \neq 0, \ \xi^p = \xi^q = 0, \ x = y = z \}.$$

Now we show that $\mathcal{Q}_{gain}^P[\cdot,\cdot]$ can be extended to the set of compactly supported conormal distributions $I_{comp}^{m_1}(N^*Y_1) \times I_{comp}^{m_2}(N^*Y_2)$, for $m_1, m_2 \in \mathbb{R}$. In fact, by [13, Corollary 1.3.8], it is sufficient to demonstrate that

$$(4.43) \qquad WF'_{\mathcal{P}M \times \mathcal{P}M}(R_A) \cap WF(f_1 \otimes f_2) = \emptyset.$$

for $(f_1, f_2) \in I_{comp}^{m_1}(N^*Y_1) \times I_{comp}^{m_2}(N^*Y_2)$. Since the wavefront set of $f_j \in I^m(N^*Y_j)$, $j = 1, 2$ is contained in N^*Y_j, we have by [13, Proposition 1.3.5] that the wavefront set of the tensor product $f_1 \otimes f_2$ satisfies

$$WF(f_1 \otimes f_2) \subset (N^*Y_1 \times \{0\}_{\mathcal{P}M}) \cup (\{0\}_{\mathcal{P}M} \times N^*Y_2) \cup (N^*Y_1 \times N^*Y_2),$$

where $\{0\}_{\mathcal{P}M} \subset T^*(\mathcal{P}M)$ is the zero bundle over $\mathcal{P}M$. By using the fact that $WF(k_R) \subset \Lambda_R$ and the equation (4.42), the set $WF'_{\mathcal{P}M \times \mathcal{P}M}(R) \subset T^*(\mathcal{P}U \times \mathcal{P}U) \setminus \{0\}$, defined in (2.13), satisfies

$$WF'_{\mathcal{P}M \times \mathcal{P}M}(R) = \{(y, z, p, q; \xi^y, \xi^z, \xi^p, \xi^q) \in T^*(\mathcal{P}M \times \mathcal{P}M) \setminus \{0\} :$$
$$\text{there is } x \in U \text{ such that } ((x; 0), (y, z, p, q; \xi^y, \xi^z, \xi^p, \xi^q)) \in WF(k_R)\}$$
$$(4.44) \qquad \subset \{(x, x, p, q; \xi^y, -\xi^y, 0, 0) \in T^*(\mathcal{P}M \times \mathcal{P}M) \setminus \{0\} : p, q \in \mathcal{P}_x U, \ \xi^y \in \mathbb{R}^n \setminus \{0\}\}.$$

Since Y_1 and Y_2 satisfy Definition 4.1, pairs of elements in the fibers of N^*Y_1 and N^*Y_2 are linearly independent. They are also non-zero by the definition of a normal bundle. Thus an element of $N^*Y_1 \times N^*Y_2$ cannot be of the form $(x, x, p, q; \xi^y, -\xi^y, 0, 0)$. We deduce that $WF'_{\mathcal{P}M \times \mathcal{P}M}(R) \cap (N^*Y_1 \times N^*Y_2) = \emptyset$. By a similar consideration, we see that the set in (4.44) does not intersect $(N^*Y_1 \times \{0\}_{\mathcal{P}M}) \cup (\{0\}_{\mathcal{P}M} \times N^*Y_2)$. In particular, we have (4.43). The set $WF'_U(R) \subset T^*U \setminus \{0\}$, defined in (2.12), satisfies

$$WF'_U(R) = \{(x, \xi^x) \in T^*U : \text{ there is}$$
$$(y, p, z, q) \in \mathcal{P}U \times \mathcal{P}U \text{ such that } ((x; \xi^x), (y, z, p, q; 0, 0, 0, 0)) \in WF(k_R)\}.$$

By using the fact that $WF(k_R) \subset \Lambda_R$ and (4.42), we see that if $((x; \xi^x), (y, z, p, q; 0, 0, 0, 0)) \in WF(k_R)$, then $\xi^x = 0$. Thus $WF'_U(R) = \emptyset$. By [13, Corollary 1.3.8] the composition $R_A \circ (f_1 \otimes f_2)$ is well defined and we obtain the desired continuous extension for $\mathcal{Q}_{gain}^P[f_1, f_2] \in \mathcal{D}'(U)$. The second claim of the proposition follows directly from [13, Corollary 1.3.8] and the facts $WF'_U(R_A) \subset WF'_U(R) = \emptyset$ and $WF(R_A) \subset \Lambda'_R$:

$$(4.45) \quad WF(\mathcal{Q}_{gain}^P[f_1, f_2]) \subset \Lambda'_R \circ \Big((N^*Y_1 \times \{0\}_{\mathcal{P}M}) \cup (\{0\}_{\mathcal{P}M} \times N^*Y_2) \cup (N^*Y_1 \times N^*Y_2)\Big).$$

By using the coordinate description (4.42), we show next that

$$(4.46) \qquad \Lambda'_R \circ (N^*Y_1 \times \{0\}_{\mathcal{P}M}) = \Lambda_1, \quad \Lambda'_R \circ (\{0\}_{\mathcal{P}M} \times N^*Y_2) = \Lambda_2.$$

By definition (see (2.14)), we have that

$$\Lambda'_R \circ (N^*Y_1 \times \{0\}_{\mathcal{P}M}) = \{(x, \xi^x) \in T^*U : \text{ such that } ((x, \xi^x), (y, p; \xi^y, \xi^p), (z, q; \xi^z, \xi^q)) \in \Lambda'_R$$
$$(4.47) \qquad\qquad\qquad \text{where } (y, p; \xi^y, \xi^p) \in N^*Y_1 \text{ and } \xi^z = \xi^q = 0\}.$$

By using (4.42), it follows that in the expression above, we must have that (x, ξ^x) is any element of the form (y, ξ^y), where $(y, p; \xi^y, 0) \in N^*Y_1$ for some p. Thus we have $\Lambda'_R \circ (N^*Y_1 \times \{0\}_{\mathcal{P}M}) = \Lambda_1$. We similarly have $\Lambda'_R \circ (\{0\}_{\mathcal{P}M} \times N^*Y_2) = \Lambda_2$. We have proven (4.46).

By using the coordinates $(x, p) = (x', x'', p', p'')$ we may also write any $(x, \xi) \in T^*U$ as $(x, \xi) = (x', x'', \xi', \xi'')$. We have that

$$\Lambda_1 = \{(x; \xi) \in T^*U : x' = 0, \ \xi'' = 0\},$$
$$\Lambda_2 = \{(x; \xi) \in T^*U : x'' = 0, \ \xi' = 0\}.$$

We calculate similarly as in (4.47)

$$\Lambda_R' \circ (N^*Y_1 \times N^*Y_2) = \{(x, \xi^y + \xi^z) \in T^*U : \text{ such that } ((x, \xi^x), (x, p; \xi^y, 0), (x, q; \xi^z, 0)) \in \Lambda_R'$$
$$\text{where } (x, p; \xi^y, 0) \in N^*Y_1 \text{ and } (x, q; \xi^z, 0) \in N^*Y_2\}$$

(4.48)
$$= \{(x', x''; \xi', \xi'') \in T^*U \setminus \{0\} : x' = x'' = 0\} = \Lambda_0.$$

Here we used again (4.42). By combining (4.45), (4.47) and (4.48), we have shown that

$$WF(\mathcal{Q}_{gain}^P[f_1, f_2]) \subset \Lambda_0 \cup \Lambda_1 \cup \Lambda_2.$$

We are left to show the last claims of the theorem. Fix arbitrary conic neighbourhoods Γ_1, Γ_2 of Λ_1 and Λ_2, respectively. Let $\epsilon > 0$ be small enough to satisfy

(4.49)
$$\left\{(x, \xi) \in TU : \|x'\| < \epsilon, \ \|\xi''\| < \epsilon\|\xi'\|\right\} \subset \Gamma_1$$

and

$$\left\{(x, \xi) \in TU : \|x''\| < \epsilon, \|\xi'\| < \epsilon\|\xi''\|\right\} \subset \Gamma_2.$$

By multiplying the amplitude in the oscillatory integral representation of $f_1 \otimes f_2$ by

$$1 = \phi + (1 - \phi),$$

where $\phi \in C^\infty(T^*(\mathcal{P}M \times \mathcal{P}M) \setminus \{0\})$ is positively homogeneous of degree 0 and equals 1 near $N^*[Y_1 \times \mathcal{P}M] = N^*Y_1 \times \{0\}_{\mathcal{P}M}$ and 0 near $N^*[\mathcal{P}M \times Y_2] = \{0\}_{\mathcal{P}M} \times N^*Y_2$, we write

$$f_1 \otimes f_2 \equiv v_1 + v_2$$

where

(4.50)
$$v_1 \in I_{comp}^{m_1+m_2+n,-m_2-n}\left(N^*Y_1 \times \{0\}_{\mathcal{P}M}, N^*[Y_1 \times Y_2]\right),$$

(4.51)
$$v_2 \in I_{comp}^{m_1+m_2+n,-m_1-n}\left(\{0\}_{\mathcal{P}M} \times N^*Y_2, N^*[Y_1 \times Y_2]\right),$$

To prove (4.39) it is sufficient to show that for both $j = 1, 2$ there is a decomposition $R_A \circ v_j = u_j + r_j$, where u_j is a Lagrangian distribution over Λ_0 and $WF(r_j) \subset \Gamma_j$. We only consider the v_1 component of $f_1 \otimes f_2$ and write $v = v_1$. The argument for the other component is similar. Fix small $\delta_1, \delta_2 \in (0, 1)$ with $\frac{2\delta_1}{1-\delta_2} < \epsilon$, and let $\psi \in C^\infty(T^*(\mathcal{P}M \times \mathcal{P}M) \setminus \{0\})$ be positively homogeneous of degree 0 such that it equals 1 on the conic neighbourhood

$$X_1 := \left\{(x, p, y, q; \xi^x, \xi^p, \xi^y, \xi^q) \in T^*(\mathcal{P}U \times \mathcal{P}U) \setminus \{0\} : \|(\xi^x)''\| < \frac{\delta_1}{2}\|(\xi^x)'\|, \right.$$
$$\left. \|x'\| < \frac{\epsilon}{2}, \ \|(\xi^y)''\| < \frac{\delta_1}{2}\|(\xi^x)'\|, \ \|(\xi^y)'\| < \frac{\delta_2}{2}\|(\xi^x)'\|\right\}$$

of $N^*Y_1 \times \{0\}_{\mathcal{P}M}$ and vanishes in the exterior of the larger neighbourhood

$$X_2 := \left\{(x, p, y, q; \xi^x, \xi^p, \xi^y, \xi^q) \in T^*(\mathcal{P}U \times \mathcal{P}U) \setminus \{0\} : \|(\xi^x)''\| < \delta_1\|(\xi^x)'\|, \right.$$
$$\left. \|x'\| < \epsilon, \ \|(\xi^y)''\| < \delta_1\|(\xi^x)'\|, \|(\xi^y)'\| < \delta_2\|(\xi^x)'\|\right\}.$$

By dividing the amplitude in the oscillatory integral according to $1 = \psi + (1 - \psi)$ we obtain the decomposition $v = v' + v''$, where $v' \in I_{comp}^{m_1+m_2+n}(N^*[Y_1 \times Y_2])$ and $WF(v'') \subset X_2$. Moreover,

$$\sigma(v') = (1 - \psi)\sigma(v) = \sigma(v) = c\phi(\sigma(f_1) \otimes \sigma(f_2)) \quad \text{on} \quad (N^*[Y_1 \times Y_2]) \setminus X_2.$$

Applying [13, Corollary 1.3.8] and $WF(v'') \subset X_2$ we deduce

$$WF(R_A \circ v'') \subset WF'(R_A) \circ WF(v'') \cup WF_U'(R_A) \subset (\pi \text{supp}A) \cap (\Lambda_R' \circ WF(v''))$$
$$\subset (\pi \text{supp}A) \cap (\Lambda_R' \circ X_2)$$

By definition, an element (x, η) in $(\Lambda_R' \circ X_2)$ satisfies $\|x'\| < \epsilon$ and

$$\eta = \xi^x + \xi^y, \quad \|(\xi^x)''\| < \delta_1\|(\xi^x)'\|, \ \|(\xi^y)''\| < \delta_1\|(\xi^x)'\|, \ \|(\xi^y)'\| < \delta_2\|(\xi^x)'\|.$$

By using the triangle inequality and the inequalities above one computes

$$\frac{\|\eta''\|}{\|\eta'\|} \leq \frac{\|(\xi^x)''\| + \|(\xi^y)''\|}{\|\|(\xi^x)'\| - \|(\xi^y)'\|\|} < \frac{2\delta_1}{1 - \delta_2} < \epsilon$$

which implies $\|\eta''\| < \epsilon\|\eta'\|$. Thus, by (4.49), we conclude

$$WF(R_A \circ v'') \subset \Gamma_1$$

To finish the proof we are left to show that the conditions of [13, Theorem 2.4.1] (cf. the global formulation [13, Theorem 4.2.2]) for the composition of v' and R_A are satisfied. The symbol identity (4.40) follows from [13, eq. (4.2.10)]. The condition [13, eq. (2.4.8)] is satisfied since v' is compactly supported. The conditions [13, eq. (2.4.9), (2.4.10)] follow from the definitions (4.31) and (4.42). The last condition [13, eq. (2.4.11)] follows from Lemma 4.33. Since the conditions of [13, Theorem 2.4.1] are met, we have that $R_A \circ v'$ is a well-defined oscillatory integral of order $m_1 + m_2 + 3n/4$ with the canonical relation $\Lambda'_R \circ (N^*(Y_1 \times Y_2)) = \Lambda_0$. In conclusion, for arbitrary conic neighbourhoods Γ_1, Γ_2 of Λ_1 and Λ_2 there is the decomposition

$$\mathcal{Q}^P_{gain}[f_1, f_2] = u + r, \quad u := R_A \circ (v'_1 + v'_2) \in I^{m_1+m_2+3n/4}_{comp}(\Lambda_0), \quad r := R_A \circ (v''_1 + v''_2),$$

such that $WF(r) \subset \Gamma_1 \cup \Gamma_2$ and the symbol identity (4.40) holds on $\Lambda_0 \setminus (\Gamma_1 \cup \Gamma_2)$. $\quad\square$

Theorem 4.3 together with the microlocal properties of the geodesic vector field yields the following corollary:

Corollary 4.4 (Extension to distributions solving Vlasov's equation)**.** *Let (M, g) be a globally hyperbolic Lorentzian manifold and let $\mathcal{C}$ be a Cauchy surface of (M, g). Let $S_1, S_2 \subset \mathcal{P}^+\mathcal{C}$ be smooth manifolds such that the geodesic flowouts $Y_1 := K_{S_1}$ and $Y_2 := K_{S_2}$ have an admissible intersection property (see Definition 4.1) in $\pi supp A$. Let U be a small neighbourhood of some $x_0 \in \pi Y_1 \cap \pi Y_2 \cap \pi supp A$ as in the definition of admissible intersection, let $P : U \to \overline{\mathcal{P}}^+U$ be a smooth section of $\overline{\mathcal{P}}^+U$ and assume additionally that $S_1, S_2 \subset \mathcal{P}M \setminus \mathcal{P}\overline{U}$. Additionally, for $f_j \in I^{l_j}_{comp}(\mathcal{P}M; N^*S_j)$, $j = 1, 2$, let $u(f_j)$ solve the Vlasov's equation with source f_j and which vanish in $\mathcal{C}^-$.*

Then, the operator $\mathcal{Q}^P_{gain}[u(\,\cdot\,), u(\,\cdot\,)]$, defined in (4.32), defines a continuous map

$$I^{l_1}_{comp}(\mathcal{P}M; N^*S_1)) \times I^{l_2}_{comp}(\mathcal{P}M; N^*S_2) \to \mathcal{D}'(U).$$

Moreover, microlocally away from both Λ_1 and Λ_2, we have that

$$(4.52) \qquad \mathcal{Q}^P_{gain}[u(f_1), u(f_2)] \in I^{l_1+l_2+3n/4-1/2}(U; \Lambda_0 \setminus (\Lambda_1 \cup \Lambda_2))$$

together with the symbol

$$(4.53) \qquad \sigma(\mathcal{Q}^P_{gain}[u(f_1), u(f_2)])(\xi', \xi'') = C\sigma(u(f_1))(0; \xi', 0)\sigma(u(f_2))(0; \xi'', 0),$$

where the constant C is given in the canonical coordinates $p = (p', p'')$ and $q = (q', q'')$ by

$$C = cA(x_0, \hat{p}(x_0), p + q - \hat{p}(x_0), p, q)\big|_{p=(1,0,\dots,0),\ q=(1,0,\dots,0)},$$

and $\hat{p}(x_0) \in \overline{\mathcal{P}}_{x_0}U$ is such that $P(x_0) = (x_0, \hat{p}(x_0))$. Here c is some non-zero constant and the manifolds $\Lambda_0 \setminus (\Lambda_1 \cup \Lambda_2)$, $T^*U \cap N^*Y_1$, and $T^*U \cap N^*Y_2$ are parametrised by the canonical coordinates (ξ', ξ''), $(x''; (\xi^x)', \xi^p)$, and $(x'; (\xi^x)'', \xi^p)$, respectively.

Proof. For $j = 1, 2$, let $f_j \in I^{l_j}_{comp}(N^*S_j; \mathcal{P}M)$. Choose a cut-off function $\chi \in C^\infty_c(M)$ so that $\chi = 1$ on $U \cap \operatorname{supp}(x \mapsto A(x, \cdot))$ and $\chi = 0$ on a neighbourhood of $\pi(S_1 \cup S_2) \subset M \setminus \overline{U}$
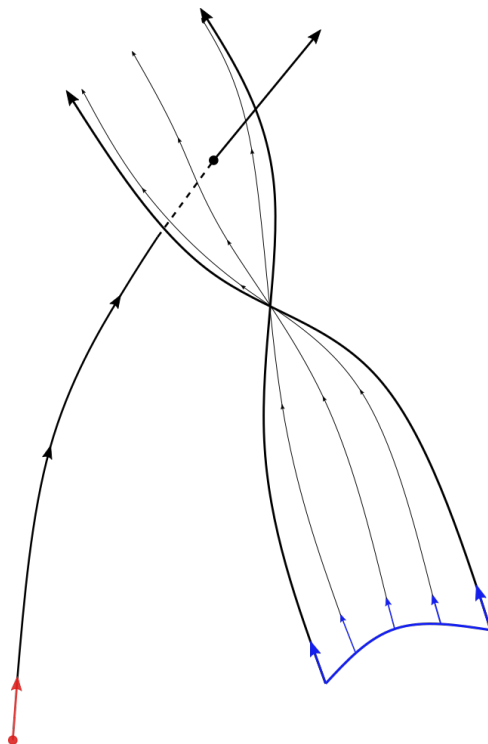
FIGURE 2. An admissible intersection of K_S and a single geodesic path $K_{(x,p)}$. The initial vector (x,p) and the manifold S are indicated in red and blue, respectively. Near the intersection point G_S behaves as a manifold transversal to $\gamma_{(x,p)}$.

Now, for each $j = 1, 2$, by Lemma B.3, there exists a solution $u(f_j)$ to the Vlasov equation with source f_j and initial data 0. Moreover, as $Y_j = K_{S_j}$, the sources f_j are compactly supported and time-like geodesics can not be trapped we get $\chi u(f_j) \in I_{comp}^{l_j-1/4}(\mathcal{P}M; N^*Y_j)$.

Substituting $u = \chi u + (1 - \chi)u$ into $\mathcal{Q}_{gain}$, we obtain that

$$\mathcal{Q}_{gain}^P[u(f_1), u(f_2)] \equiv \mathcal{Q}_{gain}^P[\chi u(f_1), \chi u(f_2)].$$

Thus we have reduced to the setting of Proposition 4.3 and obtain the desired results.

$\square$

4.1. **Existence of transversal collisions.** In this section, we construct a particular submanifold S of TM. In the proof of our main theorem, Theorem 1.3, we will construct particle sources in a common open set V of two manifolds $M = M_l$, $l = 1, 2$ such that they send information into the unknown region $W_l \subset M_l$ to create point singularities produced by using the nonlinearity. We use the source-to-solution map to study the propagation of that singularity. The idea is that for two time-like future pointing vectors with distinct base-points in V we build a manifold S around one of the vectors such that the flowouts $Y_1 = Y_{1,M_l,g_l} := K_{S;M_l,g_l}$ and $Y_2 = Y_{2,M_l,g_l} := K_{\{(x,p)\};M_l,g_l}$ of S and the other vector (x,p) will satisfy the admissible intersection property (Definition 4.1) in $\pi \text{supp} A_l \supset W_l$ in both M_l simultaneuously. Here A_l is an admissible collision kernel. The Corollary 4.5 below states that such a manifold S exists. Consequently, Corollary 4.4 will be applicable in both manifolds for sources conormal to $S_1 = S$ and $S_2 = \{x, p\}$.

Below $\mathcal{P}^+\mathcal{C} \subset TM$ stands for set of future directed time-like vectors with base points in given Cauchy surface $\mathcal{C}$.

Lemma 4.5. *Let (M_l, g_l), $l = 1, 2$ be two globally hyperbolic manifolds containing open sets $V_l \subset M_l$ and assume that there is a diffeomorphism $\Psi : V_1 \to V_2$ such that $g_1|_{V_1} = \Psi^* g_2$. For $l = 1, 2$ consider $(x_l, p_l) \in \mathcal{P}^+ V_l$ with $(x_2, p_2) = D\Psi(x_1, p_1)$. Let $\{(z_{l,j}, v_{l,j}) : j \in J_l\}$ (possibly $J_l = \emptyset$) be a countable family of vectors in*

$$\{(z, v) \in \mathcal{P}^+ M_l \mid \exists t \; : \; z = \gamma_{(x_l, p_l)}(t), \; v \in \mathcal{P}_z^+ M_l \setminus span\{\dot{\gamma}_{(x_l, p_l)}(t)\}\}.$$

Let $\mathcal{C}$ be a space-like Cauchy surface through x_l in one of the manifolds M_l, $l = 1, 2$ and copy its restriction $\mathcal{C}_l := \mathcal{C} \cap V_l$ to the other by setting $\mathcal{C}_2 = \Psi \mathcal{C}_1$. Then there exist $(2n - 2)$-dimensional submanifolds $S_l \subset \mathcal{P}^+ \mathcal{C}_l$, $l = 1, 2$ such that $(x_l, p_l) \in S_l$, $S_2 = D\Psi S_1$ and the following two conditions hold for every $j \in J_l$:

(i) *The point $z_{l,j}$ has an open neighbourhood $U_{l,j} \subset M_l$ such that the intersection $U_{l,j} \cap G_{S_l}$, where $G_{S_l} := \pi K_{S_l}$, is a $(n-1)$-dimensional submanifold of M_l.*

(ii) $v_{l,j} \notin T_{z_{l,j}} G_{S_l}$.

Corollary 4.6. *Let (M_l, g_l), $l = 1, 2$ be globally hyperbolic manifolds with a mutual open set $V \subset M_l$, $l = 1, 2$ and assume that $g_1|_V = g_2|_V$. Consider $(x, p), (y, q) \in \mathcal{P}^+ V$ with distinct base points $x \neq y$. Assume that the geodesics γ_l and $\tilde{\gamma}_l$ in M_l, defined by*

$$(4.54) \qquad\qquad (\gamma_2(0), \dot{\gamma}_2(0)) = (\gamma_1(0), \dot{\gamma}_1(0)) = (x, p),$$

$$(4.55) \qquad\qquad (\tilde{\gamma}_2(0), \dot{\tilde{\gamma}}_2(0)) = (\tilde{\gamma}_1(0), \dot{\tilde{\gamma}}_1(0)) = (y, q),$$

are distinguishable as paths (on their maximal domains), that is, (x, p) and (y, q) are not tangent to the same geodesic. Let $\mathcal{C}$ be a space-like Cauchy surface through x (resp. y) in one of the manifolds M_l and let $X_l \subset M_l$ be compact. (e.g. $X_l = \pi supp A_l$ for admissible collision kernels A_l, $l = 1, 2$) Then, there is a $(2n - 2)$-dimensional submanifold $S \subset \mathcal{P}^+(\mathcal{C} \cap V)$ containing (x, p) (resp. (y, q)) such that the pair consisting of the flowouts $K_S = K_{S;M_l, g_l}$ and $K_{(y,q)} = K_{\{(y,q)\};M_l, g_l}$ (resp. $K_{(x,p)} = K_{\{(x,p)\};M_l, g_l}$) have admissible intersection property in X_l for both $l = 1, 2$.

Proof of Lemma 4.5. Since M_l is globally hyperbolic, the space $\mathcal{P}^+ M_l$ can locally near the curve $(\gamma_{x_l, p_l}, \dot{\gamma}_{x_l, p_l})$ be written as the product $\mathcal{P}^+ \mathcal{C}_l \times \mathbb{R}$ by identifying (x, p, t) with $(\gamma_{(x,p)}(t), \dot{\gamma}_{(x,p)}(t))$. Denote by ϕ_l be the projection from the neighbourhood of $(\gamma_{x_l, p_l}, \dot{\gamma}_{x_l, p_l})$ to $\mathcal{P}^+ \mathcal{C}_l$ that in terms of the identification above equals the cartesian projection $(x, p, t) \mapsto (x, p)$. That is; ϕ_l takes each (x, p) in the neighbourhood into the unique intersection of $(\gamma_{(x,p)}, \dot{\gamma}_{(x,p)})$ and $\mathcal{P}^+ \mathcal{C}_l$. Let $\phi_l(x, p) = (z_{l,j}, p_{l,j}) \in \mathcal{P}^+ \mathcal{C}_l$, that is, $p_{l,j}$ stand for the velocity $\dot{\gamma}_{x_l, p_l}$ at $z_{l,j}$. We define $L_{j,l} \subset T_{z_{l,j}} M_l$ to be the 2-plane spanned by $p_{l,j}$ and $v_{l,j}$. Let $\pi : \mathcal{P}^+ M \to M$ be the canonical projection. We see that the linear space

$$E_{l,j} := D_{(z_{l,j}, p_{l,j})} \phi_l (D_{(z_{l,j}, p_{l,j})} \pi)^{-1} L_{l,j} \subset T_{(x_l, p_l)} \mathcal{P}^+ \mathcal{C}_l,$$

is $(n + 1)$-dimensional. There are only countable many of such manifolds so there is a small submanifold $S_1 \subset \mathcal{P}^+ \mathcal{C}_1$ through (x_1, p_1) of dimension $\dim T_{(x_1, p_1)} \mathcal{P}^+ \mathcal{C}_1 - (n + 1) = n - 2$ such that each of the spaces $E_{1,j}$, $j \in J_1$ and $D\Psi^{-1} E_{2,j}$, $j \in J_2$ intersect $T_{x_1, p_1} S_1$ transversally. By considering the dimension of these linear spaces, we observe that the intersection occurs only at the origin. This implies the analogous condition also for $S_2 := D\Psi S_1$. It is straightforward to check that

$$\ker(D_{(z_{l,j}, p_{l,j})} \pi) \cap T_{(z_{l,j}, p_{l,j})} K_{S_l} = \{0\}$$

which ensures that $D_{(z_{l,j},p_{l,j})}\pi$ defines an isomorphism from $T_{(z_{l,j},p_{l,j})}K_{S_l}$ to its image. This implies that $G_{S_l} = \pi K_{S_l}$ near $(z_{l,j}, p_{l,j})$ is a manifold of dimension $\dim K_{S_l} = \dim S_l + 1 = n - 1$. Let us deduce the condition $v_{l,j} \notin T_{z_{l,j}}G_{S_l} = \emptyset$. By the construction above,

$$T_{(x_l,p_l)}S_l \cap E_{l,j} = \{0\}.$$

Hence,

$$(D_{(z_{l,j},p_{l,j})}\phi_l)^{-1}(T_{(x_l,p_l)}S_l \cap E_{l,j}) \subset \ker(D_{(z_{l,j},p_{l,j})}\phi_l).$$

We check that

$$(D_{(z_{l,j},p_{l,j})}\phi_l)^{-1}(T_{(x_l,p_l)}S_l) = T_{(z_{l,j},p_{l,j})}K_{S_l}$$

and

$$(D_{(z_{l,j},p_{l,j})}\phi_l)^{-1}E_{l,j} = (D_{(z_{l,j},p_{l,j})}\pi)^{-1}L_{l,j}.$$

By substitution we conclude

(4.56) $$T_{(z_{l,j},p_{l,j})}K_{S_l} \cap (D_{(z_{l,j},p_{l,j})}\pi)^{-1}L_{l,j} \subset \ker(D_{(z_{l,j},p_{l,j})}\phi_l).$$

Applying $D_{(z_{l,j},p_{l,j})}\pi$ gives

$$T_{z_{l,j}}G_{S_l} \cap L_{l,j} \subset D_{(z_{l,j},p_{l,j})}\pi(\ker(D_{(z_{l,j},p_{l,j})}\phi_l)) = \mathbb{R}p_{l,j}$$

which implies $v_{l,j} \notin T_{z_{l,j}}G_{S_l}$ by the definition of $L_{l,j}$.

$\square$

5. Proof of Theorem 1.3

In this section we prove our main result Theorem 1.3. As shown in Section 3.2, the second Frechét derivative Φ'' of the source-to-solution map Φ satisfies

(5.57) $$\Phi''(0; f, h) = \mathcal{X}^{-1}\mathcal{Q}[\mathcal{X}^{-1}(f), \mathcal{X}^{-1}(h)] + \mathcal{X}^{-1}\mathcal{Q}[\mathcal{X}^{-1}(h), \mathcal{X}^{-1}(f)] \quad \text{on } \mathcal{P}^+M,$$

where f, h are any compactly supported smooth functions and $\mathcal{X}^{-1}$ is the solution operator of the linearized problem (3.21). We use the microlocal properties of the collision operator $\mathcal{Q}$ we proved in the previous section to determine the wavefront set of $\Phi''(h_1, h_2)$ for sources h_1 and h_2 with singularities.

5.1. **Delta distribution of a submanifold.** First, we construct the specific particle sources which we will use in our proofs. Let (M, g) be a C^∞ smooth globally hyperbolic manifold of dimension n. Let $\mathcal{C}$ be a Cauchy surface of (M, g). We introduce a parametrization $\mathbb{R} \times \mathcal{P}^+\mathcal{C} \to \mathcal{P}^+M$ for $\mathcal{P}^+M$ as:

(5.58) $$\Psi(s, (x, p)) = \dot{\gamma}_{(x,p)}(s), \quad s \in \mathbb{R}, \ (x, p) \in \mathcal{P}^+\mathcal{C}.$$

Here $\mathcal{P}^+\mathcal{C} \subset TM$ is the set of future-directed time-like vectors with base points in $\mathcal{C}$. We call the parametrization (5.58) the *flowout parametrization* of $\mathcal{P}^+M$. We refer to [33, Theorem 9.20] for properties of flowouts in general.

Let S be a submanifold (not necessarily closed) of $\mathcal{P}^+\mathcal{C} \subset \mathcal{P}^+M$. The delta distribution $\delta_S \in \mathcal{D}'(\mathcal{P}^+M)$ of the submanifold S on $\mathcal{P}^+M$ is defined as usual by

$$\delta_S(f) = \int_S f(x, p)\, dS, \text{ for all } f \in C_c^\infty(\mathcal{P}^+M),$$

where dS is the volume form of the submanifold S of $\mathcal{P}^+M$. For our purposes it will be convenient to consider the delta distribution on S, in the case where S is considered as a submanifold $\mathcal{P}^+\mathcal{C}$ (instead of $\mathcal{P}^+M$). We distinguish this case and denote by $\check{\delta}_S \in \mathcal{D}'(\mathcal{P}^+\mathcal{C})$ the distribution $\check{\delta}_S(\check{f}) = \int_S \check{f}(x, p)\, dS$ for all $\check{f} \in C_c^\infty(\mathcal{P}^+\mathcal{C})$. The representation of δ_S in the flowout parametrization (5.58) of $\mathcal{P}^+M$ is then

$$\delta_S = \delta_0 \otimes \check{\delta}_S,$$

where $\delta_0 \in \mathcal{D}'(\mathbb{R})$ is the delta distribution on the real line with its support at the $0 \in \mathbb{R}$, and $\check{\delta}_S \in \mathcal{D}'(\mathcal{P}^+\mathcal{C})$ is as above. We will write simply $\delta_S(s, (x, p)) = \delta_0(s)\check{\delta}_S(x, p)$, $s \in \mathbb{R}$ and $(x, p) \in \mathcal{P}^+\mathcal{C}$. We write similarly for other products of pairs of distributions supported in mutually separate variables. [1]

We would like to view δ_S as a conormal distribution of the class $I^m(\mathcal{P}^+M; N^*S)$. However, this is not strictly speaking possible due to the possible bonudary points of $\overline{S}$. To deal with the possible boundary points of the submanifold $\overline{S}$, we are going to consider the product of a cutoff function and δ_S as follows. Let $(y, q) \in S$ and let $(s', s'') \in \mathbb{R}^{\dim(\mathcal{P}^+\mathcal{C})-\dim(S)} \times \mathbb{R}^{\dim(S)}$ be coordinates on a neighborhood $B \subset \mathcal{P}^+\mathcal{C}$ of (y, q) such that S corresponds to the set $\{s' = 0\}$ and S is parametrized by the s'' variable. Let $\chi_R = \chi_R(s'')$ to be a cutoff function, which is supported in a small ball of radius R and outside a neighborhood of the boundary of submanifold $\overline{S}$. We have that $\chi_R\delta_S = \chi_R(\delta_0 \otimes \check{\delta}_S) \in \mathcal{E}'(\mathcal{P}^+M)$ in the flowout parametrization is the oscillatory integral

$$(5.59) \qquad (\chi_R\delta_S)(s, s', s'') = \chi_R(s'')a(s'')\iint e^{is\xi}e^{is'\cdot\xi'}\,d\xi d\xi'.$$

Here $a(s'')$ corresponds the volume form of S and the integration is over $\xi \in \mathbb{R}$ and $\xi' \in \mathbb{R}^{\dim(\mathcal{P}^+\mathcal{C})-\dim(S)}$.

By the above definition, $\chi_R(x, p)\delta_S(x, p)$ is a conormal distribution in the class $I^m_{comp}(\mathcal{P}^+M; N^*S)$, where the order m is

$$(5.60) \qquad m = \mathrm{codim}(S)/2 - \dim(\mathcal{P}^+M)/4,$$

see Section 2.2 for the definition of the order m. Here $\mathrm{codim}(S) = \dim(\mathcal{P}^+M) - \dim(S)$. In the proof of Theorem 1.3, the submanifold S will be either of dimension 0 or $n - 2$, and thus m will be either $n/2$ or 1 respectively. We remark that when S is of dimension 0 the cutoff function in (5.59) can be omitted.

5.1.1. *Approximate delta distributions.* We will use C^∞ smooth sources that approximate delta distributions δ_{S_1} and δ_{S_2} (multiplied by cutoff functions), where S_1 and S_2 are submanifolds of $\mathcal{P}^+\mathcal{C}$. The dimension of S_1 will be $n - 2$ and S_2 will be a point. These approximations are described next.

Let S be a submanifold of $\mathcal{P}^+\mathcal{C} \subset \mathcal{P}^+M$ and let δ_S and χ_R be as in Section 5.1 above. By using standard (Friedrichs) mollification, see e.g. [17], we have that there is a sequence (h^ϵ), $\epsilon > 0$, of functions $C_c^\infty(\mathcal{P}^+M)$ such that

$$(5.61) \qquad h^\epsilon \to \chi_R\delta_S \text{ in } \mathcal{D}'(\mathcal{P}^+M),$$

as $\epsilon \to 0$. Let $u^\epsilon = \mathcal{X}^{-1}h^\epsilon \in C^\infty(\mathcal{P}^+M)$ be the solution to

$$(5.62) \qquad \begin{aligned} \mathcal{X}u^\epsilon &= h^\epsilon \text{ on } \mathcal{P}^+M \\ u^\epsilon &= 0 \text{ on } \mathcal{P}^+\mathcal{C}^-. \end{aligned}$$

By the representation formula of solutions to (5.62) given in Theorem 3.2 we have that

$$(5.63) \qquad u^\epsilon \geq 0, \quad \mathrm{supp}(u^\epsilon) \subset K_{\mathrm{supp}(h^\epsilon)}.$$

By considering only small enough $\epsilon > 0$, the support of the solutions u^ϵ can be taken to be in any neighborhood of K_S chosen beforehand.

Let $\widetilde{\mathcal{C}}$ be another Cauchy surface of (M, g), which is in the past $\mathcal{C}^-$ of $\mathcal{C}$. Note that u^ϵ is then also a solution to $\mathcal{X}u^\epsilon = h^\epsilon$ with $u^\epsilon = 0$ on $\mathcal{P}^+\widetilde{\mathcal{C}}^-$. By Lemma B.3, u^ϵ is unique. We also

---

[1]lots of text, nothing happens

have that $WF(\chi_R \delta_S) \cap N^*(\mathcal{P}^+ \widetilde{\mathcal{C}}) \subset N^* S \cap N^*(\mathcal{P}^+ \widetilde{\mathcal{C}}) = \emptyset$. By applying Lemma B.3 again, we obtain

$$\lim_{\epsilon \to 0} u^\epsilon = u$$

in the space of distributions $\mathcal{D}'(\mathcal{P}^+ M)$. Here $u = \mathcal{X}^{-1}(\chi_R \delta_S)$ solves

(5.64)
$$\mathcal{X} u = \chi_R \delta_S \text{ on } \mathcal{P}^+ M$$
$$u = 0 \text{ on } \mathcal{P}^+ \widetilde{\mathcal{C}}.$$

Finally, if $\chi \in C_c^\infty(\mathcal{P}^+ M)$ is a cutoff such that $\text{supp}(\chi) \subset\subset \mathcal{P}^+ M \setminus S$, we have that $\chi u \in I^{m-1/4}(\mathcal{P}^+ M; N^* K_S)$, with m given in (5.60).

5.2. **Nonlinear interaction in the inverse problem.** Throughout this Section 5.2 we assume that (M, g) is a globally hyperbolic manifold of dimension n and that $\mathcal{C}$ is a Cauchy surface of (M, g). We also assume that the submanifolds $S_1 \subset \mathcal{P}^+\mathcal{C}$ and $S_2 = \{(x_0, p_0)\} \in \mathcal{P}^+\mathcal{C}$, $x_0 \notin \overline{S}_1$, are such that the flowouts K_{S_1} and K_{S_2} satisfy the admissible intersection property (Definition 4.1) in $\pi \text{supp} A$, where A is an admissible collision kernel. As shown simultaneously for two manifolds in Corollary 4.6, such a pair of submanifolds can be constructed around any distinct base points on the Cauchy surface. Further, let $\hat{\mu} : [-1, 1] \to M$ be C^∞-smooth timelike curve and $V \subset M$ be an open neighbourhood of $\hat{\mu}$.

The length of a piecewise smooth causal path $\alpha : [a, b] \to M$ is defined as

(5.65)
$$l(\alpha) := \sum_{j=0}^{m-1} \int_{a_j}^{a_{j+1}} \sqrt{-g(\dot{\alpha}(s), \dot{\alpha}(s))} ds,$$

where $a_0 < a_1 < \cdots < a_{m-1} < a_m$ are chosen such that α is smooth on each interval (a_j, a_{j+1}) for $j = 0, \ldots, m-1$. The time separation function, see e.g. [38], is denoted by $\tau : M \times M \to [0, \infty)$ and defined as

$$\tau(x, y) := \begin{cases} \sup l(\alpha), & x < y \\ 0, & \text{otherwise,} \end{cases}$$

where the supremum is taken over all piecewise smooth lightlike and timelike curves $\alpha : [0, 1] \to M$, which are smooth on each interval (b_j, b_{j+1}), and that satisfy $\alpha(0) = x$ and $\alpha(1) = y$. If $\tau(x, y) = 0$ and there is a lightlike geodesic γ connecting points $x, y \in M$, we call γ *optimal*. By [38, Proposition 14.19], we have that if (M, g) is globally hyperbolic and if $x, y \in M$ satisfy $\tau(x, y) = 0$, then an optimal lightlike geodesic γ always exists.

The main result of this Section 5.2 is the following:

Proposition 5.1. *Let (M, g) be a globally hyperbolic manifold, $A : TM^4 \to \mathbb{R}$ an admissible collision kernel with respect to a relatively compact subset $W \subset M$. Let $\mathcal{C} \subset M$ be a Cauchy surface, S_1 be a submanifold of $\mathcal{P}^+\mathcal{C}$, and $S_2 = \{(x_0, p_0)\} \in \mathcal{P}^+\mathcal{C}$, $x_0 \notin \overline{S}_1$. Assume that K_{S_1} and K_{S_2} have the admissible intersection property in $\pi \text{supp} A$ according to Definition 4.1. Assume also that $\pi(K_{S_1})$ and $\pi(K_{S_2})$ intersect in $\pi \text{supp} A$ first time at $z_1 \in W$. Let γ be an optimal future-directed light-like geodesic in M such that $\gamma(0) = z_1$ and $e := \gamma(T)$.*

Additionally, let $h_1^\epsilon, h_2^\epsilon \in C_c^\infty(\mathcal{P}^+ M)$ be the approximations of the distributions $\chi_R \delta_{S_1}$ and δ_{S_2}, where S_2 is a point, as described in Section 5.1.1.

Then, there is a section $P_e : V_e \to L^+ V_e$ on a neighborhood V_e of e such that the limit

$$\mathcal{S} := \lim_{\epsilon \to 0} (\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P_e)$$

exists and

$$singsupp(\mathcal{S}) = \gamma \cap V_e$$

In the context of the main theorem the proposition above can be used to detect rays of light propagating from the unknown set W to the measurement neighbourhood V. In that setting $z_1 \in W$ whereas $e \in V$ and $V_e \subset V$. The situation is pictured in Figure 3 below.
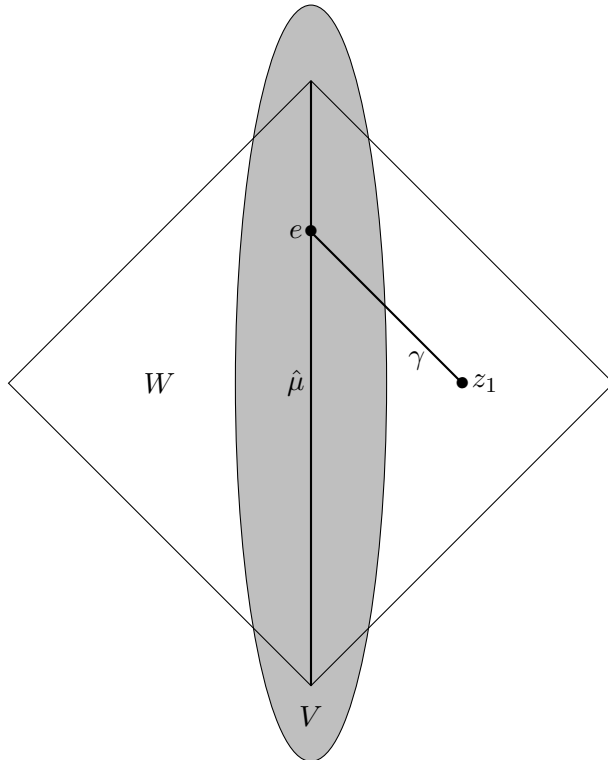

FIGURE 3. The setup of Proposition 5.1 can be used to detect singularities propagating along rays of light in V.

Before proceeding with the proof of Proposition 5.1, we state the following supporting result which follows from a simple dimension argument similar to the one used in Lemma 4.5. In other words, there is so much freedom for variation that caustic effects can be avoided at finite number of fixed points. The vector field P_e in Proposition 5.1 is constructed as a restriction of P in the lemma below for convenient choice of the vectors (x_j, p_j).

Lemma 5.2. *Let* $(x_j, p_j) \in L^+(M)$, $j = 1, \ldots, m$, *be a finite set of vectors. There is an open (possibly disconnected) neighborhood Q of* $\{x_1, \ldots, x_m\}$ *in M and a smooth local section* $P : Q \to L^+ Q$, $P(x) = (x, p(x))$, *of the bundle* $\pi : L^+ M \to M$ *such that for $x \in Q$ and $s \in \mathbb{R}$ such that* $\gamma_{(x,p(x))}(s) \in Q$

$$
\begin{aligned}
P(x_j) &= (x_j, p_j) \\
\dot\gamma_{(x,p(x))}(s) &= P(\gamma_{(x,p(x))}(s)).
\end{aligned}
$$
(5.66)

Following [38], we say a path $\alpha([t_1, t_2])$ is a pre-geodesic if $\alpha(t)$ is a C^1-smooth curve such that $\dot\alpha(t) \neq 0$ on $t \in [t_1, t_2]$, and there exists a reparametrization of $\alpha([t_1, t_2])$ so that it becomes a geodesic. Proposition 10.46 of [38] implies the existence of a shortcut path between points which are not connected by lightlike pre-geodesics:

Lemma 5.3 (Shortcut Argument). *Let (M, g) be globally hyperbolic and $x, y, z \in M$. Suppose that x can be connected to y by a future-directed lightlike geodesic $\gamma_{x \to y}$, and y can be connected*

to z by a future-directed lightlike geodesic $\gamma_{y \to z}$. Additionally, assume that $\gamma_{x \to y} \cup \gamma_{y \to z}$ is not a lightlike pre-geodesic. Then, there exists a timelike geodesic connecting x to z.

We now prove Proposition 5.1:

Proof of Proposition 5.1 . Let U be an open neighborhood of z_1, small enough so that $G_{S_1} \cap U$ and $G_{S_2} \cap U$ are manifolds and that $U \cap (\pi(S_1) \cup \pi(S_2)) = \emptyset$. Let $P : U \to L^+(U)$ be an arbitrary smooth section on U. We remark that we will choose a specific P later in the proof.

Let h_1^ϵ and h_2^ϵ be the respective approximations of the distributions $\chi_R \delta_{S_1}$ and δ_{S_2} as described in (5.61)-(5.64) compactly supported in $\mathcal{P}^+ M$. (We do not multiply δ_{S_2} with a cutoff function, since S_2 is just a point.) For $j = 1, 2$, let u_j^ϵ be the solutions corresponding to h_j^ϵ:

$$
(5.67) \qquad \begin{aligned} \mathcal{X} u_j^\epsilon &= h_j^\epsilon \text{ on } \mathcal{P}^+ M \\ u_j^\epsilon &= 0 \text{ on } \mathcal{P}^+ \mathcal{C}^-. \end{aligned}
$$

By Lemma 3.4 we have that the second linearization Φ'' of the source to solution map Φ satisfies

$$
(5.68) \qquad \Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P = \left(\mathcal{X}^{-1} \mathcal{Q}[u_1^\epsilon, u_2^\epsilon] \right) \circ P + \left(\mathcal{X}^{-1} \mathcal{Q}[u_2^\epsilon, u_1^\epsilon] \right) \circ P,
$$

where $\mathcal{X}^{-1}$ is the solution operator to the Vlasov equation (3.21). We first study the limit $\epsilon \to 0$ of

$$
\mathcal{Q}[u_1^\epsilon, u_2^\epsilon] \circ P + \mathcal{Q}[u_2^\epsilon, u_1^\epsilon] \circ P.
$$

Since h_1^ϵ and h_2^ϵ are supported on $\mathcal{P}^+ M$ and since $P(x)$ is light-like, we find that

$$
(5.69) \qquad \mathcal{Q}_{loss}[u_1^\epsilon, u_2^\epsilon] \circ P(x) = \mathcal{Q}_{loss}[u_2^\epsilon, u_1^\epsilon] \circ P(x) = 0.
$$

Therefore, the light which scatters from the collisions arises only from the terms

$$
\mathcal{Q}_{gain}^P[u_1^\epsilon, u_2^\epsilon] = \mathcal{Q}_{gain}[u_1^\epsilon, u_2^\epsilon] \circ P \quad \text{and} \quad \mathcal{Q}_{gain}^P[u_2^\epsilon, u_1^\epsilon] = \mathcal{Q}_{gain}[u_2^\epsilon, u_1^\epsilon] \circ P.
$$

By the discussion in Section 5.1.1 we have that, away from S_1 and S_2, the element $\mathcal{X}^{-1}(\chi_R \delta_{S_1})$ lies in $I^{n/2-1/4}(\mathcal{P}^+ M; N^* K_{S_1})$ and $\mathcal{X}^{-1}(\delta_{S_2})$ lies in $I^{1-1/4}(\mathcal{P}^+ M; N^* K_{S_2})$. Therefore, if Γ is any conical neighborhood of $\Lambda_1 \cup \Lambda_2$, where Λ_1 and Λ_2 are as in Property 4.1, we have by Corollary 4.4 that

$$
(5.70) \qquad \begin{aligned} &\mathcal{Q}_{gain}^P[\mathcal{X}^{-1}\delta_{S_2}, \mathcal{X}^{-1}(\chi_R \delta_{S_1})] + \mathcal{Q}_{gain}^P[\mathcal{X}^{-1}(\chi_R \delta_{S_1}), \mathcal{X}^{-1}\delta_{S_2}] \\ &\qquad \in I^l(\Lambda_0 \setminus (\Lambda_1 \cup \Lambda_2); U) + \mathcal{D}'_\Gamma(U) \subset \mathcal{D}'(U), \end{aligned}
$$

where $l = (n/2 - 1/4) + (1 - 1/4) + 3n/4 - 1/2 = 5n/4$ and U is a small neighbourhood around z_1. Here we use the standard notation to denote

$$
\mathcal{D}'_\Gamma(U) = \{ u \in \mathcal{D}'(U) : WF(u) \subset \Gamma \}.
$$

By continuity of $\mathcal{X}^{-1}$ (see Section 5.1.1) and $\mathcal{Q}_{gain}$ (Corollary 4.4), the distribution (5.70) equals the limit

$$
(5.71) \qquad \lim_{\epsilon \to 0} \left(\mathcal{Q}_{gain}^P[u_1^\epsilon, u_2^\epsilon] + \mathcal{Q}_{gain}^P[u_2^\epsilon, u_1^\epsilon] \right),
$$

in $\mathcal{D}'(\mathcal{P}^+ M)$. It also follows by definition of the collision operator and $\text{supp}(u_j^\epsilon) \subset K_{\text{supp}(h_j^\epsilon)}$ that the support of (5.71) focuses close to the intersection point z_1 as $\epsilon \to 0$. In particular, we may consider it as a compactly supported distribution in U. Now let P be a restriction of a section in Lemma 5.2 for $k = 2$, $(x_1, p_1) = (z_1, \dot{\gamma}(0))$ and $(x_2, p_2) = (e, \dot{\gamma}(T))$ where γ is the optimal geodesic connecting z_1 to e. Then the terms in (5.68) satisfy

$$
(5.72) \qquad (\mathcal{X}^{-1} \mathcal{Q}_{gain}[u_1^\epsilon, u_2^\epsilon]) \circ P = P^{-1} \mathcal{Q}_{gain}^P[u_1^\epsilon, u_2^\epsilon],
$$

$$
(5.73) \qquad (\mathcal{X}^{-1} \mathcal{Q}_{gain}[u_2^\epsilon, u_1^\epsilon]) \circ P = P^{-1} \mathcal{Q}_{gain}^P[u_2^\epsilon, u_1^\epsilon],
$$

where $P^{-1} : C_c^\infty(U) \to C^\infty(V_e)$, $P^{-1}\phi(x) := \int \phi(\gamma_{x,p(x)}(t))dt$, i.e. the integration along integral curves of the light-like field P. Here V_e is a small neighbourhood of e in the domain of the field. The canonical relation of the Fourier integral operator P^{-1} is

$$C = \{(x, \xi; y, \eta) \in T^*V_e \setminus \{0\} \times T^*U \setminus \{0\} : x = \gamma_{y,p(y)}(t), \ \xi, \eta \perp \gamma_{y,p(y)}\}$$

and the symbol is a non-vanishing constant. Since light-like geodesics intersect cleanly the time-like flow-outs $G_{S_1} = \pi\Lambda_1$ and $G_{S_2} = \pi\Lambda_2$ and a single point $\pi\Lambda_0$, it is straightforward to check that the expressions above extend also to the limit $\epsilon \to 0$ (See e.g. [13]). Hence,

$$(5.74) \qquad (\mathcal{X}^{-1}\mathcal{Q}_{gain}[u_1^\epsilon, u_2^\epsilon]) \circ P \longrightarrow P^{-1}\mathcal{Q}_{gain}^P[\mathcal{X}^{-1}(\chi_R\delta_{S_1}), \mathcal{X}^{-1}\delta_{S_2}],$$

and away from Λ_1 and Λ_2 we have the standard principal symbol formulas (cf. [13, (4.2.10)]

$$(5.75) \qquad \sigma(P^{-1}\mathcal{Q}_{gain}^P[\mathcal{X}^{-1}(\chi_R\delta_{S_1}), \mathcal{X}^{-1}\delta_{S_2}]) = \sum \sigma(P^{-1})\sigma(\mathcal{Q}_{gain}^P[\mathcal{X}^{-1}(\chi_R\delta_{S_1}), \mathcal{X}^{-1}\delta_{S_2}])$$

$$(5.76) \qquad = c\sum \sigma(\mathcal{Q}_{gain}^P[\mathcal{X}^{-1}(\chi_R\delta_{S_1}), \mathcal{X}^{-1}\delta_{S_2}]),$$

for composition of Fourier integral operators. Analogous identities hold for the other term. Thus, following the standard composition calculus for Fourier integral operators, it suffices to show that the symbol of $\mathcal{Q}_{gain}^P[\mathcal{X}^{-1}(\chi_R\delta_{S_1}), \mathcal{X}^{-1}\delta_{S_2}] + \mathcal{Q}_{gain}^P[\mathcal{X}^{-1}\delta_{S_2}, \mathcal{X}^{-1}(\chi_R\delta_{S_1})]$ is not of order $-\infty$ at any direction in $\Lambda_0 = T_{z_1}^* M$ away from Λ_1 and Λ_2. This, however, follows from Theorem 4.3 together with the formulas $\sigma(\mathcal{X}^{-1}\delta_j) \simeq \sigma(\delta_{S_2}) \simeq 1$ and $\sigma(\mathcal{X}^{-1}\chi_R\delta_{S_1}) \simeq \chi_R\sigma(\delta_j) \simeq \chi_R$ for principal symbols. □

Let S_1 and S_2 be as earlier. Then, $G_{S_1} = \pi K_{S_1}$ and $G_{S_2} = \pi K_{S_2}$ intersect in $\pi(\mathrm{supp}(A))$ only finitely many times and at discrete points. Given that the intersections exist, we can write $\{z_1, \ldots, z_k\} = \pi(K_{S_1}) \cap \pi(K_{S_2}) \cap \pi(\mathrm{supp}(A))$.

Lemma 5.4. *As above, denote the points in $\pi(K_{S_1}) \cap \pi(K_{S_2}) \cap \pi(supp(A))$ by $z_1, \ldots, z_k$ (if exist) and arrange them so that $z_1 \ll z_2 \ll \cdots \ll z_k$. For every section P of the bundle L^+V we have*

$$supp(\mathcal{S}_P) \subset \bigcup_{l=1}^k \mathcal{L}^+(z_l) \subset J^+(z_1),$$

or $supp(\mathcal{S}_P) = \emptyset$ if $\pi(K_{S_1}) \cap \pi(K_{S_2}) \cap \pi(supp(A)) = \emptyset$. Here we denote

$$\mathcal{S}_P := \lim_{\epsilon \to 0}(\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P).$$

In particular, if for some section P we have that $supp(\mathcal{S}_P) \neq \emptyset$, then the first intersection point z_1 exists.

Proof. Let us adopt the notation of the proof of Proposition 5.1. One checks that

$$\mathrm{supp}\big(\mathcal{Q}_{gain}^P[u_1^\epsilon, u_2^\epsilon] + \mathcal{Q}_{gain}^P[u_2^\epsilon, u_1^\epsilon]\big) \subset E_\epsilon$$

and

$$\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P = \int_{-\infty}^0 \mathcal{Q}_{gain}^P[u_1^\epsilon, u_2^\epsilon](\gamma_{(x,p(x))}(s)) + \mathcal{Q}_{gain}^P[u_2^\epsilon, u_1^\epsilon](\gamma_{(x,p(x))}(s))ds,$$

where for $\epsilon > 0$

$$E_\epsilon := G_{\mathrm{supp}(h_1^\epsilon)}^+ \cap G_{\mathrm{supp}(h_2^\epsilon)}^+ \cap \pi(\mathrm{supp}(A)).$$

As E_ϵ is a monotone sequence it has the limit which is $\bigcap_{\epsilon>0} E_\epsilon = \{z_1, \ldots, z_k\}$. Thus, we obtain

$$\mathrm{supp}(\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P) \subset \mathcal{L}^+ E_\epsilon$$

and hence

$$\text{supp}(\mathcal{S}_P) := \text{supp}(\lim_{\epsilon \to 0}(\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P)) \subset \bigcap_{\epsilon > 0} \mathcal{L}^+ E_\epsilon = \mathcal{L}^+\{z_l, \ldots, z_k\} = \bigcup_{l=1}^k \mathcal{L}^+(z_l)$$

It also follows from the definition of $\mathcal{L}^+$ and J^+ that $\bigcup_{l=1}^k \mathcal{L}^+(z_l) \subset J^+(z_1)$. $\qquad\square$

In our inverse problem we will observe the $\text{supp}(\mathcal{S}_P)$ and use the last claim of Lemma 5.4 to detect the collision of particles in a subset $W \subset M$.

5.3. **Separation time functions.** Let (M, g) be a C^∞-smooth, globally hyperbolic, Lorentzian manifold. Also, let $\hat{\mu} : [-1, 1] \to M$ be a given smooth, future-directed, timelike geodesic, and V be an open neighbourhood of $\hat{\mu}$. In this subsection we will use $g|_V$, $\hat{\mu}$ and V to introduce a useful representation for certain points in M. Our representation scheme will associate a point $w \in M$ to a subset $\mathcal{E}_\mathcal{U}(w) \subset \mathcal{U}$, where $\mathcal{U} \subset V$ is an open set. Loosely speaking, the subset $\mathcal{E}_\mathcal{U}(w)$ will be comprised of points in $\mathcal{U}$ which lie on a optimal lightlike geodesic emanating from w. This representation was first introduced by Kurylev, Lassas, and Uhlmann in [24]; we reproduce a summary of it here for the reader's convenience. We later will use this representation to construct the desired isometry $F : W_1 \to W_2$ described in Theorem 1.3.

To begin, as shown in [34, Section II], there exists a bounded, connected, open set $\mathcal{A} \subset \mathbb{R}^{n-1}$ and a neighbourhood $\mathcal{U} \subset V$ of $\hat{\mu}$ on which we may define coordinates

$$x \in \mathcal{U} \mapsto (s, a^1, a^2, \ldots, a^{n-1}) \in [-1, 1] \times \mathcal{A}$$

These coordinates have the property that $\hat{\mu}(s) = (s, 0, \ldots, 0)$ and for fixed $a = (a^1, a^2, \ldots, a^{n-1}) \in \mathcal{A}$ the map $\mu_a(s) = (s, a^1, a^2, \ldots, a^{n-1})$ is a C^∞-smooth timelike curve. Further, writing $\mu_{\hat{a}} = \hat{\mu}$ where $\hat{a} = (0, \ldots, 0) \in \mathcal{A}$, we have

$$\mathcal{U} = \bigcup_{a \in \mathcal{A}} \mu_a[-1, 1].$$

Let $\overline{\mathcal{A}}$ be the closure of $\mathcal{A}$ in $\mathbb{R}^{n-1}$. Below, we will assume that for all $a \in \overline{\mathcal{A}}$ we have $\hat{\mu}(s_+) \ll \mu_a(1)$ and $\mu_a(-1) \ll \hat{\mu}(s_-)$.

Given $\mathcal{U}$ and the family of curves μ_a, $a \in \overline{\mathcal{A}}$ (which may be defined by replacing $\mathcal{A}$ above by a smaller open subset if necessary), we will next define the notions of time separation functions and observation time functions. where the supremum of the length function

Consider $-1 < s^- < s^+ < 1$ and set $x^\pm := \hat{\mu}(s^\pm) \in V$. As in [24, Definition 2.1], for each $a \in \overline{\mathcal{A}}$ and corresponding path μ_a, we define the *observation time functions* $f_a^\pm : J^-(x^+) \setminus I^-(x^-) \to \mathbb{R}$ by the formulas

$$f_a^+(x) := \inf(\{s \in (-1, 1) : \tau(x, \mu_a(s)) > 0\} \cup \{1\})$$

and

$$f_a^-(x) := \sup(\{s \in (-1, 1) : \tau(\mu_a(s), x) > 0\} \cup \{-1\}).$$

Here τ is the time separation function defined in Section 5.2. We note that if $x \in M$ and $a \in \mathcal{A}$ are such that at least one point in $\mu_a(-1, 1)$ can be reached from x by a future-directed timelike curve we obtain $\tau(x, \mu_a(f_a^+(x))) = 0$, see [24]. In this case, there also exists a future-directed optimal light-like geodesic that connects x to $\mu_a(f_a^+(x))$ as discussed in Section 5.2.

The earliest time observation functions $f_a^+ : J^-(x^+) \setminus I^-(x^-) \to \mathbb{R}$ determine the set

$$(5.77) \qquad \mathcal{E}_\mathcal{U}(w) = \{\mu_a(f_a^+(w)) : a \in \mathcal{A}\} \subset U,$$

that is the *earliest light observation set* of $w \in J^-(x^+) \setminus I^-(x^-)$.

Finally, as shown in [24, Proposition 2.2.], we may construct the conformal type of the open, relatively compact set $W \subset J^-(x^+) \setminus I^-(x^-)$ when we are given the collection of all earliest light observation sets associated to points $w \in W$, that is,

$$\mathcal{E}_{\mathcal{U}}(W) = \{\mathcal{E}_{\mathcal{U}}(w) \, : \, w \in W\} \subset 2^{\mathcal{U}}.$$

5.4. **Source-to-Solution map determines earliest light observation sets.** In this section, we prove that the source-to-solution map for light observations (see (1.3)) of the Boltzmann equation on a subset V of a manifold determines the earliest light observation sets on a subset of the manifold which properly contains V. We will define such a set below. After proving this, the main result of this paper, Theorem 1.3, will follow by applying [24, Theorem 1.2], which states that the earliest light observation sets determine the Lorentzian metric structure of the manifold up to conformal class.

From this point onwards, we assume that (M_1, g_1) and (M_2, g_2) are two geodesically complete, globally hyperbolic, C^∞-smooth, Lorentzian manifolds, which contain a common open subset V and

$$g_1|_V = g_2|_V.$$

We assume that $\hat{\mu} : [-1, 1] \to V$ is a given future-directed timelike geodesic. Let $\mathcal{A} \subset \mathbb{R}^{n-1}$, the family of paths $(\mu_a)_{a \in \mathcal{A}}$, and the subset $\mathcal{U} \subset V$ be as in the Section 5.3.

For $-1 < s^- < s^+ < 1$, we set $x^\pm := \hat{\mu}(s^\pm) \in V$ and define

(5.78) $W_1 := I^-(x^+) \cap I^+(x^-) \subset M_1$ defined with respect to (M_1, g_1),

(5.79) $W_2 := I^-(x^+) \cap I^+(x^-) \subset M_2$ defined with respect to (M_2, g_2).

Additionally, for $\lambda = 1, 2$, let A_λ be an admissible collision kernel (see Definition 1.1) with respect to the space (M_λ, g_λ) and write $\Phi_{\lambda, L+V}$ for the source-to-solution map for light observations (see Equation (1.3)) associated to the relativistic Boltzmann equation (1.2) with respect to g_λ. The notation Φ_λ denotes the full source to solution map for (1.2).

In the above setting we prove:

Proposition 1.4. *Let $\Phi_{1, L+V}$ and $\Phi_{2, L+V}$ be the above source-to-solution maps for light observations. Then $\Phi_{1, L+V} = \Phi_{2, L+V}$ implies*

$$\mathcal{E}_{\mathcal{U}}^1(W_1) = \mathcal{E}_{\mathcal{U}}^2(W_2).$$

We prove Proposition 1.4 by showing that $\Phi_{1, L+V} = \Phi_{2, L+V}$ implies the existence of a diffeomorphism

$$F : W_1 \to W_2$$

which satisfies

$$\mathcal{E}_{\mathcal{U}}^1(w_1) = \mathcal{E}_{\mathcal{U}}^2(F(w_1)), \quad w_1 \in W_1.$$

To construct the map $F : W_1 \to W_2$, consider the observation time functions on (M_j, g_j), $j = 1, 2$ which we denote by $f_{a,j}^\pm$. For each

$$w_1 \in W_1 \subset M_1,$$

we define η_{w_1} to be an optimal future-directed light-like geodesic in M_1 such that

(5.80) $\eta_{w_1}(0) = \hat{\mu}(f_{\hat{a},1}^-(w_1))$ and $\eta_{w_1}(T) = w_1$ for some $T > 0$.

By the definition of W_1 such η_{w_1} exists. In the following, w_1 will be fixed and we abbreviate

$$\eta_1 := \eta_{w_1}.$$

Since $\mathcal{U}$ is open and $\dim(M_1) = n > 2$, we may choose another future-directed optimal light-like geodesic $\tilde{\eta}_1$ that is not tangential to η_1 and satisfies

$$(5.81) \qquad \tilde{\eta}_1(0) = \mu_a(f_{a,1}^-(w_1)) \text{ and } \tilde{\eta}_1(\widetilde{T}) = w_1 \text{ for some } \widetilde{T} > 0, \quad a \in \mathcal{A} \setminus \hat{a}.$$

Since both segments $\eta_1|_{[0,T]}, \tilde{\eta}_1|_{[0,\widetilde{T}]}$ are optimal, the shortcut argument implies that $\eta_1(s)$, $s > 0$ and $\tilde{\eta}_1(s')$, $s' > 0$ intersect the first time at w_1.

In the lemma below we approximate the light-like geodesics η_1 and $\tilde{\eta}_1$ by time-like geodesics

$$\gamma_{(\hat{x},\hat{p})}(s) \text{ and } \gamma_{(\hat{y},\hat{q})}(s), \quad s > 0, \quad (\hat{x},\hat{p}) \in \mathcal{P}^+V, \quad (\hat{y},\hat{q}) \in \mathcal{P}^+V$$

that intersect for the first time at w_1 as geodesics in M_1. One may always fix the geodesics such that $\hat{x}$ and $\hat{y}$ belong to a same Cauchy surface and $(\hat{x},\hat{p})$ is arbitrarily near the curve $(\eta_1, \dot{\eta}_1)$, and $(\hat{y},\hat{q})$ is arbitrarily near the curve $(\tilde{\eta}_1, \dot{\tilde{\eta}}_1)$. Notice the abuse of notation: $\gamma_{(\hat{x},\hat{p})}$, $\gamma_{(\hat{y},\hat{q})}$ may refer to a pair of geodesics either in (M_1, g_1) or (M_2, g_2). Due to global hyperbolicity we may always redefine the initial vectors by sliding them along the geodesic flow so that $\hat{x}$ and $\hat{y}$ lie in a Cauchy surface.

Let η_2 and $\tilde{\eta}_2$ be the geodesics in (M_2, g_2), which have the same initial data with η_1 and $\tilde{\eta}_1$ respectively:

$$(5.82) \qquad \dot{\eta}_2(0) = \dot{\eta}_1(0) \in T_{\eta_1(0)}\mathcal{U} \text{ and } \dot{\tilde{\eta}}_2(0) = \dot{\tilde{\eta}}_1(0) \in T_{\tilde{\eta}_1(0)}\mathcal{U}.$$

(Recall that $\mathcal{U} \subset V$ is a mutual set of M_1 and M_2 so that this makes sense.)

We thus define

$$F : W_1 \to W_2$$

as the map which assigns a given point $w_1 \in W_1$ to the first intersection of $\eta_2(s)$, $s > 0$ and $\tilde{\eta}_2(s')$, $s' > 0$ denoted by $w_2 \in M_2$. For the assignment $w_1 \mapsto w_2$ to be well-defined, we of course need to show that the first intersection w_2 exists and lies in W_2. There are also many choices for the geodesics η_1 and $\tilde{\eta}_1$ on (M_1, g_1), which are used to define η_2 and $\tilde{\eta}_2$ on (M_2, g_2). Therefore we need also to show that w_2 is independent of our choices of η_1 and $\tilde{\eta}_1$. These necessities are proven in Lemma 5.5 below.

Lemma 5.5. *Let* (M_j, g_j), $\mathcal{U} \subset V \subset M_j$, $W_j \subset M_j$, Φ_j *for* $j = 1, 2$ *be as described above. Let* $w_1 \in W_1$ *and consider light-like future-directed geodesics* $\eta_1(s)$, $s > 0$, $\tilde{\eta}_1(s')$, $s' > 0$ *in* (M_1, g_1) *with* (5.80) *and* (5.81) *intersecting the first time at* w_1. *Let* η_2 *and* $\tilde{\eta}_2$ *be the associated light-like geodesics in* (M_2, g_2) *with the initial conditions* $(\eta_2(0), \dot{\eta}_2(0)) = (\eta_1(0), \dot{\eta}_1(0))$ *and* $(\tilde{\eta}_2(0), \dot{\tilde{\eta}}_2(0)) = (\tilde{\eta}_1(0), \dot{\tilde{\eta}}_1(0))$. *Then the condition* $\Phi_{1,L+V} = \Phi_{2,L+V}$ *implies the following:*

(1) *There exists the first intersection* w_2 *of* $\eta_2(s)$, $s > 0$ *and* $\tilde{\eta}_2(s')$, $s' > 0$ *in* (M_2, g_2). *Moreover,* $w_2 \in W_2$.

(2) *The first intersection point* w_2 *is independent from the choice of the geodesics* η_1, $\tilde{\eta}_1$ *satisfying the required conditions above.*

(3) *For every pair* $(\hat{x},\hat{p}) \in \mathcal{P}^+V$ *and* $(\hat{y},\hat{q}) \in \mathcal{P}^+V$ *with the geodesics* $\gamma_{(\hat{x},\hat{p})}(s)$, $s > 0$, *and* $\gamma_{(\hat{y},\hat{q})}(s')$, $s' > 0$ *in* (M_1, g_1) *intersecting the first time at* w_1, *the associated geodesics in* (M_2, g_2) *intersect for the first time at* w_2.

Proof of Lemma 5.5. Let $w_1 \in W_1$ and let $e \in \mathcal{L}^+(w_1)$ be a first observation of w_1 in $\mathcal{U} \subset V \subset M_1$. That is, $e \in \mathcal{E}_{\mathcal{U}}^1(w_1)$. By the fact (5.77) there is a point $a \in \mathcal{A}$ and the corresponding path μ_a such that

$$(5.83) \qquad e = \mu_a(f_{a,1}^+(w_1)) \in (M_1, g_1).$$

Let γ_1 be the optimal geodesic in (M_1, g_1) as in Proposition 5.1 such that $\gamma_1(0) = w_1$ and $\gamma_1(1) = e$.
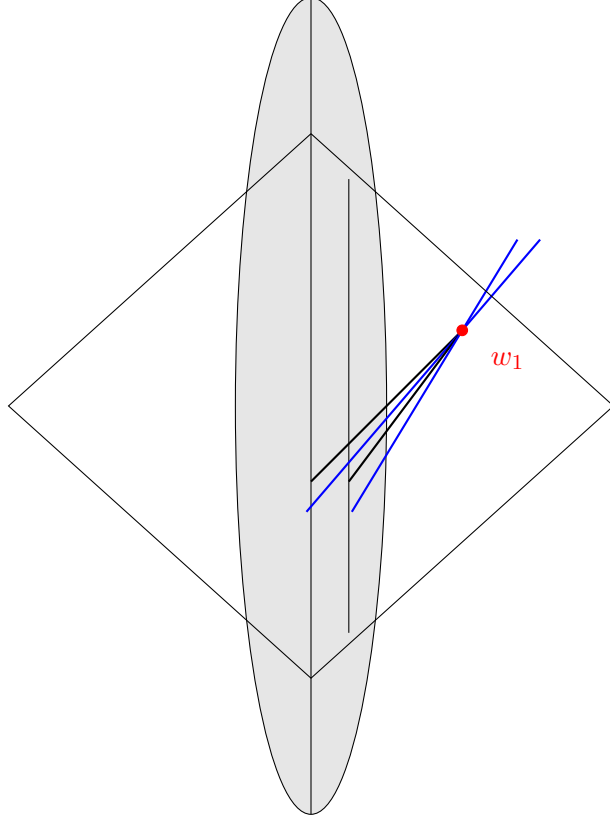
FIGURE 4. Given $w_1 \in W_1$ (in red) the light-like geodesics $\eta_1, \tilde{\eta}_1$ (in black) are chosen such that they maximize distance between their initial points in $\mathcal{U}$ (in gray) and w_1. The timelike geodesics $\gamma_{(\hat{x},\hat{p})}$ and $\gamma_{(\hat{x},\hat{p})}$ (in blue) intersect first time at w_1 and approximate the light-like segments.

We approximate the light-like geodesics η_1 and $\tilde{\eta}_1$ by time-like geodesics from $(\hat{x}, \hat{p}) \in \mathcal{P}^+ V$ and $(\hat{y}, \hat{q}) \in \mathcal{P}^+ V$ as described earlier by requiring that for the geodesics in (M_1, g_1) their first intersection for positive parameter values is at w_1. Let $S_2 = \{\hat{x}, \hat{p}\} \subset P^{m_1} V$. By Corollary 4.6 there exists a submanifold $S_1 \subset \mathcal{P}^+ V$ with $(\hat{y}, \hat{q}) \in S_1$ such that the geodesic flowouts $Y_{j,1} := K_{S_1;M_j} \subset \mathcal{P}^+ M_j$ of S_1 and $Y_{j,2} := K_{S_2;M_j} \subset \mathcal{P}^+ M_j$ of S_2 in (M_j, g_j) have an admissible intersection property in the sense of Definition 4.1 for both $j = 1, 2$. Therefore, we are in the setting for which the earlier results of this this section and Section 4 are valid. With this in mind, let us write

(5.84) $$\pi(Y_{1,1}) \cap \pi(Y_{1,2}) = \{z_{1,1}, z_{1,2}, \ldots, z_{1,k_1}\} \subset M_1,$$

where the intersection points $z_{1,l}$, $l = 1, \ldots, k_1$, of $\pi(Y_{1,1})$ and $\pi(Y_{1,2})$ are ordered causally as $z_{1,1} \ll z_{1,2} \ll \cdots \ll z_{1,k_1}$. (The index 1 in $z_{1,l}$ refers to the manifold (M_1, g_1)) Notice that $z_{1,1} = w_1$. For $\epsilon > 0$, let

$$h_1^\epsilon \in C_c^\infty(\mathcal{P}^+ V) \text{ and } h_2^\epsilon \in C_c^\infty(\mathcal{P}^+ V)$$

be the sequences of approximative delta functions of S_1 and S_2 described in (5.61).

By Proposition 5.1 there is a neighbourhood $V_e \subset V$ of e and a section $P_e : V_e \to L^+ V_e$ such that the distribution

$$\mathcal{S}_1 := \lim_{\epsilon \to 0}(\Phi_1''(0; h_1^\epsilon, h_2^\epsilon) \circ P_e) \in \mathcal{D}'(V_e),$$

satisfies

$$\text{singsupp}(\mathcal{S}_1) = \gamma_1 \cap V_e.$$

The index 2 in $Y_{2,j}$, and in $z_{2,l}$, $\mathcal{L}_2^+$ and J_2^+ below, refers to corresponding quantities on the manifold (M_2, g_2). The submanifolds of $Y_{2,1}$ and $Y_{2,2}$ have admissible intersection property by their definition above.

The condition $\Phi_{1,L+V} = \Phi_{2,L+V}$ implies

$$\mathcal{S}_1 := \lim_{\epsilon \to 0}(\Phi_1''(0; h_1^\epsilon, h_2^\epsilon) \circ P_e) = \lim_{\epsilon \to 0}(\Phi_2''(0; h_1^\epsilon, h_2^\epsilon) \circ P_e) =: \mathcal{S}_2.$$

Therefore, we have $\operatorname{singsupp}(\mathcal{S}_2) = \gamma_1 \cap V_e \subset V$ and

$$e \in \operatorname{supp}(\mathcal{S}_1) \cap \operatorname{supp}(\mathcal{S}_2).$$

Consequently, the sets $\pi(Y_{2,1})$ and $\pi(Y_{2,2})$ intersect by Lemma 5.4. Let us denote

$$\pi(Y_{2,1}) \cap \pi(Y_{2,2}) = \{z_{2,1}, z_{2,2}, \ldots, z_{2,k_2}\} \subset M_2,$$

where the intersection points of $\pi(Y_{2,1})$ and $\pi(Y_{2,2})$ are ordered as $z_{2,1} \ll z_{2,2} \ll \cdots \ll z_{2,k_2}$. At this point we do not know whether $z_{2,1}$ is w_2 or not. By applying Lemma 5.4 again, we conclude for both $j = 1, 2$ that

$$(5.85) \qquad \gamma_1 \cap V_e \subset \bigcup_{l=1}^{k_j} \mathcal{L}_2^+(z_{j,l}) \subset J_j^+(z_{j,1}) \subset (M_j, g_j).$$

In particular,

$$(5.86) \qquad e \in J_j^+(z_{j,1}) \subset (M_j, g_j).$$

Recall that $V \subset M_j$ for both $j = 1, 2$ and $g_1|_V = g_2|_V$. Let γ_2 be the geodesic in (M_2, g_2), which has the same initial condition as γ_1 at e, that is, $(\gamma_2(1), \dot\gamma_2(1)) = (\gamma_1(1), \dot\gamma_1(1))$. It follows that the geodesics γ_1 and γ_2 coincide in $V_e \subset V$. Thus,

$$(5.87) \qquad \gamma_l \cap V_e \subset \bigcup_{h=1}^{k_j} \mathcal{L}_2^+(z_{j,h}) \subset J_j^+(z_{j,1}) \subset (M_j, g_j)$$

for every combination of $l = 1, 2$ and $j = 1, 2$.

Proof of (1): We prove that η_2 and $\tilde\eta_2$ intersect in $W_2 \subset M_2$ the first time at geodesic parameter times $s > 0$ and $s' > 0$.

Fix $e := \hat\mu(f_{\hat a}^+(w_1))$, that is, $a = \hat a$ in (5.83). We approximate the light-like geodesics η_1 and $\tilde\eta_1$ with sequences $\gamma_{(\hat x_l, \hat p_l)}$ and $\gamma_{(\hat y_l, \hat q_l)}$, $l \in \mathbb{N}$ of time-like geodesics $\gamma_{(\hat x, \hat p)}$ and $\gamma_{(\hat y, \hat q)}$. In other words, we choose the geodesics such that for every $l \in \mathbb{N}$ the first intersection $z_{1,1}$ of $\gamma_{(\hat x_l, \hat p_l)}$ and $\gamma_{(\hat y_l, \hat q_l)}$ as geodesics in M_1 is w_1 and the initial values $(\hat x_l, \hat p_l)$ and $(\hat x_l, \hat p_l)$ converge to some points in $(\eta_1, \dot\eta_1) \cap \overline{\mathcal{P}}^+ V$ and $(\tilde\eta_1, \dot{\tilde\eta}_1) \cap \overline{\mathcal{P}}^+ V$, respectively. We may take $\hat x_l$ and $\hat y_l$ to lie in a fixed Cauchy surface $\mathcal{C}$ in M_2 and $\hat x_l \in W_2$ by removing the first terms in the sequence, if necessary. One applies the shortcut argument (Lemma 5.3) and convergence of $\hat x_l$ to show that $\hat x_l \in J_2^+(\hat\mu(s_0))$ for all indices l and some $s_0 \in (s^-, s^+)$. Moreover, $\gamma_{(\hat x_l, \hat p_l)}(s) \in J_2^+(\hat\mu(s_0))$ in M_2, for all $s > 0$, by a similar argument. Consequently, we have for the first intersections $z_{2,1} = z_{2,1}(l)$ of $\gamma_{(\hat x_l, \hat p_l)}$ and $\gamma_{(\hat y_l, \hat q_l)}$ in M_2 the condition

$$z_{2,1} \in \pi(Y_{2,1}) \cap \pi(Y_{2,2}) = \{\gamma_{(\hat x, \hat p)}(s) \in (M_2, g_2) : s \geq 0\} \cap \pi(Y_{2,2}) \subset J_2^+(\hat\mu(s_0)) \quad \text{in} \quad (M_2, g_2),$$

where we omitted the index l. From (5.86) we obtain

$$z_{2,1}(l) \in J_2^+(\hat\mu(s_0)) \cap J_2^-(e) \text{ in } (M_2, g_2),$$

where $z_{2,1}(l)$ is the first intersection of the geodesics $\gamma_{(\hat x_l, \hat p_l)}$ and $\gamma_{(\hat y_l, \hat q_l)}$ in (M_2, g_2). Thus, there exist sequences (s_l) and (s_l') of positive numbers such that

$$(5.88) \qquad z_{2,1}(l) = \gamma_{(\hat x_l, \hat p_l)}(s_l) = \gamma_{(\hat y_l, \hat q_l)}(s_l') \in J_2^+(\hat\mu(s_0)) \cap J_2^-(e).$$

Here $s_l > 0$ and $s'_l > 0$ are so that the geodesics $\gamma_{(\hat{x}_l,\hat{p}_l)}$, $\gamma_{(\hat{y}_l,\hat{q}_l)}$ intersect the first time at the geodesic parameter times s_l and s'_l. To finish the prove of (1), we show that a subsequence of $z_{2,1}(l) \in M_2$ converges to the first intersection of η_2 and $\tilde{\eta}_2$.

Since (M_2, g_2) is globally hyperbolic, the set $J_2^+(\hat{\mu}(s_0)) \cap J_2^-(e)$ is compact. Thus, we may pass to a subsequence so that

$$z_{2,1}(l) \text{ converges in } J_2^+(\hat{\mu}(s_0)) \cap J_2^-(e) \subset W_2 \text{ as } l \to \infty.$$

Applying the parametrisation $\overline{\mathcal{P}}^+\mathcal{C} \times \mathbb{R} \to \overline{\mathcal{P}}^+M_2$, $(x, p, t) \mapsto \gamma_{x,p}(t)$ near the curve η_2 (resp. $\tilde{\eta}_2$) to (5.88) implies that s_l and s'_l must converge. Thus, as the time-like curves in M_2 with initial values $(\hat{x}_l, \hat{p}_l)$ and $(\hat{y}_l, \hat{q}_l)$ approximate η_2 and $\tilde{\eta}_2$, respectively, it follows that there must exist the first intersection point of η_2 and $\tilde{\eta}_2$ in W_2 at the limit of $z_{2,1}(l)$. This is the point w_2.

Proof of (2) and (3): Recall that the first intersection $w_1 \in W_1$ of the time-like approximations $\gamma_{(\hat{x},\hat{p})}$ and $\gamma_{(\hat{y},\hat{q})}$ is also the first intersection w_1 of η_1 and $\tilde{\eta}_1$ in (M_1, g_1) by definition. We also know that the first intersection exist for the associated time-like geodesics in (M_2, g_2). To prove (2), let σ_1 and $\tilde{\sigma}_1$ be another pair of geodesics that satisfies the conditions of η_1 and $\tilde{\eta}_1$. The first part of the proof above applies also for σ_1 and $\tilde{\sigma}_1$ and the first intersection is obtained as a limit of first intersections for some pair of time-like geodesics $\gamma_{(\hat{x}'_l,\hat{p}'_l)}$ and $\gamma_{(\hat{y}'_l,\hat{q}'_l)}$ that as geodesics in M_1 intersect first time in w_1 for every l and approximate the light-like geodesics σ_1 and $\tilde{\sigma}_1$. Thus it suffices to show that for two pairs $(\hat{x}, \hat{p}), (\hat{y}, \hat{q}) \in \mathcal{P}^+V$ and $(\hat{x}', \hat{p}'), (\hat{y}', \hat{q}') \in \mathcal{P}^+V$ with the associated pairs of geodesics in (M_1, g_1) intersecting the first time at w_1 have the property that both pairs $\gamma_{(\hat{x},\hat{p})}$, $\gamma_{(\hat{y},\hat{q})}$ and $\gamma_{(\hat{x}',\hat{p}')}$, $\gamma_{(\hat{y}',\hat{q}')}$ of geodesics in (M_2, g_2) intersect the first time at a mutual point. Note that this point must be the limit $z_{2,1}(l) \to w_2$ of the first intersection points constructed in the proof of (1). In fact, the sequence is a constant sequence.

As earlier, let γ_1 be a light-like optimal geodesic in M_1 from $w_1 = \gamma_1(0)$ to $e = \gamma_1(1)$ and define the geodesic γ_2 in M_2 by the condition $(\gamma_2(1), \dot{\gamma}_2(1)) = (\gamma_1(1), \dot{\gamma}_1(1))$. Let us then consider three pairs of initial vectors

$$(5.89) \qquad \{(\hat{x}, \hat{p}), (\hat{y}, \hat{q})\}, \quad \{(\hat{x}, \hat{p}), (\hat{y}', \hat{q}')\}, \text{ and } \{(\hat{x}', \hat{p}'), (\hat{y}', \hat{q}')\}.$$

constructed from the pairs $(\hat{x}, \hat{p}), (\hat{y}, \hat{q}) \in \mathcal{P}^+V$ and $(\hat{x}', \hat{p}'), (\hat{y}', \hat{q}') \in \mathcal{P}^+V$ above. To each of these pairs of vectors, we may associate a pair of geodesics in (M_1, g_1) which have the vectors as initial data. Each pair of geodesics has the property that they intersect for first time (for positive geodesic parameter times) at $w_1 \in W_1$. On the other hand, to each pair of vectors in (5.89), we may achieve a pair of geodesics in (M_2, g_2) that have the vectors as initial conditions at $s = 0$. As shown earlier, each pair of geodesics has the property that they intersect in M_2 and for the first time (for positive geodesic parameter times) it happens in W_2. We label these first intersection points in W_2 by $z^{(1)}$, $z^{(2)}$ and $z^{(3)}$, respectively. These intersections points lie in $\{\gamma_2(s) : s < 1\}$ according to Lemma 5.6 below. Before proving the lemma let us assume that it holds and show that the intersection points $z^{(1)}$, $z^{(2)}$, $z^{(3)} \in M_2$ ($z^{(1)} = z_{2,1}$ in the proof of (1)) are actually identical. The claim (2) then follows from this since the first intersections of the approximative time-like geodesics accumulate arbitrarily near w_2, as shown in the proof of (1).

We argue by contradiction and suppose that $z^{(1)} \neq z^{(2)}$. That is, we suppose the first intersection of the pairs of geodesics

$$\{\gamma_{(\hat{x},\hat{p})}, \gamma_{(\hat{y},\hat{q})}\} \text{ and } \{\gamma_{(\hat{x},\hat{p})}, \gamma_{(\hat{y}',\hat{q}')}\}.$$

in (M_2, g_2) are distinct. The proof for the case $z^{(2)} \neq z^{(3)}$ is analogous. Applying Lemma 5.6 we deduce that $\gamma_{(\hat{x},\hat{p})}(s)$, $s > 0$ hits $\{\gamma_2(s) : s < 1\}$ twice, first at one of the points $z^{(1)}$, $z^{(2)}$ and then after at the other. We may assume that $\tau(z^{(1)}, z^{(2)}) > 0$ in M_2. For $\tau(z^{(2)}, z^{(1)}) > 0$ one simply swaps the roles of the points in the proof. The shortcut lemma (Lemma 5.3) implies that there is a time-like future-directed segment connecting $z^{(1)}$ to e in M_2. Thus, there is some

point e' in the curve $\hat{\mu}$ that satisfies $\tau(e', e) > 0$ and which can be reached from $z^{(1)}$ along a future-directed lightlike geodesic segment. We can now apply Corollary 4.6 and Proposition 5.1 to get $e' \in \mathrm{singsupp}(\lim_{\epsilon \to 0}(\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P_{e'}))$ in both spaces from which by Lemma 5.4 one obtains $e' \in J^+(w_1)$ in the space (M_1, g_1). Since $\tau(e', e) > 0$, we have $e \neq \hat{\mu}(f_{\hat{a}}^+(w_1))$ which is a contradiction. Thus, we have that

$$z^{(1)} = z^{(2)}.$$

In a similar way one shows that $z^{(2)} = z^{(3)}$.

$\square$

We now prove the following auxiliary lemma:

Lemma 5.6. *Let γ_1 be an optimal future-directed lightlike geodesic in (M_1, g_1) between the points $w_1 = \gamma_1(0) \in W_1$ and $e = \hat{\mu}(f_{\hat{a}}^+(w_1)) = \gamma_1(1)$. Let γ_2 be the geodesic in (M_2, g_2) with $(\gamma_2(1), \dot{\gamma}_2(1)) = (\gamma_1(1), \dot{\gamma}_1(1))$. Let $(x, p), (y, q)$ be elements in $\mathcal{P}^+V$ for both $j = 1, 2$ such that in (M_1, g_1) the geodesics $\gamma_{(x,p)}(s) = \gamma_{(x,p);M_1}(s)$, $s > 0$ and $\gamma_{(y,q)}(s') = \gamma_{(y,q);M_1}(s')$, $s' > 0$ intersect the first time at $w_1 \in W_1$. Then, $\Phi_1 = \Phi_2$ implies that the first intersection point $z = z_{2,1}$ of $\gamma_{(x,p)}(s) = \gamma_{(x,p);M_2}(s)$, $s > 0$ and $\gamma_{(y,q)}(s') = \gamma_{(y,q);M_2}(s')$, $s' > 0$ in (M_2, g_2) belongs to $\{\gamma_2(s) : s < 1\}$.*

Proof. Following the construction in page 34 one puts together Corollary 4.6, Proposition 5.1 to show that for certain sources $h_1^\epsilon, h_2^\epsilon$ and a vector field P_e on neighbourhood U_e around e we have

$$\mathrm{singsupp}(\lim_{\epsilon \to 0}(\Phi''(0; h_1^\epsilon, h_2^\epsilon) \circ P_e)) = \gamma_2 \cap U_e.$$

Moreover, Lemma 5.4 implies that $\gamma_2(1 - \delta)$ belongs to the causal future of z for sufficiently small $\delta > 0$. Hence there is a causal geodesic ν connecting z to $\gamma_2(1 - \delta)$. If $z_1 \notin \{\gamma_2(s) : s < 1\}$ we may combine ν with $\gamma|_{[1-\delta,1]}$ and apply shortcut argument to show that z can be connected to e with a timelike curve in (M_2, g_2). Thus, $\hat{\mu}$ contains a point $e' \ll e$ that can be reached from z with an optimal lightlike geodesic $\tilde{\gamma}_2$. We can then repeat the construction above with some vector field $P_{e'}$ tangent to $\tilde{\gamma}_2$ to derive that $e' \in J_1^+(w_1)$. Thus γ_1 does not define optimal segment from w_1 to e which is in conflict with the assumptions.

$\square$

The determination of the earliest light observation sets from the knowledge of the source-to-solution map data follows from Lemma 5.5:

Below, let $f_{a,1}^+$ and $f_{a,2}^+$ be the earliest observation time functions on M_1 and M_2, respectively. Lemma 5.5 shows that there is a mapping

$$(5.90) \qquad\qquad F : W_1 \to W_2, \quad F(w_1) := w_2,$$

which maps the first intersection w_1 of η_1 and $\tilde{\eta}_1$ to the first intersection w_2 of η_2 and $\tilde{\eta}_2$. Here η_j and $\tilde{\eta}_j$, $j = 1, 2$, are defined as in (5.80)–(5.82).

Proposition 5.7. *Let $\Phi_{1,L+V}$ and $\Phi_{2,L+V}$ be as earlier in Section 5 and assume that $\Phi_{1,L+V} = \Phi_{2,L+V}$. Assume also that the conditions of Theorem 1.3 are satisfied for the Lorentzian manifolds (M_1, g_1) and (M_2, g_2). Then, the map (5.90)*

$$F : W_1 \to W_2$$

is a bijection and

$$(5.91) \qquad\qquad f_{a,1}^+(w_1) = f_{a,2}^+(F(w_1)),$$

for every $w_1 \in W_1$ and $a \in \mathcal{A}$.

Proof. Let $w_1 \in W_1$ and $w_2 = F(w_1)$. We show that the earliest observation time functions $f_{a,1}^+$ on M_1 and $f_{a,2}^+$ on M_2 satisfy $f_{a,1}^+(w_1) = f_{a,2}^+(w_2)$.

Let $a \in \mathcal{A}$ and $f_{a,1}^+ : W_1 \to \mathbb{R}$ be an earliest observation time function on M_1. Moreover, let $e = \mu_a(f_{a,1}^+(w_1))$. (This means that e can be reached by a lightlike future-directed geodesic from w_1.) Let $(\hat{x}, \hat{p})$ and $(\hat{y}, \hat{q})$ be timelike initial vectors as in the last condition of Lemma 5.5. In the space (M_1, g_1) the first intersection of the associated geodesics is w_1 whereas the first intersection $z_{2,1}$ in (M_2, g_2) is w_2. As in the proof of Lemma 5.5 we deduce from $\Phi_{1,L+V} = \Phi_{2,L+V}$ that $e \in J_2^+(z_{2,1})$ in (M_2, g_2) and hence $w_2 \in J_2^-(e)$. Thus, the earliest light observation from w_2 on the path μ_a occurs either at e or at a point that is in the past of e. In particular,

$$(5.92) \qquad f_{a,2}^+(w_2) \leq f_{a,1}^+(w_1).$$

To see that F has an inverse, note that we may change the roles of (M_1, g_1) and (M_2, g_2) in Lemma 5.5 to have a mapping

$$W_2 \to W_1,$$

which maps the first intersection of η_2 and $\tilde{\eta}_2$ to the first intersection of η_1 and $\tilde{\eta}_1$. Here η_j and $\tilde{\eta}_j$, $j = 1, 2$, are defined as in (5.80)–(5.82). This mapping is the inverse of F and $F : W_1 \to W_2$ is a bijection.

By interchanging the roles of (M_1, g_1) and (M_2, g_2) and repeating the above construction we see as in (5.92) that

$$f_{a,1}^+(w_1) \leq f_{a,2}^+(w_2).$$

Thus, we have $f_{a,1}^+(w_1) = f_{a,2}^+(w_2) = f_{a,2}^+(F(w_1))$ for all $w_1 \in W_1$. Hence, $F : W_1 \to W_2$ is a bijection satisfying $f_{a,1}^+(w_1) = f_{a,2}^+(F(w_1))$ for all $w_1 \in W_1$. $\square$

This result implies Proposition 1.4:

Proof of Proposition 1.4. By definition (5.77) of the sets $\mathcal{E}_{\mathcal{U}}^j(w)$, Proposition 5.7 proves that $\mathcal{E}_{\mathcal{U}}^1(w_1) = \mathcal{E}_{\mathcal{U}}^2(w_2)$ for all $w_1 \in W_1$ and $w_2 = F(w_1)$. As $F : W_1 \to W_2$ is a bijection, this proves $\mathcal{E}_{\mathcal{U}}^1(W_1) = \mathcal{E}_{\mathcal{U}}^2(W_2)$. $\square$

5.5. **Determination of the metric from the source-to-solution map.** In this section we prove that F is in fact an isometry,

$$F^* g_2 = g_1.$$

This will prove our main theorem, Theorem 1.3.

To this end, we apply Proposition 5.7 to have the following implication of [24, Theorem 1.2]:

Theorem 5.8. *Let Φ_1 and Φ_2 be as earlier in Section 5 and assume that $\Phi_{1,L+V} = \Phi_{2,L+V}$. Assume also that the conditions of Theorem 1.3 are satisfied for the Lorentzian manifolds (M_1, g_1) and (M_2, g_2). Then, the map*

$$F : W_1 \to W_2$$

is a diffeomorphism and the metric $F^* g_2$ is conformal to g_1 in W_1.

Proof. The claim follows from the proof of [24, Theorem 1.2]. However, let us briefly explain the main steps of the proof as we need the construction in the proof to determine the conformal factor between $F^* g_2$ and g_1 in the next section.

On the manifold (M_j, g_j), let $\mathcal{F}_j : W_j \to C(\mathcal{A})$ be the map from the point q to the corresponding earliest observation time function, that is, $\mathcal{F}_j(w) = f_{a,j}^+(w)$, $j = 1, 2$. Then formula (5.91) and [24, Lemma 2.4 (ii)] imply that

$$(5.93) \qquad\qquad \mathcal{F}_1(w) = \mathcal{F}_2(F(w)), \quad \text{for every } w \in W_1.$$

In particular, the sets $\mathcal{F}_j(W_j) \subset C(\mathcal{A})$, $j = 1, 2$, satisfy $\mathcal{F}_1(W_1) = \mathcal{F}_2(W_2)$, since $F : W_1 \to W_2$ is a bijection by Proposition 5.7. Thus we can write $F = (\mathcal{F}_2)^{-1} \circ \mathcal{F}_1 : W_1 \to W_2$. By [24, Proposition 2.2], this implies that $F : W_1 \to W_2$ is a homeomorphism. Hence the sets $\mathcal{F}_j(W_j)$ can be considered as topological submanifolds of the infinite dimensional vector space $C(\mathcal{A})$. It is shown in [24, Section 5.1.2] that using the knowledge of the set $\mathcal{F}_1(W_1)$, we can construct for any point $Q \in \mathcal{F}_1(W_1)$ a neighborhood $U \subset \mathcal{F}_1(W_1)$ of Q and the values $a_1, a_2, \ldots, a_n \in \mathcal{A}$ such that the following holds: The point $q = \mathcal{F}_1^{-1}(Q) \in W_1$ has a neighborhood $\mathcal{F}_1^{-1}(U) \subset W_1$, where the map $f_{a_1,\ldots,a_n} : x \mapsto (f_{a_k,1}^+(x))_{k=1}^n$ defines smooth local coordinates on W_1. Observe that for $P \in U$, we have $f_{a_k,1}^+ \circ \mathcal{F}_1^{-1}(P) = P(a_k)$. Thus when $E_a : C(\mathcal{A}) \to \mathbb{R}$, $a \in \mathcal{A}$ are the evaluation functions $E_a(P) = P(a)$ and $E_{a_1,\ldots,a_n}(P) = (E_{a_j}(P))_{j=1}^n$, the above constructed pairs $(U, E_{a_1,\ldots,a_n})$ determine on $\mathcal{F}_1(W_1)$ an atlas of differentiable coordinates on $\mathcal{F}_1(W_1)$ which makes $\mathcal{F}_1 : W_1 \to \mathcal{F}_1(W_1)$ a diffeomorphism. As noted above, such atlas can be constructed using only the knowledge of the set $\mathcal{F}_1(W_1)$. Thus, as $\mathcal{F}_1(W_1) = \mathcal{F}_2(W_2)$, the atlases constructed on these sets coincide. This implies that the homeomorphism $F = (\mathcal{F}_2)^{-1} \circ \mathcal{F}_1 : W_1 \to W_2$ is actually a diffeomorhpsim.

Let us next consider the conformal classes of the metric tensors g_1 and g_2. Let $(x, \eta) \in L^+\mathcal{U}$ be a future directed lightlike vector. Let us consider the lightlike geodesics $\gamma_{(x,\eta)}^j(\mathbb{R}_-)$, $j = 1, 2$, on (M_j, g_j) that lie in the causal past of the point x, and let $s_1 > 0$ be such that $\gamma_{(x,\eta)}^j([-s_1, 0]) \subset \mathcal{U}$. By [24, Proposition 2.6] we have that a point $w_j \in W_j$ satisfies $w_j \in \gamma_{(x,\eta)}^j((-\infty, -s_1])$ if and only if $\gamma_{(x,\eta)}^j((-s_1, 0)) \subset \mathcal{E}_\mathcal{U}^j(w_j)$. In other words, for a point $w_j \in W_j$ the set $\mathcal{E}_\mathcal{U}^j(w_j)$ determines whether $w_j \in \gamma_{(x,\eta)}^j((-\infty, -s_1])$. By formula (5.91), this implies that F maps the lightlike geodesic $\gamma^1((-\infty, -s_1]) \cap W_1$ on to $\gamma^2((-\infty, -s_1]) \cap W_2$,

$$(5.94) \qquad\qquad F\big(\gamma^1((-\infty, -s_1]) \cap W_1\big) \subset \gamma^2((-\infty, -s_1]) \cap W_2$$

Let $w_1 \in W_1$ and $\xi_1 \in L_{w_1}^+ M_1$ be such that $\gamma_{(w_1,\xi_1)}^1(\mathbb{R}_+) \cap \mathcal{U} \neq \emptyset$, so that there is $t_1 > 0$ so that $\gamma_{(w_1,\xi_1)}^1(t_1) \in \mathcal{U}$. Let $x = \gamma_{(w_1,\xi_1)}^1(t_1)$ and $\eta = \dot{\gamma}_{(w_1,\xi_1)}^1(t_1)$. Then (5.94) implies that there is $t_0 \in (0, t_1)$ such that

$$(5.95) \quad F\big(\gamma_{(w_1,\xi_1)}^1((-t_0, t_0))\big) \subset F\big(\gamma_{(x,\eta)}^1((-\infty, -t_1 + t_0)) \cap W_1\big) \subset \gamma_{x,\eta}^{(2)}(\mathbb{R}) \cap W_2.$$

Thus for any $w_1 \in W_1$ and $\xi_1 \in L_{w_1}^+ M_1$ such that $\gamma_{w_1,\xi_1}(\mathbb{R}_+) \cap \mathcal{U} \neq \emptyset$, formula (5.95) implies that F restricted to a neighborhood of w_1 maps the lightlike geodesic $\gamma_{(w_1,\xi_1)}^1$ to a segment of a light-like geodesic of (M_2, g_2). Hence, $F_*\xi_1 \in L_{w_2} M_2$, where $w_2 = w_1$. Thus for any $w_1 \in W_1$ there are infinitely many vectors $\xi_1 \in L_{w_1}^+ M_1$ such that $\gamma_{(w_1,\xi_1)}^1(\mathbb{R}_+) \cap \mathcal{U} \neq \emptyset$, and for which $F_* w_1 \in L_{w_2} M_2$, we see that F maps $L_{w_1}^+ M_1$ to $L_{w_2} M_2$. Thus the metric tensor g_1 is conformal to $F^* g_2$ at w_1. As $w_1 \in W_1$ is arbitrary, this implies that g_1 and $F^* g_2$ are conformal $\qquad\square$

To complete the proof of Theorem 1.3 it remains to show that the conformal factor is 1.

5.6. Determination of the conformal factor.
Here we prove that also the conformal factor of the metric is the same. This is the final step in the proof of Theorem 1.3.

Proposition 5.9. *Let $(M, g), (\widetilde{M}, \widetilde{g})$ be globally hyperbolic C^∞ manifolds with metrics g and $\widetilde{g}$. Let the subsets $W \subset M$ and $\widetilde{W} \subset \widetilde{M}$ be open. Let $c \in C^\infty(W)$ be a strictly positive function*

and assume there is a diffeomorphism $F : W \to \widetilde{W}$ such that $(F^*\widetilde{g})(x) = c(x)g(x)$, $\forall x \in W$. Let $\gamma_{(x,p)}$ be the geodesic in (M,g) with initial condition $\dot{\gamma}_{(x,p)}(0) = (x,p)$, where $(x,p) \in TW$. Also let $\sigma_{(x,p)}$ be the geodesic in the manifold $(W, F^*\widetilde{g})$ with $\dot{\sigma}_{(x,p)}(0) = (x,p)$. Assume that for every $x \in W$ there exist two linearly independent timelike vectors $p_1, p_2 \in T_x W$, real numbers $\epsilon_1, \epsilon_2 > 0$ and two smooth functions $\alpha_j : (-\epsilon_j, \epsilon_j) \to \mathbb{R}$, $j = 1, 2$, such that

$$(5.96) \qquad \gamma_{(x,p_j)}(t) = \sigma_{(x,p_j)}(\alpha_j(t)), \quad \text{for every} \quad t \in (-\epsilon_j, \epsilon_j), \quad j = 1, 2.$$

Then c is a constant function on W. In particular, if the metrics g and $F^*\widetilde{g}$ equal at some point $x \in W$, then they equal everywhere in W.

Proof. Let us write $c(x) = e^{\varphi(x)}$ for some smooth function φ, and Γ_{ij}^k and $\widetilde{\Gamma}_{ij}^k$ for the Christoffel symbols of the metrics g and $F^*\widetilde{g}$ respectively. Denote the associated pair of covariant derivatives by ∇ and $\widetilde{\nabla}$. The Christoffel symbols of the conformal metrics g and $F^*\widetilde{g} = cg$ are connected to each other by

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2}\big(\delta_j^k \partial_i \varphi + \delta_i^k \partial_j \varphi - g_{ij}g^{kl}\partial_l \varphi\big).$$

(See [38, Ch. 3, Proposition 13.]). Thus, for any smooth curve γ in W,

$$
\begin{aligned}
(5.97) \quad \widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} =& \ddot{\gamma}^k \partial_k + \widetilde{\Gamma}_{ij}^k(\gamma)\dot{\gamma}^i\dot{\gamma}^j\partial_k \\
=& \ddot{\gamma}^k \partial_k + \Big(\Gamma_{ij}^k(\gamma) + \frac{1}{2}\big(\delta_j^k \partial_i\varphi|_\gamma + \delta_i^k\partial_j\varphi|_\gamma - g_{ij}(\gamma)g^{kl}(\gamma)\partial_l\varphi|_\gamma\big)\Big)\dot{\gamma}^i\dot{\gamma}^j\partial_k \\
=& \ddot{\gamma}^k\partial_k + \Gamma_{ij}^k(\gamma)\dot{\gamma}^i\dot{\gamma}^j\partial_k + \partial_j\varphi|_\gamma \dot{\gamma}^j\dot{\gamma}^k\partial_k - \frac{1}{2}g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j g^{kl}(\gamma)\partial_l\varphi|_\gamma\partial_k \\
=& \nabla_{\dot{\gamma}}\dot{\gamma} + \langle d\varphi, \dot{\gamma}\rangle\dot{\gamma} - \frac{1}{2}g(\dot{\gamma},\dot{\gamma})\nabla\varphi.
\end{aligned}
$$

Let $x \in W$ and, for $j = 1, 2$, let $p_j \in T_x W$, $\epsilon_j > 0$ and $\alpha_j : (-\epsilon_j, \epsilon_j) \to \mathbb{R}$ be as in the claim of this proposition. Let us relax the notation by omitting the indices $j = 1, 2$ from subscripts. By differentiating (5.96) in the variable t we obtain

$$(5.98) \qquad \dot{\gamma}_{(x,p)}(t) = \alpha'(t)\dot{\sigma}_{(x,p)}(\alpha(t))$$

for $t \in (-\epsilon, \epsilon)$. Since

$$\sigma_{(x,p)}(\alpha(0)) = \gamma_{(x,p)}(0) = x = \sigma_{(x,p)}(0)$$

and since the causal geodesics $\sigma_{(x,p)}$ does not have self-intersection by global hyperbolicity (recall that F identifies $\sigma_{(x,p)}$ with a causal geodesic in the globally hyperbolic manifold $\widetilde{M}$), we obtain $\alpha(0) = 0$. Substituting $\alpha(0) = 0$ to (5.98) yields

$$p = \dot{\gamma}_{(x,p)}(0) = \alpha'(0)\dot{\sigma}_{(x,p)}(\alpha(0)) = \alpha'(0)\dot{\sigma}_{(x,p)}(0) = \alpha'(0)p,$$

that is, $\alpha'(0) = 1$. Thus, $\alpha(t) = t + O(t^2)$ near $t = 0$. In particular,

$$\tilde{\nabla}_{\dot{\gamma}_{(x,p)}}\big[\dot{\sigma}_{(x,p)}(\alpha(t))\big]|_{t=0} = \tilde{\nabla}_{\dot{\sigma}_{(x,p)}}\dot{\sigma}_{(x,p)}(t)|_{t=0} = 0.$$

Substituting this to (5.97) and using that $\gamma_{(x,p)}$ is a geodesic imply

$$
\begin{aligned}
0 =& \nabla_{\dot{\gamma}_{(x,p)}}\dot{\gamma}_{(x,p)}(t)|_{t=0} = \widetilde{\nabla}_{\dot{\gamma}_{(x,p)}}[\alpha'(t)\dot{\sigma}_{(x,p)}(\alpha(t))]|_{t=0} - \langle d\varphi, \dot{\gamma}_{(x,p)}(0)\rangle\dot{\gamma}_{(x,p)}(0) \\
& + \frac{1}{2}g(\dot{\gamma}_{(x,p)}(0), \dot{\gamma}_{(x,p)}(0))\nabla\varphi(\gamma_{(x,p)}(0)) \\
=& \alpha''(0)\dot{\sigma}_{(x,p)}(0) + \widetilde{\nabla}_{\dot{\gamma}_{(x,p)}}[\dot{\sigma}_{(x,p)}(\alpha(t))]|_{t=0} - \langle d\varphi(x), p\rangle p + \frac{1}{2}g(p,p)\nabla\varphi(x) \\
=& \big(\alpha''(0) - \langle d\varphi(x), p\rangle\big)p + \frac{1}{2}\underbrace{g(p,p)}_{\neq 0}\nabla\varphi(x).
\end{aligned}
$$

In particular, $\nabla\varphi(x)$ lies in the line spanned by p in the space T_xW. Since this holds for every $x \in W$ and for the corresponding two linearly independent $p_1, p_2 \in T_xW$ (recall that we omitted the subscripts in the calculation above), we conclude that $\nabla\varphi(x) = 0$ for every $x \in W$. That is, the function φ, and hence $c = e^\varphi$ is a constant function on W. $\square$

We finish the proof Theorem 1.3 by showing that the conformal factor is 1; that is, $g_1 = F^*g_2$ on W_1.

Theorem 5.10. *Assume that the conditions of Theorem 1.3 are satisfied and $\Phi_{1,L+V} = \Phi_{2,L+V}$ for Lorentzian manifolds (M_j, g_j), $j = 1, 2$. Equip the associated spaces W_j, $j = 1, 2$ with the canonical metrics $g_j|_{W_j}$ induced by the trivial inclusions $W_j \hookrightarrow M_j$. Then, the map*

$$F : W_1 \to W_2$$

is an isometry.

Proof. Let $w_1 \in W_1$ be arbitrary and choose two time-like vectors $(\hat{x}, \hat{p}), (\hat{y}, \hat{q})$ based in $\mathcal{U}$ such that the geodesics $\gamma_{(\hat{x},\hat{p})}$ and $\gamma_{(\hat{y},\hat{q})}$ intersect first time at w_1 in (M_1, g_1) as earlier. By Lemma 5.5 we may fix them such that the corresponding geodesics in (M_2, g_2) intersect first time at $F(w_1) = w_2$. For $(x, p) \in TW_1$ let us denote by $\sigma_{(x,p)}$ the geodesic in (W_1, F^*g_2) that satisfies $\dot{\sigma}_{(x,p)}(0) = (x, p)$. While keeping (x, p) fixed, we can vary the initial direction (y, q) so that the intersection point of the geodesics $\gamma_{(y,q)}$ and $\gamma_{(x,p)}$ varies along the geodesic $\gamma_{(x,p)}$ through an open geodesic segment. Since the geodesics $\gamma_{(\hat{x},\hat{p})}$ and $\gamma_{(\hat{y},\hat{q})}$ intersect in both geometries, this variation of $w_2 = F(w_1)$ corresponds to a path through w_2 that is some reparametrisation of the geodesic $\gamma_{(\hat{x},\hat{p})}$ in (W_2, g_2). Mapping this reparametrised geodesic segment to M_1 using F^{-1} implies that there is a vector $p_1 \in T_{w_1}W_1$ and a smooth reparametrisation $\alpha_1(s)$ such that the geodesic $\gamma_{(w_1,p_1)}$ in (M_1, g_1) satisfies $\gamma_{(w_1,p_1)}(s) = \sigma_{(w_1,p_1)}(\alpha_1(s))$ on some interval $s \in (-\epsilon_1, \epsilon_1)$. Further, by exchanging the roles of $(\hat{x}, \hat{p})$ and $(\hat{y}, \hat{q})$ and then repeating the construction above one obtains another linearly independent vector $p_2 \in T_{w_1}W_1$ and a smooth reparametrisation $\alpha_2(s)$, $s \in (-\epsilon_2, \epsilon_2)$ such that $\gamma_{(w_1,p_2)}(s) = \sigma_{(w_1,p_2)}(\alpha_2(s))$, for $s \in (-\epsilon_2, \epsilon_2)$. By Theorem 5.8, we have $g_1|_{W_1} = cF^*g_2$. Consequently by Proposition 5.9 we have that $c > 0$ is constant. Since $F^*g_2 = g_1$ on V we conclude that $c = 1$. Thus F is an isometry. $\square$

Appendix A. Auxiliary lemmas

A.1. **Lemmas used in the proof of Proposition 4.3.**

Lemma 4.2. *Let Y_1 and Y_2 and U be as in Definition 4.1 and adopt also the associated notation. Define*

$$\Lambda_R = N^*(\bigcup_{x \in U} \{x\} \times \mathcal{P}_x U \times \mathcal{P}_x U).$$

The submanifold Λ_R of $T^*(U \times \mathcal{P}U \times \mathcal{P}U) \setminus \{0\}$ equals the set

$$\Lambda_R = \{(x, y, z, p, q \,;\, \xi^x, \xi^y, \xi^z, \xi^p, \xi^q) \in \; T^*(U \times \mathcal{P}U \times \mathcal{P}U) \setminus \{0\} :$$
$$\xi^x + \xi^y + \xi^z = 0, \; \xi^p = \xi^q = 0, \; x = y = z\}.$$

Then we have

$$\Lambda'_R = \{((x; \xi^x), (y, z, p, q \,;\, \xi^y, \xi^z, \xi^p, \xi^q)) \in \; T^*U \times T^*(\mathcal{P}U \times \mathcal{P}U) \setminus \{0\} :$$
(A.99)
$$\xi^x = \xi^y + \xi^z \neq 0, \; \xi^p = \xi^q = 0, \; x = y = z\}.$$

The spaces $\Lambda'_R \times (N^*[Y_1 \times Y_2])$ and $T^*U \times \mathrm{diag}\,T^*(\mathcal{P}M \times \mathcal{P}M)$ intersect transversally in $T^*U \times T^*(\mathcal{P}M \times \mathcal{P}M) \times T^*(\mathcal{P}M \times \mathcal{P}M)$.

Proof. First notice that the manifold $T\left(\bigcup_{x \in U} \{x\} \times \mathcal{P}_x U \times \mathcal{P}_x U\right)$ can be written by using local coordinates as

(A.100) $$\mathcal{V}_1 := \{(x, y, z, p, q \,;\, \dot{x}, \dot{y}, \dot{z}, \dot{p}, \dot{q}) \in (U^3 \times \mathbb{R}^{2n}) \times \mathbb{R}^{5n} : \dot{x} = \dot{y} = \dot{z}, \; x = y = z\}.$$

Note the $\mathcal{V}_1$ is a space of dimension $6n$. Let us consider the subspace of $T^*(U \times \mathcal{P}U \times \mathcal{P}U)$

$$\mathcal{V}_2 := \{(x, y, z, p, q \,;\, \xi^x, \xi^y, \xi^z, \xi^p, \xi^q) : x = y = z, \; \xi^x + \xi^y + \xi^z = 0, \; \xi^p = \xi^q = 0\}.$$

Note that a vector in $\mathcal{V}_1$ paired with a covector in $\mathcal{V}_2$ yields zero since

$$(\xi^x, \xi^y, \xi^z, \xi^p, \xi^q) \cdot (\dot{x}, \dot{y}, \dot{z}, \dot{p}, \dot{q}) = \dot{x}\,\xi^x + \dot{y}\,\xi^y + \dot{z}\,\xi^z = \dot{x}(\xi^x + \xi^y + \xi^z) = 0.$$

Thus we have that $\mathcal{V}_2 \subset \Lambda_R$. Note that the fibers of $\mathcal{V}_1$ are of dimension $5n - 2n = 3n$. Note also that the fibers of $\mathcal{V}_2$ have dimension $5n - 3n = 2n$. We thus have $\mathcal{V}_2 = \Lambda_R$, since dimensions of a fiber of the conormal bundle of $\cup_{x \in U}\{x\} \times \mathcal{P}_x U \times \mathcal{P}_x U$ is the same as the codimension of a fiber of $\mathcal{V}_1$. In conclusion, the coordinate expressions for Λ_R and Λ'_R hold.

Next we prove the transversality claim. First, fix the notation

$$X = T^*U \times T^*(\mathcal{P}M \times \mathcal{P}M) \times T^*(\mathcal{P}M \times \mathcal{P}M)$$
$$L_1 = \Lambda'_R \times (N^*[Y_1 \times Y_2])$$
$$L_2 = T^*U \times \mathrm{diag}\,T^*(\mathcal{P}M \times \mathcal{P}M).$$

We want to show that the linear spaces L_1 and L_2 intersect transversally in X; that is for all $\lambda \in L_1 \cap L_2$,

(A.101) $$\dim(T_\lambda L_1 + T_\lambda L_2) = \dim T_\lambda X.$$

Let $\lambda \in L_1 \cap L_2$. Since Y_1 and Y_2 satisfy the intersection property, there exists coordinates $x = (x', x'')$ on U such that

(A.102)
$$Y_1 \cap TU = \{(x, p) \in TU : x' = 0, \; p' = 0, \; p'' = (1, 0, \ldots, 0)\},$$
$$Y_2 \cap TU = \{(x, p) \in TU : x'' = 0, \; p'' = 0, \; p' = (1, 0, \ldots, 0)\},$$

in the associated canonical coordinates $(x, p) = (x', x'', p', p'')$ of TU. We next compute the induced local expressions of L_1, L_2 and $L_1 \cap L_2$ with respect to the coordinate system in (A.102).

Above, we showed that Λ'_R has the local expression (A.99). Therefore if $\lambda \in L_1$, λ has the local coordinate form

$$\text{(A.103)} \qquad \lambda = ((x, \xi^x, y, p, \xi^y, 0, z, q, \xi^z, 0), \alpha, \beta),$$

where $(\alpha, \beta) \in N^*[Y_1 \times Y_2]$, $(x, \xi^x) \in T^*U$, and $(y, p, \xi^y, 0), (z, q, \xi^z, 0) \in T^*(\mathcal{P}U)$ are such that $y = z = x$ and $\xi^y + \xi^z = \xi^x$.

If $(\alpha, \beta) \in N^*[Y_1 \times Y_2]$, then (α, β) is nonzero and must satisfy

$$\text{(A.104)} \qquad \alpha = (x', 0, p', 0, 0, (\xi^x)'', 0, (\xi^p)'')$$

$$\text{(A.105)} \qquad \beta = (0, y'', 0, q'', (\xi^y)', 0, (\xi^q)', 0),$$

where one (but not both) of the components is allowed to be zero.

The local expression for $\lambda \in L_2$ induced by (A.102) is

$$\text{(A.106)} \qquad \lambda = (x, \xi^x, y, p, z, q, \xi^y, \xi^p, \xi^z, \xi^q, y, p, z, q, \xi^y, \xi^p, \xi^z, \xi^q).$$

where $x, y, z \in U$.

Using (A.103), (A.104), and (A.106), we obtain that points $\lambda \in L_1 \cap L_2$ are described by

$$\text{(A.107)} \qquad \lambda = (0, \xi^x, \gamma, \gamma),$$

where

$$\xi^x = ((\xi^x)', (\xi^x)''),$$
$$\gamma = (0, (p', 0), (0, (\xi^x)''), 0, 0, (0, p''), (0, (\xi^x)'), 0),$$
$$p' = (1, 0, \dots, 0),$$
$$p'' = (1, 0, \dots, 0).$$

Indeed, if $\lambda \in L_1 \cap L_2$, we must have $(x', x'') = (y', 0) = (0, z'')$ which is only satisfied if $x = y = z = 0$. From (A.103), we have $\xi^p = 0 = \xi^q$ in (A.106). Since $\xi^x = \xi^y + \xi^z$ in (A.103), from (A.104) and (A.106) we find $((\xi^x)', (\xi^x)'') = ((\xi^z)', (\xi^y)'')$. From (A.102), $(p', p'') = (1, 0, \dots, 0, 1, 0, \dots, 0)$.

From the above coordinate expressions, we now compute the dimensions of L_1, L_2, and $L_1 \cap L_2$. First note that

$$\dim(X) = \dim(T^*U) + \dim(T^*(\mathcal{P}M \times \mathcal{P}M)) + \dim(T^*(\mathcal{P}M \times \mathcal{P}M)) = 18n$$

Similarly one computes $\dim(L_2) = 10n$. The expression (A.107) shows that $\dim(L_1 \cap L_2) = n$.

Therefore, for $\lambda \in L_1 \cap L_2$,

$$\dim(T_\lambda L_1) + \dim(T_\lambda L_2) - \dim T_\lambda(L_1 \cap L_2) = 9n + 10n - n = 18n = \dim(T_\lambda X).$$

This shows that L_1 is transverse to L_2 in X. $\qquad \square$

APPENDIX B. EXISTENCE THEOREMS FOR VLASOV AND BOLTZMANN CAUCHY PROBLEMS

In this section, the space (M, g) is assumed to be globally hyperbolic C^∞-smooth Lorentzian manifold. By $\gamma_{(x,p)} : (-T_1, T_2) \to M$ we denote the inextendible geodesic which satisfies

$$\text{(B.108)} \qquad \gamma_{(x,p)}(0) = x \text{ and } \dot{\gamma}_{(x,p)}(0) = p.$$

We do not assume that (M, g) is necessarily geodesically complete. Therefore, we might have that $T_1 < \infty$ or $T_2 < \infty$. We will repeatedly use the fact that if $f(x, p)$ is a smooth function on $\overline{\mathcal{P}}^+ M$ whose support on the base variable $x \in M$ is compact, then the map

$$\text{(B.109)} \qquad (x, p) \mapsto \int_{-\infty}^0 f(\gamma_{(x,p)}(t), \dot{\gamma}_{(x,p)}(t)) dt \quad \text{on} \quad (x, p) \in \overline{\mathcal{P}}^+ M$$

is well defined. This is because on a globally hyperbolic Lorentzian manifold any causal geodesic γ exits a given compact set K_π permanently after finite parameter times. That is, there are parameter times t_1, t_2 such that $\gamma(\{t < t_1\}), \gamma(\{t > t_2\}) \subset M \setminus K_\pi$. Thus the integral above is actually an integral of a smooth function over a finite interval. Further, since f and the geodesic flow on (M, g) are smooth the map in (B.109) is smooth. If (M, g) is not geodesically complete and if $\gamma_{(x,p)} : (-T_1, T_2) \to M$, we interpret the integral above to be over $(-T_1, 0]$. We interpret similarly for all similar integrals in this section without further notice.

We record the following lemma.

Lemma B.1. *Let (M, g) be a globally hyperbolic Lorentzian manifold, let X be a compact subset of $\overline{\mathcal{P}}^+ M$ and let K_π be a compact subset of M. Then the function $\ell : \overline{\mathcal{P}}^+ M \to \mathbb{R}$*

$$\ell(x, p) = \sup\{s \geq 0 : \gamma_{(x,p)}(-s) \in K_\pi\}.$$

is upper semi-continuous and

(B.110)
$$l_0 = \max\left\{\ell(y, q) : (y, q) \in X\right\} < \infty.$$

In addition, if $\lambda > 0$ then $\ell(x, \lambda p) = \lambda^{-1}\ell(x, p)$.

Proof. To see that l is upper semi-continuous, consider a geodesic touching the closed set K_π at a single point. Since the globally hyperbolic manifold (M, g) is disprisoning and causally pseudoconvex, any of its causal geodesics exits the compact set K_π permanently after finite parameter times, see e.g. [5, Proposition 1] and [4, Lemma 11.19]. It follows that $\ell(x, p) < \infty$ for all $(x, p) \in \overline{\mathcal{P}}^+ M$. The maximum in (B.110) exists since ℓ is upper semi-continuous and X is compact. Since $\gamma_{(x,\lambda p)}(s) = \gamma_{(x,p)}(\lambda s)$, for all $\lambda \in \mathbb{R}$, we have that $\ell(x, \lambda p) = \lambda^{-1}\ell(x, p)$, for $\lambda > 0$. $\qquad\square$

In the next theorem we consider sources which are compactly supported in $\overline{\mathcal{P}}^+ M$. Note that this especially means that f is supported outside a neighbourhood of the zero section of TM, since by our convention $\{0\} \notin \overline{\mathcal{P}}^+ M$. Recall that if $\mathcal{C}$ is a Cauchy surface of M, we use the notation $\mathcal{C}^\pm$ for the causal future $(+)$ or the causal past $(-)$ of $\mathcal{C}$:

$$\mathcal{C}^\pm = J^\pm(\mathcal{C}).$$

In particular, the notation $\overline{\mathcal{P}}^+ \mathcal{C}^\pm$ refers to the subset of TM of future directed causal vectors on the causal future $(+)$ or past $(-)$ of the Cauchy surface $\mathcal{C}$. That is, we abbreviate $\overline{\mathcal{P}}^+ \mathcal{C}^\pm = \overline{\mathcal{P}}^+ [J^\pm(\mathcal{C})]$.

If $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ is compact and $k \geq 0$ is integer, we define

$$C_K^k(\overline{\mathcal{P}}^+ \mathcal{C}^+) := \{f \in C^k(\overline{\mathcal{P}}^+ \mathcal{C}^+) : \operatorname{supp}(f) \subset K\}.$$

and

$$C_b(\overline{\mathcal{P}}^+ \mathcal{C}^+) := \{f \in C(\overline{\mathcal{P}}^+ \mathcal{C}^+) : f \text{ is bounded}\},$$

Notice that the spaces $C_K^k(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ and $C_b(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ equipped with the norm of C^k and sup-norm respectively are Banach spaces. We define the C^k norm of a function by fixing a partition of unity and summing up the C^k norms of the local coordinate representations of the function.

Theorem 3.2. *Assume that (M, g) is a globally hyperbolic C^∞-smooth Lorentzian manifold. Let $\mathcal{C}$ be a Cauchy surface of (M, g), $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ be compact and $k \geq 0$. Let also $f \in C_K^k(\overline{\mathcal{P}}^+ M)$. Then, the problem*

$$\mathcal{X}u(x, p) = f(x, p) \quad on \quad \overline{\mathcal{P}}^+ M$$

(B.111)
$$u(x, p) = 0 \quad\quad on \quad \overline{\mathcal{P}}^+ \mathcal{C}^-$$

has a unique solution u in $C^k(\overline{\mathcal{P}}^+ M)$. In particular, if $Z \subset \overline{\mathcal{P}}^+ M$ is compact, there is a constant $c_{k,K,Z} > 0$ such that

(B.112)
$$\|u|_Z\|_{C^k(Z)} \le c_{k,K,Z} \|f\|_{C^k(\overline{\mathcal{P}}^+ M)}.$$

If $k = 0$, the estimate above is independent of Z:

$$\|u\|_{C(\overline{\mathcal{P}}^+ M)} \le c_K \|f\|_{C(\overline{\mathcal{P}}^+ M)}.$$

Proof. Let us denote by $K_\pi = \pi(K)$ the compact set containing $\pi(\mathrm{supp}\,(f))$. Let $(x,p) \in \overline{\mathcal{P}}^+ M$ and $f \in C_c^k(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. Evaluating (B.111) at $(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))$ reads

$$(\mathcal{X}u)(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) = f(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)).$$

Since $\mathcal{X}$ is the geodesic vector field we have for all s that

$$(\mathcal{X}u)(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) = \frac{d}{ds} u(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)).$$

By integrating in s, we obtain

(B.113)
$$u(x,p) = \int_{-\infty}^0 f(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) ds.$$

Here we used that $f(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))$ vanishes for $s < -\ell(x,p)$ by Lemma B.1, where

$$\ell(x,p) = \sup\{s \ge 0 : \gamma_{(x,p)}(-s) \in K_\pi\}.$$

We verify that u is a solution to (B.111). Note that if $(y,q) = \big(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)\big)$, then

$$\gamma_{(y,q)}(z) = \gamma_{(x,p)}(z+s) \text{ and } \dot{\gamma}_{(y,q)}(z) = \dot{\gamma}_{(x,p)}(z+s).$$

It follows that

$$u(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) = \int_{-\infty}^0 f\big(\gamma_{(x,p)}(z+s), \dot{\gamma}_{(x,p)}(z+s)\big) dz = \int_{-\infty}^s f\big(\gamma_{(x,p)}(z), \dot{\gamma}_{(x,p)}(z)\big) dz$$

and consequently

$$\mathcal{X}u(x,p) = \frac{d}{ds}\Big|_{s=0} u(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) = f\big(\gamma_{(x,p)}(0), \dot{\gamma}_{(x,p)}(0)\big) = f(x,p).$$

If $(x,p) \in \overline{\mathcal{P}}^+ \mathcal{C}^-$, then $u(x,p) = 0$ by the integral formula (B.113) and the fact that $f \in C_c^k(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. We have now shown that a solution u to (B.111) exists. The solution u is unique since it was obtained by integrating the equation B.111.

Next we prove the estimate (3.22). We have by the representation formula (B.113) for the solution u that

(B.114)
$$\sup_{(x,p) \in \overline{\mathcal{P}}^+ M} |u(x,p)| = \sup_{(x,p) \in \overline{\mathcal{P}}^+ K} |u(x,p)|.$$

The equation (B.114) holds since $\pi(\mathrm{supp}\,(f))$ is properly contained in K_π. Let e be some auxiliary smooth Riemannian metric on M and let $SK_\pi \subset TM$ be the unit sphere bundle with respect to e over K_π. Let us also denote

$$X = SK_\pi \cap \overline{\mathcal{P}}^+ K_\pi$$

the bundle of future directed causal (with respect to g) vectors that have unit length in the Riemannian metric e. Since X is a closed subset of the compact set SK_π, we have that X is compact. By Lemma B.1 we have that

$$l_0 = \max \big\{\ell(y,q) : (y,q) \in X\big\}$$

exists.

Let us continue to estimate $|u(x,p)|$ for $(x,p) \in \overline{\mathcal{P}}^+ K_\pi$. If $(x,p) \in \overline{\mathcal{P}}^+ K_\pi$, then there is $\lambda > 0$ such that $(x, \lambda^{-1} p) \in X$. Let us denote $q = \lambda^{-1} p \in SK_\pi$. We have that

$$
\begin{aligned}
u(x,p) &= \int_{-\infty}^0 f(\gamma_{(x,\lambda q)}(s), \dot{\gamma}_{(x,\lambda q)}(s)) ds = \int_{-\infty}^0 f(\gamma_{(x,q)}(\lambda s), \lambda \dot{\gamma}_{(x,q)}(\lambda s)) ds \\
&= \frac{1}{\lambda} \int_{-\infty}^0 f(\gamma_{(x,q)}(s), \lambda \dot{\gamma}_{(x,q)}(s)) ds = \frac{1}{\lambda} \int_{-l_0}^0 f(\gamma_{(x,q)}(s), \lambda \dot{\gamma}_{(x,q)}(s)) ds.
\end{aligned}
$$

(B.115)

Here we used

$$
\gamma_{(x,\lambda p)}(z) = \gamma_{(x,p)}(\lambda z) \quad \text{and} \quad \frac{d}{dz} \gamma_{(x,\lambda p)}(z) = \lambda \dot{\gamma}_{(x,p)}(\lambda z).
$$

Let us define two positive real numbers

$$
C = \max_{s \in [0,l_0]} \max_{(x,q) \in X} |\dot{\gamma}_{(x,q)}(-s)|_e < \infty
$$

$$
R = \inf\{r > 0 : f|_{B_e(0,r) \subset T_x M} = 0 \text{ for all } x \in K_\pi\} > 0.
$$

Here for $x \in K_\pi$, the set $B_e(0,r)$ is the unit ball of radius r with respect to the Riemannian metric e in the tangent space $T_x M$. The constant R is positive since f has compact support in $\overline{\mathcal{P}}^+ M$ by assumption. Let us define

$$
\lambda_{\min} := \frac{R}{C} > 0.
$$

Then, if $\lambda < \lambda_{\min} = \frac{R}{C}$, we have for $(x,q) \in X$ that

$$
\int_{-l_0}^0 f(\gamma_{(x,q)}(s), \lambda \dot{\gamma}_{(x,q)}(s)) ds = 0,
$$

since in this case $|\lambda \dot{\gamma}_{(x,q)}(s)| < R$ for all $s \in [-l_0, 0]$. It follows that for all $\lambda > 0$ we have that

$$
\frac{1}{\lambda} \left| \int_{-l_0}^0 f(\gamma_{(x,q)}(s), \lambda \dot{\gamma}_{(x,q)}(s)) ds \right| \leq \frac{c}{\lambda_{\min}} \|f\|_{C(\overline{\mathcal{P}}^+ M)}.
$$

Finally, combining the above with (B.114) and (B.115) shows that

$$
\begin{aligned}
\|u\|_{C(\overline{\mathcal{P}}^+ M)} &= \sup_{(x,p) \in \overline{\mathcal{P}}^+ M} |u(x,p)| \leq \sup_{\lambda > 0} \sup_{(x,q) \in X} \frac{1}{\lambda} \left| \int_{-l_0}^0 f(\gamma_{(x,q)}(s), \lambda \dot{\gamma}_{(x,q)}(s)) ds \right| \\
&\leq \frac{c}{\lambda_{\min}} \|f\|_{C(\overline{\mathcal{P}}^+ M)}.
\end{aligned}
$$

Let $Z \subset \overline{\mathcal{P}}^+ M$ be a compact set. We next show that

$$
\|u\|_{C^k(Z)} \leq c_{k,K,Z} \|f\|_{C^k(\overline{\mathcal{P}}^+ M)}
$$

for $k \geq 1$.

We have proven that this estimate holds for $k = 0$. We prove the claim for $k > 0$. Let ∂ denote any of the partial differentials ∂_{x^a} or ∂_{p^a} in canonical coordinates of the bundle TM. We apply ∂ to the formula (B.113) of the solution u to obtain

$$
\partial u(x,p) = \int_{-\infty}^0 \left[\frac{\partial f}{\partial x^\alpha}(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) \partial \gamma_{(x,p)}^\alpha(s) + \frac{\partial f}{\partial p^\alpha}(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)) \partial \dot{\gamma}_{(x,p)}^\alpha(s) \right] ds.
$$

Since $\frac{\partial f}{\partial x^\alpha}$ and $\frac{\partial f}{\partial p^\alpha}$ have the same properties as f, and the smooth coefficients $\partial \gamma_{(x,p)}^\alpha$ and $\partial \dot{\gamma}_{(x,p)}^\alpha$ are uniformly bounded for $(x,p) \in Z \subset \overline{\mathcal{P}}^+ M$, we may apply the proof above to show that

$$
\|u\|_{C^1(Z)} \leq c_{1,K,Z} \|f\|_{C^1(\overline{\mathcal{P}}^+ M)}.
$$

The proof for $k \geq 2$ is similar. $\square$

By using the solution formula (B.113),

$$u(x,p) = \int_{-\infty}^{0} f(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))ds,$$

in the proof of Theorem 3.2, we have the following result for Cauchy problems for the equation $\mathcal{X}u = f$ restricted to $\mathcal{P}^+M$ and L^+M. We denote by $L^+\mathcal{C}^\pm$ the bundle of future-directed lightlike vectors in the future $\mathcal{C}^+$ or past $\mathcal{C}^-$ of a Cauchy surface $\mathcal{C}$.

Corollary B.2. *Assume as in Theorem 3.2 and adopt its notation. Then for the compact set $K \subset \overline{\mathcal{P}}^+\mathcal{C}^+$, the Cauchy problems*

(B.116)
$$\begin{aligned} \mathcal{X}u(x,p) &= f(x,p) && on \quad \mathcal{P}^+M \\ u(x,p) &= 0 && on \quad \mathcal{P}^+\mathcal{C}^-, \end{aligned}$$

and

(B.117)
$$\begin{aligned} \mathcal{X}u(x,p) &= f(x,p) && on \quad L^+M \\ u(x,p) &= 0 && on \quad L^+\mathcal{C}^-, \end{aligned}$$

have continuous solution operators $\mathcal{X}^{-1} : C_K^k(\mathcal{P}^+M) \to C^k(\mathcal{P}^+M)$ and $\mathcal{X}_L^{-1} : C_K^k(L^+M) \to C^k(L^+M)$ respectively.

Note that we slightly abused notation by denoting by $\mathcal{X}^{-1}$ the solution operator to both Cauchy problems (B.111) and (B.116). Here $C_K^k(\mathcal{P}^+M)$ and $C_K^k(L^+M)$ are defined similarly as $C_K^k(\overline{\mathcal{P}}^+M)$. Since $\mathcal{P}^+M$ and L^+M are manifolds without boundary, we are able to use standard results to extend the problems (B.116) and (B.117) for a class of distributional sources f.

Lemma B.3. *Assume that (M,g) is a globally hyperbolic C^∞-smooth Lorentzian manifold. Let $\mathcal{C}$ be a Cauchy surface of (M,g)*

(1) The solution operator $\mathcal{X}^{-1}$ to the Cauchy problem (B.116) on $\mathcal{P}^+M$ has a unique continuous extension to $f \in \{h \in \mathcal{D}'(\mathcal{P}^+M) : WF(h) \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset, h = 0 \text{ in } \mathcal{P}^+\mathcal{C}^-\}$. If S is a submanifold of $\mathcal{P}^+\mathcal{C}^+$, $f \in I^l(\mathcal{P}^+M; N^*S)$, $l \in \mathbb{R}$, then we have that $u = \mathcal{X}^{-1}f$ satisfies $\chi u \in I^{l-1/4}(\mathcal{P}^+M; N^*K_S)$ for any $\chi \in C_c^\infty(\mathcal{P}^+M)$ with $supp(\chi) \subset\subset \mathcal{P}^+M \setminus S$.

(2) The solution operator $\mathcal{X}_L^{-1}$ to the Cauchy problem (B.117) on L^+M has a unique continuous extension to $\{h \in \mathcal{D}'(L^+M) : WF(h) \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset, h = 0 \text{ in } L^+\mathcal{C}^-\}$. If S is a submanifold of $L^+\mathcal{C}^+$, $f \in I^l(L^+M; N^*S)$, $l \in \mathbb{R}$, then we have that $u = \mathcal{X}^{-1}f$ satisfies $\chi u \in I^{l-1/4}(L^+M; N^*K_S)$ for any $\chi \in C_c^\infty(L^+M)$ with $supp(\chi) \subset\subset L^+M \setminus S$.

Proof. Let us first consider the solution operator $\mathcal{X}^{-1}$ to (B.116). We will refer to [13, Theorem 5.1.6]. To do that, we consider $\mathcal{P}^+M$ as $\mathbb{R} \times \mathcal{P}^+\mathcal{C}$ by using the flowout parametrization $\mathbb{R} \times \mathcal{P}^+\mathcal{C} \to \mathcal{P}^+M$ given by

$$(s, (x,p)) \mapsto \dot{\gamma}_{(x,p)}(s), \quad s \in \mathbb{R}, \ (x,p) \in \mathcal{P}^+\mathcal{C}.$$

Also, by reviewing the proof of Lemma 3.1, we conclude that $\mathcal{X}$ is strictly hyperbolic with respect to $\mathcal{P}^+\mathcal{C}$. Then, by [13, Theorem 5.1.6], the problem (B.116) has a unique solution for $f \in \{h \in \mathcal{D}'(\mathcal{P}^+M) : WF(h) \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset, h = 0 \text{ in } \mathcal{P}^+\mathcal{C}^-\}$. By [13, Remarks after Theorem 5.1.6] the solution operator $\mathcal{X}^{-1}$ to (B.116) extends continuously to $\{h \in \mathcal{D}'(\mathcal{P}^+M) : WF(h) \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset, h = 0 \text{ in } \mathcal{P}^+\mathcal{C}^-\}$.

If $S \subset \mathcal{P}^+\mathcal{C}^+$ and $f \in I^l(\mathcal{P}^+M; N^*S)$, then we have $f \in \{h \in \mathcal{D}'(\mathcal{P}^+M) : WF(h) \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset\}$, because

$$WF(f) \cap N^*(\mathcal{P}^+\mathcal{C}) \subset N^*S \cap N^*(\mathcal{P}^+\mathcal{C}) = \emptyset.$$

By using the definitions of the sets C_0 and R_0, which appear in [13, Theorem 5.1.6], we obtain

$$C_0 \circ R_0 = \{((x,p;\xi),(z,w;\lambda)) \in T^*(\mathcal{P}^+M) \times T^*(\mathcal{P}^+M) \setminus \{0\} : (x,p;\xi) \text{ on the bicharacteristic}$$
$$\text{strip through } \lambda \in T^*_{(z,w)}\mathcal{P}^+M \text{ with } (z,w) \in \mathcal{P}^+\mathcal{C} \text{ and } \sigma_{\mathcal{X}}(z,w;\lambda) = 0\}.$$

Here $\sigma_{\mathcal{X}}$ is the principal symbol of $\mathcal{X}$, see (3.17). Let $\chi_1 \in C_c^\infty(\mathcal{P}^+M)$ be such that $\mathrm{supp}(\chi_1) \subset\subset \mathcal{P}^+M \setminus S$. We choose $\chi_2 \in C_c^\infty(\mathcal{P}^+M)$ such that χ_2 equals 1 on a neighborhood of S and $\mathrm{supp}(\chi_2) \subset \mathcal{P}^+\mathcal{C}^+$ and such that $\mathrm{supp}(\chi_1) \cap \mathrm{supp}(\chi_2) = \emptyset$. Let us denote $A = \chi_1\mathcal{I}$ and $B = \chi_2\mathcal{I}$, where $\mathcal{I}$ is the identity operator. If we consider A and B as pseudodifferential operators of class $\Psi_{1,0}^0(\mathcal{P}^+M)$, we have that $(WF(A) \times WF(B)) \cap [\mathrm{diag}(T^*(\mathcal{P}^+M)) \cup (C_0 \circ R_0)] = \emptyset$. We write

$$\chi_1 u = \chi_1\mathcal{X}^{-1}\chi_2 f + \chi_1\mathcal{X}^{-1}(1-\chi_2)f = A\mathcal{X}^{-1}Bf,$$

where we used $(1-\chi_2)f = 0$ so that $\mathcal{X}^{-1}(1-\chi_2)f = 0$. By [13, Theorem 5.1.6], we have that

$$A\mathcal{X}^{-1}B \in I^{-1/4}(\mathcal{P}^+M, \mathcal{P}^+M; \Lambda_{\mathcal{X}}).$$

The flowout of the conormal bundle over S under $\mathcal{X}$ is the conormal bundle of the geodesic flowout of S. That is

$$\Lambda_{\mathcal{X}} \circ N^*S = N^*K_S.$$

Finally, by applying [13, Theorem 2.4.1, Theorem 4.2.2], we have

(B.118) $$\chi_1 v \in I^{l-1/4}(\mathcal{P}^+M; N^*K_S).$$

Renaming χ_1 as χ concludes the proof of **(1)**. The proof of **(2)** is a similar application of [13, Theorem 5.1.6] by using the flowout parametrization for L^+M given by $(s,(x,p)) \mapsto \dot{\gamma}_{(x,p)}(s)$, $(x,p) \in L^+\mathcal{C}$ and $s \in \mathbb{R}$. $\square$

Next we prove that the Boltzmann equation has unique small solutions for small enough sources. Before that, we give an estimate regarding the collision operator in the following lemma. Following our convention of this section, the integral in the statement of the lemma over a geodesic parameter is interpreted to be over the largest interval of the form $(-T, 0]$, $T > 0$, where the geodesic exists.

Lemma B.4. *Let (M, g) be a globally hyperbolic Lorentzian manifold and let $\mathcal{Q}$ be a collision operator with an admissible collision kernel $A : \Sigma \to \mathbb{R}$ in the sense of Definition 1.1. Then there exists a constant $C_A > 0$ such that*

$$\left| \int_{-\infty}^0 \mathcal{Q}[v,u](\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))ds \right| \leq C_A \|u\|_{C(\overline{\mathcal{P}}^+M)} \|v\|_{C(\overline{\mathcal{P}}^+M)}$$

for every $(x,p) \in \overline{\mathcal{P}}^+M$ and $u, v \in C_b(\overline{\mathcal{P}}^+M)$.

Proof. Let us define a compact set $K_\pi := \pi(\mathrm{supp}(A))$. Let e be some auxiliary Riemannian metric on M. Let us denote

$$X = SK_\pi \cap \overline{\mathcal{P}}^+K_\pi$$

the bundle of future directed causal (with respect to g) vectors who have unit length in the Riemannian metric e. Since X is a closed subset of the compact set SK_π, we have that X is compact. By Lemma B.1, we have that there exists the maximum

$$l_0 = \max\{\ell(y,q) : (y,q) \in X\},$$

where $\ell(x,p) = \sup\{s \geq 0 : \gamma_{(x,p)}(-s) \in K_\pi\}$.

Let us define another compact set $\mathcal{K}$ as

$$\mathcal{K} := \{(y,r) \in \overline{\mathcal{P}}^+M : (y,r) = (\gamma_{(x,q)}(s), \dot{\gamma}_{(x,q)}(s)), \ s \in [-\ell(x,q), 0], \ (x,q) \in X\}.$$

To see that $\mathcal{K}$ is compact, note that it is the image of the compact set $\{(s, (x,q)) : s \in [-\ell(x,q), 0], (x,q) \in X\}$ under the geodesic flow. The set $\{(s, (x,q)) : s \in [-\ell(x,q), 0], (x,q) \in X\}$ is compact since it is bounded by Lemma B.1 and closed by the upper semi-continuity of ℓ. Since the collision kernel is admissible, the function

$$\lambda \mapsto F_{x,p}(\lambda) = \|A(x, \lambda p, \,\cdot\,, \,\cdot\,, \,\cdot\,)\|_{L^1(\Sigma_{x,\lambda p})},$$

is by assumption continuously differentiable in λ and attains its minimum value zero at $\lambda = 0$. Thus, for any $(x,p) \in \overline{\mathcal{P}}^+ M$, we have that

$$\frac{d}{d\lambda}\Big|_{\lambda=0} F_{x,p}(\lambda) = \lambda^{-1} F_{x,p}(\lambda) \longrightarrow 0$$

as $\lambda \to 0$. Since the continuous function $(x,p) \mapsto \frac{d}{d\lambda}\big|_{\lambda=0} F_{x,p}(\lambda)$ on the compact set $\mathcal{K}$ is uniformly continuous, there is a constant $\lambda_0 > 0$ such that

(B.119) $$\lambda^{-1} F_{y,r}(\lambda) \leq 1$$

for $0 < \lambda < \lambda_0$ and for (y,r) in the compact set $\mathcal{K}$.

Let $(x,p) \in \overline{\mathcal{P}}^+ M$ and write $(x,p) = (x, \lambda q)$. Recall from Lemma B.1 that $\lambda \ell(x,p) = \ell(x, \lambda^{-1}p) = \ell(x,q)$. We have that

$$
\begin{aligned}
0 &\leq \int_{-\infty}^0 \mathcal{Q}[u,v]\big(\gamma_{(x,p)}(s), \dot\gamma_{(x,p)}(s)\big) ds \\
&= \int_{-\ell(x,p)}^0 \mathcal{Q}[u,v]\big(\gamma_{(x,\lambda q)}(s), \dot\gamma_{(x,\lambda q)}(s)\big) ds \\
&= \int_{-\ell(x,p)}^0 \mathcal{Q}[u,v]\big(\gamma_{(x,q)}(\lambda s), \lambda\dot\gamma_{(x,q)}(\lambda s)\big) ds \\
\text{(B.120)} \quad &= \frac{1}{\lambda} \int_{-\lambda\ell(x,p)}^0 \mathcal{Q}[u,v]\big(\gamma_{(x,q)}(s'), \lambda\dot\gamma_{(x,q)}(s')\big) ds' \\
&= \frac{1}{\lambda} \int_{-\ell(x,q)}^0 \mathcal{Q}[u,v]\big(\gamma_{(x,q)}(s'), \lambda\dot\gamma_{(x,q)}(s')\big) ds' \\
&\leq 2\|u\|_{C(\overline{\mathcal{P}}^+ M)} \|v\|_{C(\overline{\mathcal{P}}^+ M)} \frac{1}{\lambda} \int_{-l_0}^0 \|A(\gamma_{(x,q)}(s'), \lambda\dot\gamma_{(x,q)}(s'), \,\cdot\,, \,\cdot\,, \,\cdot\,)\|_{L^1(\Sigma_{x,p})} ds' \\
&\leq \begin{cases} 2l_0 \|u\|_{C(\overline{\mathcal{P}}^+ M)} \|v\|_{C(\overline{\mathcal{P}}^+ M)} & \text{if} \quad \lambda < \lambda_0 \\ 2\lambda_0^{-1} l_0 \|u\|_{C(\overline{\mathcal{P}}^+ M)} \|v\|_{C(\overline{\mathcal{P}}^+ M)} & \text{if} \quad \lambda \geq \lambda_0 \end{cases}
\end{aligned}
$$

Here, for $\lambda \geq \lambda_0$, we used the condition (4) of the assumptions in the definition of an admissible kernel. For $\lambda \leq \lambda_0$ we used (B.119). We also did a change of the variable in the integration as $s' = \lambda s$. This proves the claim. $\qquad\square$

Theorem 1.2. *Let (M, g) be a globally hyperbolic C^∞-smooth Lorentzian manifold. Let also $\mathcal{C}$ be a Cauchy surface of M and $K \subset \overline{\mathcal{P}}^+ \mathcal{C}^+$ be compact. Assume that $A : \Sigma \to \mathbb{R}$ is an admissible collision kernel in the sense of Definition 1.1.*

There are open neighbourhoods $B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ and $B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$ of the respective origins such that if $f \in B_1$, the Cauchy problem

$$\mathcal{X}u(x,p) - \mathcal{Q}[u,u](x,p) = f(x,p) \quad \text{on } \overline{\mathcal{P}}^+ M$$

(B.121) $$u(x,p) = 0 \quad \text{on } \overline{\mathcal{P}}^+ \mathcal{C}^-$$

has a unique solution $u \in B_2$. There is a constant $c_{A,K} > 0$ such that

$$\|u\|_{C(\overline{\mathcal{P}}^+ M)} \leq c_{A,K} \|f\|_{C(\overline{\mathcal{P}}^+ M)}.$$

Proof. We integrate the equation B.121 along the flow of $\mathcal{X}$ in TM and then use the implicit function theorem in Banach spaces for the resulting equation. (Integrating the equation (B.121) avoids some technicalities regarding Banach spaces, which there would be in the application of the implicit function theorem without the integration.) We define the mapping

$$F : C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+) \times C_b(\overline{\mathcal{P}}^+ M) \to C_b(\overline{\mathcal{P}}^+ M),$$

by
(B.122)

$$F(f, u)(x, p) = u(x, p) - \int_{-\infty}^0 \mathcal{Q}[u, u]\big(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)\big)ds - \int_{-\infty}^0 f(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))ds.$$

Let us denote

$$Z = \pi[\operatorname{supp}(A)] \cup \pi[\operatorname{supp}(f)],$$

where π is the canonical projection. We have by Lemma B.1 that the integrals in the definition of F above are actually over $[-\ell(x, p), 0]$ where $\ell(x, p) = \sup\{s \geq 0 : \gamma_{(x,p)}(-s) \in Z\}$. In combination with Lemma B.4, we have that F is well-defined.

We apply the implicit function theorem for Banach spaces (see e.g. [39, Theorem 9.6]) to F to obtain a solution u if the source $f \in C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ is small enough. First note that $F(0,0) = 0$. Additionally, observe that for $u, v \in C_b(\overline{\mathcal{P}}^+ M)$ we have

(B.123) $$\mathcal{Q}[u + v, u + v] = \mathcal{Q}[u, u] + \mathcal{Q}[v, u] + \mathcal{Q}[u, v] + \mathcal{Q}[v, v]$$

since $\mathcal{Q}$ is linear in both of its arguments.

Next, we argue that F is continuously Frechét differentiable (in the sense of [39, Definition 9.2]). Let $u, v \in C_b(\overline{\mathcal{P}}^+ M)$ and $f, h \in C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. We have that

$$F(f + h, u + v) - F(f, u) - L(f, u)(h, v) = -\int_{-\infty}^0 \mathcal{Q}[v, v]\big(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)\big)ds,$$

where

$$L(f, u)(h, v) := v(x, p) - \int_{-\infty}^0 h(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s))ds$$
$$- \int_{-\infty}^0 \mathcal{Q}[u, v]\big(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)\big)ds - \int_{-\infty}^0 \mathcal{Q}[v, u]\big(\gamma_{(x,p)}(s), \dot{\gamma}_{(x,p)}(s)\big)ds.$$

It thus follows from Lemma B.4 that

$$\|F(f + h, u + v) - F(f, u) - L(f, u)(h, v)\|_{C_b(\overline{\mathcal{P}}^+ M)}^2 \leq C_A \|v\|_{C_b(\overline{\mathcal{P}}^+ M)}^2.$$

We conclude that the Frechét derivative of F at (f, u) is given by $DF(f, u)(h, v) = L(f, u)(h, v)$. We have that $DF(f, u)$ is continuous

$$DF(f, u) : C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+) \times C_b(\overline{\mathcal{P}}^+ M) \to C_b(\overline{\mathcal{P}}^+ M),$$

by Lemma B.4. Finally, note that the Frechét differential in the second variable of F at $(0, 0)$

$$DF_2(0, 0) : C_b(\overline{\mathcal{P}}^+ M) \to C_b(\overline{\mathcal{P}}^+ M),$$

given by $DF_2(0, 0) = DF(0, 0)(0, \cdot)$, is just the identity map.

By the implicit function theorem in Banach spaces there exist open neighbourhoods $B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ and $B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$ of the respective origins and a continuously (Frechét) differentiable map $T : V \to U$ such that for $f \in B_1$, the function $u = T(f) \in B_2$ is the unique solution to $F(f, u) = 0$. Applying $\mathcal{X}$ to (B.122) shows that u solves (B.121). Further, since T is continuously differentiable, there exists $c_{A,K} > 0$ such that

$$\|u\|_{C_b(\overline{\mathcal{P}}^+ M)} \leq c_{A,K} \|f\|_{C_K(\overline{\mathcal{P}}^+ M)}.$$

This concludes the proof. □

Next we show that the source-to-solution mapping of the Boltzmann equation can be used to compute the source-to-solution mappings of the first and second linearizations of the Boltzmann equation.

Lemma 3.4. *Assume as in Theorem 1.2 and adopt its notation. Let $\Phi : B_1 \to B_2 \subset C_b(\overline{\mathcal{P}}^+ M)$, $B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$, be the source-to-solution map of the Boltzmann equation.*

The map Φ is twice Frechét differentiable at the origin of $C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. If $f, h \in B_1$, then we have:

(1) The first Frechét derivative Φ' of the source-to-solution map Φ at the origin satisfies

$$\Phi'(0; f) = \Phi^L(f),$$

where Φ^L is the source-to-solution map of the Vlasov equation (3.21).

(2) The second Frechét derivative Φ'' of the source-to-solution map Φ at the origin satisfies

$$\Phi''(0; f, h) = \Phi^{2L}(f, h),$$

where $\Phi^{2L}(f, h) \in C(\overline{\mathcal{P}}^+ M)$ is the unique solution to the equation

$$\mathcal{X}\Phi^{2L}(f, h) = \mathcal{Q}[\Phi^L(f), \Phi^L(h)] + \mathcal{Q}[\Phi^L(h), \Phi^L(f)], \quad on \ \overline{\mathcal{P}}^+ M,$$

(B.124)
$$\Phi^{2L}(f, h) = 0, \quad\quad\quad\quad\quad\quad on \ \overline{\mathcal{P}}^+ \mathcal{C}^-.$$

Proof of Lemma 3.4. Proof of (1). We adopt the notation of Theorem 1.2. Let $f \in C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$ and let $f_0 \in B_1$. Then by Theorem 1.2 there exists a neighbourhood B_2 of the origin in $C_b(\overline{\mathcal{P}}^+ M)$ and $\epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$ the problem

(B.125)
$$\mathcal{X}u_\epsilon - \mathcal{Q}[u_\epsilon, u_\epsilon] = \epsilon f \quad on \ \overline{\mathcal{P}}^+ M$$
$$u_\epsilon = 0 \quad on \ \overline{\mathcal{P}}^+ \mathcal{C}^-,$$

has a unique solution $u_\epsilon \in B_2$ satisfying $\|u_\epsilon\|_{C_b(\overline{\mathcal{P}}^+ M)} \leq c_{A,K}\epsilon \|f\|_{C_K(\overline{\mathcal{P}}^+ M)}$. Let us define functions $r_\epsilon \in C_b(\overline{\mathcal{P}}^+ M)$, for $\epsilon < \epsilon_0$, by

$$u_\epsilon = \epsilon v_0 + r_\epsilon,$$

where $v_0 \in C_b(\overline{\mathcal{P}}^+ M)$ solves

(B.126)
$$\mathcal{X}v_0 = f \quad on \ \overline{\mathcal{P}}^+ M$$
$$v_0 = 0 \quad on \ \overline{\mathcal{P}}^+ \mathcal{C}^-.$$

We show that $r_\epsilon = \mathcal{O}(\epsilon^2)$ in $C_b(\overline{\mathcal{P}}^+ M)$. To show this, we first calculate

$$\mathcal{X}r_\epsilon = \mathcal{X}(u_\epsilon - \epsilon v_0) = \mathcal{Q}[u_\epsilon, u_\epsilon] + \epsilon f - \epsilon \mathcal{X}v_0 = \mathcal{Q}[u_\epsilon, u_\epsilon].$$

We integrate this equation along the flow of $\mathcal{X}$ to obtain

$$r_\epsilon(x, p) = \int_{-\infty}^0 \mathcal{Q}[u_\epsilon, u_\epsilon]\big(\gamma_{(x,p)}(s), \dot\gamma_{(x,p)}(s)\big)ds.$$

(As before, the integral is actually over a finite interval since u_ϵ vanishes in $\pi^{-1}(\mathcal{C}^-)$.) By Lemma B.4, we have that the right hand side is at most

$$C_1\|u_\epsilon\|^2_{C_b(\overline{\mathcal{P}}^+ M))}.$$

Since $\|u_\epsilon\|_{C_b(\overline{\mathcal{P}}^+ M)} \leq C\epsilon\|f\|_{C_K(\overline{\mathcal{P}}^+ M)}$ by Theorem 1.2, we have that $r_\epsilon = \mathcal{O}(\epsilon^2)$ in $C_b(\overline{\mathcal{P}}^+ M)$ as claimed. Consequently, we have that

$$\lim_{\epsilon \to 0} \frac{\Phi(\epsilon f) - \Phi(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{u_\epsilon - 0}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon v_0 + r_\epsilon}{\epsilon} = v_0 = \Phi^L(f),$$

where the limit is in $C_b(\overline{\mathcal{P}}^+ M)$. This proves Part (1).

Proof of (2). Let f, u_ϵ and v_0 be as before. We first prove that

$$u_\epsilon = \epsilon v_0 + \epsilon^2 w + \mathcal{O}(\epsilon^3)$$

in $C_b(\overline{\mathcal{P}}^+ M)$, where w is the unique solution to

(B.127)
$$\mathcal{X} w = \mathcal{Q}[v_0, v_0] \quad \text{on } \overline{\mathcal{P}}^+ M$$
$$w = 0, \quad\quad \text{on } \overline{\mathcal{P}}^+ \mathcal{C}^-.$$

A unique solution to (B.127) exists by using the formula (B.113) and noting the $\pi(\text{supp}(A))$ is compact. To show this, we define $R_\epsilon \in C_b(\overline{\mathcal{P}}^+ M)$ by

(B.128)
$$u_\epsilon = \epsilon v_0 - \epsilon^2 w + R_\epsilon.$$

To show that $R_\epsilon = \mathcal{O}(\epsilon^3)$ in $C_b(\overline{\mathcal{P}}^+ M)$ we apply $\mathcal{X}$ to the equation B.128 above. We have that

(B.129) $\mathcal{X} R_\epsilon = \mathcal{X}(u_\epsilon - \epsilon v_0 - \epsilon^2 w) = \mathcal{Q}[u_\epsilon, u_\epsilon] + \epsilon f - \epsilon \mathcal{X} v_0 - \epsilon^2 \mathcal{Q}[v_0, v_0] = \mathcal{Q}[u_\epsilon, u_\epsilon] - \epsilon^2 \mathcal{Q}[v_0, v_0].$

Since the collision operator $\mathcal{Q}$ is linear in both of its arguments, we have that

(B.130)
$$\mathcal{Q}[u_\epsilon, u_\epsilon] - \epsilon^2 \mathcal{Q}[v_0, v_0] = \mathcal{Q}[(u_\epsilon - \epsilon v_0), u_\epsilon] - \mathcal{Q}[\epsilon v_0, (u_\epsilon - \epsilon v_0)].$$

We integrate the equation (B.129) for R_ϵ along the flow of $\mathcal{X}$ to obtain

$$R_\epsilon(x, p) \leq C_1 \|u_\epsilon - \epsilon v_0\|_{C_b(\overline{\mathcal{P}}^+ M)} \|u_\epsilon\|_{C_b(\overline{\mathcal{P}}^+ M)} + C_1 \|\epsilon v_0\|_{C_b(\overline{\mathcal{P}}^+ M)} \|u_\epsilon - \epsilon v_0\|_{C_b(\overline{\mathcal{P}}^+ M)}.$$

Here we used (B.130) and Lemma B.4. By using the estimate $\|u_\epsilon - \epsilon v_0\|_{C(\overline{\mathcal{P}}^+ M)} \leq C\epsilon^2$ from Part (1) of this lemma, and by using that $\|u_\epsilon\|_{C_b(\overline{\mathcal{P}}^+ M)} \leq C\epsilon\|f\|_{C_K(\overline{\mathcal{P}}^+ M)}$ and that $\|v_0\|_{C_b(\overline{\mathcal{P}}^+ M)} \leq C_2\|f\|_{C_K(\overline{\mathcal{P}}^+ M)}$ we obtain

$$\|R_\epsilon(x, p)\|_{C_b(\overline{\mathcal{P}}^+ M)} \leq C_3 \epsilon^3.$$

We have shown that

$$u_\epsilon = \epsilon v_0 - \epsilon^2 w + \mathcal{O}(\epsilon^3).$$

It follows that Φ is twice Frechét differentiable at the origin in $C_b(\overline{\mathcal{P}}^+ M)$.

Let $f, h \in C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$. To prove Part (2) of the claim, we use "polarization identity of differentiation", which says that any function F, which is twice differentiable at 0, satisfies

$$\frac{\partial^2}{\partial \epsilon_1 \epsilon_2}\Big|_{\epsilon_1 = \epsilon_2 = 0} F(\epsilon_1 f_1 + \epsilon_2 f_2) = \frac{1}{4} \frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} [F(\epsilon(f_1 + f_2)) - F(\epsilon(f_1 - f_2))].$$

For $f \in B_1 \subset C_K(\overline{\mathcal{P}}^+ \mathcal{C}^+)$, we denote by u_f the solution to the Boltzmann equation (B.125) with source f. We also denote by v_f the solution to the Vlasov equation (B.126) with source f, and we denote similarly for w_f, where w_f solves (B.127) where v_0 is replaced by v_f.

We need to show that

(B.131)
$$\frac{1}{\epsilon_1 \epsilon_2} \left[\Phi(\epsilon_1 f_1 + \epsilon_2 f_2) - \Phi(\epsilon_2 f_2) - \Phi(\epsilon_1 f_1) + \Phi(0) \right] \longrightarrow w,$$

as $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, where w solves

$$
\begin{aligned}
\text{(B.132)} \qquad \mathcal{X}w &= \mathcal{Q}[v_{f_1}, v_{f_2}] + \mathcal{Q}[v_{f_2}, v_{f_1}], \quad \text{on } \overline{\mathcal{P}}^+ M \\
w &= 0, \qquad\qquad\qquad\qquad\quad \text{on } \overline{\mathcal{P}}^+ \mathcal{C}^-.
\end{aligned}
$$

By using the polarization identity (B.131) and the expansion of $u_{\epsilon(f_1 \pm f_2)}$ for ϵ small, which we have already proven, we obtain

$$
\begin{aligned}
\lim_{\epsilon_1, \epsilon_2 \to 0} \frac{\Phi(\epsilon_1 f_1 + \epsilon_2 f_2) - \Phi(\epsilon_2 f_2) - \Phi(\epsilon_1 f_1) + \Phi(0)}{\epsilon_1 \epsilon_2} &= \frac{1}{4} \frac{\partial^2}{\partial \epsilon^2}\Big|_{\epsilon=0} [u_{\epsilon(f_1+f_2)} - u_{\epsilon(f_1-f_2)}] \\
&= \frac{1}{2} w_{f_1+f_2} - \frac{1}{2} w_{f_1-f_2}.
\end{aligned}
$$

By denoting $w_{f_1-f_2} - w_{f_1+f_2} = 2w$ and by using the linearity of $\mathcal{Q}$ in both of its argument, we finally have have that

$$
\mathcal{X}w = \frac{1}{2} \mathcal{Q}[v_{f_1+f_2}, v_{f_1+f_2}] - \frac{1}{2} \mathcal{Q}[v_{f_1-f_2}, v_{f_1-f_2}] = \mathcal{Q}[v_{f_1}, v_{f_2}] + \mathcal{Q}[v_{f_2}, v_{f_1}].
$$

Renaming f_1 and f_2 as f and h respectively proves the claim. $\qquad\square$

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