Calc. Var. (2022) 61:180
https://doi.org/10.1007/s00526-022-02284-7

Calculus of Variations

Rigidity and almost rigidity of Sobolev inequalities on compact spaces with lower Ricci curvature bounds

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Received: 5 August 2021 / Accepted: 7 June 2022

Abstract

We prove that if M is a closed n-dimensional Riemannian manifold, $n \geq 3$, with $\mathrm{Ric} \geq n-1$ and for which the optimal constant in the critical Sobolev inequality equals the one of the n-dimensional sphere $\mathbb{S}^n$, then M is isometric to $\mathbb{S}^n$. An almost-rigidity result is also established, saying that if equality is almost achieved, then M is close in the measure Gromov–Hausdorff sense to a spherical suspension. These statements are obtained in the RCD-setting of (possibly non-smooth) metric measure spaces satisfying synthetic lower Ricci curvature bounds. An independent result of our analysis is the characterization of the best constant in the Sobolev inequality on any compact CD space, extending to the non-smooth setting a classical result by Aubin. Our arguments are based on a new concentration compactness result for mGH-converging sequences of RCD spaces and on a Pólya–Szegő inequality of Euclidean-type in CD spaces. As an application of the technical tools developed we prove both an existence result for the Yamabe equation and the continuity of the generalized Yamabe constant under measure Gromov–Hausdorff convergence, in the RCD-setting.

Mathematics Subject Classification 53C24 · 46E36

Contents

Communicated by A. Mondino.

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Published online: 12 July 2022

1 Introduction

The standard Sobolev inequality in sharp form reads as

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le Eucl(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad \forall u \in W^{1,p}(\mathbb{R}^n), \tag{1.1}$$

where $p \in (1,n)$, $p^* := \frac{pn}{n-p}$ is the Sobolev conjugate exponent and $Eucl(n,p)$ is the smallest positive constant for which the inequality (1.1) is valid. Its precise value (see (2.2) below) was computed independently by Aubin [20] and Talenti [94] (see also [41]).

In the setting of compact Riemannian manifolds, the presence of constant functions in the Sobolev space immediately shows that an inequality of the kind of (1.1) must fail. Yet, Sobolev embeddings are certainly valid also in this context and they can be expressed by calling into play the full Sobolev norm:

$$\|u\|^p_{L^{p^*}(M)} \le A\|\nabla u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)}, \qquad \forall u \in W^{1,p}(M), \tag{$\star$}$$

where M is a compact n-dimensional Riemannian manifold and $A, B > 0$. From the presence of the two parameters A, B, it is not straightforward which is the notion of best constants in this case. The issue of defining and determining the best constants in ($\star$) has been the central role of the celebrated AB-program, we refer to [59] for a thorough presentation of this topic (see also [46]). The starting point of this program is the definition of the following two different notions of "best Sobolev constants":

$$\alpha_p(M) := \inf\{A\ :\ (\star) \text{ holds for some } B\}, \quad \beta_p(M) := \inf\{B\ :\ (\star) \text{ holds for some } A\}.$$

Then the first natural problem is to determine the value of $\alpha_p(M)$ and $\beta_p(M)$. It is rather easy to see that

$$\beta_p(M) = Vol(M)^{p/p^*-1},$$

indeed constant functions give automatically $\beta_p(M) \geq Vol(M)^{p/p^*-1}$, while the other inequality follows from the Sobolev–Poincaré inequality (see, e.g. [59,Sect. 4.1]). It is instead more subtle to determine whether $\beta_p(M)$ is attained, in the sense that the infimum in its definition is actually a minimum. This is true for $p = 2$ and due to Bakry [23] (see also Proposition 5.1), but actually false for $p > 2$ (see e.g. [59,Prop. 4.1]).

Concerning instead the value of $\alpha_p(M)$, it turns out to be related to the sharp constant in the Euclidean Sobolev inequality (1.1). More precisely Aubin in [20] (see also [59]) showed that on any compact n-dimensional Riemannian manifold M with $n \geq 2$, we have

$$\alpha_p(M) = Eucl(n, p)^p \qquad \forall p \in (1, n). \tag{1.2}$$

We point out that it is a hard task to show that $\alpha_p(M)$ is attained, namely that there exists some $B > 0$ for which $(\star)$ holds with $A = \alpha_p(M)$ and B. This has been verified for $p = 2$ in [60], answering affirmatively to a conjecture of Aubin.

On the other hand, knowing the value of $\beta_p(M)$ (and that it is attained for $p = 2$), we can define a further notion of optimal-constant A, "relative" to $B = \beta_2(M)$. More precisely we define

$$A^{\text{opt}}_{2^*}(M) := \text{Vol}(M)^{1-2/2^*} \cdot \inf\{A\ :\ (\star) \text{ for } p = 2 \text{ holds with } A \text{ and } B = \text{Vol}(M)^{2/2^*-1}\}.$$

For the sake of generality we will actually consider A^{opt} also in the so-called *subcritical case*, meaning that we enlarge the class of Sobolev inequalities and consider for every $q \in (2, 2^*]$

$$\|u\|^2_{L^q(M)} \leq A\|\nabla u\|^2_{L^2(M)} + \text{Vol}(M)^{2/q-1}\|u\|^2_{L^2(M)}, \qquad \forall u \in W^{1,2}(M), \tag{$\star\star$}$$

for some constant $A \geq 0$. Then we define

$$A^{\text{opt}}_q(M) := \text{Vol}(M)^{1-2/q} \cdot \inf\{A\ :\ (\star\star) \text{ holds}\}.$$

Note that the infimum above is always a minimum and that $\text{Vol}(M)^{2/q-1}$ is the "minimal B" that we can take in $(\star\star)$.

Remark 1.1 We bring to the attention of the reader the renormalization factor $\text{Vol}(M)^{1-2/q}$ in the definition of $A^{\text{opt}}_q(M)$. This is usually not present in the literature concerning the AB-program (see e.g. [59]), however this choice will allow us to have cleaner inequalities. This also makes A^{opt}_q invariant under rescalings of the volume measure of M. □

One of the main questions that we will investigate in this note concerns the value of $A^{\text{opt}}_q(M)$. So far $A^{\text{opt}}_q(M)$ is known explicitly only in the case of $\mathbb{S}^n$ and was firstly computed by Aubin in [19] in the case of $q = 2^*$ and by Beckner in [27] for a general q:

$$A^{\text{opt}}_q(\mathbb{S}^n) = \frac{q-2}{n}, \qquad \forall n \geq 3. \tag{1.3}$$

Aubin also exhibited a family of non-constant functions that achieve equality in (⋆⋆) with $A = A_{2^*}^{\text{opt}}(\mathbb{S}^n)$. For a general manifold M instead it can be proved that

$$A_q^{\text{opt}}(M) \le C(K, D, N), \tag{1.4}$$

where $K \in \mathbb{R}$ is a lower bound on the Ricci curvature of M, N is an upper bound on the dimension and $D \in \mathbb{R}^+$ an upper bound on its diameter. This follows from the Sobolev–Poincaré inequality combined with an inequality by Bakry (see e.g. [46,Theorem 4.4] and also Sect. 5.1). On the other hand, for positive Ricci curvature we have the following celebrated comparison result originally proven in [66] (see also [24, 76] for the case of a general q):

Theorem 1.2 *Let M be an n-dimensional Riemannian manifold, $n \ge 3$, with* $\text{Ric} \ge n-1$. *Then, for every $q \in (2, 2^*]$, it holds*

$$A_q^{\text{opt}}(M) \le A_q^{\text{opt}}(\mathbb{S}^n). \tag{1.5}$$

One of the main consequence of the results in this note is the characterization of the equality in (1.5), in particular we show:

Theorem 1.3 *Equality in* (1.5) *holds for some $q \in (2, 2^*]$ if and only if M is isometric to $\mathbb{S}^n$.*

It is important to point out that the novelty of the above result is that it covers the case $q = 2^*$. Indeed, for $q < 2^*$, Theorem 1.3 was already established (see e.g. [24,Remark 6.8.5]) and follows from an improvement (only for $q < 2^*$) of (1.5) due to [50] involving the spectral gap (see Remark 6.9 for more details). On the other hand, up to our knowledge, this is the first time that it appears in the critical case $q = 2^*$.

It is also worth to compare Theorem 1.3 with the rigidity result in [75] for the Sobolev inequality on manifolds with non-negative Ricci curvature (and later improved in [97], see also [26]). In [75] it is proved that if (1.1) is valid on a non-compact manifold with non-negative Ricci curvature, then the manifold must be the Euclidean space. Here instead we consider compact manifolds and the rigidity is obtained in comparison with the Sobolev inequality on the sphere. For this reason, our arguments will also be substantially different from the ones in [75, 97]. Nevertheless, we will also deal with the former types of rigidity in Corollary 1.14 below.

Theorem 1.3 will be proved in the context of metric measure spaces with synthetic Ricci curvature bounds. One of the main reasons to approach the problem in this more general setting is that it will allow us to characterize also the "almost-equality" in (1.5) (see Theorem 1.10 below). Indeed, as we will see, in this case we need to compare the manifold M to a class of singular spaces, rather than to the round sphere.

1.1 Best constant in the Sobolev inequality on compact CD spaces

The notion of metric measure spaces with synthetic Ricci curvature bounds originated in the independent seminal works of [92, 93] and [80], where the celebrated *curvature-dimension condition* $\text{CD}(K, N)$ was introduced. Here $K \in \mathbb{R}$ is a lower bound for the Ricci curvature and $N \in [1, \infty]$ is an upper bound on the dimension. The definition is given via optimal transport, by requiring some convexity properties of entropy functionals (see Definition 2.5 below).

The proof of the rigidity (and almost rigidity) of A_q^{opt} in the case $q = 2^*$, will force us to study also the value of α_p in the context of CD-spaces. The connection of this with the proof of Theorem 1.9 will be explained towards the end of Sect. 1.4, where we provide a sketch of the proof yielding the main rigidity theorem.

Let then $(\mathrm{X}, d, \mathfrak{m})$ be a $\mathrm{CD}(K, N)$ space with $N \in (1, \infty)$. For any $p \in (1, N)$ set $p^* := \frac{Np}{N-p}$ and, in the same fashion of ($\star$), we consider:

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} \le A\||Du|\|^p_{L^p(\mathfrak{m})} + B\|u\|^p_{L^p(\mathfrak{m})}, \qquad \forall u \in W^{1,p}(\mathrm{X}). \tag{1.6}$$

We are then interested in the minimal A for which (1.6) holds. In other words we set (with the usual convention that the inf is ∞ when no A exists):

$$\alpha_p(\mathrm{X}) := \inf\{A \ : \ (1.6) \text{ holds for some } B\}. \tag{1.7}$$

We will be able to compute the value of $\alpha_p(\mathrm{X})$ for every compact $\mathrm{CD}(K, N)$ space X, extending the result of Aubin for Riemannian manifolds (see (1.2) above). Before passing to the actual statement, it is useful to explain first the intuition behind it and the geometrical meaning of the constant $\alpha_p(\mathrm{X})$. The rough idea is that its value is tightly linked to the local structure of the space. Indeed, the key observation is that $\alpha_p(\mathrm{X})$ is invariant under rescaling of the form $(\mathrm{X}, d/r, \mathfrak{m}/r^N)$. For example, since manifolds are locally Euclidean, it is not surprising that in (1.2) the optimal Euclidean–Sobolev constant appears. On the other hand, $\mathrm{CD}(K, N)$ spaces have a more singular local behavior and additional parameters must be taken into account. In particular the value of $\alpha_p(\mathrm{X})$ turns out to be related to the Bishop–Gromov density:

$$(0, +\infty] \ni \theta_N(x) := \lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}, \quad x \in \mathrm{X},$$

where ω_N is the volume of the Euclidean unit ball (see (2.1) for non integer N). Our result is then the following:

Theorem 1.4 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a compact* $\mathrm{CD}(K, N)$ *space for some* $K \in \mathbb{R}$ *and* $N \in (1, \infty)$. *Then for every* $p \in (1, N)$

$$\alpha_p(\mathrm{X}) = \left(\frac{Eucl(N, p)}{\min_{x \in \mathrm{X}} \theta_N(x)^{\frac{1}{N}}} \right)^p. \tag{1.8}$$

We point out that, since X is compact, $\min_{x\in \mathrm{X}} \theta_N(x)$ always exists because θ_N is lower semicontinuous (see Sect. 2.3.1).

Remark 1.5 Note that if X is a n-dimensional Riemannian manifold, $\theta_n(x) = 1$ for every $x \in \mathrm{X}$, hence in this case (1.8) (with $N = n$) is exactly Aubin's result in (1.2). Recall also that here N needs not to be an integer and thus $Eucl(N, p)$ has to be defined for arbitrary $N \in (1, \infty)$ (see (2.2)). □

Remark 1.6 We are not assuming $(\mathrm{X}, d, \mathfrak{m})$ to be renormalized. In particular observe that if we rescale the reference measure $\mathfrak{m}$ as $c \cdot \mathfrak{m}$, then α_p gets multiplied by $c^{-p/N}$, which is in accordance with the scaling in (1.8). □

Remark 1.7 Theorem 1.4 gives non-trivial information even in the "collapsed" case, i.e. when $\theta_N = +\infty$ in a set of positive (or even full) measure. Indeed, to have $\alpha_p(\mathrm{X}) > 0$ it is sufficient that $\theta_N(x) < +\infty$ at a single point $x \in \mathrm{X}$. As an example, consider the model space $([0, \pi], |.|, \sin^{N-1} \mathscr{L}^1)$ which is $\mathrm{CD}(N-1, N)$ with $\theta_N(x) < +\infty$ only for $x \in \{0, \pi\}$. □

Theorem 1.4 will be proved in two steps, by the combination of an upper bound (Theorem 3.13), obtained via local Sobolev inequalities (Theorem 3.8), and a lower bound (Theorem 4.4) derived with a blow-up analysis.

We end this part with a question that naturally arises from the validity of Theorem 1.4: **Question:** Let $(\mathrm{X}, d, \mathfrak{m})$ be a compact $\mathrm{CD}(K, N)$ (or $\mathrm{RCD}(N, K)$) space with $N \in (1, \infty)$ and suppose that $\alpha_{2^*}(\mathrm{X}) \in (0, \infty)$. Is there a constant $\overline{B} < +\infty$ such that

$$\|u\|^p_{L^{2^*}(\mathfrak{m})} \leq \alpha_2(\mathrm{X})\||Du|\|^2_{L^2(\mathfrak{m})} + \overline{B}\|u\|^2_{L^2(\mathfrak{m})}, \qquad \forall u \in W^{1,2}(\mathrm{X}) \ ? \tag{1.9}$$

This has positive answer in the smooth setting [60]. However in [59,Proposition 5.1] it is shown that on a Riemannian manifold M of dimension $n \geq 4$, the scalar curvature of M is bounded above by $c_n\overline{B}$, for a dimensional constant $c_n > 0$. This points to a negative answer, since we are assuming only a Ricci lower bound on the space, however it is not clear to us how to prove or disprove (1.9).

1.2 Main rigidity and almost rigidity results in compact RCD spaces

Even if some of our results will hold for the general class of $\mathrm{CD}(K, N)$ spaces, our main focus will be the smaller class of spaces satisfying the *Riemannian curvature-dimension condition* $\mathrm{RCD}(K, N)$, which adds to the CD class the linearity of the heat flow (see Definition 2.7 below). This notion appeared first in the infinite dimensional case ($N = \infty$) in [11] (see also [9] in the case of σ-finite reference measure) while, in the finite dimensional case ($N < \infty$), it was introduce in [51]. We also mention the slightly weaker $\mathrm{RCD}^*(K, N)$ condition (coming from the *reduced curvature-dimension condition* $\mathrm{CD}^*(K, N)$ introduced in [22]) which has been proved in [15, 47] to be equivalent to the validity of a weak N-dimensional Bochner-inequality (see also [12] for the same result in the infinite dimensional case). We recall that in the compact case (or more generally for finite reference measure) which will be the main setting of this note, the $\mathrm{RCD}^*(K, N)$ and the $\mathrm{RCD}(K, N)$ conditions turn out to be perfectly equivalent after the work in [35]. The main advantage for us to work in the RCD class, as opposed to the more general CD class, is that it enjoys rigidity and stability properties that are analogous to the Riemannian manifolds setting.

To state our main results for metric measure spaces we need to define first the notion of optimal constant in the Sobolev inequality in the non-smooth setting. Given a (compact) $\mathrm{RCD}(K, N)$ space (or more generally a $\mathrm{CD}(K, N)$ space) $(\mathrm{X}, d, \mathfrak{m})$, for some $K \in \mathbb{R}$, $N \in (2, \infty)$, we set $2^* := 2N/(N-2)$ and consider the analogous of $(\star\star)$:

$$\|u\|^2_{L^q(\mathfrak{m})} \leq A\||Du|\|^2_{L^2(\mathfrak{m})} + \mathfrak{m}(\mathrm{X})^{2/q-1}\|u\|^2_{L^2(\mathfrak{m})}, \qquad \forall u \in W^{1,2}(\mathrm{X}), \tag{1.10}$$

for $q \in (2, 2^*]$ and a constant $A \geq 0$. Then we define

$$A^{\mathrm{opt}}_q(\mathrm{X}) := \mathfrak{m}(\mathrm{X})^{1-2/q} \cdot \inf\{A \ : \ (1.10) \text{ holds}\},$$

with the convention that $A^{\mathrm{opt}}_q(\mathrm{X}) = \infty$ when no A exists. Note that $A^{\mathrm{opt}}_q(X)$, when is finite, is actually a minimum. Observe also that, as in the smooth case, there is a renormalization factor $\mathfrak{m}(\mathrm{X})^{1-2/q}$ in the definition. However, being not restrictive, we will mainly work asking $\mathfrak{m}(\mathrm{X}) = 1$ so that the value of $A^{\mathrm{opt}}_q(\mathrm{X})$ is equivalent to the non-renormalized one.

Remarkably in this more general framework, a comparison analogous to (1.5) holds.

Theorem 1.8 ([36]) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an essentially non-branching* $\mathrm{CD}(N-1, N)$ *space,* $N \in (2, \infty)$. *Then, for every* $q \in (2, 2^*]$

$$A_q^{\mathrm{opt}}(\mathrm{X}) \leq \frac{q-2}{N}. \tag{1.11}$$

The essentially nonbranching condition is a technical property of mass transportation that, roughly said, requires a suitable nonbranching property of transportation geodesics. It was introduced in [88] where it was shown that it is satisfied in the $\mathrm{RCD}(K, N)$-class. We also mention that Theorem 1.8 in the RCD case was previously obtained in [86]. Observe also that, whenever N is an integer and thanks to (1.3), for a N-dimensional Riemannian manifolds (1.11) is exactly (1.5) and in particular Theorem 1.8 generalizes Theorem 1.2.

We can now state our main rigidity result in the setting of metric measure spaces.

Theorem 1.9 (Rigidity of A_q^{opt}) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *space for some* $N \in (2, \infty)$ *and let* $q \in (2, 2^*]$. *Then, equality holds in* (1.11) *if and only if* $(\mathrm{X}, d, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathrm{RCD}(N-2, N-1)$ *space* $(\mathrm{Z}, d_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *such that* $(\mathrm{X}, d, \mathfrak{m}) \simeq [0, \pi] \times_{\sin}^{N-1} \mathrm{Z}$.

Differently from the smooth case, in the more abstract setting of RCD spaces the above result is instead new for all q. As anticipated above, we can also prove an "almost-rigidity" statement linked to the almost-equality case in (1.11) (see Sect. 2.3.3 for the notion of measure-Gromov–Hausdorff convergence and distance d_{mGH}.).

Theorem 1.10 (Almost-rigidity of A_q^{opt}) *For every* $N \in (2, \infty)$, $q \in (2, 2^*]$ *and every* $\varepsilon > 0$, *there exists* $\delta := \delta(N, \varepsilon, q) > 0$ *such that the following holds. Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *space with* $\mathfrak{m}(\mathrm{X}) = 1$ *and suppose that*

$$A_q^{\mathrm{opt}}(\mathrm{X}) \geq \frac{(q-2)}{N} - \delta,$$

Then, there exists a spherical suspension $(\mathrm{Y}, d_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ *(i.e. there exists an* $\mathrm{RCD}(N-2, N-1)$ *space* $(\mathrm{Z}, d_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *so that* Y *is isomorphic as a metric measure space to* $[0, \pi] \times_{\sin}^{N-1} \mathrm{Z}$*) such that*

$$d_{mGH}((\mathrm{X}, d, \mathfrak{m}), (\mathrm{Y}, d_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})) < \varepsilon.$$

Remark 1.11 We briefly point out two important facts concerning the two above statements.

(i) In the smooth setting, for $q < 2^*$, the almost rigidity follows "directly" from the sharper version of (1.5) cited above (see Remark 6.9 for the explicit statement) and using the almost-rigidity of the 2-spectral gap [36, 38]. Nevertheless, we are not aware of any such statement in the literature and anyhow, our proof does not rely on any improved version of (1.5).

(ii) The key feature of Theorems 1.9 and 1.10 is that they include the "critical" exponent. Indeed, the difference between the "subcritical" case $q < 2^*$ and $q = 2^*$ is not only technical but a major issue linked to the lack of compactness in the Sobolev embedding. As it will be clear in the sequel, the proof of the critical case requires several additional arguments that constitute the heart of this note. □

The almost-rigidity result contained in Theorem 1.10 will be actually a consequence of a stronger statement, that is the continuity of A_q^{opt} under measure Gromov–Hausdorff convergence. More precisely we will prove the following:

Theorem 1.12 (Continuity of $A_q^{\rm opt}$ under mGH-convergence) *Let* $(\mathrm{X}_n, d_n, \mathfrak{m}_n)$, $n \in \mathbb{N}\cup\{\infty\}$, *be a sequence of compact* $\mathrm{RCD}(K, N)$*-spaces with* $\mathfrak{m}_n(\mathrm{X}_n) = 1$ *and for some* $K \in \mathbb{R}$, $N \in (2, \infty)$ *so that* $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$. *Then,* $A_q^{\rm opt}(\mathrm{X}_\infty) = \lim_n A_q^{\rm opt}(\mathrm{X}_n)$, *for every* $q \in (2, 2^*]$.

1.3 Additional results and application to the Yamabe equation

Euclidean-type Pólya–Szegő inequality on $\mathrm{CD}(K, N)$ *spaces.* We will develop a Pólya–Szegő inequality (see Sect. 3.1), which is roughly a Euclidean-variant of the Pólya–Szegő inequality for $\mathrm{CD}(K, N)$ spaces, $K > 0$, derived in [84]. The main feature of this inequality is that it holds on arbitrary $\mathrm{CD}(K, N)$ spaces, $K \in \mathbb{R}$, but assumes the validity of an isoperimetric inequality of the type

$$\mathrm{Per}(E) \geq C_{Isop}\mathfrak{m}(E)^{\frac{N-1}{N}}, \qquad \forall E \subset \Omega \text{ Borel},$$

for some $\Omega \subset \mathrm{X}$ open and where C_{Isop} is a positive constant independent of E. For our purposes this Pólya–Szegő inequality will be used to derive local Sobolev inequalities of Euclidean-type (see Theorem 3.8), however it allows us to obtain also sharp Sobolev inequalities under Euclidean-volume growth assumption.
Sharp and rigid Sobolev inequalities under Euclidean-volume growth. As a by-product of our analysis, we achieve sharp Sobolev inequalities on $\mathrm{CD}(0, N)$ spaces with Euclidean-volume growth. We recall that a $\mathrm{CD}(0, N)$ space $(\mathrm{X}, d, \mathfrak{m})$ has Euclidean-volume growth if

$$AVR(\mathrm{X}) := \lim_{R\to+\infty} \frac{\mathfrak{m}(B_R(x_0))}{\omega_N R^N} > 0,$$

for some (and thus any) $x_0 \in \mathrm{X}$. We will prove the following.

Theorem 1.13 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(0, N)$ *space for some* $N \in (1, \infty)$ *and with Euclidean volume growth. Then, for every* $p \in (1, N)$, *it holds*

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq Eucl(N, p)AVR(\mathrm{X})^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}_c(\mathrm{X}). \tag{1.12}$$

Moreover (1.12) *is sharp.*

This extends a result recently derived in [26] in the case of Riemannian manifolds and answers positively to a question posed in [26,Sec. 5.2].

Combining Theorem 1.13 with the volume rigidity for *non-collapsed* RCD spaces in [53] and the results in [40,Appendix A] (see also [68,Theorem 3.5]) we immediately get the following topological rigidity which extends to the non-smooth setting the results for Riemannian manifolds in [75, 97]. Recall that an $\mathrm{RCD}(K, N)$ space $(\mathrm{X}, d, \mathfrak{m})$ is said to be non-collapsed (see Definition 2.13) if $\mathfrak{m} = \mathscr{H}^N$, the N-dimensional Hausdorff measure (this notion has been introduced in [53], see also [72] and inspired by [40]).

Corollary 1.14 (Topological-rigidity of Sobolev embeddings) *For every* $N \in \mathbb{N}$, $p \in (1, N)$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that the following holds. Let* $(\mathrm{X}, d, \mathscr{H}^N)$ *be an* $\mathrm{RCD}(0, N)$ *space with Euclidean volume growth and such that*

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq (Eucl(N, p) + \delta)\||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}_c(\mathrm{X}). \tag{1.13}$$

Then X *is homeomorphic to* $\mathbb{R}^N$ *and* $d_{GH}(B_r(x), B_r(0^N)) \leq \varepsilon r$ *for every* $x \in \mathrm{X}$ *and* $r > 0$.

To deduce the above result, a lower bound on the optimal constant in (1.12) is actually sufficient (see Theorem 4.6).

Concentration compactness and mGH-convergence. As often happens for almost-rigidity results in RCD spaces, Theorem 1.10 will be proved by compactness. However, in the case $q = 2^*$ we have a strong lack of compactness, hence for the proof we will need an additional tool, which is a concentration compactness result under mGH-convergence of compact RCD-spaces. In particular, we will prove a concentration-compactness dichotomy principle (see Lemma 6.6 and Theorem 6.1 below) in the spirit of [79] (see also the monograph [91]), but under varying underlying measure. As far as we know, this is the first result of this type dealing with varying spaces and we believe it to be interesting on its own.

Existence for the Yamabe equation and mGH-continuity of Yamabe constant on RCD *spaces* As an application of Theorem 1.4 we show that on a compact $\mathrm{RCD}(K, N)$ space a (non-negative and non-zero) solution to the so-called Yamabe equation

$$-\Delta u + \mathrm{S}\, u = \lambda u^{2^*-1}, \qquad \text{for } \lambda \in \mathbb{R},\ \mathrm{S} \in L^p(\mathfrak{m}),\ p > N/2, \tag{1.14}$$

exists provided

$$\lambda_{\mathrm{S}}(\mathrm{X}) := \inf_{u \in W^{1,2}(\mathrm{X})\setminus\{0\}} \frac{\int |Du|^2 + \mathrm{S}|u|^2 \,\mathrm{dVol}}{\|u\|^2_{L^{2^*}(M)}} < \frac{\min \theta_N^{N/2}}{Eucl(N,2)^2},$$

where λ_{S} is called generalized Yamabe constant (see Theorem 8.2). This extends a classical result on smooth Riemannian manifolds (see Sect. 8 for more details and references).

We also show the continuity of the generalized Yamabe constant under measure Gromov–Hausdorff convergence. More precisely for a sequence X_n of compact $\mathrm{RCD}(K, N)$ spaces such that $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$ with X_∞ a compact $\mathrm{RCD}(K, N)$ space, we show that

$$\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) = \lambda_{\mathrm{S}}(\mathrm{X}_\infty),$$

where S_n converges L^p-weak to S for some $p > N/2$. See Theorem 8.6 for a precise statement and Sect. 2.3.3 for the definition of L^p-weak convergence with varying spaces. This result extends and sharpens an analogous statement proved for Ricci-limits in [64], where an additional boundedness assumption on the sequence $\lambda_{\mathrm{S}_n}(\mathrm{X}_n)$ is required.

1.4 Proof-outline of the rigidity of A_q^{opt}

Here we explain the scheme of the proof of the rigidity result in Theorem 1.9.

We consider only the case $q = 2^*$, since it is the most interesting one and we also restrict to the case of manifolds, which already contains all the main ideas.

Let M be a compact n-manifold M, with $\mathrm{Ric} \geq n-1$ and $A_{2^*}^{\mathrm{opt}}(M) = A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n)$, $n \geq 3$. This is equivalent to the existence of a sequence $(u_i) \subset W^{1,2}(M)$ of non-constant functions satisfying $\|u_i\|^2_{L^{2^*}(\mathfrak{m})} = 1$ and

$$\mathcal{Q}(u_i) := \frac{\|u_i\|^2_{L^{2^*}} - \mathrm{Vol}(M)^{-2/n}\|u_i\|^2_{L^2}}{\mathrm{Vol}(M)^{-2/n}\|\nabla u_i\|^2_{L^2}} \to A_{2^*}^{\mathrm{opt}}(\mathbb{S}^n). \tag{1.15}$$

In a nutshell, the strategy of the proof consists in a fine investigation of these sequences.

We will show (Theorem 6.1) that (u_i) up to a subsequence can have only three possible behaviors. For each case the conclusion will follow applying a different rigidity theorem.

CASE 1 (Convergence to extremal). The sequence u_i converges in L^{2^*} to a *non constant* extremal function u such that $\mathcal{Q}(u) = A^{\rm opt}_{2^*}(M) = A^{\rm opt}_{2^*}(\mathbb{S}^n)$. This forces the monotone rearrangement $u^* : \mathbb{S}^n \to \mathbb{R}$ (as defined in Sect. 2.4) to achieve equality in the Pólya–Szegő inequality. Since u is assumed not constant, the rigidity case of the Pólya–Szegő inequality (see Theorem 2.22) ensures $M = \mathbb{S}^n$.
CASE 2 (Convergence to constant). The sequence u_i converges in L^{2^*} to a *constant* function $u \equiv c$. Up to renormalization (of the volume measure), it can be assumed that $\int u_i = 1$ and $u \equiv 1$. In this case the rigidity follows exploiting that the linearization of the Sobolev inequality is the Poincaré inequality. More precisely we write $u_i = 1+v_i$, so that $v_i := u_i - 1$ has zero mean. Then it can be shown that:

$$\frac{2^*-2}{n} = A^{\rm opt}_{2^*}(\mathbb{S}^n) = \lim_{i\to\infty} \mathcal{Q}(u_i) = \lim_{i\to\infty} \frac{(2^*-2)\|v_i\|^2_{L^2}}{\|\nabla v_i\|^2_{L^2}} \le \frac{2^*-2}{\lambda_1(M)},$$

where $\lambda_1(M)$ is the spectral gap. This forces $\lambda_1(M) = n$ and the conclusion follows by classical Obata's rigidity theorem.
CASE 3 (Concentration in a single point). The sequence u_i vanishes, i.e. $\|u_i\|_{L^2} \to 0$ (in fact the following concentration happens: $|u_i|^{2^*} \rightharpoonup \delta_p$ for some point $p \in M$). Here is where the constant $\alpha_2(M)$ enters into play. Indeed, by definition of $\alpha_2(M)$, for every $\varepsilon > 0$ there exists B_ε such that

$$1 = \|u_i\|^2_{L^{2^*}} \le (\alpha_2(M)+\varepsilon)\|\nabla u_i\|^2_{L^2} + B_\varepsilon \|u_i\|^2_{L^2}, \qquad \forall i \in \mathbb{N}.$$

Moreover, from $\|u_i\|_{L^2} \to 0$ we must have $\underline{\lim}_i \|\nabla u_i\|^2_{L^2} > 0$. Combining these two observation we obtain that

$$\overline{\lim_i} \frac{\|u_i\|^2_{L^{2^*}}}{\|\nabla u_i\|^2_{L^2}} \le (\alpha_2(M)+\varepsilon).$$

By assumption $\mathcal{Q}(u_i) \to A^{\rm opt}_{2^*}(\mathbb{S}^n)$, which implies

$$\underline{\lim_i} \frac{\|u_i\|^2_{L^{2^*}}}{\|\nabla u_i\|^2_{L^2(M)}} \ge \mathrm{Vol}(M)^{-2/n} A^{\rm opt}_{2^*}(\mathbb{S}^n).$$

Therefore $\alpha_2(M) \ge \mathrm{Vol}(M)^{-2/n} A^{\rm opt}_{2^*}(\mathbb{S}^n)$. However combining (1.2) with

$$Eucl(n,2)^2 = A^{\rm opt}_{2^*}(\mathbb{S}^n)\mathrm{Vol}(\mathbb{S}^n)^{-2/n},$$

we have $\alpha_2(M) = A^{\rm opt}_{2^*}(\mathbb{S}^n)\mathrm{Vol}(\mathbb{S}^n)^{-2/n}$, that coupled with the previous observation yields $\mathrm{Vol}(M) \ge \mathrm{Vol}(\mathbb{S}^n)$. This and the Bishop–Gromov volume ratio implies that $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{S}^n)$, which forces $\mathrm{diam}(M) = \pi$ and the required rigidity follows from Cheng's diameter rigidity theorem.
Structure of the paper. This note is organized as follows:

We begin in Sect. 2 with the necessary preliminaries concerning Sobolev calculus on metric measure spaces and the main properties of CD/RCD spaces.

Section 3 is devoted to show the upper bound of $\alpha_p(\mathrm{X})$ in (1.8). This upper bound is obtained from a class of local Euclidean Sobolev inequalities (Theorem 3.8). To prove these inequalities we develop, in the general framework of $\mathrm{CD}(K,N)$ spaces, a Euclidean Pólya–Szegő inequality (Sect. 3.1), which is then coupled with a local isoperimetric inequality of Euclidean type (Theorem 3.9).

Section 4 is devoted to achieve the lower bound of $\alpha_p(\mathrm{X})$ in (1.8) and, combined with the previous section, the proof of Theorem 1.4. Here we also derive, as an application, sharp Sobolev inequalities on $\mathrm{CD}(0, N)$ spaces (Sect. 4.2).

In Sect. 5, we consider three different geometric bounds on the optimal constant A_q^{opt} in the Sobolev inequality (1.10): an upper bound depending on the Ricci curvature bounds (Sect. 5.1), a lower bound in terms of the first eigenvalue (Sect. 5.2) and a lower bound in terms of the diameter (Sect. 5.3).

In Sect. 6, we prove our main rigidity result on A_q^{opt}, namely Theorem 1.9. To this aim we develop a concentration compactness dichotomy principle under mGH-convergence, in the $\mathrm{RCD}(K, N)$ setting (Theorem 6.1). The second ingredient for the rigidity is instead a quantitative linearization lemma for the Sobolev inequality that we prove in Sect. 6.2.

In Sect. 7, we prove the main almost-rigidity result of this note stated in Theorem 1.10. This will be obtained as a consequence of the continuity of the constant A_q^{opt} under mGH-convergence (Sect. 7.2). For this result we will need to fully exploit the concentration compactness tools under mGH-convergence developed in the previous section.

Finally, in Sect. 8 we conclude this note by studying the so-called generalize Yamabe equation on $\mathrm{RCD}(K, N)$ spaces. We will prove a classical existence result in Sect. 8.1 while in Sect. 8.2 we will show a continuity result for the generalized Yamabe constant under mGH-convergence.

2 Preliminaries

2.1 Basic notations

We collect once and for all the key constants appearing in this note.

For all $N \in [1, \infty)$, $p \in (1, N)$, we define the generalized[1] unit ball and unit sphere volumes by

$$\omega_N := \frac{\pi^{N/2}}{\Gamma(N/2+1)}, \quad \sigma_{N-1} := N\omega_N, \tag{2.1}$$

where Γ is the Gamma-function, and the sharp Euclidean Sobolev constant by

$$Eucl(N, p) := \frac{1}{N}\Big(\frac{N(p-1)}{N-p}\Big)^{\frac{p-1}{p}}\Big(\frac{\Gamma(N+1)}{N\omega_N\Gamma(N/p)\Gamma(N+1-N/p)}\Big)^{\frac{1}{N}}. \tag{2.2}$$

For $N > 2$ and $p = 2$, the above reduces to

$$Eucl(N, 2) = \Big(\frac{4}{N(N-2)\sigma_N^{2/N}}\Big)^{\frac{1}{2}}. \tag{2.3}$$

We will sometimes need also the following identity:

$$\int_0^{\pi} \sin^{N-1}(t)\,\mathrm{d}t = \frac{\sigma_N}{\sigma_{N-1}}, \quad \forall N > 1. \tag{2.4}$$

Throughout this note a metric measure space will be a triple $(\mathrm{X}, d, \mathfrak{m})$, where

(X, d) is a complete and separable metric space,

$\mathfrak{m} \neq 0$ is non negative and boundedly finite Borel measure.

[1] For an integer N, ω_N is the volume of the unit ball in $\mathbb{R}^N$ and σ_N is the volume of the N-sphere $\mathbb{S}^N$.

To avoid technicalities, we will work under the assumption that $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$.

We will denote by $\mathrm{LIP}(\mathrm{X})$, $\mathrm{LIP}_b(\mathrm{X})$, $\mathrm{LIP}_{bs}(\mathrm{X})$, $\mathrm{LIP}_c(\mathrm{X})$, $C(\mathrm{X})$, $C_b(\mathrm{X})$ and $C_{bs}(\mathrm{X})$ respectively the spaces of Lipschitz functions, Lipschitz and bounded functions, Lipschitz functions with bounded support, Lipschitz functions with compact support, continuous functions, continuous and bounded functions and continuous functions with bounded support on X. We will also denote by $\mathrm{LIP}_c(\Omega)$ and $\mathrm{LIP}_{loc}(\Omega)$, for $\Omega \subset \mathrm{X}$ open, the spaces of Lipschitz functions with compact support and locally Lipschitz functions in Ω. Moreover, if $f \in \mathrm{LIP}(\mathrm{X})$, we denote by $\mathrm{Lip}(f)$ its Lipschitz constant, and we say that f is L-Lipschitz, for $L > 0$, if $\mathrm{Lip}(f) \le L$. Also, we recall the notion of local Lipschitz constant for a locally Lipschitz function f:

$$\mathrm{lip}\, f(x) := \varlimsup_{y\to x} \frac{|f(y) - f(x)|}{d(x, y)},$$

taken to be 0 if x is isolated.

We will denote by $\mathscr{M}_b^+(\mathrm{X})$ and $\mathscr{P}(\mathrm{X})$ respectively the space of Borel non-negative finite measure and Borel probability measures on X. By $\mathscr{P}_2(\mathrm{X})$, we denote the class of probability measures with finite second moment, that is the space of all $\mu \in \mathscr{P}(\mathrm{X})$ so that $\int d^2(x, x_0)\,\mathrm{d}\mu(x) < \infty$ for some (and thus, any) $x_0 \in \mathrm{X}$. Given two complete metric spaces (X, d), $(\mathrm{Y}, d_\mathrm{Y})$ a Borel measure μ on X and a Borel map $\varphi \colon \mathrm{X} \to \mathrm{Y}$, the *pushforward* of μ via φ, is the measure $(\varphi)_\sharp\mu$ on Y defined by $(\varphi)_\sharp\mu(E) := \mu(\varphi^{-1}(E))$ for every $E \subset \mathrm{Y}$. Then two metric measure spaces $(\mathrm{X}_i, d_i, \mathfrak{m}_i)_{i=1,2}$ are said to be *isomorphic*, $\mathrm{X}_1 \simeq \mathrm{X}_2$ in short, if there exists an isometry $\iota : \mathrm{X}_1 \to \mathrm{X}_2$ such that $(\iota)_\sharp\mathfrak{m}_1 = \mathfrak{m}_2$.

For $B \subset \mathrm{X}$ we will denote by $\mathrm{diam}(B)$ the quantity $\sup_{x,y\in B} d(x, y)$. We say that a metric measure space $(\mathrm{X}, d, \mathfrak{m})$ is locally doubling if for every $R > 0$, there exists a constant $C := C(R)$ so that

$$\mathfrak{m}(B_{2r}(x)) \le C\mathfrak{m}(B_r(x)), \qquad \forall x \in \mathrm{X}, r \in (0, R).$$

Whenever $C(R)$ can be taken independent of R we say that $(\mathrm{X}, d, \mathfrak{m})$ is doubling.

A geodesic for us will denote a constant speed length-minimizing curve between its endpoints and defined on $[0, 1]$, i.e. a curve $\gamma : [0, 1] \to \mathrm{X}$ so that $d(\gamma_t, \gamma_s) = |t - s|d(\gamma_0, \gamma_1)$, for every $t, s \in [0, 1]$. Also, we denote by $\mathrm{Geo}(\mathrm{X})$ the set of all geodesics and call X a geodesic metric space, provided for any two couple of points, there exists a geodesic linking the two as already discussed. To conclude, we define the evaluation map e_t, $t \in [0, 1]$, as the assignment $e_t \colon C([0, 1], \mathrm{X}) \to \mathrm{X}$ defined via $e_t(\gamma) := \gamma_t$.

2.2 Calculus on metric measure spaces

2.2.1 Sobolev spaces

We start recalling the notion of Sobolev spaces in a metric measure space. We refer to [52, 56, 61] for more details on this topic.

The concept of Sobolev space for a metric measure space was introduced in the seminal works of Cheeger [39] and of Shanmugalingam [90], while here we adopt the approach via Cheeger energy developed in [10] and proved there to be equivalent with the notions in [39, 90].

Let $p \in (1, \infty)$ and $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure spaces. The p-Cheeger energy $\mathrm{Ch}_p \colon L^p(\mathfrak{m}) \to [0, \infty]$ is defined as the convex and lower semicontinuous functional

$$\mathrm{Ch}_p(f) := \inf\Big\{ \varliminf_{n\to\infty} \int \mathrm{lip}^p f_n \, d\mathfrak{m} \colon (f_n) \subset L^p(\mathfrak{m}) \cap \mathrm{LIP}(\mathrm{X}), \lim_n \|f - f_n\|_{L^p(\mathfrak{m})} = 0 \Big\}.$$

The p-Sobolev space is then defined as the space $W^{1,p}(\mathrm{X}) := \{\mathrm{Ch}_p < \infty\}$ equipped with the norm $\|f\|^p_{W^{1,p}(\mathrm{X})} := \|f\|^p_{L^p(\mathfrak{m})} + \mathrm{Ch}_p(f)$, which makes it a Banach space. Under the assumption that $(\mathrm{X}, d, \mathfrak{m})$ is doubling, $W^{1,p}(\mathrm{X})$ is reflexive as proven in [5] and in particular the class $\mathrm{LIP}_{bs}(\mathrm{X})$ is dense in $W^{1,p}(\mathrm{X})$ (see also the more recent [48]). Finally, exploiting the definition by relaxation given for the p-Cheeger energy, it can be proved (see [10]) that whenever $f \in W^{1,p}(\mathrm{X})$, then there exists a minimal $\mathfrak{m}$-a.e. object $|Df|_p \in L^p(\mathfrak{m})$ called minimal p-weak upper gradient so that

$$\mathrm{Ch}_p(f) := \int |Df|_p^p \, d\mathfrak{m}.$$

In general, the dependence on p of such object is hidden and not trivial (that is why we introduced the p-subscript in the object $|Df|_p$), as shown for example in the analysis [44]. Nevertheless, in this note, we are mainly concerned in working on a class of spaces, which will be later discussed, where such dependence is ruled out (see Remark 2.8). In this case, the subscript will be automatically omitted.

We will often need to consider the case when $(\mathrm{X}, d, \mathfrak{m})$ is a weighted interval with a weight that is bounded away from zero, i.e. $(\mathrm{X}, d, \mathfrak{m}) = ([a, b], |.|, h\mathscr{L}^1)$, $a, b \in \mathbb{R}$ with $a < b$ where $h \in L^1([a, b])$ and for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ so that $h \geq c_\varepsilon$ $\mathscr{L}^1$-a.e. in $[a + \varepsilon, b - \varepsilon]$. In this case, we denote by $W^{1,p}([a, b], |.|, h\mathscr{L}^1)$ the p-Sobolev space over the weighted interval according to the metric definition relying on the Cheeger energy, while simply write $W^{1,p}(a, b)$ for the classical definition via integration by parts. It can be shown that (for example using [10,Remark 4.10])

$$f \in W^{1,p}([a, b], |.|, h\mathscr{L}^1) \quad \Longleftrightarrow \quad f \in W^{1,1}_{loc}(a, b) \text{ with } f, f' \in L^p(h\mathscr{L}^1), \tag{2.5}$$

in which case $|Df|_p = |f'|$ $\mathscr{L}^1$-a.e..

For every $\Omega \subset \mathrm{X}$ we define also the local Sobolev space $W^{1,p}_0(\Omega) \subset L^p(\Omega)$ as

$$W^{1,p}_0(\Omega) := \overline{\mathrm{LIP}_c(\Omega)}^{W^{1,p}(\mathrm{X})}.$$

From the previous discussion, if X is locally compact and locally doubling, then $W^{1,p}_0(\mathrm{X}) = W^{1,p}(\mathrm{X})$.

Next, according to the definition given in [51], we say that $(\mathrm{X}, d, \mathfrak{m})$ is *infinitesimally Hilbertian* if $W^{1,2}(\mathrm{X})$ is a Hilbert space. This property reflects that the underlying geometry looks Riemannian at small scales and can equivalently be characterized via the validity of the following parallelogram identity

$$|D(f + g)|_2^2 + |D(f - g)|_2^2 = 2|Df|_2^2 + |Dg|_2^2 \qquad \mathfrak{m}\text{-a.e.}, \quad \forall f, g \in W^{1,2}(\mathrm{X}). \tag{2.6}$$

This allows to give a notion of scalar product between gradients of Sobolev functions

$$L^1(\mathfrak{m}) \ni \langle \nabla f, \nabla g \rangle := \lim_{\varepsilon\to 0} \frac{|D(f + \varepsilon g)|_2^2 - |Df|_2^2}{\varepsilon}, \qquad \forall f, g \in W^{1,2}(\mathrm{X}), \tag{2.7}$$

where the limit exists and is bilinear on its entries, as it can be directly checked using (2.6). Notice that the symbol $\langle \nabla f, \nabla g \rangle$ is purely formal. Nevertheless, by introducing the right framework to discuss gradients ∇f, it can be made rigorous (see [52]), but we will never need this fact.

In the infinitesimal Hilbertian class, we can give a notion of a *measure-valued Laplacian* via integration by parts. Since it will be enough for our purposes, we will only consider the compact case.

Definition 2.1 *(Measure-valued Laplacian,* [51]*)* Let $(\mathrm{X}, d, \mathfrak{m})$ be a compact infinitesimally Hilbertian metric measure space. We say that $f \in W^{1,2}(\mathrm{X})$ has a measure-valued Laplacian, and we write $f \in D(\mathbf{\Delta})$, provided there exists a Radon measure μ such that

$$\int g \, \mathrm{d}\mu = -\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}\mathfrak{m}, \qquad \forall g \in \mathrm{LIP}(\mathrm{X}).$$

In this case the we will denote (the unique) μ by $\mathbf{\Delta} f$.

From the bilinearity of the pointwise inner product we see that $D(\mathbf{\Delta})$ is a vector space and the assignment $f \mapsto \mathbf{\Delta} f$ is linear.

2.2.2 Functions of bounded variations and sets of finite perimeter

We introduce the space of functions of bounded variation and sets finite perimeter following [6, 82].

Definition 2.2 *(BV-functions)* A function $f \in L^1(\mathfrak{m})$ is of bounded variation, and we write $f \in BV(\mathrm{X})$, provided there exists a sequence of locally Lipschitz functions $f_n \to f$ in $L^1(\mathfrak{m})$ such that

$$\varlimsup_{n\to\infty} \int \mathrm{lip}\, f_n \, \mathrm{d}\mathfrak{m} < \infty.$$

By localizing this definition, we can define accordingly

$$|\boldsymbol{D} f|(A) := \inf\left\{ \varliminf_{n\to\infty} \int_A \mathrm{lip}\, f_n \, \mathrm{d}\mathfrak{m} \colon f_n \subset \mathrm{LIP}_{loc}(A),\ f_n \to f \text{ in } L^1(A) \right\},$$

for every open $A \subset \mathrm{X}$. It turns out (see [6] and also [82] for locally compact spaces) that the map $A \mapsto |\boldsymbol{D} f|(A)$ is the restriction to open sets of a non-negative finite Borel measure called the *total variation* of f, which we will still denote by $|\boldsymbol{D} f|$.

For every $f \in \mathrm{LIP}_{bs}(\mathrm{X})$ we clearly have that $|\boldsymbol{D} f| \le \mathrm{lip}\, f \mathfrak{m}$ and in particular that $|\boldsymbol{D} f| \ll \mathfrak{m}$. In this case we call $|Df|_1$ the density of $|\boldsymbol{D} f|$ with respect to $\mathfrak{m}$.

If we suitably modify Definition 2.2 for functions in $L^1_{loc}(\mathfrak{m})$ we can choose $f = \chi_E$ for any $E \subset \mathrm{X}$ Borel and define:

Definition 2.3 *(Perimeter and finite perimeter sets)* Let E be Borel and A open subset of X. The perimeter of E in A, written $\mathrm{Per}(E, A)$ is defined as

$$\mathrm{Per}(E, A) := \inf\left\{ \varliminf_{n\to\infty} \int_A \mathrm{lip}\, u_n \, \mathrm{d}\mathfrak{m} \colon u_n \subset \mathrm{LIP}_{loc}(A), u_n \to \chi_E \text{ in } L^1_{loc}(A) \right\}.$$

Moreover, we say that E is a set of finite perimeter if $\mathrm{Per}(E, \mathrm{X}) < \infty$.

Again, (see, e.g. [4, 6, 82]), when E has finite perimeter, it holds that $A \mapsto \mathrm{Per}(E, A)$ is the restriction of a non-negative finite Borel measure to open sets, which we denote by $\mathrm{Per}(E, \cdot)$. Moreover, as a common convention, when $A = \mathrm{X}$ we simply write $\mathrm{Per}(E)$ instead of $\mathrm{Per}(E, \mathrm{X})$.

For a Borel set $E \subset \mathrm{X}$ of finite measure we also define its Minkowski content as:

$$\mathfrak{m}^+(E) = \varliminf_{\delta \to 0^+} \frac{\mathfrak{m}(E^\delta) - \mathfrak{m}(E)}{\delta},$$

where $E^\delta := \{x \in \mathrm{X} \,:\, d(x, E) < \delta\}$. In general we only have $\mathrm{Per}(E) \le \mathfrak{m}^+(E)$.

We recall that the following *coarea formula* is valid after [82,Proposition 4.2].

Theorem 2.4 (Coarea formula) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a locally compact metric measure space and* $f \in BV(\mathrm{X})$. *Then the set* $\{f > t\}$ *is of finite perimeter for a.e.* $t \in \mathbb{R}$ *and given any Borel function* $g : \mathrm{X} \to [0, \infty)$*, it holds that*

$$\int_{\{s \le u < t\}} g \,\mathrm{d}|\boldsymbol{D} f| = \int_s^t \int g \,\mathrm{dPer}(\{f > t\}, \cdot)\,\mathrm{d}t, \qquad \forall s, t \in [0, \infty),\ s < t. \tag{2.8}$$

2.3 CD(K, N) and RCD(K, N) spaces

2.3.1 Main definitions and properties

In this note, as anticipated in the introduction, we will work in the general framework of metric measure spaces $(\mathrm{X}, d, \mathfrak{m})$ satisfying synthetic Ricci curvature lower bounds. For completeness, we briefly recall the definition and the key properties that we will need.

The first notion of synthetic Ricci lower bounds was given independently in the seminal papers [80] and [92, 93] where the authors introduced the celebrated *curvature dimension condition*. We report here its definition only in finite dimension $N \in [1, \infty)$, given in term of convexity properties of the *N-Rényi-entropy* functional $\mathcal{U}_N : \mathscr{P}_2(\mathrm{X}) \to [-\infty, 0]$ defined by

$$\mathcal{U}_N(\mu|\mathfrak{m}) := -\int \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}, \qquad \text{if } \mu = \rho\mathfrak{m} + \mu^s,$$

where $\mu \in \mathscr{P}_2(\mathrm{X})$ and μ^s is singular with respect to $\mathfrak{m}$. In this note, since optimal transportation plays a minor role, we shall assume the reader to be familiar with Optimal Transport and the Wasserstein Space $(\mathscr{P}_2(\mathrm{X}), W_2)$ and we refer to [96] for a systematic discussion (see also [8]).

We start recalling the definition of distortion coefficients. For every $K \in \mathbb{R}, N \in [0, \infty), t \in [0, 1]$ set

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} +\infty, & \text{if } K\theta^2 \ge N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})}, & \text{if } 0 < K\theta^2 < N\pi^2, \\ t, & \text{if } K\theta^2 < 0 \text{ and } N = 0 \text{ or if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}, & \text{if } K\theta^2 \le 0 \text{ and } N > 0. \end{cases}$$

Set also, for $N > 1$, $\tau^{(t)}_{K,N}(\theta) := t^{\frac{1}{N}}\sigma^{(t)}_{K,N-1}(\theta)^{1-\frac{1}{N}}$ while $\tau^{(t)}_{K,1}(\theta) = t$ if $K \le 0$ and $\tau^{(t)}_{K,1}(\theta) = \infty$ if $K > 0$.

Definition 2.5 *(CD(K, N)-spaces)* Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space $(\mathrm{X}, d, \mathfrak{m})$ satisfies the *curvature dimension condition* CD(K, N) if, for every $\mu_0, \mu_1 \in \mathscr{P}_2(\mathrm{X})$ absolutely continuous with bounded supports, there exists a dynamical optimal transference plan $\pi \in \mathscr{P}(\mathrm{Geo}(\mathrm{X}))$ between μ_0, μ_1 so that: for every $t \in [0, 1]$ and $N' \ge N$, we have $\mu_t := (e_t)_\sharp\pi = \rho_t\mathfrak{m}$ and

$$\mathcal{U}_{N'}(\mu_t|\mathfrak{m}) \le -\int \left(\tau^{(1-t)}_{K,N'}(d(\gamma_1,\gamma_0))\rho_0(\gamma_0)^{-\frac{1}{N}} + \tau^{(t)}_{K,N'}(d(\gamma_1,\gamma_0))\rho_1(\gamma_1))^{-\frac{1}{N}}\right) \mathrm{d}\pi(\gamma). \tag{2.9}$$

We recall the also the notion of one-dimensional model space for the CD($N-1, N$) condition:

Definition 2.6 *(One dimensional model space)* For every $N > 1$ we define $I_N := ([0,\pi], |.|, \mathfrak{m}_N)$, where $|.|$ is the Euclidean distance restricted on $[0,\pi]$ and

$$\mathfrak{m}_N := \tfrac{1}{c_N}\sin^{N-1}\mathscr{L}^1|_{[0,\pi]},$$

with $c_N := \int_{[0,\pi]} \sin(t)^{N-1}\,\mathrm{d}t$.

To encode a more "Riemannian" behavior of the space, and to rule out Finsler spaces which are allowed by the CD condition, it was introduced in [11] the so-called RCD condition in the infinite dimensional case (see also [55] for the case of σ-finite reference measure). In this note however we will only work in finite dimensional RCD-spaces introduced in [51].

Definition 2.7 *(RCD(K, N)-spaces)* Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A metric measure space $(\mathrm{X}, d, \mathfrak{m})$ is an RCD(K, N)-space, provided it is an infinitesimal Hilbertian CD(K, N)-space.

Remark 2.8 Spaces satisfying the CD(K, N) (and thus also the RCD(K, N)) condition, support a $(1,1)$-local Poincaré inequality (see [87]) and by the Bishop–Gromov inequality below they are locally-doubling, therefore from the results in [39] we know that the minimal weak upper gradient is independent on the exponent p (see also [54]). For this reason, to lighten the notation, in this setting we will simply write $|Df|$ for $f \in W^{1,p}(\mathrm{X})$ and call it simply minimal weak upper gradient of f. □

We start by recalling some useful properties about these spaces that are going to be used in the sequel.

On CD(K, N) spaces the Bishop–Gromov inequality holds (see [93]):

$$\frac{\mathfrak{m}(B_R(x))}{v_{K,N}(R)} \le \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}, \quad \text{for any } 0 < r < R \le \pi\sqrt{\frac{N-1}{K^+}} \text{ and any } x \in \mathrm{X}, \tag{2.10}$$

where the quantities $v_{K,N}(r)$, $N \in [1,\infty)$ $K \in \mathbb{R}$ are defined as

$$v_{K,N}(r) := \sigma_{N-1}\int_0^r |s_{K,N}(t)|^{N-1}\,\mathrm{d}t,$$

and $s_{K,N}(t)$ is defined as $\sin\left(t\sqrt{\frac{K}{N-1}}\right)$, if $K > 0$, $\sinh\left(t\sqrt{\frac{|K|}{N-1}}\right)$, if $K < 0$ and t if $K = 0$. In particular CD(K, N) spaces are uniformly locally doubling and thus proper, i.e. closed and bounded sets are also compact. We also note that in the case $K = 0$ this implies that the limit

$$AVR(\mathrm{X}) := \lim_{r\to+\infty} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}$$

exists finite and does not depend on the point $x \in \mathrm{X}$. We call the quantity $AVR(\mathrm{X})$ *asymptotic volume ratio* of X and if $AVR(\mathrm{X}) > 0$ we say that X has *Euclidean-volume growth*. A key role in the note will be played by the following quantities:

$$\theta_{N,r}(x) := \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}, \quad \theta_N(x) := \lim_{r \to 0^+} \theta_{N,r}(x), \qquad \forall r > 0, x \in \mathrm{X}.$$

Observe that the above limit exists thanks to the Bishop–Gromov inequality and the fact that $\lim_{r \to 0^+} \frac{\omega_N r^N}{v_{K,N}(r)} = 1$ for every $K \in \mathbb{R}$, $N \in [1, \infty)$, which in particular grants that

$$\theta_N(x) = \lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} = \sup_{r>0} \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}. \tag{2.11}$$

This and the fact that $\mathfrak{m}(\partial B_r(x)) = 0$ for every $r > 0$ and $x \in \mathrm{X}$ (which follows from the Bishop–Gromov inequality), implies that $\theta_N(x)$ is a *lower-semicontinuous* function of x. Therefore, when X is compact, there exists $\min_{x \in \mathrm{X}} \theta_N(x)$.

Next we recall the Brunn–Minkowski inequality.

Theorem 2.9 ([93]) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(K, N)$ *space with* $N \in [1, \infty)$, $K \in \mathbb{R}$. *For any couple of Borel sets* $A_0, A_1 \subset \mathrm{X}$ *it holds that*

$$\mathfrak{m}(A_t)^{\frac{1}{N}} \geq \sigma_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{\frac{1}{N}} + \sigma_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{\frac{1}{N}}, \qquad \forall t \in [0, 1], \tag{2.12}$$

where $A_t := \{\gamma_t \ : \ \gamma$ *geodesic such that* $\gamma_0 \in A_0,\ \gamma_1 \in A_1\}$ *and*

$$\theta := \begin{cases} \inf_{(x_0,x_1) \in A_0 \times A_1} d(x_0, x_1), & \text{if } K \geq 0, \\ \sup_{(x_0,x_1) \in A_0 \times A_1} d(x_0, x_1), & \text{if } K < 0, \end{cases}$$

We remark that (2.12) is actually weaker than the statement appearing in [93] and it holds for the (a priori) larger class of $\mathrm{CD}^*(K, N)$ spaces (see [22]).

We report the Bonnet–Myers diameter-comparison theorem for CD-spaces from [93]:

$$(\mathrm{X}, d, \mathfrak{m}) \text{ is a } \mathrm{CD}(K, N) \text{ space, for some } K > 0 \quad \Rightarrow \quad \mathrm{diam}(\mathrm{X}) \leq \pi\sqrt{\tfrac{N-1}{K}}, \tag{2.13}$$

The Lichnerowitz 2-spectral gap inequality is valid also in the CD-setting. To state it we recall the notion of first non-trivial Neumann eigenvalue of the Laplacian (or 2-spectral gap) in metric measure spaces.

Definition 2.10 Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space with finite measure. We define the first non trivial 2-eigenvalue $\lambda^{1,2}(\mathrm{X})$ as the non-negative number given by

$$\lambda^{1,2}(\mathrm{X}) := \inf \left\{ \frac{\int |Df|_2^2 \, \mathrm{d}\mathfrak{m}}{\int |f|^2 \, \mathrm{d}\mathfrak{m}} \ : \ f \in \mathrm{LIP}(\mathrm{X}) \cap L^2(\mathfrak{m}),\ f \neq 0,\ \int f \, \mathrm{d}\mathfrak{m} = 0 \right\}. \tag{2.14}$$

Clearly, in light of [10], in the above definition one can equivalently take the infimum among all $f \in W^{1,2}(\mathrm{X})$. In the sequel will use this fact without further notice.

Then the spectral-gap inequality as proven in [80] (see also [67]) says that:

$$\lambda^{1,2}(\mathrm{X}) \geq N, \qquad \text{for every } \mathrm{CD}(N-1, N)\text{-space X},$$

with N ranging in $(1, \infty)$.

We conclude this part recalling some rigidity and stability statements for $\mathrm{RCD}(K, N)$ spaces and to this goal we need to define the notion of spherical suspension over a metric measure space. For any $N \in [1, \infty)$ the N-spherical suspension over a metric measure space

$(\mathrm{Z}, \mathfrak{m}_{\mathrm{Z}}, d_{\mathrm{Z}})$ is defined to be the space $([0,\pi] \times_{\sin}^N \mathrm{Z}) := \mathrm{Z} \times [0,\pi]/(\mathrm{Z} \times \{0,\pi\})$ endowed with the following distance and measure

$$d((t,z),(s,z')) := \cos^{-1}\Big(\cos(s)\cos(t) + \sin(s)\sin(t)\cos\big(d_{\mathrm{Z}}(z,z') \wedge \pi\big)\Big),$$
$$\mathfrak{m} := \sin^{N-1}(t)\mathrm{d}t \otimes \mathfrak{m}_{\mathrm{Z}}.$$

It turns out that the RCD condition is stable under the action of taking spherical suspensions, more precisely it has been proven in [71] that

$$\begin{aligned}&[0,\pi] \times_{\sin}^N \mathrm{Z},\ N \geq 2 \text{ is a } \mathrm{RCD}(N-1,N) \text{ space if and only if}\\ &\mathrm{diam}(\mathrm{Z}) \leq \pi \text{ and Z is an } \mathrm{RCD}(N-2,N-1) \text{ space,}\end{aligned} \tag{2.15}$$

We can now recall the two main rigidity statements that we will use in the note: the maximal diameter theorem and the Obata theorem for $\mathrm{RCD}(K,N)$ spaces:

Theorem 2.11 ([70]) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1,N)$ *space with and* $N \in [2,\infty)$ *and suppose that* $\mathrm{diam}(\mathrm{X}) = \pi$*. Then* $(\mathrm{X}, d, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathrm{RCD}(N-2,N-1)$ *space* $(\mathrm{Z}, d_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with* $\mathrm{diam}(\mathrm{Z}) \leq \pi$ *satisfying* $\mathrm{X} \simeq [0,\pi] \times_{\sin}^N \mathrm{Z}$.

Theorem 2.12 ([71]) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1,N)$ *space with and* $N \in [2,\infty)$ *and suppose that* $\lambda^{1,2}(\mathrm{X}) = N$*. Then* $(\mathrm{X}, d, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathrm{RCD}(N-2,N-1)$ *space* $(\mathrm{Z}, d_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with* $\mathrm{diam}(\mathrm{Z}) \leq \pi$ *satisfying* $\mathrm{X} \simeq [0,\pi] \times_{\sin}^N \mathrm{Z}$.

We end this part by recalling the definition of "non-collapsed" RCD-spaces, which extends the notion of non-collapsed Ricci-limits introduced in [40].

Definition 2.13 *([53])* We say that $(\mathrm{X}, d, \mathfrak{m})$ is a *non-collapsed* $\mathrm{RCD}(K,N)$ space, for some $K \in \mathbb{R}$, $N \in \mathbb{N}$, provided it is $\mathrm{RCD}(K,N)$ and $\mathfrak{m} = \mathscr{H}^N$, where $\mathscr{H}^N$ is the N-dimensional Hausdorff measure.

This class of spaces enjoys extra regularity with respect to the general RCD-class and are a suitable setting to derive the *topological rigidity* results of this note. Here we just mention that if θ_N is finite $\mathfrak{m}$-a.e. (or equivalently if $\mathfrak{m} \ll \mathscr{H}^N$), then up to a constant multiplicative factor, $\mathfrak{m}$ equals $\mathscr{H}^N$ and the space is non-collapsed. This has been proved first in [62] for compact spaces and then in [31] in the general case solving a conjecture of [53] (see also [65] for an account on further conjectures around this topic).

2.3.2 Sobolev–Poincaré inequality on CD(K, N) spaces

A well-established fact which goes back to the seminal work [58], is that a $(1,p)$-Poincaré inequality on a doubling metric measure space, improves to a (q,p)-Poincaré inequality with $q > 1$. On $\mathrm{CD}(K,N)$ spaces this translates in the following result.

Theorem 2.14 ((p^*,p)-Poincaré inequality) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(K,N)$ *space for some* $N \in (1,\infty)$*,* $K \in \mathbb{R}$*. Fix also* $p \in (1,N)$ *and* $r_0 > 0$*. Then, for every* $B_r(x) \subset \mathrm{X}$ *with* $r \leq r_0$ *it holds*

$$\left(\fint_{B_r(x)} |u - u_{B_r(x)}|^{p^*}\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p^*}} \leq C(K,N,p,r_0)r\left(\fint_{B_{2r}(x)} |Du|^p\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}, \quad \forall u \in \mathrm{LIP}(\mathrm{X}), \tag{2.16}$$

where $p^* := pN/(N-p)$ *and* $u_{B_r(x)} := \fint_{B_r(x)} u\,\mathrm{d}\mathfrak{m}$.

Proof From [87] we have that X supports a strong (1, 1)-Poincaré inequality, in particular it also supports a strong $(1, p)$-Poincaré inequality for every $p \in [1, \infty)$, by Hölder inequality. Moreover, for every $x_0 \in \mathrm{X}$, $r \leq r_0$ and $x \in B_{r_0}(x_0)$, from the Bishop–Gromov inequality (2.10) it holds that

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_{r_0}(x_0))} \geq C(K, N, r_0)\Big(\frac{r}{r_0}\Big)^N.$$

Then (2.16) follows from [58,Theorem 5.1] (see also [29,Theorem 4.21]). □

We end this part recalling the sharp Sobolev-inequality on the N model space I_N (see Def. 2.6) for $N \in (2, \infty)$ (see e.g. [76]):

$$\|u\|^2_{L^q(\mathfrak{m}_N)} \leq \frac{q-2}{N}\,\||Du|\|^2_{L^2(\mathfrak{m}_N)} + \|u\|^2_{L^2(\mathfrak{m}_N)}, \quad \forall u \in W^{1,2}([0,\pi], |.|, \mathfrak{m}_N), \tag{2.17}$$

for every $q \in (2, 2^*]$, with $2^* = 2N/(N-2)$.

2.3.3 Convergence and compactness under mGH-convergence

We recall here the notion of *pointed-measure Gromov Hausdorff convergence* (pmGH convergence for short). Let us say that the definition we will adopt is not the classical one (see e.g. [34, 57]), but it is equivalent in the case of a sequence of uniformly locally doubling metric measure spaces, thanks to the results in [55]. It will be convenient to consider in this section the set $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Recall also that a pointed metric measure space is a quadruple $(\mathrm{X}, d, \mathfrak{m}, x)$ consisting of a metric measure space $(\mathrm{X}, d, \mathfrak{m})$ and a point $x \in \mathrm{X}$.

Definition 2.15 *(Pointed measure Gromov–Hausdorff convergence)* We say that the sequence $(\mathrm{X}_n, d_n, \mathfrak{m}_n, x_n)$, $n \in \mathbb{N}$, of pointed metric measure spaces, *pointed measure Gromov–Hausdorff*-converges (pmGH-converges in short) to $(\mathrm{X}_\infty, d_\infty, \mathfrak{m}_\infty, x_\infty)$, if there exist isometric embeddings $\iota_n : \mathrm{X}_n \to (\mathrm{Z}, d_\mathrm{Z})$, $n \in \bar{\mathbb{N}}$, into a common metric space $(\mathrm{Z}, d_\mathrm{Z})$ such that

$$(\iota_n)_\sharp \mathfrak{m}_n \rightharpoonup (\iota_\infty)_\sharp \mathfrak{m}_\infty \text{ in duality with } C_{bs}(\mathrm{Z}) \text{ and } \iota_n(x_n) \to \iota_\infty(x_\infty).$$

In the case of a sequence of uniformly locally doubling spaces (as in the case of $\mathrm{CD}(K, N)$-spaces for fixed $K \in R$, $N < \infty$) we can also take $(\mathrm{Z}, d_\mathrm{Z})$ to be proper. Moreover, again for a class of uniformly locally doubling spaces, in [55] it is proven that the pmGH-convergence is metrizable with a distance which we call d_{pmGH}.

It will be also convenient to adopt, thanks to Definition 2.15, the so-called *extrinsic approach*, where the spaces X_n are identified as subsets of a common *proper* metric space $(\mathrm{Z}, d_\mathrm{Z})$, $\mathrm{X}_n \subset \mathrm{Z}$, $\mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$, $d_{\mathrm{Z}|\mathrm{X}_n \times \mathrm{X}_n} = d_n$ for all $n \in \bar{\mathbb{N}}$, and $d_\mathrm{Z}(x_n, x_\infty) \to 0$, $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C_{bs}(\mathrm{Z})$. Any such space $(\mathrm{Z}, d_\mathrm{Z})$ (together with an the identification of $\mathrm{X}_n \subset \mathrm{Z}$) is called *realization of the convergence* and (in the case of geodesic uniformly locally doubling spaces) can be taken so that $d^\mathrm{Z}_H(B^{\mathrm{X}_n}_R(x_n), B^{\mathrm{X}_\infty}_R(x_\infty)) \to 0$ for every $R > 0$, where d^Z_H is the Hausdorff distance in Z. To avoid confusion when dealing with this identification, we shall sometimes write $B^{\mathrm{X}_n}_r(x)$ with $x \in \mathrm{X}_n$, $r > 0$, to denote the set $B^\mathrm{Z}_r(x) \cap \mathrm{X}_n$.

After the works in [11, 55, 80, 92, 93] and thanks to the Gromov's precompactness theorem [57] we have the following precompactness result.

Theorem 2.16 *Let* $(\mathrm{X}_n, d_n, \mathfrak{m}_n, x_n)$ *be a sequence of pointed* $\mathrm{CD}(K_n, N_n)$ *(resp.* $\mathrm{RCD}(K_n, N_n)$*) spaces,* $n \in \bar{\mathbb{N}}$*, with* $\mathfrak{m}(B_1(x_n)) \in [v^{-1}, v]$*, for* $v > 1$ *and* $K_n \to K \in \mathbb{R}$, $N_n \to N \in [1, \infty)$*. Then, there exists a subsequence* (n_k) *and a pointed* $\mathrm{CD}(K, N)$ *(resp.* $\mathrm{RCD}(K, N)$*) space* $(\mathrm{X}_\infty, d_\infty, \mathfrak{m}_\infty, x_\infty)$ *satisfying*

$$\lim_{k\to\infty} d_{pmGH}\big((\mathrm{X}_{n_k}, d_{n_k}, \mathfrak{m}_{n_k}, x_{n_k}), (\mathrm{X}_\infty, d_\infty, \mathfrak{m}_\infty, x_\infty)\big) = 0.$$

We will be frequently consider the case of compact (with uniformly bounded diameter) metric measure spaces which is the natural setting for the Sobolev embedding of this note, for which we can reduce the above convergence to the so-called *measure Gromov Hausdorff convergence*, mGH-convergence for short, where we simply ignore the convergence of the base points. Also in this case, on every class of uniformly doubling metric measure spaces with uniformly bounded diameter, the mGH-convergence can be metrized by a distance that we denote by d_{mGH}. The extrinsic approach applies verbatim as well, with the exception that the common ambient space Z can be also taken to be compact.

We now recall some stability and convergence results of functions along pmGH-convergence. For additional details and analogous results we refer to [13, 55, 63]. For brevity reasons in what follows we fix a sequence of pointed $\mathrm{CD}(K, N)$ spaces $(\mathrm{X}_n, d_n, \mathfrak{m}_n, x_n)$, for $n \in \bar{\mathbb{N}}$, so that $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$.

Definition 2.17 Let $p \in (1, \infty)$, we say that

(i) $f_n \in L^p(\mathfrak{m}_n)$ *converges* L^p*-weak* to $f_\infty \in L^p(\mathfrak{m}_\infty)$, provided $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\mathfrak{m}_n)} < \infty$ and $f_n\mathfrak{m}_n \rightharpoonup f_\infty\mathfrak{m}_\infty$ in $C_{bs}(\mathrm{Z})$,
(ii) $f_n \in L^p(\mathfrak{m}_n)$ *converges* L^p*-strong* to $f_\infty \in L^p(\mathfrak{m}_\infty)$, provided it converges L^p-weak and $\overline{\lim}_n \|f_n\|_{L^p(\mathfrak{m}_n)} \le \|f_\infty\|_{L^p(\mathfrak{m}_\infty)}$,
(iii) $f_n \in W^{1,2}(\mathrm{X}_n)$ *converges* $W^{1,2}$*-weak* to $f_\infty \in W^{1,2}(\mathrm{X})$ provided it converges L^2-weak and $\sup_{n\in\mathbb{N}} \||Df_n|\|_{L^2(\mathfrak{m}_n)} < \infty$,
(iv) $f_n \in W^{1,2}(\mathrm{X}_n)$ *converges* $W^{1,2}$*-strong* to $f_\infty \in W^{1,2}(\mathrm{X})$ provided it converges L^2-strong and $\||Df_n|\|_{L^2(\mathfrak{m}_n)} \to \||Df_\infty|\|_{L^2(\mathfrak{m}_\infty)}$.

Moreover, we say that f_n is uniformly bounded in L^p if $\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} < \infty$. In the following statement we collect a list of useful properties of L^p-convergence.

Proposition 2.18 (Properties of L^p-convergence) *For all* $p \in (1, \infty)$*, it holds*

(i) *If* f_n *converges* L^p*-strong to* f_∞*, then* $\varphi(f_n)$ *converges* L^p*-strong to* $\varphi(f_\infty)$ *for every* $\varphi \in \mathrm{LIP}(\mathbb{R})$ *with* $\varphi(0) = 0$,
(ii) *If* f_n *(resp.* g_n*) converges* L^p*-strong to* f_∞ *(resp.* g_∞*), then* $f_n + g_n$ *converges* L^p*-strong to* $f_\infty + g_\infty$,
(iii) *if* f_n *converges* L^p*-weak to* f*, then* $\|f_\infty\|_{L^p(\mathfrak{m}_\infty)} \le \underline{\lim}_n \|f_n\|_{L^p(\mathfrak{m}_n)}$,
(iv) *suppose that* $\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} < +\infty$*, then up to a subsequence* f_n *converges* L^p*-weak to some* $f_\infty \in L^p(\mathfrak{m}_\infty)$,
(v) *If* f_n *converges* L^p*-strong (resp.* L^p*-weak) to* f_∞*, then* φf_n *converges* L^p*-strong (resp.* L^p*-weak) to* φf_∞*, for all* $\varphi \in C_b(\mathrm{Z})$,
(vi) *for every* $f \in L^p(\mathfrak{m}_\infty)$ *there exists a sequence* $f_n \in L^p(\mathfrak{m}_n)$ *converging* L^p*-strong to* f,
(vii) *if* f_n *are non-negative and converge in* L^p*-strong to* f*, then for every* $q \in (1, \infty)$, $f_n^{p/q}$ *converge* L^q*-strong to* $f^{p/q}$,
(viii) *Fix* $p, q \in (1, \infty]$ *so that* $p < q$*. If the sequence* (f_n) *is uniformly bounded in* L^q *and converges* L^p*-strong to* f_∞*, then it converges also* L^r*-strong to* f_∞ *for every* $r \in [p, q)$,

Proof For the proof of the items (i)–(v) we refer to [13,Prop. 3.3]. (vi) can instead be found in [55] (see also [63]). (vii) follows immediately from the characterization of L^p-strong convergence via convergence of graph (see e.g. [13,Remark 3.2]). For $(viii)$, the case $q = \infty$ follows immediately from item (i) (see also [13,e) of Prop. 3.3]), hence we can assume $q < +\infty$. Fix $r \in [p, q)$. Clearly from the Hölder inequality f_n is uniformly bounded in L^r, hence by definition f_n converges L^r-weakly to f_∞. Moreover from item (iii) we known that $f_\infty \in L^r(\mathfrak{m}_\infty)$, therefore by truncation and diagonalization we can suppose that $f \in L^\infty(\mathfrak{m}_\infty)$. From (vi) then there exists a sequence $g_n \in L^r(\mathfrak{m}_n)$ converging to f_∞ in L^r-strong and by item $i)$ we can also assume that g_n are uniformly bounded in L^∞. Then, from $(viii)$ in the case $q = \infty$ we have that g_n converge also in L^p-strong to f_∞. Then by (ii) we have that $g_n - f_n$ converges to 0 in L^p-strong and in particular $\|f_n - g_n\|_{L^p(\mathfrak{m}_n)} \to 0$. Finally by the Hölder inequality (since f_n, g_n are both uniformly bounded in L^q) we have that $\|f_n - g_n\|_{L^r(\mathfrak{m}_n)} \to 0$. In particular $\lim_n \|f_n\|_{L^r(\mathfrak{m}_n)} = \lim_n \|g_n\|_{L^r(\mathfrak{m}_n)} = \|f_\infty\|_{L^r(\mathfrak{m}_\infty)}$, which concludes the proof. □

We now pass to some convergence and stability results related to Sobolev spaces. We start with the following generalized version of the compact embedding of $W^{1,2} \hookrightarrow L^2$ (reported here specifically for compact metric measure spaces):

Proposition 2.19 ([55]) *Suppose that* X_n*,* $n \in \bar{\mathbb{N}}$ *are compact and assume that* $(f_n) \in W^{1,2}(\mathrm{X}_n)$ *are uniformly bounded in* $W^{1,2}$*, i.e.* $\sup_n \|f_n\|_{W^{1,2}(\mathrm{X}_n)} < +\infty$*. Then* (f_n) *has a* L^2*-strongly convergent subsequence.*

We recall the Γ-convergences of the 2-Cheeger energies proven in [55]:

- Γ-$\underline{\lim}$: for every $f_n \in L^2(\mathfrak{m}_n)$ L^2-strong converging to $f_\infty \in L^2(\mathfrak{m}_\infty)$, it holds

$$\int |Df_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty \le \varliminf_{n\to\infty} \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n; \tag{2.18}$$

- Γ-$\overline{\lim}$: for every $f_\infty \in L^2(\mathfrak{m}_\infty)$, there exists a sequence $f_n \in L^2(\mathfrak{m}_n)$ converging L^2-strong to f_∞ so that

$$\varlimsup_{n\to\infty} \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n \le \int |Df_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty. \tag{2.19}$$

We will also need the Γ-$\overline{\lim}$ inequality also for the p-Cheeger energies as proved in [13,Theorem 8.1]: for every $p \in (1, \infty)$ and every $f_\infty \in L^p(\mathfrak{m}_\infty)$, there exists $f_n \in L^p(\mathfrak{m}_n)$ converging L^p-strong to f_∞ so that

$$\varlimsup_{n\to\infty} \int |Df_n|^p \,\mathrm{d}\mathfrak{m}_n \le \int |Df_\infty|^p \,\mathrm{d}\mathfrak{m}_\infty.$$

The above is stated in [13] only for a sequence of $\mathrm{RCD}(K, \infty)$ spaces, but it easily seen that the proof works without modification also in the case of $\mathrm{CD}(K, \infty)$ spaces.

We end this part recalling a well known continuity result of the spectral gap (see [55] and [14]): if X_n, $n \in \bar{\mathbb{N}}$, are all compact it holds

$$\lambda^{1,2}(\mathrm{X}_\infty) = \lim_{n\to\infty} \lambda^{1,2}(\mathrm{X}_n). \tag{2.20}$$

We mention that the continuity of the spectral gap was previously obtained in the setting of Ricci-limit spaces by Cheeger and Colding [40].

2.4 Pólya–Szego inequality

The Pólya–Szegő inequality, namely the fact that the Dirichlet energy decreases under *decreasing rearrangements*, dates back to Faber and Krahn and was successively formalized in [85]. Later, in [28], this collection of ideas was brought to the context of manifolds with Ricci lower bounds to achieve applications concerning the rigidity of the 2-spectral gap. Concerning the topic of this manuscript, the said inequality has revealed effective in [66] in the proof of Theorem 1.2.

In this part we recall the Pólya–Szegő inequality for essentially nonbranching $\mathrm{CD}(K, N)$ spaces proven in [84]. We will also collect some additional technical results and definitions from [84] that will be used in Sect. 3.1 to prove a Euclidean-variant of this inequality.

Definition 2.20 (Distribution function) Let $(\mathrm{X}, d, \mathfrak{m})$ be a compact metric measure space, $\Omega \subseteq \mathrm{X}$ an open set with $\mathfrak{m}(\Omega) < +\infty$ and $u : \Omega \to [0, +\infty)$ a non-negative Borel function. We define $\mu : [0, +\infty) \to [0, \mathfrak{m}(\Omega)]$, the distribution function of u, as

$$\mu(t) := \mathfrak{m}(\{u > t\}). \tag{2.21}$$

For u and μ as above, we let $u^{\#}$ be the generalized inverse of μ, defined by

$$u^{\#}(s) := \begin{cases} \operatorname{ess\,sup} u & \text{if s=0,} \\ \inf\{t : \mu(t) < s\} & \text{if s>0.} \end{cases}$$

It can be checked that $u^{\#}$ is non-increasing and left-continuous.

Then, given $\Omega \subseteq X$ an open set and $u : \Omega \to [0, +\infty)$ a non-negative Borel function, we define the *monotone rearrangement* into $I_N = ([0, \pi], |.|, \mathfrak{m}_N)$ (see Definition 2.6) as follows: first, we consider $r > 0$ so that $\mathfrak{m}(\Omega) = \mathfrak{m}_N([0, r])$ and define $\Omega^* := [0, r]$, then we define the monotone rearrangement function $u^*_N : \Omega^* \to \mathbb{R}^+$ as

$$u^*_N(x) := u^{\#}(\mathfrak{m}_N([0, x])), \qquad \forall x \in [0, r].$$

In the sequel, whenever u and Ω are fixed, Ω^* and u^*_N will be implicitly defined as above.

Theorem 2.21 (Pólya–Szegő inequality, [84]) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an essentially non braching* $\mathrm{CD}(N-1, N)$ *space for some* $N \in (1, \infty)$ *and* $\Omega \subseteq \mathrm{X}$ *be open. Then, for every* $p \in (1, \infty)$*, the monotone rearrangement in* I_N *maps* $L^p(\Omega)$ *(resp.* $W^{1,p}_0(\Omega)$*) into* $L^p(\Omega^*)$ *(resp.* $W^{1,p}(\Omega^*)$*) and satisfies:*

$$\|u\|_{L^p(\Omega)} = \|u^*_N\|_{L^p(\Omega^*)}, \qquad \forall u \in L^p(\Omega) \tag{2.22}$$

$$\int_\Omega |Du|^p \, d\mathfrak{m} \geq \int_{\Omega^*} |Du^*_N|^p \, d\mathfrak{m}_N, \qquad \forall u \in W^{1,p}_0(\Omega). \tag{2.23}$$

We will also need the following rigidity of the Pólya–Szegő inequality proven in [84,Theorem 5.4].

Theorem 2.22 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *space for some* $N \in [2, \infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$ *and* $p \in (1, \infty)$*. Let* $\Omega \subset \mathrm{X}$ *be an open set and assume that there exists a non-negative and non-constant function* $u \in W^{1,p}_0(\Omega)$ *achieving equality in* (2.23).

Then $(\mathrm{X}, d, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathrm{RCD}(N-2, N-1)$ *space* $(\mathrm{Z}, d_\mathrm{Z}, \mathfrak{m}_\mathrm{Z})$ *with* $\mathfrak{m}_\mathrm{Z}(\mathrm{Z}) = 1$ *so that* $\mathrm{X} \simeq [0, \pi] \times^N_{\sin} \mathrm{Z}$.

Remark 2.23 Observe that in Theorem 2.22 we did not assume that $\mathfrak{m}(\Omega) < 1$, assumption that is actually present in Theorem 5.4 of [84]. This is intentional, since we will need to apply Theorem 2.22 precisely in the case $\Omega = \mathrm{X}$. This is possible since the arguments in [84] work also in the case $\Omega = \mathrm{X}$ without modification. The only part where the argument does not cover explicitly the case $\Omega = \mathrm{X}$ is the proof of the approximation Lemma 3.6 in [84], which however can be easily adapted (see Lemma 2.24 below). □

The following technical result will be needed in Sect. 3.1. We include a sketch of the argument in the case $\Omega = \mathrm{X}$, to further justify the validity of Theorem 2.22 also in this case (see the above Remark).

Lemma 2.24 (Approximation with non-vanishing gradients) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(K, N)$ *metric measure space with* $N < +\infty$*, and let* $\Omega \subset \mathrm{X}$ *be open with* $\mathfrak{m}(\Omega) < +\infty$*. Then for any non-negative* $u \in \mathrm{LIP}_c(\Omega)$ *there exists a sequence of non-negative* $u_n \in \mathrm{LIP}_c(\Omega)$ *satisfying* $|Du_n|_1 \neq 0$ $\mathfrak{m}$*-a.e. in* $\{u_n > 0\}$ *and such that* $u_n \to u$ *in* $W^{1,p}(\mathrm{X})$*.*

Proof The case $\Omega \neq \mathrm{X}$ has been proven in [84,Lemma 3.6 and Corollary 3.7]. The proof presented there, as it is written, does not cover the case $\Omega = \mathrm{X}$ with X compact and $\mathrm{supp}(u) = \mathrm{X}$. However, the argument can be easily adapted by considering a sequence $\varepsilon_n \to 0$ such that $\mathfrak{m}(\{\mathrm{lip}(u_n) = \varepsilon_n\}) = 0$ and taking

$$u_n := u + \varepsilon_n v,$$

with $v(x) := d(x_0, x)$, for an arbitrary fixed point $x_0 \in \mathrm{X}$. Since $v \in \mathrm{LIP}(\mathrm{X})$ and $\mathrm{lip}(v) = 1$ $\mathfrak{m}$-a.e. in X, arguing exactly as in [84,Lemma 3.6] we get that $u_n \to u$ in $W^{1,p}(\mathrm{X})$ and $\mathrm{lip}(u_n) \neq 0$ $\mathfrak{m}$-a.e. in $\{u_n > 0\}$. To get the claimed non-vanishing of $|Du_n|_1$, as in [84,Corollary 3.7] we use the existence of a constant $c > 0$ such that

$$|Du|_1 \geq c\,\mathrm{lip}(u), \quad \mathfrak{m}\text{-a.e.},$$

for every $u \in \mathrm{LIP}_{loc}(\mathrm{X})$, which holds from the results in [16] and the fact that $\mathrm{CD}(K, N)$ spaces are locally doubling and supports a local-Poincaré inequality. □

Lemma 2.25 (Derivative of the distribution function, ([84])) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space and let* $\Omega \subseteq X$ *be an open subset with* $\mathfrak{m}(\Omega) < +\infty$*. Assume that* $u \in \mathrm{LIP}_c(\Omega)$ *is non-negative and* $|Du|_1(x) \neq 0$ *for* $\mathfrak{m}$*-a.e.* $x \in \{u > 0\}$*. Then its distribution function* $\mu : [0, +\infty) \to [0, \mathfrak{m}(\Omega)]$*, defined in* (2.21)*, is absolutely continuous. Moreover it holds*

$$\mu'(t) = -\int \frac{1}{|Du|_1}\,\mathrm{dPer}(\{u > t\}, \cdot) \qquad a.e., \tag{2.24}$$

where the quantity $1/|Du|_1$ *is defined to be* 0 *whenever* $|Du|_1 = 0$*.*

3 Upper bound for α_p

To prove an upper bound of α_p we will need to derive a Sobolev inequality of the type (1.6) for some explicit A. This will be achieved by proving first a class of local Sobolev-inequalities (see Theorem 3.8) and then "patch" them together (see Theorem 1.8) to obtain the desired global inequality. The local-Sobolev inequalities will be achieved through a Euclidean Pólya–Szegő symmetrization inequality (Theorem 3.6).

3.1 Pólya–Szego inequality of Euclidean-type

The goal of this section is to prove a Euclidean-variant of the Pólya–Szegő inequality for $\mathrm{CD}(K,N)$ spaces derived in [84] (under essentially nonbranching assumption, see also Sect. 2.4). The main difference is that our inequality holds for arbitrary $K \in \mathbb{R}$ and assumes the a priori validity of a Euclidean-type isoperimetric inequality, while the one in [84] requires $K > 0$ and it is based on the Lévy-Gromov isoperimetric inequality for the $\mathrm{CD}(K,N)$ condition. As opposed to Sect. 2.4, where the symmetrization has as target the model space for the $\mathrm{CD}(K,N)$ condition with $K > 0$, we will use a notion of symmetrization that lives in the weighted half line $([0,\infty), |.|, t^{N-1}\mathscr{L}^1)$. It should be remarked that, in general, there is not a natural curvature model space to symmetrize functions defined on an arbitrary $\mathrm{CD}(K,N)$-space with $K \le 0$. This is because there is not a unique model-space for the Lévy–Gromov isoperimetric inequality in the case $K \le 0$ (see [81]). Therefore, it is unclear in this high-generality where the rearrangements should live. For this reason we will equip the metric measure spaces under consideration with a (possibly local) isoperimetric inequality of Euclidean-type:

$$\mathrm{Per}(E) \ge C\mathfrak{m}(E)^{\frac{N-1}{N}},$$

for $N > 1$ and C a non-negative constant.

We start with the definition of *Euclidean model space* $(I_{0,N}, |.|, \mathfrak{m}_{0,N})$, $N \in (1,\infty)$:

$$I_{0,N} := [0,\infty), \qquad \mathfrak{m}_{0,N} := \sigma_{N-1} t^{N-1}\mathscr{L}^1,$$

where $|.|$ is the Euclidean distance. Next, we define the *Euclidean monotone rearrangement*.

Definition 3.1 *(Euclidean monotone rearrangement)* Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space and $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. For any Borel function $u\colon \Omega \to \mathbb{R}^+$, we define $\Omega^* := [0,r]$ with $\mathfrak{m}_{0,N}([0,r]) = \mathfrak{m}(\Omega)$ (i.e. $r^N = \omega_N^{-1}\mathfrak{m}(\Omega)$) and the monotone rearrangement $u^*_{0,N} : \Omega^* \to \mathbb{R}^+$ by

$$u^*_{0,N}(x) := u^\#(\mathfrak{m}_{0,N}([0,x])) = u^\#(\omega_N x^N), \qquad \forall x \in \Omega^*,$$

where $u^\#$ is the generalized inverse of the distribution function of u, as defined in Sect. 2.4.

In the sequel, whenever we fix Ω and $u\colon \Omega \to [0,\infty)$, the set Ω^* and the rearrangement $u^*_{0,N}$ are automatically defined as above.

Proposition 3.2 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space and* $\Omega \subset \mathrm{X}$ *be open and bounded with* $\mathfrak{m}(\Omega) < +\infty$. *Let* $u : \Omega \to [0,+\infty)$ *be Borel and let* $u^*_{0,N} : \Omega^* \to [0,+\infty)$ *be its monotone rearrangement.*
Then, u and $u^*_{0,N}$ have the same distribution function. Moreover

$$\|u\|_{L^p(\Omega)} = \|u^*_{0,N}\|_{L^p(\Omega^*)}, \qquad \forall\, 1 \le p < +\infty, \tag{3.1}$$

and the radial decreasing rearrangement operator $L^p(\Omega) \ni u \mapsto u^*_{0,N} \in L^p(\Omega^*)$ *is continuous.*

The proof of the above proposition is classical, following e.g. [69], with straightforward modification for the metric measure setting (see also [84]). Observe also that, given $u \in L^p(\Omega)$, its monotone rearrangement must be defined by fixing a Borel representative of u. However, this choice does not affect the outcome object $u^*_{0,N}$, as clearly the distribution function $\mu(t)$ of u is independent of the representative.

We now introduce the additional assumption that will make this section meaningful. For some open set $\Omega \subset \mathrm{X}$ and a number $N \in (1, \infty)$, we require the validity of the following local Euclidean-isoperimetric inequality

$$\mathrm{Per}(E) \geq C_{Isop}\mathfrak{m}(E)^{\frac{N-1}{N}}, \qquad \forall\, E \subset \Omega \text{ Borel}. \tag{3.2}$$

where C_{Isop} is a positive constant independent of E.

Remark 3.3 There is a rich literature about Euclidean-type isoperimetric inequalities in metric measure spaces. Inequalities as in (3.2) have been proven to hold, at least on balls, in the general setting of locally doubling metric measure spaces satisfying a weak local (1, 1)-Poincaré inequality (see, e.g., [4, 82]). In this setting we also mention the recent [18], where a global Euclidean-type isoperimetric inequality for small volumes is proved. In the context of $\mathrm{CD}(K, N)$ spaces, local almost-Euclidean isoperimetric inequalities have been derived in [37], while in the recent [26], a global version of (3.2) is proven to hold in $\mathrm{CD}(0, N)$ spaces with Euclidean-volume growth. For us, the validity of (3.2) will come from Theorem 3.9. □

Proposition 3.4 (Lipschitz to Lipschitz property of the rearrangement) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $\Omega \subset \mathrm{X}$ *be open with* $\mathfrak{m}(\Omega) < +\infty$. *Assume furthermore that, for some* $N \in (1, \infty)$ *and* $C_{Isop} > 0$, *the isoperimetric inequality in (3.2) holds in* Ω. *Finally, let* $u \in \mathrm{LIP}_c(\Omega)$ *be non-negative with Lipschitz constant* $L \geq 0$ *and such that* $|Du|_1(x) \neq 0$ *for* $\mathfrak{m}$*-a.e.* $x \in \{u > 0\}$. *Then* $u^*_{0,N} \in \mathrm{LIP}(\Omega^*)$ *with* $\mathrm{Lip}(u^*_{0,N}) \leq N\omega_N^{\frac{1}{N}} L/C_{Isop}$.

Proof We closely follow [84]. Let μ be the distribution function associated to u and denote by $M := \sup u < +\infty$. The assumptions grant that μ is continuous and strictly decreasing. Therefore for any $s, k \geq 0$ such that $s + k \leq \mathfrak{m}(\mathrm{supp}(u))$ we can find $0 \leq t - h \leq t \leq M$ in such a way that $\mu(t - h) = s + k$ and $\mu(t) = s$. Then from the coarea formula (2.8) and the L-Lipschitzianity of u we get

$$\int_{t-h}^{t} \mathrm{Per}(\{u > r\}, \cdot)\,\mathrm{d}r = \int_{\{t-h<u\leq t\}} |Du|_1\,\mathrm{d}\mathfrak{m} \leq L(\mu(t-h) - \mu(t)) = kL. \tag{3.3}$$

Observe that $\{u > r\} \subset \Omega$ for every $r > 0$, therefore we can apply the isoperimetric inequality (3.2) and obtain that

$$\mathrm{Per}(\{u > r\}) \geq C_{Isop}\mu(r)^{\frac{N-1}{N}}, \qquad \forall r > 0.$$

Therefore from (3.3) and the monotonicity of μ we obtain

$$kL \geq C_{Isop}\int_{t-h}^{t} \mu(r)^{\frac{N-1}{N}}\,\mathrm{d}r \geq C_{Isop}h\mu(t)^{\frac{N-1}{N}},$$

from which, observing that in this case $u^\#$ is the inverse of μ, we reach

$$u^\#(s) - u^\#(s+k) \leq s^{-1+1/N}C_{Isop}^{-1}kL.$$

In particular $u^\#$ is Lipschitz in $(\varepsilon, \mathrm{supp}(u)]$ (and thus in $(\varepsilon, \mathfrak{m}(\Omega)]$) for every $\varepsilon > 0$ and at every one of its differentiability points $s \in (0, \mathfrak{m}(\Omega))$ it holds that

$$-\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s) \leq s^{1-1/N}C_{Isop}^{-1}L.$$

Fix now two arbitrary and distinct points $x, y \in \Omega^*$ and assume without loss of generality that $y > x$. Recalling the definition of $u^*_{0,N}$ we have that $u^*_{0,N}(x) \geq u^*_{0,N}(y)$ and

$$u^*_{0,N}(x) - u^*_{0,N}(y) = u^\#(\omega_N x^N) - u^\#(\omega_N y^N) = \int_{\omega_N x^N}^{\omega_N y^N} -\frac{\mathrm{d}}{\mathrm{d}s} u^\#(s)\,\mathrm{d}s$$
$$\leq \int_{\omega_N x^N}^{\omega_N y^N} \frac{s^{-1+1/N}}{C_{Isop}} L\,\mathrm{d}s = \omega_N^{\frac{1}{N}} \frac{NL}{C_{Isop}} |x - y|,$$

which proves that $u^*_{0,N} : \Omega^* \to [0, \infty)$ is $N\omega_N^{\frac{1}{N}} L/C_{Isop}$-Lipschitz. □

The proof of the following result is exactly the same as in Lemma 3.11 of [84], since the only relevant fact for the proof is that $\mathfrak{m}_{0,N} = h_N \mathscr{L}^1$ with weight h_N which is bounded away from zero out of the origin (recall also (2.5)).

Lemma 3.5 *Let $p \in (1, \infty)$. Let $u \in W^{1,p}([0, r], |.|, \mathfrak{m}_{0,N})$, with $r \in (0, \infty)$, be monotone. Then $u \in W^{1,1}_{loc}(0, r)$ and it holds that*

$$|Du|_1(t) = |u'|(t) = |Du|(t), \quad \textit{for a.e. } t \in [0, r].$$

Theorem 3.6 (Euclidean Pólya–Szegő inequality) *Let $(\mathrm{X}, d, \mathfrak{m})$ be a $\mathrm{CD}(K, N')$ space, $K \in \mathbb{R}$ $N' \in (1, \infty)$ and let $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. Assume furthermore that, for some $N \in (1, \infty)$ and $C_{Isop} > 0$, the isoperimetric inequality in (3.2) holds in Ω. Then the Euclidean-rearrangement maps $W^{1,p}_0(\Omega)$ to $W^{1,p}(\Omega^*, |.|, \mathfrak{m}_{0,N})$ for any $1 < p < +\infty$. Moreover for any $u \in W^{1,p}_0(\Omega)$ it holds*

$$\int_\Omega |Du|^p \mathrm{d}\mathfrak{m} \geq \left(\frac{C_{Isop}}{N\omega_N^{1/N}}\right)^p \int_{\Omega^*} |Du^*_{0,N}|^p \mathrm{d}\mathfrak{m}_{0,N}. \tag{3.4}$$

Proof The proof is a standard argument and we follow [84] for its adaptation to the non-smooth setting. We first prove the result assuming that $u \in \mathrm{LIP}_c(\Omega)$ and $|Du|_1(x) \neq 0$ for $\mathfrak{m}$-a.e. $x \in \{u > 0\}$, then the general case will follow by approximation. Set $M := \sup u$. From the coarea formula (2.8) and the assumed isoperimetric inequality (3.2) we can obtain (see e.g. the proof of Prop. 3.12 in [84])

$$\int_\Omega |Du|_1^p \,\mathrm{d}\mathfrak{m} \geq \int_0^M \frac{C^p_{Isop}\mu(t)^{\frac{(N-1)p}{N}}}{(-\mu'(t))^{p-1}}\,\mathrm{d}t, \tag{3.5}$$

where $\mu'(t)$ exists a.e. since from Lemma 2.25 μ is absolutely continuous. Recall now from Proposition 3.2 that $\mu(t) = \mathfrak{m}(\{u^*_{0,N} > t\})$, where $u^*_{0,N} : \Omega^* \to \mathbb{R}^+$ is the Euclidean monotone rearrangement. Moreover, thanks to the non-vanishing assumptions on $|Du|_1$, we have from Proposition 3.4 that $u^*_{0,N} \in \mathrm{LIP}(\Omega^*)$. Additionally $u^*_{0,N}$ is strictly decreasing in $(0, \mathfrak{m}(\mathrm{supp}(u)))$ and in particular $\{u^*_{0,N} > t\} = [0, r_t)$ (and $\{u^*_{0,N} = t\} = \{r_t\}$) for some $r_t \in [0, \mathfrak{m}(\Omega)]$, for every $t \in (0, M)$. Note that r_t can be computed explicitly as $r_t = (\omega_N^{-1}\mu(t))^{1/N}$, which also shows that $t \mapsto r_t$ is a locally absolutely continuous map. Combining these observations with Lemma 2.25 and recalling Lemma 3.5 we have following expression for the derivative of μ:

$$-\mu'(t) = \int_{\{u^*_{0,N}=t\}} |Du^*_{0,N}|_1^{-1} \mathrm{dPer}(\{u^*_{0,N} > t\}, \cdot) = \frac{\mathrm{Per}(\{u^*_{0,N} > t\})}{|(u^*_{0,N})'|(r_t)} \quad \text{for a.e. } t \in (0, M),$$

where r_t is as above. It is clear that $\mathrm{Per}([0,r)) = \sigma_{N-1}r^{N-1}$ for every $r \in (0,\infty)$ (where the perimeter is computed in the space $(I_{0,N}, |.|, \mathfrak{m}_{0,N})$, therefore $\mathrm{Per}(\{u^*_{0,N} > t\}) = N\omega_N^{\frac{1}{N}}\mu(t)^{\frac{N-1}{N}}$, from which we deduce

$$-\mu'(t) = N\omega_N^{\frac{1}{N}} \frac{\mu(t)^{\frac{N-1}{N}}}{|(u^*_{0,N})'|(r_t)} \quad \text{for a.e. } t \in [0, M].$$

Plugging this identity in (3.5) and recalling also Lemma 3.5 we reach

$$\int_\Omega |Du|_1^p \,\mathrm{d}\mathfrak{m} \geq C^p_{Isop}(N\omega_N^{1/N})^{1-p}$$
$$\int_0^M (|(u^*_{0,N})'|(r_t))^{p-1}\mu(t)^{\frac{(N-1)}{N}}\,\mathrm{d}t = \Big(\frac{C_{Isop}}{N\omega_N^{1/N}}\Big)^p \int_{\Omega^*} |Du^*_{0,N}|^p \,\mathrm{d}\mathfrak{m}.$$

Recalling that $|Du|_1 \leq \mathrm{lip}\, u$ $\mathfrak{m}$-a.e., $u \in \mathrm{LIP}_{bs}(\mathrm{X})$, we obtain (3.4). For general $u \in W^{1,p}_0(\Omega)$ the result follows via approximation via Lemma 2.24 exactly as in the proof of [84,Theorem 1.4]. □

Remark 3.7 It follows from its proof, that Theorem 3.6 holds with the weaker assumption that $(\mathrm{X}, d, \mathfrak{m})$ is uniformly locally doubling and supports a weak local $(1,1)$-Poincaré inequality. Recall also from Remark 3.3 that under these assumptions an isoperimetric inequality as in (3.2) is available. □

3.2 Local Sobolev inequality

The main goal of this section is to prove the following local Sobolev inequality of Euclidean-type.

Theorem 3.8 (Local Euclidean–Sobolev inequality) *For every $\varepsilon > 0$, $N \in (1,\infty)$ and $D > 0$ there exists $\delta = \delta(\varepsilon, D, N) > 0$ such that the following holds. Let $(\mathrm{X}, d, \mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space, $K \in \mathbb{R}$. Let $r, R \in (0, \frac{1}{2}\sqrt{N/K^-})$ and $x \in \mathrm{X}$ be such that $r < \delta R$, $R < \delta\sqrt{N/K^-}$ (with $\sqrt{N/K^-} := +\infty$ if $K \geq 0$) and $\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} \leq D(r/R)^N$. Then*

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq (1+\varepsilon)Eucl(N,p)\left(\frac{\mathfrak{m}(B_R(x))}{R^N\omega_N}\right)^{-\frac{1}{N}} \||Du|\|_{L^p(\mathfrak{m})}, \quad \forall u \in \mathrm{LIP}_c(B_r(x)). \tag{3.6}$$

We mention that local "almost-Euclidean" Sobolev inequalities as in the above result are well known on Riemannian manifolds, however they usually depend on double sided bounds on the sectional curvature or on Ricci lower bounds coupled with a lower bound on the injectivity radius (see e.g. [21,Lemma 2.24] and [59,Lemma 7.1, Sec. 7.1]). Instead in our case we only need a lower bound on the Ricci curvature and bounds on the measure of small balls, for this reason Theorem 3.8 appears interesting also in the smooth setting.

We face now a necessary step for the proof of Theorem 3.8 starting with the following local isoperimetric inequality of Euclidean type to be used in conjunction with Pólya–Szegő inequality developed in the previous section. The proof relies on the Brunn–Minkowski inequality and it is mainly inspired by [26], where sharp global isoperimetric inequalities for $\mathrm{CD}(0,N)$ spaces have been proved (see also [17] for a refinement and the previous [32] and [49] for the smooth case). It is worth mentioning that a class of "almost-Euclidean" isoperimetric inequalities in essentially nonbranching CD-spaces, similar to the following ones, were proved in [37] via localization-technique. However, the results in [37] present a set of assumptions that are not suitable for our purposes. Moreover our arguments are different and do not assume the space to be essentially non-branching.

Theorem 3.9 (Almost–Euclidean isoperimetric inequality) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(K, N)$ *space for some* $N \in (1, \infty)$, $K \in \mathbb{R}$. *Then for every* $0 < r < R < \frac{1}{2}\sqrt{N/K^-}$ *(where* $\sqrt{N/K^-} = +\infty$ *for* $K \geq 0$*) and* $x \in \mathrm{X}$ *we have*

$$\mathrm{Per}(E) \geq \mathfrak{m}(E)^{\frac{N-1}{N}} N\omega_N^{\frac{1}{N}}\theta_{N,R}^{\frac{1}{N}}(x)(1-(2C_{r,R}^{1/N}+1)\delta - \eta), \quad \forall\, E \subset B_r(x), \tag{3.7}$$

where $\delta := \frac{r}{R}$, $\eta := 1 - \frac{2R\sqrt{K^-/N}}{\sinh(2R\sqrt{K^-/N})}$ *(taken to be zero when* $K \geq 0$*) and* $C_{r,R} := \theta_{N,r}(x)/\theta_{N,R}(x)$.

Proof It is sufficient to prove (3.7) with the Minkowski content $\mathfrak{m}(E)^+$ instead of the perimeter. Indeed we could then apply the approximation result in Proposition 3.10 below to deduce that for every $r' \in (r, R)$, (3.7) holds with $r = r'$ (this time with $\mathrm{Per}(E)$). Noticing that $\theta_{N,r'}(x) \to \theta_{N,r}(x)$ as $r' \downarrow r$, sending $r' \to r$ would give the conclusion.

Let $r, R \in \mathbb{R}^+$ with $r < R$ and fix $E \subset B_r(x_0)$ with $\mathfrak{m}(E) > 0$. We aim to apply the Brunn–Minkowski inequality to the sets $A_0 := E$, $A_1 := B_R(x_0)$. The triangle inequality easily yields that $A_t \subset E^{t(r+R)}$ for every $t \in (0, 1)$ (recall that E^ε is the ε-enlargement of the set E, while A_t is the set of t-midpoint between A_0, A_1). We consider first the case $K \geq 0$. From the Brunn–Minkowski applied with $K = 0$ we obtain

$$\begin{aligned}
\mathfrak{m}^+(E) &= \varliminf_{\varepsilon\to 0^+} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon} = \varliminf_{t\to 0^+} \frac{\mathfrak{m}(E^{t(r+R)}) - \mathfrak{m}(E)}{t(r+R)} \\
&\overset{(2.12)}{\geq} \varliminf_{t\to 0^+} \frac{\left(t\mathfrak{m}(B_R(x_0))^{1/N} + (1-t)\mathfrak{m}(E)^{1/N}\right)^N - \mathfrak{m}(E)}{t(r+R)} \\
&= N\mathfrak{m}(E)^{\frac{N-1}{N}} \frac{\mathfrak{m}(B_R(x_0))^{1/N} - \mathfrak{m}(E)^{1/N}}{r+R} \\
&\geq N\mathfrak{m}(E)^{\frac{N-1}{N}} \frac{\mathfrak{m}(B_R(x_0))^{1/N} - \mathfrak{m}(B_r(x_0))^{1/N}}{r+R},
\end{aligned}$$

where we have used that $E \subset B_r(x_0)$. If instead $K < 0$, arguing analogously we obtain

$$\begin{aligned}
\mathfrak{m}^+(E) \geq \frac{N\mathfrak{m}(E)^{\frac{N-1}{N}}}{r+R} \Big(&\frac{\theta\sqrt{-K/N}}{\sinh(\theta\sqrt{-K/N})} \mathfrak{m}(B_R(x_0))^{\frac{1}{N}} \\
&- \frac{\theta\sqrt{-K/N}\cosh(\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} \mathfrak{m}(B_r(x_0))^{\frac{1}{N}} \Big),
\end{aligned}$$

where θ is the maximal length of geodesics from A_0 to A_1. It is clear that $\theta \leq r + R$. Note also that $\frac{t}{\sinh(t)}$ is decreasing and less or equal than one for $t \geq 0$, moreover for $t \leq 1$ we have $\cosh(t) \leq 1 + t$. In particular if $R \leq \frac{1}{2}\sqrt{-N/K}$ we obtain that

$$\begin{aligned}
\mathfrak{m}^+(E) \geq\ & \frac{N\mathfrak{m}(E)^{\frac{N-1}{N}}}{r+R} \\
&\left(\frac{\sqrt{-K/N}(r+R)\mathfrak{m}(B_R(x_0))^{1/N}}{\sinh\left(\sqrt{-K/N}(r+R)\right)} - \left(1 + \sqrt{-K/N}(r+R)\right) \mathfrak{m}(B_r(x_0))^{1/N} \right).
\end{aligned}$$

Going back to the case of a general $K \in \mathbb{R}$, combining the above estimates and rearranging the terms we reach

$$\begin{aligned}\mathfrak{m}^+(E) \geq & \frac{\mathfrak{m}(E)^{\frac{N-1}{N}} N\omega_N^{\frac{1}{N}} \theta_{N,R}(x)^{\frac{1}{N}}}{1+r/R} \\ & \Big(\frac{\sqrt{K^-/N}(r+R)}{\sinh(\sqrt{K^-/N}(r+R))} - (1+\sqrt{\tfrac{K^-}{N}}(r+R))\tfrac{r}{R}\Big(\frac{\theta_{N,r}(x)}{\theta_{N,R}(x)}\Big)^{\frac{1}{N}}\Big),\end{aligned}$$

provided $R \leq \frac{1}{2}\sqrt{N/K^-}$ and taking $\frac{t}{\sinh(t)} = 1$ for $t = 0$. Setting $\delta := \frac{r}{R}$, $\eta := 1 - \frac{2R\sqrt{K^-/N}}{\sinh(2R\sqrt{K^-/N})}$ and $C := \theta_{N,r}(x)/\theta_{N,R}(x)$, the above gives (recalling that $\frac{t}{\sinh(t)}$ is decreasing for $t \geq 0$)

$$\mathfrak{m}^+(E) \geq \mathfrak{m}(E)^{\frac{N-1}{N}} N\omega_N^{\frac{1}{N}} \theta_{N,R}(x)^{\frac{1}{N}} \frac{1}{1+\delta}\big(1-\eta-2\delta C^{\frac{1}{N}}\big),$$

that easily implies the conclusion. □

In the above proof was used the following approximation result.

Proposition 3.10 ([7]) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space and let* $E \subset B_r(x)$ *be Borel with finite perimeter and* $\mathfrak{m}(E) < +\infty$. *Then for every* $r' > r$ *there exists a sequence* $E_n \subset B_{r'}(x)$ *of closed sets such that* $\chi_{E_n} \to \chi_E$ *in* $L^1(\mathfrak{m})$ *and*

$$\mathrm{Per}(E) = \lim_{n\to\infty} \mathfrak{m}^+(E_n).$$

Proof The result is contained in [7], however since it does not appear in this exact form we provide some details. The result follows observing that there exists a sequence $f_n \in \mathrm{LIP}(\mathrm{X})$ with $\mathrm{supp}(f_n) \subset B_{r'}(x)$ so that $f_n \to \chi_E$ in $L^1(\mathfrak{m})$ and $\mathrm{Per}(E) = \lim_n \int \mathrm{lip} f_n \,\mathrm{d}\mathfrak{m}$. Indeed from this fact, the conclusion follows arguing as in the end of the proof of [7,Theorem 3.6].

To construct the sequence (f_n) we known that from the definition of perimeter there exist $g_n \in \mathrm{LIP}_{loc}(\mathrm{X})$ so that $g_n \to \chi_E$ in $L^1(\mathfrak{m})$ and $\mathrm{Per}(E) = \lim_n \int \mathrm{lip}\, g_n \,\mathrm{d}\mathfrak{m}$. Moreover we can build a cut-off function $\eta \in \mathrm{LIP}(X)$ such that $\eta = 1$ in $B_r(x)$, $0 \leq \eta \leq 1$, $\mathrm{supp}(\eta) \subset B_{r'}(\mathrm{X})$ and $\mathrm{Lip}(\eta) \leq 2(r'-r)^{-1}$. Then we simply take $f_n := g_n\eta$. Clearly $f_n \to \chi_E$. Moreover

$$\mathrm{Per}(E) \leq \varliminf_{n\to\infty} \int \mathrm{lip}\, f_n \,\mathrm{d}\mathfrak{m} \leq \varliminf_{n\to\infty} \int \mathrm{lip}\, g_n \,\mathrm{d}\mathfrak{m} + \frac{2}{r'-r}\int_{E^c} g_n \,\mathrm{d}\mathfrak{m} = \mathrm{Per}(E),$$

that is what we wanted. □

Next, we recall the following classical one-dimensional inequality by Bliss [30] (see also [21, 94]).

Lemma 3.11 (Bliss inequality) *Let* $u\colon [0,\infty) \to \mathbb{R}$ *be locally absolutely continuous. Then for any* $1 < p < N$ *it holds*

$$\Big(\sigma_{N-1}\int_0^\infty |u|^{p^*} t^{N-1}\,\mathrm{d}t\Big)^{\frac{1}{p^*}} \leq \mathrm{Eucl}(N,p)\Big(\sigma_{N-1}\int_0^\infty |u'|^p t^{N-1}\,\mathrm{d}t\Big)^{\frac{1}{p}}, \tag{3.8}$$

whenever one side is finite and where $p^* := pN/(N-p)$. *Moreover the functions* $v_b(r) := (1+br^{\frac{p}{p-1}})^{\frac{p-N}{p}}$, $b > 0$, *satisfy* (3.8) *with equality.*

With the above local isoperimetric inequality and the Euclidean Pólya–Szegő inequality, the strategy is now to symmetrize functions on the space and exploit the Bliss inequality to deduce the desired local-Sobolev inequalities.

Proof of Theorem 3.8 We start observing that it is enough to prove (3.6) for non-negative functions. Fix $u \in \mathrm{LIP}_c(B_r(x))$ non-negative and consider $u^*_{0,N} : B_r(x)^* \to [0,\infty)$ be the Euclidean-rearrangement of u as in Definition 3.1, where $B_r(x)^* = [0,t]$ for some $t > 0$. The local Euclidean-isoperimetric inequality given by Theorem 3.9 implies that the hypothesis of Proposition 3.4 and Theorem 3.6 are fulfilled with $\Omega = B_r(x)$ and $C_{Isop} = (1-(2D^{1/N}+1)\delta' - 2\eta)N\omega_N^{\frac{1}{N}}\theta_{N,R}(x)^{\frac{1}{N}}$, with $\delta' := \frac{r}{R}$, $\eta := 1 - \frac{2R\sqrt{K^-/N}}{\sinh(2R\sqrt{K^-/N})}$ ($=0$ if $K \geq 0$) and $D := \theta_{N,r}(x)/\theta_{N,R}(x)$. In particular it holds that $u^*_{0,N} \in W^{1,p}([0,t], |.|, \mathfrak{m}_{0,N})$, which implies (recall (2.5)) that $u^*_{0,N} \in W^{1,1}_{loc}(0,t)$ with $(u^*_{0,N})' \in L^p(\mathfrak{m}_{0,N})$ and $|Du^*_{0,N}| = |(u^*_{0,N})'|$ a.e.. Moreover, since $\mathfrak{m}_{0,N}$ is bounded away from 0 far from the origin, $u^*_{0,N} \in W^{1,1}(\varepsilon, t]$ for every $\varepsilon > 0$ and by definition $u^*_{0,N}(t) = 0$. Therefore $u^*_{0,N}$ (extended by 0 in (t,∞)) satisfies the assumptions for the Bliss inequality. Recall also from Proposition 3.2 that $\|u^*_{0,N}\|_{L^p(\mathfrak{m}_{0,N})} = \|u\|_{L^p(\mathfrak{m})}$ for every $p \in [1,\infty)$. Therefore we are in position to apply the Euclidean Pólya–Szegő inequality given by (3.4), that combined with the Bliss-inequality (3.8) gives

$$\|u\|_{L^{p^*}(\mathfrak{m})} = \|u^*_{0,N}\|_{L^{p^*}(\mathfrak{m}_{0,N})} \overset{(3.8.)}{\leq} \mathrm{Eucl}(N,p)\||Du^*_{0,N}|\|_{L^p(\mathfrak{m}_{0,N})}$$
$$\overset{(3.4)}{\leq} \frac{Eucl(N,p)\theta_{N,R}(x)^{-\frac{1}{N}}}{(1-(2D^{1/N}+1)\delta' - 2\eta)}\||Du|\|_{L^p(\mathfrak{m})}.$$

Finally from the above and observing that $\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} = D(r/R)^N$, we immediately see that there exists $\delta := \delta(\varepsilon, D, N)$ so that, provided $\delta', \eta < \delta$, (3.6) holds. □

We end this section with another simpler variant of local Sobolev inequality. It will be needed to deal with cases where $\theta_N(x) = +\infty$, where Theorem 3.8 does not give the right information.

Proposition 3.12 (Local Sobolev embedding) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(K,N)$ *space for some* $N \in (1,\infty)$*,* $K \in \mathbb{R}$*. Then, for every* $p \in (1,N)$ *and every* $B_r(x) \subset \mathrm{X}$ *with* $r \leq 1$*, it holds*

$$\left(\int_{B_r(x)} |u|^{p^*}\,\mathrm{d}\mathfrak{m}\right)^{\frac{p}{p^*}} \leq \left(\frac{C\mathfrak{m}(B_r(x))}{r^N}\right)^{-\frac{p}{N}} \int_{B_{2r}(x)} |Du|^p\,\mathrm{d}\mathfrak{m} + 2^p\mathfrak{m}(B_r(x))^{-\frac{p}{N}} \int_{B_r(x)} |u|^p\,\mathrm{d}\mathfrak{m}, \tag{3.9}$$

for every $u \in \mathrm{LIP}(\mathrm{X})$*, where* $p^* = pN/(N-p)$ *and* $C = C(K,N,p)$*.*

Proof Applying (2.16) and the Bishop–Gromov inequality

$$\left(\int_{B_r(x)} |u|^{p^*}\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p^*}} \leq C_1 r \frac{\mathfrak{m}(B_r(x))^{1/p^*}}{\mathfrak{m}(B_{2r}(x))^{1/p}} \left(\int_{B_{2r}(x)} |Du|^p\right)^{\frac{1}{p}} + \mathfrak{m}(B_r(x))^{1/p^*}|u_{B_r(x)}|$$
$$\leq C_2 r \frac{\mathfrak{m}(B_r(x))^{1/p^*}}{\mathfrak{m}(B_r(x))^{1/p}} \left(\int_{B_{2r}(x)} |Du|^p\right)^{\frac{1}{p}} + \mathfrak{m}(B_r(x))^{\frac{1}{p^*}-\frac{1}{p}} \left(\int_{B_r(x)} |u|^p\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}},$$

for suitable positive constants C_1, C_2 depending only on K, N, p. The desired conclusion follows raising to the p in the above inequality. □

3.3 Proof of the upper bound

The strategy of the proof of the following result is by-now classical and combines local-Sobolev inequalities with a partition of unity argument (see [20], [21,Chp. 2 Sec. 7], [59,Theorem 4.5] and also [2,Prop. 3.3]).

Theorem 3.13 (Upper bound on α_p) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a compact* $\mathrm{CD}(K, N)$ *space, for some* $N \in (1, \infty)$*,* $K \in \mathbb{R}$*. Then, for every* $\varepsilon > 0$ *and every* $p \in (1, N)$*, there exists a constant* $B = B(\varepsilon, p, \mathrm{X}) > 0$ *such that*

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} \le \Big(\frac{Eucl(N,p)^p}{\min_{\mathrm{X}} \theta_N(x)^{p/N}} + \varepsilon\Big)\||Du|\|^p_{L^p(\mathfrak{m})} + B\|u\|^p_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}(\mathrm{X}). \tag{3.10}$$

Proof We start claiming that the following local version of (3.10) holds: for any $x \in \mathrm{X}$ and every $\varepsilon > 0$ there exists $r = r(\varepsilon, x) > 0$ and $C = C(\varepsilon, p, x) < +\infty$ such that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} \le \Big(\frac{Eucl(N,p)^p}{\min_{y\in\mathrm{X}} \theta_N(y)^{p/N}} + \varepsilon\Big)\||Du|\|^p_{L^p(\mathfrak{m})} + C\|u\|^p_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}_c(B_r(x)). \tag{3.11}$$

To show the above we observe first that in the case that $\theta_N(x) = +\infty$, (3.11) follows immediately from (3.9) for r small enough. We are left with the case $0 < \theta_N(x) < +\infty$. We start by fixing $\varepsilon \in (0, 1/2)$. From the definition of $\theta_N(x)$, there exists $r' = r'(x, \varepsilon)$ so that for every $r \in (0, r')$ it holds $\theta_{N,r}(x) \in ((1-\varepsilon)\theta_N(x), (1+\varepsilon)\theta_N(x))$. In particular we have that $\frac{\theta_{N,r}(x)}{\theta_{N,R}(x)} \le 4$ for every $r, R \in (0, r')$. We are therefore in position to apply Theorem 3.8 and deduce that there exists $\delta = \delta(\varepsilon, N)$ so that for every $r, R \in (0, r' \wedge \delta\sqrt{N/K^-})$, with $r < \delta R$, the following inequality holds for every $u \in \mathrm{LIP}_c(B_r(x))$

$$\begin{aligned}\|u\|^p_{L^{p^*}(\mathfrak{m})} &\overset{(3.6)}{\le} (1+\varepsilon)^p \frac{Eucl(N,p)^p}{\theta_{N,R}(x)^{p/N}}\||Du|\|^p_{L^p(\mathfrak{m})}\\ &\le \frac{(1+\varepsilon)^p}{(1-\varepsilon)^{p/N}} \frac{Eucl(N,p)^p}{\min_{\mathrm{X}} \theta_N(x)^{p/N}}\||Du|\|^p_{L^p(\mathfrak{m})},\end{aligned}$$

where in the second inequality we have used $\theta_{N,R}(x) \ge (1-\varepsilon)\theta_N(x)$. Therefore (3.11) (with $C = 0$) follows from the above provided we choose ε small enough.

Since X is compact we can extract a finite covering of balls $\{B_i\}_{i=1}^M$ from the covering $\cup_{x\in\mathrm{X}} B_{r(\varepsilon,x)/2}(x)$. We also set $C := \max_i C_i$ and

$$A := \frac{Eucl(N,p)^p}{\min_{\mathrm{X}} \theta_N(x)^{p/N}} + \varepsilon.$$

We claim that there exists a partition of unity made of functions $\{\varphi_i\}_{i=1}^M$ such that $\varphi_i \in \mathrm{LIP}_c(2B_i)$, $0 \le \varphi_i \le 1$ and $\varphi_i^{1/p} \in \mathrm{LIP}_c(2B_i)$ for all i, having denoted $2B_i$, the ball of twice the radius. To build such partition of unity we can argue as follows: start considering functions $\psi_i \in \mathrm{LIP}_c(2B_i)$, such that $0 \le \psi_i \le 1$ and $\psi_i \ge 1$ in B_i. Then we fix $\beta > p$ and take

$$\varphi_i := \frac{\psi_i^\beta}{\sum_{j=1}^M \psi_j^\beta}.$$

Since by construction $\sum_{j=1}^M \psi_j^\beta \ge 1$ everywhere on X, we have that $\varphi_i^{1/p} \in \mathrm{LIP}_c(2B_i)$. Finally it is clear that $\sum_{i=1}^M \varphi_i = 1$.

We are now ready to prove (3.10). Fix $u \in \mathrm{LIP}(\mathrm{X})$ and observe that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} = \Big\|\sum_i \varphi_i|u|^p\Big\|_{L^{p^*/p}(\mathfrak{m})} \le \sum_i \big\|\varphi_i|u|^p\big\|_{L^{p^*/p}(\mathfrak{m})} = \sum_i \|\varphi_i^{1/p}|u|\|^p_{L^{p^*}(\mathfrak{m})}. \tag{3.12}$$

Since $\varphi_i^{1/p}|u| \in \mathrm{LIP}_c(2B_i)$ we can apply (3.11) to obtain

$$\begin{aligned}\|u\|^p_{L^{p^*}(\mathfrak{m})} &\le \sum_{i=1}^M A \int \left(|D\varphi_i^{1/p}||u| + |Du|\varphi_i^{1/p}\right)^p \,\mathrm{d}\mathfrak{m} + C\int \varphi_i |u|^p \,\mathrm{d}\mathfrak{m}\\ &\le \sum_{i=1}^M A \int \varphi_i |Du|^p + c_1|Du|^{p-1}\varphi_i^{\frac{p-1}{p}}|D\varphi_i^{1/p}||u| + c_2|D\varphi_i^{1/p}|^p|u|^p \,\mathrm{d}\mathfrak{m}\\ &\quad + C\int \varphi_i |u|^p \,\mathrm{d}\mathfrak{m},\end{aligned}$$

where $c_1, c_2 \ge 0$ are such that $(1+t)^p \le 1 + c_1 t + c_2 t^p$ for all $t \ge 0$. Recalling that the functions $0 \le \varphi_i^{1/p} \le 1$ are Lipschitz we obtain

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} \le A\int |Du|^p \,\mathrm{d}\mathfrak{m} + \tilde{C}\int |Du|^{p-1}|u| \,\mathrm{d}\mathfrak{m} + \tilde{C}\int |u|^p \,\mathrm{d}\mathfrak{m},$$

where $\tilde{C} = \tilde{C}(p, M, L)$, L begin the maximum of the Lipschitz constants of the functions $\varphi_i^{1/p}$. Finally from the Young inequality we have for every $\delta > 0$

$$\int |Du|^{p-1}|u| \,\mathrm{d}\mathfrak{m} \le \frac{p\delta^{\frac{p}{p-1}}}{p-1}\int |Du|^p \,\mathrm{d}\mathfrak{m} + \frac{1}{p\delta^p}\int |u|^p \,\mathrm{d}\mathfrak{m}, \quad \forall \delta > 0$$

and plugging this estimate above, choosing δ small enough (but independent of u), we obtain that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})} \le (A+\varepsilon)\int |Du|^p \,\mathrm{d}\mathfrak{m} + C'\int |u|^p \,\mathrm{d}\mathfrak{m},$$

for some $C' = C'(\varepsilon, L, M, p)$. Since $\varepsilon > 0$ and $u \in \mathrm{LIP}(\mathrm{X})$ were arbitrary, this concludes the proof. □

4 Lower bound on α_p

the rough idea of the lower bound on α_p is that, when $\theta_N(x) < +\infty$ the space near x has a conical structure, hence the constant in the Sobolev inequality cannot be better than the one of the tangent structures of the underlying space. This will be formalized with a blow-up argument combined with a stability result for the Sobolev constants.

4.1 Blow-up analysis of Sobolev constants

For convenience, we introduce the following notation: whenever in a metric measure space $(\mathrm{X}, d, \mathfrak{m})$ it holds that

$$\|u\|^p_{L^q(\mathfrak{m})} \le A\||Du|_p\|^p_{L^p(\mathfrak{m})} + B\|u\|^p_{L^p(\mathfrak{m})}, \qquad \forall u \in W^{1,p}(\mathrm{X}).$$

for some constants $A, B > 0$ and exponents $1 < p < q$, we will say that X **supports a** (q, p)**-Sobolev inequality with constants** A, B. This convention will be used often here, and sometimes in the subsequent sections, without further notice.

We make precise the scaling enjoyed by the Sobolev inequalities under consideration. It is immediate to check that if a space $(\mathrm{X}, d, \mathfrak{m})$ supports a (p^*, p)-Sobolev for $p \in (1, N)$

and $p^* := \frac{pN}{N-p}$ with constants A, B, then for every $r > 0$ we have

$$(\mathrm{X}, d/r, \mathfrak{m}/r^N) \text{ supports a } (p^*, p) - \text{Sobolev with constants } A, Br^p. \tag{4.1}$$

We pass to the stability of Sobolev embeddings under pmGH-convergence (see also [64,Thm. 3.1] for a similar result for Ricci-limits).

Lemma 4.1 (pmGH-Stability of Sobolev constants) *Let* $(\mathrm{X}_n, d_n, \mathfrak{m}_n, x_n)$, $n \in \bar{\mathbb{N}}$, *be a sequence of* $\mathrm{CD}(K, N)$ *spaces for some* $K \in \mathbb{R}$, $N \in (1, \infty)$ *with* $\mathrm{X}_n \overset{pmGH}{\to} X_\infty$. *Suppose* X_n *support a* (q, p)*-Sobolev inequality for* $1 < p < q$ *with constants* A, B. *Then also* X_∞ *supports a* (q, p)*-Sobolev inequality with the same constants* A, B.

Proof Fix $u \in \mathrm{LIP}_c(\mathrm{X}_\infty)$, from the Γ-$\overline{\lim}$ inequality of the Ch_p energy, there exists a sequence $u_n \in W^{1,p}(\mathrm{X}_\infty)$ such that u_n converges in L^p-strong to u and $\overline{\lim}_n \int |Du|^p \,\mathrm{d}\mathfrak{m}_n \leq \int |Du|^p \,\mathrm{d}\mathfrak{m}_\infty$. In particular

$$\begin{aligned}\overline{\lim_n} \|u_n\|^p_{L^q(\mathfrak{m}_n)} &\leq \overline{\lim_{n\to\infty}} A\||Du_n|\|^p_{L^p(\mathfrak{m}_n)} + B\|u_n\|^p_{L^p(\mathfrak{m}_n)} \\ &\leq A\||Du|\|^p_{L^p(\mathfrak{m}_\infty)} + B\|u\|^p_{L^p(\mathfrak{m}_\infty)} < +\infty.\end{aligned}$$

Therefore u_n converge also L^q-weak to u. From the lower semicontinuity of the L^q-norm with respect to L^q-weak convergence and the arbitrariness of $u \in \mathrm{LIP}_c(\mathrm{X}_\infty)$ the conclusion follows. □

The following result is a consequence of the existence of the disintegration and can be found for example in [42,Corollary 3.8].

Lemma 4.2 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(0, N)$ *space with* $N \in [1, \infty)$. *Suppose that for some* $x_0 \in \mathrm{X}$ *it holds that* $\frac{\mathfrak{m}(B_r(x_0))}{\omega_N r^N} = 1$ *for every* $r \in (0, \infty)$, *then*

$$\int \varphi(d(x_0, x)) \,\mathrm{d}\mathfrak{m} = \sigma_{N-1} \int_0^\infty \varphi(r) r^{N-1} \,\mathrm{d}r, \qquad \forall \varphi \in C_c([0, \infty]).$$

Lemma 4.3 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(0, N)$ *space,* $N \in (1, \infty)$, $p \in (1, N)$ *and set* $p^* := \frac{pN}{N-p}$. *Suppose that for some* $x_0 \in \mathrm{X}$ *it holds that* $\frac{\mathfrak{m}(B_r(x_0))}{\omega_N r^N} = 1$ *for every* $r \in (0, \infty)$. *Then there exists a sequence of non-constant functions* $u_n \in \mathrm{LIP}_c(\mathrm{X})$ *satisfying*

$$\lim_n \frac{\|u_n\|_{L^{p^*}(\mathfrak{m})}}{\||Du_n|\|_{L^p(\mathfrak{m})}} \geq Eucl(N, p).$$

Proof Let $v : [0, \infty) \to [0, \infty)$, $v \in C^\infty(0, \infty)$, be an extremal function for the Bliss inequality (3.8) as given by Lemma 3.11. It can be easily shown that we can approximate v with functions $v_n \in \mathrm{LIP}_c([0, \infty))$ so that $\|v_n\|_{L^{p^*}(h_N\mathscr{L}^1)} \to \|v\|_{L^{p^*}(h_N n\mathscr{L}^1)}$ and $\|v_n'\|_{L^p(h_N\mathscr{L}^1)} \to \|v'\|_{L^p(h_N\mathscr{L}^1)}$, where $h_N \mathscr{L}^1 = \sigma_{N-1} t^{N-1} \mathscr{L}^1$. For example we can take $v_n := \varphi_n(u_b)$ with $\varphi_n \in \mathrm{LIP}[0, \infty)$, $\varphi_n \geq 0$, $\varphi_n(t) \leq |t|$, $\mathrm{Lip}(\varphi_n) \leq 2$, $\varphi_n(t) = t$ in $[2/n, \infty)$ and $\mathrm{supp}(\varphi_n) \subset [1/n, \infty)$. The claimed approximation of the norms then follows immediately from the fact that v is decreasing and vanishing at infinity. Therefore we have

$$\lim_n \frac{\|v_n\|_{L^{p^*}(h_n\mathscr{L}^1)}}{\|v_n'\|_{L^p(h_n\mathscr{L}^1)}} = Eucl(N, p). \tag{4.2}$$

We can now define $u_n := v_n \circ d_{x_0}$, where $d_{x_0}(\cdot) := d(x_0, \cdot)$. We clearly have that $u_n \in \mathrm{LIP}_c(\mathrm{X})$ and from the chain rule also that $|Du_n| = |v_n'| \circ d_{x_0} |Dd_{x_0}| \leq |v_n'| \circ d_{x_0}$ $\mathfrak{m}$-a.e., since

d_{x_0} is 1-Lipschitz. Hence applying Lemma 4.2 we obtain $\|u_n\|_{L^{p^*}(\mathfrak{m})} = \|v_n\|_{L^{p^*}(h_N\mathscr{L}^1)}$ and $\||Du_n|\|_{L^p(\mathfrak{m})} \le \|v_n'\|_{L^p(h_N\mathscr{L}^1)}$. This combined with (4.2) (up to passing to a subsequence) gives the conclusion. □

Theorem 4.4 (Lower bound on the Sobolev constant) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a* $\mathrm{CD}(K, N)$ *space,* $K \in \mathbb{R}$, $N \in (1, \infty)$ *that supports a* (p^*, p)*-Sobolev inequality for* $p \in (1, N)$ *with constants* A, B, *where* $p^* = pN/(N-p)$. *Then*

$$A \ge \frac{\mathrm{Eucl}(N, p)^p}{\theta_N(x)^{\frac{p}{N}}}, \qquad \forall x \in \mathrm{X}. \tag{4.3}$$

Proof If $\theta_N(x) = \infty$, there is nothing to prove. Hence we can assume that $\theta_N(x) < +\infty$. From the compactness and stability of the $\mathrm{CD}(K, N)$ condition, there exists a sequence $r_i \to 0$ such that $\mathrm{X}_i := (\mathrm{X}, d/r_i, \mathfrak{m}/{r_i}^N, x)$ pmGH-converge to a $\mathrm{CD}(0, N)$ space $(\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \boldsymbol{o}_\mathrm{Y})$. Moreover, from (4.1) we have that X_i supports a (p^*, p)-Sobolev inequality with constants $A, r_i^p B$. This combined with Lemma 4.1 shows that $(\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ supports a (p^*, p)-Sobolev inequality with constants $A, 0$. However we clearly have that $\mathfrak{m}_\mathrm{Y}$ satisfies $\frac{\mathfrak{m}_\mathrm{Y}(B_r(\boldsymbol{o}_\mathrm{Y}))}{\omega_N r^N} = \theta_N(x)$ for every $r > 0$. Therefore Lemma 4.3, after a rescaling, ensures that $A \ge \frac{\mathrm{Eucl}(N,p)^p}{\theta_N(x)^{\frac{p}{N}}}$, which is what we wanted. □

The above, together with Theorem 3.13, proves our main result Theorem 1.4 concerning $\alpha_p(\mathrm{X})$.

Using Theorem 4.4 we can also prove the topological rigidity of the Sobolev inequality on non-collapsed RCD spaces. More precisely combining the volume rigidity for non-collapsed RCD spaces ([53,Theorem 1.6]) and the Cheeger–Colding's metric Reifenberg's theorem ([40,Theorem A.1.2]) (see also [68]) we can obtain the following result.

Corollary 4.5 (Manifold-regularity from almost Euclidean–Sobolev inequality) *For every* $K \in \mathbb{R}$, $N \in \mathbb{N}$, $p \in (1, N)$, $\alpha \in (0, 1)$, $\varepsilon > 0$ *there exists* $\delta = \delta(K, N, \varepsilon, \alpha)$ *such that the following holds. Suppose that* $(\mathrm{X}, d, \mathscr{H}^N)$ *is a compact* $\mathrm{RCD}(K, N)$ *space satisfying the following Sobolev inequality*

$$\|u\|^p_{L^{p^*}(\mathscr{H}^N)} \le (Eucl(N, p)^p + \delta)\||Du|\|^p_{L^p(\mathscr{H}^N)} + B\|u\|^p_{L^p(\mathscr{H}^N)}, \quad \forall u \in W^{1,p}(\mathrm{X}), \tag{4.4}$$

for some constant $B > 0$, *where* $p^* := pN/(N-p)$.

Then, there exists a smooth N*-dimensional Riemannian manifold* M *and an* α*-biHölder homeomorphism* $F : M \to \mathrm{X}$.

Proof The argument is analogous to [68,Theorem 3.1], however for completeness we include the details.

We start fixing $\varepsilon > 0$, $N \in \mathbb{N}$, $K \in \mathbb{R}$, $p \in (1, N)$ and two numbers $\bar\delta = \bar\delta(K, N, p, \varepsilon) > 0$ $\bar r = \bar r(K, N, p, \varepsilon)$ small enough to be chosen later.

Suppose that $(\mathrm{X}, d, \mathscr{H}^N)$ is a compact $\mathrm{RCD}(K, N)$ space that supports a (p^*, p)-Sobolev inequality with constant $Eucl(N, p)^p + \delta$, B, for some $\delta \le \bar\delta$ and $B > 0$ (i.e. such that (4.4) holds). Then from (4.3), if $\bar\delta \le Eucl(N, p)^p/4$, we have that

$$\theta_N(x) \ge 1 - 2\delta, \quad \forall x \in \mathrm{X}.$$

Therefore for every $x \in \mathrm{X}$ there exists $r_x \in (0, \bar r)$ such that $\mathscr{H}^N(B_{r_x}(x)) \ge (1 - 3\delta)r_x^N\omega_N$. Moreover from the Bishop–Gromov inequality, for every $y \in B_{\delta r_x}(x)$ and every $s \in (0, r_x)$

it holds that

$$\frac{\mathscr{H}^N(B_s(y))}{v_{K,N}(s)} \geq \frac{\mathscr{H}^N(B_{(1+\delta)r_x}(y))}{v_{K,N}((1+\delta)r_x)} \geq \frac{\mathscr{H}^N(B_{r_x}(x))}{v_{K,N}((1+\delta)r_x)} \geq \frac{(1-3\delta)r_x^N\omega_N}{v_{K,N}((1+\delta)r_x)}. \tag{4.5}$$

Recalling that $\lim_{r\to 0^+}\frac{\omega_N r^N}{v_{K,N}(r)} = 1$, from (4.5) we deduce that if both $\bar{r}$ and $\bar{\delta}$ are small enough, with respect to K, N, p, ε, then

$$\mathscr{H}^N(B_s(y)) \geq (1-\varepsilon)s^N\omega_N, \quad \forall y \in B_{r_x}(x),\ s \in (0, r_x).$$

Finally from the compactness of X there exists a finite number of points x_i, $i = 1, \dots, m$ such that $\mathrm{X} \subset \cup_i B_{r_{x_i}}(x_i)$. Taking $R := \min_i r_{x_i} < \bar{r}$ we then have

$$\mathscr{H}^N(B_s(y)) \geq (1-\varepsilon)s^N\omega_N, \quad \forall y \in \mathrm{X},\ s \in (0, R).$$

From this the conclusion follows combining the volume rigidity theorem for non-collapsed RCD spaces ([53,Theorem 1.6]) and the intrinsic metric-Reifenberg's theorem ([40,Theorem A.1.2]). □

4.2 Sharp and rigid Sobolev inequalities under Euclidean volume growth

Here we prove the sharp Sobolev inequalities on $\mathrm{CD}(0, N)$ spaces contained Theorem 1.13. The validity of the inequality (1.12) will be derived as a consequence of the local-Sobolev inequalities in Theorem 3.8. The sharpness instead follows from a well known principle for which the validity of a Euclidean–Sobolev inequality implies certain growth on the measure of balls. In particular we have the following result:

Theorem 4.6 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an* $\mathrm{CD}(0, N)$*,* $N \in (1, \infty)$ *such that for some* $p \in (1, N)$ *and* $A > 0$

$$\|u\|_{L^{p^*}(\mathfrak{m})} \leq A\||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}_c(\mathrm{X}), \tag{4.6}$$

where $p^* := \frac{pN}{N-p}$*. Then* X *has Euclidean volume-growth and*

$$AVR(\mathrm{X}) \geq \left(\frac{Eucl(N,p)}{A}\right)^N. \tag{4.7}$$

On the general setting of CD spaces Theorem 4.6 is proved in [73] (see also [74] for the case $p = 2$), extending to non-smooth setting the same results for Riemannian manifolds due to Ledoux [75] and improved by Xia [97]. We mention also [45] and [98] for analogous statements related to different class of inequalities. In all the cited works the arguments depend on rather intricate ODE-comparison (originated in [75] and inspired by the previous [25]) and heavily rely on the explicit knowledge of the extremal functions for the inequalities. However, using the results in Sect. 4 we are able to give a short proof of Theorem 4.6, which uses a more direct blow-down procedure, that we believe being interesting on its own. The main advantage of this approach is that we will never need, as opposed to the ODE-comparison approach, the explicit expression of extremals functions in the Euclidean Sobolev inequality (1.1).

Proof of Theorem 4.6 The fact that $\mathfrak{m}(\mathrm{X}) = +\infty$ can be immediately seen by plugging in the Sobolev inequality functions $u_R \in \mathrm{LIP}_c(\mathrm{X})$ so that $u_R = 1$ in $B_R(x_0)$ $\mathrm{supp}(u_R) \subset B_{2R}(x_0)$ and $\mathrm{Lip}(u_R) \leq 1/R$ and sending $R \to +\infty$. The fact that X has Euclidean volume growth follows by considering instead functions $u_R(\cdot) := (R - d_{x_0}(\cdot))^+$ as $R \to +\infty$ with fixed $x_0 \in \mathrm{X}$ and using the Bishop–Gromov inequality.

It remains to prove (4.7). We argue via blow-down. Let $R_i \to +\infty$. From the Euclidean volume-growth property, up to passing to a non relabeled subsequence, the rescaled spaces $(\mathrm{X}, d/R_i, \mathfrak{m}/R_i^N, x_0)$, $x_0 \in \mathrm{X}$, pmGH-converge to an $\mathrm{CD}(0, N)$ space $(\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y}, \boldsymbol{o}_\mathrm{Y})$ satisfying $\frac{\mathfrak{m}_\mathrm{Y}(B_R(\boldsymbol{o}_\mathrm{Y}))}{\omega_N r^N} = \mathrm{AVR}(\mathrm{X})$. Moreover combining (4.6) with Lemma 4.1 proves that Y satisfy a (p^*, p)-Sobolev inequality with constants $A, 0$. Then (4.7) follows from Lemma 4.3. □

We can now move to the proof of the sharp Sobolev inequalities under the Euclidean volume growth assumption.

Proof of Theorem 1.13 Fix $x \in \mathrm{X}$. From the definition of $AVR(\mathrm{X})$, for every r big enough $\theta_{N,r}(x) \le 2AVR(\mathrm{X})$. Fix one of such $r > 0$. From the Bishop–Gromov inequality we also have that $\theta_{N,R}(x) \ge AVR(\mathrm{X})$ for every $R > 0$. In particular $\theta_{N,r}(x)/\theta_{N,R}(x) \le 2$ for every $R > 0$. Hence by Theorem 3.8 (for $K = 0$) we have that for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ so that for every $R > r/\delta$ the following local Euclidean Sobolev inequality holds:

$$\|u\|_{L^{p^*}(\mathfrak{m})} \le (1+\varepsilon) Eucl(N, p) \theta_{N,R}(x)^{-\frac{1}{N}} \\ \||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}_c(B_r(x)).$$

Taking $R \to \infty$ we achieve

$$\|u\|_{L^{p^*}(\mathfrak{m})} \le (1+\varepsilon) Eucl(N, p) AVR(\mathrm{X})^{-\frac{1}{N}} \\ \||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u \in \mathrm{LIP}_c(B_r(x)).$$

Since ε was chosen arbitrarily and independent of $r > 0$, we can first send $\varepsilon \to 0^+$ and then $r \to +\infty$ to achieve the first part of the statement.

The sharpness of (1.12) instead follows immediately from Theorem 4.6. □

5 The constant A_q^{opt} in metric measure spaces

In this section we will prove some upper and lower bounds on A_q^{opt} in the case of metric measure spaces. Some of the results contained here (more precisely, Sect. 5.3) are actually not used in other parts of the note, however we chose to include them here for completeness and to give a more clear picture around the value of A_q^{opt}. Let us also remark that the results of this part are valid for a general lower bound $K \in \mathbb{R}$.

We start recalling the definition of A_q^{opt}. In this section we assume that $(\mathrm{X}, d, \mathfrak{m})$ is a metric measure space with $\mathfrak{m}(\mathrm{X}) = 1$. For every $q \in (2, +\infty)$ we define $A_q^{\mathrm{opt}}(\mathrm{X}) \in [0, +\infty]$ as the minimal constant satisfying

$$\|u\|^2_{L^q(\mathfrak{m})} \le A_q^{\mathrm{opt}}(\mathrm{X}) \, \||Du|_2\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \quad \forall u \in W^{1,2}(\mathrm{X}), \tag{5.1}$$

with the convention that $A := +\infty$ if no such A exists. Note that, since $\mathfrak{m}(\mathrm{X}) = 1$, this is the same definition given right after (1.10). In the following sections we will prove three type of bounds on $A_q^{\mathrm{opt}}(\mathrm{X})$: an upper bound in the case of synthetic Ricci curvature and dimension bounds; a lower bound in terms of the first non-trivial eigenvalue; a lower bound related to the diameter.

5.1 Upper bound on A_q^{opt} in terms of Ricci bounds

Here we prove a generalization to the non-smooth setting of a well known estimate on A_q^{opt} valid on manifolds (recall (1.4)). The two key ingredients for the proof are the Sobolev–Poincaré inequality and an inequality due to Bakry:

Proposition 5.1 *For every $K \in \mathbb{R}$, $N \in (2, \infty)$ and $D > 0$ there exists a constant $A = A(K, N, D) > 0$ such that the following holds. Let $(\mathrm{X}, d, \mathfrak{m})$ be a compact $\mathrm{CD}(K, N)$ space with $N \in (1, \infty)$, $K \in \mathbb{R}$, $\mathfrak{m}(\mathrm{X}) = 1$ and $\mathrm{diam}(\mathrm{X}) \le D$. Then for every $q \in (2, 2^*]$ we have*

$$\|u\|^2_{L^q(\mathfrak{m})} \le A\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \qquad \forall u \in W^{1,2}(\mathrm{X}) \tag{5.2}$$

and in particular $A_q^{\mathrm{opt}}(\mathrm{X}) \le A(K, N, D)$.

Proof The proof is based on the following inequality: for every $q \in (2, \infty)$

$$\Big(\int |u|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} \le (u_\mathrm{X})^2 + (q-1)\Big(\int |u - u_\mathrm{X}|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} \qquad \forall u \in L^q(\mathfrak{m}), \tag{5.3}$$

where $u_\mathrm{X} = \int u \,\mathrm{d}\mathfrak{m}$. See ([23] or [24,Prop. 6.2.2]) for a proof of this fact. Then (5.2) follows combining (5.3) with (2.16) and the Jensen inequality. □

Recall that for $K > 0$ an explicit and sharp upper bound on A_q^{opt} exists and has been proven in [36] (see Theorem 1.8). The argument in [36] relies on the powerful *localization technique*. However, it is worth to point out that Theorem 1.8 can also be deduced from the Pólya–Szegő inequality proved in [84] (see Theorem 2.21) and the Sobolev inequality on the model space (2.17).

5.2 Lower bound on A_q^{opt} in terms of the first eigenvalue

It is well known that a “tight-Sobolev inequality” as in (5.1) (i.e. with a constant 1 in front of $\|u\|_{L^2}$ when X is normalized with unit volume) implies a Poincaré-inequality (see e.g. [24,Prop. 6.2.2]). This can be rephrased as a lower bound on A_q^{opt} in terms of the first non-trivial eigenvalue:

Proposition 5.2 *Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathrm{X}) = 1$. Then for every $q \in (2, +\infty)$ it holds*

$$A_q^{\mathrm{opt}}(\mathrm{X}) \ge \frac{q-2}{\lambda^{1,2}(\mathrm{X})}, \tag{5.4}$$

(meaning that if $\lambda^{1,2}(\mathrm{X}) = 0$, then $A_q^{\mathrm{opt}}(\mathrm{X}) = +\infty$).

We will give a detailed proof of this result, which amounts to a linearization procedure. Indeed a refinement of the same argument will also play a key role on the rigidity and almost-rigidity results in the sequel (see Sect. 6.2).

We start with an elementary linearization-Lemma.

Lemma 5.3 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space with* $\mathfrak{m}(\mathrm{X}) = 1$ *and fix* $q \in (2, \infty)$. *Let* $f \in L^2 \cap L^q(\mathfrak{m})$ *with* $\int f \,\mathrm{d}\mathfrak{m} = 0$. *Then*

$$\Big|\Big(\int |1+f|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} - \int (1+f)^2 \,\mathrm{d}\mathfrak{m} - (q-2)\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big| \\ \le C_q\Big(\int |f|^{3\wedge q} + |f|^q \,\mathrm{d}\mathfrak{m} + \Big(\int |f|^q \,\mathrm{d}\mathfrak{m}\Big)^2 + \Big(\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big)^2\Big), \tag{5.5}$$

where C_q *is a constant depending only on* q.

Proof We start defining $I := \int |1+f|^q \,\mathrm{d}\mathfrak{m} - 1$ and observe that

$$\Big|\Big(\int |1+f|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} - 1 - \frac{2}{q} I\Big| \le c_q |I|^2, \tag{5.6}$$

which follows from the inequality $||1+t|^{2/q} - 1 - 2t/q| \le c_q t^2$, $t \ge 0$. It remains to investigate the behavior of I. Exploiting the inequality $||1+t|^q - 1 - qt| \le \tilde{c}_q(|t|^2 + |t|^q)$, $t \ge 0$, and the fact that f has zero mean we have the following simple bound

$$|I| \le \tilde{c}_q \int |f|^2 + |f|^q \,\mathrm{d}\mathfrak{m}. \tag{5.7}$$

We will also need a more precise estimate of I, which will follow from the following inequality

$$\Big||1+t|^q - 1 - qt - \frac{q(q-1)}{2} t^2\Big| \le C_q(|t|^{3\wedge q} + |t|^q), \quad \forall t \in \mathbb{R}, \tag{5.8}$$

that can be seen using Taylor expansion when $|t| \le 1/2$ and elementary estimates in the case $|t| \ge 1/2$. Using (5.8) we obtain that

$$\Big|I - \int qf + \frac{q(q-1)}{2}|f|^2 \,\mathrm{d}\mathfrak{m}\Big| \le C_q \int |f|^{3\wedge q} + |f|^q \,\mathrm{d}\mathfrak{m}$$

and since we are assuming that f has zero mean, we deduce

$$\Big|I - \frac{q(q-1)}{2}\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big| \le C_q \int |f|^{3\wedge q} + |f|^q \,\mathrm{d}\mathfrak{m}. \tag{5.9}$$

Combining (5.6), (5.7) and (5.9), noting that $\int (1+f)^2 \,\mathrm{d}\mathfrak{m} = 1 + \int f^2 \,\mathrm{d}\mathfrak{m}$, we deduce (5.5). □

Exploiting the above linearization, we can now prove the lower bound on A_q^{opt} in terms of the first eigenvalue.

Proof of Proposition 5.2 If $A_q^{\mathrm{opt}}(\mathrm{X}) = +\infty$ there is nothing to prove, hence we assume that $A_q^{\mathrm{opt}}(\mathrm{X}) < +\infty$. Let $f \in \mathrm{LIP}(\mathrm{X}) \cap L^2(\mathfrak{m})$ with $\int f \,\mathrm{d}\mathfrak{m} = 0$ and $\|f\|_{L^2(\mathfrak{m})} = 1$. Observe also that, since $A_q^{\mathrm{opt}}(\mathrm{X}) < +\infty$, $f \in L^q(\mathrm{X})$. Therefore applying (5.5) we obtain

$$\Big(\int |1+\varepsilon f|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} - \int (1+\varepsilon f)^2 \,\mathrm{d}\mathfrak{m} - (q-2)\int |\varepsilon f|^2 \,\mathrm{d}\mathfrak{m} = o(\varepsilon^2),$$

which combined with (5.1) gives

$$A_q^{\mathrm{opt}}(\mathrm{X})\varepsilon^2 \int |Df|_2^2 \,\mathrm{d}\mathfrak{m} - (q-2)\int |\varepsilon f|^2 \,\mathrm{d}\mathfrak{m} \ge o(\varepsilon^2).$$

Dividing by ε^2 and sending $\varepsilon \to 0$ gives that $\lambda^{1,2}(\mathrm{X}) \geq \frac{q-2}{A_q^{\mathrm{opt}}(\mathrm{X})}$, which concludes the proof. □

5.3 Lower bound on A_q^{opt} in terms of the diameter

We start recalling the following result, which was proved in [25] in the context of Markov-triple and which proof works with straightforward modifications also in the setting of metric measure spaces (see also [59] for an exposition of the argument on Riemannian manifolds). For this reason we shall omit its proof. We stress that, since this result and its consequences are used only on this section, the exposition of the rest of the note remains self-contained.

Theorem 5.4 *Let $q \in (2, \infty)$ and define $N(q) := \frac{2q}{q-2}$. Let $(\mathrm{X}, d, \mathfrak{m})$ be a compact metric measure with $\mathrm{diam}(\mathrm{X}) = \pi$, $\mathfrak{m}(\mathrm{X}) = 1$ and suppose that*

$$\|u\|_{L^q(\mathfrak{m})} \leq \frac{q-2}{N(q)}\||Du|\|^2_{L^2} + \|u\|^2_{L^2(\mathfrak{m})}, \quad \forall u \in W^{1,2}(\mathrm{X}). \tag{5.10}$$

Then there exists a non-constant function $f \in \mathrm{LIP}(\mathrm{X})$ realizing equality in (5.10).

Note that $q = 2N(q)/(N(q)-2)$, so that in a sense "$q = 2^*(N(q))$". With Theorem 5.4 we can now prove the following lower bound on $A_q^{\mathrm{opt}}(\mathrm{X})$. The proof uses a scaling argument due to Hebey [59,Proposition 5.11].

Proposition 5.5 *Let $(\mathrm{X}, d, \mathfrak{m})$ be a compact metric measure space with $\mathfrak{m}(\mathrm{X}) = 1$ and $\mathrm{diam}(\mathrm{X}) \leq \pi$. Then for every $q \in (2, \infty)$ it holds*

$$A_q^{\mathrm{opt}}(\mathrm{X}) \geq \left(\frac{\mathrm{diam}(\mathrm{X})}{\pi}\right)^2 \frac{q-2}{N(q)}, \tag{5.11}$$

where $N(q) = \frac{2q}{q-2}$.

Proof Set $D := \mathrm{diam}(X)$ and, by contradiction, suppose that $A_q^{\mathrm{opt}}(\mathrm{X}) < (\frac{D}{\pi})^2 \frac{q-2}{N(q)}$. Define the scaled metric measure space

$$(\mathrm{X}', d', \mathfrak{m}') := (\mathrm{X}, \tfrac{1}{D/\pi} d, \mathfrak{m}).$$

It can be directly checked that X' satisfies the hypotheses of Theorem 5.4. Hence there exists a non-constant function $u \in \mathrm{LIP}(\mathrm{X})$ satisfying (5.10) with equality (in the space X'), which rewritten on the the original space X reads as

$$\|u\|_{L^q(\mathfrak{m})} = \left(\frac{D}{\pi}\right)^2 \frac{q-2}{N(q)}\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})},$$

which however contradicts the assumption $A_{2^*}^{\mathrm{opt}}(\mathrm{X}) < (\frac{D}{\pi})^2 \frac{q-2}{N(q)}$. □

Remark 5.6 Arguing exactly as in [25], it is possible to prove that under the assumptions of Theorem 5.4 and assuming X to be also infinitesimal Hilbertian, there exists a function satisfying $\Delta u = N(q)u$. From this, it directly follows that equality in (5.11) (in the case of an Infinitesimally Hilbertian space) implies the existence of a function satisfying:

$$\Delta u = \left(\frac{\pi}{\mathrm{diam}(\mathrm{X})}\right)^2 N(q)u.$$

Since this is not relevant in the present note, we will not provide the details of such result. □

6 Rigidity of A_q^{opt}

6.1 Concentration Compactness

In this section we assume that $(X_n, d_n, \mathfrak{m}_n)$ is a sequence of compact $\mathrm{RCD}(K, N)$ spaces, for some fixed $K \in \mathbb{R}, N \in (2, \infty)$, which converges in mGH-topology to a compact $\mathrm{RCD}(K, N)$ space $(X_\infty, d_\infty, \mathfrak{m}_\infty)$. We will also adopt the extrinsic approach [55] identifying X_n, X_∞ as subset of a common compact metric space (Z, d_Z), with $\mathrm{supp}(\mathfrak{m}_n) = X_n$, $\mathrm{supp}(\mathfrak{m}_\infty) = X_\infty$, $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C_b(Z)$ and $X_n \to X_\infty$ in the Hausdorff topology of Z. To lighten the discussion, we shall not recall in the following statements these facts and assume $(X_n, d_n, \mathfrak{m}_n)$, $n \in \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and (Z, d) to be fixed as just explained. Also, we will set $2^* := 2N/(N-2)$ without recalling its expression in the statements.

Our main goal then is to prove the following dichotomy for the behavior of extremizing sequence for the Sobolev inequalities, on varying metric measure spaces.

Theorem 6.1 (Concentration-compactness for Sobolev-extremals) *Suppose that $\mathfrak{m}_n(X_n)$, $\mathfrak{m}_\infty(X_\infty) = 1$ and that X_n supports a $(2^*, 2)$-Sobolev inequality*

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)} \le A\||Du|\|^2_{L^2(\mathfrak{m}_n)} + B\|u\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall u \in W^{1,2}(X_n),$$

for some constants $A, B > 0$. Suppose that $u_n \in W^{1,2}(X_n)$ is a sequence of non-zero functions satisfying

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \ge A_n\||Du_n|\|^2_{L^2(\mathfrak{m}_n)} + B_n\|u_n\|^2_{L^2(\mathfrak{m}_n)},$$

for some sequences $A_n \to A$, $B_n \to B$.

Then, setting $\tilde{u}_n := u_n\|u_n\|^{-1}_{L^{2^*}(\mathfrak{m}_n)}$, there exists a non relabeled subsequence such that only one of the following holds:

(I) *$\tilde{u}_n$ converges L^{2^*}-strong to a function $u_\infty \in W^{1,2}(X_\infty)$;*
(II) *$\|\tilde{u}_n\|_{L^2(\mathfrak{m}_n)} \to 0$ and there exists $x_0 \in X_\infty$ so that $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{x_0}$ in duality with $C_b(Z)$.*

The principle behind the concentration compactness technique is very general and was originated in [78, 79]. In our case, since we will work in a compact setting, the lack of compactness is formally due to dilations or rescalings (and not to translations) and the fact that we deal with the critical exponent in the Sobolev embedding. The main idea behind the principle is first to prove that in general the failure of compactness can only be realized by concentration on a countable number of points. The second step is then to exploit a strict sub-additivity property of the minimization problem to show that either we have full concentration at a single point or we do not have concentration at all and thus compactness.

We start by proving necessary results towards the proof of Theorem 6.1.

A variant of the following appears also in [63,Prop. 3.27]. For the sake of completeness, we provide here a complete proof.

Proposition 6.2 *Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that u_n converges L^q-strong to u_∞ and that v_n converges L^p-weak to v_∞, then*

$$\lim_{n\to\infty} \int u_n v_n \, d\mathfrak{m}_n = \int u_\infty v_\infty \, d\mathfrak{m}_\infty.$$

Proof It is sufficient to consider the case $u_n \geq 0, u_\infty \geq 0$, then the conclusion will follow recalling that $u_n^+ \to u_\infty^+, u_n^- \to u_\infty^-$ strongly in L^q.

The argument is similar to the one for the case $p = 2$ (see, e.g., in [13]), except that we need to consider the functions $u_n^{q/p} + tv_n$, $t \in \mathbb{R}$. Observe first that $u_n^{p/q} \to u_\infty^{q/p}$ strongly in L^p (by (vii) of Prop. 2.18). In particular $u_n^{q/p} + tv_n$ converges to $u_\infty^{q/p} + tv_\infty$ weakly in L^p and in particular from $iii)$ of Prop. 2.18 we have

$$\|u_\infty^{q/p} + tv_\infty\|_{L^p(\mathfrak{m}_\infty)} \leq \varliminf_n \|u_n^{q/p} + tv_n\|_{L^p(\mathfrak{m}_n)}. \tag{6.1}$$

The second ingredient is the following inequality

$$\big||a+|b||^p - |b|^p - pa|b|^{p-1}\big| \leq C_p(|a|^{p\wedge 2}|b|^{p-p\wedge 2} + |a|^p), \quad \forall a, b \in \mathbb{R}, \tag{6.2}$$

which is easily derived from $\big||1+t|^p - 1 - pt\big| \leq C_p(|t|^{p\wedge 2} + |t|^p)$, $\forall t \in \mathbb{R}$. Combining (6.2) and (6.1) we have

$$\begin{aligned}
&\int |u_\infty|^q \, \mathrm{d}\mathfrak{m}_\infty + pt\int u_\infty v_\infty \, \mathrm{d}\mathfrak{m}_\infty - C_p t^{p\wedge 2}\int |v_\infty|^{p\wedge 2}|u_\infty^{q/p}|^{p-p\wedge 2} \, \mathrm{d}\mathfrak{m}_\infty \\
&\quad - C_p t^p \int |v_\infty|^p \, \mathrm{d}\mathfrak{m}_\infty \leq \|u_\infty^{q/p} + tv_\infty\|^p_{L^p(\mathfrak{m}_\infty)} \leq \varliminf_n \|u_n^{q/p} + tv_n\|_{L^p(\mathfrak{m}_n)} \\
&\leq \varliminf_n \int |u_n|^q \, \mathrm{d}\mathfrak{m}_n + pt\int u_n v_n \, \mathrm{d}\mathfrak{m}_n \\
&\quad + C_p t^{p\wedge 2}\int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2} \, \mathrm{d}\mathfrak{m}_n + C_p t^p \int |v_n|^p \, \mathrm{d}\mathfrak{m}_n
\end{aligned}$$

Observe that in the case $p < 2$ we have

$$\varlimsup_n \int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2} = \varlimsup_n \int |v_n|^p \, \mathrm{d}\mathfrak{m}_n < +\infty,$$

while for $p \geq 2$ using the Hölder inequality

$$\varlimsup_n \int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2} \leq \varlimsup_n \|v_n\|^2_{L^p(\mathfrak{m}_n)} \|u_n\|^{q(p-2)/p}_{L^q(\mathfrak{m}_n)} < +\infty.$$

In particular, recalling that $\int |u_n|^q \mathrm{d}\mathfrak{m}_n \to \int |u_\infty|^q \mathrm{d}\mathfrak{m}_\infty$ and choosing first $t \downarrow 0$ and then $t \uparrow 0$ above we obtain the desired conclusion. □

The following is a version for varying-measure of the famous Brezis–Lieb Lemma [33]. The key difference with the classical version of this result, is that in our setting it does not makes sense to write "$|u_\infty - u_n|$", since u_∞ and u_n will be integrated with respect to different measures. Hence we need to replace this term in (6.3) with $|v_n - u_n|$, where v_n is sequence approximating u_∞ in a strong sense.

Lemma 6.3 (Brezis–Lieb type Lemma) *Suppose that* $\mathfrak{m}_n(\mathrm{X}_n), \mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$, *let* $q \in [2, \infty)$ *and* $q' \in (1, q)$. *Suppose that* $u_n \in L^q(\mathfrak{m}_n)$ *satisfy* $\sup_n \|u_n\|_{L^q(\mathfrak{m}_n)} < +\infty$ *and that* u_n *converges to* u_∞ *strongly in* $L^{q'}$ *to some* $u_\infty \in L^{q'} \cap L^q(\mathfrak{m}_\infty)$. *Then for any sequence* $v_n \in L^q(\mathfrak{m}_n)$ *such that* $v_n \to u_\infty$ *strongly both in* $L^{q'}$ *and* L^q, *it holds*

$$\lim_{n\to\infty} \int |u_n|^q \, \mathrm{d}\mathfrak{m}_n - \int |u_n - v_n|^q \, \mathrm{d}\mathfrak{m}_n = \int |u_\infty|^q \, \mathrm{d}\mathfrak{m}_\infty. \tag{6.3}$$

Proof The proof is based on the following inequality:

$$\big||a+b|^q - |b|^q - |a|^q\big| \le C_p(|a||b|^{q-1} + |a|^{q-1}|b|), \quad \forall a, b \in \mathbb{R}. \tag{6.4}$$

Indeed, if $a = v_n - u_n$ and $b = v_n$, we get from the above

$$\int \big||u_n|^q - |v_n - u_n|^q - |v_n|^q\big| \, \mathrm{d}\mathfrak{m}_n \le C_q \int |v_n - u_n||v_n|^{q-1} + |v_n - u_n|^{q-1}|v_n| \, \mathrm{d}\mathfrak{m}_n. \tag{6.5}$$

Since $\int |v_n|^q \, \mathrm{d}\mathfrak{m}_n \to \int |u_\infty|^q \, \mathrm{d}\mathfrak{m}_\infty$, to conclude it is sufficient to show that the right hand side of (6.5) vanishes as $n \to +\infty$. We wish to apply Proposition 6.2. It follows from our assumptions that $|v_n| \to |v_\infty|$ strongly in L^q and $|v_n|^{q-1} \to |v_\infty|^{q-1}$ strongly in L^p, with $p := q/(q-1)$ (recall Prop. 2.18). Hence it remains only to show that $|v_n - u_n|, |v_n - u_n|^{q-1}$ converges to 0 weakly in L^q and weakly in L^p respectively. We have that $\sup_n \|u_n - v_n\|_{L^q(\mathfrak{m}_n)} < +\infty$, hence by $iv)$ in Prop. 2.18 up to a subsequence $|u_n - v_n|$ converge weakly in L^q to a function $w \in L^q(\mathfrak{m})$. However by assumption the sequences $(v_n), (u_n)$ both converge strongly in $L^{q'}$ to u, hence $v_n - u_n \to 0$ strongly in $L^{q'}$ (recall $ii)$ in Prop. 2.18) and in particular by from $i)$ of Prop. 2.18 we have that $|v_n - u_n| \to 0$ strongly in $L^{q'}$, which implies that $w = 0$. Analogously we also get that up to a subsequence $|u_n - v_n|^{q-1}$ converge weakly in L^p to a non-negative function $w' \in L^p(\mathfrak{m})$. Suppose first that $q' \le q - 1$. taking $t \in [0, 1]$ such that $q - 1 = tq' + (1-t)q$ we have

$$\int w' \, \mathrm{d}\mathfrak{m}_\infty = \lim_n \int |u_n - v_n|^{q-1} \, \mathrm{d}\mathfrak{m}_n \le \|v_n - u_n\|^{tq'}_{L^{q'}(\mathfrak{m}_n)} \|v_n - u_n\|^{(1-t)q}_{L^q(\mathfrak{m}_n)} \to 0,$$

where we have used again that $u_n - v_n \to 0$ strongly in $L^{q'}$ and that $u_n - v_n$ is uniformly bounded in L^q. If instead $q' \ge q - 1$ by Hölder inequality we have

$$\int w' \, \mathrm{d}\mathfrak{m}_\infty = \lim_n \int |u_n - v_n|^{q-1} \, \mathrm{d}\mathfrak{m}_n \le \left(\int |u_n - v_n|^{q'} \, \mathrm{d}\mathfrak{m}_n\right)^{(q-1)/q'} \to 0.$$

In both cases we deduce that $w' = 0$, which concludes the proof. □

Lemma 6.4 *Let $q \in [2, \infty)$ and let $u_\infty \in W^{1,2}(\mathrm{X}_\infty) \cap L^q(\mathfrak{m}_\infty)$. Then, there exists a sequence $u_n \in W^{1,2}(\mathrm{X}_n) \cap L^q(\mathrm{X}_n)$ that converges both L^q-strong and $W^{1,2}$-strong to u_∞.*

Proof By truncation and a diagonal argument we can assume that $u_\infty \in L^\infty(\mathfrak{m}_\infty)$. By the Γ-$\overline{\lim}$ inequality of the Ch_2 energy there exists a sequence $v_n \in W^{1,2}(\mathrm{X}_n)$ converging strongly in $W^{1,2}$ to u_∞. Defining $u_n := (v_n \wedge C) \vee -C$, with $C \ge \|u_\infty\|_{L^\infty(\mathfrak{m}_\infty)}$, we have by (i) of Proposition 2.18 that u_n converges in L^2-strong to u_∞. Moreover $|Du_n| \le |Dv_n|$ $\mathfrak{m}_n$-a.e., therefore $\overline{\lim}_n \int |Du_n|^2 \, \mathrm{d}\mathfrak{m}_n \le \overline{\lim}_n \int |Dv_n|^2 = \int |Du_\infty|^2 \, \mathrm{d}\mathfrak{m}$, which grants that u_n converges also $W^{1,2}$-strongly to u_∞. Finally, the sequence u_n is uniformly bounded in L^∞ and converges to u_∞ in L^2-strong, hence by $(viii)$ of Proposition 2.18. we have that that u_n is also L^q-strongly convergent to u_∞. □

The following statement is the analogous in metric measure spaces of [79,Lemma I.1]. We shall omit its proof since the arguments presented there in $\mathbb{R}^n$ extend to this setting with obvious modifications (see also Remark I.5 in [79]).

Lemma 6.5 *Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space and $\mu, \nu \in \mathscr{M}_b^+(\mathrm{X})$. Suppose that*

$$\left(\int |\varphi|^q \, \mathrm{d}\nu\right)^{1/q} \le C \left(\int |\varphi|^p \, \mathrm{d}\mu\right)^{1/p}, \qquad \forall \varphi \in \mathrm{LIP}_b(\mathrm{X}),$$

for some $1 \le p < q < +\infty$ *and* $C \ge 0$. *Then there exists a countable set of indices* J, *points* $(x_j)_{j\in J} \subset \mathrm{X}$ *and positive weights* $(\nu_j)_{j\in J} \subset \mathbb{R}^+$ *so that*

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \ge C^{-p} \sum_{j\in J} \nu_j^{p/q} \delta_{x_j}. \tag{6.6}$$

Next, we present a generalized Concentration–Compactness principle, with underlying varying ambient space. For the sake of generality and for an application to the Yamabe equation in Sect. 8, we will be working with a slightly more general Sobolev inequality containing an arbitrary L^q-norm (apart from Sect. 8, we will use this statement only with $q = 2$).

Lemma 6.6 (Concentration–Compactness Lemma) *Suppose that* $\mathfrak{m}_n(\mathrm{X}_n), \mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$ *and that for some fixed* $q \in (1, \infty)$ *the spaces* X_n *satisfy the following Sobolev-type inequalities*

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)} \le A_n \||Du|\|^2_{L^2(\mathfrak{m}_n)} + B_n \|u\|^2_{L^q(\mathfrak{m}_n)}, \qquad \forall u \in W^{1,2}(\mathrm{X}_n), \tag{6.7}$$

with uniformly bounded positive constants A_n, B_n. *Let also* $u_n \in W^{1,2}(\mathrm{X}_n)$ *be* $W^{1,2}$-*weak and both* L^2-*strong and* L^q-*strong converging to* $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$ *and suppose that* $|Du_n|^2\mathfrak{m}_n \rightharpoonup \mu$, $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$ *in duality with* $C_b(\mathrm{Z})$ *for two given measures* $\mu, \nu \in \mathscr{M}_b^+(\mathrm{Z})$.
Then,

(i) *there exists a countable set of indices* J, *points* $(x_j)_{j\in J} \subset \mathrm{X}_\infty$ *and positive weights* $(\nu_j)_{j\in J} \subset \mathbb{R}^+$ *so that*

$$\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j \delta_{x_j};$$

(ii) *there exist* $(\mu_j)_{j\in J} \subset \mathbb{R}^+$ *satisfying* $\nu_j^{2/2^*} \le (\overline{\lim}_n A_n)\mu_j$ *and such that*

$$\mu \ge |Du_\infty|^2\mathfrak{m}_\infty + \sum_{j\in J} \mu_j \delta_{x_j}.$$

In particular, we have $\sum_j \nu_j^{2/2^*} < \infty$.

Proof We subdivide the proof in two steps.
Step 1. We assume that $u_\infty = 0$. Let $\varphi \in \mathrm{LIP}_b(\mathrm{Z})$ and consider the sequence $(\varphi u_n) \in W^{1,2}(\mathrm{X}_n)$ which plugged in the Sobolev inequality for each X_n gives

$$\left(\int |\varphi|^{2^*}|u_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n\right)^{1/2^*} \le \left(A_n \int |D(\varphi u_n)|^2\,\mathrm{d}\mathfrak{m}_n + B_n\Big(\int |\varphi|^q u_n^q\,\mathrm{d}\mathfrak{m}_n\Big)^{2/q}\right)^{1/2}, \ \forall n \in \mathbb{N}.$$

It is clear that, by weak convergence, the left hand side of the inequality tends to $(\int |\varphi|^{2^*}\,\mathrm{d}\nu)^{1/2^*}$. While for the right hand side we discuss the two terms separately. First, by L^q-strong convergence, we have $\int \varphi^q u_n^q\,\mathrm{d}\mathfrak{m}_n \to 0$, while an an application of the Leibniz rule gives $\int |D(\varphi u_n)|\,\mathrm{d}\mathfrak{m}_n \le \int |D\varphi||u_n| + |\varphi||Du_n|\,\mathrm{d}\mathfrak{m}_n$. Moreover again by strong convergence $\int |D\varphi|^2|u_n|^2\mathrm{d}\mathfrak{m}_n \to 0$. Combining these observations we reach

$$\left(\int |\varphi|^{2^*}\mathrm{d}\nu\right)^{1/2^*} \le \big(\overline{\lim_n}\, A_n\big)^{1/2}\left(\int |\varphi|^2\,\mathrm{d}\mu\right)^{1/2}, \qquad \forall \varphi \in \mathrm{LIP}_b(\mathrm{Z}).$$

Thus, Lemma 6.5 (applied in the space $(\mathrm{Z}, d_\mathrm{Z})$) gives (i)–(ii), for the case $u_\infty = 0$, except for the fact that we currently do no know whether the points $(x_j)_{j\in J}$ are in X_∞. This last simple fact can be seen as follows. Fix $j \in J$. From the weak convergence $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$, there must be a sequence $y_n \in \mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$ such that $d_Z(y_n, x_j) \to 0$. Then the GH-convergence of X_n to X_∞ ensures that $x_j \in \mathrm{X}_\infty$, which is what we wanted.

STEP 2. We now consider the case of a general u_∞. Observe that from Lemma 4.1 X_∞ supports a $(2^*, 2)$-Sobolev inequality hence, $u_\infty \in L^{2^*}(\mathfrak{m}_\infty)$. From Lemma 6.4 there exists a sequence $\tilde{u}_n \in W^{1,2}(\mathrm{X}_n)$ such that $\tilde{u}_n$ converges to u_∞ both strongly in $W^{1,2}$ and strongly in L^{2^*}. Consider now the sequence $v_n := u_n - \tilde{u}_n$. Clearly v_n converges to zero both in L^2-strong and in $W^{1,2}$-weak. Moreover the measures $|v_n|^{2^*}\mathfrak{m}_n$ and $|Dv_n|^2\mathfrak{m}_n$ have uniformly bounded mass. Since (Z, d) is compact, passing to a non-relabeled subsequence we have $|v_n|^{2^*}\mathfrak{m}_n \rightharpoonup \bar{\nu}$ and $|Dv_n|^2\mathfrak{m}_n \rightharpoonup \bar{\mu}$ in duality with $C_b(\mathrm{Z})$ for some $\bar{\nu}, \bar{\mu} \in \mathscr{M}_b^+(\mathrm{Z})$. Therefore we can apply Step 1 to the sequence v_n to get $\bar{\nu} = \sum_{j\in J} \nu_j \delta_{x_j}$, $\bar{\mu} \ge \sum_{j \in J} \mu_j \delta_{x_j}$ for a suitable countable family J, $(x_j) \subset \mathrm{X}_\infty$ and weights (ν_j), (μ_j) satisfying $\nu_j^{2/2^*} \le (\overline{\lim}_n A_n)\mu_j$. To carry the properties of v_n to the sequence u_n we invoke Lemma 6.3 (with $q' = 2$ and $q = 2^*$) to deduce that

$$\lim_{n\to\infty} \int |\varphi|^{2^*}|u_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n - \int |\varphi|^{2^*}|v_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n = \int |\varphi|^{2^*}|u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty, \tag{6.8}$$

and, taking into account the weak convergence, this implies that

$$\int \varphi^{2^*}\,\mathrm{d}\nu - \int \varphi^{2^*}\,\mathrm{d}\bar{\nu} = \int |u_\infty|^{2^*}\varphi^{2^*}\,\mathrm{d}\mathfrak{m}_\infty,$$

for every non-negative $\varphi \in C_b(\mathrm{Z})$. In particular, this is equivalent to say that $\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \bar{\nu} = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\delta_{x_j}$, which proves i). Next, we claim that $\mu \ge \sum_{j\in J}\mu_j\delta_{x_j}$ and, to do so, we consider for each $j \in J$ and $\varepsilon > 0$, $\chi_\varepsilon \in \mathrm{LIP}_b(\mathrm{Z})$, $0 \le \chi_\varepsilon \le 1$, $\chi_\varepsilon(x_j) = 1$ and supported in $B_\varepsilon(x_j)$. The key ingredient is the following estimate

$$\begin{aligned}
\left| \int \chi_\varepsilon |Du_n|^2\,\mathrm{d}\mathfrak{m}_n - \int \chi_\varepsilon|Dv_n|^2\,\mathrm{d}\mathfrak{m}_n \right| &\le \int \chi_\varepsilon \big||Du_n| - |Dv_n|\big|\big(|Du_n| + |Dv_n|\big)\,\mathrm{d}\mathfrak{m}_n \\
&\le \int \chi_\varepsilon |D\tilde{u}_n|\big(|Du_n| + |Dv_n|\big)\,\mathrm{d}\mathfrak{m}_n \\
&\le \Big(\int \chi_\varepsilon^2 |D\tilde{u}_n|^2\,\mathrm{d}\mathfrak{m}_n\Big)^{1/2}\Big(\||Du_n|\|_{L^2(\mathfrak{m}_n)} \\
&\quad + \||Dv_n|\|_{L^2(\mathfrak{m}_n)}\Big).
\end{aligned}$$

Observe now that from [13,Theorem 5.7] $|D\tilde{u}_n| \to |Du_\infty|$ strongly in L^2 and in particular $\int \chi_\varepsilon^2|D\tilde{u}_n|^2\,\mathrm{d}\mathfrak{m}_n \to \int \chi_\varepsilon^2|Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty$. Moreover $\int \chi_\varepsilon^2|Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty \to 0$ as $\varepsilon \to 0^+$ and u_n, v_n are uniformly bounded in $W^{1,2}(\mathrm{X}_n)$. Therefore taking in the above inequality first $n \to +\infty$ and afterwards $\varepsilon \to 0^+$ we ultimately deduce that

$$\mu(\{x_i\}) = \bar{\mu}(\{x_i\}) \ge \mu_j, \quad \forall\, j \in J.$$

In particular, since μ is non-negative, $\mu \ge \sum_{j\in J}\mu_j\delta_{x_j}$, as claimed. Finally, by the weak lower semicontinuity result in [13,Lemma 5.8], we have

$$\int \varphi|Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty \le \underline{\lim_n} \int \varphi|Du_n|^2\,\mathrm{d}\mathfrak{m}_n = \int \varphi\,\mathrm{d}\mu$$

for every $\varphi \in C_b(\mathrm{Z})$. Therefore, we get $\mu \ge |Du_\infty|^2\mathfrak{m}_\infty$ and, by mutual singularity of the two lower bounds, we have (ii) and the proof is now concluded. □

We are finally ready to prove the main result of this section.

Proof of Theorem 6.1 Set $\tilde{u}_n := u_n \|u_n\|^{-1}_{L^q(\mathfrak{m}_n)}$. By assumption

$$1 \geq A_n \||D\tilde{u}_n|\|^2_{L^2(\mathfrak{m}_n)} + B_n \|\tilde{u}_n\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall n \in \mathbb{N}. \tag{6.9}$$

Moreover again by hypothesis $A_n \to A > 0$, $B_n \to B > 0$, therefore the sequences A_n, B_n are bounded away from zero and thus $\sup_n \|\tilde{u}_n\|_{W^{1,2}(\mathrm{X}_n)} < \infty$. Hence, up to passing to a non relabeled subsequence, Proposition 2.19 grants that $\tilde{u}_n$ converges L^2-strongly to a function $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$. Moreover, the measures $|D\tilde{u}_n|^2\mathfrak{m}_n$, $|\tilde{u}_n|^{2^*}\mathfrak{m}_n$ have uniformly bounded mass. In particular up to a further not relabeled subsequence, there exists $\mu, \nu \in \mathscr{M}^+_b(\mathrm{Z})$ so that $|D\tilde{u}_n|^2\mathfrak{m}_n \rightharpoonup \mu$ and $|\tilde{u}_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$ in duality with $C_b(\mathrm{Z})$. We are in position to apply Lemma 6.5 to get the existence of at most countably many points $(x_j)_{j\in J}$ and weights $(\nu_j)_{j\in J}$, so that $\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j \delta_{x_j}$ and $\mu \geq |Du_\infty|^2\mathfrak{m}_\infty + \sum_{j\in J}\mu_j\delta_{x_j}$, with $A\mu_j \geq \nu_j^{2/2^*}$ and in particular $\sum_j \nu_j^{2/2^*} < \infty$. Finally from Lemma 4.1 we have that X_∞ supports a $(2^*, 2)$-Sobolev inequality with constants A, B. Therefore we can perform the following estimates

$$\begin{aligned}
1 = \lim_n \|\tilde{u}_n\|^2_{L^{2^*}(\mathfrak{m}_n)} &\geq \lim_n A_n \||D\tilde{u}_n|\|^2_{L^2(\mathfrak{m}_n)} + B\|\tilde{u}_n\|^2_{L^2(\mathfrak{m}_n)}\\
&= A\mu(\mathrm{X}_\infty) + B\int |u_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty\\
&\geq A\int |Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty + B\int |u_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j^{2/2^*}\\
&\geq \Big(\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty\Big)^{2/2^*} + \sum_{j\in J}\nu_j^{2/2^*}\\
&\geq \Big(\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\Big)^{2/2^*} = \nu(\mathrm{X}_\infty)^{2/2^*} = 1,
\end{aligned}$$

where in the last inequality we have used the concavity of the function $t^{2/2^*}$. In particular all the inequalities must be equalities and, since $t^{2/2^*}$ is strictly concave, we infer that every term in the sum $\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j^{2/2^*}$ must vanish except for one that must be equal to 1. If $\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty = 1$ then I) must hold. If instead $\nu_j = 1$ for some $j \in J$, then $u_\infty = 0$ and by definition of ν, $|\tilde{u}_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{x_j}$, which is exactly II). □

6.2 Quantitative linearization

A key point in our argument for the rigidity, and especially for the almost-rigidity, of A^{opt}_q will be a more "quantitative" version of the elementary linearization of the Sobolev inequality contained in Lemma 5.3. To state our result, given $q \in (2, \infty)$ and $u \in W^{1,2}(\mathrm{X})$ with $\int |Du|_2, \mathrm{d}\mathfrak{m} > 0$, it is convenient to define the Sobolev ratio associated to u as the quantity

$$\mathcal{Q}^{\mathrm{X}}_q(u) := \frac{\|u\|^2_{L^q(\mathfrak{m})} - \|u\|^2_{L^2(\mathfrak{m})}}{\||Du|_2\|^2_{L^2(\mathfrak{m})}}. \tag{6.10}$$

Observe that, if $\lambda^{1,2}(\mathrm{X}) > 0$, $\int |Du|_2\mathrm{d}\mathfrak{m} > 0$ as soon as u is not ($\mathfrak{m}$-a.e. equal to a) constant.

Lemma 6.7 (Quantitative linearization) *For all numbers $A, B \geq 0$, $q > 2$ and $\lambda > 0$ there exists a constant $C = C(q, A, B, \lambda)$ such that the following holds. Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric* *measure space with* $\mathfrak{m}(\mathrm{X}) = 1$, $\lambda^{1,2}(\mathrm{X}) \geq \lambda$ *and supporting a* $(q, 2)$*-Sobolev inequality with constants* A, B*. Then, for every non-constant* $f \in W^{1,2}(\mathrm{X})$ *satisfying* $\|f\|_{L^2(\mathrm{X})} \leq 1/2$*, it holds*

$$\Big|\mathcal{Q}_q^{\mathrm{X}}(1+f) - \frac{(q-2)\int \big(f - \int f \,\mathrm{d}\mathfrak{m}\big)^2 \,\mathrm{d}\mathfrak{m}}{\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}}\Big|$$
$$\leq C\big(\|f\|_{W^{1,2}(\mathrm{X})}^{3\wedge q-2} + \|f\|_{W^{1,2}(\mathrm{X})}^{q-2} + \|f\|_{W^{1,2}(\mathrm{X})}^{2q-2}\big). \tag{6.11}$$

Proof We claim that it is enough to prove the statement for functions $f \in W^{1,2}(\mathrm{X})$ with zero mean (and arbitrary L^2-norm). Indeed for a generic $f \in W^{1,2}(\mathrm{X})$ satisfying $\|f\|_{L^2(\mathrm{X})} \leq 1/2$, we can take $\tilde{f} := \frac{f - \int f \,\mathrm{d}\mathfrak{m}}{1+\int f \,\mathrm{d}\mathfrak{m}}$, which clearly has zero mean. Then the conclusion would follow observing that the left hand side of (6.11) computed at $\tilde{f}$ coincides with the left hand side of (6.11) computed at f and from the fact that

$$\|\tilde{f}\|_{W^{1,2}(\mathrm{X})} \leq \|f\|_{W^{1,2}(\mathrm{X})}\Big(1 + \int f \,\mathrm{d}\mathfrak{m}\Big)^{-1} \leq \|f\|_{W^{1,2}(\mathrm{X})}(1 - \|f\|_{L^2(\mathrm{X})})^{-1} \leq 2\|f\|_{W^{1,2}(\mathrm{X})}.$$

Therefore we can now fix $f \in W^{1,2}(\mathrm{X})$ with $\int f \,\mathrm{d}\mathfrak{m} = 0$. We start with a basic estimate of the L^r norm of f for $r \in [1, q]$. Combining the Hölder and the $(q, 2)$-Sobolev inequalities we have

$$\int |f|^r \,\mathrm{d}\mathfrak{m} \leq \Big(\int |f|^q \,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{q}} \leq (A^{r/2} + B^{r/2})\|f\|_{W^{1,2}(\mathrm{X})}^r \tag{6.12}$$

In the case $r \in (2, q]$ the following refined estimate holds:

$$\frac{\int |f|^r \,\mathrm{d}\mathfrak{m}}{\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}} \leq C_q A^{r/2}\Big(\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{2}-1} + C_q B^{r/2}\Big(\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{2}-1} \frac{\int |f|^2 \,\mathrm{d}\mathfrak{m}}{\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}}$$
$$\leq C_q(A^{r/2} + B^{r/2}\lambda^{-1})\|f\|_{W^{1,2}(\mathrm{X})}^{r-2}. \tag{6.13}$$

We now apply (5.3) to f, which we rewrite here for the convenience of the reader:

$$\Big|\Big(\int |1+f|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} - \int (1+f)^2 \,\mathrm{d}\mathfrak{m} - (q-2)\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big|$$
$$\leq \tilde{C}_q\Big(\int |f|^{3\wedge q} + |f|^q \,\mathrm{d}\mathfrak{m} + \Big(\int |f|^q \,\mathrm{d}\mathfrak{m}\Big)^2 + \Big(\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big)^2\Big),$$

where $\tilde{C}_q$ is a constant depending only on q. Dividing by $\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}$ the above inequality and rearranging terms, using the definition of $\lambda^{1,2}(\mathrm{X})$ and the estimates (6.12), (6.13) we obtain (6.11). □

6.3 Proof of the rigidity

Here we prove Theorem 1.9. This result will follow from the following theorem, which characterizes the behavior of extremal sequences for the Sobolev inequality and which combines the tools of concentration compactness and linearization, developed in the previous sections. This result can be summarized as: either there exist non-constant extremals, or we have information on the first eigenvalue $\lambda^{1,2}(\mathrm{X})$, or we have information on the density θ_N.

Theorem 6.8 (The Sobolev-alternative) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$, $N \in (2, \infty)$ *and with* $\mathfrak{m}(\mathrm{X}) = 1$*. Let* $q \in (2, 2^*]$*, with* $2^* := 2N/(N-2)$*. Then at least one of the following holds:*

(i) *there exists a non-constant function* $u \in W^{1,2}(\mathrm{X})$ *satisfying*

$$\|u\|^2_{L^q(\mathfrak{m})} = A^{\mathrm{opt}}_q(\mathrm{X})\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \tag{6.14}$$

(ii) $A^{\mathrm{opt}}_q(\mathrm{X}) = \frac{q-2}{\lambda^{1,2}(\mathrm{X})}$,

(iii) $q = 2^*$ *and* $A^{\mathrm{opt}}_{2^*}(\mathrm{X}) = \alpha_2(\mathrm{X}) = \frac{Eucl(N,2)^2}{\min \theta_N^{2/N}}$ *(see the introduction and* (2.2) *for the definition of* $\alpha_2(\mathrm{X})$ *and* $Eucl(N,2)$*).*

Proof By definition of $A^{\mathrm{opt}}_q(\mathrm{X})$ there exists a sequence of non-constant functions $u_n \in \mathrm{LIP}(\mathrm{X})$ such that $\mathcal{Q}^{\mathrm{X}}_q(u_n) \to A^{\mathrm{opt}}_q(\mathrm{X})$ (recall (6.10)). By scaling we can suppose that $\|u_n\|_{L^{2^*}(\mathfrak{m})} \equiv 1$. In particular (u_n) is bounded in $W^{1,2}(\mathrm{X})$. We distinguish two cases.
SUBCRITICAL: $q < 2^*$. By compactness (see Proposition 2.19), up to passing to a subsequence, $u_n \to u$ strongly in L^q to some function $u \in W^{1,2}(\mathrm{X})$ such that, from the lower semicontinuity of the Cheeger energy, $\mathcal{Q}^{\mathrm{X}}_q(u) = A^{\mathrm{opt}}_q(\mathrm{X})$. If u is non-constant (i) holds and we are done, so suppose that u is constant. Then from the renormalization we must have $u \equiv 1$. Moreover, since $\|u_n\|_{L^q(\mathfrak{m})}, \|u_n\|_{L^2(\mathfrak{m})} \to 1$ and $\mathcal{Q}^{\mathrm{X}}_q(u_n) \to A^{\mathrm{opt}}_q(\mathrm{X})$, we deduce that $\||Du|\|^2_{L^2(\mathfrak{m})} \to 0$. Consider now the functions $f_n := u_n - 1 \in \mathrm{LIP}(X)$, which are non-constant and such that $f_n \to 0$ in $W^{1,2}(\mathrm{X})$. We are therefore in position to apply Lemma 6.7 and deduce that

$$A^{\mathrm{opt}}_q(\mathrm{X}) = \lim_{n\to\infty} \mathcal{Q}^{\mathrm{X}}_q(u_n) = \lim_n \frac{(q-2)\int \left(f_n - \int f_n \mathrm{d}\mathfrak{m}\right)^2 \mathrm{d}\mathfrak{m}}{\int |Df_n|^2 \mathrm{d}\mathfrak{m}} \le \frac{q-2}{\lambda^{1,2}(\mathrm{X})}.$$

Combining this with (5.4), we get that $A^{\mathrm{opt}}_q(\mathrm{X}) = \frac{q-2}{\lambda^{1,2}(\mathrm{X})}$, i.e. (ii) is true and we conclude the proof in this case.
CRITICAL: $q = 2^*$. We apply the concentration-compactness result in Theorem 6.1 and deduce that up to a subsequence: either $u_n \to u$ in $L^{2^*}(\mathfrak{m})$ to some $u \in W^{1,2}(\mathrm{X})$ or $\|u_n\|_{L^2(\mathfrak{m})} \to 0$. In the first case we argue exactly as above using Lemma 6.7 and deduce that either (i) or (ii) holds. Hence we are left to deal with the case $\|u_n\|_{L^2(\mathfrak{m})} \to 0$. From the definition of $\alpha_2(\mathrm{X})$, for every ε there exits B_ε so that a $(2^*, 2)$-Sobolev inequality with constants $\alpha_2(\mathrm{X}) + \varepsilon$ and B_ε is valid. Hence we have

$$\mathcal{Q}^{\mathrm{X}}_{2^*}(u_n)\||Du_n|\|^2_{L^2(\mathfrak{m})} + \|u_n\|^2_{L^2(\mathfrak{m})} = \|u_n\|_{L^{2^*}(\mathfrak{m})} \le (\alpha_2(\mathrm{X}) + \varepsilon)\||Du_n|\|^2_{L^2(\mathfrak{m})} + B_\varepsilon \|u_n\|^2_{L^2(\mathfrak{m})},$$

which gives

$$\mathcal{Q}^{\mathrm{X}}_{2^*}(u_n) \le (\alpha_2(\mathrm{X}) + \varepsilon) + B_\varepsilon \|u_n\|^2_{L^2(\mathfrak{m})}(\||Du_n|\|^2_{L^2(\mathfrak{m})})^{-1}.$$

Observing that $\underline{\lim}_n \||Du_n|\|^2_{L^2(\mathfrak{m})} > 0$ (which follows from the Sobolev inequality, $\|u_n\|^2_{L^2(\mathfrak{m})} \to 0$ and $\|u_n\|_{L^{2^*}(\mathfrak{m})} = 1$) and letting $n \to +\infty$ we arrive at $A^{\mathrm{opt}}_{2^*}(\mathrm{X}) \le (\alpha_2(\mathrm{X}) + \varepsilon)$. From the arbitrariness ε we deduce that $A^{\mathrm{opt}}_{2^*}(\mathrm{X}) \le \alpha_2(\mathrm{X})$ and the proof is concluded (indeed by definition $\alpha_2(\mathrm{X}) \ge A^{\mathrm{opt}}_{2^*}(\mathrm{X})$ is always true). □

We can finally come to the proof of the principal result of this note.

Proof of Theorem 1.9 The "if" implication is direct as any N-spherical suspension, X is so that $A_q^{\mathrm{opt}}(\mathrm{X}) = \frac{q-2}{N}$. This can be seen from the lower bound in Proposition 5.2 (recall also Theorem 2.11) and the upper bound given in Theorem 1.8.

For the "only if' implication, the result will follow from three different rigidity results, one for each of the alternatives in Theorem 6.8. Up to scaling the reference measures, we can suppose $\mathfrak{m}(\mathrm{X}) = 1$.

Case 1: i) in Theorem 6.8 holds. Let u be the non-constant function satisfying (6.14). Observe that we can assume that u is non-negative. We aim to apply the Pólya–Szegő inequality with the model space I_N as in Sect. 2.4. Let $u_N^* : I_N \to [0, \infty]$ be the monotone-rearrangement of u. From the Pólya–Szegő inequality in Theorem 2.21 we have that $u_N^* \in W^{1,2}(I_N, |.|, \mathfrak{m}_N)$, $\|u\|_{L^p(\mathfrak{m})} = \|u_N^*\|_{L^p(\mathfrak{m}_N)}$ for both $p \in \{q, 2\}$ and that $\||Du_N^*|\|_{L^2(\mathfrak{m}_N)} \le \||Du|\|_{L^2(\mathfrak{m})}$. Combining this with (2.17) we have

$$\begin{aligned}\|u\|_{L^q(\mathfrak{m})}^2 = \|u_N^*\|_{L^q(\mathfrak{m}_N)}^2 &\le \tfrac{q-2}{N}\||Du_N^*|\|_{L^2(\mathfrak{m}_N)}^2 + \|u_N^*\|_{L^2(\mathfrak{m}_N)}^2 \\ &\le \tfrac{q-2}{N}\||Du|\|_{L^2(\mathfrak{m})}^2 + \|u\|_{L^2(\mathfrak{m})}^2 = \|u\|_{L^q(\mathfrak{m})}^2.\end{aligned}$$

Therefore $\||Du_N^*|\|_{L^2(\mathfrak{m}_N)} = \||Du|\|_{L^2(\mathfrak{m})}$ and, since u is non-constant, we are in position to apply the rigidity of the Pólya–Szegő inequality of Theorem 2.22 and conclude the proof in this case.

Case 2: ii) in Theorem 6.8 holds. We immediately deduce that $\lambda^{1,2}(\mathrm{X}) = N$ and the conclusion follows from the Obata's rigidity (Theorem 2.11).

Case 3: iii) in Theorem 6.8 holds. From Theorem 3.13 and the explicit expression for $Eucl(N, 2)$ (see (2.3)) we have that

$$\frac{2^* - 2}{N} = A_{2^*}^{\mathrm{opt}}(\mathrm{X}) = \alpha_2(\mathrm{X}) = \frac{Eucl(N,2)^2}{\min_{x\in \mathrm{X}} \theta_N(x)^{2/N}} = \frac{2^* - 2}{N\sigma_N^{2/N} \min_{x\in \mathrm{X}} \theta_N(x)^{2/N}},$$

therefore $\min_{x\in\mathrm{X}} \theta_N = \sigma_N^{-1}$. On the other hand by the Bishop–Gromov inequality and identity (2.11)

$$\frac{1}{\sigma_N} = \inf_{\mathrm{X}} \theta_N(x) \ge \frac{\mathfrak{m}(\mathrm{X})}{v_{N-1,N}(\mathrm{diam}(\mathrm{X}))} = \frac{1}{v_{N-1,N}(\mathrm{diam}(\mathrm{X}))},$$

which, from the definition of $v_{N-1,N}$ and (2.4) forces $\mathrm{diam}(\mathrm{X}) = \pi$. The conclusion then follows by the rigidity of the maximal diameter (Theorem 2.12). □

Remark 6.9 The rigidity result for $A_q^{\mathrm{opt}}(M)$ in the subcritical range $q < 2^*$ was already observed in [76] as a consequence of the following sharper estimate due to [50]: for any n-dimensional Riemannian manifolds M, $n \ge 3$, with $\mathrm{Ric} \ge n - 1$ it holds

$$A_q^{\mathrm{opt}}(M) \le \frac{(q-2)}{\kappa(\theta)}, \qquad \forall q \in (2, 2^*), \tag{6.15}$$

where $\kappa(\theta) := \theta n + (1-\theta)\lambda^{1,2}(M)$, $\lambda^{1,2}(M)$ being the first non trivial eigenvalue and $\theta = \theta(q) \in [0, 1]$ is a suitable interpolation parameter. The spectral gap inequality $\lambda^{1,2}(M) \ge n$ grants that the bound (6.15) improves the one of (1.5). For every $q \in (2, 2^*)$, the condition $A_q^{\mathrm{opt}}(M) = A_q^{\mathrm{opt}}(\mathbb{S}^n)(= (q-2)/n)$ forces $\kappa(\theta) = n$ which in turn implies $\lambda^{1,2}(M) = n$. By appealing to the classical Obata's Theorem, this argument covers the rigidity of Theorem 1.3 for $q < 2^*$. Nevertheless, this does not extend to the critical exponent: more precisely $\theta(q) \to 1$ as $q \to 2^*$, hence the quantity $\kappa(\theta)$ carries no information on the spectral gap in this case. □

7 Almost rigidity of A^{opt}

7.1 Behavior at concentration points

The following technical result will be needed for the almost-rigidity result and has the role of replacing in the varying-space case, the Sobolev inequality with constants $\alpha_2(\mathrm{X}) + \varepsilon$, B_ε which we used in the fixed-space case of the rigidity (see the proof of Theorem 6.8). Indeed it is not clear how to control the constant B_ε in a sequence of mGH-converging spaces. Therefore we need a more precise local analysis that fully exploits the local Sobolev inequalities in Theorem 3.8 and Proposition 3.12.

Lemma 7.1 (Behavior at concentration points) *Let $(\mathrm{X}_n, d_n, \mathfrak{m}_n, x_n)$, $n \in \bar{\mathbb{N}}$, be a sequence of* $\mathrm{RCD}(K, N)$ *spaces $K \in \mathbb{R}$, $N \in (1, \infty)$, so that $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$. Fix $p \in (1, N)$, set $p^* := pN/(N-p)$ and assume that $u_n \in \mathrm{LIP}_c(\mathrm{X}_n)$ is a sequence satisfying*

$$\|u_n\|^p_{L^{p^*}(\mathfrak{m}_n)} \geq A_n \||Du_n|\|^p_{L^p(\mathfrak{m}_n)} - B_n \|u_n\|^p_{L^s(\mathfrak{m}_n)}, \tag{7.1}$$

for some constants $A_n, B_n \geq 0$ uniformly bounded and $s > 0$ so that $s \in [p, p^*)$. Assume furthermore that $u_n \to 0$ strongly in L^p, $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$ and that $|u_n|^{p^*}\mathfrak{m}_n \rightharpoonup \delta_{y_0}$ for some $y_0 \in \mathrm{X}_\infty$ in duality with $C_{bs}(\mathrm{Z})$ (where $(\mathrm{Z}, d_\mathrm{Z})$ is a proper space realizing the convergence in the extrinsic approach). Then

$$\theta_N(y_0) \leq Eucl(N, p)^N (\overline{\lim_n} A_n)^{-N/p}, \tag{7.2}$$

meaning that if $\theta_N(y_0) = +\infty$, then $\overline{\lim}_n A_n = 0$.

Proof We subdivide the proof in two cases.

Case 1: $\theta_N(y_0) < +\infty$.

Fix $\varepsilon < \theta_N(y_0)/4$ arbitrary. Since $\theta_{N,r}(y_0) \to \theta_N(y_0)$ as $r \to 0^+$ there exists $\bar{r} = \bar{r}(\varepsilon)$ such that

$$|\theta_{N,r}(y_0) - \theta_N(y_0)| \leq \varepsilon, \quad \forall r < \bar{r}. \tag{7.3}$$

Let $\delta := \delta(2\varepsilon, D, N)$, with $D = 4$, be the constant given by Theorem 3.8 and fix two radii $r, R \in (0, \bar{r})$ such that $R < \delta\sqrt{N/K^-}$ and $r < \delta R$. Consider now a sequence $y_n \in \mathrm{X}_n$ such that $y_n \to y_0$. From the convergence of the measures $\mathfrak{m}_n$ to $\mathfrak{m}_\infty$ we have that $\theta_{N,r}(y_n) \to \theta_{N,r}(y_0)$ and $\theta_{N,R}(y_n) \to \theta_{N,R}(y_0)$. In particular by (7.3) there exists $\bar{n} = \bar{n}(r, R, \varepsilon)$ such that

$$|\theta_{N,R}(y_n) - \theta_N(y_0)|, |\theta_{N,r}(y_n) - \theta_N(y_0)| \leq 2\varepsilon, \qquad \forall n \geq \bar{n}. \tag{7.4}$$

From the initial choice of ε this also implies that $\theta_{N,r}(y_n)/\theta_{N,R}(y_n) \leq 4$ for every $n \geq \bar{n}$. We are in position to apply Theorem 3.8 and get that for every $n \geq \bar{n}$

$$\|f\|_{L^{p^*}(\mathfrak{m}_n)} \leq \frac{(1+2\varepsilon)Eucl(N, p)}{(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}} \||Df|\|_{L^p(\mathfrak{m}_n)}, \qquad \forall f \in \mathrm{LIP}_c(B_r(y_n)). \tag{7.5}$$

Choose $\varphi \in \mathrm{LIP}(\mathrm{Z})$ such that $\varphi = 1$ in $B^\mathrm{Z}_{r/8}(y_0)$, $\mathrm{supp}(\varphi) \subset B^\mathrm{Z}_{r/4}(y_0)$ and $0 \leq \varphi \leq 1$. From the assumptions, we have that $\int \varphi |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n \to 1$, in particular up to increasing $\bar{n}$ it holds that $\int \varphi |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n \geq 1 - \varepsilon$ for all $n \geq \bar{n}$. Moreover, again up to increasing $\bar{n}$, we have that $d_\mathrm{Z}(y_n, y_0) \leq r/4$ for all $n \geq \bar{n}$, therefore

$$1 - \varepsilon \leq \int_{B_{r/2}(y_n)} |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n, \qquad \forall n \geq \bar{n}. \tag{7.6}$$

For every n we choose a cut-off function $\varphi_n \in \mathrm{LIP}(\mathrm{X}_n)$ such that $\varphi_n = 1$ in $B_{r/2}(y_n)$, $0 \le \varphi_n \le 1$, $\mathrm{supp}(\varphi_n) \subset \mathrm{LIP}_c(B_r(y_n))$ and $\mathrm{Lip}(\varphi_n) \le 2/r$. Plugging the function $u_n\varphi_n \in \mathrm{LIP}_c(B_r(y_n))$ in (7.5) and using (7.6) we obtain

$$(1-\varepsilon)^{\frac{1}{p^*}} \le \|u_n\varphi_n\|_{L^{p^*}(\mathfrak{m}_n)} \le \frac{(1+2\varepsilon)Eucl(N,p)}{(\theta_N(y_0)-2\varepsilon)^{\frac{1}{N}}}\big(\||Du_n|\|_{L^p(\mathfrak{m}_n)} + \tfrac{2}{r}\|u_n\|_{L^p(\mathfrak{m}_n)}\big). \tag{7.7}$$

Moreover recalling that $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$ and the assumption (7.1), from (7.7) we reach

$$\begin{aligned}&(1-\varepsilon)^{\frac{1}{p^*}}\big(A_n^{1/p}\||Du_n|\|_{L^p(\mathfrak{m}_n)} - B_n\|u_n\|^p_{L^s(\mathfrak{m}_n)}\big)\\ &\quad\le \frac{(1+2\varepsilon)Eucl(N,p)}{(\theta_N(y_0)-2\varepsilon)^{\frac{1}{N}}}\big(\||Du_n|\|_{L^p(\mathfrak{m}_n)} + \tfrac{2}{r}\|u_n\|_{L^p(\mathfrak{m}_n)}\big).\end{aligned}$$

We also observe that from the assumption $\|u_n\|_{L^p(\mathfrak{m}_n)} \to 0$ and the fact that $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$, we have by $(viii)$ in Proposition 2.18 that $\|u_n\|_{L^s(\mathfrak{m}_n)} \to 0$. Finally by (7.7) and the assumption $\|u_n\|_{L^p(\mathfrak{m}_n)} \to 0$ it holds that $\underline{\lim}_n \||Du_n|\|_{L^p(\mathfrak{m}_n)} > 0$. In particular for n big enough we can divide by $\||Du_n|\|_{L^p(\mathfrak{m}_n)}$ the above inequality and letting $n \to +\infty$ we get

$$\overline{\lim_n}\, A_n^{1/p} \le \frac{(1+2\varepsilon)Eucl(N,p)}{(1-\varepsilon)^{1/p^*}(\theta_N(y_0)-2\varepsilon)^{\frac{1}{N}}}.$$

From the arbitrariness of ε, the conclusion follows.

Case 2: $\theta_N(y_0) = \infty$.

The argument is similar to Case 1, but we will use Proposition 3.12 instead of Theorem 3.8. Let $M > 0$ be arbitrary. There exists $r \le 1$ such that $\theta_{N,r}(y_0) \ge 2M$. As above we choose a sequence $y_n \to y_0$. For n big enough we have that

$$\theta_{N,r}(y_n) \ge M. \tag{7.8}$$

Applying Proposition 3.12, from (7.8) we get that for every n big enough

$$\|f\|^p_{L^{p^*}(B_r(y_n))} \le \frac{C_{K,N,p}}{M^{\frac{p}{N}}}\||Df|\|^p_{L^p(B_r(y_n))} + \frac{C_{p,N}\|f\|^p_{L^p(B_r(y_n))}}{r^{p/N}M^{\frac{p}{N}}}, \qquad \forall f \in \mathrm{LIP}(\mathrm{X}_n). \tag{7.9}$$

Observing that (7.6) is still satisfied with $\varepsilon = 1/M$ and n big enough, we can repeat the above argument, using (7.1) and plugging $\varphi_n u_n$ in (7.9), where φ_n is as above. This leads us to

$$\overline{\lim_n}\, A_n^{1/p} \le \frac{C_{K,N,p}}{(1-1/M)^{1/p^*}M^{\frac{1}{N}}},$$

which from the arbitrariness M implies the conclusion. □

7.2 Continuity of A^{opt} under mGH-convergence

In Lemma 4.1, we proved that Sobolev embeddings are stable with respect to pmGH-convergence. A much more involved task it to prove that *optimal* constants are also continuous: indeed, if $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$, in general Lemma 4.1 ensures only that $A_q^{\mathrm{opt}}(\mathrm{X}_\infty) \le \underline{\lim}_n A_q^{\mathrm{opt}}(\mathrm{X}_n)$. With the concentration compactness tools developed in Sect. 6.1, the "quantitative-linearization" result in Lemma 6.7 and the technical tool developed in the previous section we can now prove the mGH-continuity of $A_q^{\mathrm{opt}}(\mathrm{X}_n)$ as stated in Theorem 1.12, that we restate here for convenience of the reader.

Theorem 7.2 (Continuity of A_q^{opt} under mGH-convergence) *Let* $(\mathrm{X}_n, d_n, \mathfrak{m}_n)$ *be a sequence,* $n \in \mathbb{N} \cup \{\infty\}$*, of compact* $\mathrm{RCD}(K, N)$*-spaces with* $\mathfrak{m}_n(\mathrm{X}_n) = 1$ *and for some* $K \in \mathbb{R}$*,* $N \in (2, \infty)$ *so that* $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$*. Then,* $A_q^{\mathrm{opt}}(\mathrm{X}_\infty) = \lim_n A_q^{\mathrm{opt}}(\mathrm{X}_n)$*, for every* $q \in (2, 2^*]$.

Proof By definition of $A_q^{\mathrm{opt}}(\mathrm{X}_n)$, there exists sequence of non-negative and non-constant functions $u_n \in \mathrm{LIP}(\mathrm{X}_n)$ satisfying

$$\|u_n\|^2_{L^q(\mathfrak{m}_n)} \geq A_n \||Du_n|\|^2_{L^2(\mathfrak{m}_n)} + \|u_n\|^2_{L^2(\mathfrak{m}_n)}, \tag{7.10}$$

having set $A_n := A_q^{\mathrm{opt}}(\mathrm{X}_n) - \frac{1}{n}$. By scaling invariance, it is not restrictive to suppose $\|u_n\|_{L^q(\mathfrak{m}_n)} = 1$ for every $n \in \mathbb{N}$. Observe that thanks to Lemma 4.1 we already have that $0 < A_q^{\mathrm{opt}}(\mathrm{X}_\infty) \leq \underline{\lim}_n A_q^{\mathrm{opt}}(\mathrm{X}_n)$, hence we only need to show that $A_q^{\mathrm{opt}}(\mathrm{X}) \geq \overline{\lim}_n A_q^{\mathrm{opt}}(\mathrm{X}_n)$. To this aim, we distinguish two cases.

Subcritical: $q < 2^*$. It is clear that A_n is uniformly bounded from below whence the sequence u_n has uniformly bounded $W^{1,2}$ norms. Then, by Proposition 2.19 and the Γ-$\underline{\lim}$ inequality of the Ch_2 energy, there exists a (not relabeled) subsequence L^2-strongly converging to some $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$. Moreover, since u_n are bounded in L^{2^*}, they also converge to u_∞ in L^q-strong and in particular $\|u_\infty\|^2_{L^q(\mathfrak{m}_\infty)} = 1$. Suppose first that the function u_∞ is not constant, then we get

$$\begin{aligned}
1 = \|u_\infty\|^2_{L^q(\mathfrak{m}_\infty)} &\geq \overline{\lim_{n\to\infty}}\, A_n \||Du_n|\|^2_{L^2(\mathfrak{m}_n)} + \|u_n\|^2_{L^2(\mathfrak{m}_n)} \\
(2.18) + L^2\text{-strong} \qquad &\geq \overline{\lim_{n\to\infty}}\, A_q^{\mathrm{opt}}(\mathrm{X}_n) \||Du_\infty|\|^2_{L^2(\mathfrak{m}_\infty)} + \|u_\infty\|^2_{L^2(\mathfrak{m}_\infty)}.
\end{aligned}$$

Since u_∞ is not constant this in turn yields $\overline{\lim}_n A_q^{\mathrm{opt}}(\mathrm{X}_n) \leq A_q^{\mathrm{opt}}(\mathrm{X}_\infty)$ which is what we wanted.

Suppose now that u_∞ is constant. Then, necessarily $u_\infty = 1$. Define now $f_n := 1 - u_n$ and observe that $\|f_n\|_{W^{1,2}(\mathrm{X}_n)} \to 0$, which follows from (7.10) and the fact that $\|u_n\|_{L^2(\mathfrak{m}_n)} \to 1$. Moreover from (2.20) we have that $\lambda^{1,2}(\mathrm{X}_n)$ are uniformly bounded below away from zero. Therefore we can apply Lemma 6.7 to deduce (recall (6.10) for the def. of $\mathcal{Q}_q^{\mathrm{X}}$)

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, A_q^{\mathrm{opt}}(\mathrm{X}_n) &= \overline{\lim_{n\to\infty}}\, \mathcal{Q}_q^{\mathrm{X}_n}(u_n) \\
&= \overline{\lim_{n\to\infty}}\, \frac{(q-2)\int \left|f_n - \int f_n \, \mathrm{d}\mathfrak{m}_n\right|^2 \mathrm{d}\mathfrak{m}_n}{\int |Df_n|^2 \, \mathrm{d}\mathfrak{m}_n} \\
&\leq \overline{\lim_{n\to\infty}}\, \frac{(q-2)}{\lambda^{1,2}(\mathrm{X}_n)} = \frac{(q-2)}{\lambda^{1,2}(\mathrm{X}_\infty)},
\end{aligned} \tag{7.11}$$

having used, in the last inequality, the continuity of the 2-spectral gap (2.20). This combined with (5.4) gives that $\overline{\lim}_n A_q^{\mathrm{opt}}(\mathrm{X}_n) \leq A_q^{\mathrm{opt}}(\mathrm{X}_\infty)$.

Critical exponent: $q = 2^*$. Observe that we are now in position to invoke Theorem 6.1 and, up to a further not relabeled subsequence, we just need to handle one of the two different situations I),II) occurring in Theorem 6.1. If the case I) occurs, we argue exactly as in the Subcritical: $q < 2^*$ case, to conclude that $\overline{\lim}_n A_q^{\mathrm{opt}}(\mathrm{X}_n) \leq A_q^{\mathrm{opt}}(\mathrm{X}_\infty)$. Hence we are left with situation II), where the sequence u_n develops a concentration point $y_0 \in \mathrm{X}_\infty$. Recalling

Lemma 7.1, either $\theta_N(y_0) = \infty$ and $\overline{\lim}_n A^{\text{opt}}_{2^*}(\mathrm{X}_n) = 0$ or $\theta_N(y_0) < \infty$. The first situation cannot happen, since $A^{\text{opt}}_{2^*}(\mathrm{X}_\infty) > 0$. In the second one rearranging in (7.2) we have

$$\overline{\lim_{n\to\infty}}\, A^{\text{opt}}_{2^*}(\mathrm{X}_n) \overset{(7.2)}{\leq} \frac{Eucl(N,2)^2}{\theta_N(y_0)^{2/N}} \overset{(1.7)}{\leq} \alpha_2(\mathrm{X}_\infty) \leq A^{\text{opt}}_{2^*}(\mathrm{X}_\infty).$$

□

7.3 Proof of the almost-rigidity

Combining the rigidity result for A^{opt}_q with the continuity result proved in the previous part we can now prove the almost-rigidity result for A^{opt}_q.

Proof of Theorem 1.10 We argue by contradiction, and suppose that there exists $\varepsilon > 0$, $q \in (2, 2^*]$ and a sequence $(\mathrm{X}_n, d_n, \mathfrak{m}_n)$ of $\mathrm{RCD}(N-1, N)$-spaces with $\mathfrak{m}_n(\mathrm{X}_n) = 1$ so that

$$d_{mGH}((\mathrm{X}_n, d_n, \mathfrak{m}_n), (\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})) > \varepsilon, \tag{7.12}$$

for every spherical suspension $(\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ and $\lim_n A^{\text{opt}}_q(\mathrm{X}_n) = \frac{q-2}{N}$. Theorem 2.16 (recall that $\mathfrak{m}_n(\mathrm{X}_n) = 1$) ensures that up to passing to a non-relabeled subsequence we have $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$, for some $\mathrm{RCD}(N-1, N)$-space $(\mathrm{X}_\infty, d_\infty, \mathfrak{m}_\infty)$ with $\mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$. Hence (7.12) implies

$$d_{mGH}((\mathrm{X}_\infty, d_\infty, \mathfrak{m}_\infty), (\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})) \geq \varepsilon, \tag{7.13}$$

for every spherical suspension $(\mathrm{Y}, d_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$. Finally, by Theorem 1.12 we deduce

$$A^{\text{opt}}_q(\mathrm{X}_\infty) = \lim_n A^{\text{opt}}_q(\mathrm{X}_n) = \frac{q-2}{N}.$$

Therefore, by invoking the rigidity Theorem 1.9, we get that $(\mathrm{X}_\infty, d_\infty, \mathfrak{m}_\infty)$ is isomorphic to a spherical suspension. This contradicts (7.13) and concludes the proof. □

Remark 7.3 The results of Theorem 1.10 (and therefore of Theorem 1.9) extend directly to the class of $\mathrm{RCD}(K, N)$ spaces for some $K > 0$ and $N \geq 2$ with normalized volume. Consider an $\mathrm{RCD}(K, N)$ space $(\mathrm{X}, d, \mathfrak{m})$ and define $(\mathrm{X}', d', \mathfrak{m}') := (\mathrm{X}, \sqrt{\frac{K}{N-1}}d, \mathfrak{m})$ which is $\mathrm{RCD}(N-1, N)$. Then, since $A^{\text{opt}}_q(\mathrm{X}') = \frac{K}{N-1} A^{\text{opt}}_q(\mathrm{X})$, it is straightforward to set $\delta = \delta(K, N, \varepsilon, q) := \frac{N-1}{K}\delta(N, \varepsilon, q)$ and extend the aforementioned results also for arbitrary $K > 0$. □

8 Application: The Yamabe equation on RCD(K, N) spaces

In this section we apply Theorem 1.4 and the concentration compactness results of Sect. 6.1 to study the Yamabe equation to the $\mathrm{RCD}(K, N)$ setting. In particular, we prove an existence result for the Yamabe equation and continuity of the generalized Yamabe constants under mGH-convergence, extending and improving some of the results proved in [64] in the case of Ricci limits. For results concerning the Yamabe problem and the Yamabe constant in non-smooth spaces see also [1–3, 83, 83].

We recall that the Yamabe problem [99] asks if a compact Riemannian manifold admits a conformal metric with constant scalar curvature. This has been completely solved and shown to be true after the works of Trudinger, Aubin and Schoen [19, 89, 95]. We also refer to [77] for an introduction to this problem and for a complete and self-contained proof of this result.

The Yamabe problem turns out to be linked to the so-called Yamabe equation:

$$-\Delta u + \mathrm{S}\, u = \lambda u^{2^*-1}, \qquad \lambda \in \mathbb{R},\ \mathrm{S} \in L^\infty(M), \tag{8.1}$$

where $2^* = \frac{2n}{n-2}$. Indeed solving the Yamabe problem is equivalent to find a non-negative and non-zero solution to (8.1) for some $\lambda \in \mathbb{R}$ and with $\mathrm{S} = \mathrm{Scal}$, the scalar curvature of M. In this direction, it is relevant to see that the Yamabe equation is the Euler–Lagrange equation of the following functional:

$$Q(u) := \frac{\int |Du|^2 + \mathrm{S}|u|^2 \,\mathrm{dVol}}{\|u\|^2_{L^{2^*}}}, \qquad u \in W^{1,2}(M) \setminus \{0\},$$

where Vol is the volume measure of M. One then defines the Yamabe constant as the infimum of the above functional:

$$\lambda_{\mathrm{S}}(M) := \inf_{u \in W^{1,2}(M)\setminus\{0\}} Q(u).$$

A crucial step in the solution of the Yamabe problem is:

Theorem 8.1 ([19, 95, 99]) *Let M be a compact n-dimensional Riemannian manifold satisfying $\lambda_{\mathrm{S}}(M) < Eucl(n,2)^{-2}$. Then there is a non-zero solution to (8.1) with $\lambda = \lambda_{\mathrm{S}}(M)$.*

Recall that $Eucl(n,2)$ denotes the optimal constant in the sharp Euclidean Sobolev inequality (1.1). It has also been proven by Aubin [20] (see also [77]) that

$$\lambda_{\mathrm{S}}(M) \le Eucl(n,2)^{-2} \tag{8.2}$$

always holds.

The relevant point for our discussion is that Theorem 8.1 turns out to be linked to the notion of optimal Sobolev constant $\alpha_2(M)$, in particular it is actually a corollary of the fact that $\alpha_2(M) = Eucl(n,2)^2$ (recall (1.2)). Since we generalized this last result to setting of compact $\mathrm{RCD}(K,N)$-spaces (see Theorem 1.4), it is natural to ask if an analogue of Theorem 8.1 holds also in this singular framework. We will positively address this in this part of the note.

Capacity and quasi continuous functions

In the next section we will use the notions of capacity and quasi continuous functions. We briefly recall here the needed definitions and properties.

Given a metric measure space $(\mathrm{X}, d, \mathfrak{m})$, the *capacity* of a set $E \subset \mathrm{X}$ is defined as

$$\mathrm{Cap}(E) := \inf\{\|f\|^2_{W^{1,2}(\mathrm{X})} : f \in W^{1,2}(\mathrm{X}),\ \ f \ge 1\ \mathfrak{m}\text{-a.e. in a neighborhood of } E\}. \tag{8.3}$$

It turns out (see, e.g., [43,Proposition 1.7]) that Cap is a submodular outer measure on X and satisfies $\mathfrak{m}(E) \le \mathrm{Cap}(E)$ for every Borel set $E \subset \mathrm{X}$.

A function $f : \mathrm{X} \to \mathbb{R}$ is said to be *quasi-continuous* if for every $\varepsilon > 0$ there exists a set $E \subset \mathrm{X}$ such that $\mathrm{Cap}(E) < \varepsilon$ and $f_{|\mathrm{X}\setminus E}$ is continuous. We denote by $\mathcal{QC}(\mathrm{X})$ the set of all equivalence classes-up to Cap-a.e. equality-of quasi-continuous functions.

In [43] it has been proven that, in situations where continuous functions are dense in $W^{1,2}(\mathrm{X})$, there exists a unique map

$$QCR : W^{1,2}(\mathrm{X}) \to L^0(\mathrm{Cap})$$

that is linear and such that $QCR(f)$ is (the Cap-a.e. equivalence class of) a function which is *quasi continuous* and coincides $\mathfrak{m}$-a.e. with f. Recall that when X is reflexive, then Lipschitz functions are dense in $W^{1,2}(\mathrm{X})$ (see, e.g., [5,Proposition 7.6]), hence the map QCR is available.

We conclude with the following convergence result contained in [43]:

$$f_n \to f \text{ strongly in } W^{1,2}(\mathrm{X}) \implies \text{ up to subsequence } QCR(f_n) \to QCR(f) \text{ Cap-a.e..} \tag{8.4}$$

8.1 Existence of solutions to the Yamabe equation on compact RCD spaces

We start by clarifying in which sense (8.1) is intended and, to this aim, we fix $(\mathrm{X}, d, \mathfrak{m})$ a compact $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in (2, \infty)$ with $\mathfrak{m}(\mathrm{X}) = 1$. We will also denote by 2^* the Sobolev-exponent defined as $2^* := 2N/(N-2)$. We fix a radon measure S in X so that, for some $p > N/2$, it satisfies

$$\mathrm{S} \geq g\mathfrak{m}, \quad g \in L^p(\mathfrak{m}) \quad \text{and} \quad \mathrm{S} \ll \mathrm{Cap}, \tag{8.5}$$

where Cap denotes the capacity of X as defined above. We also denote by $|\mathrm{S}|$ the total variation of S which for instance can be characterized by the formula $\mathrm{S} = \mathrm{S}^+ + \mathrm{S}^-$, being $\mathrm{S}^\pm$ the Hahn's decomposition of a general signed σ-additive measure. The reason for this more general choice of S is the fact that on $\mathrm{RCD}(K, N)$ spaces a "scalar curvature" that is bounded is not natural (recall that to solve the Yamabe problem one would like to take $\mathrm{S} = \mathrm{Scal}$). Indeed, requiring only a synthetic lower bound on the Ricci curvature, it is more desirable to impose only lower bounds on S.

Recall that every function $u \in W^{1,2}(\mathrm{X})$ has a well defined and unique quasi continuous representative $QCR(u)$ defined Cap-a.e.. In particular, thanks to (8.5), the object $QCR(u)$ is also defined S or $|\mathrm{S}|$-a.e.. To avoid heavy notation, for any $u \in W^{1,2}(\mathrm{X})$, we shall denote in the sequel by u its quasi-continuous representative without further notice.

The goal is then to discuss positive solutions $u \in D(\mathbf{\Delta}) \cap L^2(|\mathrm{S}|)$ of

$$-\mathbf{\Delta}u = \lambda u^{2^*-1}\mathfrak{m} - u\mathrm{S}, \qquad \lambda \in \mathbb{R}. \tag{8.6}$$

Observe that if $u \in D(\mathbf{\Delta}) \subset W^{1,2}(\mathrm{X})$, by the Sobolev embedding we have that $u \in L^{2^*}(\mathfrak{m})$ and thus, the right hand side of (8.6) is a well defined Radon measure on X. A solution for this equation will be deduced with a variational approach as described above. More precisely we define the functional $Q_\mathrm{S} \colon W^{1,2}(\mathrm{X}) \setminus \{0\} \to \mathbb{R}$ defined as

$$u \mapsto Q_\mathrm{S}(u) := \frac{\int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int |u|^2 \,\mathrm{dS}}{\|u\|^2_{L^{2^*}(\mathfrak{m})}}.$$

Observe that since $\mathrm{S} \geq g\mathfrak{m}$, with $g \in L^p(\mathfrak{m})$, $p > N/2$, the integral $\int |u|^2 \,\mathrm{dS}$ exists, i.e. its value is well defined. We then define

$$\begin{aligned} \lambda_\mathrm{S}(\mathrm{X}) &:= \inf\{Q_\mathrm{S}(u) \ : \ u \in W^{1,2}(\mathrm{X}) \setminus \{0\}\} \\ &= \inf\{Q_\mathrm{S}(u) \ : \ u \in W^{1,2}(\mathrm{X}), \ \|u\|_{L^{2^*}(\mathfrak{m})} = 1\}, \end{aligned} \tag{8.7}$$

and claim that

$$\lambda_{\mathrm{S}}(\mathrm{X}) \in (-\infty, +\infty). \tag{8.8}$$

Indeed, $\lambda_{\mathrm{S}}(\mathrm{X}) < +\infty$ as can be seen considering constant functions. On the other hand for every $u \in W^{1,2}(\mathrm{X})$ with $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$, Hölder inequality yields

$$Q_{\mathrm{S}}(u) \geq -\|g\|_{L^p(\mathfrak{m})}\|u\|_{L^{2^*}(\mathfrak{m})} = -\|g\|_{L^p(\mathfrak{m})}.$$

The ultimate goal of this section is to prove the following:

Theorem 8.2 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$, $N \in (2, \infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$ *and let* S *as in* (8.5). *If*

$$\lambda_{\mathrm{S}}(\mathrm{X}) < \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{Eucl(N, 2)^2}, \tag{8.9}$$

then there exists a non-negative and non-zero $u \in D(\mathbf{\Delta}) \cap L^2(|\mathrm{S}|)$ *which is a minimum for* (8.7) *and satisfies* (8.6).

We start by showing that (8.6) is the Euler–Lagrange equation for the minimization problem (8.7).

Proposition 8.3 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$*-space for some* $K \in \mathbb{R}$, $N \in (2, \infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$ *and let* S *be as in* (8.5). *Suppose* $u \in W^{1,2}(\mathrm{X}) \cap L^2(|\mathrm{S}|)$ *is a minimizer for* (8.7) *satisfying* $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$. *Then*

$$\int \langle \nabla u, \nabla v \rangle \, \mathrm{d}\mathfrak{m} = -\int uv \, \mathrm{dS} + \lambda_{\mathrm{S}}(\mathrm{X}) \int u^{2^*-1} v \, \mathrm{d}\mathfrak{m}, \qquad \forall v \in \mathrm{LIP}(\mathrm{X}). \tag{8.10}$$

Proof We consider for every $\varepsilon \in (-1, 1)$ and $v \in \mathrm{LIP}(\mathrm{X})$, the function $u^\varepsilon := \|u + \varepsilon v\|^{-1}_{L^{2^*}(\mathfrak{m})}(u + \varepsilon v)$, whenever $\|u + \varepsilon v\|_{L^{2^*}(\mathfrak{m})}$ is not zero. It can be seen that for a fixed v then u^ε is well defined at least for ε close to zero. Indeed, the fact that $\int |u|^{2^*}, \mathrm{d}\mathfrak{m} = 1$ grants that $\|u + \varepsilon v\|_{L^{2^*}(\mathfrak{m})} \to 1$ as $\varepsilon \to 0$ (see below) and in particular $\|u + \varepsilon v\|_{L^{2^*}(\mathfrak{m})}$ does not vanish for $|\varepsilon|$ small enough. By minimality we have (recall also (2.7))

$$0 \leq \lim_{\varepsilon \downarrow 0} \frac{Q_{\mathrm{S}}(u^\varepsilon) - Q_{\mathrm{S}}(u)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(\frac{1}{I_\varepsilon^2} - 1\right)\lambda_{\mathrm{S}}(\mathrm{X}) + \frac{2}{I_\varepsilon^2}\int \langle \nabla u, \nabla v \rangle \, \mathrm{d}\mathfrak{m} + \int uv \, \mathrm{dS},$$

where $I_\varepsilon := \|u + \varepsilon v\|_{L^{2^*}(\mathfrak{m})}$. Furthermore, from the elementary estimate $\big||a + \varepsilon b|^q - |a|^q\big| \leq q|\varepsilon b|\big||a + \varepsilon b|^{q-1} + |a|^{q-1}\big|$, with $q = 2^*$, and the fact that $u, v \in L^{2^*}(\mathfrak{m})$, we have that $\int |u + \varepsilon v|^q \, \mathfrak{m} \to 1$ as $\varepsilon \to 0$. Thanks to the same estimates, the dominated convergence theorem grants that

$$\lim_{\varepsilon \downarrow 0} \frac{1 - I_\varepsilon^2}{\varepsilon} = \frac{2}{2^*} \lim_{\varepsilon \downarrow 0} \int \frac{|u|^{2^*} - |u + \varepsilon v|^{2^*}}{\varepsilon} \, \mathrm{d}\mathfrak{m} = -2 \int u^{2^*-1} v \, \mathrm{d}\mathfrak{m}.$$

Arguing analogously considering $\varepsilon \uparrow 0$ gives (8.10). □

We can now prove Theorem 8.2 which, thanks to the previous proposition, amounts to the existence of a minimizer for (8.7). We will do so using the concentration-compactness tools developed in Sect. 6.1, here employed with a fixed space X.

Proof (Proof of Theorem 8.2) Let $u_n \in W^{1,2}(\mathrm{X})$ be such that $Q_{\mathrm{S}}(u_n) \to \lambda_{\mathrm{S}}(\mathrm{X})$ and $\|u_n\|_{L^{2^*}(\mathfrak{m})} = 1$. We claim that u_n are uniformly bounded in $W^{1,2}(\mathrm{X})$. Indeed, this can be seen from the estimate

$$\int |Du_n|^2 + |u_n|^2 \,\mathrm{d}\mathfrak{m} \le \int |Du_n|^2 \,\mathrm{d}\mathfrak{m} + \int |u_n|^2 \,\mathrm{dS}$$
$$+(1 + \|g\|_{L^p(\mathfrak{m})})\|u_n\|_{L^{2^*}(\mathfrak{m})} = 1 + Q_{\mathrm{S}}(u_n) + \|g\|_{L^p(\mathfrak{m})},$$

obtained combining the Hölder inequality with (8.5). Hence, by compactness (see Proposition 2.19), up to a not relabeled subsequence, we have $u_n \to u$ in $L^2(\mathfrak{m})$ for some $u \in W^{1,2}(\mathrm{X})$. Observe that, since $u \in W^{1,2}(\mathrm{X})$, u admits a quasi-continuous representative (still denoted by u) and thus thanks to (8.5) it makes sense to integrate u^2 against $|\mathrm{S}|$. We claim that $u \in L^2(|\mathrm{S}|)$ and

$$\int u^2 \,\mathrm{dS} \le \varliminf_n \int u_n^2 \,\mathrm{dS}. \tag{8.11}$$

Observe first that, by (8.5), we have $\mathrm{S}^- \le |g|\mathfrak{m}$. In particular by the Hölder inequality, denoted by p' the conjugate exponent to p, $\int u^2 \mathrm{dS}^- \le \|g\|_{L^p(\mathfrak{m})}\|u\|^2_{L^{2p'}} < +\infty$, since $u \in L^{2^*}(\mathfrak{m})$ by the Sobolev embedding, hence $u \in L^2(\mathrm{S}^-)$. Moreover, again by the Hölder inequality, since $u_n \to u$ in $L^2(\mathfrak{m})$, we get that and $u_n \to u$ also in $L^2(\mathrm{S}^-)$. To prove (8.11) it remains to prove that $\int u^2 \,\mathrm{dS}^+ \le \varliminf_n \int u_n^2 \,\mathrm{dS}^+$. Observe first that up to passing to a further non-relabeled subsequence we can assume that the right hand side is actually a limit. From Mazur's lemma there exists a sequence $(N_n) \subset \mathbb{N}$ and numbers $(\alpha_{n,i})_{i=n}^{N_n} \subset [0, 1]$ such that $\sum_{i=n}^{N_n} \alpha_{n_i} = 1$ for every $n \in N$ and $v_n := \sum_{i=n}^{N_n} \alpha_{n_i} u_i$ converges to u strongly in $W^{1,2}(\mathrm{X})$. In particular from (8.4) up to a subsequence $v_n \to u$ also Cap-a.e. and thus, since $\mathrm{S}^+ \ll \mathrm{Cap}$ (recall (8.5)), also S^+-a.e.. Therefore, from Fatou's Lemma and the convexity of the L^2-norm we have

$$\|u\|_{L^2(\mathrm{S}^+)} \le \varliminf_n \|v_n\|_{L^2(\mathrm{S}^+)} \le \sum_{i=n}^{N_n} \alpha_{n_i} \|u_i\|_{L^2(\mathrm{S})} \le \lim_n \|u_n\|_{L^2(\mathrm{S}^+)},$$

since we are assuming that the last limit exists. This proves the claim.

We now distinguish two cases:

Case 1. $\lambda_{\mathrm{S}}(\mathrm{X}) < 0$. By lower semicontinuity of the Cheeger-energy and (8.11) we have

$$0 > \lambda_{\mathrm{S}}(\mathrm{X}) = \lim_n Q_{\mathrm{S}}(u_n) \ge \int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int u^2 \mathrm{dS}.$$

In particular u is not identically zero and by the lower semicontinuity of the $L^{2^*}(\mathfrak{m})$-norm we have $0 < \|u\|_{L^{2^*}(\mathfrak{m})} \le 1$. Moreover, from the above we have that $\int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int u^2 \mathrm{dS}$ is negative, hence

$$\lambda_{\mathrm{S}}(\mathrm{X}) \ge \|u\|^{-2}_{L^{2^*}(\mathfrak{m})} \left(\int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int u^2 \mathrm{dS} \right) = Q_{\mathrm{S}}(\|u\|^{-1}_{L^{2^*}(\mathfrak{m})} u).$$

Therefore $\|u\|^{-1}_{L^{2^*}(\mathfrak{m})} u$ is a minimizer for $Q_{\mathrm{S}}(u)$.

Case 2. $\lambda_{\mathrm{S}}(\mathrm{X}) \ge 0$. Recall that the sequence (u_n) is uniformly bounded both in $L^{2^*}(\mathfrak{m})$ and in $W^{1,2}(\mathrm{X})$. Therefore since X is compact, again up to a subsequence, $|Du_n|^2\mathfrak{m} \rightharpoonup \mu$ and $|u_n|^{2^*} \rightharpoonup \nu$ for some $\mu \in \mathscr{M}_b^+(\mathrm{X})$ and $\nu \in \mathscr{P}(\mathrm{X})$ in duality with $C(\mathrm{X})$. By assumption there exists $\varepsilon > 0$ such that $\lambda_{\mathrm{S}}(\mathrm{X}) < \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{Eucl(N,2)^2+\varepsilon} =: \lambda_\epsilon$. We fix one of such $\varepsilon > 0$ and define

$A_\varepsilon = \lambda_\varepsilon^{-1}$. From Theorem 1.4 there exists a constant $B_\varepsilon > 0$ so that

$$\|u\|^2_{L^{2^*}(\mathfrak{m})} \le A_\varepsilon \||Du|\|^2_{L^2(\mathfrak{m})} + B_\varepsilon \|u\|^2_{L^2(\mathfrak{m})}, \quad \forall u \in W^{1,2}(\mathrm{X}).$$

Hence we are in position to apply Lemma 6.6 (with fixed space X) to deduce that there exists a countable set of indices J, points $(x_j)_{j\in J} \subset \mathrm{X}$ and weights $(\mu_j) \subset \mathbb{R}^+$, $(\nu_j) \subset \mathbb{R}^+$ such that $\mu_j \ge \lambda_\varepsilon \nu_j^{2/2^*}$ for every $j \in J$ and

$$\nu = |u|^{2^*}\mathfrak{m} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \ge |Du|^2\mathfrak{m} + \sum_{j\in J} \mu_j \delta_{x_j}.$$

We now observe that

$$\int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int u^2 \,\mathrm{dS} \ge \|u\|^2_{L^{2^*}(\mathfrak{m})} \lambda_{\mathrm{S}}(\mathrm{X}). \tag{8.12}$$

Indeed, this is obvious if $u = 0$ $\mathfrak{m}$-a.e., hence we assume that $u \ne 0$ $\mathfrak{m}$-a.e.. In this case, (8.12) follows noticing that $\lambda_{\mathrm{S}}(\mathrm{X}) \le Q_{\mathrm{S}}(u\|u\|^{-1}_{L^{2^*}(\mathfrak{m})}) = \|u\|^{-2}_{L^{2^*}(\mathfrak{m})} \big(\int |Du|^2 \mathrm{d}\mathfrak{m} + \int u^2 \,\mathrm{dS}\big)$. Therefore using again (8.11) we have

$$\begin{aligned}
\lambda_{\mathrm{S}}(\mathrm{X}) = \lim_n Q_{\mathrm{S}}(u_n) &\ge \mu(\mathrm{X}) + \int u^2 \,\mathrm{dS} \ge \int |Du|^2 \mathrm{d}\mathfrak{m} + \lambda_\varepsilon \sum_{j\in J} \nu_j^{2/2^*} + \int u^2 \,\mathrm{dS} \\
&\overset{(8.12)}{\ge} \|u\|^2_{L^{2^*}(\mathfrak{m})} \lambda_{\mathrm{S}}(\mathrm{X}) + \lambda_\varepsilon \sum_{j\in J} \nu_j^{2/2^*} \ge \lambda_{\mathrm{S}}(\mathrm{X}) \left(\|u\|^2_{L^{2^*}(\mathfrak{m})} + \sum_{j\in J} \nu_j^{2/2^*} \right) \\
&\ge \lambda_{\mathrm{S}}(\mathrm{X}) \Big(\int |u|^{2^*} \,\mathrm{d}\mathfrak{m} + \sum_{j\in J} \nu_j \Big)^{2/2^*} = \lambda_{\mathrm{S}}(\mathrm{X})\nu(\mathrm{X}) = \lambda_{\mathrm{S}}(\mathrm{X}),
\end{aligned}$$

where in the last line, we used the concavity of the function $t^{2/2^*}$, the fact that $\nu \in \mathscr{P}(\mathrm{X})$ and finally that $\lambda_{\mathrm{S}}(\mathrm{X}) \ge 0$. Hence all the inequalities are equalities and in particular from the strict concavity of $t^{2/2^*}$ we deduce that either $\int |u|^{2^*} \mathrm{d}\mathfrak{m} = 1$ or $u = 0$ (and the numbers ν_j are all zero except one that is equal to one). In the second case, plugging $u = 0$ in the above chain of inequalities, we infer that $\lambda_\varepsilon = \lambda_{\mathrm{S}}(\mathrm{X})$ which is a contradiction. Hence, we must have $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$ and $u_n \to u$ strongly in $L^{2^*}(\mathfrak{m})$ and in particular u is a minimizer for (8.7). This together with Proposition 8.3 concludes the proof. □

We conclude by extending the classical upper bound (8.2) to the setting of $\mathrm{RCD}(K, N)$ spaces. This in particular shows that (8.9) is a reasonable assumption. Unfortunately, at present, we are able to prove this comparison only by adding integrability conditions on S.

Proposition 8.4 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$, $N \in (2, \infty)$ *and let* $\mathrm{S} \in L^p(\mathfrak{m})$*, with* $p > \frac{N}{2}$*. Then*

$$\lambda_{\mathrm{S}}(\mathrm{X}) \le \frac{\min_{\mathrm{X}} \theta_N^{2/N}}{Eucl(N, 2)^2}.$$

Proof The argument is almost the same as for Theorem 4.4. We start noticing that in the case $\min_{\mathrm{X}} \theta_N = +\infty$, evidently there is nothing to prove. We are left then to deal with the case $0 < \min_{\mathrm{X}} \theta_N < +\infty$. Let $x \in \mathrm{X}$ such that $\theta_N(x) = \min_{\mathrm{X}} \theta_N$. Then there exists a sequence $r_i \to 0$ such that the sequence of metric measure spaces $(\mathrm{X}_i, d_i, \mathfrak{m}_i, x_i) := (\mathrm{X}, d/r_i, \mathfrak{m}/r_i^N, x)$ pmGH-converges to an $\mathrm{RCD}(0, N)$ space $(\mathrm{Y}, d_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}}, \boldsymbol{o}_{\mathrm{Y}})$ satisfying $\mathfrak{m}_{\mathrm{Y}}(B_r(\boldsymbol{o}_{\mathrm{Y}})) = \omega_N \theta_N(x) r^N$ for every $r > 0$ (this space is actually a cone by [53]). In particular from Lemma 4.3 for every $\varepsilon > 0$ there exists a non-zero $u \in \mathrm{LIP}_c(\mathrm{Y})$ such that $\frac{\|u\|^2_{L^{2^*}(\mathfrak{m}_\mathrm{Y})}}{\||Du|\|^2_{L^2(\mathfrak{m}_\mathrm{Y})}} \geq \frac{Eucl(N,2)^2-\varepsilon}{\theta_N(x)^{2/N}}$. Then by the Γ-convergences of the 2-Cheeger energies there exists a sequence $u_i \in W^{1,2}(\mathrm{X}_i)$ such that $u_i \to u$ strongly in $W^{1,2}$. Moreover, since u_i are uniformly bounded in $W^{1,2}$ (meaning in $W^{1,2}(\mathrm{X}_i)$), by the Sobolev embedding (recall also the scaling property in (4.1)) we have $\sup_i \|u_i\|_{L^{2^*}(\mathfrak{m}_i)} < +\infty$. In particular from the lower semicontinuity of the L^{2^*}-norm we get

$$\varliminf_i \frac{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}}{\||Du_i|\|^2_{L^2(\mathfrak{m})}} = \varliminf_i \frac{\|u_i\|^2_{L^{2^*}(\mathfrak{m}_i)}}{\||Du_i|_i\|^2_{L^2(\mathfrak{m}_i)}} \geq \frac{\|u\|^2_{L^{2^*}(\mathfrak{m}_\mathrm{Y})}}{\||Du|\|^2_{L^2(\mathfrak{m}_\mathrm{Y})}} \geq \frac{Eucl(N,2)^2-\varepsilon}{\min_\mathrm{X} \theta_N^{2/N}}, \tag{8.13}$$

where $|Du_i|_i$ denotes the weak upper gradient computed in the space X_i.

Denote by $p' := p/(p-1)$ the conjugate exponent of p and observe that by hypothesis $2p' < 2^*$. This and the fact that u_i are bounded in L^{2^*}, by Proposition 2.18 $(viii)$ imply that u_i converges in $L^{2p'}$-strong to u. Finally using the Hölder inequality we can write

$$\begin{aligned}
\varlimsup_i Q_\mathrm{S}(u_i) &\leq \varlimsup_i \frac{\int |Du_i|^2\,\mathrm{d}\mathfrak{m}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}} + \varlimsup_i \frac{\int \mathrm{S}|u_i|^2\,\mathrm{d}\mathfrak{m}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}} \\
&\overset{(8.13)}{\leq} \frac{\min_\mathrm{X}\theta_N^{2/N}}{Eucl(N,2)^2-\varepsilon} + \varlimsup_i \|\mathrm{S}\|_{L^p(\mathfrak{m})} \frac{\left(\int |u_i|^{2p'}\,\mathrm{d}\mathfrak{m}\right)^{1/p'}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}}, \\
&= \frac{\min_\mathrm{X}\theta_N^{2/N}}{Eucl(N,2)^2-\varepsilon} + \varlimsup_i \|\mathrm{S}\|_{L^p(\mathfrak{m})} r_i^{N\left(\frac{1}{p'}-\frac{2}{2^*}\right)} \frac{\|u_i\|^2_{L^{2p'}(\mathfrak{m}_i)}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m}_i)}} = \frac{\min_\mathrm{X}\theta_N^{2/N}}{Eucl(N,2)^2-\varepsilon}.
\end{aligned}$$

where we have used that $1/p' < 2/2^*$, that $\varliminf_i \|u_i\|_{L^{2^*}(\mathfrak{m}_i)} \geq \|u\|_{L^{2^*}(\mathfrak{m}_\mathrm{Y})} > 0$ and as observed above $\|u_i\|_{L^{2p'}(\mathfrak{m}_i)} \to \|u\|_{L^{2p'}(\mathfrak{m}_\mathrm{Y})}$. From the arbitrariness of $\varepsilon > 0$ the proof is now concluded. □

8.2 Continuity of λ_S under mGH-convergence

In [64] it has been proven in the setting of Ricci-limits a result about mGH-continuity of the generalized Yamabe constant, under some additional boundedness assumption on the sequence. In the following result we extend this fact in the setting of RCD-spaces and we remove such extra assumption.

We start proving that λ_S is upper semicontinuous under mGH-convergence.

Lemma 8.5 *Let $(\mathrm{X}_n, d_n, \mathfrak{m}_n)$ be a sequence of compact $\mathrm{RCD}(K,N)$-spaces with $\mathfrak{m}(\mathrm{X}_n) = 1$, $n \in \bar{\mathbb{N}}$, for some $K \in \mathbb{R}$, $N \in (2,\infty)$ and satisfying $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$. Let also $\mathrm{S}_n \in L^p(\mathfrak{m}_n)$ be L^p-weak convergent to S, for some $p > N/2$. Then,*

$$\varlimsup_{n\to\infty} \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \leq \lambda_\mathrm{S}(\mathrm{X}_\infty). \tag{8.14}$$

Proof Fix a non-zero $u \in W^{1,2}(\mathrm{X}_\infty)$. By the Sobolev embedding on X_∞ we know that $u \in L^{2^*}(\mathfrak{m}_\infty)$, therefore by Lemma 6.4 there exists a sequence $u_n \in W^{1,2}(\mathrm{X}_n)$ that converge $W^{1,2}$-strong and L^{2^*}-strong to u. By definition of $\lambda_{\mathrm{S}_n}(\mathrm{X}_n)$, we have

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \leq \int |Du_n|^2\,\mathrm{d}\mathfrak{m}_n + \int \mathrm{S}_n|u_n|^2\,\mathrm{d}\mathfrak{m}_n, \qquad \forall n \in \mathbb{N}.$$

From the assumption that $p > N/2$, we have that its conjugate exponent p' satisfies $2p' < 2^*$, therefore from (vii), $(viii)$ in Proposition 2.18 we have that $|u_n|^2$ $L^{p'}$-strongly converges to u^2. Recalling Proposition 6.2, we get that all the above quantities pass to the limit and thus we reach

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_\infty)} \overline{\lim_{n\to\infty}} \lambda_{S_n}(X_n) \le \int |Du|^2 \, d\mathfrak{m}_\infty + \int S|u|^2 \, d\mathfrak{m}_\infty.$$

By arbitrariness of u, we conclude. □

We shall now come to the main continuity result.

Theorem 8.6 (mGH-continuity of λ_S) *Let $(X_n, d_n, \mathfrak{m}_n)$ be a sequence of compact* $\mathrm{RCD}(K, N)$*-spaces with $\mathfrak{m}(X_n) = 1$, $n \in \bar{\mathbb{N}}$, for some $K \in \mathbb{R}$, $N \in (2, \infty)$ satisfying $X_n \overset{mGH}{\to} X_\infty$. Let also $S_n \in L^p(\mathfrak{m}_n)$ be L^p-weak convergent to $S \in L^p(\mathfrak{m}_\infty)$, for a given for $p > N/2$. Then,*

$$\lim_{n\to\infty} \lambda_{S_n}(X_n) = \lambda_S(X_\infty).$$

Proof In light of Lemma 8.5, we only have to prove that

$$\underline{\lim_{n\to\infty}} \lambda_{S_n}(X_n) \ge \lambda_S(X_\infty).$$

It is not restrictive to assume that the $\underline{\lim}$ is actually a limit. For every $n \in \mathbb{N}$, we take $u_n \in W^{1,2}(X_n)$ non-zero so that $Q_{S_n}(u_n) - \lambda_{S_n}(X_n) \le n^{-1}$. In other words

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)}\big(\lambda_{S_n}(X_n) + \tfrac{1}{n}\big) \ge \int |Du_n|^2 \, d\mathfrak{m}_n + \int S_n|u_n|^2 \, d\mathfrak{m}_n. \tag{8.15}$$

It is also clearly not restrictive to suppose that $u_n \in \mathrm{LIP}_c(X_n)$ are non-negative and such that $\|u_n\|_{L^{2^*}(\mathfrak{m}_n)} \equiv 1$. Hence, arguing as in the proof of Theorem 8.2 (using also (8.14)), we get that u_n is uniformly bounded in $W^{1,2}$. Then, by compactness (see Proposition 2.19), up to a not relabeled subsequence, we have that u_n converge L^2-strong and $W^{1,2}$-weak to some $u_\infty \in W^{1,2}(X_\infty)$. From $\|u_n\|_{L^{2^*}(\mathfrak{m}_n)} \equiv 1$ and the assumption $p > N/2$, Proposition 2.18 implies that u_n^2 converges $L^{p/(p-1)}$-strongly to u_∞^2 and that u_n converges $L^{2p/(p-1)}$-strongly to u_∞. From this point we subdivide the proof in three cases to be handled separately.
CASE 1: $\lim_n \lambda_{S_n}(X_n) < 0$. In this case, by (8.15) we know by lower semicontinuity of the 2-Cheeger energy and Proposition 6.2, we have that

$$0 > \lim_n \lambda_{S_n}(X_n) \ge \int |Du_\infty|^2 \, d\mathfrak{m}_\infty + \int S u_\infty^2 \, d\mathfrak{m}_\infty.$$

In particular, u_∞ is not $\mathfrak{m}_\infty$-a.e. equal to zero and by weak-lower semicontinuity, we have that $0 < \|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)} \le 1$. Therefore

$$\begin{aligned}\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)} \lim_n \lambda_{S_n}(X_n) \ge \lim_n \lambda_{S_n}(X_n) &\ge \int |Du_\infty|^2 \, d\mathfrak{m}_\infty \\ &+ \int S u_\infty^2 \, d\mathfrak{m}_\infty \ge \lambda_S(X_\infty)\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)},\end{aligned}$$

which concludes the proof in this case.
CASE 2: $\lim_n \lambda_{S_n}(X_n) > 0$. Before starting, notice that by using the Hölder inequality, for any $n \in \mathbb{N}$ and any $u \in W^{1,2}(X_n)$ we have by the definition of $\lambda_{S_n}(X_n)$ that

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)} \le \lambda_{S_n}(X_n)^{-1} \int |Du|^2 \, d\mathfrak{m}_n + \lambda_{S_n}(X_n)^{-1}\|S_n\|_{L^p(\mathfrak{m}_n)}\|u\|^2_{L^{2p/p-1}(\mathfrak{m}_n)}. \tag{8.16}$$

Moreover, since all X_n are compact and renormalized, there are $\mu \in \mathscr{M}_b^+(\mathrm{Z})$, $\nu \in \mathscr{P}(\mathrm{Z})$ so that, up to a not relabeled subsequence, $|Du_n|^2\mathfrak{m}_n \rightharpoonup \mu$ and $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$ in duality with $C(\mathrm{Z})$ as n goes to infinity, where $(\mathrm{Z}, d_{\mathrm{Z}})$ is a (compact) space realizing the convergences via extrinsic approach. Since we are assuming that $\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) > 0$, the constant in (8.16) are uniformly bounded (for n big enough) and we are in position to apply Lemma 6.6. In particular we get the existence of an at most countable set J, points $(x_j)_{j\in J} \subset \mathrm{X}_\infty$ and weights $(\mu_j), (\nu_j) \subset \mathbb{R}^+$, so that $\mu_j \geq \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)\nu_j^{2/2^*}$ with $j \in J$ and

$$\nu = |u_\infty|^{2^*}\mathfrak{m} + \sum_{j\in J}\nu_j\delta_{x_j}, \quad \mu \geq |Du_\infty|^2\mathfrak{m} + \sum_{j\in J}\mu_j\delta_{x_j}.$$

Moreover, recalling Proposition 6.2 we have

$$\mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty = \lim_{n\to\infty} Q_{\mathrm{S}_n}(u_n) \overset{(8.15)}{\leq} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n), \tag{8.17}$$

and, arguing as in the proof of (8.12), u_∞ is so that $\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\lambda_{\mathrm{S}}(\mathrm{X}_\infty) \leq \int |Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty + \int \mathrm{S}|u_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty$. Finally, we can perform the chain of estimates

$$\begin{aligned}
\lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n) &\overset{(8.17)}{\geq} \mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty \geq \int |Du_\infty|^2\mathrm{d}\mathfrak{m}_\infty + \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n) \\
&\quad \sum_{j\in J}\nu_j^{2/2^*} + \int \mathrm{S}u_\infty^2\,\mathrm{d}\mathfrak{m}_\infty \\
&\geq \lambda_{\mathrm{S}}(\mathrm{X}_\infty)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\sum_{j\in J}\nu_j^{2/2^*} \\
&\overset{(8.14)}{\geq} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\Big(\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + \sum_{j\in J}\nu_j^{2/2^*}\Big) \\
&\geq \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\Big(\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\Big)^{2/2^*} \geq \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n),
\end{aligned}$$

where in the last line, we used the concavity of $t^{2/2^*}$ and the fact that $\nu \in \mathscr{P}(\mathrm{X})$. In particular, all inequalities must be equalities and by the strict concavity of $t^{2/2^*}$ either $\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)} = 1$ and all $\nu_j = 0$, or $u_\infty = 0$ $\mathfrak{m}_\infty$-a.e. and all the weights are zero except one $\nu_j = 1$. The first situation is the easiest one, as in this case the above inequalities which are actually equalities imply that $\lambda_{\mathrm{S}}(\mathrm{X}_\infty) = \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)$, which is what we wanted. Therefore we suppose that we are in the second case, i.e. that there exists a point $y_0 \in \mathrm{X}_\infty$ so that $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{y_0}$ in duality with $C(\mathrm{Z})$ and that u_n converges in L^2-strong to zero. Moreover, from (8.15) and Hölder inequality we get

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \geq \big(\lambda_{\mathrm{S}_n}(\mathrm{X}_n) + \tfrac{1}{n}\big)^{-1}\Big(\int |Du_n|^2\,\mathrm{d}\mathfrak{m}_n - \|\mathrm{S}_n\|_{L^p(\mathfrak{m}_n)}\|u_n\|^2_{L^{2p/(p-1)}(\mathfrak{m}_n)}\Big), \ \forall n \in \mathbb{N}.$$

We can therefore apply Lemma 7.1 to get that $\theta_N(y_0) \leq Eucl(N,2)^N \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)^{N/2}$. Finally, we can rearrange and invoke Proposition 8.4 to get

$$\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \geq \frac{\theta_N(y_0)^{2/N}}{Eucl(N,2)^2} \geq \lambda_{\mathrm{S}}(\mathrm{X}_\infty).$$

Case 3: $\lim_n \lambda_{S_n}(X_n) = 0$. The argument is the same as in the previous case, only that we replace (8.16) with the Sobolev inequality given in Proposition 5.1:

$$\|u\|^2_{L^q(\mathfrak{m})} \le A(K,N,D)\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall u \in W^{1,2}(X_n), \tag{8.18}$$

where $D > 0$ is constant such that $\mathrm{diam}(X_n) \le D$. Then we can apply exactly as in the previous case Lemma 6.6, except that in this case we obtain $\mu_j \ge A(K,N,D)^{-1}\nu_j^{2/2^*}$ for every $j \in J$. Then the above chain of estimates becomes

$$\begin{aligned} 0 = \lim_{n\to\infty} \lambda_{S_n}(X_n) &\overset{(8.17)}{\ge} \mu(X) + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty \\ &\ge \int |Du_\infty|^2 \mathrm{d}\mathfrak{m}_\infty + A(K,N,D)^{-1} \sum_{j\in J} \nu_j^{2/2^*} + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty \\ &\ge \lambda_S(X_\infty)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} \\ &\overset{(8.14)}{\ge} \lim_{n\to\infty} \lambda_{S_n}(X_n)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} \ge 0. \end{aligned}$$

Therefore we must have that $\nu_j = 0$ for every $j \in J$. This forces $\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} = 1$ giving in turn that $\lambda_S(X_\infty) = 0$. Having examined all the three cases, the proof is now concluded. □

Acknowledgements We wish to thank A. Mondino and D. Semola for their useful comments and suggestions on a preliminary version of this manuscript.

Funding Open Access funding provided by University of Jyväskylä (JYU).

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