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Research Article

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Asymptotic mean-value formulas for solutions of general second-order elliptic equations

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Abstract: We obtain asymptotic mean-value formulas for solutions of second-order elliptic equations. Our approach is very flexible and allows us to consider several families of operators obtained as an infimum, a supremum, or a combination of both infimum and supremum, of linear operators. The families of equations that we consider include well-known operators such as Pucci, Issacs, and k-Hessian operators.

Keywords: mean-value formulas, viscosity solutions, k-Hessian equation, Issacs equation

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1 Introduction

It is well-known that mean-value formulas characterize harmonic functions; in fact, a weaker statement, known as the asymptotic mean-value property, is enough to characterize harmonicity (see [7,18,31]). Furthermore, we have

$$\Delta u(x) = \lim_{\varepsilon \to 0} \frac{2(n+2)}{\varepsilon^2} \left(\fint_{B_\varepsilon(x)} u(y) \mathrm{d}y - u(x) \right),$$

from which we conclude that, when f is continuous, a function u satisfies the classical Poisson equation $\Delta u(x) = f(x)$ in Ω if and only if

$$u(x) = \fint_{B_\varepsilon(x)} u(y) \mathrm{d}y - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \tag{1.1}$$

for each $x \in \Omega$. We use the standard notation $o(\varepsilon^2)$ to denote a quantity such that $o(\varepsilon^2)/\varepsilon^2 \to 0$ as $\varepsilon \to 0$. Analogous results hold for sub- and supersolutions replacing equalities by appropriate inequalities.

The classic mean-value property can be seen as a nonlocal formulation of the local Laplace and Poisson equations. It has deep connections with other fundamental properties such as the maximum principle, which allows proving existence results by Perron's method or symmetry of solutions by the moving-planes method.

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In this sense, the mean-value property and the maximum principle are nothing more than a quantitative expression of the monotonicity inherent to the equation.

Over the past 10 years, there has been increasing interest in whether the mean value property holds in some form for nonlinear operators. This question was in part motivated by the surprising connection between harmonious extensions, the normalized infinity Laplacian, and dynamic programming principles for Random Tug-of-Wag games, discovered in [19,20,29]. Lately, a partial differential equation (PDE) revisitation of the dynamic programming principle for the p-Laplacian, introduced in [30], has been performed in [13], see also [25].

In our recent paper [4], we established asymptotic mean-value properties in the viscosity sense for solutions to the Monge-Ampère equation,

$$\det D^2u(x) = f(x), \quad \text{in } \Omega, \tag{1.2}$$

which is elliptic only in the class of convex functions and, consequently, requires $f \geq 0$. Our results in [4] are based on the formula

$$n(\det D^2u(x))^{1/n} = \inf_{\det A=1} \text{trace}(A^t D^2u(x)A),$$

which holds whenever $D^2u(x) \geq 0$. Mean-value properties in the viscosity sense require more geometrical arguments adapted to the operator. In particular, our methods do not require explicit representation formulas, making them flexible enough to be applied to various nonlinear problems.

In this article, we establish asymptotic mean-value formulas for a wide array of fully nonlinear equations that include degenerate operators such as the k-Hessians, [11], truncated Laplacians [3], and prescribed eigenvalues of the Hessian [5,28]. The operators we discuss can be written as an infimum of linear operators with coefficients chosen from a given set, see below for examples. Of course, there are corresponding statements for equations involving a supremum or combinations of infimum and supremum.

Mean-value formulas hold under more lenient regularity conditions than the corresponding PDEs and can provide a simple and unified approach to nonlinear equations in nonEuclidean contexts such as Carnot groups. As a proof of concept, we prove mean-value formulas for Monge-Ampère operators in the Heisenberg group in Section 6. Other directions for applications of the asymptotic mean-value formulas below concern game-theoretic interpretations of the corresponding PDEs and their numerical analysis. In this regard, there are convergent difference schemes for the normalized infinity Laplacian and p-Laplacian using mean-value formulas, see [27]. Hence, the mean-value formulas that we develop here could provide new numerical methods for the corresponding nonlinear equations.

Let us now describe our results. First, we consider differential operators of the form $F(D^2u)$, with $F : S^n(\mathbb{R}) \to \mathbb{R} \cup \{-\infty\}$ given by

$$F(M) = \inf_{A\in\mathcal{A}} \text{trace}(A^t MA). \tag{1.3}$$

Here, $\mathcal{A}$ is a subset of $S^n_+(\mathbb{R})$, the set of symmetric positive semi-definite matrices. We observe in Lemma 4.1 that the set $\mathcal{A}$ is bounded if and only if the operator $F(M)$ is well-defined (finite) for all $M \in S^n(\mathbb{R})$. Therefore, $\mathcal{A}$ determines the set of admissible solutions, functions for which $F(D^2u)$ is finite. We say that

$$u \in C^2(\Omega) \text{ is } \mathcal{A}\text{-admissible in } \Omega \text{ if } D^2u(x) \in \Gamma_{\mathcal{A}} \text{ for every } x \in \Omega, \tag{1.4}$$

where we define the cone

$$\Gamma_{\mathcal{A}} = \{M \in S^n(\mathbb{R}) : F(M) > -\infty\}$$

(see [11] for a related notion of admissibility). This condition plays an analogous role to the convexity for the Monge-Ampère equation; in fact,

$$\inf_{A\in\mathcal{A}} \text{trace}(A^t D^2u(x)A) = \begin{cases} n(\det D^2u(x))^{1/n} & \text{if } D^2u(x) \geq 0 \\ -\infty & \text{otherwise}, \end{cases} \tag{1.5}$$

for the unbounded set $\mathcal{A} = \{A \in S^n_+(\mathbb{R}) : \det(A) = 1\}$, see [4].

Observe that F is an infimum of linear functions, which are continuous; therefore, it is upper semi-continuous. Even more, F is concave and continuous in $\Gamma^\circ_{\mathcal{A}}$. We assume that

$$F \text{ is lower semi-continuous in } \Gamma_{\mathcal{A}} \setminus \Gamma^\circ_{\mathcal{A}}, \tag{1.6}$$

so that we have that

$$F \text{ is continuous in } \Gamma_{\mathcal{A}}. \tag{1.7}$$

Condition (1.6) is not satisfied in Example 4.6 and the mean-value formula fails.

To deal with operators of the form (1.3), where the class of matrices $\mathcal{A}$ is unbounded, we consider matrices A such that $A \in \mathcal{A}$ and $A \le \phi(\varepsilon)I$ with $\phi(\varepsilon)$ a positive function such that

$$\lim_{\varepsilon\to 0} \phi(\varepsilon) = +\infty \quad \text{and} \quad \lim_{\varepsilon\to 0} \varepsilon\ \phi(\varepsilon) = 0 \tag{1.8}$$

(an example of such a function is $\phi(\varepsilon) = \varepsilon^{-\alpha}$ for $0 < \alpha < 1$). Note that the condition $A \le \phi(\varepsilon)I$ becomes less restrictive as $\varepsilon \to 0$ but is still enough to make the mean-value formula local, see Section 4. We obtain the following result.

Theorem 1.1. *Let $\phi(\varepsilon)$ be a positive function that satisfies* (1.8), *and $F : S^n(\mathbb{R}) \to \mathbb{R} \cup \{-\infty\}$ an operator of the form* (1.3) *that satisfies* (1.6). *Let the function $u \in C^2(\Omega)$ be $\mathcal{A}$-admissible. Then, for every $x \in \Omega$ we have*

$$F(D^2u(x)) = 2(n+2) \inf_{\substack{A\in\mathcal{A}\\ A\le\phi(\varepsilon)I}} \fint_{B_\varepsilon(0)} \frac{u(x+Ay) - u(x)}{\varepsilon^2} \mathrm{d}y + o(1), \quad \text{as } \varepsilon \to 0. \tag{1.9}$$

As a consequence, we can characterize viscosity solutions to the equation $F(D^2u(x)) = f(x)$ in terms of an asymptotic mean-value formula in the viscosity sense. For the precise definition of the notion of viscosity solutions and viscosity mean-value formulas, see Section 2, [12], and [25]. Informally, an equation or a mean-value property holds in the viscosity sense when they hold with an appropriate inequality (instead of an equality) for smooth test functions that touch u from above or below at x.

The concept of a mean-value formula in the viscosity sense is weaker than a mean-value formula that holds pointwise. For the infinity Laplacian and the Monge-Ampère equation, there are instances of asymptotic mean-value formulas that hold in the viscosity sense but do not hold pointwise, see [25] and [4], respectively. An interesting question is, under what circumstances do the viscosity mean-value formulas outlined above hold pointwise. This question has an obvious affirmative answer for all equations for which viscosity solutions are known to be classical since, in that case, we can apply the pointwise formula for regular functions in (1.9). Nevertheless, there are examples of nonclassical viscosity solutions for which mean-value formulas hold pointwise; see [2,4,23].

Theorem 1.2. *Let $\phi(\varepsilon)$ be a positive function that satisfies* (1.8). *Consider $f \in C(\Omega)$ and $F : S^n(\mathbb{R}) \to \mathbb{R} \cup \{-\infty\}$ defined as in* (1.3) *that satisfies* (1.6). *Then, a function $u \in C(\Omega)$ is a viscosity subsolution* (*respectively, super-solution*) *of*

$$F(D^2u(x)) = f(x) \quad \text{in } \Omega,$$

if and only if

$$u(x) \le \inf_{\substack{A\in\mathcal{A}\\ A\le\phi(\varepsilon)I}} \fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0$$

(*respectively*, $\ge$) *in the viscosity sense.*

Remark 1.3. Analogous results hold for supremum operators

$$F(D^2u(x)) = \sup_{A\in\mathcal{A}} \operatorname{trace}(A^t D^2u(x) A)$$

and operators with a combination of an infimum and a supremum.

A fundamental example of application of Theorems 1.1 and 1.2 is the k-Hessian operators, given by the elementary symmetric polynomials

$$\sigma_k(\lambda_1,\dots,\lambda_n) = \sum_{1\le i_1<i_2<\dots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_k}$$

evaluated in the eigenvalues of the Hessian, $\{\lambda_i(D^2u)\}_{1\le i\le n}$. Here, $k = 1, 2, \dots, n$ and the cases $k = 1$ and $k = n$ correspond to the Laplacian and Monge-Ampère, respectively. For these operators to fit our framework we need to write them in the form

$$F_k(D^2u(x)) = k\sigma_k(\lambda_1(D^2u(x)),\dots,\lambda_n(D^2u(x)))^{\frac{1}{k}},$$

according to the following characterization, which we prove in Section 5.

Lemma 1.4. *Let $k \ge 2$ and*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \quad \text{for all } j = 1, \dots, k\}.$$

Then, for every $M \in S^n(\mathbb{R})$ we have

$$\inf_{A\in\mathcal{A}_k} \operatorname{trace}(A^tMA) = \begin{cases} k\sigma_k(\lambda(M))^{\frac{1}{k}} & \text{if } M \in \overline{\Gamma}_k,\\ -\infty & \text{otherwise},\end{cases} \tag{1.10}$$

where

$$\mathcal{A}_k = \{A \in S^n_+(\mathbb{R}) : \lambda_i^2(A) = \sigma_{k-1,i}(\gamma) \text{ with } \gamma = (\gamma_1,\dots,\gamma_n) \in \Gamma_k \text{ and } \sigma_k(\gamma) = 1\},$$

and $\sigma_{k-1,i}(\gamma_1,\dots,\gamma_n) = \sigma_{k-1}(\gamma_1,\dots,\gamma_{i-1}, 0, \gamma_{i+1},\dots,\gamma_n)$.

An important consequence of Lemma 1.4 is that for the k-Hessian operators, $\mathcal{A}$-admissibility is equivalent to the notion of k-convexity in [33–35], see Corollary 5.3. This means that

$$\Gamma_{\mathcal{A}_k} = \{M \in S^n(\mathbb{R}) : F(M) > -\infty\} = \{M \in S^n(\mathbb{R}) : F(M) \ge 0\} = \overline{\Gamma}_k$$

and

$$F \equiv 0 \quad \text{on } \partial\Gamma_{\mathcal{A}_k}. \tag{1.11}$$

A condition such as (1.11) could be used in place of (1.6) to prove the mean-value property. In fact, (1.11) may even appear more natural at first; for instance, it was used in [15] in relation to the existence and uniqueness of solutions to general fully nonlinear second-order PDEs. However, condition (1.11) implies (1.6), but not vice versa (see Remark 4.4 and Example 4.5), making condition (1.6) more general.

In this case Theorem 1.1 reads as follows, see Section 5 for details.

Theorem 1.5. *Let $\phi(\varepsilon)$ be a positive function that satisfies* (1.8) *and assume that $u \in C^2(\Omega)$ is k-convex, that is, $\sigma_j(\lambda(D^2u(x))) \ge 0$ for all $j = 1,\dots, k$, for every $x \in \Omega$. Then, for every $x \in \Omega$ we have*

$$u(x) = \inf_{\substack{A\in\mathcal{A}_k\\ A\le\phi(\varepsilon)I}} \fint_{B_\varepsilon(0)} u(x + Ay)dy - \frac{\varepsilon^2}{2(n+2)} k(\sigma_k(D^2u(x)))^{\frac{1}{k}} + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0, \tag{1.12}$$

with a corresponding viscosity characterization as in Theorem 1.2.

It is also possible to consider operators where the class of matrices $\mathcal{A}$ depends on the point x. This happens naturally when we consider mean-value formulas for Monge-Ampère operators in the Heisenberg group in Section 6. In that case, the sets $\mathcal{A}$ depend on the point x and are unbounded.

We also obtain a rich family of examples when considering operators for which $\mathcal{A}_x \subset S^n_+(\mathbb{R})$ is bounded for each $x \in \Omega$. In fact, we consider sup-inf operators where for every x, the supremum is taken over a subset $\mathbb{A}_x$ of $\mathcal{P}(S_+^n(\mathbb{R}))$, the power set of $S_+^n(\mathbb{R})$. This is not done only for the sake of generality. It is motivated by examples that cannot be covered otherwise, such as prescribed eigenvalues of the Hessian and Isaacs operators. These operators can be degenerate elliptic since we do not impose any lower bounds on the eigenvalues of the matrices $A \in \mathcal{A}_x$. We provide a list of examples below, which shows the flexibility of the approach.

Therefore, let us now consider differential operators of the form $F(x, D^2u(x))$ with $F : \Omega \times S^n(\mathbb{R}) \to \mathbb{R}$ given by

$$F(x, M) = \sup_{\mathcal{A} \in \mathbb{A}_x} \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t MA). \tag{1.13}$$

Here, $\mathbb{A}_x \subset \mathcal{P}(S_+^n(\mathbb{R}))$ (the power set of $S_+^n(\mathbb{R})$) is a nonempty subset for each $x \in \mathbb{R}^n$ and we assume that

$$\bigcup \mathbb{A}_x = \{A \in S_+^n(\mathbb{R}) : A \in \mathcal{A} \quad \text{for some } \mathcal{A} \in \mathbb{A}_x\} \quad \text{is bounded}. \tag{1.14}$$

Observe that if $\mathbb{A}_x$ contains only one element $\mathcal{A}_x$ for each x, then (1.13) is equivalent to

$$F(x, M) = \inf_{A \in \mathcal{A}_x} \operatorname{trace}(A^t MA), \tag{1.15}$$

with $\mathcal{A}_x \subset S_+^n(\mathbb{R})$ bounded for each $x \in \mathbb{R}^n$. On the other hand, if every $\mathbb{A}_x$ is a set of singletons, (1.13) is equivalent to

$$F(x, M) = \sup_{A \in \mathcal{A}_x} \operatorname{trace}(A^t MA), \tag{1.16}$$

where $\mathcal{A}_x = \{A \in S_+^n(\mathbb{R}) : A \in \mathcal{A} \text{ for some } \mathcal{A} \in \mathbb{A}_x\}$. One can also consider inf-sup operators of the form

$$F(x, M) = \inf_{\mathcal{A} \in \mathbb{A}_x} \sup_{A \in \mathcal{A}} \operatorname{trace}(A^t MA) \tag{1.17}$$

with straightforward adaptations in the statements and proofs.

By Lemma 4.1, (1.14) is equivalent to the operator $M \mapsto F(x, M)$ being well-defined and finite for every $M \in S^n(\mathbb{R})$; in particular, every $u \in C^2(\Omega)$ is admissible. Moreover, we prove in Lemma 3.2 that $M \mapsto F(x, M)$ is Lipschitz continuous for every $x \in \Omega$.

We obtain the following counterparts of Theorems 1.1 and 1.2. Observe that when $\cup \mathbb{A}_x$ is bounded, the condition $A \leq \phi(\varepsilon)I$ in (1.9) becomes unnecessary because it is satisfied at every point for ε small enough.

Theorem 1.6. *Consider $u \in C^2(\Omega)$ and let $F : \Omega \times S^n(\mathbb{R}) \to \mathbb{R}$ be an operator of the form* (1.13) *that satisfies* (1.14). *Then, for every $x \in \Omega$ we have*

$$F(x, D^2u(x)) = 2(n+2) \sup_{\mathcal{A} \in \mathbb{A}_x} \inf_{A \in \mathcal{A}} \fint_{B_\varepsilon(0)} \frac{u(x + Ay) - u(x)}{\varepsilon^2} dy + o(1), \quad \text{as } \varepsilon \to 0. \tag{1.18}$$

We have a corresponding result in the line of Theorem 1.2, characterizing viscosity solutions to the equation $F(x, D^2u(x)) = f(x)$ by an asymptotic mean-value formula in the viscosity sense.

Theorem 1.7. *Consider $f \in C(\Omega)$ and $F : S^n(\mathbb{R}) \to \mathbb{R}$ defined as in* (1.13) *that satisfies* (1.14). *Then, a function $u \in C(\Omega)$ is a viscosity subsolution* (*respectively, supersolution*) *of*

$$F(D^2u(x)) = f(x) \quad \text{in } \Omega,$$

if and only if

$$u(x) \leq \sup_{\mathcal{A} \in \mathbb{A}_x} \inf_{A \in \mathcal{A}} \fint_{B_\varepsilon(0)} u(x + Ay) dy - \frac{\varepsilon^2}{2(n+2)} f(x) + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0$$

(*respectively*, $\geq$) *in the viscosity sense.*

Let us now provide some examples of operators of the form (1.13), (1.15), and (1.16) and the corresponding sets $\mathbb{A}_x$ and $\mathcal{A}_x$. In the sequel, we consider the eigenvalues of a matrix $M \in S^n(\mathbb{R})$ arranged in increasing order, that is, $\lambda_1(M) \le \lambda_2(M) \le \cdots \le \lambda_n(M)$.

(1) Operators of the form (1.13), (1.17) include the usual Isaacs operators (see [8]) defined as

$$F(x, M) = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \operatorname{trace}(A_{\alpha\beta}^t M A_{\alpha\beta}) \quad \text{and} \quad F(x, M) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \operatorname{trace}(A_{\alpha\beta}^t M A_{\alpha\beta})$$

for a given family of matrices $\{A_{\alpha\beta}\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$ (and similarly in the inf-sup case). Here we take $\mathbb{A} = \{\{A_{\alpha\beta}\}_{\beta\in\mathcal{B}}\}_{\alpha\in\mathcal{A}}$ in (1.13), with $\mathbb{A}$ independent of the point x. In fact, operators of the form (1.13) could be seen as Issacs operators, but the usual Isaacs condition

$$\inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \operatorname{trace}(A_{\alpha\beta}^t M A_{\alpha\beta}) = \sup_{\beta\in\mathcal{B}} \inf_{\alpha\in\mathcal{A}} \operatorname{trace}(A_{\alpha\beta}^t M A_{\alpha\beta})$$

does not seem to have a clear counterpart. A typical requirement in the literature is that the family $\{A_{\alpha\beta}\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$ is uniformly elliptic, i.e., it is uniformly bounded between two constants $0 < \theta < \Theta$. In this case, the resulting Issacs operator is uniformly elliptic. We want to emphasize that here we are only requiring $\cup\mathbb{A}_x$ to be bounded and, therefore, the operators (1.13) may be degenerate elliptic.

(2) We consider the truncated Laplacians [3], defined as

$$\mathcal{P}_k^-(D^2u) = \sum_{i=1}^k \lambda_i(D^2u) \quad \text{and} \quad \mathcal{P}_k^+(D^2u) = \sum_{i=1}^k \lambda_{n+1-i}(D^2u),$$

for $k = 1, 2, \ldots, n-1$ (for $k = n$ these operators coincide with the Laplacian). These degenerate operators appear naturally in geometric problems, when considering manifolds of partially positive curvature [32,37], or mean curvature flow in arbitrary codimension [1], see also [3,10,15,16] and references therein. The operators $\mathcal{P}_k^-$ and $\mathcal{P}_k^+$ are of the form (1.15) and (1.16), respectively, for the set

$$\mathcal{A} = \{A \in S_+^n(\mathbb{R}) : \lambda_1 = \cdots = \lambda_{n-k} = 0 \text{ and } \lambda_{n-k+1} = \cdots = \lambda_n = 1\}.$$

(3) Operators of the form (1.13) allow us to consider degenerate operators such as the kth smallest eigenvalue of the Hessian, given by the Courant-Fischer min–max principle

$$\lambda_k(D^2u(x)) = \max_V \left\{ \min_{v\in V, |v|=1} \langle D^2u(x)v, v\rangle : V \subset \mathbb{R}^n \quad \text{subspace}, \quad \dim(V) = n-k+1 \right\}. \tag{1.19}$$

We can write the operator $\lambda_k(D^2u)$ in the form (1.13) for the set

$$\mathbb{A} = \{\{A \in S_+^n(\mathbb{R}) : \lambda_i(A) = 0 \text{ for } i \ne n, \lambda_n(A) = 1, v_n \in V\} : V \subset \mathbb{R}^n \text{ subspace of dimension } n-k+1\},$$

where v_n is the eigenvector corresponding to $\lambda_n(A)$. The cases $k = 1$ and $k = n$ were studied in [5,15,28] in connection with the convex and concave envelope of a function; i.e., the unique viscosity solutions of $\lambda_1(D^2u(x)) = 0$ and $\lambda_n(D^2u(x)) = 0$ are, respectively, the convex and concave envelopes of $u|_{\partial\Omega}$ in Ω. These operators are of the form (1.15), (1.16) with the set of matrices

$$\mathcal{A} = \{A \in S_+^n(\mathbb{R}) : \lambda_1(A) = \cdots = \lambda_{n-1}(A) = 0 \text{ and } \lambda_n(A) = 1\}.$$

(4) When the mean-value formula involves averages over balls that are not centered at 0 but at $\varepsilon^2 v$ with $|v| = 1$, we obtain operators with first-order terms. For example, we have

$$\inf_{A\in\mathcal{A}_x} \fint_{B_\varepsilon(\varepsilon^2 v)} u(x+Ay)\mathrm{d}y - u(x) = \varepsilon^2 \inf_{A\in\mathcal{A}_x} \left\{ \frac{1}{2(n+2)} \operatorname{trace}(A^t D^2u(x)A) + \langle Du(x), Av\rangle \right\} + o(\varepsilon^2),$$

as $\varepsilon \to 0$. We can also look for zero-order terms and consider mean-value properties like

$$\inf_{A\in\mathcal{A}_x} (1-\alpha\varepsilon^2) \fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y - u(x) = \varepsilon^2 \left\{ \frac{1}{2(n+2)} \inf_{A\in\mathcal{A}_x} \operatorname{trace}(A^t D^2u(x)A) - \alpha u(x) \right\} + o(\varepsilon^2),$$

as $\varepsilon \to 0$.

(5) Finally, we consider extremal Pucci operators for given ellipticity constants $0 < \theta < \Theta$, defined as

$$\mathcal{M}^-_{\theta,\Theta}(D^2u) = \theta \sum_{\lambda_i(D^2u)>0} \lambda_i(D^2u) + \Theta \sum_{\lambda_i(D^2u)<0} \lambda_i(D^2u)$$

and

$$\mathcal{M}^+_{\theta,\Theta}(D^2u) = \Theta \sum_{\lambda_i(D^2u)>0} \lambda_i(D^2u) + \theta \sum_{\lambda_i(D^2u)<0} \lambda_i(D^2u).$$

These operators are of the form (1.15), (1.16) with

$$\mathcal{A}_{\theta\Theta} = \{A \in S^n_+(\mathbb{R}) : \sqrt{\theta} \le \lambda_i(A) \le \sqrt{\Theta}\},$$

which is bounded uniformly in x. From the definition of $\mathcal{A}_{\theta\Theta}$, it is clear that $\mathcal{M}^\pm_{\theta,\Theta}$ are uniformly elliptic operators. We apply similar ideas to general fully nonlinear uniformly elliptic operators $F(x, D^2u(x))$ (with ellipticity constants $0 < \theta \le \Theta$) by means of the characterization

$$F(x, M) = \sup_{N \in S^n(\mathbb{R})} \inf_{A \in \mathcal{A}_{\theta\Theta}} \{\operatorname{trace}(A^tMA) + F(x, N) - \operatorname{trace}(A^tNA)\},$$

see Lemma 3.4.

The paper is organized as follows. In Section 2, we gather some definitions and preliminary results, and in Section 3 we deal with concave and Isaacs operators, when the sets of coefficients are bounded. These include general uniformly elliptic operators and operators including lower-order terms. We consider bounded coefficients first for the sake of clarity, to establish the techniques and ideas before we dive into the unbounded case in Section 4. In Section 5, we study the k-Hessians. Finally, Section 6 is devoted to examples in the Heisenberg group, where the class $\mathcal{A}_x$ is unbounded and naturally depends on x.

2 Preliminaries

We begin by stating the definition of a viscosity solution to a fully nonlinear, second-order, elliptic PDE. We refer to [12] for general results on viscosity solutions. Given a continuous function

$$\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n(\mathbb{R}) \to \mathbb{R},$$

we consider the PDE

$$\mathcal{F}(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in \Omega. \tag{2.1}$$

Viscosity solutions use the monotonicity of $\mathcal{F}$ in D^2u (ellipticity) in order to “pass derivatives to smooth test functions.”

Definition 2.1. A lower semi-continuous function u is a viscosity supersolution of (2.1) if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below (i.e., $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$), we have

$$\mathcal{F}(x, \phi(x), D\phi(x), D^2\phi(x)) \le 0.$$

An upper semi-continuous function u is a subsolution of (2.1) if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from above (i.e., $u - \phi$ has a strict maximum at x with $u(x) = \phi(x)$), we have

$$\mathcal{F}(x, \phi(x), D\phi(x), D^2\phi(x)) \ge 0.$$

Finally, u is a viscosity solution of (2.1) if it is both a super- and a subsolution.

We will also need the definition of an asymptotic mean-value formula in the viscosity sense. In the next definition, $M(u,\varepsilon)(x)$ stands for a mean-value operator (that depends on the parameter ε) applied to u at the point x. For example, we can take

$$M(u,\varepsilon)(x) = \inf_{A\in\mathcal{A}_x} \fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y - \frac{\varepsilon^2}{2(n+2)} f(x)$$

as in Theorem 1.2 and the next section.

Definition 2.2. A continuous function u verifies

$$u(x) \geq M(u,\varepsilon)(x) + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

in the viscosity sense if for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$ (i.e., ϕ touches u at $x \in \Omega$ strictly from below), we have

$$\phi(x) \geq M(\phi,\varepsilon)(x) + o(\varepsilon^2).$$

Similarly, a continuous function u verifies

$$u(x) \leq M(u,\varepsilon)(x) + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

in the viscosity sense if for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$ (i.e., ϕ touches u at $x \in \Omega$ strictly from above), we have

$$\phi(x) \leq M(\phi,\varepsilon)(x) + o(\varepsilon^2).$$

Next, we include a lemma that is related to the simplest mean-value property for the usual Laplacian. A proof can be found in [4].

Lemma 2.3. *Let M be a square matrix of dimension n. Then,*

$$\operatorname{trace}(M) = \frac{n}{\varepsilon^2} \fint_{\partial B_\varepsilon(0)} \langle My, y\rangle \mathrm{d}\mathcal{H}^{n-1}(y) = \frac{n+2}{\varepsilon^2} \fint_{B_\varepsilon(0)} \langle My, y\rangle \mathrm{d}y.$$

Remark 2.4. We will use the solid mean

$$\operatorname{trace}(M) = \frac{n+2}{\varepsilon^2} \fint_{B_\varepsilon(0)} \langle My, y\rangle \mathrm{d}y,$$

in our proofs of the mean-value formulas. However, if one uses the mean on spheres

$$\operatorname{trace}(M) = \frac{n}{\varepsilon^2} \fint_{\partial B_\varepsilon(0)} \langle My, y\rangle \mathrm{d}\mathcal{H}^{n-1}(y),$$

one can obtain mean-value formulas of the type

$$\inf_{A\in\mathcal{A}_x} \fint_{\partial B_\varepsilon(0)} u(x+Ay)\mathrm{d}y - u(x) = \frac{\varepsilon^2}{2n} F(x, D^2u(x)) + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0, \tag{2.2}$$

for $u \in C^2(\Omega)$ and $F(x, D^2u)$ defined as in (1.15) (and accordingly for the rest of the cases).

3 Bounded operators

In this section, we prove Theorems 1.6 and 1.7. We consider differential operators given by

$$F(x, D^2u(x)) = \sup_{\mathcal{A}\in\mathbb{A}_x} \inf_{A\in\mathcal{A}} \operatorname{trace}(A^t D^2u(x) A), \tag{3.1}$$

where $\mathbb{A}_x \subset \mathcal{P}(S^n_+(\mathbb{R}))$ (the power set of $S^n_+(\mathbb{R})$) is a nonempty subset for each $x \in \mathbb{R}^n$ and

$$\bigcup \mathbb{A}_x = \{A \in S^n_+(\mathbb{R}) : A \in \mathcal{A} \quad \text{for some } \mathcal{A} \in \mathbb{A}_x\} \text{ is bounded}. \tag{3.2}$$

Due to (3.2), the operator $F(x, M)$ is finite for every $M \in S^n(\mathbb{R})$.

Moreover, since the set of matrices $\cup\mathbb{A}_x$ is bounded, the mean-value formula (1.18) is local. In fact, for every $x \in \Omega$ and $y \in B_\varepsilon(0)$ there exists $C_x > 0$ such that $A \le C_x I$ for every $A \in \cup\mathbb{A}_x$. We obtain

$$\operatorname{dist}(x + Ay, x) = |Ay| \le C_x\varepsilon \to 0$$

as $\varepsilon \to 0$. In particular, observe that for ε small enough $x + Ay \in \Omega$ for every $y \in B_\varepsilon(0)$ and hence the integrals

$$\fint_{B_\varepsilon(0)} u(x + Ay)\mathrm{d}y$$

are well-defined for integrable functions $u : \Omega \to \mathbb{R}$.

Remark 3.1. We can assume that the matrices $A \in \cup\mathbb{A}_x$ are symmetric and positive semi-definite without loss of generality. This is because, given $A \in M^{n\times n}(\mathbb{R})$, we can write its left polar decomposition $A = SQ$ with Q orthogonal and S positive semi-definite and symmetric. Then, for every $M \in M^{n\times n}(\mathbb{R})$, we have

$$\operatorname{trace}(A^tMA) = \operatorname{trace}(Q^tS^tMSQ) = \operatorname{trace}(S^tMSQQ^t) = \operatorname{trace}(S^tMS)$$

and

$$\fint_{B_\varepsilon(0)} u(x + Ay)\mathrm{d}y = \fint_{B_\varepsilon(0)} u(x + SQy)\mathrm{d}y = \fint_{B_\varepsilon(0)} u(x + Sz)\mathrm{d}z.$$

In the next lemma, we prove an explicit continuity estimate that we use in the proof of Theorem 1.6.

Lemma 3.2. *Consider the differential operator $F(x, D^2u(x))$ given by* (3.1). *If $\cup\mathbb{A}_x$ is bounded, then the mapping $M \mapsto F(x, M)$ is Lipschitz continuous in M. In particular,*

$$\sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \operatorname{trace}(A^t(M \pm \eta I)A) \to \sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \operatorname{trace}(A^tMA) \tag{3.3}$$

as $\eta \to 0$, for every $M \in S^n(\mathbb{R})$.

Proof. We fix $x \in \mathbb{R}^n$. For every $A \in \cup\mathbb{A}_x$, there exists $C_x > 0$ such that $A \le C_x I$. Given $M, N \in S^n(\mathbb{R})$, we have

$$\begin{aligned}
&\Big|\sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \operatorname{trace}(A^tMA) - \sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \operatorname{trace}(A^tNA)\Big| \\
&\quad\le \sup_{A\in\cup\mathbb{A}_x} |\operatorname{trace}(A^tMA) - \operatorname{trace}(A^tNA)| \le \sup_{A\in\cup\mathbb{A}_x} |\operatorname{trace}(A^t(M - N)A)| \le nC_x^2\|M - N\|,
\end{aligned}$$

and the result follows. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Given $x \in \Omega$, let us consider the paraboloid

$$P(z) = u(x) + \langle\nabla u(x), z - x\rangle + \frac{1}{2}\langle D^2u(x)(z - x), (z - x)\rangle. \tag{3.4}$$

Since $u \in C^2(\Omega)$, we have

$$u(z) - P(z) = o(|z - x|^2) \quad \text{as } z \to x,$$

which means that for every $\eta > 0$, there exists $\delta > 0$ such that for every $z \in B_\delta(x)$,

$$P(z) - \frac{\eta}{2}|z - x|^2 \le u(z) \le P(z) + \frac{\eta}{2}|z - x|^2, \tag{3.5}$$

with equality only when $z = x$. For convenience, let us denote

$$P_\eta^\pm(z) = P(z) \pm \frac{\eta}{2}|z-x|^2.$$

Then,

$$\begin{aligned}\fint_{B_\varepsilon(0)} (P_\eta^\pm(x+Ay) - P_\eta^\pm(x))dy &= \frac{1}{2}\fint_{B_\varepsilon(0)} (\langle A^t D^2u(x)Ay, y\rangle \pm \eta|Ay|^2)dy \\ &= \frac{1}{2}\fint_{B_\varepsilon(0)} \langle A^t(D^2u(x) \pm \eta I)Ay, y\rangle dy \\ &= \frac{\varepsilon^2}{2(n+2)}\ \mathrm{trace}(A^t(D^2u(x) \pm \eta I)A),\end{aligned} \tag{3.6}$$

by Lemma 2.3.

On the other hand, since $\cup\mathbb{A}_x$ is bounded there exists $C_x > 0$ such that $A \le C_x I$ for every $A \in \cup\mathbb{A}_x$. Then, $x + Ay \in B_\delta(x)$ for every $|y| \le \varepsilon$ and $\varepsilon < \varepsilon_0$, where $\varepsilon_0 C_x \le \delta$. Therefore, by (3.5), if $\varepsilon < \varepsilon_0$, then

$$P_\eta^-(x+Ay) \le u(x+Ay) \le P_\eta^+(x+Ay) \quad \text{for every } y \in B_\varepsilon. \tag{3.7}$$

Then,

$$\frac{\varepsilon^2}{2(n+2)}\mathrm{trace}(A^t(D^2u(x) - \eta I)A) \le \fint_{B_\varepsilon(0)} u(x+Ay)dy - u(x) \le \frac{\varepsilon^2}{2(n+2)}\ \mathrm{trace}(A^t(D^2u(x) + \eta I)A)$$

and the result follows by (3.3). □

We now prove Theorem 1.7.

Proof of Theorem 1.7. First, assume that u is a viscosity solution and take a C^2 function ϕ that touches u at $x \in \Omega$ strictly from below. Then, as $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$, we have

$$F(x, D^2\phi(x)) = \sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \mathrm{trace}(A^t D^2\phi(x)A) \le f(x).$$

Now, since ϕ is C^2, Theorem 1.6 gives

$$\sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \fint_{B_\varepsilon(0)} \phi(x+Ay)dy - u(x) + o(\varepsilon^2) = \frac{\varepsilon^2}{2(n+2)}\sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \mathrm{trace}(A^t D^2\phi(x)A) \le \frac{\varepsilon^2}{2(n+2)}f(x),$$

proving that u satisfies

$$u(x) \ge \sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \fint_{B_\varepsilon(0)} u(x+Ay)dy - \frac{\varepsilon^2}{2(n+2)}f(x) + o(\varepsilon^2),$$

in the viscosity sense. An analogous computation reversing the inequalities shows that when a C^2 function ϕ touches u at $x \in \Omega$ strictly from above, u satisfies

$$u(x) \le \sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \fint_{B_\varepsilon(0)} u(x+Ay)dy - \frac{\varepsilon^2}{2(n+2)}f(x) + o(\varepsilon^2)$$

in the viscosity sense.

Now, assume that the mean-value property holds and take a C^2 function ϕ that touches u at $x \in \Omega$ strictly from below. Then, as $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$, Theorem 1.6 yields

$$\frac{\varepsilon^2}{2(n+2)}F(x, D^2\phi(x)) + o(\varepsilon^2) = \sup_{\mathcal{A}\in\mathbb{A}_x}\inf_{A\in\mathcal{A}} \fint_{B_\varepsilon(0)} \phi(x+Ay)dy - u(x) \le \frac{\varepsilon^2}{2(n+2)}f(x) + o(\varepsilon^2).$$

Dividing by ε^2 and letting $\varepsilon \to 0$ we obtain that u is a viscosity supersolution to

$$F(x, D^2u(x)) = f(x).$$

A similar argument reversing the inequalities shows that u is a viscosity subsolution. □

We devote the rest of the section to discuss the examples mentioned in Section 1 in more detail.

3.1 Isaacs operators

As mentioned in Section 1, hypotheses (3.1) and (3.2) also cover degenerate operators such as the kth smallest eigenvalue of the Hessian, given by the Courant-Fischer min–max principle

$$\lambda_k(D^2u(x)) = \max_V \left\{ \min_{v \in V, |v|=1} \langle D^2u(x)v, v\rangle : V \subset \mathbb{R}^n \text{ subspace of dimension } n-k+1 \right\}. \tag{3.8}$$

To write this operator in the form (3.1), let $A_v = v \otimes v$ and note that $A_v^2 = A_v$ when $|v| = 1$. With this notation we have trace $(A_v^t M A_v) = \langle Mv, v\rangle$. Let G_{n-k+1} denote the set of all subspaces $V \subset \mathbb{R}^n$ of dimension $n-k+1$. For each $V \in G_{n-k+1}$, we set $\mathcal{A}_V = \{v \otimes v : v \in V \text{ such that } |v| = 1\}$. Finally, we define $\mathbb{A}_k = \{\mathcal{A}_V : V \in G_{n-k+1}\}$, or equivalently,

$$\mathbb{A}_k = \{\{A \in S_+^n(\mathbb{R}) : \lambda_i(A) = 0 \text{ for } i \neq n, \lambda_n(A) = 1, \text{ and } v_n \in V\} : V \subset \mathbb{R}^n \text{ subspace of dim. } n-k+1\},$$

where v_n is the eigenvector corresponding to $\lambda_n(A)$. We can then write

$$\lambda_k(M) = \sup_V \left\{ \inf_{v \in V, |v|=1} \langle Mv, v\rangle \right\} = \sup_{\mathcal{A}_V \in \mathbb{A}_k} \inf_{A \in \mathcal{A}_V} \operatorname{trace}(A^t M A).$$

This allows us to prove mean-value formulas for (3.8). In fact, Theorem 1.6 gives

$$\sup_{\mathcal{A}_V \in \mathbb{A}_k} \inf_{A \in \mathcal{A}_V} \fint_{B_\varepsilon(0)} u(x + Ay)\,dy - u(x) = \frac{\varepsilon^2}{2(n+2)} \lambda_k(D^2u(x)) + o(\varepsilon^2), \tag{3.9}$$

and the corresponding viscosity analogue following Theorem 1.7.

A variant of (3.9) is contained in [6],

$$\sup_{\dim(V)=n-k+1} \inf_{v \in V, |v|=1} \left\{ \frac{u(x+\varepsilon v) + u(x - \varepsilon v)}{2} \right\} - u(x) = \frac{\varepsilon^2}{2} \lambda_k(D^2u(x)) + o(\varepsilon^2). \tag{3.10}$$

In order to make the connection between both mean-value formulas, (3.9) and (3.10), let $A_v = v \otimes v$ with $|v| = 1$ as before and observe that

$$2(n+2)\left(\fint_{B_\varepsilon(0)} u(x + A_v y)\,dy - u(x) \right) = \varepsilon^2 \langle D^2u(x)v, v\rangle + o(\varepsilon^2) = u(x+\varepsilon v) + u(x-\varepsilon v) - 2u(x) + o(\varepsilon^2),$$

where the error estimate is uniform in v.

3.2 Uniformly elliptic fully nonlinear operators

Let us state the definition of a uniformly elliptic operator for completeness.

Definition 3.3. An operator $F : \Omega \times S^n(\mathbb{R}) \to \mathbb{R}$ is uniformly elliptic with constants $0 < \theta \le \Theta$ if and only if for every $M, N \in S^n(\mathbb{R})$ with $N \ge 0$,

$$\theta \cdot \operatorname{trace}(N) \le F(x, M+N) - F(x, M) \le \Theta \cdot \operatorname{trace}(N) \tag{3.11}$$

for all $x \in \Omega$.

Here we modify the formulas from the previous section to obtain a mean-value formula for general uniformly elliptic operators. This is possible because every uniformly elliptic operator can be written as an Isaacs operator. We include a proof for the reader's convenience, see also [9].

Lemma 3.4. *Let*

$$\mathcal{A}_{\theta\Theta} = \{A \in S^n(\mathbb{R}) : \sqrt{\theta}|\xi|^2 \le \langle A\xi, \xi\rangle \le \sqrt{\Theta}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n\}.$$

An operator $F : \Omega \times S^n(\mathbb{R}) \to \mathbb{R}$ is uniformly elliptic with constants $0 < \theta \le \Theta$ if and only if

$$\begin{aligned} F(x, M) &= \inf_{N \in S^n(\mathbb{R})} \sup_{A \in \mathcal{A}_{\theta\Theta}} \{\operatorname{trace}(A^t M A) + F(x, N) - \operatorname{trace}(A^t N A)\}, \\ F(x, M) &= \sup_{N \in S^n(\mathbb{R})} \inf_{A \in \mathcal{A}_{\theta\Theta}} \{\operatorname{trace}(A^t M A) + F(x, N) - \operatorname{trace}(A^t N A)\}. \end{aligned} \tag{3.12}$$

Proof. It is well-known that an operator $F : \Omega \times S^n(\mathbb{R}) \to \mathbb{R}$ is uniformly elliptic if and only if,

$$\inf_{A \in \mathcal{A}_{\theta\Theta}} \operatorname{trace}(A^t(M - N)A) \le F(x, M) - F(x, N) \le \sup_{A \in \mathcal{A}_{\theta\Theta}} \operatorname{trace}(A^t(M - N)A)$$

for every $M, N \in S^n(\mathbb{R})$ and $x \in \mathbb{R}$. Since we have equalities when $M = N$, in particular (3.12) holds. □

With this characterization, we can prove the following.

Theorem 3.5. *Consider $F : \Omega \times S^n(\mathbb{R}) \to \mathbb{R}$, uniformly elliptic and let $u \in C^2(\Omega)$. Then, for every $x \in \Omega$ we have*

$$\inf_{N \in S^n(\mathbb{R})} \sup_{A \in \mathcal{A}} \left(\fint_{B_\varepsilon(0)} u(x + Ay)\mathrm{d}y + F(x, N) - \operatorname{trace}(A^t N A) \right) - u(x) = \frac{\varepsilon^2}{2(n+2)} F(x, D^2 u(x)) + o(\varepsilon^2),$$

as $\varepsilon \to 0$, where $\mathcal{A} = \{A \in S^n_+(\mathbb{R}) : \sqrt{\theta}|\xi|^2 \le \langle A\xi, \xi\rangle \le \sqrt{\Theta}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n\}$.

Proof of Theorem 3.5. Since the family of matrices $\mathcal{A}_{\theta\Theta}$ is uniformly elliptic, in particular it is bounded. Then, an analogous result to Lemma 3.2 can be obtained for the operators considered here and the proof follows in the same way as the one of Theorem 1.6. From the expression

$$F(x, D^2 u(x)) = \inf_{N \in S^n(\mathbb{R})} \sup_{A \in \mathcal{A}_{\theta\Theta}} \{\operatorname{trace}(A^t D^2 u(x) A) + F(x, N) - \operatorname{trace}(A^t N A)\},$$

we obtain that for a smooth function u it holds that

$$\inf_{N \in S^n(\mathbb{R})} \sup_{A \in \mathcal{A}} \left(\fint_{B_\varepsilon(0)} u(x + Ay)\mathrm{d}y + F(x, N) - \operatorname{trace}(A^t N A) \right) - u(x) = \frac{\varepsilon^2}{2(n+2)} F(x, D^2 u(x)) + o(\varepsilon^2),$$

as $\varepsilon \to 0$, where $\mathcal{A} = \{A \in S^n_+(\mathbb{R}) : \sqrt{\theta}|\xi|^2 \le \langle A\xi, \xi\rangle \le \sqrt{\Theta}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n\}$. □

Remark 3.6. Observe that for some operators both Theorems 1.6 and 3.5 apply. Even then, the formulas that we obtain are different. For example, for Issacs operators of the form

$$F(x, D^2 u(x)) = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \operatorname{trace}(A^t_{\alpha\beta} D^2 u(x) A_{\alpha\beta}),$$

we obtain two possible mean-value formulas, namely,

$$\inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \fint_{B_\varepsilon(0)} u(x + A_{\alpha\beta} y)\mathrm{d}y - u(x) = \frac{\varepsilon^2}{2(n+2)} F(x, D^2 u(x)) + o(\varepsilon^2)$$

and

$$\inf_{N\in S^n(\mathbb{R})}\sup_{A\in\mathcal{A}_{\theta\Theta}}\left(\fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y + F(x,N) - \operatorname{trace}(A^tNA)\right) - u(x) = \frac{\varepsilon^2}{2(n+2)}F(x,D^2u(x)) + o(\varepsilon^2).$$

3.3 Off-center means and equations involving lower-order terms

When the mean-value formula involves integrals in balls that are not centered at 0 but at $\varepsilon^2 v$ with $|v| = 1$, we obtain second-order operators with first-order terms,

$$\inf_{A\in\mathcal{A}_x}\fint_{B_\varepsilon(\varepsilon^2 v)}\frac{u(x+Ay)-u(x)}{\varepsilon^2}\mathrm{d}y = \inf_{A\in\mathcal{A}_x}\left\{\frac{1}{2(n+2)}\operatorname{trace}(A^tD^2u(x)A) + \langle Du(x), Av\rangle\right\} + o(1), \tag{3.13}$$

as $\varepsilon \to 0$. Here we assume for simplicity that $\mathcal{A}_x \subset S^n_+(\mathbb{R})$ is bounded for each $x \in \Omega$; whenever $\mathcal{A}_x$ is not bounded, we can restrict the argument to matrices that satisfy $A \le \phi(\varepsilon)I$ for $\phi(\varepsilon)$ satisfying (1.8).

The mean-value formula (3.13) is a consequence of the fact that for a C^2 function we have

$$\begin{aligned}
\fint_{B_\varepsilon(\varepsilon^2 v)} u(x+Ay)\mathrm{d}y - u(x) &= \fint_{B_\varepsilon(0)} u(x+A(z+\varepsilon^2 v))\mathrm{d}z - u(x)\\
&= \fint_{B_\varepsilon(0)}\left(\langle Du(x), A(z+\varepsilon^2 v)\rangle + \frac{1}{2}\langle D^2u(x)A(z+\varepsilon^2 v), A(z+\varepsilon^2 v)\rangle\right)\mathrm{d}z + o(\varepsilon^2)\\
&= \varepsilon^2\langle Du(x), Av\rangle + \frac{1}{2}\fint_{B_\varepsilon(0)}\langle D^2u(x)Az, Az\rangle\mathrm{d}z + o(\varepsilon^2)\\
&= \varepsilon^2\left\{\langle Du(x), Av\rangle + \frac{1}{2(n+2)}\operatorname{trace}(A^tD^2u(x)A)\right\} + o(\varepsilon^2).
\end{aligned}$$

Note that the remainder $o(\varepsilon^2)$ is independent of A because $\mathcal{A}_x$ is a bounded set for each $x \in \Omega$.

In addition, we observe that when we center the average at $\varepsilon^\alpha v$ we obtain: for $\alpha > 2$ a pure second-order operator,

$$\inf_{A\in\mathcal{A}_x}\fint_{B_\varepsilon(\varepsilon^\alpha v)} u(x+Ay)\mathrm{d}y - u(x) = \frac{\varepsilon^2}{2(n+2)}\inf_{A\in\mathcal{A}_x}\operatorname{trace}(A^tD^2u(x)A) + o(\varepsilon^2),$$

as $\varepsilon \to 0$; and for $0 < \alpha < 2$ a pure first-order operator

$$\inf_{A\in\mathcal{A}_x}\fint_{B_\varepsilon(\varepsilon^\alpha v)} u(x+Ay)\mathrm{d}y - u(x) = \varepsilon^\alpha\inf_{A\in\mathcal{A}_x}\langle Du(x), Av\rangle + o(\varepsilon^\alpha),$$

as $\varepsilon \to 0$.

Also, we can look for zero-order terms and consider mean-value properties like

$$(1-\alpha\varepsilon^2)\inf_{A\in\mathcal{A}_x}\fint_{B_\varepsilon(0)}\frac{u(x+Ay)-u(x)}{\varepsilon^2}\mathrm{d}y = \frac{1}{2(n+2)}\inf_{A\in\mathcal{A}_x}\operatorname{trace}(A^tD^2u(x)A) - \alpha u(x) + o(1),$$

as $\varepsilon \to 0$. Arguing as before, we have

$$\begin{aligned}
(1-\alpha\varepsilon^2)\fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y - u(x) &= \fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y - u(x) - \alpha\varepsilon^2\fint_{B_\varepsilon(0)} u(x+Ay)\mathrm{d}y\\
&= \varepsilon^2\left\{\frac{1}{2(n+2)}\operatorname{trace}(A^tD^2u(x)A) - \alpha u(x)\right\} + o(\varepsilon^2).
\end{aligned}$$

Similar mean-value formulas also hold for sup-inf operators with lower-order terms. We leave the details to the reader.

4 Unbounded operators

In this section, we deal with unbounded operators $F : S^n(\mathbb{R}) \to \mathbb{R} \cup \{-\infty\}$ given by

$$F(M) = \inf_{A\in\mathcal{A}} \operatorname{trace}(A^t MA). \tag{4.1}$$

As mentioned in Section 1, we restrict the set of matrices where we compute the infimum in the mean-value property by considering matrices $A \in \mathcal{A}$ such that $A \le \phi(\varepsilon)I$, with $\phi(\varepsilon)$ a positive function satisfying

$$\lim_{\varepsilon\to 0} \phi(\varepsilon) = +\infty \quad \text{and} \quad \lim_{\varepsilon\to 0} \varepsilon\phi(\varepsilon) = 0. \tag{4.2}$$

The condition $A \le \phi(\varepsilon)I$ becomes less restrictive as $\varepsilon \to 0$, but is still enough to make the mean-value formula (1.9) local. In fact, for every $x \in \Omega$ and $|y| \le \varepsilon$,

$$\operatorname{dist}(x + Ay, x) = |Ay| \le \varepsilon\, \phi(\varepsilon) \le \operatorname{dist}(x, \partial\Omega)$$

for ε sufficiently small (since $\varepsilon\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$).

Recall that we have assumed (1.6) and therefore

$$F \text{ is continuous in the cone } \Gamma_{\mathcal{A}} = \{M \in S^n(\mathbb{R}) : F(M) > -\infty\}.$$

Next, we prove that, as long as $\mathcal{A}$ is unbounded, there are matrices $M \in S^n(\mathbb{R})$ for which $F(M) = -\infty$.

Lemma 4.1. *The operator*

$$F(x, M) = \inf_{A\in\mathcal{A}_x} \operatorname{trace}(A^t MA)$$

is finite for every $M \in S^n(\mathbb{R})$ if and only if $\mathcal{A}_x$ is bounded.

Proof. It is clear that if $\mathcal{A}_x$ is bounded then $F(x, M)$ is finite for every $M \in S^n(\mathbb{R})$. To prove the converse, suppose that $F(x, M)$ is finite for every $M \in S^n(\mathbb{R})$ and $\mathcal{A}_x$ is not bounded. Then there exists a sequence of matrices $A_k \in \mathcal{A}_x$ such that their largest eigenvalues $\lambda_n(A_k)$ diverge as $k \to \infty$. Write $A_k = Q_k^t D_k Q_k$, where Q_k is an orthogonal matrix and D_k is a diagonal matrix with diagonal entries $(D_k)_{jj} = \lambda_j(A_k)$. Let us assume that the eigenvalues of A_k satisfy $0 \le \lambda_1(A_k) \le \lambda_2(A_k) \le \ldots \le \lambda_n(A_k)$. Extracting a subsequence, if needed, we may also assume that $Q_k \to Q_\infty$. Set $M = Q_\infty^t J Q_\infty$, where J is the $n \times n$ diagonal matrix with diagonal entries $\{0, 0, \ldots, 0, -1\}$. We then have

$$\begin{aligned}
\operatorname{trace}(A_k^t M A_k) &= \operatorname{trace}(J Q_\infty Q_k^t D_k^2 Q_k Q_\infty^t)\\
&= -(Q_\infty Q_k^t D_k^2 Q_k Q_\infty^t)_{nn}\\
&= -\sum_{j=1}^{n} (Q_\infty Q_k^t)_{nj} [\lambda_j(A_k)]^2 (Q_k Q_\infty^t)_{jn}\\
&= -\sum_{j=1}^{n} (Q_\infty Q_k^t)_{nj}^2 [\lambda_j(A_k)]^2\\
&\le -(Q_\infty Q_k^t)_{nn}^2 [\lambda_n(A_k)]^2,
\end{aligned}$$

which tends to $-\infty$ since $(Q_\infty Q_k^t)_{nn} \to 1$ and $\lambda_n(A_k) \to \infty$ as $k \to \infty$ contradicting the fact that $F(x, M)$ is finite. □

We can now proceed with the proof of Theorem 1.1. First, we prove a continuity lemma, analogous to Lemma 3.2. Here, however, we have the extra restriction $A \le \phi(\varepsilon)I$ and the continuity of F from only one side. This is because the cone $\Gamma_{\mathcal{A}}$ is, in principle, neither open nor closed, and when $D^2u(x) \in \partial\Gamma_{\mathcal{A}}$ we can only use perturbations of the form $D^2u(x) + \eta I$.

Lemma 4.2. *For every $M \in \Gamma_{\mathcal{A}}$, we have*

$$\inf_{\substack{A \in \mathcal{A} \\ A \le \phi(\varepsilon) I}} \operatorname{trace}(A^t(M + \eta I)A) \to \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t M A) \quad \text{as } \varepsilon, \eta \searrow 0. \tag{4.3}$$

Proof. Since $\operatorname{trace}(A^t(M + \eta I)A) \ge \operatorname{trace}(A^t M A)$, we have

$$\inf_{\substack{A \in \mathcal{A} \\ A \le \phi(\varepsilon) I}} \operatorname{trace}(A^t(M + \eta I)A) \ge \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t M A).$$

Let us fix $M \in \Gamma_{\mathcal{A}}$ and $\delta > 0$. We consider $A_0 \in \mathcal{A}$ such that

$$\inf_{A \in \mathcal{A}} \operatorname{trace}(A^t M A) + \delta/2 \ge \operatorname{trace}(A_0^t M A_0).$$

Since $\phi(\varepsilon) \to \infty$ as $\varepsilon \to 0$ there exists ε_0 such that $A_0 \le \phi(\varepsilon) I$ for every $\varepsilon < \varepsilon_0$. Let $\eta_0 > 0$ be such that $\operatorname{trace}(A_0^t A_0)\eta_0 < \delta/2$, we obtain

$$\begin{aligned} \inf_{\substack{A \in \mathcal{A} \\ A \le \phi(\varepsilon) I}} \operatorname{trace}(A^t(M + \eta I)A) &\le \operatorname{trace}(A_0^t(M + \eta I)A_0) \\ &= \operatorname{trace}(A_0^t M A_0) + \operatorname{trace}(A_0^t A_0)\eta \\ &\le \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t M A) + \delta/2 + \delta/2 \end{aligned}$$

for every $\varepsilon < \varepsilon_0$ and $\eta < \eta_0$. We have proved (4.3). □

Lemma 4.2 allows us to obtain an upper bound for

$$\inf_{\substack{A \in \mathcal{A} \\ A \le \phi(\varepsilon) I}} \fint_{B_\varepsilon(0)} u(x + Ay)\,dy - u(x). \tag{4.4}$$

To conclude the lower bound we need a different argument and it is at this point that we use that F is continuous in $\Gamma_{\mathcal{A}}$. The continuity of F is a necessary condition in order to have a mean-value property, see Example 4.6.

Proof of Theorem 1.1. From (4.2) it follows that $A \le \phi(\varepsilon) I$ and $|y| \le \varepsilon$ imply $x + Ay \in B_\delta(x)$ for $\varepsilon < \varepsilon_0$, where $\varepsilon\phi(\varepsilon) < \delta$ for every $\varepsilon < \varepsilon_0$ and δ is as in (3.5). Therefore, by (3.5), if $\varepsilon < \varepsilon_0$, then

$$u(x + Ay) \le P_\eta^+(x + Ay) \quad \text{for every } y \in B_\varepsilon(0), \tag{4.5}$$

for

$$P_\eta^+(z) = P(z) + \frac{\eta}{2}|z - x|^2$$

with P given by (3.4). Then, following the proof of Theorem 1.6, we have

$$\fint_{B_\varepsilon(0)} u(x + Ay)\,dy - u(x) \le \frac{\varepsilon^2}{2(n+2)} \operatorname{trace}(A^t(D^2 u(x) + \eta I)A). \tag{4.6}$$

By (4.3) this gives us an upper bound of (4.4).

To obtain a lower bound we use the continuity of F. Given x and $\eta > 0$ there exists $\delta > 0$ such that

$$F(D^2 u(z)) \ge F(D^2 u(x)) - \eta$$

for every $z \in B_\delta(x)$. Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$, $y \in B_\varepsilon(0)$, and $A \in \mathcal{A}$ with $A \le \phi(\varepsilon) I$ we have $x + Ay \in B_\delta(x)$. Let us fix $A \in \mathcal{A}$. We have

$$\operatorname{trace}(A^t D^2 u(x + Ay)A) \ge F(D^2 u(x + Ay)) \ge F(D^2 u(x)) - \eta.$$

We consider $v(y) = u(x + Ay)$, and we observe that

$$\Delta v(y) = \operatorname{trace}(A^t D^2 u(x + Ay)A) \geq F(D^2 u(x)) - \eta.$$

Therefore, from the mean-value formula for the Laplacian, we obtain

$$\frac{\varepsilon^2}{2(n+2)}(F(D^2u(x)) - \eta) \leq \fint_{B_\varepsilon(0)} v(y)\mathrm{d}y - v(0) = \fint_{B_\varepsilon(0)} u(x + Ay)\mathrm{d}y - u(x).$$

With this lower bound, taking infima and the limit as $\eta \to 0$, we have completed the proof. □

Remark 4.3. Our assumptions imply that $F(D^2u(x)) > -\infty$ (since u is assumed to be $\mathcal{A}$-admissible). Observe that in the case that $F(D^2u(x)) = -\infty$ by the upper bound in (4.6) we also obtain

$$\inf_{A \in \mathcal{A}} \fint_{B_\varepsilon(0)} u(x + Ay)\mathrm{d}y - u(x) = -\infty.$$

Observe that condition (1.7) is necessary. In Example 4.6, this assumption is not satisfied and the mean-value formula fails.

Remark 4.4. In some important examples, we have

$$F(M) = 0 \quad \text{for every } M \in \partial\Gamma_{\mathcal{A}} \tag{4.7}$$

and

$$\Gamma_{\mathcal{A}} = \{M \in S^n(\mathbb{R}) : \operatorname{trace}(A^t M A) \geq 0 \text{ for all } A \in \mathcal{A}\}.$$

This holds for the Monge-Ampère equation, see (1.5), and more generally for the k-Hessians, see Lemma 5.1 and Remark 5.2. Condition (4.7) was used in [15] in relation to existence and uniqueness of solutions to general fully nonlinear second-order PDEs. We observe that condition (4.7) implies the continuity of F in $\Gamma_{\mathcal{A}}$ and therefore Theorem 1.1 can be proved assuming (4.7) instead of (1.7). In fact, if there exists $C \in \mathbb{R}$ such that the cone $\Gamma_{\mathcal{A}} = \{M : F(M) > -\infty\}$ can be written as

$$\Gamma_{\mathcal{A}} = \{M : F(M) \geq C\}$$

and $F \equiv C$ in $\partial\Gamma_{\mathcal{A}}$, then F is continuous, that is, (1.7) holds. This can be easily deduced from the fact that F is lower semi-continuous on $\partial\Gamma_{\mathcal{A}}$ because F attains its minimum at every point of the boundary.

In the following example, we show that condition (4.7) is not necessary by exhibiting an operator that does not satisfy (4.7) for which the mean-value property holds. Here, F is not constant on the boundary of the associated cone; nevertheless, Theorem 1.1 applies since F is continuous.

Example 4.5. If we consider

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} : n = 0, 1, 2, \dots \right\},$$

then we have

$$F(D^2u) = \inf_{A \in \mathcal{A}} \operatorname{trace}(A^t D^2 u A) = \begin{cases} u_{x_1x_1} & \text{if } u_{x_2x_2} \geq 0 \\ -\infty & \text{if } u_{x_2x_2} < 0. \end{cases}$$

The equation $F(D^2u) = f$ is equivalent to have $u_{x_1x_1} = f$ and $u_{x_2x_2} \geq 0$. It is not a nice equation, in the sense that it is overdetermined. Observe, however, that F is continuous where it is finite and therefore our result applies to this case and we have a mean value formula for this operator.

Associated with this F, we have the closed cone

$$\Gamma = \{M : M_{22} \geq 0\} = \{M : F(M) > -\infty\}.$$

It is interesting that in this case F is not constant in the boundary of Γ. Observe that both

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

belong to Γ. Even more, they belong to the boundary of Γ, in fact

$$\begin{bmatrix} 0 & 0 \\ 0 & -\delta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & -\delta \end{bmatrix}$$

are not in Γ for every $\delta > 0$. Hence, F is not constant on the boundary of the cone since we have

$$F\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 \quad \text{and} \quad F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1.$$

The second example shows that continuity of F is necessary for the validity of a mean-value formula.

Example 4.6. We provide an example showing that the continuity of F is a necessary condition for Theorem 1.1. We consider the set $\mathcal{A} = \{A_\delta\}_{\delta>0}$ where

$$A_\delta = \begin{bmatrix} \sqrt{2}/\sqrt{\delta} & 0 \\ 0 & 1/\delta \end{bmatrix}$$

and define

$$F(D^2u) = \inf_{A\in\mathcal{A}} \operatorname{trace}(A^t D^2uA) = \inf_{\delta>0}\left(\frac{2}{\delta}u_{x_1x_1} + \frac{1}{\delta^2}u_{x_2x_2}\right).$$

We have

$$F(D^2u) = \begin{cases} -\infty & \text{if } u_{x_2x_2} < 0 \\ -\infty & \text{if } u_{x_2x_2} = 0 \text{ and } u_{x_1x_1} < 0 \\ 0 & \text{if } u_{x_2x_2} \geq 0 \text{ and } u_{x_1x_1} \geq 0 \\ -\dfrac{u_{x_1x_1}}{u_{x_2x_2}} & \text{if } u_{x_2x_2} > 0 \text{ and } u_{x_1x_1} < 0 \end{cases}$$

(Figure 1). Observe that F is not continuous at the origin, even if we restrict the domain to the set where it is finite.

Consider

$$u(x_1, x_2) = -|x_1|^{5/2} + x_1^2x_2^2 + x_2^{10}.$$

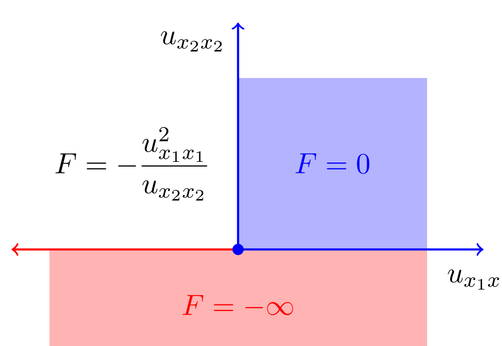

Figure 1: The operator F in Example 4.6.

Observe that $u \in C^2$ and $\mathcal{A}$-admissible. In fact, we have

$$u_{x_2x_2}(x_1, x_2) = 2x_1^2 + 90x_2^8 \geq 0,$$

with equality only at the origin, with $u_{x_1x_1}(0,0) = 0$ (note that $D^2u(0,0) = 0$). Therefore, we have that $F(D^2u(x_1, x_2)) > -\infty$ for every $(x_1, x_2) \in \mathbb{R}^2$.

Let us show that the mean-value formula in Theorem 1.1 is not satisfied at $(0,0)$, i.e.,

$$\inf_{\delta>0} \fint_{B_\varepsilon(0)} u(A_\delta y)dy - u(0,0) \neq \frac{\varepsilon^2}{2(n+2)} F(D^2u(0,0)) + o(\varepsilon^2).$$

Since $D^2u(0,0) = 0$, we have $F(D^2u(0,0)) = 0$ and we only need to prove that

$$\inf_{\delta>0} \fint_{B_\varepsilon(0)} u(A_\delta y)dy \neq o(\varepsilon^2). \tag{4.8}$$

An explicit computation shows that

$$\fint_{B_\varepsilon(0)} u(A_\delta y)dy = \fint_{B_\varepsilon(0)} \left(\frac{-2^{5/4}|y_1|^{5/2}}{\delta^{5/4}} + \frac{2y_1^2y_2^2}{\delta^3} + \frac{y_2^{10}}{\delta^{10}} \right) dy = -C_1 \frac{\varepsilon^{5/2}}{\delta^{5/4}} + C_2 \frac{\varepsilon^4}{\delta^3} + C_3 \frac{\varepsilon^{10}}{\delta^{10}}.$$

For $\delta = \varepsilon^{1/2}$ we have

$$\frac{\varepsilon^{5/2}}{\delta^{5/4}} = \varepsilon^{15/8}, \quad \frac{\varepsilon^4}{\delta^3} = \varepsilon^{5/2}, \quad \text{and} \quad \frac{\varepsilon^{10}}{\delta^{10}} = \varepsilon^5,$$

and we obtain

$$\inf_{\delta>0} \fint_{B_\varepsilon(0)} u(A_\delta y)dy \leq -C\varepsilon^{15/8},$$

which proves (4.8). Therefore, the mean-value formula in Theorem 1.1 does not hold in this case.

5 *k*-Hessian operators

A relevant example where Theorem 1.1 applies is the k-Hessian operators. For $k = 2, \ldots, n$ the k-Hessian operators are given by elementary symmetric polynomials in the eigenvalues of the Hessian, i.e.,

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2} \ldots \lambda_{i_k}.$$

We write the operators in the form

$$F_k(D^2u(x)) = k\sigma_k(\lambda(D^2u(x)))^{\frac{1}{k}}, \tag{5.1}$$

where $\lambda(D^2u(x)) = (\lambda_1(D^2u(x)), \ldots, \lambda_n(D^2u(x)))$, see Lemma 1.4.

We recall some definitions and properties of elementary symmetric polynomials, see for example [36]. We define the cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \quad \text{for all } j = 1, \ldots, k\}.$$

With a slight abuse of notation we write $M \in \Gamma_k$ to denote that $\lambda(M) \in \Gamma_k$. We have

$$\overline{\Gamma}_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) \geq 0 \quad \text{for all } j = 1, \ldots, k\}$$

and $\overline{\Gamma}_k^\circ = \Gamma_k$. Let us define

$$\sigma_{k-1,i}(y_1, \ldots, y_n) = \sigma_{k-1}(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n)$$

and

$$\mathcal{A}_k = \{A : \lambda_i^2(A) = \sigma_{k-1,i}(\gamma) \text{ with } \gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma_k \text{ and } \sigma_k(\gamma) = 1\}.$$

Then, we have $\mathcal{A}_k \subset S_+^n(\mathbb{R})$ since

$$\sigma_{k-1,i}(\gamma) > 0, \quad \forall \gamma \in \Gamma_k,$$

see [36]. Also observe that by the continuity of $\sigma_{k-1,i}(\gamma)$ we obtain $\sigma_{k-1,i}(\gamma) \geq 0$ for every $\gamma \in \overline{\Gamma}_k$.

Our goal is to find the k-Hessian counterpart of formula (1.5), which holds for the Monge-Ampère operator ($k = n$). We do that in two steps, first show the result for numbers, then for matrices.

Lemma 5.1. *For every $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ we have*

$$\inf_{\substack{\gamma \in \Gamma_k \\ \sigma_k(\gamma)=1}} \sum_{i=1}^n \mu_i \sigma_{k-1,i}(\gamma) = \begin{cases} k\sigma_k(\mu)^{\frac{1}{k}} & \text{if } \mu \in \overline{\Gamma}_k, \\ -\infty & \text{otherwise}. \end{cases}$$

Remark 5.2. Note that

$$\inf_{\substack{\gamma \in \Gamma_k \\ \sigma_k(\gamma)=1}} \sum_{i=1}^n \mu_i \sigma_{k-1,i}(\gamma) = 0 \quad \text{if } \mu \in \partial\Gamma_k,$$

since

$$\partial\Gamma_k = \{M \in S^n(\mathbb{R}) : \sigma_k(\lambda(M)) = 0 \quad \text{and} \quad \sigma_j(\lambda(M)) \geq 0 \quad \text{for all } j = 1, \dots, k-1\}. \tag{5.2}$$

Proof of Lemma 5.1. From formula (1.3) in [11] (see also (xi) in [36]), we have that for all $\mu, \gamma \in \Gamma_k$,

$$k\sigma_k(\mu)^{\frac{1}{k}} \sigma_k(\gamma)^{\frac{k-1}{k}} \leq \sum_{i=1}^n \mu_i \sigma_{k-1,i}(\gamma),$$

with an equality for $\gamma^* = \sigma_k(\mu)^{-\frac{1}{k}}\mu$, i.e.,

$$\sum_{i=1}^n \mu_i \sigma_{k-1,i}(\gamma^*) = \sum_{i=1}^n \mu_i \frac{\sigma_{k-1,i}(\mu)}{\sigma_k(\mu)^{\frac{k-1}{k}}} = \frac{k\sigma_k(\mu)}{\sigma_k(\mu)^{\frac{k-1}{k}}} = k\sigma_k(\mu)^{\frac{1}{k}}.$$

For $\mu \in \partial\Gamma_k$, observe that $(\mu_1 + \varepsilon, \dots, \mu_n + \varepsilon) \in \Gamma_k$ for every $\varepsilon > 0$. We consider

$$\gamma_i^\varepsilon = (\mu_i + \varepsilon)\sigma_k(\mu_1 + \varepsilon, \dots, \mu_n + \varepsilon)^{-\frac{1}{k}},$$

and we have

$$k\sigma_k(\mu_1 + \varepsilon, \dots, \mu_n + \varepsilon)^{\frac{1}{k}} = \sum_{i=1}^n (\mu_i + \varepsilon)\sigma_{k-1,i}(\gamma^\varepsilon) \geq \sum_{i=1}^n \mu_i \sigma_{k-1,i}(\gamma^\varepsilon),$$

since $\sigma_{k-1,i}(\gamma^\varepsilon) \geq 0$. To conclude observe that

$$k\sigma_k(\mu_1 + \varepsilon, \dots, \mu_n + \varepsilon)^{\frac{1}{k}} \to k\sigma_k(\mu)^{\frac{1}{k}}$$

as $\varepsilon \to 0$.

The only remaining case is when $\mu \notin \overline{\Gamma}_k$. Note that if $a > 0$ is large enough such that $\mu_i + a > 0$ for every $i \in \{1, \dots, n\}$, we obtain $(\mu_1 + a, \dots, \mu_n + a) \in \Gamma_k$. Thus, there exists $a > 0$ such that $(\mu_1 + a, \dots, \mu_n + a) \in \partial\Gamma_k$. Observe that $\sigma_k(\mu_1 + a, \dots, \mu_n + a) = 0$ by (5.2). Let us fix one such value a and consider

$$\gamma_i^\varepsilon = (\mu_i + a + \varepsilon)\sigma_k(\mu_1 + a + \varepsilon, \dots, \mu_n + a + \varepsilon)^{-\frac{1}{k}}.$$

As before, we have

$$\sum_{i=1}^n (\mu_i + a + \varepsilon)\sigma_{k-1,i}(\gamma^\varepsilon) = k\sigma_k(\mu_1 + a + \varepsilon, \dots, \mu_n + a + \varepsilon)^{\frac{1}{k}},$$

and therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\mu_i\sigma_{k-1,i}(\gamma^{\varepsilon}) &= -\lim_{\varepsilon\to 0}(a+\varepsilon)\sum_{i=1}^{n}\sigma_{k-1,i}(\gamma^{\varepsilon}) + k\lim_{\varepsilon\to 0}\sigma_k(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon)^{\frac{1}{k}}\\ &= -a(n-k+1)\lim_{\varepsilon\to 0}\sigma_{k-1}(\gamma^{\varepsilon})\\ &= -a(n-k+1)\lim_{\varepsilon\to 0}\left(\frac{\sigma_{k-1}(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon)}{\sigma_k(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon)^{\frac{k-1}{k}}}\right).\end{aligned}$$

Recall that

$$\lim_{\varepsilon\to 0}\sigma_k(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon) = \sigma_k(\mu_1+a,\ldots,\mu_n+a) = 0,$$

and if we have

$$\lim_{\varepsilon\to 0}\sigma_{k-1}(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon) = \sigma_{k-1}(\mu_1+a,\ldots,\mu_n+a) > 0$$

we can conclude that

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\mu_i\sigma_{k-1,i}(\gamma^{\varepsilon}) = -\infty$$

as desired. If this is not the case we have to look at the rate of convergence in more detail.

We consider two cases. First we assume that $\mu_i + a \neq 0$ for some $i \in \{1,\ldots,n\}$. In this case, we have that $\sigma_j(\mu_1+a,\ldots,\mu_n+a) > 0$ for some $j \in \{1,\ldots,k-1\}$ because otherwise

$$\sum_{i=1}^{n}(\mu_n+a)^2 = \sigma_1^2(\mu_1+a,\ldots,\mu_n+a) - 2\sigma_2(\mu_1+a,\ldots,\mu_n+a) = 0,$$

a contradiction. Let $l \in \{1,\ldots,k-1\}$ be the largest integer such that $\sigma_l(\mu_1+a,\ldots,\mu_n+a) > 0$.

We have

$$\begin{aligned}&\sigma_k(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon)\\ &= \sum_{i=0}^{k}\binom{n-i}{n-k}\sigma_i(\mu_1+a,\ldots,\mu_n+a)\varepsilon^{k-i},\\ &= \varepsilon^{k-l}\left[\binom{n}{n-k}\varepsilon^{l} + \binom{n-1}{n-k}\sigma_1(\mu_1+a,\ldots,\mu_n+a)\varepsilon^{l-1} + \cdots + \binom{n-l}{n-k}\sigma_l(\mu_1+a,\ldots,\mu_n+a)\right]\\ &= \varepsilon^{k-l}\left[\binom{n-l}{n-k}\sigma_l(\mu_1+a,\ldots,\mu_n+a) + O(\varepsilon)\right]\end{aligned}$$

as $\varepsilon \to 0$ (in the first equality we have defined $\sigma_0(\mu_1+a,\ldots,\mu_n+a) = 1$). Similarly,

$$\sigma_{k-1}(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon) = \varepsilon^{k-1-l}\left[\binom{n-l}{n-k+1}\sigma_l(\mu_1+a,\ldots,\mu_n+a) + O(\varepsilon)\right]$$

as $\varepsilon \to 0$. Therefore,

$$\frac{\sigma_{k-1}(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon)}{\sigma_k(\mu_1+a+\varepsilon,\ldots,\mu_n+a+\varepsilon)^{\frac{k-1}{k}}} = \left(\frac{\binom{n-l}{n-k+1}\sigma_l(\mu_1+a,\ldots,\mu_n+a) + O(\varepsilon)}{\left[\binom{n-l}{n-k}\sigma_l(\mu_1+a,\ldots,\mu_n+a) + O(\varepsilon)\right]^{\frac{k-1}{k}}}\right)\varepsilon^{-\frac{l}{k}}$$

as $\varepsilon \to 0$ and we conclude that

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{n}\mu_i\sigma_{k-1,i}(\gamma^{\varepsilon}) = -\infty.$$

Finally, in the case that $\mu_i = -a$ for every $i \in \{1, \dots, n\}$ we consider $y_b = (1, \dots, 1, b, 1/b, 0, \dots, 0)$ where $b > 0$ and $k - 2$ coordinates are equal to 1. We have $y_b \in \Gamma_k$, $\sigma_k(y_b) = 1$ and

$$\sum_{i=1}^{n} \mu_i \sigma_{k-1,i}(y_b) = -a \sum_{i=1}^{n} \sigma_{k-1,i}(y_b) = -a(n-k+1)\sigma_{k-1}(y_b) = -a(n-k+1)\left(k - 2 + b + \frac{1}{b}\right)$$

which goes to $-\infty$ as $b \to \infty$. □

We are now ready to show Lemma 1.4, the matrix counterpart of Lemma 5.1.

Proof of Lemma 1.4. Given $M \in S^n(\mathbb{R})$ we consider $A \in \mathcal{A}_k$ such that both matrices are diagonal in the same basis. We have

$$\operatorname{trace}(A^t M A) = \operatorname{trace}(A A^t M) = \sum_{i=1}^{n} \lambda_i(AA^t)\lambda_i(M) = \sum_{i=1}^{n} \lambda_i^2(A)\lambda_i(M).$$

Then, by the definition of $\mathcal{A}_k$ and Lemma 5.1 we obtain that

$$\inf_{A \in \mathcal{A}_k} \operatorname{trace}(A^t M A) \le \inf_{\substack{y \in \Gamma_k \\ \sigma_k(y)=1}} \sum_{i=1}^{n} \lambda_i(M)\sigma_{k-1,i}(y) = \begin{cases} k\sigma_k(\lambda(M))^{\frac{1}{k}} & \text{if } M \in \overline{\Gamma}_k, \\ -\infty & \text{otherwise}. \end{cases}$$

Therefore, it only remains to prove that

$$\operatorname{trace}(A^t M A) \ge k\sigma_k(\lambda(M))^{\frac{1}{k}} \tag{5.3}$$

for every $A \in \mathcal{A}_k$ and $M \in \overline{\Gamma}_k$. To that end, we recall the following inequality by Marcus (see [26])

$$\operatorname{trace}(XM) \ge \min_p \sum_{i=1}^{n} \lambda_i(M)\lambda_{p(i)}(X),$$

where p ranges over the permutations of the numbers $\{1, \dots, n\}$. Recalling that $A \in \mathcal{A}_k$ we obtain

$$\operatorname{trace}(A^t M A) \ge \min_p \sum_{i=1}^{n} \lambda_i(M)\sigma_{k-1,p(i)}(y)$$

for some $y \in \Gamma_k$ such that $\sigma_k(y) = 1$. Let $\tilde{p}$ be the permutation where the minimum is attained and observe that

$$\sigma_{k-1,\tilde{p}(i)}(y) = \sigma_{k-1,i}(\tilde{y}),$$

where $\tilde{y}$ is such that $\tilde{y}_i = y_{\tilde{p}(i)}$. We have

$$\operatorname{trace}(A^t M A) \ge \sum_{i=1}^{n} \lambda_i(M)\sigma_{k-1,i}(\tilde{y}).$$

Observe that $\tilde{y} \in \Gamma_k$ and $\sigma_k(\tilde{y}) = 1$, hence, by Lemma 5.1 we have

$$\sum_{i=1}^{n} \lambda_i(M)\sigma_{k-1,i}(\tilde{y}) \ge k\sigma_k(\lambda(M))^{\frac{1}{k}}$$

and (5.3) follows. □

Recall that a function $u \in C^2(\Omega)$ is called k-convex whenever $D^2u(x) \in \overline{\Gamma}_k$ for every $x \in \Omega$, see [33–35]. As a consequence of Lemma 1.4 we have the following result.

Corollary 5.3. *A function $u \in C^2(\Omega)$ is k-convex if and only if it is $\mathcal{A}_k$-admissible. In other words, we have*

$$\Gamma_{\mathcal{A}_k} = \overline{\Gamma}_k.$$

As a consequence of Lemma 1.4 we can obtain Theorem 1.5 from Theorem 1.1.

Proof of Theorem 1.5. From Corollary 5.3, we obtain $\Gamma_{\mathcal{A}_k} = \bar{\Gamma}_k$, i.e., it is equivalent to be k-convex and $\mathcal{A}_k$-admissible. Also observe that F_k is continuous in $\mathcal{A}_k$. Therefore, we are under the hypothesis of Theorem 1.1 and the result follows. □

6 Examples in the Heisenberg group

Let $\mathbb{H} = (\mathbb{R}^3, *)$ be the Heisenberg group. We write a point $q \in \mathbb{H}$ as $q = (x, y, z)$. The point $\bar{q} = (x, y)$ is the horizontal projection of q. When convenient, we will also use the notation $q = (q_1, q_2, q_3) = (\bar{q}, q_3)$. The group operation is

$$q * q' = (x, y, z) * (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')\right).$$

The Korániy gauge is given by

$$|q|_K = ((x^2 + y^2)^2 + 16z^2)^{1/4}.$$

It induces a left-invariant metric $d(q, q') = |q^{-1} * q'|_K$ in $\mathbb{H}$. We also have a family of anisotropic dilations:

$$\rho_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z), \quad \lambda > 0$$

that are group homomorphisms. The Koràniy gauge and the Koràniy metric are homogeneous with respect to the dilations

$$|\rho_\lambda(q)|_K = \lambda |q|_K, \quad d(\rho_\lambda(q), \rho_\lambda(q')) = \lambda \mathrm{d}(q, q').$$

The open ball centered at q with radius $r > 0$ is a translation and dilation of the open ball centered at 0 of radius 1

$$B_r(q) = \{q' \in \mathbb{H}; d(q, q') < r\} = q * B_r(0) = q * \rho_r(B_1(0)).$$

Euclidean balls in two dimensions will be denoted by B_r^2.

The vector fields

$$X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = \partial_z$$

are left-invariant and form a basis for the Lie algebra of $\mathbb{H}$. The only nontrivial commuting relation is $Z = [X, Y]$. The horizontal tangent space at the point q is the plane generated by $X(q)$ and $Y(q)$

$$T_q = \operatorname{span}\left\{\left(1, 0, -\frac{y}{2}\right), \left(0, 1, \frac{x}{2}\right)\right\} = \left(\frac{y}{2}, -\frac{x}{2}, 1\right)^{\perp}. \tag{6.1}$$

The horizontal gradient and the sub-Laplacian of a function $v : \mathbb{H} \to \mathbb{R}$ are, respectively, the vector field and function

$$\nabla_{\mathbb{H}} v = (Xv)X + (Yv)Y, \quad \Delta_{\mathbb{H}} v = (X^2 + Y^2)v.$$

The horizontal second derivatives are given by the non necessarily symmetric 2×2 matrix

$$\nabla^2_{\mathbb{H}} v(q) = \begin{pmatrix} X^2 v(q) & XYv(q) \\ YXv(q) & Y^2 v(q) \end{pmatrix}.$$

The symmetrized horizontal second derivatives are $(\nabla^2_{\mathbb{H}} v(q))^* = \frac{1}{2}(\nabla^2_{\mathbb{H}} v(q) + (\nabla^2_{\mathbb{H}} v(q))^t)$. Our starting point is the Taylor expansion for a function $v \in C^2(\mathbb{H})$ at a point $q = (x, y, z) \in \mathbb{H}$ adapted to the Heisenberg group, which we take from Section 3 in [22],

$$v(p) = v(q) + \langle (\nabla_{\mathbb{H}}, Z)v(q), q^{-1} * p \rangle + \frac{1}{2}\langle (\nabla^2_{\mathbb{H}} v(q))^* \overline{q^{-1} * p}, \overline{q^{-1} * p} \rangle + o(|q^{-1} * p|_K^2). \tag{6.2}$$

Next, we consider the horizontal average operator

$$\mathcal{A}_2(v,\varepsilon)(q) = \fint_{B^2_\varepsilon(0)} v(q * (a,b,0))\,\mathrm{d}a\mathrm{d}b = \fint_{B^2_1(0)} v(q + \varepsilon(a,b,\tfrac{1}{2}(xb - ya)))\,\mathrm{d}a\mathrm{d}b.$$

From Proposition 2.3 in [22] we obtain

Lemma 6.1.

$$\mathcal{A}_2(v,\varepsilon)(q) - v(q) = \frac{\varepsilon^2}{8}\Delta_{\mathbb{H}} v(q) + o(\varepsilon^2), \text{ as } \varepsilon \to 0.$$

By changing variables we obtain averages over general horizontal planes.

Lemma 6.2. *Let A be a 2×2 matrix and let $\varepsilon > 0$. We have*

$$\fint_{B^2_1(0)} v(q * \varepsilon(A\cdot(a,b),0))\,\mathrm{d}a\mathrm{d}b - v(q) = \frac{\varepsilon^2}{8}\mathrm{trace}(A^t(\nabla^2_{\mathbb{H}} v(q))^* A) + o(\varepsilon^2|A|^2)$$

as $\varepsilon \to 0$.

Proof. Set $p = q * \varepsilon(A\cdot(a,b),0)$, where $(a,b)\in B^2_1(0)$. Observe that $q^{-1} * p = \varepsilon(A\cdot(a,b),0)$ so that $\overline{q^{-1} * p} = \varepsilon A\cdot(a,b)$. Writing the Taylor expansion (6.2) we obtain

$$\begin{aligned} v(p) &= v(q) + \langle(\nabla_{\mathbb{H}}, Z)v(q), (\varepsilon A\cdot(a,b),0)\rangle + \frac{\varepsilon^2}{2}\langle(\nabla^2_{\mathbb{H}} v(q))^* \cdot A\cdot(a,b), A\cdot(a,b)\rangle + o(|\varepsilon A\cdot(a,b),0)|^2_K \\ &= v(q) + \varepsilon\langle\nabla_{\mathbb{H}} v(q), A\cdot(a,b)\rangle + \frac{\varepsilon^2}{2}\langle(A^t\cdot\nabla^2_{\mathbb{H}} v(q))^* \cdot A\cdot(a,b),(a,b)\rangle + o(|\varepsilon A\cdot(a,b)|^2). \end{aligned}$$

□

We now state a version of Theorem 1.1 in the Heisenberg group for the expression

$$F(D^2 v(q)) = 2(\det(\nabla^2_{\mathbb{H}} v(q))^*)^{1/2}.$$

Recall that we have

$$\inf_{A\in\mathcal{A}} \mathrm{trace}(A^t M A) = \begin{cases} 2(\det M)^{1/2} & \text{if } M \geq 0 \\ -\infty & \text{otherwise}, \end{cases}$$

for the unbounded set $\mathcal{A} = \{A \in S^2_+(\mathbb{R}) : \det(A) = 1\}$, for example see [4].

We say that a function $v \in C^2$ is horizontally convex if

$$(\nabla^2_{\mathbb{H}} v(q))^* \geq 0. \tag{6.3}$$

Note that this condition can be extended in the viscosity sense, when v is only upper semicontinuous, by asking that the test functions $\phi \in C^2$ that touch v from above at a point q satisfy (6.3). This condition is equivalent to asking that the function of one real variable

$$t \mapsto v(p\cdot\delta_t(h))$$

is convex for t near zero. Here $h = (h_1, h_2, 0)$ is a horizontal vector. For this, and many other characterizations of convexity on the Heisenberg group, and Carnot groups in general, see [24], [17], and [14].

Theorem 6.3. *Let $v \in C^2$ be horizontally convex and $\phi(\varepsilon)$ a positive function satisfying* (1.8). *Then, we have*

$$\inf_{\substack{\det(A)=1 \\ A\leq\phi(\varepsilon)I}} \fint_{B^2_1(0)} v(q * (A(a,b),0))\,\mathrm{d}a\mathrm{d}b - v(q) = \frac{\varepsilon^2}{4}\left(\det(\nabla^2_{\mathbb{H}} v(q))^*\right)^{1/2} + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

The proof of this theorem is a straightforward adaptation of the proof of Theorem 1.1 in [4]. Note that the two-dimensional averages are taken over ellipsoids in the horizontal tangent plane T_q defined in (6.1).

As in the Euclidean case, we can use this theorem to characterize horizontally convex viscosity solutions of the equation

$$\det(\nabla^2_{\mathbb{H}} v(q))^* = f(q), \tag{6.4}$$

where f is continuous and nonnegative.

Theorem 6.4. *Let v be horizontally convex and $\phi(\varepsilon)$ a positive function satisfying* (1.8). *Then, we have that v is a viscosity subsolution* (*respectively, supersolution*) *of equation* (6.4) *if and only if the expansion*

$$v(q) \leq \inf_{\substack{\det(A)=1 \\ A \leq \phi(\varepsilon) I}} \fint_{B_1^2(0)} v(q * (A(a,b),0)) \mathrm{d}a\mathrm{d}b - \frac{\varepsilon^2}{4}(f(q))^{1/2} + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0. \tag{6.5}$$

(*respectively* $\geq$) *holds in the viscosity sense; that is, whenever a horizontal paraboloid*

$$P(p) = a + \langle k, q^{-1} * p\rangle + \frac{1}{2}\langle A\overline{q^{-1} * p}, \overline{q^{-1} * p}\rangle,$$

where $k \in \mathbb{R}^3$ and A is 2×2 *symmetric matrix, touches u from above* (*respectively, below*) *at q, the expansion* (6.5) *holds for P at the point q.*

The proof is a straightforward adaptation of the viscosity proof in [4].

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