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#### **Research Article**

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## Non-Parametric Mean Curvature Flow with Prescribed Contact Angle in Riemannian Products

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**Abstract:** Assuming that there exists a translating soliton  $u_{\infty}$  with speed *C* in a domain  $\Omega$  and with prescribed contact angle on  $\partial\Omega$ , we prove that a graphical solution to the mean curvature flow with the same prescribed contact angle converges to  $u_{\infty} + Ct$  as  $t \to \infty$ . We also generalize the recent existence result of Gao, Ma, Wang and Weng to non-Euclidean settings under suitable bounds on convexity of  $\Omega$  and Ricci curvature in  $\Omega$ .

Keywords: Mean curvature flow; prescribed contact angle; translating graphs

MSC: Primary 53C21, 53E10

## **1** Introduction

We study a non-parametric mean curvature flow in a Riemannian product  $N \times \mathbb{R}$  represented by graphs

$$M_t := \left\{ \left( x, u(x, t) \right) \colon x \in \overline{\Omega} \right\}$$
(1.1)

with prescribed contact angle with the cylinder  $\partial \Omega \times \mathbb{R}$ .

We assume that *N* is a Riemannian manifold and  $\Omega \Subset N$  is a relatively compact domain with smooth boundary  $\partial \Omega$ . We denote by  $\gamma$  the inward pointing unit normal vector field to  $\partial \Omega$ . The boundary condition is determined by a given smooth function  $\phi \in C^{\infty}(\partial \Omega)$ , with  $|\phi| \le \phi_0 < 1$ , and the initial condition by a smooth function  $u_0 \in C^{\infty}(\overline{\Omega})$ .

The function u above in (1.1) is a solution to the following evolution equation

$$\begin{cases} \frac{\partial u}{\partial t} = W \operatorname{div} \frac{\nabla u}{W} & \text{in } \Omega \times [0, \infty), \\ \frac{\partial_{\gamma} u}{W} := \frac{\langle \nabla u, \gamma \rangle}{W} = \phi & \text{on } \partial \Omega \times [0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \bar{\Omega}, \end{cases}$$
(1.2)

where  $W = \sqrt{1 + |\nabla u|^2}$  and  $\nabla u$  denotes the gradient of u with respect to the Riemannian metric on N at  $x \in \overline{\Omega}$ . The boundary condition above can be written as

$$\langle v, \gamma \rangle = \phi,$$
 (1.3)

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where v is the downward pointing unit normal to the graph of u, i.e.

$$u(x) = rac{
abla u(x, \cdot) - \partial_t}{\sqrt{1 + |
abla u(x, \cdot)|^2}}, \ x \in \overline{\Omega}.$$

The longtime existence of the solution  $u_t := u(\cdot, t)$  to (1.2) and convergence as  $t \to \infty$  have been studied under various conditions on  $\Omega$  and  $\phi$ . Huisken [5] proved the existence of a smooth solution in a  $C^{2,\alpha}$ -smooth bounded domain  $\Omega \subset \mathbb{R}^n$  for  $u_0 \in C^{2,\alpha}(\overline{\Omega})$  and  $\phi \equiv 0$ . Moreover, he showed that  $u_t$  converges to a constant function as  $t \to \infty$ . In [1] Altschuler and Wu complemented Huisken's results for prescribed contact angle in case  $\Omega$  is a smooth bounded strictly convex domain in  $\mathbb{R}^2$ . Guan [4] proved a priori gradient estimates and established longtime existence of solutions in case  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain. Recently, Zhou [8] studied mean curvature type flows in a Riemannian product  $M \times \mathbb{R}$  and proved the longtime existence of the solution for relatively compact smooth domains  $\Omega \subset M$ . Furthermore, he extended the convergence result of Altschuler and Wu to the case M is a Riemannian surface with nonnegative curvature and  $\Omega \subset M$  is a smooth bounded strictly convex domain; see [8, Theorem 1.4].

The key ingredient, and at the same time the main obstacle, for proving the uniform convergence of  $u_t$  has been a difficulty to obtain a time-independent gradient estimate. We circumvent this obstacle by modifying the method of Korevaar [6], Guan [4] and Zhou [8] and obtain a uniform gradient estimate in an arbitrary relatively compact smooth domain  $\Omega \subset N$  provided there exists a translating soliton with speed *C* and with the prescribed contact angle condition (1.3).

Towards this end, let *d* be a smooth bounded function defined in some neighborhood of  $\overline{\Omega}$  such that  $d(x) = \min_{y \in \partial \Omega} \operatorname{dist}(x, y)$ , the distance to the boundary  $\partial \Omega$ , for points  $x \in \Omega$  sufficiently close to  $\partial \Omega$ . Thus  $\gamma = \nabla d$  on  $\partial \Omega$ . We assume that  $0 \le d \le 1$ ,  $|\nabla d| \le 1$  and  $|\operatorname{Hess} d| \le C_d$  in  $\overline{\Omega}$ . We also assume that the function  $\phi \in C^{\infty}(\partial \Omega)$  is extended as a smooth function to the whole  $\overline{\Omega}$ , satisfying the condition  $|\phi| \le \phi_0 < 1$ .

Our main theorem is the following:

**Theorem 1.1.** Suppose that there exists a solution  $u_{\infty}$  to the translating soliton equation

$$\begin{cases} \operatorname{div} \frac{\nabla u_{\infty}}{\sqrt{1+|\nabla u_{\infty}|^{2}}} = \frac{C_{\infty}}{\sqrt{1+|\nabla u_{\infty}|^{2}}} & \text{in } \Omega, \\ \frac{\partial_{\gamma} u_{\infty}}{\sqrt{1+|\nabla u_{\infty}|^{2}}} = \phi & \text{on } \partial\Omega, \end{cases}$$
(1.4)

where  $C_{\infty}$  is given by

$$C_{\infty} = \frac{-\int_{\partial\Omega} \phi \,\mathrm{d}\sigma}{\int_{\Omega} \left(1 + |\nabla u_{\infty}|^2\right)^{-1/2} \,\mathrm{d}x}.$$
(1.5)

Then the equation (1.2) has a smooth solution  $u \in C^{\infty}(\overline{\Omega}, [0, \infty))$  with  $W \leq C_1$ , where  $C_1$  is a constant depending on  $\phi$ ,  $u_0$ ,  $C_d$ , and the Ricci curvature of  $\Omega$ . Moreover, u(x, t) converges uniformly to  $u_{\infty}(x) + C_{\infty}t$  as  $t \to \infty$ .

Notice that the existence of a solution  $u \in C^{\infty}(\overline{\Omega} \times [0, \infty))$  to (1.2) is given by [8, Corollary 4.2].

**Remark 1.2.** Very recently, Gao, Ma, Wang, and Weng [3] proved the existence of such  $u_{\infty}$  and obtained Theorem 1.1 for smooth, bounded, strictly convex domains  $\Omega \subset \mathbb{R}^n$  for sufficiently small  $|\phi|$ ; see [3, Theorem 1.1, Theorem 3.1]. It turns out that their proof can be generalized beyond the Euclidean setting under suitable bounds on the convexity of  $\Omega$  and the Ricci curvature in  $\Omega$ .

More precisely, let  $\Omega \Subset N$  be a relatively compact, strictly convex domain with smooth boundary admitting a smooth defining function h such that h < 0 in  $\Omega$ , h = 0 on  $\partial \Omega$ ,

$$(h_{i;j}) \ge k_1(\delta_{ij}) \tag{1.6}$$

for some constant  $k_1 > 0$  and  $\sup_{\Omega} |\nabla h| \le 1$ ,  $h_{\gamma} = -1$  and  $|\nabla h| = 1$  on  $\partial \Omega$ . Furthermore, by strict convexity of  $\Omega$ , the second fundamental form of  $\partial \Omega$  satisfies

$$\left(\kappa_{ij}\right)_{1\leq i,j\leq n-1}\geq\kappa_0\left(\delta_{ij}\right)_{1\leq i,j\leq n-1},\tag{1.7}$$

where  $\kappa_0 > 0$  is the minimal principal curvature of  $\partial \Omega$ . In the Euclidean case,  $N = \mathbb{R}^n$ , such functions *h* are constructed in [2]. We give some simple examples at the end of Section 3.

**Theorem 1.3.** Let  $\Omega \in N$  be a smooth, strictly convex, relatively compact domain associated with constants  $k_1 > 0$  and  $\kappa_0 > 0$  as in (1.6) and (1.7). Let  $\alpha < \min{\{\kappa_0, k_1(n-1)/2\}}$  and assume that the Ricci curvature in  $\Omega$  satisfies  $|\operatorname{Ric}| < \alpha(k_1(n-1)-\alpha)/(n+1)$ . Then there exists  $\varepsilon_0 > 0$  such that if  $\phi =: \cos \theta \in C^3(\overline{\Omega})$  satisfies  $|\cos \theta| \le \varepsilon_0 \le 1/4$  and  $||\nabla \theta||_{C^1(\overline{\Omega})} \le \varepsilon_0$  in  $\overline{\Omega}$ , there exist a unique constant  $C_{\infty}$  and a solution  $u_{\infty}$  to (1.4). Furthermore,  $u_{\infty}$  is unique up to an additive constant.

We will sketch the proof of Theorem 1.3 in Section 3.

## 2 Proof of Theorem 1.1

Let *u* be a solution to (1.2) in  $\overline{\Omega} \times \mathbb{R}$ . Given a constant  $C_{\infty} \in \mathbb{R}$  we define, following the ideas of Korevaar [6], Guan [4] and Zhou [8], a function  $\eta : \overline{\Omega} \times \mathbb{R} \to (0, \infty)$  by setting

$$\eta = e^{K(u-C_{\infty}t)} \left( Sd + 1 - \frac{\phi}{W} \langle \nabla u, \nabla d \rangle \right), \qquad (2.1)$$

where *K* and *S* are positive constants to be determined later. We start with a gradient estimate.

**Proposition 2.1.** Let u be a solution to (1.2) and define  $\eta$  as in (2.1). Then, for a fixed T > 0, letting

$$(W\eta)(x_0, t_0) = \max_{x \in \bar{\Omega}, \ t \in [0,T]} (W\eta)(x, t),$$

there exists a constant  $C_0$  only depending on  $C_d$ ,  $\phi$ ,  $C_{\infty}$ , and the lower bound for the Ricci curvature in  $\Omega$  such that  $W(x_0, t_0) \leq C_0$ .

*Proof.* Let  $g = g_{ij} dx^i dx^j$  be the Riemannian metric of *N*. We denote by  $(g^{ij})$  the inverse of  $(g_{ij})$ ,  $u_j = \partial u / \partial x^j$ , and  $u_{i;j} = u_{ij} - \Gamma_{ij}^k u_k$ . We set

$$a^{ij} = g^{ij} - \frac{u^i u^j}{W^2}$$

and define an operator L by  $Lu = a^{ij}u_{i;j} - \partial_t u$ . Observe that (1.2) can be rewritten as Lu = 0. In all the following, computations will be done at the maximum point  $(x_0, t_0)$  of  $\eta W$ . We first consider the case where  $x_0 \in \partial \Omega$ . We choose normal coordinates at  $x_0$  such that  $g_{ij} = g^{ij} = \delta^{ij}$  at  $x_0, \partial_n = \gamma$ ,

$$u_1 \ge 0$$
,  $u_i = 0$  for  $2 \le i \le n - 1$ .

This implies that

$$d_i = 0$$
 for  $1 \le i \le n - 1$ ,  $d_n = 1$ , and  $d_{i:n} = 0$  for  $1 \le i \le n$ .

We have

$$0 \ge (W\eta)_{n} = W_{n}\eta + W\eta_{n}$$

$$= e^{K(u-C_{\infty}t)} \left( SW_{n}d + W_{n} - \frac{\phi W_{n}}{W} g^{ij} u_{i}d_{j} + SWd_{n} - \frac{W}{W} \phi_{n}g^{ij} u_{i}d_{j} - \frac{W}{W} \phi g^{ij} (u_{i;n}d_{j} + u_{i}d_{j;n}) + W \frac{W_{n}}{W^{2}} \phi g^{ij} u_{i}d_{j} + KWu_{n}(Sd + 1 - \frac{\phi}{W} g^{ij} u_{i}d_{j}) \right)$$

$$= e^{K(u-C_{\infty}t)} \left( W_{n} + SW - \phi_{n}u_{n} - \phi u_{n;n} + KWu_{n}(1 - \phi^{2}) \right).$$
(2.2)

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#### Using our coordinate system, we get

$$0 \ge \frac{W_n}{W} + S - \frac{\phi_n u_n}{W} - \frac{\phi_{u_{n;n}}}{W} + K u_n (1 - \phi^2)$$
  
=  $S - \frac{u_1^2 d_{1;1}}{W^2} + \frac{u_1 \phi_1}{W} \left( 1 + \frac{2\phi^2}{1 - \phi^2} \right) - \frac{\phi_{u_1}}{W} K u_1$   
 $- \frac{\phi_n u_n}{W} + K u_n (1 - \phi^2)$   
 $\ge S - C - \frac{K \phi u_1^2}{W} + K u_n (1 - \phi^2)$   
 $= S - C - \frac{K \phi}{W} \ge S - C - \frac{K}{W},$ 

for some constant *C* depending only on  $C_d$  and  $\phi$ . So choosing  $S \ge C + 1$ , we get that

$$W(x_0, t_0) \le K. \tag{2.3}$$

Next we assume that  $x_0 \in \Omega$  and that  $S \ge C + 1$ , where *C* is as above. Let us recall from [8, Lemma 3.5] that

$$LW = \frac{2}{W}a^{ij}W_iW_j + \operatorname{Ric}(v_N, v_N)W + |A|^2W,$$

where  $v_N = \nabla u/W$  and  $|A|^2 = a^{ij} a^{\ell k} u_{i;k} u_{j;\ell}/W^2$  is the squared norm of the second fundamental form of the graph  $M_t$ . Since  $0 = W_i \eta + W \eta_i$ , for every i = 1, ..., n, we deduce that

$$0 \ge L(W\eta) = WL\eta + \eta \left( LW - 2a^{ij} \frac{W_i W_j}{W} \right) = WL\eta + \eta W \left( |A|^2 + \operatorname{Ric}(v_N, v_N) \right).$$

This yields to

$$\frac{1}{\eta}L\eta + |A|^2 + \operatorname{Ric}(\nu_N, \nu_N) \le 0.$$
(2.4)

To simplify the notation, we set

$$h = Sd + 1 - \phi u^k d_k / W = Sd + 1 - \phi v^k d_k$$

So we have

$$\frac{1}{\eta}L\eta = K^2 a^{ij} u_i u_j + KL(u - C_{\infty}t) + \frac{2K}{h} a^{ij} u_i h_j + \frac{1}{h}Lh.$$
(2.5)

We can compute *Lh* as

$$Lh = a^{ij} \left( Sd_{i;j} - (\phi d_k)_{i;j} v^k - (\phi d_k)_i v^k_j - (\phi d_k)_j v^k_i - \phi d_k L v^k \right) \ge -C - 2a^{ij} (\phi d_k)_i v^k_j - \phi d_k L v^k.$$

Since, by [8, Lemma 3.5],

$$Lv^k = \operatorname{Ric}(a^{k\ell}\partial_\ell, v_N) - |A|^2v^k$$

and, by Young's inequality for matrices,

$$a^{ij}(\phi d_k)_i v_j^k = \frac{1}{W} (\phi d_k)_i a^{ij} a^{\ell k} u_{\ell;j} \le \frac{|A|^2}{6} + C_{\ell}$$

we get the estimate

$$Lh \ge -C - |A|^2 / 3 + \phi d_k v^k |A|^2$$
(2.6)

by using the assumption that Ric is bounded.

Next we turn our attention to the other terms in (2.5). We have

$$a^{ij}u_i = \frac{u^j}{W^2}$$
 and  $a^{ij}u_iu_j = 1 - \frac{1}{W^2}$ . (2.7)

Then we note that by the assumptions, we clearly have

$$KL(u - C_{\infty}t) = KC_{\infty} \ge -KC, \qquad (2.8)$$

and we are left to consider

$$a^{ij}u_{i}h_{j} = \frac{u^{j}h_{j}}{W^{2}} = \frac{u^{j}(Sd_{j} - (\phi d_{k})_{j}v^{k} - \phi d_{k}v^{k}_{j})}{W^{2}}$$

$$\geq -C - \frac{\phi d_{k}u^{j}v^{k}_{j}}{W^{2}}$$

$$= -C + \frac{K\phi a^{\ell k}d_{k}u_{\ell}}{W} + \frac{\phi}{hW}a^{\ell k}d_{k}h_{\ell}$$

$$= -C + \frac{K\phi a^{\ell k}d_{k}u_{\ell}}{W}$$

$$+ \frac{S\phi a^{\ell k}d_{k}d_{\ell}}{hW} - \frac{\phi a^{\ell k}d_{k}(\phi d_{s})_{\ell}v^{s}}{hW} - \frac{\phi^{2}a^{\ell k}d_{k}d_{s}a^{sm}u_{m;\ell}}{hW^{2}}$$

$$\geq -C - \frac{CK}{W^{2}} - \frac{|A|^{2}}{3K}.$$
(2.9)

Plugging the estimates (2.6), (2.7), (2.8), and (2.9) into (2.5) and using (2.4) with the Ricci lower bound we obtain

$$0 \ge K^{2} \left(1 - \frac{1}{W^{2}}\right) - CK - \frac{2K}{h} \left(C + \frac{CK}{W} + \frac{CK}{W^{2}} + \frac{|A|^{2}}{3K}\right) - \frac{1}{h} \left(C + |A|^{2}/3 - \phi d_{k}v^{k}|A|^{2}\right) + |A|^{2} - CK = K^{2} \left(1 - \frac{1}{W^{2}} - \frac{C}{hW^{2}}\right) - KC \left(1 + \frac{1}{h}\right) - \frac{|A|^{2}}{h} + \frac{\phi d_{k}v^{k}|A|^{2}}{h} - \frac{C}{h} + |A|^{2} - C.$$

Then collecting the terms including  $|A|^2$  and noticing that

$$1 - \frac{1}{h} + \frac{\phi d_k v^k}{h} = \frac{Sd}{h} \ge 0$$

we have

$$0 \geq K^2 \left(1 - \frac{1}{W^2} - \frac{C}{hW^2}\right) - CK \left(1 + \frac{1}{h}\right) - C.$$

Now choosing *K* large enough, we obtain  $W(x_0, t_0) \leq C_0$ , where  $C_0$  depends only on  $C_{\infty}$ , *d*,  $\phi$ , the lower bound of the Ricci curvature in  $\Omega$ , and the dimension of *N*. We notice that the constant  $C_0$  is independent of *T*.

Since

$$e^{K\left(u(\cdot,t)-C_{\infty}t\right)}(1-\phi_0) \leq \eta \leq e^{K\left(u(\cdot,t)-C_{\infty}t\right)}(S+2),$$

we have

$$W(x, t) \leq \frac{(W\eta)(x_0, t_0)}{\eta(x, t)} \leq \frac{C_0\eta(x_0, t_0)}{\eta(x, t)}$$

$$\leq \frac{C_0(S+2)}{1-\phi_0} e^{K\left(u(x_0, t_0) - C_{\infty}t_0 - u(x, t) + C_{\infty}t\right)}$$
(2.10)

for every  $(x, t) \in \overline{\Omega} \times [0, T]$ .

We observe that the function  $u_{\infty}(x) + Ct$  solves the equation (1.2) with the initial condition  $u_0 = u_{\infty}$  if  $u_{\infty}$  is a solution to the elliptic equation (1.4) and *C* is given by (1.5). As in [1, Corollary 2.7], applying a parabolic maximum principle ([7]) we obtain:

**Lemma 2.2.** Suppose that (1.4) admits a solution  $u_{\infty}$  with the unique constant *C* given by (1.5). Let *u* be a solution to (1.2). Then, we have

$$|u(x,t)-Ct|\leq c_2,$$

for some constant  $c_2$  only depending on  $u_0$ ,  $\phi$ , and  $\Omega$ .

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*Proof.* Let  $V(x, t) = u(x, t) - u_{\infty}(x)$ , where  $u_{\infty}$  is a solution to (1.4). We see that V satisfies

$$\begin{cases} \frac{\partial V}{\partial t} = \tilde{a}^{ij} V_{i;j} + b^i V_i + C & \text{in } \Omega \times [0, T) \\ \tilde{c}^{ij} V_i v_j = 0 & \text{on } \partial \Omega \times [0, T), \end{cases}$$

where  $\tilde{a}^{ij}$ ,  $\tilde{c}^{ij}$  are positive definite matrices and  $b^i \in \mathbb{R}$ . Then the proof of the lemma follows by applying the maximum principle.

In view of Lemma 2.2, taking  $C_{\infty} = C$ , and observing that the constant  $C_0$  is independent of T, we get from (2.10) a uniform gradient bound.

**Lemma 2.3.** Suppose that (1.4) admits a solution  $u_{\infty}$  with the unique constant *C* given by (1.5). Let *u* be a solution to (1.2). Then  $W(x, t) \leq C_1$  for all  $(x, t) \in \overline{\Omega} \times [0, \infty)$  with a constant  $C_1$  depending only on  $\phi_0$ ,  $u_0$ , and  $\Omega$ .

Having a uniform gradient bound in our disposal, applying once more the strong maximum principle for linear uniformly parabolic equations, we obtain:

**Theorem 2.4.** Suppose that (1.4) admits a solution  $u_{\infty}$  with the unique constant *C* given by (1.5). Let  $u_1$  and  $u_2$  be two solutions of (1.2) with the same prescribed contact angle as  $u_{\infty}$ . Let  $u = u_1 - u_2$ . Then *u* converges to a constant function as  $t \to \infty$ . In particular, if *C* is given by (1.5), then  $u_1(x, t) - u_{\infty}(x) - Ct$  converges uniformly to a constant as  $t \to \infty$ .

*Proof.* The proof is given in [1, p. 109]. We reproduce it for the reader's convenience. One can check that *u* satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \tilde{a}^{ij} u_{i;j} + b^i u_i & \text{ in } \Omega \times [0, \infty) \\ \tilde{c}^{ij} u_i v_j = 0 & \text{ on } \partial \Omega \times [0, \infty), \end{cases}$$

where  $\tilde{a}^{ij}$ ,  $\tilde{c}^{ij}$  are positive definite matrices and  $b^i \in \mathbb{R}$ . By the strong maximum principle, we get that the function  $F_u(t) = \max u(\cdot, t) - \min u(\cdot, t) \ge 0$  is either strictly decreasing or u is constant. Assuming on the contrary that  $\lim_{t\to\infty} u$  is not a constant function, setting  $u_n(\cdot, t) = u(\cdot, t - t_n)$  for some sequence  $t_n \to \infty$ , we would get a non-constant solution, say v, defined on  $\Omega \times (-\infty, +\infty)$  for which  $F_v$  would be constant. We get a contradiction with the maximum principle.

Theorem 1.1 now follows from Lemma 2.3 and Theorem 2.4.

#### **3** Proof of Theorem 1.3

Theorem 1.3 is essentially proven in [3, Theorem 2.1, 3.1]. The only extra ingredient we must take into account in our non-flat case is the following Ricci identity for the Hessian  $\varphi_{i:i}$  of a smooth function  $\varphi$ 

$$\varphi_{k;ij} = \varphi_{i;kj} = \varphi_{i;jk} + R^{\ell}_{kji} \varphi_{\ell}.$$
(3.1)

For the convenience of the reader, we mostly use the same notations as in [3]. Thus let *h* be a smooth defining function of  $\Omega$  such that h < 0 in  $\Omega$ , h = 0 on  $\partial\Omega$ ,  $(h_{i;j}) \ge k_1(\delta_{ij})$  for some constant  $k_1 > 0$  and  $\sup_{\Omega} |\nabla h| \le 1$ ,  $h_{\gamma} = -1$  and  $|\nabla h| = 1$  on  $\partial\Omega$ . Furthermore, by strict convexity of  $\Omega$ , the second fundamental form of  $\partial\Omega$  satisfies

$$(\kappa_{ij})_{1\leq i,j\leq n-1}\geq \kappa_0(\delta_{ij})_{1\leq i,j\leq n-1},$$

where  $\kappa_0 > 0$  is the minimal principal curvature of  $\partial \Omega$ .

We consider the equation

$$\begin{cases} a^{ij}u_{i;j} := \left(g^{ij} - \frac{u^i u^j}{1 + |\nabla u|^2}\right) u_{i;j} = \varepsilon u & \text{in } \Omega\\ \partial_{\gamma} u = \phi \sqrt{1 + |\nabla u|^2} & \text{on } \partial\Omega \end{cases}$$
(3.2)

for small  $\varepsilon > 0$ . Writing  $\phi = -\cos \theta$ ,  $v = \sqrt{1 + |\nabla u|^2}$  and

 $\Phi(x) = \log w(x) + \alpha h(x),$ 

where  $w(x) = v - u^{\ell} h_{\ell} \cos \theta$  and  $\alpha > 0$  is a constant to be determined, we assume that the maximum of  $\Phi$  is attained in a point  $x_0 \in \overline{\Omega}$ . If  $x_0 \in \partial \Omega$ , we can proceed as in [3, pp. 34-36]. Thus choosing  $0 < \alpha < \kappa_0$  and  $0 < \varepsilon_0 \le \varepsilon_\alpha < 1$  such that

$$\kappa_0 - \alpha > \frac{\varepsilon_\alpha (M_1 + 3)}{1 - \varepsilon_\alpha^2},\tag{3.3}$$

where  $M_1 = \sup_{\bar{\Omega}} |\nabla^2 h|$ , yields an upper bound

$$|\nabla' u(x_0)|^2 \leq \frac{\frac{\varepsilon_0(M_1+3)}{1-\varepsilon_0^2} + \alpha}{\kappa_0 - \alpha - \frac{\varepsilon_0(M_1+3)}{1-\varepsilon_0^2}} < \frac{\kappa_0}{\kappa_0 - \alpha - \frac{\varepsilon_\alpha(M_1+3)}{1-\varepsilon_\alpha^2}}$$

for the tangential component of  $\nabla u$  on  $\partial \Omega$ . Combining this with the boundary condition  $u_{\gamma} = -v \cos \theta$  gives an upper bound for  $|\nabla u(x_0)|$  and hence for  $\Phi(x_0)$ .

The only difference to the Euclidean case occurs when  $x_0 \in \Omega$ , i.e. is an interior point of  $\Omega$ . At this point we have, using the same notations as in [3, p. 42],

$$0 = \Phi_i(x_0) = \frac{w_i}{w} + \alpha h_i$$

and

$$0 \ge a^{ij} \Phi_{i;j}(x_0) = \frac{a^{ij} w_{i;j}}{w} - \alpha^2 a^{ij} h_i h_j + \alpha a^{ij} h_{i;j} =: I + II + III.$$

We choose normal coordinates at  $x_0$  such that  $u_1(x_0) = |\nabla u(x_0)|$  and  $(u_{i;j}(x_0))_{2 \le i,j \le n}$  is diagonal. Then at  $x_0$ , we have

$$II + III \ge -\alpha^2(1 + 1/\nu^2) + \alpha k_1(n - 1 + 1/\nu^2).$$

We denote  $J = a^{ij}w_{i;j} = J_1 + \tilde{J}_2 + J_3 + J_4$ , where  $J_1, J_3$  and  $J_4$  are as in [3, (2.19)]. We have, by [3, (2.22)],

$$J_3+J_4 \ge -C(|\cos\theta|+|\nabla\theta|+|\nabla^2\theta|)u_1-C(|\cos\theta|+|\nabla\theta|)\sum_{i=2}^n |u_{ii}|,$$

where *C* depends only on *n*,  $M_1$  and  $\sup_{\bar{\Omega}} |\nabla^3 h|$ . Writing  $S^{\ell} = \frac{u_{\ell}}{v} - h_{\ell} \cos \theta$  and using the Ricci identity

$$a^{ij}u_{k;ij} = a^{ij}u_{i;jk} + \operatorname{Ric}(\partial_k, \nabla u)$$

(see [8, (2.28)]) and (3.2), we get

$$\begin{split} \tilde{J}_2 &= a^{ij} \left( \frac{u^k u_{k;ij}}{v} - u_{k;ij} h^k \cos \theta \right) = S^k a^{ij} u_{i;jk} + S^k \operatorname{Ric}(\partial_k, \nabla u) \\ &= -S^k a^{ij}_{;k} u_{i;j} + S^k (\varepsilon u)_k + S^k \operatorname{Ric}(\partial_k, \nabla u) \\ &= J_2 + \varepsilon u_1 S^1 + S^k \operatorname{Ric}(\partial_k, \partial_1) |\nabla u|, \end{split}$$

where  $J_2$  is as in [3, (2.19)]. Since  $|S^1| \le 2$  and  $|S^k| \le 1$  for  $k \ge 2$ , we obtain

$$\tilde{J}_2 \ge J_2 - (n+1)|\operatorname{Ric}_{\Omega}||\nabla u|, \tag{3.4}$$

$$J_1+J_2\geq \sum_{i=2}^n \frac{u_{ii}^2}{2\nu}.$$

So combining the previous estimates, we find

$$I = \frac{J}{w} \ge -C(|\cos\theta| + |\nabla\theta| + |\nabla^2\theta|) - (n+1)|\operatorname{Ric}_{\Omega}|.$$

Hence we obtain

$$0 \ge I + II + III \ge -C(|\cos\theta| + |\nabla\theta| + |\nabla^2\theta|) - (n+1)|\operatorname{Ric}_{\Omega}| - \alpha^2(1+1/\nu^2) + \alpha k_1(n-1+1/\nu^2) = C_1 + C_2/\nu^2,$$

where

$$C_1 = -C\varepsilon_0 - (n+1)|\operatorname{Ric}_{\Omega}| + \alpha (k_1(n-1) - \alpha)$$

and  $C_2 = \alpha(k_1 - \alpha)$ . If  $C_1 > 0$  and  $C_2 > 0$ , we get a contradiction, and therefore the maximum of  $\Phi$  is attained on  $\partial \Omega$ . If  $C_1 > 0$  and  $C_2 < 0$ , then  $\nu^2 \le -C_2/C_1$  and again we have an upper bound for  $\Phi(x_0)$ . To have  $C_1 > 0$ we need

$$|\operatorname{Ric}_{\Omega}| < (\alpha(k_1(n-1)-\alpha)-C\varepsilon_0)/(n+1).$$
(3.5)

Fixing  $\alpha < \min\{\kappa_0, k_1(n-1)/2\}$  and assuming that

$$|\operatorname{Ric}_{\Omega}| < (\alpha(k_1(n-1)-\alpha)/(n+1))$$
(3.6)

and, finally, choosing  $0 < \varepsilon_0 \le \min{\{\varepsilon_{\alpha}, 1/4\}}$  small enough so that (3.5) holds, we end up again with a contradiction, and therefore the maximum of  $\Phi$  is attained on  $\partial \Omega$ . All in all, we have obtained a uniform gradient bound for a solution *u* to (3.2) that is independent of  $\varepsilon$ . Once the uniform gradient bound is established the rest of the proof goes as in [1] (or [3]).

In some special cases we get sharper estimates than those above.

**Example 3.1.** As the first example let us consider the hyperbolic space  $\mathbb{H}^n$  and a geodesic ball  $\Omega = B(o, R)$ . Furthermore, we choose

$$h(x)=\frac{r(x)^2}{2R}-\frac{R}{2}$$

as a defining function for  $\Omega$ . Here  $r(\cdot) = d(\cdot, o)$  is the distance to the center o. Then  $\kappa_0 = \coth R$  and we may choose  $k_1 = 1/R$ . Since  $\operatorname{Ric}(\partial_k, \partial_1) = -(n-1)\delta_{k1}$ , (3.4) can be replaced by

 $\tilde{J}_2 \ge J_2 - 2(n-1)|\nabla u|$ 

and consequently (3.6) can be replaced by

$$2(n-1) < \alpha((n-1)/R - \alpha),$$

where  $\alpha < \min\{\operatorname{coth} R, \frac{n-1}{2R}\}$ . Hence we obtain an upper bound for the radius *R*. For instance, if n = 2, then  $\alpha < \frac{1}{2R}$  and we need  $R < \frac{1}{2\sqrt{2}}$ . For all dimensions,  $\alpha = 1$  and  $R < \frac{n-1}{2n-1}$  will do.

**Example 3.2.** As a second example let *N* be a Cartan-Hadamard manifold with sectional curvatures bounded from below by  $-K^2$ , with K > 0. Again we choose  $\Omega = B(o, R)$  and

$$h(x)=\frac{r(x)^2}{2R}-\frac{R}{2}.$$

Now  $1/R \le \kappa_0 \le K \operatorname{coth}(KR)$  and again we may choose  $k_1 = 1/R$ . This time  $\operatorname{Ric}(\partial_1, \partial_1) \ge -(n-1)K^2$  and  $\operatorname{Ric}(\partial_k, \partial_1) \ge -\frac{1}{2}(n-1)K^2$  for k = 2, ..., n, and therefore instead of (3.4) and (3.6) we have

$$\tilde{J}_2 \ge J_2 - K^2 ((n+1)^2/2 - 2) |\nabla u|$$

and

$$K^{2}((n+1)^{2}/2-2) < \alpha((n-1)/R-\alpha),$$

where  $\alpha < \min\{1/R, \frac{n-1}{2R}\}$ . Again we obtain upper bounds for the radius *R*. If  $n \ge 3$  we need

$$R < \left(\frac{n-2}{K^2 \left((n+1)^2/2 - 2\right)}\right)^{1/2}$$

whereas for n = 2 the bound

$$R < \frac{1}{2\sqrt{2}K}$$

is enough since now  $\operatorname{Ric}(\partial_2, \partial_1) = 0$ .

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