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Title: Non-Parametric Mean Curvature Flow with Prescribed Contact Angle in Riemannian Products

Year: 2022

Version: Published version

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## Please cite the original version:

Casteras, J.-B., Heinonen, E., Holopainen, I., \& De Lira, J. H. (2022). Non-Parametric Mean Curvature Flow with Prescribed Contact Angle in Riemannian Products. Analysis and Geometry in Metric Spaces, 10(1), 31-39. https://doi.org/10.1515/agms-2020-0132

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# Non-Parametric Mean Curvature Flow with Prescribed Contact Angle in Riemannian Products 

https://doi.org/10.1515/agms-2020-0132
Received July 27, 2021; accepted January 20, 2022
Abstract: Assuming that there exists a translating soliton $u_{\infty}$ with speed $C$ in a domain $\Omega$ and with prescribed contact angle on $\partial \Omega$, we prove that a graphical solution to the mean curvature flow with the same prescribed contact angle converges to $u_{\infty}+C t$ as $t \rightarrow \infty$. We also generalize the recent existence result of Gao, Ma, Wang and Weng to non-Euclidean settings under suitable bounds on convexity of $\Omega$ and Ricci curvature in $\Omega$.

Keywords: Mean curvature flow; prescribed contact angle; translating graphs
MSC: Primary 53C21, 53E10

## 1 Introduction

We study a non-parametric mean curvature flow in a Riemannian product $N \times \mathbb{R}$ represented by graphs

$$
\begin{equation*}
M_{t}:=\{(x, u(x, t)): x \in \bar{\Omega}\} \tag{1.1}
\end{equation*}
$$

with prescribed contact angle with the cylinder $\partial \Omega \times \mathbb{R}$.
We assume that $N$ is a Riemannian manifold and $\Omega \Subset N$ is a relatively compact domain with smooth boundary $\partial \Omega$. We denote by $\gamma$ the inward pointing unit normal vector field to $\partial \Omega$. The boundary condition is determined by a given smooth function $\phi \in C^{\infty}(\partial \Omega)$, with $|\phi| \leq \phi_{0}<1$, and the initial condition by a smooth function $u_{0} \in C^{\infty}(\bar{\Omega})$.

The function $u$ above in (1.1) is a solution to the following evolution equation

$$
\begin{cases}\frac{\partial u}{\partial t}=W \operatorname{div} \frac{\nabla u}{W} & \text { in } \Omega \times[0, \infty)  \tag{1.2}\\ \frac{\partial_{\gamma} u}{W}:=\frac{\langle\nabla u, \gamma\rangle}{W}=\phi & \text { on } \partial \Omega \times[0, \infty) \\ u(\cdot, 0)=u_{0} & \text { in } \bar{\Omega},\end{cases}
$$

where $W=\sqrt{1+|\nabla u|^{2}}$ and $\nabla u$ denotes the gradient of $u$ with respect to the Riemannian metric on $N$ at $x \in \bar{\Omega}$. The boundary condition above can be written as

$$
\begin{equation*}
\langle v, \gamma\rangle=\phi, \tag{1.3}
\end{equation*}
$$

[^0]where $v$ is the downward pointing unit normal to the graph of $u$, i.e.
$$
v(x)=\frac{\nabla u(x, \cdot)-\partial_{t}}{\sqrt{1+\mid \nabla u\left(x,\left.\cdot \cdot\right|^{2}\right.}}, x \in \bar{\Omega} .
$$

The longtime existence of the solution $u_{t}:=u(\cdot, t)$ to (1.2) and convergence as $t \rightarrow \infty$ have been studied under various conditions on $\Omega$ and $\phi$. Huisken [5] proved the existence of a smooth solution in a $C^{2, \alpha}$-smooth bounded domain $\Omega \subset \mathbb{R}^{n}$ for $u_{0} \in C^{2, \alpha}(\bar{\Omega})$ and $\phi \equiv 0$. Moreover, he showed that $u_{t}$ converges to a constant function as $t \rightarrow \infty$. In [1] Altschuler and Wu complemented Huisken's results for prescribed contact angle in case $\Omega$ is a smooth bounded strictly convex domain in $\mathbb{R}^{2}$. Guan [4] proved a priori gradient estimates and established longtime existence of solutions in case $\Omega \subset \mathbb{R}^{n}$ is a smooth bounded domain. Recently, Zhou [8] studied mean curvature type flows in a Riemannian product $M \times \mathbb{R}$ and proved the longtime existence of the solution for relatively compact smooth domains $\Omega \subset M$. Furthermore, he extended the convergence result of Altschuler and Wu to the case $M$ is a Riemannian surface with nonnegative curvature and $\Omega \subset M$ is a smooth bounded strictly convex domain; see [8, Theorem 1.4].

The key ingredient, and at the same time the main obstacle, for proving the uniform convergence of $u_{t}$ has been a difficulty to obtain a time-independent gradient estimate. We circumvent this obstacle by modifying the method of Korevaar [6], Guan [4] and Zhou [8] and obtain a uniform gradient estimate in an arbitrary relatively compact smooth domain $\Omega \subset N$ provided there exists a translating soliton with speed $C$ and with the prescribed contact angle condition (1.3).

Towards this end, let $d$ be a smooth bounded function defined in some neighborhood of $\bar{\Omega}$ such that $d(x)=\min _{y \in \partial \Omega} \operatorname{dist}(x, y)$, the distance to the boundary $\partial \Omega$, for points $x \in \Omega$ sufficiently close to $\partial \Omega$. Thus $\gamma=\nabla d$ on $\partial \Omega$. We assume that $0 \leq d \leq 1,|\nabla d| \leq 1$ and $\mid$ Hess $d \mid \leq C_{d}$ in $\bar{\Omega}$. We also assume that the function $\phi \in C^{\infty}(\partial \Omega)$ is extended as a smooth function to the whole $\bar{\Omega}$, satisfying the condition $|\phi| \leq \phi_{0}<1$.

Our main theorem is the following:
Theorem 1.1. Suppose that there exists a solution $u_{\infty}$ to the translating soliton equation

$$
\begin{cases}\operatorname{div} \frac{\nabla u_{\infty}}{\sqrt{1+\left|\nabla u_{\infty}\right|^{2}}}=\frac{C_{\infty}}{\sqrt{1+\left|\nabla u_{\infty}\right|^{2}}} & \text { in } \Omega  \tag{1.4}\\ \frac{\partial_{\gamma} u_{\infty}}{\sqrt{1+\left|\nabla u_{\infty}\right|^{2}}}=\phi & \text { on } \partial \Omega\end{cases}
$$

where $C_{\infty}$ is given by

$$
\begin{equation*}
C_{\infty}=\frac{-\int_{\partial \Omega} \phi \mathrm{d} \sigma}{\int_{\Omega}\left(1+\left|\nabla u_{\infty}\right|^{2}\right)^{-1 / 2} \mathrm{~d} x} . \tag{1.5}
\end{equation*}
$$

Then the equation (1.2) has a smooth solution $u \in C^{\infty}(\bar{\Omega},[0, \infty))$ with $W \leq C_{1}$, where $C_{1}$ is a constant depending on $\phi, u_{0}, C_{d}$, and the Ricci curvature of $\Omega$. Moreover, $u(x, t)$ converges uniformly to $u_{\infty}(x)+C_{\infty} t$ as $t \rightarrow \infty$.

Notice that the existence of a solution $u \in C^{\infty}(\bar{\Omega} \times[0, \infty))$ to (1.2) is given by [8, Corollary 4.2].
Remark 1.2. Very recently, Gao, Ma, Wang, and Weng [3] proved the existence of such $u_{\infty}$ and obtained Theorem 1.1 for smooth, bounded, strictly convex domains $\Omega \subset \mathbb{R}^{n}$ for sufficiently small $|\phi|$; see [3, Theorem 1.1, Theorem 3.1]. It turns out that their proof can be generalized beyond the Euclidean setting under suitable bounds on the convexity of $\Omega$ and the Ricci curvature in $\Omega$.

More precisely, let $\Omega \Subset N$ be a relatively compact, strictly convex domain with smooth boundary admitting a smooth defining function $h$ such that $h<0$ in $\Omega, h=0$ on $\partial \Omega$,

$$
\begin{equation*}
\left(h_{i ; j}\right) \geq k_{1}\left(\delta_{i j}\right) \tag{1.6}
\end{equation*}
$$

for some constant $k_{1}>0$ and $\sup _{\Omega}|\nabla h| \leq 1, h_{\gamma}=-1$ and $|\nabla h|=1$ on $\partial \Omega$. Furthermore, by strict convexity of $\Omega$, the second fundamental form of $\partial \Omega$ satisfies

$$
\begin{equation*}
\left(\kappa_{i j}\right)_{1 \leq i, j \leq n-1} \geq \kappa_{0}\left(\delta_{i j}\right)_{1 \leq i, j \leq n-1}, \tag{1.7}
\end{equation*}
$$

where $\kappa_{0}>0$ is the minimal principal curvature of $\partial \Omega$. In the Euclidean case, $N=\mathbb{R}^{n}$, such functions $h$ are constructed in [2]. We give some simple examples at the end of Section 3.

Theorem 1.3. Let $\Omega \Subset N$ be a smooth, strictly convex, relatively compact domain associated with constants $k_{1}>0$ and $\kappa_{0}>0$ as in (1.6) and (1.7). Let $\alpha<\min \left\{\kappa_{0}, k_{1}(n-1) / 2\right\}$ and assume that the Ricci curvature in $\Omega$ satisfies $\mid$ Ric $\mid<\alpha\left(k_{1}(n-1)-\alpha\right) /(n+1)$. Then there exists $\varepsilon_{0}>0$ such that if $\phi=: \cos \theta \in C^{3}(\bar{\Omega})$ satisfies $|\cos \theta| \leq \varepsilon_{0} \leq 1 / 4$ and $\|\nabla \theta\|_{C^{1}(\bar{\Omega})} \leq \varepsilon_{0}$ in $\bar{\Omega}$, there exist a unique constant $C_{\infty}$ and a solution $u_{\infty}$ to (1.4). Furthermore, $u_{\infty}$ is unique up to an additive constant.

We will sketch the proof of Theorem 1.3 in Section 3.

## 2 Proof of Theorem 1.1

Let $u$ be a solution to (1.2) in $\bar{\Omega} \times \mathbb{R}$. Given a constant $C_{\infty} \in \mathbb{R}$ we define, following the ideas of Korevaar [6], Guan [4] and Zhou [8], a function $\eta: \bar{\Omega} \times \mathbb{R} \rightarrow(0, \infty)$ by setting

$$
\begin{equation*}
\eta=e^{K\left(u-C_{\infty} t\right)}\left(S d+1-\frac{\phi}{W}\langle\nabla u, \nabla d\rangle\right), \tag{2.1}
\end{equation*}
$$

where $K$ and $S$ are positive constants to be determined later. We start with a gradient estimate.
Proposition 2.1. Let $u$ be a solution to (1.2) and define $\eta$ as in (2.1). Then, for a fixed $T>0$, letting

$$
(W \eta)\left(x_{0}, t_{0}\right)=\max _{x \in \bar{\Omega}, t \in[0, T]}(W \eta)(x, t),
$$

there exists a constant $C_{0}$ only depending on $C_{d}, \phi, C_{\infty}$, and the lower bound for the Ricci curvature in $\Omega$ such that $W\left(x_{0}, t_{0}\right) \leq C_{0}$.

Proof. Let $g=g_{i j} d x^{i} d x^{j}$ be the Riemannian metric of $N$. We denote by $\left(g^{i j}\right)$ the inverse of $\left(g_{i j}\right), u_{j}=\partial u / \partial x^{j}$, and $u_{i ; j}=u_{i j}-\Gamma_{i j}^{k} u_{k}$. We set

$$
a^{i j}=g^{i j}-\frac{u^{i} u^{j}}{W^{2}}
$$

and define an operator $L$ by $L u=a^{i j} u_{i ; j}-\partial_{t} u$. Observe that (1.2) can be rewritten as $L u=0$. In all the following, computations will be done at the maximum point ( $x_{0}, t_{0}$ ) of $\eta W$. We first consider the case where $x_{0} \in \partial \Omega$. We choose normal coordinates at $x_{0}$ such that $g_{i j}=g^{i j}=\delta^{i j}$ at $x_{0}, \partial_{n}=\gamma$,

$$
u_{1} \geq 0, \quad u_{i}=0 \quad \text { for } 2 \leq i \leq n-1 .
$$

This implies that

$$
d_{i}=0 \text { for } 1 \leq i \leq n-1, d_{n}=1 \text {, and } d_{i ; n}=0 \text { for } 1 \leq i \leq n .
$$

We have

$$
\begin{align*}
0 \geq & (W \eta)_{n}=W_{n} \eta+W \eta_{n} \\
= & e^{K\left(u-C_{\infty} t\right)}\left(S W_{n} d+W_{n}-\frac{\phi W_{n}}{W} g^{i j} u_{i} d_{j}+S W d_{n}-\frac{W}{W} \phi_{n} g^{i j} u_{i} d_{j}\right. \\
& \quad-\frac{W}{W} \phi g^{i j}\left(u_{i ; n} d_{j}+u_{i} d_{j ; n}\right)+W \frac{W_{n}}{W^{2}} \phi g^{i j} u_{i} d_{j} \\
& \left.\quad+K W u_{n}\left(S d+1-\frac{\phi}{W} g^{i j} u_{i} d_{j}\right)\right) \\
= & e^{K\left(u-C_{\infty} t\right)}\left(W_{n}+S W-\phi_{n} u_{n}-\phi u_{n ; n}+K W u_{n}\left(1-\phi^{2}\right)\right) . \tag{2.2}
\end{align*}
$$

Using our coordinate system, we get

$$
\begin{aligned}
0 \geq & \frac{W_{n}}{W}+S-\frac{\phi_{n} u_{n}}{W}-\frac{\phi u_{n ; n}}{W}+K u_{n}\left(1-\phi^{2}\right) \\
= & S-\frac{u_{1}^{2} d_{1 ; 1}}{W^{2}}+\frac{u_{1} \phi_{1}}{W}\left(1+\frac{2 \phi^{2}}{1-\phi^{2}}\right)-\frac{\phi u_{1}}{W} K u_{1} \\
& \quad-\frac{\phi_{n} u_{n}}{W}+K u_{n}\left(1-\phi^{2}\right) \\
\geq & S-C-\frac{K \phi u_{1}^{2}}{W}+K u_{n}\left(1-\phi^{2}\right) \\
= & S-C-\frac{K \phi}{W} \geq S-C-\frac{K}{W}
\end{aligned}
$$

for some constant $C$ depending only on $C_{d}$ and $\phi$. So choosing $S \geq C+1$, we get that

$$
\begin{equation*}
W\left(x_{0}, t_{0}\right) \leq K . \tag{2.3}
\end{equation*}
$$

Next we assume that $x_{0} \in \Omega$ and that $S \geq C+1$, where $C$ is as above. Let us recall from [8, Lemma 3.5] that

$$
L W=\frac{2}{W} a^{i j} W_{i} W_{j}+\operatorname{Ric}\left(v_{N}, v_{N}\right) W+|A|^{2} W
$$

where $v_{N}=\nabla u / W$ and $|A|^{2}=a^{i j} a^{\ell k} u_{i ; k} u_{j ; \ell} / W^{2}$ is the squared norm of the second fundamental form of the graph $M_{t}$. Since $0=W_{i} \eta+W \eta_{i}$, for every $i=1, \ldots, n$, we deduce that

$$
0 \geq L(W \eta)=W L \eta+\eta\left(L W-2 a^{i j} \frac{W_{i} W_{j}}{W}\right)=W L \eta+\eta W\left(|A|^{2}+\operatorname{Ric}\left(v_{N}, v_{N}\right)\right)
$$

This yields to

$$
\begin{equation*}
\frac{1}{\eta} L \eta+|A|^{2}+\operatorname{Ric}\left(v_{N}, v_{N}\right) \leq 0 \tag{2.4}
\end{equation*}
$$

To simplify the notation, we set

$$
h=S d+1-\phi u^{k} d_{k} / W=S d+1-\phi v^{k} d_{k}
$$

So we have

$$
\begin{equation*}
\frac{1}{\eta} L \eta=K^{2} a^{i j} u_{i} u_{j}+K L\left(u-C_{\infty} t\right)+\frac{2 K}{h} a^{i j} u_{i} h_{j}+\frac{1}{h} L h \tag{2.5}
\end{equation*}
$$

We can compute $L h$ as

$$
L h=a^{i j}\left(S d_{i ; j}-\left(\phi d_{k}\right)_{i ; j} v^{k}-\left(\phi d_{k}\right)_{i} v_{j}^{k}-\left(\phi d_{k}\right)_{j} v_{i}^{k}-\phi d_{k} L v^{k}\right) \geq-C-2 a^{i j}\left(\phi d_{k}\right)_{i} v_{j}^{k}-\phi d_{k} L v^{k}
$$

Since, by [8, Lemma 3.5],

$$
L v^{k}=\operatorname{Ric}\left(a^{k \ell} \partial_{\ell}, v_{N}\right)-|A|^{2} v^{k}
$$

and, by Young's inequality for matrices,

$$
a^{i j}\left(\phi d_{k}\right)_{i} v_{j}^{k}=\frac{1}{W}\left(\phi d_{k}\right)_{i} a^{i j} a^{\ell k} u_{\ell ; j} \leq \frac{|A|^{2}}{6}+C
$$

we get the estimate

$$
\begin{equation*}
L h \geq-C-|A|^{2} / 3+\phi d_{k} v^{k}|A|^{2} \tag{2.6}
\end{equation*}
$$

by using the assumption that Ric is bounded.
Next we turn our attention to the other terms in (2.5). We have

$$
\begin{equation*}
a^{i j} u_{i}=\frac{u^{j}}{W^{2}} \quad \text { and } \quad a^{i j} u_{i} u_{j}=1-\frac{1}{W^{2}} \tag{2.7}
\end{equation*}
$$

Then we note that by the assumptions, we clearly have

$$
\begin{equation*}
K L\left(u-C_{\infty} t\right)=K C_{\infty} \geq-K C \tag{2.8}
\end{equation*}
$$

and we are left to consider

$$
\begin{align*}
a^{i j} u_{i} h_{j} & =\frac{u^{j} h_{j}}{W^{2}}=\frac{u^{j}\left(S d_{j}-\left(\phi d_{k}\right)_{j} v^{k}-\phi d_{k} v_{j}^{k}\right)}{W^{2}} \\
& \geq-C-\frac{\phi d_{k} u^{j} v_{j}^{k}}{W^{2}} \\
& =-C+\frac{K \phi a^{\ell k} d_{k} u_{\ell}}{W}+\frac{\phi}{h W} a^{\ell k} d_{k} h_{\ell} \\
& =-C+\frac{K \phi a^{\ell k} d_{k} u_{\ell}}{W} \\
& +\frac{S \phi a^{\ell k} d_{k} d_{\ell}}{h W}-\frac{\phi a^{\ell k} d_{k}\left(\phi d_{s}\right)_{\ell} v^{s}}{h W}-\frac{\phi^{2} a^{\ell k} d_{k} d_{s} a^{s m} u_{m ; \ell}}{h W^{2}} \\
& \geq-C-\frac{C K}{W^{2}}-\frac{|A|^{2}}{3 K} \tag{2.9}
\end{align*}
$$

Plugging the estimates (2.6), (2.7), (2.8), and (2.9) into (2.5) and using (2.4) with the Ricci lower bound we obtain

$$
\begin{aligned}
0 & \geq K^{2}\left(1-\frac{1}{W^{2}}\right)-C K-\frac{2 K}{h}\left(C+\frac{C K}{W}+\frac{C K}{W^{2}}+\frac{|A|^{2}}{3 K}\right)-\frac{1}{h}\left(C+|A|^{2} / 3-\phi d_{k} v^{k}|A|^{2}\right)+|A|^{2}-C \\
& =K^{2}\left(1-\frac{1}{W^{2}}-\frac{C}{h W^{2}}\right)-K C\left(1+\frac{1}{h}\right)-\frac{|A|^{2}}{h}+\frac{\phi d_{k} v^{k}|A|^{2}}{h}-\frac{C}{h}+|A|^{2}-C
\end{aligned}
$$

Then collecting the terms including $|A|^{2}$ and noticing that

$$
1-\frac{1}{h}+\frac{\phi d_{k} v^{k}}{h}=\frac{S d}{h} \geq 0
$$

we have

$$
0 \geq K^{2}\left(1-\frac{1}{W^{2}}-\frac{C}{h W^{2}}\right)-C K\left(1+\frac{1}{h}\right)-C .
$$

Now choosing $K$ large enough, we obtain $W\left(x_{0}, t_{0}\right) \leq C_{0}$, where $C_{0}$ depends only on $C_{\infty}, d, \phi$, the lower bound of the Ricci curvature in $\Omega$, and the dimension of $N$. We notice that the constant $C_{0}$ is independent of $T$.

Since

$$
e^{K\left(u(\cdot, t)-C_{\infty} t\right)}\left(1-\phi_{0}\right) \leq \eta \leq e^{K\left(u(\cdot, t)-C_{\infty} t\right)}(S+2)
$$

we have

$$
\begin{align*}
W(x, t) & \leq \frac{(W \eta)\left(x_{0}, t_{0}\right)}{\eta(x, t)} \\
& \leq \frac{C_{0} \eta\left(x_{0}, t_{0}\right)}{\eta(x, t)}  \tag{2.10}\\
& \leq \frac{C_{0}(S+2)}{1-\phi_{0}} e^{K\left(u\left(x_{0}, t_{0}\right)-C_{\infty} t_{0}-u(x, t)+C_{\infty} t\right)}
\end{align*}
$$

for every $(x, t) \in \bar{\Omega} \times[0, T]$.
We observe that the function $u_{\infty}(x)+C t$ solves the equation (1.2) with the initial condition $u_{0}=u_{\infty}$ if $u_{\infty}$ is a solution to the elliptic equation (1.4) and $C$ is given by (1.5). As in [1, Corollary 2.7], applying a parabolic maximum principle ([7]) we obtain:

Lemma 2.2. Suppose that (1.4) admits a solution $u_{\infty}$ with the unique constant $C$ given by (1.5). Let $u$ be $a$ solution to (1.2). Then, we have

$$
|u(x, t)-C t| \leq c_{2},
$$

for some constant $c_{2}$ only depending on $u_{0}, \phi$, and $\Omega$.

Proof. Let $V(x, t)=u(x, t)-u_{\infty}(x)$, where $u_{\infty}$ is a solution to (1.4). We see that $V$ satisfies

$$
\begin{cases}\frac{\partial V}{\partial t}=\tilde{a}^{i j} V_{i ; j}+b^{i} V_{i}+C & \text { in } \Omega \times[0, T) \\ \tilde{c}^{i j} V_{i} v_{j}=0 & \text { on } \partial \Omega \times[0, T)\end{cases}
$$

where $\tilde{a}^{i j}, \tilde{c}^{i j}$ are positive definite matrices and $b^{i} \in \mathbb{R}$. Then the proof of the lemma follows by applying the maximum principle.

In view of Lemma 2.2, taking $C_{\infty}=C$, and observing that the constant $C_{0}$ is independent of $T$, we get from (2.10) a uniform gradient bound.

Lemma 2.3. Suppose that (1.4) admits a solution $u_{\infty}$ with the unique constant $C$ given by (1.5). Let $u$ be a solution to (1.2). Then $W(x, t) \leq C_{1}$ for all $(x, t) \in \bar{\Omega} \times[0, \infty)$ with a constant $C_{1}$ depending only on $\phi_{0}, u_{0}$, and $\Omega$.

Having a uniform gradient bound in our disposal, applying once more the strong maximum principle for linear uniformly parabolic equations, we obtain:

Theorem 2.4. Suppose that (1.4) admits a solution $u_{\infty}$ with the unique constant $C$ given by (1.5). Let $u_{1}$ and $u_{2}$ be two solutions of (1.2) with the same prescribed contact angle as $u_{\infty}$. Let $u=u_{1}-u_{2}$. Then $u$ converges to a constant function as $t \rightarrow \infty$. In particular, if $C$ is given by (1.5), then $u_{1}(x, t)-u_{\infty}(x)-C t$ converges uniformly to a constant as $t \rightarrow \infty$.

Proof. The proof is given in [1, p. 109]. We reproduce it for the reader's convenience. One can check that $u$ satisfies

$$
\begin{cases}\frac{\partial u}{\partial t}=\tilde{a}^{i j} u_{i ; j}+b^{i} u_{i} & \text { in } \Omega \times[0, \infty) \\ \tilde{c}^{i j} u_{i} v_{j}=0 & \text { on } \partial \Omega \times[0, \infty)\end{cases}
$$

where $\tilde{a}^{i j}$, $\tilde{c}^{i j}$ are positive definite matrices and $b^{i} \in \mathbb{R}$. By the strong maximum principle, we get that the function $F_{u}(t)=\max u(\cdot, t)-\min u(\cdot, t) \geq 0$ is either strictly decreasing or $u$ is constant. Assuming on the contrary that $\lim _{t \rightarrow \infty} u$ is not a constant function, setting $u_{n}(\cdot, t)=u\left(\cdot, t-t_{n}\right)$ for some sequence $t_{n} \rightarrow \infty$, we would get a non-constant solution, say $v$, defined on $\Omega \times(-\infty,+\infty)$ for which $F_{v}$ would be constant. We get a contradiction with the maximum principle.

Theorem 1.1 now follows from Lemma 2.3 and Theorem 2.4.

## 3 Proof of Theorem 1.3

Theorem 1.3 is essentially proven in [3, Theorem 2.1, 3.1]. The only extra ingredient we must take into account in our non-flat case is the following Ricci identity for the Hessian $\varphi_{i ; j}$ of a smooth function $\varphi$

$$
\begin{equation*}
\varphi_{k ; i j}=\varphi_{i ; k j}=\varphi_{i ; j k}+R_{k j i}^{\ell} \varphi_{\ell} . \tag{3.1}
\end{equation*}
$$

For the convenience of the reader, we mostly use the same notations as in [3]. Thus let $h$ be a smooth defining function of $\Omega$ such that $h<0$ in $\Omega, h=0$ on $\partial \Omega,\left(h_{i ; j}\right) \geq k_{1}\left(\delta_{i j}\right)$ for some constant $k_{1}>0$ and $\sup _{\Omega}|\nabla h| \leq 1$, $h_{\gamma}=-1$ and $|\nabla h|=1$ on $\partial \Omega$. Furthermore, by strict convexity of $\Omega$, the second fundamental form of $\partial \Omega$ satisfies

$$
\left(\kappa_{i j}\right)_{1 \leq i, j \leq n-1} \geq \kappa_{0}\left(\delta_{i j}\right)_{1 \leq i, j \leq n-1}
$$

where $\kappa_{0}>0$ is the minimal principal curvature of $\partial \Omega$.

We consider the equation

$$
\begin{cases}a^{i j} u_{i ; j}:=\left(g^{i j}-\frac{u^{i} u^{j}}{1+|\nabla u|^{2}}\right) u_{i ; j}=\varepsilon u & \text { in } \Omega  \tag{3.2}\\ \partial_{\gamma} u=\phi \sqrt{1+|\nabla u|^{2}} & \text { on } \partial \Omega\end{cases}
$$

for small $\varepsilon>0$. Writing $\phi=-\cos \theta, v=\sqrt{1+|\nabla u|^{2}}$ and

$$
\Phi(x)=\log w(x)+\alpha h(x)
$$

where $w(x)=v-u^{\ell} h_{\ell} \cos \theta$ and $\alpha>0$ is a constant to be determined, we assume that the maximum of $\Phi$ is attained in a point $x_{0} \in \bar{\Omega}$. If $x_{0} \in \partial \Omega$, we can proceed as in [3, pp. 34-36]. Thus choosing $0<\alpha<\kappa_{0}$ and $0<\varepsilon_{0} \leq \varepsilon_{\alpha}<1$ such that

$$
\begin{equation*}
\kappa_{0}-\alpha>\frac{\varepsilon_{\alpha}\left(M_{1}+3\right)}{1-\varepsilon_{\alpha}^{2}} \tag{3.3}
\end{equation*}
$$

where $M_{1}=\sup _{\bar{\Omega}}\left|\nabla^{2} h\right|$, yields an upper bound

$$
\left|\nabla^{\prime} u\left(x_{0}\right)\right|^{2} \leq \frac{\frac{\varepsilon_{0}\left(M_{1}+3\right)}{1-\varepsilon_{0}^{2}}+\alpha}{\kappa_{0}-\alpha-\frac{\varepsilon_{0}\left(M_{1}+3\right)}{1-\varepsilon_{0}^{2}}}<\frac{\kappa_{0}}{\kappa_{0}-\alpha-\frac{\varepsilon_{\alpha}\left(M_{1}+3\right)}{1-\varepsilon_{\alpha}^{2}}}
$$

for the tangential component of $\nabla u$ on $\partial \Omega$. Combining this with the boundary condition $u_{\gamma}=-v \cos \theta$ gives an upper bound for $\left|\nabla u\left(x_{0}\right)\right|$ and hence for $\Phi\left(x_{0}\right)$.

The only difference to the Euclidean case occurs when $x_{0} \in \Omega$, i.e. is an interior point of $\Omega$. At this point we have, using the same notations as in [3, p. 42],

$$
0=\Phi_{i}\left(x_{0}\right)=\frac{w_{i}}{w}+\alpha h_{i}
$$

and

$$
0 \geq a^{i j} \Phi_{i ; j}\left(x_{0}\right)=\frac{a^{i j} w_{i ; j}}{w}-\alpha^{2} a^{i j} h_{i} h_{j}+\alpha a^{i j} h_{i ; j}=: I+I I+I I I .
$$

We choose normal coordinates at $x_{0}$ such that $u_{1}\left(x_{0}\right)=\left|\nabla u\left(x_{0}\right)\right|$ and $\left(u_{i ; j}\left(x_{0}\right)\right)_{2 \leq i, j \leq n}$ is diagonal. Then at $x_{0}$, we have

$$
I I+I I I \geq-\alpha^{2}\left(1+1 / v^{2}\right)+\alpha k_{1}\left(n-1+1 / v^{2}\right)
$$

We denote $J=a^{i j} w_{i ; j}=J_{1}+\tilde{J}_{2}+J_{3}+J_{4}$, where $J_{1}, J_{3}$ and $J_{4}$ are as in [3, (2.19)]. We have, by [3, (2.22)],

$$
J_{3}+J_{4} \geq-C\left(|\cos \theta|+|\nabla \theta|+\left|\nabla^{2} \theta\right|\right) u_{1}-C(|\cos \theta|+|\nabla \theta|) \sum_{i=2}^{n}\left|u_{i i}\right|
$$

where $C$ depends only on $n, M_{1}$ and $\sup _{\bar{\Omega}}\left|\nabla^{3} h\right|$. Writing $S^{\ell}=\frac{u_{\ell}}{v}-h_{\ell} \cos \theta$ and using the Ricci identity

$$
a^{i j} u_{k ; i j}=a^{i j} u_{i ; j k}+\operatorname{Ric}\left(\partial_{k}, \nabla u\right)
$$

(see $[8,(2.28)]$ ) and (3.2), we get

$$
\begin{aligned}
\tilde{J}_{2} & =a^{i j}\left(\frac{u^{k} u_{k ; i j}}{v}-u_{k ; i j} h^{k} \cos \theta\right)=S^{k} a^{i j} u_{i ; j k}+S^{k} \operatorname{Ric}\left(\partial_{k}, \nabla u\right) \\
& =-S^{k} a_{; k}^{i j} u_{i ; j}+S^{k}(\varepsilon u)_{k}+S^{k} \operatorname{Ric}\left(\partial_{k}, \nabla u\right) \\
& =J_{2}+\varepsilon u_{1} S^{1}+S^{k} \operatorname{Ric}\left(\partial_{k}, \partial_{1}\right)|\nabla u|
\end{aligned}
$$

where $J_{2}$ is as in [3, (2.19)]. Since $\left|S^{1}\right| \leq 2$ and $\left|S^{k}\right| \leq 1$ for $k \geq 2$, we obtain

$$
\begin{equation*}
\tilde{J}_{2} \geq J_{2}-(n+1)\left|\operatorname{Ric}_{\Omega} \| \nabla u\right|, \tag{3.4}
\end{equation*}
$$

where $\left|\operatorname{Ric}_{\Omega}\right|$ is the bound for the Ricci curvature in $\Omega$, i.e. $|\operatorname{Ric}(x)| \leq\left|\operatorname{Ric}_{\Omega}\right|$ for all unit vectors $x \in T \Omega$. At this point, we can proceed as in [3] to get that

$$
J_{1}+J_{2} \geq \sum_{i=2}^{n} \frac{u_{i i}^{2}}{2 v}
$$

So combining the previous estimates, we find

$$
I=\frac{J}{w} \geq-C\left(|\cos \theta|+|\nabla \theta|+\left|\nabla^{2} \theta\right|\right)-(n+1)\left|\operatorname{Ric}_{\Omega}\right|
$$

Hence we obtain

$$
\begin{aligned}
0 & \geq I+I I+I I I \geq-C\left(|\cos \theta|+|\nabla \theta|+\left|\nabla^{2} \theta\right|\right)-(n+1)\left|\operatorname{Ric}_{\Omega}\right|-\alpha^{2}\left(1+1 / v^{2}\right)+\alpha k_{1}\left(n-1+1 / v^{2}\right) \\
& =: C_{1}+C_{2} / v^{2}
\end{aligned}
$$

where

$$
C_{1}=-C \varepsilon_{0}-(n+1)\left|\operatorname{Ric}_{\Omega}\right|+\alpha\left(k_{1}(n-1)-\alpha\right)
$$

and $C_{2}=\alpha\left(k_{1}-\alpha\right)$. If $C_{1}>0$ and $C_{2}>0$, we get a contradiction, and therefore the maximum of $\Phi$ is attained on $\partial \Omega$. If $C_{1}>0$ and $C_{2}<0$, then $v^{2} \leq-C_{2} / C_{1}$ and again we have an upper bound for $\Phi\left(x_{0}\right)$. To have $C_{1}>0$ we need

$$
\begin{equation*}
\left|\operatorname{Ric}_{\Omega}\right|<\left(\alpha\left(k_{1}(n-1)-\alpha\right)-C \varepsilon_{0}\right) /(n+1) \tag{3.5}
\end{equation*}
$$

Fixing $\alpha<\min \left\{\kappa_{0}, k_{1}(n-1) / 2\right\}$ and assuming that

$$
\begin{equation*}
\left|\operatorname{Ric}_{\Omega}\right|<\left(\alpha\left(k_{1}(n-1)-\alpha\right) /(n+1)\right. \tag{3.6}
\end{equation*}
$$

and, finally, choosing $0<\varepsilon_{0} \leq \min \left\{\varepsilon_{\alpha}, 1 / 4\right\}$ small enough so that (3.5) holds, we end up again with a contradiction, and therefore the maximum of $\Phi$ is attained on $\partial \Omega$. All in all, we have obtained a uniform gradient bound for a solution $u$ to (3.2) that is independent of $\varepsilon$. Once the uniform gradient bound is established the rest of the proof goes as in [1] (or [3]).

In some special cases we get sharper estimates than those above.
Example 3.1. As the first example let us consider the hyperbolic space $\mathbb{H}^{n}$ and a geodesic ball $\Omega=B(o, R)$. Furthermore, we choose

$$
h(x)=\frac{r(x)^{2}}{2 R}-\frac{R}{2}
$$

as a defining function for $\Omega$. Here $r(\cdot)=d(\cdot, o)$ is the distance to the center $o$. Then $\kappa_{0}=\operatorname{coth} R$ and we may choose $k_{1}=1 / R$. Since $\operatorname{Ric}\left(\partial_{k}, \partial_{1}\right)=-(n-1) \delta_{k 1}$, (3.4) can be replaced by

$$
\tilde{J}_{2} \geq J_{2}-2(n-1)|\nabla u|
$$

and consequently (3.6) can be replaced by

$$
2(n-1)<\alpha((n-1) / R-\alpha),
$$

where $\alpha<\min \left\{\operatorname{coth} R, \frac{n-1}{2 R}\right\}$. Hence we obtain an upper bound for the radius $R$. For instance, if $n=2$, then $\alpha<\frac{1}{2 R}$ and we need $R<\frac{1}{2 \sqrt{2}}$. For all dimensions, $\alpha=1$ and $R<\frac{n-1}{2 n-1}$ will do.

Example 3.2. As a second example let $N$ be a Cartan-Hadamard manifold with sectional curvatures bounded from below by $-K^{2}$, with $K>0$. Again we choose $\Omega=B(o, R)$ and

$$
h(x)=\frac{r(x)^{2}}{2 R}-\frac{R}{2}
$$

Now $1 / R \leq \kappa_{0} \leq K \operatorname{coth}(K R)$ and again we may choose $k_{1}=1 / R$. This time $\operatorname{Ric}\left(\partial_{1}, \partial_{1}\right) \geq-(n-1) K^{2}$ and $\operatorname{Ric}\left(\partial_{k}, \partial_{1}\right) \geq-\frac{1}{2}(n-1) K^{2}$ for $k=2, \ldots, n$, and therefore instead of (3.4) and (3.6) we have

$$
\tilde{J}_{2} \geq J_{2}-K^{2}\left((n+1)^{2} / 2-2\right)|\nabla u|
$$

and

$$
K^{2}\left((n+1)^{2} / 2-2\right)<\alpha((n-1) / R-\alpha)
$$

where $\alpha<\min \left\{1 / R, \frac{n-1}{2 R}\right\}$. Again we obtain upper bounds for the radius $R$. If $n \geq 3$ we need

$$
R<\left(\frac{n-2}{K^{2}\left((n+1)^{2} / 2-2\right)}\right)^{1 / 2}
$$

whereas for $n=2$ the bound

$$
R<\frac{1}{2 \sqrt{2} K}
$$

is enough since now $\operatorname{Ric}\left(\partial_{2}, \partial_{1}\right)=0$.
Conflict of interest: Authors state no conflict of interest.

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