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Rigidity, counting and equidistribution of quaternionic Cartan chains

Jouni Parkkonen
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Abstract

In this paper, we prove an analog of Cartan's theorem, saying that the chain-preserving transformations of the boundary of the quaternionic hyperbolic spaces are projective transformations. We give a counting and equidistribution result for the orbits of arithmetic chains in the quaternionic Heisenberg group.

Rigidité, comptage et équidistribution de chaînes de Cartan quaternioniennes

Résumé

Dans ce papier, nous montrons un analogue d'un théorème de Cartan, disant que les transformations du bord des espaces hyperboliques quaternioniens qui préservent les chaînes sont des transformations projectives. Nous donnons un résultat de comptage et d'équidistribution pour les orbites de chaînes arithmétiques dans le groupe de Heisenberg quaternionien.

1. Introduction

The sphere at infinity $\partial_\infty X$ of a negatively curved symmetric space X carries many rich structures, from the geometric, analytic and arithmetic points of view. When the sectional curvature is not constant, the possibilities are particularly rich, for instance with the Carnot–Carathéodory, sub-Riemannian or (hyper) CR structures (see for instance [4, 10, 12, 14, 17]), leading to strong rigidity properties, as Pansu's rigidity theorem for quasi-isometries [18]. Arithmetic subgroups of the isometry group of X endow the sphere at infinity of X with arithmetic structures, and problems of equidistribution of rational points or subvarieties in $\partial_\infty X$, as well as in other homogeneous manifolds, have been intensively studied (see for instance [1, 2, 6, 8, 9, 11, 15, 22] and many others).

In this paper, we study the quaternionic hyperbolic spaces X, whose extreme rigidity is exemplified by the Margulis–Gromov–Schoen theorem in [13], proving, contrarily to the real or complex case, the arithmeticity of lattices in the isometry group of X. As announced in [22], we prove a von Staudt–Cartan type of rigidity result for the family of all 3-sphere chains in the sphere at infinity of X, and, analogously to the complex hyperbolic case treated in [20], an effective equidistribution result for the arithmetic

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chains in orbits of arithmetic groups built using maximal orders in rational quaternion algebras.

More precisely, let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$, with $x \mapsto \bar{x}$ its conjugation, $\mathtt{n} : x \mapsto x\bar{x}$ its reduced norm, $\mathtt{tr} : x \mapsto x + \bar{x}$ its reduced trace. Let q be the quaternionic Hermitian form on the right vector space $\mathbb{H}^3$ over $\mathbb{H}$ defined by

$$q(z_0, z_1, z_2) = -\,\mathtt{tr}(\overline{z_0}z_2) + \mathtt{n}(z_1),$$

and PU_q its projective unitary group. It is the isometry group of the quaternionic hyperbolic plane $\mathbf{H}^2_{\mathbb{H}}$, realised as the negative cone of q in the right projective plane $\mathbb{P}^2_{\mathrm{r}}(\mathbb{H})$, and normalised to have maximal sectional curvature -1. See Section 2 for a more complete description.

The boundary at infinity $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$ of $\mathbf{H}^2_{\mathbb{H}}$ is the isotropic cone of q in $\mathbb{P}^2_{\mathrm{r}}(\mathbb{H})$, and the intersections with $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$ of the quaternionic projective lines meeting $\mathbf{H}^2_{\mathbb{H}}$ are called *chains*. We study them, giving their elementary properties and complete geometric descriptions in Section 3. Our first result is similar to Cartan's theorem (see [7, 10]) in the complex hyperbolic case. See Theorem 3.3 for a version in any dimension.

Theorem 1.1. *A chain-preserving transformation from the boundary at infinity of the quaternionic hyperbolic plane to itself is a projective unitary transformation.*

The boundary at infinity $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$ of $\mathbf{H}^2_{\mathbb{H}}$, with the point $\infty = [1 : 0 : 0]$ removed, identifies by the map $(w_0, w) \mapsto [w_0 : w : 1]$ with the quaternionic Heisenberg group

$$\mathbb{H}\mathrm{eis}_7 = \{(w_0, w) \in \mathbb{H} \times \mathbb{H} : \mathtt{tr}\, w_0 = \mathtt{n}(w)\},$$

with group law

$$(w_0, w)(w_0', w') = (w_0 + w_0' + \overline{w}w', w + w'). \tag{1.1}$$

We endow the metabelian simply connected real Lie group $\mathbb{H}\mathrm{eis}_7$ with its Cygan distance d_{Cyg}, which is the unique left-invariant distance such that $d_{\mathrm{Cyg}}((w_0, w), (0, 0)) = (4\,\mathtt{n}(w_0))^{\frac{1}{4}}$. The chains C contained in $\mathbb{H}\mathrm{eis}_7$ are ellipsoids, and have a natural center $\mathrm{cen}(C)$ and radius (see Section 3).

Let A be a definite ($A \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$) quaternion algebra over $\mathbb{Q}$, with discriminant D_A. Let $\mathscr{O}$ be a maximal order in A. We refer for instance to [25] for background on quaternion algebras and orders. The group $\mathrm{PU}_q(\mathscr{O})$ of elements of PU_q represented by matrices with coefficients in $\mathscr{O}$ is a (necessarily arithmetic) lattice in PU_q. A chain C_0 is said to be *arithmetic* over $\mathscr{O}$ if the orbit of some point of C_0 under the stabiliser of C_0 in $\mathrm{PU}_q(\mathscr{O})$ is dense in C_0. The stabiliser $\mathrm{PU}_q(\mathscr{O})_\infty$ of $[1 : 0 : 0]$ in $\mathrm{PU}_q(\mathscr{O})$ preserves the diameters of the chains for d_{Cyg}. The following result (see Theorem 4.2 for an explicit and more general version) is an asymptotic counting result of the arithmetic chains in an orbit under the arithmetic group $\mathrm{PU}_q(\mathscr{O})$ when their Cygan diameter tends to 0.

Theorem 1.2. *Let C_0 be an arithmetic chain in $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$. There exists a constant $\kappa > 0$ and an explicit constant $c > 0$ such that, as $\epsilon \to 0$, the number of chains modulo $\mathrm{PU}_q(\mathscr{O})_\infty$ in the $\mathrm{PU}_q(\mathscr{O})$-orbit of C_0, with Cygan diameter at least ϵ, is equal to $c\epsilon^{-10}(1 + \mathrm{O}(\epsilon^\kappa))$.*

An arithmetic chain C_0 bounds in $\mathbf{H}^2_{\mathbb{H}}$ a homothetic copy of the real hyperbolic space of dimension 4. We denote by $\mathrm{Covol}(C_0)$ the volume of the quotient of this real hyperbolic space, normalised to have sectional curvature -1, by the stabiliser $\mathrm{PU}_q(\mathscr{O})_{C_0}$ of C_0 in $\mathrm{PU}_q(\mathscr{O})$, and by m_0 the order of the pointwise stabiliser of this real hyperbolic space in $\mathrm{PU}_q(\mathscr{O})$. We endow the real Lie group $\mathbb{H}\mathrm{eis}_7$ with its Haar measure $\mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7}$ normalised in such a way that the total mass of the induced measure on the quotient of $\mathbb{H}\mathrm{eis}_7$ by its (uniform) lattice $\mathbb{H}\mathrm{eis}_7 \cap (\mathscr{O} \times \mathscr{O})$ is $\frac{D_A^2}{4}$ (see for instance [22, Lem. 8·4] for an explanation of this normalisation). Let $m_A = 72$ if D_A is even, and $m_A = 1$ otherwise. Finally, we denote by Δ_x the unit Dirac mass at any point x. The following result proves that the centers of the arithmetic chains in an orbit under the arithmetic group $\mathrm{PU}_q(\mathscr{O})$ equidistribute in the quaternionic Heisenberg group.

Theorem 1.3. *For the weak-star convergence of measures on $\mathbb{H}\mathrm{eis}_7$, we have*

$$\frac{m_0 m_A \pi^6 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)}{25515\ 2^{24}\ \mathrm{Covol}(C_0)}\,\epsilon^{10} \sum_{\substack{[g]\in \mathrm{PU}_q(\mathscr{O})/\mathrm{PU}_q(\mathscr{O})_{C_0} \\ \epsilon \leq \mathrm{diam}_{d_{\mathrm{Cyg}}}(gC_0) < \infty}} \Delta_{\mathrm{cen}(gC_0)} \overset{*}{\rightharpoonup} \mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7}\,.$$

We refer to Section 4 for a version with congruences and error terms, and a more developped study of explicit examples of arithmetic chains.

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2. Quaternionic hyperbolic spaces and Heisenberg groups

In this section, we briefly recall some background on the quaternionic hyperbolic spaces and quaternionic Heisenberg groups, as mostly contained in [22, §3 and §6], see also [16, 23] (with different choices of quaternionic Hermitian form and normalisation of the curvature).

Let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$, with $x \mapsto \bar{x}$ its conjugation, $\mathtt{n} : x \mapsto x\bar{x}$ its reduced norm, $\mathtt{tr} : x \mapsto x + \bar{x}$ its reduced trace and $\operatorname{Im} : x \mapsto \frac{1}{2}(x - \bar{x})$ its imaginary part map. We denote by $(1, i, j, k)$ the canonical basis of $\mathbb{H}$ as a real vector space, so that $\overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k$. Let

$$\operatorname{Im}\mathbb{H} = \{x \in \mathbb{H} : \mathtt{tr}\, x = 0\} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

be the $\mathbb{R}$-subspace of purely imaginary quaternions of $\mathbb{H}$. For all $w = (w_1, \ldots, w_N)$ and $w' = (w'_1, \ldots, w'_N)$ in the right vector space $\mathbb{H}^N$ over $\mathbb{H}$, we denote by $\overline{w} \cdot w' = \sum_{p=1}^{N} \overline{w_p} w'_p$ their standard quaternionic Hermitian product, and we define $\mathtt{n}(w) = \overline{w} \cdot w = \sum_{p=1}^{N} \mathtt{n}(w_p)$. We endow $\mathbb{H}^N$ with the standard Euclidean structure $(w, w') \mapsto \frac{1}{2}\, \mathtt{tr}(\overline{w} \cdot w')$.

We fix $n \in \mathbb{N} - \{0, 1\}$. On the right vector space $\mathbb{H} \times \mathbb{H}^{n-1} \times \mathbb{H}$ over $\mathbb{H}$ with coordinates (z_0, z, z_n), let q be the nondegenerate quaternionic Hermitian form

$$q(z_0, z, z_n) = -\,\mathtt{tr}(\overline{z_0} z_n) + \mathtt{n}(z) \tag{2.1}$$

of Witt signature $(1, n)$, and let $\Phi : \mathbb{H}^{n+1} \times \mathbb{H}^{n+1} \to \mathbb{H}$, defined by

$$\Phi : \big((z_0, z, z_n), (z'_0, z', z'_n)\big) \mapsto -\overline{z_0} z'_n - \overline{z_n} z'_0 + \bar{z} \cdot z', \tag{2.2}$$

be the associated quaternionic sesquilinear form.

The *Siegel domain* model of the quaternionic hyperbolic n-space $\mathbf{H}^n_{\mathbb{H}}$ is

$$\big\{(w_0, w) \in \mathbb{H} \times \mathbb{H}^{n-1} : \mathtt{tr}\, w_0 - \mathtt{n}(w) > 0\big\},$$

endowed with the Riemannian metric

$$\mathrm{d}s^2_{\mathbf{H}^n_{\mathbb{H}}} = \frac{1}{(\mathtt{tr}\, w_0 - \mathtt{n}(w))^2}\big(\mathtt{n}(\mathrm{d}w_0 - \overline{\mathrm{d}w} \cdot w) + (\mathtt{tr}\, w_0 - \mathtt{n}(w))\, \mathtt{n}(\mathrm{d}w)\big).$$

Its boundary at infinity is

$$\partial_\infty \mathbf{H}^n_{\mathbb{H}} = \big\{(w_0, w) \in \mathbb{H} \times \mathbb{H}^{n-1} : \mathtt{tr}\, w_0 - \mathtt{n}(w) = 0\big\} \cup \{\infty\}.$$

A *quaternionic geodesic line* in $\mathbf{H}^n_{\mathbb{H}}$ is the image by an isometry of $\mathbf{H}^n_{\mathbb{H}}$ of the intersection of $\mathbf{H}^n_{\mathbb{H}}$ with the quaternionic line $\mathbb{H} \times \{0\}$. With our normalisation of the metric, a quaternionic geodesic line is a totally geodesic submanifold of real dimension 4 and constant sectional curvature -4.

The closed horoballs in $\mathbf{H}^n_{\mathbb{H}}$ centred at $\infty \in \partial_\infty \mathbf{H}^n_{\mathbb{H}}$ are the subsets

$$\mathscr{H}_s = \{(w_0, w) \in \mathbf{H}^n_{\mathbb{H}} : \mathtt{tr}\, w_0 - \mathtt{n}(w) \geq s\}, \tag{2.3}$$

and the horospheres centred at ∞ are their boundaries $\partial \mathscr{H}_s$, where s ranges in $]0, +\infty[$. Note that, for every $s \in\,]0, 1]$, we have

$$d(\partial \mathscr{H}_1, \partial \mathscr{H}_s) = -\frac{\ln s}{2}. \tag{2.4}$$

The Siegel domain $\mathbf{H}^n_{\mathbb{H}}$ embeds in the right quaternionic projective n-space $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ by the map (using homogeneous coordinates)

$$(w_0, w) \mapsto [w_0 : w : 1].$$

By this map, we identify $\mathbf{H}^n_{\mathbb{H}}$ with its image, which when endowed with the isometric Riemannian metric, is called the *projective model* of $\mathbf{H}^n_{\mathbb{H}}$. Note that this image is the *negative cone* of the quaternionic Hermitian form q defined in Equation (2.1) : we have $\mathbf{H}^n_{\mathbb{H}} = \big\{[z_0 : z : z_n] \in \mathbb{P}^n_{\mathrm{r}}(\mathbb{H}) : q(z_0, z, z_n) < 0\big\}$. This embedding extends continuously to the boundary at infinity, by mapping the point $(w_0, w) \in \partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}$ to $[w_0 : w : 1]$ and ∞ to $[1 : 0 : 0]$, so that the image of $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ is the *isotropic cone* of q: we have $\partial_\infty \mathbf{H}^n_{\mathbb{H}} = \big\{[z_0 : z : z_n] \in \mathbb{P}^n_{\mathrm{r}}(\mathbb{H}) : q(z_0, z, z_n) = 0\big\}$. A projective point $[z_0 : z : z_n] \in \mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ is *positive* if $q(z_0, z, z_n) > 0$.

For every $N \in \mathbb{N}$, let I_N be the identity $N \times N$ matrix. Let

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_{n-1} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The *conjugate-transpose* matrix of a quaternionic matrix $X = (x_{p,p'})_{1 \le p \le r, 1 \le p' \le s}$ in $\mathscr{M}_{r,s}(\mathbb{H})$ is $X^* = (x^*_{p,p'} = \overline{x_{p',p}})_{1 \le p \le s, 1 \le p' \le r} \in \mathscr{M}_{s,r}(\mathbb{H})$. Let

$$\mathrm{U}_q = \{g \in \mathrm{GL}_{n+1}(\mathbb{H}) : q \circ g = q\} = \{g \in \mathrm{GL}_{n+1}(\mathbb{H}) : g^* J g = J\}$$

be the *unitary group* of q. Its left linear action on $\mathbb{H}^{n+1}$ induces a projective action on $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ with kernel its center, which is reduced to $\{\pm I_{n+1}\}$. The *projective unitary group*

$$\mathrm{PU}_q = \mathrm{U}_q / \{\pm I_{n+1}\}$$

of q acts faithfully on $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$, preserving $\mathbf{H}^n_{\mathbb{H}}$, and its restriction to $\mathbf{H}^n_{\mathbb{H}}$ is the full isometry group of $\mathbf{H}^n_{\mathbb{H}}$.

A matrix

$$X = \begin{pmatrix} a & \gamma^* & b \\ \alpha & M & \beta \\ c & \delta^* & d \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{H}),$$

with $a, b, c, d \in \mathbb{H}, \alpha, \beta, \gamma, \delta \in \mathbb{H}^{n-1}$ (identified with their column matrices in $\mathscr{M}_{n-1,1}(\mathbb{H})$) and $M \in \mathscr{M}_{n-1,n-1}(\mathbb{H})$, belongs to U_q if and only if

$$\begin{cases} \bar{c}a - \alpha^*\alpha + \bar{a}c = 0 \\ \bar{d}b - \beta^*\beta + \bar{b}d = 0 \\ -\delta\gamma^* + M^*M - \gamma\delta^* = I_{n-1} \\ \bar{d}a - \beta^*\alpha + \bar{b}c = 1 \\ \delta a - M^*\alpha + \gamma c = 0 \\ \delta b - M^*\beta + \gamma d = 0. \end{cases} \tag{2.5}$$

With $\mathrm{Sp}(n-1) = \{g \in \mathrm{GL}_{n+1}(\mathbb{H}) : g^*g = I_{n-1}\}$, an easy computation shows that the block upper triangular subgroup of U_q is

$$\mathrm{B}_q = \left\{ \begin{pmatrix} \mu r & \zeta^* & \frac{1}{2r}(\mathrm{n}(\zeta)+u)\mu \\ 0 & U & \frac{1}{r}U\zeta\mu \\ 0 & 0 & \frac{\mu}{r} \end{pmatrix} : \begin{array}{l} \zeta \in \mathbb{H}^{n-1}, u \in \mathrm{Im}\,\mathbb{H}, \\ U \in \mathrm{Sp}(n-1), \mu \in \mathrm{Sp}(1), r > 0 \end{array} \right\}.$$

Its image $\mathrm{PB}_q = \mathrm{B}_q/\{\pm I_{n+1}\}$ in PU_q is equal to the stabiliser of ∞ in PU_q.

The *quaternionic Heisenberg group* $\mathbb{H}\mathrm{eis}_{4n-1}$ of dimension $4n-1$ is the real Lie group structure on $\mathbb{H}^{n-1} \times \mathrm{Im}\,\mathbb{H}$ with law

$$(\zeta, u)(\zeta', u') = (\zeta + \zeta', u + u' + 2\,\mathrm{Im}\,\bar{\zeta} \cdot \zeta')$$

and inverses $(\zeta, u)^{-1} = (-\zeta, -u)$. It identifies with the punctured boundary at infinity $\partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}$ by the map $(\zeta, u) \mapsto (w_0, w)$ where

$$(w_0, w) = \left(\frac{\mathrm{n}(\zeta) + u}{2}, \zeta\right) \quad \text{hence} \quad (\zeta, u) = (w, 2\,\mathrm{Im}\,w_0), \tag{2.6}$$

and with a subgroup of $\mathrm{PB}_q \subset \mathrm{PU}_q$, preserving every horoball $\mathscr{H}_s$ for $s > 0$, by the map

$$(\zeta, u) \mapsto \pm \begin{pmatrix} 1 & \zeta^* & \frac{\mathrm{n}(\zeta)+u}{2} \\ 0 & I_{n-1} & \zeta \\ 0 & 0 & 1 \end{pmatrix}.$$

Equation (2.6) allows to recover the definition of $\mathbb{H}\mathrm{eis}_7$ given in the Introduction, for which the inverses are $(w_0, w)^{-1} = (-w_0 + \mathrm{n}(w), -w)$.

For every $(\zeta, u) \in \mathbb{H}\mathrm{eis}_{4n-1}$, the map $(\zeta', u') \mapsto (\zeta, u)(\zeta', u')$ is the *Heisenberg translation* by (ζ, u). For every $\zeta \in \mathbb{H}^{n-1}$, the Heisenberg translation by $(\zeta, 0)$ is called a *horizontal (Heisenberg) translation*. For every $u \in \mathrm{Im}\,\mathbb{H}$, the Heisenberg translation by $(0, u)$ is called a *vertical (Heisenberg) translation*. The canonical map $\Pi_v : \mathbb{H}\mathrm{eis}_{4n-1} \to \mathbb{H}^{n-1}$ defined by $(\zeta, u) \mapsto \zeta$ is a real Lie group morphism, called the *vertical projection*, whose kernel is the center of $\mathbb{H}\mathrm{eis}_{4n-1}$. For every $U \in \mathrm{Sp}(n-1)$, the map $(\zeta, u) \mapsto (U\zeta, u)$ is the *Heisenberg rotation* by U. For every $\lambda > 0$, the map $h_\lambda : (\zeta, u) \mapsto (\lambda\zeta, \lambda^2 u)$ is the *Heisenberg dilation* by λ.

The *Cygan distance* $d_{\rm Cyg}$ on $\mathbb{H}\mathrm{eis}_{4n-1}$ is the unique left-invariant distance on the real Lie group $\mathbb{H}\mathrm{eis}_{4n-1}$ such that

$$d_{\rm Cyg}((\zeta, u), (0, 0)) = \big(\mathtt{n}(\zeta)^2 + \mathtt{n}(u)\big)^{1/4}, \tag{2.7}$$

or equivalently $d_{\rm Cyg}((w_0, w), (0, 0)) = (4\,\mathtt{n}(w_0))^{\frac{1}{4}}$ by Equation (2.6). We introduce (see [19, 20] in the complex case) the *modified Cygan distance* $d''_{\rm Cyg}$, as the unique left-invariant map from $\mathbb{H}\mathrm{eis}_{4n-1} \times \mathbb{H}\mathrm{eis}_{4n-1}$ to $[0, +\infty[$ such that

$$d''_{\rm Cyg}((\zeta, u), (0, 0)) = \frac{(\mathtt{n}(\zeta)^2 + \mathtt{n}(u))^{1/2}}{\big((\mathtt{n}(\zeta)^2 + \mathtt{n}(u))^{1/2} + \mathtt{n}(\zeta)\big)^{1/2}}, \tag{2.8}$$

or equivalently by Equation (2.6)

$$d''_{\rm Cyg}((w_0, w), (0, 0)) = \frac{2\,\mathtt{n}(w_0)^{1/2}}{(2\,\mathtt{n}(w_0)^{1/2} + \mathtt{n}(w))^{1/2}}.$$

Though not actually a distance, the map $d''_{\rm Cyg}$ is symmetric and satisfies

$$\frac{1}{\sqrt{2}} d_{\rm Cyg} \leq d''_{\rm Cyg} \leq d_{\rm Cyg}.$$

For every nonempty bounded subset E of $\mathbb{H}\mathrm{eis}_{4n-1}$, we define the *diameter* of E for this almost distance as

$$\mathrm{diam}_{d''_{\rm Cyg}}(E) = \sup_{x, y \in E} d''_{\rm Cyg}(x, y).$$

Note that the Cygan distance and the modified Cygan distance are invariant under Heisenberg translations and rotations, and that for every $\lambda > 0$, the Heisenberg dilation h_λ is a homothety of ratio λ for both distances.

Lemma 2.1. *For every geodesic line* $]x, y[$ *in* $\mathbf{H}^n_{\mathbb{H}}$ *disjoint from the horoball* $\mathscr{H}_1$*, the distance in* $\mathbf{H}^n_{\mathbb{H}}$ *between* $\mathscr{H}_1$ *and* $]x, y[$ *is equal to*

$$d(\mathscr{H}_1,]x, y[) = -\ln\left(\frac{1}{\sqrt{2}} d''_{\rm Cyg}(x, y)\right).$$

Proof. By the invariance under Heisenberg translations of $\mathscr{H}_1$, of the distance in $\mathbf{H}^n_{\mathbb{H}}$ and of the modified Cygan distance, we may assume that $x = (w_0, w) \in \partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty, (0, 0)\}$ and $y = (0, 0) \in \partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}$. By [22, Lem. 6·4], the geodesic line from (w_0, w) to $(0, 0)$ is, up to translation at the source, the map

$$\gamma_{w_0, w} : t \mapsto \big(w_0(1 + e^{2t} w_0)^{-1}, w(1 + e^{2t} w_0)^{-1}\big).$$

The point $\gamma_{w_0,w}(t)$ belongs to the horosphere $\mathscr{H}_{s(t)}$, where, since $\mathtt{tr}\, w_0 = \mathtt{n}(w)$,

$$s(t) = \mathtt{tr}(w_0(1+e^{2t}w_0)^{-1}) - \mathtt{n}(w(1+e^{2t}w_0)^{-1}) = \frac{2e^{2t}\,\mathtt{n}(w_0)}{\mathtt{n}(1+e^{2t}w_0)}.$$

Let $r = \mathtt{n}(w_0)^{1/2}$ be the norm of the vector w_0 and θ the angle between the vectors 1 and w_0 in the Euclidean space $\mathbb{H}$. Then the map

$$t \mapsto s(t) = \frac{2e^{2t}r^2}{e^{4t}r^2 + 2re^{2t}\cos\theta + 1}$$

reaches its maximum at $e^{2t} = \frac{1}{r}$. Since $\mathtt{tr}\, w_0 = \mathtt{n}(w)$, the value of this maximum is

$$s_{\max} = \frac{2\,\mathtt{n}(w_0)^{1/2}}{2+\mathtt{tr}(w_0\,\mathtt{n}(w_0)^{-1/2})} = \frac{2\,\mathtt{n}(w_0)}{2\,\mathtt{n}(w_0)^{1/2}+\mathtt{n}(w)} = \frac{1}{2}d''_{\mathrm{Cyg}}((w_0,w),(0,0))^2.$$

The result then follows from Equation (2.4). □

3. **Chains**

In this section, we define the quaternionic Cartan chains and give their elementary geometric properties, see also [24]. In the complex case, the notion of chain is attributed to von Staudt by [7]. The exposition follows the one of [10] in the complex case. We fix $m \in \{1, \dots, n-1\}$.

3.1. **A vocabulary of chains**

An *m-chain* C in $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ is the intersection with $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ of a quaternionic projective space L_C of dimension m meeting $\mathbf{H}^n_{\mathbb{H}}$. Note that C determines L_C and conversely. A *chain* is a 1-chain, and a *hyperchain* is an $(n-1)$-chain. An m-chain is *vertical* if it contains $\infty = [1:0:0]$, and *finite* otherwise.

If $P = [z_0 : z : z_n] \in \mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$, let

$$P^\perp = \big\{[z'_0 : z' : z'_n] \in \mathbb{P}^n_{\mathrm{r}}(\mathbb{H}) : \Phi\big((z_0, z, z_n), (z'_0, z', z'_n)\big) = 0\big\}$$

be the orthogonal quaternionic projective subspace of P. The map $P \mapsto P^\perp$, from the set of positive projective points to the set of quaternionic projective hyperplanes in $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ meeting $\mathbf{H}^n_{\mathbb{H}}$, is a PU_q-equivariant bijection. Therefore, the map

$$P \mapsto C_P = P^\perp \cap \partial_\infty \mathbf{H}^n_{\mathbb{H}}$$

is a PU_q-equivariant bijection from the set of positive projective points to the set of hyperchains. The point P is called the *polar point* of the hyperchain C_P, or of the quaternionic projective hyperplane $P^\perp$. If $P = [z_0 : z : z_n]$, we have

$$C_P \cap (\partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}) = \left\{[w_0 : w : 1] : -\left(\frac{\mathtt{n}(w)}{2} - \operatorname{Im} w_0\right) z_n + \overline{w} \cdot z - z_0 = 0\right\}. \quad (3.1)$$

This hyperchain C_P is hence vertical if and only if $z_n = 0$, in which case $C_P \cap (\partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\})$ is the preimage by the vertical projection $\Pi_v : \mathbb{H}\mathrm{eis}_{4n-1} \longrightarrow \mathbb{H}^{n-1}$ of the quaternionic affine hyperplane of $\mathbb{H}^{n-1}$ with equation $\overline{z} \cdot w = \overline{z_0}$ in the unknown w. Similarly a vertical chain is the preimage of a point of $\mathbb{H}^{n-1}$ by the vertical projection Π_v.

When $C = C_P$ is a finite hyperchain, that is, when $z_n \neq 0$, then C is a codimension 4 ellipsoid in the Euclidean space $\mathbb{H}^{n-1} \times \operatorname{Im} \mathbb{H}$, whose vertical projection is the Euclidean sphere in $\mathbb{H}^{n-1}$ with real codimension 1 and equation $\mathtt{n}(w) - \mathtt{tr}(\overline{w} \cdot (z z_n^{-1})) + \mathtt{tr}(z_0 z_n^{-1}) = 0$ in the unknown w, with center $z z_n^{-1}$ and radius

$$R_C = \frac{q(z_0, z, z_n)^{1/2}}{\mathtt{n}(z_n)^{1/2}}.$$

This radius R_C of the Euclidean sphere $\Pi_v(C)$ is called the *radius* of the finite hyperchain C. The map $\Pi_v|_C$ from C to $\Pi_v(C)$ is a homeomorphism. When $z = 0$ and $z_0 z_n^{-1} \in \mathbb{R}$, the hyperchain $C = C_P$ is contained in the horizontal subspace $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im} \mathbb{H} : u = 0\}$ of $\mathbb{H}\mathrm{eis}_{4n-1}$, by Equation (2.6).

Similarly, a finite chain is a 3-dimensional ellipsoid in the Euclidean space $\mathbb{H}^{n-1} \times \operatorname{Im} \mathbb{H}$, whose vertical projection is a Euclidean 3-sphere in $\mathbb{H}^{n-1}$. In particular, any chain is homeomorphic to the 3-sphere $\mathbb{S}^3$.

3.2. Transitivity properties of PU_q on chains

Through any two distinct projective points belonging to $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ passes one and only one quaternionic projective line, and this projective line meets $\mathbf{H}^n_{\mathbb{H}}$. Hence through two distinct points of $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ passes one and only one chain. By Witt's theorem, the group PU_q acts transitively on the set of quaternionic projective spaces L of dimension m meeting $\mathbf{H}^n_{\mathbb{H}}$, hence it acts transitively on the set of m-chains.

Note that two m-chains having the same vertical projection differ by a vertical Heisenberg translation, that the group generated by Heisenberg translations and rotations acts transitively on the set of vertical m-chains, and that PB_q (that contains the Heisenberg dilations, rotations and translations) acts transitively on the set of finite m-chains.

The next result gives the topological structure of a family of chains, called a *fan* in the complex hyperbolic case (see for instance [10, p. 131]).

Proposition 3.1. *The union F of all chains containing a given point $P \in \partial_\infty \mathbf{H}^n_{\mathbb{H}}$ and passing through an m-chain C of $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ not containing P is homeomorphic to the* *topological quotient space* $(\mathbb{S}^3 \times \mathbb{S}^{4m-1})/\!\sim$ *where* $\sim$ *is the equivalence relation generated by* $(x_0, x) \sim (x_0, y)$ *for all* $x, y \in \mathbb{S}^{4m-1}$*, where* x_0 *is any fixed point in* $\mathbb{S}^3$.

Proof. By the transitivity properties of PU_q, we may assume that $P = \infty$. Hence C is a finite chain, and by the transitivity properties of the Heisenberg translations, we may assume that C is a Euclidean sphere of dimension $4m-1$ contained in the horizontal space $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : u = 0\}$. Thus $F = \bigcup_{(\zeta,u)\in C} \Pi_v^{-1}(\zeta, u)$ is clearly homeomorphic to the above quotient of $\mathbb{S}^3 \times \mathbb{S}^{4m-1}$. □

3.3. **Reflexions on chains**

The chains are fixed point sets at infinity of natural isometries of $\mathbf{H}^n_{\mathbb{H}}$, that we now describe.

If L is a proper quaternionic projective subspace of $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ meeting $\mathbf{H}^n_{\mathbb{H}}$, there exists a unique involution ι_L in PU_q with fixed point set L, called the *reflexion* on L. Note that the set of fixed points of ι_L in $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ is the m-chain $L \cap \partial_\infty \mathbf{H}^n_{\mathbb{H}}$, where m is the quaternionic dimension of L, assuming that $m \neq 0$.

For instance, $C = \big\{[z_0 : z_1 : \cdots : z_n] \in \partial_\infty \mathbf{H}^n_{\mathbb{H}} : z_m = 0, \dots, z_{n-1} = 0\big\} \cup \{\infty\}$ is a vertical m-chain, called the *standard vertical m-chain* and the reflexion ι_{L_C} is the map

$$[z_0 : z_1 : \cdots : z_n] \mapsto [z_0 : z_1 : \cdots : z_{m-1} : -z_m : \cdots : -z_{n-1} : z_n].$$

The vertical m-chains are the images of the standard vertical m-chain by the Heisenberg translations and Heisenberg rotations: they are the subspaces

$$(E \times \operatorname{Im}\mathbb{H}) \cup \{\infty\}$$

where E is a quaternionic affine subspace of $\mathbb{H}^{n-1}$ with dimension $m-1$ (hence a point when $m = 1$).

Lemma 3.2. *Let L and L' be quaternionic projective subspaces of* $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ *meeting* $\mathbf{H}^n_{\mathbb{H}}$ *such that one is not contained in the other, whose sum of dimensions is n. The following assertions are equivalent.*

(1) *The reflexions ι_L and $\iota_{L'}$ commute.*

(2) *The reflexion ι_L preserves L'.*

(3) *The reflexion $\iota_{L'}$ preserves L.*

(4) *We have* $(\iota_L \circ \iota_{L'})^2 = \mathrm{id}$.

(5) *The totally geodesic subspaces* $L \cap \mathbf{H}^n_{\mathbb{H}}$ *and* $L' \cap \mathbf{H}^n_{\mathbb{H}}$ *intersect perpendicularly in the Riemannian manifold* $\mathbf{H}^n_{\mathbb{H}}$.

(6) *The subspace* $L \cap \mathbf{H}^n_{\mathbb{H}}$ *is a fiber of the orthogonal projection on* $L' \cap \mathbf{H}^n_{\mathbb{H}}$ *in* $\mathbf{H}^n_{\mathbb{H}}$.

(7) *The subspace* $L' \cap \mathbf{H}^n_{\mathbb{H}}$ *is a fiber of the orthogonal projection on* $L \cap \mathbf{H}^n_{\mathbb{H}}$ *in* $\mathbf{H}^n_{\mathbb{H}}$.

Proof. The proof is similar to the one of [10, Lem. 4.3.1] in the complex hyperbolic case. Note that $L \cap \mathbf{H}^n_{\mathbb{H}}$, being the set of fixed points of the isometry ι_L of the negatively curved Riemannian manifold $\mathbf{H}^n_{\mathbb{H}}$, is indeed totally geodesic.

Two involutions commute if and only if their composition is an involution or the identity, hence Assertions (1) and (4) are equivalent. Since the centralizer of a projective transformation preserves its fixed point set, Assertion (1) implies Assertions (2) and (3). If Assertion (2) is satisfied, then $\iota_L \circ \iota_{L'} \circ \iota_L^{-1} = \iota_{\iota_L(L')} = \iota_{L'}$, so that Assertion (1) is satisfied. Similarly, Assertion (3) implies Assertion (1). Finally, the totally geodesic subspaces $L' \cap \mathbf{H}^n_{\mathbb{H}}$ and $L \cap \mathbf{H}^n_{\mathbb{H}}$ in $\mathbf{H}^n_{\mathbb{H}}$

- either have disjoint closures in $\mathbf{H}^n_{\mathbb{H}} \cup \partial_\infty \mathbf{H}^n_{\mathbb{H}}$, or
- are disjoint and have closures meeting in $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$, or
- meet in $\mathbf{H}^n_{\mathbb{H}}$.

In the first two cases, the composition $\iota_L \circ \iota_{L'}$ has infinite order, and in the last case, $\iota_L \circ \iota_{L'}$ can be an involution if and only if $L' \cap \mathbf{H}^n_{\mathbb{H}}$ and $L \cap \mathbf{H}^n_{\mathbb{H}}$ are perpendicular. □

An m-chain C and an $(n-m)$-chain C' are *orthogonal* if neither of the corresponding quaternionic projective subspaces L_C and $L_{C'}$ contains the other and if they satisfy one of the equivalent assertions of Lemma 3.2. For instance, the hyperchains orthogonal to the standard vertical chain $(\{0\} \times \operatorname{Im} \mathbb{H}) \cup \{\infty\}$ are exactly the Euclidean spheres centered at $(0, u_0)$ in the horizontal subspace $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im} \mathbb{H} : u = u_0\}$ of $\mathbb{H}\mathrm{eis}_{4n-1}$, for some u_0 in $\operatorname{Im} \mathbb{H}$.

3.4. Description of the center and radius of chains

We now define and study the centers of chains, whose equidistribution we will prove in Section 4.

The *center* of an m-chain C is $\operatorname{cen}(C) = \iota_{L_C}(\infty)$. In particular, $\operatorname{cen}(C) = \infty$ if and only if C is vertical. For every element $\gamma \in \mathrm{PB}_q$ (which fixes ∞), the reflexion on the m-chain γC is $\gamma \iota_{L_C} \gamma^{-1}$, so that the center of γC is

$$\operatorname{cen}(\gamma C) = \gamma \operatorname{cen}(C). \tag{3.2}$$

When $P_0 = [-\frac{1}{2} : 0 : 1]$, the hyperchain C_{P_0} with polar point P_0 is, by Equation (3.1), the sphere centered at $(0, 0)$ with radius 1 in the horizontal codimension 3 Euclidean subspace $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : u = 0\}$ in $\mathbb{H}\mathrm{eis}_{4n-1}$. The reflexion on $L = L_{C_{P_0}}$ is the involutive map $\iota_L : (w_0, w) \mapsto (\frac{1}{4} w_0^{-1}, \frac{1}{2} w w_0^{-1})$, induced by

$$\pm \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & I_{n-1} & 0 \\ 2 & 0 & 0 \end{pmatrix} \in \mathrm{PU}_q \,.$$

Thus, $\operatorname{cen}(C_{P_0}) = \iota_L(\infty) = (0, 0)$.

Let $P = [z_0 : z : z_n]$ be a positive projective point with $z_n \neq 0$. An easy computation shows that the Heisenberg translation γ by

$$\left[\frac{\mathtt{n}(z)}{2\,\mathtt{n}(z_n)} - \operatorname{Im}(z_0 z_n^{-1}) : -z z_n^{-1} : 1 \right]$$

maps P to $\left[-\frac{R^2}{2} : 0 : 1\right]$ where $R = R_{C_P} = \frac{q(z_0,z,z_n)^{1/2}}{\mathtt{n}(z_n)^{1/2}}$ is the radius of the finite hyperchain C_P, and the Heisenberg dilation

$$h_R : (w_0, w) \mapsto (R^2 w_0, R w)$$

maps P_0 to $\left[-\frac{R^2}{2} : 0 : 1\right]$. Hence the center of the finite hyperchain C_P with polar point P is, by Equation (3.2), equal to

$$\operatorname{cen}(C_P) = \gamma^{-1} h_R \operatorname{cen}(C_{P_0}) = \gamma^{-1}(0,0) = \left[\frac{2\operatorname{Im}(z_0 \overline{z_n}) + \mathtt{n}(z)}{2\,\mathtt{n}(z_n)} : z z_n^{-1} : 1 \right],$$

or $\operatorname{cen}(C_P) = \big(z z_n^{-1}, 2\operatorname{Im}(z_0 z_n^{-1})\big)$ in the (ζ, u)-coordinates of $\mathbb{H}\mathrm{eis}_{4n-1}$ by Equation (2.6). Thus, by Equation (3.1), if C is a finite hyperchain in $\mathbb{H}\mathrm{eis}_{4n-1}$ with center (ζ_0, u_0) and radius r_0, then

$$C = \big\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : \mathtt{n}(\zeta - \zeta_0) = r_0^2 \ \text{ and } \ u = u_0 + 2\operatorname{Im}(\overline{\zeta_0}\zeta)\big\}.$$

In particular, a finite hyperchain is uniquely determined by its center and its radius, and the hyperchains contained in the horizontal Euclidean space $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : u = 0\}$ are exactly the Euclidean spheres centered at $(0, 0)$.

3.5. A von Staudt–Cartan rigidity theorem

The following theorem shows that the chain-preserving transformations of the boundary of the quaternionic hyperbolic spaces are projective transformations. This is a quaternionic version of the result of Cartan in the complex case (see for instance [10, Thm. 4.3.12]), close to von Staudt's fundamental theorem of real projective geometry.

Theorem 3.3. *A bijection f from $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ to itself, mapping chains to chains, is (the restriction to $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ of) an element of* PU_q.

Proof. Up to composing by an element of PU_q, we may assume that f fixes $\infty = [1:0:0]$. Hence f preserves the set of vertical chains, which are the ones containing ∞. The set of vertical chains identifies with the horizontal space $\mathbb{H}^{n-1}$ of the quaternionic Heisenberg group by the vertical projection Π_v, which sends a vertical chain C to the unique point of $\mathbb{H}^{n-1}$ whose preimage by Π_v is C. Hence f induces a bijection $\overline{f}$ from $\mathbb{H}^{n-1}$ to itself, which sends the vertical projections of the finite chains to the vertical projections of the finite chains.

The vertical projections of the finite chains are exactly all the Euclidean 3-spheres in $\mathbb{H}^{n-1}$. Given two distinct points x, y in $\mathbb{H}^{n-1}$, the complement of the union of all the Euclidean 3-spheres containing x and y is the real affine line containing x and y, with x and y removed. Hence $\overline{f}$ is a bijection of $\mathbb{H}^{n-1}$ sending real affine lines to real affine lines. By the fundamental theorem of real affine geometry, this map is an affine transformation of $\mathbb{H}^{n-1}$. Since the affine transformations of $\mathbb{H}^{n-1}$ are vertical projections of elements of the stabiliser PB_q of ∞ in PU_q, up to composing f by an element of PB_q, we may assume that $\overline{f}$ is the identity map of $\mathbb{H}^{n-1}$, and also that $f(0) = 0$.

Let $x \in \partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}$, and let us prove that $f(x) = x$. First assume that $\Pi_v(x) \neq 0$. Then the unique chain C_x passing through 0 and x is a finite chain, and the vertical projections of C_x and $f(C_x)$ coincide, since $\overline{f} = \mathrm{id}$. By the uniqueness of a chain with given vertical projection up to a vertical translation, since $f(0) = 0$, we have $f(C_x) = C_x$. But if $f(x) \neq x$, then since $f(x)$ and x have the same vertical projections, the chains C_x and $f(C_x)$ through 0 would be different. Hence $f(x) = x$. This is in particular true for any given $x = x_0 \neq 0$ in the horizontal space $\mathbb{H}^{n-1} \times \{0\}$. Replacing 0 by such an x_0 in the above argument allows to prove that $f(x) = x$ when $\Pi_v(x) = 0$. □

A similar proof shows that an injective map f from $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ to itself, such that any three points belong to a same chain if and only if their images by f belong to a same chain, is the restriction of an element of PU_q.

3.6. **Relation with the hyper CR structure**

In this subsection, we give a characterisation of the chains in terms of the natural hyper CR structure on $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$. We refer for example to [3] and [14] for background on hyperkähler manifolds and hyper CR manifolds, respectively.

We endow the manifold $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ with its natural hyperkähler structure, and we denote by $(\mathbb{I}, \mathbb{J}, \mathbb{K})$ the corresponding triple of almost complex structures. The boundary at infinity $W = \partial_\infty \mathbf{H}^n_{\mathbb{H}}$ is a smooth real hypersurface in the real manifold $\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})$ of real dimension $4n$, and $E = TW \cap \mathbb{I}\,TW \cap \mathbb{J}\,TW \cap \mathbb{K}\,TW$ is a real codimension 3 subbundle of the real tangent bundle $T\mathbb{P}^n_{\mathrm{r}}(\mathbb{H})\,|_W$, invariant under PU_q, defining a hyper CR structure on W. When x is the point $(0,0)$ in the (ζ, u)-coordinates of $\mathbb{H}\mathrm{eis}_{4n-1} = \partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}$, then, identifying

$\mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H}$ with its real tangent space at x, the fiber E_x of E over x is the horizontal subspace $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : u = 0\}$.

A *calibration* of E is a 1-form ω on W with values in $\operatorname{Im}\mathbb{H}$ such that $E = \ker \omega$. Its *Levi form* is $\mathrm{d}\omega$. For instance, in the (ζ, u)-coordinates of $\mathbb{H}\mathrm{eis}_{4n-1}$, the form

$$\omega = \mathrm{d}u - 2\operatorname{Im}(\overline{\zeta} \cdot \mathrm{d}\zeta)$$

is a calibration of E (when restricted to $\partial_\infty \mathbf{H}^n_{\mathbb{H}} - \{\infty\}$). An easy computation shows that this calibration is invariant under Heisenberg translations and rotations: For every such transformation γ, we have $\gamma^*\omega = \omega$. The fact that ω is indeed a calibration follows by invariance since $\ker \mathrm{d}u = \{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : u = 0\}$. This calibration ω is scaled by the Heisenberg dilations as follows : for every $\lambda > 0$, we have $(h_\lambda)^*\omega = \lambda^2\omega$.

In the following result, we denote by $v = v_1 i + v_2 j + v_3 k$ the standard coordinate in $\operatorname{Im}\mathbb{H}$, and by $\mathrm{d}v$ the tautological $(\operatorname{Im}\mathbb{H})$-valued 1-form on $\operatorname{Im}\mathbb{H}$, so that for every $x \in \operatorname{Im}\mathbb{H}$, the map $\mathrm{d}v_x : T_x \operatorname{Im}\mathbb{H} = \operatorname{Im}\mathbb{H} \to \operatorname{Im}\mathbb{H}$ is the identity map. We denote by $\omega_1, \omega_2, \omega_3$ the standard coordinates of the calibration ω, so that

$$\omega = \omega_1 i + \omega_2 j + \omega_3 k.$$

Given a chain C in $\partial \mathbf{H}^n_{\mathbb{H}}$, let $\mu = \mu_C$ be the (Borel positive) measure on $\mathbb{H}\mathrm{eis}_{4n-1}$ with support $C \cap \mathbb{H}\mathrm{eis}_{4n-1}$ associated with the volume form $\omega_1 \wedge \omega_2 \wedge \omega_3$ on C. For instance, if $C = \{(\zeta, u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : \zeta = 0\} \cup \{\infty\}$ is the standard vertical chain, then $\omega|_C = \mathrm{d}u|_C$, so that μ_C is the (infinite) measure

$$\mu_C = \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3,$$

whose restriction to the Euclidean space $C - \{\infty\} = \{0\} \times \operatorname{Im}\mathbb{H}$ is the standard Lebesgue measure.

Given a nonzero measure μ with compact support on a finite dimensional real affine space V, the *barycenter* (or *centroid*) of μ is the point $\operatorname{bar}(\mu)$ of V defined by

$$\operatorname{bar}(\mu) = \frac{1}{\mu(V)} \int_{x \in V} x \, \mathrm{d}\mu(x).$$

For instance, when μ is supported on a finite set S, then $\operatorname{bar}(\mu)$ is the usual affine barycenter of the weighted family of points $\big\{\big(s, \frac{\mu(\{s\})}{\mu(S)}\big)\big\}_{s \in S}$.

We denote the open ball of center 0 and radius r in the Euclidean space $\operatorname{Im}\mathbb{H}$ by $B(r)$. Recall that the radius of a finite chain C is denoted by R_C.

Proposition 3.4. *Let C be a chain in $\partial_\infty \mathbf{H}^n_{\mathbb{H}}$ and $c \in C$.*

(1) *If C is a finite chain, then the center of the chain C is equal to the barycenter of the measure μ_C:*

$$\operatorname{cen}(C) = \operatorname{bar}(\mu_C).$$

(2) *If C is a vertical chain, there is a diffeomorphism $\tau = \tau_C : \operatorname{Im}\mathbb{H} \to C - \{\infty\}$ such that $\tau^*\omega = \mathrm{d}v$, unique up to postcomposition by a vertical Heisenberg translation. For every Heisenberg translation or rotation γ, we have $\tau_{\gamma C} = \gamma \circ \tau_C$.*

(3) *If C is a finite chain, there exists a smooth diffeomorphism $\tau = \tau_{C,c}$ from $B(2\pi R_C^2)$ to $C - \{c\}$, admitting a continuous extension to $\partial B(2\pi R_C^2)$ sending this sphere to c, such that $\tau^*\omega = \mathrm{d}v$. This mapping is unique up to postcomposition by a Heisenberg rotation preserving C and c, and $2\pi R_C^2$ is the unique radius for which such a mapping exists.*

For every Heisenberg translation or rotation γ, we have $\tau_{\gamma C,\gamma c} = \gamma \circ \tau_{C,c}$.

Proof. (1). Note that $\mathbb{H}\mathrm{eis}_{4n-1} = \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H}$ has a natural structure of a real affine space, and that the elements of PB_q act by affine transformations on $\mathbb{H}\mathrm{eis}_{4n-1}$. This can be seen for instance by saying that $\mathbb{H}\mathrm{eis}_{4n-1}$, identified with the boundary of the projective model of $\mathbf{H}^n_{\mathbb{H}}$ minus $\{\infty\}$, is a PB_q-invariant affine subspace of the affine chart of the quaternionic projective space defined by the quaternionic projective hyperplane $\big\{[z_0 : z : z_n] \in \mathbb{P}^n_{\mathrm{r}}(\mathbb{H}) : z_n = 0\big\}$, and that the quaternionic projective transformations preserving this hyperplane act by affine transformations on the associated affine chart. Another way is to check, by an easy computation, that the Heisenberg translations, rotations and dilations preserve the barycenters in the real affine space $\mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H}$: For instance, for all $(\zeta_0, u_0), (\zeta, u), (\zeta', u') \in \mathbb{H}\mathrm{eis}_{4n-1}$ and $t \in [0, 1]$, we have

$$(\zeta_0, u_0) \cdot \big(t(\zeta, u) + (1-t)(\zeta', u')\big) = t(\zeta_0, u_0) \cdot (\zeta, u) + (1-t)(\zeta_0, u_0) \cdot (\zeta', u').$$

In particular, the barycenters of measures μ with compact support on $\mathbb{H}\mathrm{eis}_{4n-1}$ are equivariant under the Heisenberg translations, rotations and dilations: For every such transformation γ, we have

$$\mathrm{bar}(\gamma_*\mu) = \gamma\,\mathrm{bar}(\mu). \tag{3.3}$$

In order to prove Assertion (1), by Equations (3.2) and (3.3), and by the transitivity properties of the Heisenberg translations and dilations on chains, we may assume that $n = 2$ and that C is a Euclidean sphere with center $(0, 0)$ and radius 1 in the horizontal subspace $\{(\zeta, u) \in \mathbb{H}^{n-1} \times \mathbb{H} : u = 0\}$. Since the $\operatorname{Im}\mathbb{H}$-valued 1-form $\omega|_C$ is invariant under the Heisenberg rotations, the volume form $\omega_1 \wedge \omega_2 \wedge \omega_3$ on C is invariant under the Heisenberg rotations. Since the only measure on C invariant under the Heisenberg rotations is, up to a scalar multiple, the Lebesgue measure on the Euclidean sphere C, the measure μ_C is a multiple of the Lebesgue measure on C. This can also be proved by a direct computation: On the Euclidean sphere C, with $\zeta = \zeta_0 + \zeta_1 i + \zeta_2 j + \zeta_3 k$, we have

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = -8 \sum_{i=0}^{3} (-1)^i \zeta_i \,\mathrm{d}\zeta_0 \wedge \cdots \wedge \widehat{\mathrm{d}\zeta_i} \wedge \cdots \wedge \mathrm{d}\zeta_3.$$

Since the barycenter of this measure is exactly the origin $(0,0)$, which is the center of the finite chain C, this proves Assertion (1).

(2). First assume that C is the standard vertical chain

$$C_\infty = \{(\zeta,u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : \zeta = 0\} \cup \{\infty\}.$$

Let $\tau = \tau_{C_\infty} : v \mapsto (0,v)$. Then τ is a diffeomorphism from $\operatorname{Im}\mathbb{H}$ onto $C_\infty - \{\infty\}$, such that $\tau^*(\mathrm{d}u - 2\operatorname{Im}(\overline{\zeta}\mathrm{d}\zeta)) = \mathrm{d}v$. For every vertical Heisenberg translation γ, the map $\gamma \circ \tau$ is also a diffeomorphism from $\operatorname{Im}\mathbb{H}$ onto $C_\infty - \{\infty\}$, and since ω is invariant under the Heisenberg translations, we also have $(\gamma\circ\tau)^*\omega = \mathrm{d}v$.

If $\sigma : \operatorname{Im}\mathbb{H} \to C_\infty - \{\infty\}$ is another diffeomorphism such that $\sigma^*\omega = \mathrm{d}v$, then for every $v \in \operatorname{Im}\mathbb{H}$, we have $\sigma'(v) - \tau'(v) \in TC_\infty \cap \ker\omega = \{0\}$, thus the maps σ and τ differ by an element of the vector subspace C_∞. Therefore there exists a vertical Heisenberg translation γ such that $\sigma = \gamma\circ\tau$.

Now, if C is another vertical chain, there exists a composition γ of Heisenberg translations and rotations such that $C = \gamma C_\infty$. Defining $\tau_C = \gamma \circ \tau_{C_\infty}$ gives a diffeomorphism from $\operatorname{Im}\mathbb{H}$ onto $C - \{\infty\}$ such that ${\tau_C}^*\omega = \mathrm{d}v$, by the invariance of ω under the Heisenberg translations and rotations. This proves Assertion (2).

(3). First assume that C is the Euclidean 3-sphere

$$\left\{(\zeta,u) \in \mathbb{H}^{n-1} \times \operatorname{Im}\mathbb{H} : \mathfrak{n}(\zeta_1) = R^2 \text{ and } u = \zeta_2 = \cdots = \zeta_{n-1} = 0\right\},$$

and that $c = (\zeta_c = (-R,0,\dots,0), u_c = 0)$. Note that R is the radius of the finite chain C. By the properties of the exponential map of the Lie group of unit quaternions, whose tangent space at the identity element 1 is $\operatorname{Im}\mathbb{H}$, the smooth map

$$\tau = \tau_{C,c} : v \mapsto \big(\zeta = (Re^{-v/(2R^2)}, 0, \dots, 0), u = 0\big)$$

from $\operatorname{Im}\mathbb{H}$ to C is a diffeomorphism from $B(2\pi R^2)$ onto $C - \{c\}$. It extends continuously (and even smoothly) to the sphere $\partial B(2\pi R^2)$, mapping this sphere to c. Considering ζ as a function of v, we have $\mathrm{d}\zeta = (-\frac{1}{2R}e^{-v/(2R^2)}\mathrm{d}v, 0, \dots, 0)$. Hence, since v and $\mathrm{d}v$ are purely imaginary quaternions, we have

$$\tau^*\omega = -2\operatorname{Im}(\overline{\zeta}\cdot\mathrm{d}\zeta) = -2\operatorname{Im}\Big(\big(Re^{-\overline{v}/(2R^2)}\big)\Big(-\frac{1}{2R}e^{-v/(2R^2)}\mathrm{d}v\Big)\Big) = \mathrm{d}v.$$

The uniqueness of τ up to postcomposition by a Heisenberg rotation preserving C and c, and the extension to the other chains, follow as previously from the fact that the chains are transverse to the quaternionic contact structure on $\mathbb{H}\mathrm{eis}_{4n-1}$ and by invariance of the calibration ω under the Heisenberg translations and rotations. □

4. Counting and equidistribution of arithmetic chains in hyperspherical geometry

In this section, we prove (generalised versions of) Theorems 1.2 and 1.3 of the introduction. We start by recalling a general statement, coming from a special case of the main results of [21], that has been made explicit in [22].

Let Γ be a lattice in PU_q. Let D^- and D^+ be nonempty proper closed convex subsets of $\mathbf{H}^n_{\mathbb{H}}$, with stabilisers Γ_{D^-} and Γ_{D^+} in Γ respectively, such that the families $(\gamma D^-)_{\gamma\in\Gamma/\Gamma_{D^-}}$ and $(\gamma D^+)_{\gamma\in\Gamma/\Gamma_{D^+}}$ are locally finite in $\mathbf{H}^n_{\mathbb{H}}$. For all γ,γ' in Γ, the convex sets γD^- and $\gamma' D^+$ have a common perpendicular if and only if their closures $\overline{\gamma D^-}$ and $\overline{\gamma' D^+}$ in $\mathbf{H}^n_{\mathbb{H}} \cup \partial_\infty \mathbf{H}^n_{\mathbb{H}}$ do not intersect. We denote by $\alpha_{\gamma,\gamma'}$ this common perpendicular, starting from γD^- at time $t=0$, and by $\ell(\alpha_{\gamma,\gamma'})$ its length. The *multiplicity* of $\alpha_{\gamma,\gamma'}$ is

$$m_{\gamma,\gamma'} = \frac{1}{\operatorname{card}(\gamma\Gamma_{D^-}\gamma^{-1} \cap \gamma'\Gamma_{D^+}\gamma'^{-1})},$$

which equals 1 for all $\gamma,\gamma' \in \Gamma$ when Γ acts freely on $T^1\mathbf{H}^n_{\mathbb{H}}$ (for instance when Γ is torsion-free). For all $s>0$ and $x \in \partial D^-$, let

$$m_s(x) = \sum_{\gamma\in\Gamma/\Gamma_{D^+}\,:\,\overline{D^-}\cap\overline{\gamma D^+}=\emptyset,\ \alpha_{e,\gamma}(0)=x,\ \ell(\alpha_{e,\gamma})\le s} m_{e,\gamma}$$

be the multiplicity of x as the origin of common perpendiculars with length at most s from D^- to the elements of the Γ-orbit of D^+.

For every $s>0$, let

$$\mathcal{N}_{D^-,D^+}(s) = \sum_{(\gamma,\gamma')\in\Gamma\backslash((\Gamma/\Gamma_{D^-})\times(\Gamma/\Gamma_{D^+}))\,:\,\overline{\gamma D^-}\cap\overline{\gamma' D^+}=\emptyset,\ \ell(\alpha_{\gamma,\gamma'})\le s} m_{\gamma,\gamma'},$$

where Γ acts diagonally on $\Gamma\times\Gamma$. When Γ has no torsion, $\mathcal{N}_{D^-,D^+}(s)$ is the number (with multiplicities coming from the fact that $\Gamma_{D^\pm}\backslash D^\pm$ is not assumed to be embedded in $\Gamma\backslash\mathbf{H}^n_{\mathbb{H}}$) of the common perpendiculars of length at most s between the images of D^- and D^+ in $\Gamma\backslash\mathbf{H}^n_{\mathbb{H}}$.

The following statement is a special case of [22, Thm. 8·1]. We denote by Δ_x the unit Dirac mass at a point x.

Theorem 4.1. *Let D^- be a horoball in $\mathbf{H}^n_{\mathbb{H}}$ centred at a parabolic fixed point of Γ and let D^+ be a quaternionic geodesic line in $\mathbf{H}^n_{\mathbb{H}}$ such that $\Gamma_{D^+}\backslash D^+$ has finite volume. Let m^+ be the order of the pointwise stabiliser of D^+ in Γ and let*

$$c(D^-,D^+) = \frac{2(n-1)(2n-1)}{\pi^2 m^+}\,\frac{\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)\operatorname{Vol}(\Gamma_{D^+}\backslash D^+)}{\operatorname{Vol}(\Gamma\backslash\mathbf{H}^n_{\mathbb{H}})}.$$

There exists $\kappa > 0$ such that, as $s \to +\infty$,

$$\mathscr{N}_{D^-,D^+}(s) = c(D^-, D^+)e^{(4n+2)s}\big(1 + \mathrm{O}(e^{-\kappa s})\big).$$

Furthermore, the origins of the common perpendiculars from D^- to the images of D^+ under the elements of Γ equidistribute in ∂D^- to the induced Riemannian measure: As $s \to +\infty$, we have

$$\frac{2(2n+1)\,\mathrm{Vol}(\Gamma_{D^-}\backslash D^-)}{c(D^-, D^+)} e^{-(4n+2)s} \sum_{x \in \partial D^-} m_s(x)\,\Delta_x \overset{*}{\rightharpoonup} \mathrm{vol}_{\partial D^-}\,. \tag{4.1}$$

For smooth functions ψ with compact support on ∂D^-, there is an error term in the equidistribution claim of Theorem 4.1 when the measures on both sides are evaluated on ψ, of the form $\mathrm{O}(e^{-\kappa s}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

From now on, we assume that $n = 2$. Let A, D_A, m_A and $\mathscr{O}$ be as in the Introduction. We denote by $|\mathscr{O}^\times|$ the order of the unit group of $\mathscr{O}$, equal to 24 if $D_A = 2$, to 12 if $D_A = 3$, or else to 2, 4 or 6. See for instance [25]. As usual, by $\prod_{p|D_A}$, we mean a product where p ranges over the prime positive numbers dividing D_A.

For every chain C in $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$, let L_C be the quaternionic projective line in $\mathbb{P}^2_{\mathrm{r}}(\mathbb{H})$ such that $C = L_C \cap \partial_\infty \mathbf{H}^2_{\mathbb{H}}$, and let $D_C = L_C \cap \mathbf{H}^2_{\mathbb{H}}$ be the associated quaternionic geodesic line. For every finite index subgroup G of the arithmetic lattice $\mathrm{PU}_q(\mathscr{O})$, we denote by G_C the stabiliser of C in G, by G_∞ the stabiliser of ∞ in G, and by $\mathrm{Covol}_G(C)$ the volume of the orbifold $G_C\backslash D_C$ for the Riemannian metric of constant sectional curvature -1 on the real hyperbolic 4-space D_C. Recall that a chain C is arithmetic over $\mathscr{O}$ if and only if the stabiliser in $\mathrm{PU}_q(\mathscr{O})$ (or equivalently in G) of the quaternionic geodesic line D_C has finite covolume on D_C.

Theorem 4.2. *Let C_0 be an arithmetic chain over a maximal order $\mathscr{O}$ in a definite quaternion algebra over $\mathbb{Q}$. Let G be a finite index subgroup of $\mathrm{PU}_q(\mathscr{O})$. Then there exists a constant $\kappa > 0$ such that, as $\epsilon > 0$ tends to 0, the number $\psi_{C_0,G}(\epsilon)$ of chains modulo G_∞ in the G-orbit of C_0 with d_{Cyg}-diameter at least ϵ is equal to*

$$\frac{35\; 2^{23}\; 3^6 D_A^2\, \mathrm{Covol}_G(C_0)\,[\mathrm{PU}_q(\mathscr{O})_\infty : G_\infty]}{\pi^6 m_{C_0,G}\, m_A |\mathscr{O}^\times|^2 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)\,[\mathrm{PU}_q(\mathscr{O}) : G]}\,\epsilon^{-10}\big(1 + \mathrm{O}(\epsilon^\kappa)\big),$$

where $m_{C_0,G}$ is the order of the pointwise stabiliser of D_{C_0} in G.

Recall that the center $\mathrm{cen}(C)$ of a finite chain C is the image of $\infty = [1 : 0 : 0]$ under the reflexion on L_C. The following result is an equidistribution result in the quaternionic Heisenberg group of the centers of the arithmetic chains in a given orbit under (a finite index subgroup of) $\mathrm{PU}_q(\mathscr{O})$.

Theorem 4.3. *Let C_0, G and $m_{C_0,G}$ be as in Theorem 4.2. As $\epsilon > 0$ tends to 0, we have*

$$\frac{m_{C_0,G}\, m_A \pi^6 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)[\mathrm{PU}_q(\mathscr{O}) : G]}{35\ 2^{24}\ 3^6\ \mathrm{Covol}_G(C_0)}\,\epsilon^{10} \sum_{\substack{C \in G\cdot C_0\\ \mathrm{diam}_{d_{\mathrm{Cyg}}}(C) \geq \epsilon}} \Delta_{\mathrm{cen}(C)} \overset{*}{\rightharpoonup} \mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7}\,.$$

As in Theorem 4.1, there exist $\kappa > 0$ and $\ell \in \mathbb{N}$ such that for every smooth function ψ with compact support on $\mathbb{H}\mathrm{eis}_7$, there is an error term in this equidistribution result when the measures on both sides are evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\|\psi\|_\ell$ is the Sobolev norm of ψ.

We begin by a technical result used in the proofs of the above theorems, which does not require the assumption $n = 2$. Recall that d''_{Cyg} is the modified Cygan distance defined in Section 2.

Lemma 4.4. *For every m-chain C in $\mathbf{H}^n_{\mathbb{H}}$, we have* $\mathrm{diam}_{d_{\mathrm{Cyg}}}(C) = \sqrt{2}\,\mathrm{diam}_{d''_{\mathrm{Cyg}}}(C)$.

Proof. If C is a vertical m-chain, then both diameters are $+\infty$. We hence assume that C is finite. Since the Heisenberg translations and rotations preserve d_{Cyg} and d''_{Cyg}, and by the transitivity properties of the Heisenberg translations and rotations on the set of m-chains (see Section 3.2), we may assume that C is a Euclidean sphere centered at $(0,0)$ with dimension $4m-1$, contained in the horizontal plane $\mathbb{H}^{n-1}\times\{0\}$ of $\mathbb{H}\mathrm{eis}_{4n-1}$. Since the Heisenberg dilations $(\zeta,u)\mapsto(\lambda\zeta,\lambda^2 u)$ with $\lambda > 0$ are homotheties of ratio λ for d_{Cyg} and d''_{Cyg}, we may assume that the radius of C is equal to 1.

For every $(\zeta,0)\in C$, we thus have $d_{\mathrm{Cyg}}((\zeta,0),(0,0)) = 1$ by Equation (2.7), hence $\mathrm{diam}_{d_{\mathrm{Cyg}}}(C) \leq 2$ by the triangle inequality. Since

$$d_{\mathrm{Cyg}}((\zeta,0),(-\zeta,0)) = d_{\mathrm{Cyg}}((\zeta,0)\cdot(\zeta,0),(0,0)) = d_{\mathrm{Cyg}}((2\zeta,0),(0,0)) = 2,$$

we have $\mathrm{diam}_{d_{\mathrm{Cyg}}}(C) = 2$.

Using the transitivity properties of $\mathrm{Sp}(n-1)$ on the unit sphere C of the Euclidean space $\mathbb{H}^{n-1}$ in the same way as in the proof of [20, Lem. 8] in the complex hyperbolic case, we may assume that $n = 3$, and that

$$\mathrm{diam}_{d''_{\mathrm{Cyg}}}(C) = \sup_{u\in\mathbb{H},\,\phi\in[0,\pi]\,:\,\mathrm{n}(u)=1} d''_{\mathrm{Cyg}}\big((1,0,0),(u\cos\phi,\sin\phi,0)\big).$$

By a computation similar to the one in [20, Lem. 8], using Equation (2.8) and the fact that $4\,\mathtt{n}(\operatorname{Im} u) = 4 - (\mathtt{tr}\, u)^2$ for any unit quaternion u, we have

$$\begin{aligned}
&d''_{\mathrm{Cyg}}\big((1,0,0),(u\cos\phi,\sin\phi,0)\big)^2\\
&\qquad = d''_{\mathrm{Cyg}}\big((0,0,0),(-1,0,0)\cdot(u\cos\phi,\sin\phi,0)\big)^2\\
&\qquad = d''_{\mathrm{Cyg}}\big((0,0,0),\big(u\cos\phi-1,\sin\phi,-2\cos\phi\operatorname{Im}u\big)\big)^2\\
&\qquad = \frac{(2-\cos\phi\,\mathtt{tr}\,u)^2+4\cos^2\phi\;\mathtt{n}(\operatorname{Im}u)}{\big((2-\cos\phi\,\mathtt{tr}\,u)^2+4\cos^2\phi\;\mathtt{n}(\operatorname{Im}u)\big)^{\frac12}+(2-\cos\phi\,\mathtt{tr}\,u)}\\
&\qquad = \frac{2}{\frac{1}{(1+\cos^2\phi-\cos\phi\,\mathtt{tr}\,u)^{\frac12}}+\frac{2-\mathtt{tr}\,u\cos\phi}{2(1+\cos^2\phi-\cos\phi\,\mathtt{tr}\,u)}}.
\end{aligned}$$

As $1+\cos^2\phi-\cos\phi\,\mathtt{tr}\,u \leq 2-\cos\phi\,\mathtt{tr}\,u \leq 4$, we have

$$d''_{\mathrm{Cyg}}\big((1,0,0),(u\cos\phi,\sin\phi,0)\big)^2 \leq 2.$$

Furthermore, the equality holds when $u = 1$ and $\phi = \pi$. This proves the result. $\square$

Proof of Theorem 4.2 and Theorem 4.3. The diameter of a chain for the Cygan distance is invariant under the stabiliser in PU_q of the horosphere $\partial\mathscr{H}_1$, hence is invariant under G_∞. The counting function $\psi_{C_0,G}$ is thus well defined.

Note that $\mathscr{H}_1$ is a horoball centered at the fixed point of a parabolic element in $\mathrm{PU}_q(\mathscr{O})$ (take the vertical Heisenberg translation by $(0,2u)$ for any nonzero $u\in\mathscr{O}\cap\operatorname{Im}\mathbb{H}$). We will apply Theorem 4.1 with $\Gamma = G$, with $D^- = \mathscr{H}_1$, which is hence a horoball centered at the fixed point of a parabolic element in G, and with $D^+ = D_{C_0}$, which is the quaternionic geodesic line in $\mathbf{H}^2_{\mathbb{H}}$ with boundary at infinity equal to C_0. In particular $m^+ = m_{C_0,G}$.

Let us compute the constant $c(D^-,D^+)$ appearing in the statement of Theorem 4.1. We have $\operatorname{Vol}(G\backslash\mathbf{H}^2_{\mathbb{H}}) = [\mathrm{PU}_q(\mathscr{O}):G]\operatorname{Vol}(\mathrm{PU}_q(\mathscr{O})\backslash\mathbf{H}^2_{\mathbb{H}})$, where, by [22, Thm. 1·4],

$$\operatorname{Vol}(\mathrm{PU}_q(\mathscr{O})\backslash\mathbb{H}^2_{\mathbb{H}}) = \frac{\pi^4 m_A}{175\;2^{13}\;3^5}\prod_{p|D_A}(p-1)(p^2+1)(p^3-1),$$

and by [22, Lem. 8·4],

$$\operatorname{Vol}(\Gamma_{D^-}\backslash D^-) = [\mathrm{PU}_q(\mathscr{O})_\infty:G_\infty]\operatorname{Vol}(\mathrm{PU}_q(\mathscr{O})_{\mathscr{H}_1}\backslash\mathscr{H}_1) = \frac{D_A^2\,[\mathrm{PU}_q(\mathscr{O})_\infty:G_\infty]}{160\,|\mathscr{O}^\times|^2}. \quad (4.2)$$

By definition, we have

$$\operatorname{Vol}(\Gamma_{D^+}\backslash D^+) = 16\operatorname{Covol}_G(C_0),$$

since the sectional curvature of D^+ is constant -4 and D^+ has real dimension 4. We hence have

$$c(D^-, D^+) = \frac{35\ 2^{13}\ 3^6 D_A^2 \operatorname{Covol}_G(C_0)[\mathrm{PU}_q(\mathscr{O})_\infty : G_\infty]}{\pi^6 m_{C_0,G} m_A |\mathscr{O}^\times|^2 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)[\mathrm{PU}_q(\mathscr{O}) : G]}. \quad (4.3)$$

Let $g \in G$ be such that the quaternionic geodesic line gD^+ is disjoint from $\mathscr{H}_1$ (which is the case except for g in finitely many double classes in $G_{\mathscr{H}_1}\backslash G/G_{D^+}$). Let δ_g be the common perpendicular from $\mathscr{H}_1$ to gD^+. Its length $\ell(\delta_g)$ is the minimum of the distances from $\mathscr{H}_1$ to a geodesic line between two points of $\partial_\infty(gD^+) = gC_0$. Hence, by Lemmas 2.1 and 4.4, we have

$$\begin{aligned}\ell(\delta_g) &= \min_{x,y\in gC_0,\, x\neq y} d(\mathscr{H}_1,]x,y[) = -\max_{x,y\in gC_0,\, x\neq y} \ln \frac{d''_{\mathrm{Cyg}}(x,y)}{\sqrt{2}}\\ &= -\ln \frac{\operatorname{diam}_{d''_{\mathrm{Cyg}}}(gC_0)}{\sqrt{2}} = -\ln \frac{\operatorname{diam}_{d_{\mathrm{Cyg}}}(gC_0)}{2}.\end{aligned} \quad (4.4)$$

Respectively by the definition of the counting function $\psi_{C_0,G}$ in the statement of Theorem 4.2, since the stabiliser of C_0 in G is equal to $G_{D_{C_0}} = G_{D^+}$, by Equation (4.4), by Theorem 4.1, and by Equation (4.3), we have, as $\epsilon > 0$ tends to 0,

$$\begin{aligned}&\psi_{C_0,G}(\epsilon)\\ &= \operatorname{card} G_\infty\backslash\{C \in G\cdot C_0 : \operatorname{diam}_{d_{\mathrm{Cyg}}}(C) \geq \epsilon\}\\ &= \operatorname{card}\{[g] \in G_\infty\backslash G/G_{D_{C_0}} : \operatorname{diam}_{d_{\mathrm{Cyg}}}(gC_0) \geq \epsilon\}\\ &= \operatorname{card}\Big\{[g] \in G_{\mathscr{H}_1}\backslash G/G_{D_{C_0}} : \ell(\delta_g) \leq -\ln\frac{\epsilon}{2}\Big\} + \mathrm{O}(1)\\ &= \mathscr{N}_{D^-,D^+}\Big(-\ln\frac{\epsilon}{2}\Big) + \mathrm{O}(1) = c(D^-,D^+)e^{-10\ln\frac{\epsilon}{2}}\big(1+\mathrm{O}(e^{\kappa\ln\frac{\epsilon}{2}})\big)\\ &= \frac{35\ 2^{23}\ 3^6 D_A^2 \operatorname{Covol}_G(C_0)[\mathrm{PU}_q(\mathscr{O})_\infty : G_\infty]}{\pi^6 m_{C_0,G} m_A |\mathscr{O}^\times|^2 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)[\mathrm{PU}_q(\mathscr{O}) : G]}\epsilon^{-10}\big(1+\mathrm{O}(\epsilon^\kappa)\big).\end{aligned}$$

This proves Theorem 4.2. Let us now prove Theorem 4.3.

We apply the equidistribution result in Equation (4.1) of the origins $\mathrm{or}(\delta_g)$ of the common perpendiculars δ_g from $D^- = \mathscr{H}_1$ to the images gD^+ for $g \in G$. As $s \to +\infty$, we hence have, using Equations (4.3) and (4.2),

$$\frac{m_{C_0,G} m_A \pi^6 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)[\mathrm{PU}_q(\mathscr{O}) : G]}{35\ 2^{17}\ 3^6 \operatorname{Covol}_G(C_0)} e^{-10s} \sum_{[g]\in G/G_{D^+} :\, \ell(\delta_g)\leq s} \Delta_{\mathrm{or}(\delta_g)} \overset{*}{\rightharpoonup} \mathrm{vol}_{\partial\mathscr{H}_1}. \quad (4.5)$$

Let $f : \partial_\infty \mathbf{H}^2_{\mathbb{H}} - \{\infty\} = \mathbb{H}\mathrm{eis}_7 \to \partial\mathscr{H}_1$ be the orthogonal projection map, which is the homeomorphism $(w_0, w) \mapsto (w_0 + \frac{1}{2}, w)$. The pushforward of the Haar measure $\mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7}$ by f is

$$f_* \,\mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7} = 8\,\mathrm{vol}_{\partial\mathscr{H}_1}, \tag{4.6}$$

see for example the end of the proof of Theorem 8·3 in [22].

Note that, for every chain C, if r_C is the reflexion on the quaternionic projective line containing C, then the geodesic line from ∞ to $\mathrm{cen}(C) = r_C(\infty)$, being invariant under r_C, is orthogonal to the quaternionic geodesic line with boundary at infinity C. Hence for every $g \in G$, we have

$$f^{-1}(\mathrm{or}(\delta_g)) = \mathrm{cen}(gC_0).$$

Let us use in Equation (4.5) the change of variables $s = -\ln \frac{\epsilon}{2}$ and the continuity of the pushforward of measures by f^{-1}. By Equations (4.4) and (4.6), as $\epsilon > 0$ tends to 0, we obtain that the measures

$$\frac{m_{C_0,G}\, m_A \pi^6 \prod_{p|D_A}(p-1)(p^2+1)(p^3-1)[\mathrm{PU}_q(\mathscr{O}) : G]}{35\; 2^{24}\; 3^6\, \mathrm{Covol}_G(C_0)}\,\epsilon^{10} \sum_{\substack{[g]\in G/G_{D^+} \\ \mathrm{diam}_{d_{\mathrm{Cyg}}}(gC_0)\geq \epsilon}} \Delta_{\mathrm{cen}(gC_0)}$$

weak-star converge to the Haar measure $\mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7}$. This proves Theorem 4.3. □

Example 4.5. Let $C_0 = \big\{[w_0 : 0 : 1] \in \mathbb{P}^2_{\mathrm{r}}(\mathbb{H}) : \mathtt{tr}\, w_0 = 0\big\}$ be the standard vertical chain in $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$, which is the intersection of $\partial_\infty \mathbf{H}^2_{\mathbb{H}}$ with the quaternionic projective line $L_{C_0} = \big\{[z_0 : z_1 : z_2] \in \mathbb{P}^2_{\mathrm{r}}(\mathbb{H}) : z_1 = 0\big\}$.

An element

$$\pm\begin{pmatrix} a & \gamma^* & b \\ \alpha & M & \beta \\ c & \delta^* & d \end{pmatrix}$$

of PU_q preserving the quaternionic geodesic line $L_{C_0} \cap \mathbf{H}^2_{\mathbb{H}}$ satisfies $\alpha w_0 + \beta = 0$ for all $w_0 \in \mathbb{H}$ with $\mathtt{tr}\, w_0 > 0$. Thus, $\alpha = \beta = 0$, and Equations (2.5) (or rather the similar equations obtained by the formula $XX^* = I_{n+1}$ instead of $X^*X = I_{n+1}$) imply that $\gamma = \delta = 0$. Using again Equations (2.5), we see that the stabiliser of L_{C_0} consists of the elements

$$\begin{pmatrix} a & 0 & b \\ 0 & M & 0 \\ c & 0 & d \end{pmatrix}$$

such that $\mathtt{tr}(\bar{c}a) = \mathtt{tr}(\bar{d}b) = 0$, $\bar{c}b + \bar{a}d = 1$ and $M \in \mathscr{O}^\times$. Thus,

$$\mathrm{Covol}_{\mathrm{PU}_q(\mathscr{O})}(C_0) = \frac{\pi^2}{1080} \prod_{p|D_A} (p-1)(p^2+1)$$

by [5, Thm. 2.5].

The pointwise stabiliser of C_0 in $\mathrm{PU}_q(\mathscr{O})$ consists of the diagonal elements with $a = d = \pm 1$ and $M \in \mathscr{O}^\times$, giving $m_{C_0,\mathrm{PU}_q(\mathscr{O})} = |\mathscr{O}^\times|$.

Theorems 4.2 and 4.3 then give

$$\psi_{C_0,\mathrm{PU}_q(\mathscr{O})}(\epsilon) = \frac{189\ 2^{20} D_A^2}{\pi^4 m_A |\mathscr{O}^\times|^3 \prod_{p|D_A}(p^3-1)} \epsilon^{-10}\big(1 + \mathrm{O}(\epsilon^\kappa)\big),$$

and

$$\frac{\pi^4 m_A |\mathscr{O}^\times| \prod_{p|D_A}(p^3-1)}{189\ 2^{21}} \epsilon^{10} \sum_{C \in \mathrm{PU}_q(\mathscr{O})\cdot C_0 \colon \mathrm{diam}_{d_{\mathrm{Cyg}}} C \geq \epsilon} \Delta_{\mathrm{cen}(C)} \overset{*}{\rightharpoonup} \mathrm{Haar}_{\mathbb{H}\mathrm{eis}_7}\,.$$

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Jouni Parkkonen
Department of Mathematics and Statistics
P.O. Box 35
40014 University of Jyväskylä
FINLAND
jouni.t.parkkonen@jyu.fi

Frédéric Paulin
Laboratoire de mathématique d'Orsay
UMR 8628 CNRS
Université Paris-Saclay
91405 ORSAY Cedex
FRANCE
frederic.paulin@universite-paris-saclay.fr