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Title: Free boundary methods and non-scattering phenomena

Year: 2021

Version: Published version

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Please cite the original version:

Salo, M., & Shahgholian, H. (2021). Free boundary methods and non-scattering phenomena. *Research in the Mathematical Sciences*, 8(4), Article 58. <https://doi.org/10.1007/s40687-021-00294-z>

RESEARCH

Free boundary methods and non-scattering phenomena



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Abstract

We study a question arising in inverse scattering theory: given a penetrable obstacle, does there exist an incident wave that does not scatter? We show that every penetrable obstacle with real-analytic boundary admits such an incident wave. At zero frequency, we use quadrature domains to show that there are also obstacles with inward cusps having this property. In the converse direction, under a nonvanishing condition for the incident wave, we show that there is a dichotomy for boundary points of any penetrable obstacle having this property: either the boundary is regular, or the complement of the obstacle has to be very thin near the point. These facts are proved by invoking results from the theory of free boundary problems.

1 Introduction

1.1 Motivation

In this article, we discuss some examples of non-scattering phenomena based on methods from free boundary problems. The connection between these fields was recently pointed out in [18], and we invoke further ideas from free boundary problems to obtain stronger results. The methods are relevant both for inverse scattering problems and inverse boundary value problems. We first describe the boundary case and state the main results in that setting, and discuss the scattering case afterward. All functions will be assumed real valued unless mentioned otherwise.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $q \in L^\infty(\Omega)$ be a potential in Ω . Assuming that 0 is not a Dirichlet eigenvalue for $\Delta + q$ in Ω , for any $f \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$(\Delta + q)u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f.$$

We assume that we can fix a Dirichlet data f and measure the corresponding Neumann data $\partial_\nu u|_{\partial\Omega}$ (interpreted weakly as an element of $H^{-1/2}(\partial\Omega)$) on the boundary. This kind of situation arises in diffuse optical tomography [5]. It is also relevant in electrical impedance tomography, i.e., Calderón problem [51], where the underlying conductivity equation $\operatorname{div}(\gamma \nabla v) = 0$ can often be reduced to the equation $(\Delta + q)u = 0$ with $q = -\gamma^{-1/2} \Delta(\gamma^{1/2})$ by using the Liouville transformation $v = \gamma^{-1/2} u$.

In inverse boundary value problems of this type, one often assumes the knowledge of the full Dirichlet-to-Neumann map

$$\Lambda_q : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega), \quad \Lambda_q f = \partial_\nu u|_{\partial\Omega}.$$

This corresponds to an idealized case where we can perform infinitely many measurements. However, in practice, only finitely many measurements are possible. Moreover, the idealized problem is formally overdetermined when $n \geq 3$, in the sense that the unknown q is a function of n variables, whereas the measurements (Schwartz kernel of Λ_q) depend on $2n - 2$ variables. This suggests that fewer measurements might be sufficient. We are interested in the *single measurement inverse problem*: which properties of q can be determined from the measurement $\partial_\nu u|_{\partial\Omega}$ corresponding to a fixed Dirichlet data $f \in H^{1/2}(\partial\Omega)$?

In general, it is not possible to determine a potential q from a single measurement. This is indicated by a heuristic dimension count argument: the measurement $\partial_\nu u|_{\partial\Omega}$ is a function of $n - 1$ variables, whereas the unknown potential is a function of n variables. Thus the inverse problem of determining q from a single measurement is formally underdetermined. A related problem is to determine the shape of a *penetrable obstacle* from a single measurement. This corresponds to potentials of the form

$$q = h\chi_D, \quad \bar{D} \subset \Omega,$$

where χ_D is the characteristic function of D (a bounded open set, i.e., the obstacle), and h satisfies a nonvanishing condition at ∂D . We will sometimes assume the following conditions for D and h .

Definition A bounded open set $D \subset \mathbb{R}^n$ is called a *solid domain* if D and $\mathbb{R}^n \setminus \bar{D}$ are connected, and $\text{int}(\bar{D}) = D$. We say that h is a *contrast* for D if

$$h \in L^\infty(\mathbb{R}^n), \quad \text{and } |h| \geq c > 0 \quad \text{a.e. in some neighborhood of } \partial D.$$

Since the potential is $q = h\chi_D$, the values of the contrast outside D will not play any role. The inverse problem is to determine the shape of the obstacle, i.e., ∂D , or some properties of ∂D from a single measurement $\partial_\nu u|_{\partial\Omega}$. If ∂D is (say) a Lipschitz domain, then it is locally the graph of a function of $n - 1$ variables and thus the inverse problem is formally well-determined.

There are various results in the literature for determining ∂D from a single measurement. For a related Calderón-type problem corresponding to the equation $\text{div}(\gamma \nabla u) = 0$ where $\gamma = 1 + h\chi_D$ and h is a nonzero constant, there are partial results when D has special geometry, such as D being a convex polygon or polyhedron, a ball, or a cylinder. There are also local uniqueness results (if D and D' are close enough in some sense then they can be distinguished by a single measurement) and estimates for the size of D . See [1] for a survey of classical results, and [37] and references therein for more recent results. The results in [2,6] are of particular interest to us: they invoke methods from free boundary problems to show that if D is, e.g., a Lipschitz domain and it is not determined by a single measurement, then part of ∂D is necessarily real-analytic. We refer to [24,30,31] for recent related results.

It turns out that such results are closely connected to a certain non-scattering phenomenon in inverse scattering theory. These problems involve a fixed frequency $\lambda \geq 0$.

Given a bounded open set $\Omega \subset \mathbb{R}^n$, we ask if there exist nontrivial solutions of

$$(\Delta + \lambda^2 + q)u_q = 0 \quad \text{in } \Omega, \quad (\Delta + \lambda^2)u_0 = 0 \quad \text{in } \Omega \tag{1.1}$$

that satisfy

$$u_q, u_0 \in H^1(\Omega), \quad u_q - u_0 \in H_0^2(\Omega). \tag{1.2}$$

This problem is rather similar to the interior transmission problem (see [16, 17]) but note that we require u_q, u_0 to be H^1 instead of L^2 . The above problem can in fact be considered as a matching problem as in [25]. One could also accommodate the condition $u_q - u_0 - g \in H_0^2(\Omega)$ for some smooth enough g , which would be close to the inhomogeneous interior transmission problem.

If the boundary of Ω is smooth enough, the condition (1.2) can be written as

$$u_q|_{\partial\Omega} = u_0|_{\partial\Omega}, \quad \partial_\nu u_q|_{\partial\Omega} = \partial_\nu u_0|_{\partial\Omega}.$$

Thus, if one fixes the Dirichlet data $f = u_0|_{\partial\Omega} = u_q|_{\partial\Omega}$, this would mean that the measurement $\partial_\nu u_q|_{\partial\Omega}$ corresponding to q is identical to the measurement $\partial_\nu u_0|_{\partial\Omega}$ for the zero potential. Thus the potential q is invisible for this particular measurement and looks like empty space. In the terminology of scattering theory, if this happens we say that “the incident wave u_0 does not scatter.”

We specialize the above question to the case of an obstacle D . The following is the main question studied in this article:

Given a bounded open set $D \subset \mathbb{R}^n$ with $\bar{D} \subset \Omega$, is there a contrast h for D such that there exist nontrivial solutions u_q and u_0 with $q = h\chi_D$ satisfying (1.1)–(1.2)?

If the answer is positive, then there is some contrast h for D that admits an incident wave that does not scatter (thus D will be invisible with respect to this measurement). On the other hand, if the answer is negative, then the obstacle D scatters every incident wave nontrivially.

1.2 Main results

There are various results stating that if ∂D is piecewise smooth and has a corner singularity, then every incident wave will scatter nontrivially. We will give precise references in Sect. 1.4. On the other hand, there seem to be few examples in the literature of penetrable obstacles admitting incident waves that do not scatter. Balls have this property [19, Sections 10.3 and 8.4], and [26] gives examples of potentials $q \in C_c^\infty(\mathbb{R}^n)$ having this property whose supports are unions of balls. See [53] for some related results.

Our first result states that any obstacle with real-analytic boundary admits incident waves that do not scatter:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\lambda \geq 0$, and let $D \subset \mathbb{R}^n$ be a bounded open set with real-analytic boundary such that $\bar{D} \subset \Omega$ and $\mathbb{R}^n \setminus \bar{D}$ is connected. Suppose that λ is not a Dirichlet eigenvalue for $-\Delta$ in D . Then, there is a contrast for D that admits an incident wave that does not scatter.*

While corner singularities typically scatter every incident wave, we show that at least for $\lambda = 0$ there also exist obstacles with inward cusp singularities admitting incident

waves that do not scatter. We say that a connected open set $D \subset \mathbb{R}^n$ is a *quadrature domain* (for harmonic functions) if there is a compactly supported distribution μ in \mathbb{R}^n with $\text{supp}(\mu) \subset D$ such that

$$\int_D H dx = \int_D H d\mu \quad (1.3)$$

whenever $H \in L^1(D)$ is harmonic in D . A basic example is a ball $B(a, r) \subset \mathbb{R}^n$ with $\mu = |B_r| \delta_a$, so that (1.3) holds by the mean value theorem. There exist many examples of quadrature domains, and their boundaries can exhibit inward cusps (see [21, Chapter 14] or [46] for examples). One example is the cardioid domain $D = \{w + \frac{1}{2}w^2 : w \in \mathbb{D}\} \subset \mathbb{R}^2$ that has an inward cusp, see [44, Figure 0.1] (though note that ∂D is the image of S^1 by an analytic map).

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\lambda = 0$, and let $D \subset \mathbb{R}^n$ be a quadrature domain such that $\bar{D} \subset \Omega$. Then there is a contrast for D that admits an incident wave that does not scatter.*

As non-scattering incident waves in Theorems 1.1 and 1.2, one can choose any solution of

$$(\Delta + \lambda^2)u_0 = 0 \quad \text{in } \Omega \quad (1.4)$$

such that u_0 is positive on ∂D . A nonvanishing condition for u_0 on ∂D will be important for many results in this article (with the exception of Theorem 1.6), and it is of interest to determine if such solutions u_0 exist. They always do when $\lambda = 0$ (take $u_0 \equiv 1$) or when u_0 is allowed to be complex valued (take $u_0 = e^{i\lambda x_1}$). However, by Lemma 3.1 any real-valued solution of (1.4) has a zero in any ball of radius $\geq c_n/\lambda$ and the nonvanishing condition on ∂D is nontrivial in this case. In fact, if λ is a Dirichlet eigenvalue of $-\Delta$ in D then solutions u_0 satisfying the nonvanishing condition may not exist (see Remark 3.2). On the positive side, we will show the following result.

Theorem 1.3 *Let D be a bounded C^1 domain, or Lipschitz domain when $n = 2, 3$, with $\mathbb{R}^n \setminus \bar{D}$ connected. Suppose that $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta$ in D . Then, there is a real-valued solution u_0 of $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n such that u_0 is positive on ∂D .*

Theorems 1.1 and 1.2 are not difficult to prove, and they are analogous to certain facts in the theory of free boundary problems. As mentioned above, the connection between single measurement inverse problems and free boundary methods is classical in the Calderón problem [2, 6]. Curiously, it seems that for non-scattering phenomena, this connection was only pointed out very recently in [18]. The main point is the following: if an obstacle D admits an incident wave that does not scatter, then ∂D can be understood as a free boundary in a certain obstacle-type problem. This observation was used in [18] to show that if D has Lipschitz boundary and the incident wave u_0 is nonvanishing on ∂D , then necessarily ∂D must be real-analytic (resp. $C^{k+1, \alpha}$) if the contrast is real-analytic (resp. $C^{k, \alpha}$).

We prove a corresponding result where the a priori assumption that D has Lipschitz boundary is removed. However, as indicated by Theorem 1.2, one must then allow for the

possibility that D has inward cusps. We first need to introduce the concept of minimal diameter.

For any set K , we define $\text{MD}(K)$ to be the minimal diameter of K , i.e., the infimum of distances between pairs of parallel planes such that K is contained in the strip determined by the planes. For any ball $B(z, r)$, we also define the thickness function

$$\delta_r(K, z) := \frac{\text{MD}(K \cap B(z, r))}{r}.$$

To illustrate this notion, note that if $D \subset \mathbb{R}^n$ is a bounded Lipschitz domain, then there are $c, r_0 > 0$ such that

$$\delta_r(\mathbb{R}^n \setminus D, x^0) \geq c \quad \text{whenever } x^0 \in \partial D \text{ and } 0 < r < r_0.$$

On the other hand, if $D \cap B(0, 1) = \{x_n < |x'|^{1/\gamma}\} \cap B(0, 1)$ where $\gamma > 1$, so that D has an inward cusp at 0, one can check that

$$\delta_r(\mathbb{R}^n \setminus D, 0) \leq Cr^{\gamma-1}, \quad 0 < r < 1.$$

We first state the following result showing that if D admits an incident wave u_0 that does not scatter, and if both the contrast h and u_0 are nonvanishing at a point $x^0 \in \partial D$, then there are two possibilities: either D is regular near x^0 , or the complement of D is thin near x^0 .

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and suppose that $u_q, u_0 \in H^1(\Omega)$ satisfy (1.1)–(1.2) where $q = h\chi_D$ for some solid domain D with $\overline{D} \subset \Omega$. Assume that h is Dini continuous (i.e., h has a modulus of continuity ω with $\int_0^1 \omega(r) d(\log r) < \infty$). For any $x^0 \in \partial D$ such that*

$$h(x^0)u_0(x^0) \neq 0,$$

one of the following conditions holds:

- (a) $\limsup_{r \rightarrow 0} \delta_r(\Omega \setminus D, x^0) > 0$, and D is locally a C^1 domain near x^0 ; or
- (b) $\Omega \setminus D$ is thin near x^0 in the sense that $\lim_{r \rightarrow 0} \delta_r(\Omega \setminus D, x^0) = 0$.

If h is additionally assumed to be Lipschitz (resp. $C^{k,\alpha}$ where $k \geq 1$ and $0 < \alpha < 1$, or real-analytic) near x^0 and if (a) holds, then D is locally a $C^{1,\alpha}$ (resp. $C^{k+1,\alpha}$, or real-analytic) domain near x^0 .

The case where h is Dini continuous is a consequence of the following more precise result from [4, Theorem 1.3]:

Theorem 1.5 *Retain the hypotheses of Theorem 1.4, with h being a Dini continuous function. Suppose for some $r_1 > 0$, $x^0 \in \partial D$, we have*

$$h(x)u_0(x) \neq 0 \quad \text{for } x \in B(x^0, r_1). \tag{1.5}$$

Then, there exists a modulus of continuity $\sigma(r)$, and a universal constant $\tau > 0$, such that if for some $0 < r_0 < r_1$, we have $\sigma(r_0) < \delta_{r_0}(\{u_0 = u_q\}, x^0)$ then $\partial D \cap B(x^0, \tau r_0)$ is a C^1 -graph.

If h is Lipschitz, or if h is $C^{1,1}$ and u_0 vanishes on ∂D but ∇u_0 is nonvanishing, we also have the following result.

Theorem 1.6 *Retain the hypotheses of Theorem 1.4, and suppose that either of the following conditions is satisfied:*

1. $h \in C^{0,1}(B(x^0, r))$,
2. $h \in C^{1,1}(B(x^0, r))$ and condition (1.5) is replaced by

$$|h| > 0 \quad \text{and} \quad u_0 = 0 \quad \text{and} \quad |\nabla u_0| > 0 \quad \text{on} \quad \partial D \cap B(x^0, r). \quad (1.6)$$

Then, there exists $r_0 > 0$ such that one of the following holds:

- (a) ∂D is a $C^{1,\alpha}$ graph in $B(x^0, r_0)$.
- (b) In a translated and rotated system of coordinates

$$\partial D \cap B(x^0, r_0) \subset \{x : |x_1| < k(x')\},$$

where $x' = (x_2, \dots, x_n)$, and $k(x') \geq 0$ is a C^1 -function with $k(0') = 0$.

It is noteworthy that Theorem 1.6 can be “calibrated” to the case where u_0 vanishes to a fixed higher order on some part of ∂D , by asking higher-order regularity for the right-hand side. Notwithstanding this, it remains a tantalizing problem when the higher-order vanishing of u_0 takes place on isolated points of ∂D . This remains to be studied in the future. See [56] for partial results in this direction.

1.3 Connection to free boundary problems

We now describe more precisely how the existence of an incident wave that does not scatter leads to a free boundary problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and suppose that $u_q, u_0 \in H^1(\Omega)$ satisfy (1.1)–(1.2) where $q = h\chi_D$ for some solid domain D with $\bar{D} \subset \Omega$ and for $h \in C(\mathbb{R}^n)$. Then, u_0 is real-analytic, and also $u_q \in W_{\text{loc}}^{2,p}(\Omega)$ for any $p < \infty$ by elliptic regularity. We write $u := u_q - u_0 \in H_0^2(\Omega)$ and extend u by zero to \mathbb{R}^n . Then, $u \in H^2(\mathbb{R}^n)$ satisfies

$$(\Delta + \lambda^2)u = f_0\chi_D \quad \text{in } \mathbb{R}^n \quad (1.7)$$

where $f_0 = -hu_q$ near \bar{D} . Note also that since u solves $(\Delta + \lambda^2)u = 0$ in $\mathbb{R}^n \setminus \bar{D}$ and $u|_{\mathbb{R}^n \setminus \bar{\Omega}} = 0$, unique continuation implies that $u|_{\mathbb{R}^n \setminus \bar{D}} = 0$ using that $\mathbb{R}^n \setminus \bar{D}$ is connected.

Suppose that

$$h(z)u_0(z) \neq 0 \quad \text{at some } z \in \partial D. \quad (1.8)$$

Since $u_q = u_0$ outside \bar{D} and u_q is continuous, we also have $f_0(z) \neq 0$. We claim that

$$\bar{D} \cap B(z, r) = \text{supp}(u) \cap B(z, r) \quad (1.9)$$

whenever $r > 0$ is such that $|hu_q| > 0$ in $B(z, r)$. Since $u|_{\mathbb{R}^n \setminus \bar{D}} = 0$, one always has $\text{supp}(u) \cap B(z, r) \subset \bar{D} \cap B(z, r)$. Conversely, let $x \in \bar{D} \cap B(z, r)$. If $x \notin \text{supp}(u)$, then $u = 0$ near x , which by (1.7) implies that $f_0 = 0$ near x which is impossible since $|f_0| > 0$ in $B(z, r)$. This proves (1.9).

Since D is a solid domain, it follows from (1.9) that

$$D \cap B(z, r) = \text{int}(\text{supp}(u)) \cap B(z, r).$$

Thus, (1.7) implies that

$$(\Delta + \lambda^2)u = f_0 \chi_{\text{int}(\text{supp}(u))} \text{ in } B(z, r), \quad z \in \partial(\text{int}(\text{supp}(u))). \tag{1.10}$$

Writing $f = f_0 - \lambda^2 u$, so that $f(z) = f_0(z) \neq 0$, we may further write this as

$$\Delta u = f \chi_{\text{int}(\text{supp}(u))} \text{ in } B(z, r), \quad z \in \partial(\text{int}(\text{supp}(u))). \tag{1.11}$$

The last equation only involves u and not D . Thus, we have reduced our original problem to an *obstacle problem* in free boundary theory, where locally near z the obstacle is the set $\text{int}(\text{supp}(u))$ and its boundary $\partial(\text{int}(\text{supp}(u)))$ can be understood as a free boundary. In the free boundary literature, it is more customary to work with equations like

$$\Delta u = f \chi_{\{u \neq 0\}} \text{ in } B(z, r), \quad z \in \partial(\{u = |\nabla u| = 0\}). \tag{1.12}$$

However, the methods for proving regularity of the free boundary in (1.12) also apply to (1.11). See [44] for even more general equations when f is assumed Lipschitz, or [4] for f Dini.

The standard obstacle problem corresponds to (1.12) for solutions $u \geq 0$ (and $f \equiv 1$ in the most classical case). In our case, $u = u_q - u_0$, and it is not possible to assume that u is nonnegative. This means that (1.12) corresponds to a *no-sign obstacle problem*. The no-sign assumption on u makes the analysis of this problem extremely hard, and one has to resort to advanced tools such as monotonicity formulas along with strong geometric analysis. On the other hand, if ∂D is Lipschitz close to z , then it follows by standard free boundary techniques that u has a sign in a vicinity of z ; see the beginning of the proof of Theorem 1.3 in [4]. Thus, the case of Lipschitz domains falls back to the regularity theory for the standard obstacle problem, which is rather classical [13].

There is by now also a well-developed theory for no-sign obstacle problems when f is nonvanishing at the point z of interest, i.e., when (1.8) holds. We refer to [44] for an account of this theory. After the reduction to (1.11), Theorems 1.4–1.6 follow rather directly from this theory. One can also accommodate the possibility that f vanishes to some fixed order at each point of $\partial D \cap B(z, r)$ (see Theorem 1.6 for an example result). If f , or u_0 , only vanishes at z (or in a set of dimension $\leq n - 2$), then the problem becomes non-standard and is more or less untouched in the free boundary literature.

1.4 Inverse scattering

Finally, we discuss the case of inverse scattering problems where the bounded domain Ω is replaced by \mathbb{R}^n . In this subsection, we allow functions to be complex valued.

Let $\lambda > 0$ be a fixed frequency, and let u_0 be a solution of $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n . We consider u_0 as an incident wave that is used to probe a medium whose scattering properties are described by a compactly supported potential $q \in L^\infty(\mathbb{R}^n)$. The incident wave u_0 induces a total wave $u_q = u_0 + v$ that solves

$$(\Delta + \lambda^2 + q)u_q = 0 \quad \text{in } \mathbb{R}^n.$$

The solution is unique if we require that the scattered wave v is *outgoing* in the sense that

$$v = (-\Delta - (\lambda + i0)^2 - q)^{-1}(qu_0)$$

where $(-\Delta - (\lambda + i0)^2 - q)^{-1}$ is the outgoing resolvent. Since qu_0 is compactly supported, v has the asymptotics

$$v(r\theta) = e^{i\lambda r} r^{-\frac{n-1}{2}} u_q^\infty(\theta) + o(r^{-\frac{n-1}{2}}) \quad \text{as } r \rightarrow \infty,$$

where $\theta \in S^{n-1}$. The function u_q^∞ on S^{n-1} is called the *far field pattern* corresponding to incident wave u_0 , and it can be measured from the knowledge of u_q as $|x| \rightarrow \infty$. We refer to [19,55] for these basic facts.

A commonly used class of incident waves is given by the Herglotz waves, which are solutions of $(\Delta + \lambda^2)u_0 = 0$ having the form

$$u_0(x) = \int_{S^{n-1}} e^{i\lambda x \cdot \omega} f(\omega) d\omega, \quad f \in L^2(S^{n-1}). \quad (1.13)$$

A scattering analogue of the Dirichlet-to-Neumann map is given by the far field operator

$$A_q(\lambda) : L^2(S^{n-1}) \rightarrow L^2(S^{n-1}), \quad A_q(\lambda)f = u_q^\infty.$$

A standard fixed frequency inverse problem is to determine q from the knowledge of $A_q(\lambda)$, which corresponds to infinitely many measurements. However, we wish consider the *single measurement problem* in (fixed frequency) inverse scattering: determine some properties of q from knowledge of the far field pattern u_q^∞ corresponding to a fixed incident wave u_0 . If $u_q^\infty \equiv 0$, we say that the incident wave u_0 does not scatter. Again, in order to obtain a formally well-determined problem, we consider the case of *penetrable obstacles*, so that

$$q = h\chi_D,$$

where $D \subset \mathbb{R}^n$ is a bounded open set (the obstacle) and h is a contrast for D .

In the imaging community, it has been understood for a long time that if ∂D has corner singularities, one often has strong scattering effects. A rigorous analysis of this phenomenon was initiated in the important work [11] which showed that if part of ∂D is part of a cube, then every incident wave scatters nontrivially for every frequency $\lambda > 0$. In two dimensions, this was extended to sectors with angle $< 90^\circ$ and single measurement results in [27,43]. The analysis was based on studying Laplace transforms of characteristic functions of cones via complex geometrical optics solutions as in the Calderón problem. There are several related results including quantitative bounds even when corners are replaced by high curvature points, see [7–10,12] and the survey [36] (which also discusses results for electromagnetic and elastic scattering). Another important approach to non-scattering problems, introduced in [23] (see [22,35] for related work), is based on the theory of boundary value problems in corner domains and can be used to produce similar results even for curvilinear polyhedra or when h vanishes to finite order at ∂D . We mention that these results related to corner singularities are most complete for $n = 2$, and even when $n = 3$ they become more limited and mostly apply to edge or circular cone singularities. Finally, the results in [18], already discussed before, show regularity of the free boundary if the obstacle is a Lipschitz domain, and the incident wave is nonvanishing on its boundary.

In [11], a frequency $\lambda > 0$ was called a *non-scattering wavenumber* if there is some incident wave u_0 that does not scatter. The results mentioned above show that if ∂D has corner singularities, then there are no non-scattering wavenumbers (i.e., every incident wave

scatters nontrivially independent of the frequency). This notion is connected with *interior transmission eigenvalues* (see [16,17]) in the sense that a non-scattering wavenumber is also an interior transmission eigenvalue. The converse is not true: for a given potential, there are typically infinitely many interior transmission eigenvalues whereas the set of non-scattering frequencies may be empty.

The free boundary approach described above extends directly to the scattering case. If u_0 is an incident wave solving $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n , let u_q be the outgoing solution of $(\Delta + \lambda^2 + q)u_q = 0$ in \mathbb{R}^n defined above where $q = h\chi_D$, and D is a bounded solid domain. Suppose that u_0 does not scatter, i.e., $u_q^\infty \equiv 0$. Then, the Rellich uniqueness theorem and unique continuation imply that $u_q = u_0$ in $\mathbb{R}^n \setminus \bar{D}$. We may thus take Ω to be some large ball containing \bar{D} , and we are back in the situation of (1.1)–(1.2). Moreover, if h is assumed to be real valued, and if for some $z \in \partial D$, one has $h(z)u_0(z) \neq 0$, then $\text{Re}(u_0)(z) \neq 0$ or $\text{Im}(u_0)(z) \neq 0$. Thus by taking real or imaginary parts of the solutions, one can reduce to a situation where the functions involved are real valued.

If the obstacle has real-analytic boundary, combining Theorem 2.1 (with $\Omega = \mathbb{R}^n$) and Theorem 1.3 leads to an analogue of Theorem 1.1 in the scattering setting. This provides examples of real-valued contrasts and incident Herglotz waves that do not scatter.

Structure of the article

In Sect. 2, we will prove Theorems 1.1 and 1.2 by using a simple extension argument, the Cauchy–Kowalevski theorem and the defining property of quadrature domains. In Sect. 3, we discuss Helmholtz solutions and prove Theorem 1.3, which follows by combining a result in D with a Runge approximation argument. In Sect. 4, we discuss how Theorems 1.4–1.6 follow from arguments in the theory of free boundaries.

2 Examples of free boundaries

In this section, we show that any real-analytic boundary, or the boundary of any quadrature domain when $\lambda = 0$, can be realized as a free boundary. The following results are more precise versions of Theorems 1.1 and 1.2, since they also give information on the kinds of incident waves and contrasts for which one has no scattering. Note that Theorem 1.1 follows by combining Theorems 2.1 and 1.3, and 1.2 follows from Theorem 2.2 by taking $u_0 \equiv 1$.

Theorem 2.1 *Let $\lambda \geq 0$, let $D \subset \mathbb{R}^n$ be a bounded open set with real-analytic boundary, let $\Omega \subset \mathbb{R}^n$ be an open set with $\bar{D} \subset \Omega$, and let u_0 be any solution of $(\Delta + \lambda^2)u_0 = 0$ in Ω that is positive on ∂D . Let also h_0 be a real-analytic function near ∂D . Then, there is $h \in C^\infty(\bar{\Omega})$ with $h = h_0$ near ∂D so that with the choice $q = h\chi_D$, the equation*

$$(\Delta + \lambda^2 + q)u_q = 0 \quad \text{in } \Omega$$

has a solution $u_q \in H^2(\Omega)$ satisfying $u_q = u_0$ in $\Omega \setminus \bar{D}$.

Theorem 2.2 *Let $D \subset \mathbb{R}^n$ be a bounded quadrature domain, let $\Omega \subset \mathbb{R}^n$ be an open set with $\bar{D} \subset \Omega$, and let u_0 be any solution of $\Delta u_0 = 0$ in Ω that is positive on ∂D . Then, there is $h \in L^\infty(\Omega)$ with $|h| \geq c > 0$ near ∂D so that with the choice $q = h\chi_D$, the equation*

$$(\Delta + q)u_q = 0 \quad \text{in } \Omega$$

has a solution $u_q \in H^2(\Omega)$ satisfying $u_q = u_0$ in $\Omega \setminus \bar{D}$.

We mention that the existence of solutions of $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n that are positive on ∂D is proved later in Proposition 3.3. It follows from the proof that u_0 can be chosen to be a Herglotz wave, i.e., of the form (1.13), which is relevant for applications in scattering theory.

The proofs of Theorems 2.1 and 2.2 involve the following simple result. It begins with a solution of $(\Delta + \lambda^2)u_0 = 0$ in Ω that is positive on ∂D and with a local solution v_0 , for some potential h_0 that extends u_0 slightly inside D . The result gives a solution v that extends u_0 all the way into D and corresponds to some potential $h\chi_D$, where h extends h_0 into D . The point is that one first chooses a suitable extension v of v_0 , and then constructs the potential h depending on v .

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^n$ be open, let $D \subset \mathbb{R}^n$ be a bounded open set with $\bar{D} \subset \Omega$, let $\lambda \geq 0$, and assume that $(\Delta + \lambda^2)u_0 = 0$ in Ω with u_0 positive on ∂D . Suppose that U is a neighborhood of ∂D in Ω and that $h_0 \in L^\infty(U)$ and $v_0 \in C^{1,1}(U)$ satisfy*

$$(\Delta + \lambda^2 + h_0)v_0 = 0 \text{ in } D \cap U, \quad v_0|_{U \setminus \bar{D}} = u_0|_{U \setminus \bar{D}}.$$

Then, there are $h \in L^\infty(\Omega)$ and $v \in C^{1,1}(\Omega)$, with $h = h_0$ and $v = v_0$ near ∂D , so that

$$(\Delta + \lambda^2 + h\chi_D)v = 0 \text{ in } \Omega, \quad v|_{\Omega \setminus \bar{D}} = u_0|_{\Omega \setminus \bar{D}}.$$

If additionally $h_0 \in C^\infty(U)$ and $v_0|_{\bar{D} \cap U} \in C^\infty(\bar{D} \cap U)$, then $h \in C^\infty(\bar{\Omega})$.

Proof Note that v_0 is positive in some neighborhood $U_1 \subset U$ of ∂D , since $v_0 \in C^{1,1}(U)$ and $v_0|_{U \setminus \bar{D}} = u_0|_{U \setminus \bar{D}}$ and u_0 is positive on ∂D . Let $\psi \in C_c^\infty(U_1)$ satisfy $0 \leq \psi \leq 1$ and $\psi = 1$ near ∂D , and define

$$v = \begin{cases} v_0\psi + (1 - \psi) & \text{in } \bar{D}, \\ u_0 & \text{in } \Omega \setminus \bar{D}. \end{cases}$$

Then, $v \in C^{1,1}(\Omega)$ is positive near \bar{D} and satisfies $v = v_0$ near ∂D . One can now define a function $h \in L^\infty(\Omega)$ by

$$h = \begin{cases} -\frac{(\Delta + \lambda^2)v}{v} & \text{in } \bar{D}, \\ \psi h_0 & \text{in } \Omega \setminus \bar{D}. \end{cases}$$

The functions h and v will have the required properties.

If additionally $h_0 \in C^\infty(U)$ and $v_0|_{\bar{D} \cap U} \in C^\infty(\bar{D} \cap U)$, it follows that $v|_{\bar{D}} \in C^\infty(\bar{D})$ and $h|_{\bar{D}} \in C^\infty(\bar{D})$. Since $h = h_0$ near ∂D , we have $h \in C^\infty(\bar{\Omega})$.

By Lemma 2.3, the proofs of Theorems 2.1 and 2.2 are reduced to finding a local solution v_0 that extends u_0 a little bit inside D . In the real-analytic case, this can be done by solving a Cauchy problem using the Cauchy–Kowalevski theorem.

Proof of Theorem 2.1 Note that u_0 is real-analytic in Ω . For any $x \in \partial D$, we may use the Cauchy–Kowalevski theorem to find a real-analytic solution of

$$(\Delta + \lambda^2 + h_0)v_x = 0 \text{ in } U_x, \quad v_x|_{U_x \cap \partial D} = u_0|_{U_x \cap \partial D}, \quad \partial_\nu v_x|_{U_x \cap \partial D} = \partial_\nu u_0|_{U_x \cap \partial D},$$

where U_x is an open set of the form $\{z + t\nu(z) : z \in V_x, |t| < \varepsilon_x\}$ with V_x a neighborhood of x in ∂D and $\varepsilon_x > 0$. Any two solutions v_x and v_y agree on their overlap $U_x \cap U_y$ by

the unique continuation principle. Thus, for some neighborhood U of ∂D in Ω , there is a real-analytic function v_0 in U so that

$$(\Delta + \lambda^2 + h_0)v_0 = 0 \text{ in } U, \quad v_0|_{\partial D} = u_0|_{\partial D}, \quad \partial_\nu v_0|_{\partial D} = \partial_\nu u_0|_{\partial D}.$$

We may redefine $v_0 = u_0$ in $U \setminus \bar{D}$, so that $v_0 \in C^{1,1}(U)$ will satisfy

$$(\Delta + \lambda^2 + h_0)v_0 = 0 \text{ in } D \cap U, \quad v_0|_{U \setminus \bar{D}} = u_0.$$

By Lemma 2.3, there are $v \in C^{1,1}(\Omega)$ and $h \in C^\infty(\bar{\Omega})$ that satisfy $v = v_0$ and $h = h_0$ near ∂D , such that

$$(\Delta + \lambda^2 + h\chi_D)v = 0 \text{ in } \Omega, \quad v_0|_{\Omega \setminus \bar{D}} = u_0.$$

It remains to set $u_q = v$ in D and $u_q = u_0$ in $\Omega \setminus \bar{D}$. Then, $u_q \in H^2(\Omega)$ has the required properties.

In the case of quadrature domains, we instead use (1.3) to produce the required local solution.

Proof of Theorem 2.2 Let G be the fundamental solution for $-\Delta$ in \mathbb{R}^n , i.e. $G(x) = c_2 \log |x|$ when $n = 2$ and $G(x) = c_n |x|^{2-n}$ when $n \geq 3$. Let μ be the distribution with $\text{supp}(\mu) \subset D$ appearing in the definition of the quadrature domain D , and define

$$u = G * (\chi_D - \mu).$$

Since $\chi_D - \mu$ is a compactly supported distribution, u is a distribution in \mathbb{R}^n , and it satisfies

$$\Delta u = \chi_D - \mu \text{ in } \mathbb{R}^n.$$

Moreover, if $x \in \mathbb{R}^n \setminus \bar{D}$, we may take $H(y) = G(x - y)$ in (1.3) to obtain that

$$u|_{\mathbb{R}^n \setminus \bar{D}} = 0.$$

In particular, since $\text{supp}(\mu) \subset D$, there is a neighborhood U of ∂D in \mathbb{R}^n such that

$$\Delta u = \chi_D \text{ in } U, \quad u|_{U \setminus \bar{D}} = 0. \tag{2.1}$$

Note that $u \in C^{1,1}(U)$ using the $C^{1,1}$ regularity results for the no-sign obstacle problem [4].

Define $v_0 = u + u_0$ in U . We first claim that there is h_0 near ∂D with $|h_0| \geq c > 0$ near ∂D so that

$$(\Delta + h_0\chi_D)v_0 = 0 \text{ near } \partial D.$$

In fact, using the equations for u and u_0 , for any h_0 one has

$$(\Delta + h_0\chi_D)v_0 = \chi_D + h_0\chi_D v_0 \text{ in } U.$$

This quantity vanishes near ∂D , if we set

$$h_0 = -\frac{1}{v_0} \text{ near } \partial D.$$

The denominator is nonvanishing near ∂D since u is continuous with $u|_{\partial D} = 0$, and since u_0 is positive on ∂D . We have thus found the required function h_0 near ∂D .

By Lemma 2.3, there are $h \in L^\infty(\Omega)$ and $v \in C_{loc}^{1,1}(\Omega)$, with $h = h_0$ and $v = v_0$ near ∂D , so that

$$(\Delta + h\chi_D)v = 0 \text{ in } \Omega, \quad v|_{\Omega \setminus \bar{D}} = u_0|_{\Omega \setminus \bar{D}}.$$

Setting $u_q = v$ in \bar{D} and $u_q = u_0$ in $\Omega \setminus \bar{D}$ gives the required solution.

Remark 2.4 The only property of quadrature domains needed in the proof of Theorem 2.2 was the existence of a function u satisfying (2.1). The conclusion of Theorem 2.2, also with a frequency $\lambda \geq 0$, would hold for any domain D that admits a function u satisfying

$$(\Delta + \lambda^2)u = f\chi_D \text{ in } U, \quad u|_{U \setminus \bar{D}} = 0,$$

where f is nonvanishing on ∂D . In Theorem 2.2, we produced such a function as $u = G * (\chi_D - \mu)$, which can be understood as continuing the potential $G * \chi_D$ smoothly a little bit inside D as the function $G * \mu$.

3 Zero sets of Helmholtz solutions

We complement the previous results by showing that for any bounded open set D , there is a solution u_0 of $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n which is positive on ∂D , under some restrictions on D and λ . In the case $\lambda = 0$, one can take $u_0 \equiv 1$, so we will assume $\lambda > 0$. If one allows complex-valued solutions, the function $u_0 = e^{i\lambda x_1}$ is a nonvanishing solution in \mathbb{R}^n . However, real-valued solutions always have zeros. This is already seen in the case $n = 1$ where any solution of $(\Delta + \lambda^2)u_0 = 0$ takes the form $u_0(x) = a \sin(\lambda x) + b \cos(\lambda x)$, and such a function has a zero in any closed interval of length π/λ . A similar result holds for Laplace eigenfunctions in compact manifolds (see the survey [38]). The following version of this result shows that any real solution of $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n has a zero in any ball of radius $\geq c_n/\lambda$.

Lemma 3.1 *Let $\lambda > 0$ and $x^0 \in \mathbb{R}^n$. There is a nonvanishing real function $u \in C^\infty(B(x^0, r))$ solving $(\Delta + \lambda^2)u = 0$ in $B(x^0, r)$ if and only if $r < c_n/\lambda$, where c_n denotes the first positive zero of the Bessel function $J_{\frac{n-2}{2}}$.*

Proof By translation invariance, we may assume $x^0 = 0$, and replacing $u(x)$ by $u(\lambda x)$, we may assume $\lambda = 1$. We consider radial solutions $v = v(r)$ of $(\Delta + 1)v = 0$ in \mathbb{R}^n . Writing the Laplacian in polar coordinates, we see that v should satisfy

$$v''(r) + \frac{n-1}{r}v'(r) + v(r) = 0 \quad \text{in } \mathbb{R}_+.$$

The substitution $v(r) = r^{\frac{2-n}{2}}w(r)$ leads to the Bessel equation

$$r^2w''(r) + rw'(r) + \left(r^2 - \left(\frac{n-2}{2}\right)^2\right)w(r) = 0 \quad \text{in } \mathbb{R}_+.$$

Since $v(r)$ should be bounded near $r = 0$, one must have $v(r) = r^{\frac{2-n}{2}}J_{\frac{n-2}{2}}(r)$ (up to a scalar multiple). Thus, there is a positive solution in $B(0, c_n)$.

For the converse, we argue by contradiction and suppose that $w \in C^\infty(\bar{B}_r)$ is positive and solves $(\Delta + 1)w = 0$ in B_r for some $r \geq c_n$. We now use the fact that if a Schrödinger equation has a nonvanishing solution, then it can be reduced to a divergence form equation with no zero-order term. Writing $\gamma = w^2$, we see that

$$(\Delta + 1)u = 0 \quad \text{in } B_r \iff \operatorname{div}(\gamma \nabla(\gamma^{-1/2}u)) = 0 \quad \text{in } B_r.$$

Now since $r \geq c_n$, taking u to be the radial solution v above shows that $\gamma^{-1/2}u$ is positive in $B(0, c_n)$ but becomes zero on $\partial B(0, c_n)$. By the maximum principle for the equation $\operatorname{div}(\gamma \nabla \cdot) = 0$, the maximum of $\gamma^{-1/2}u$ in $\bar{B}(0, c_n)$ should be attained at the boundary. This is a contradiction.

We now turn to the question of determining if there is a real-valued solution of $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n that is nonvanishing on ∂D . The following remark shows that this may be false when $\lambda > 0$ is an eigenvalue of $-\Delta$ in D .

Remark 3.2 Let $D = B(0, c_n)$ where c_n is as in Lemma 3.1, and suppose u_0 is real and solves $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n . Let $v \in H_0^1(D)$ be the radial solution in the proof of Lemma 3.1 with $v > 0$ in D . Since $v|_{\partial D} = 0$, one has

$$\int_{\partial D} u_0 \partial_\nu v \, dS = \int_D (u_0 \Delta v - (\Delta u_0)v) \, dx = 0.$$

However, $\partial_\nu v < 0$ on ∂D , which implies that u_0 must change sign on ∂D . The same argument works for any sufficiently regular D if λ is the first Dirichlet eigenvalue and ∂D is connected.

The next result shows that if λ is not a Dirichlet eigenvalue in D , there is a solution u_0 which is positive on ∂D .

Proposition 3.3 *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain if $n = 2, 3$ (resp. C^1 domain if $n \geq 4$) so that $\mathbb{R}^n \setminus \bar{D}$ is connected. Suppose that $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta$ in D . Then, there is a real-valued u_0 solving $(\Delta + \lambda^2)u_0 = 0$ in \mathbb{R}^n so that u_0 is positive on ∂D .*

We will prove the above result in two steps. First, we show that there is $v \in W^{1,p}(D)$ for some $p > n$ such that $(\Delta + \lambda^2)v = 0$ in D and $v|_{\partial D}$ is positive. Then, we apply a Runge approximation property, showing that v can be approximated by functions $u|_D$ where u solves $(\Delta + \lambda^2)u = 0$ in \mathbb{R}^n . This kind of property is classical for second-order elliptic equations, and it follows from the unique continuation property. See [34, 39] for the original results in bounded domains and [45] for further references.

Approximation by Helmholtz solutions in \mathbb{R}^n has been used in scattering theory at least with respect to L^2 norms, see e.g., [54] and references therein. We need a corresponding approximation result in the $C(\bar{D})$ norm. This is more involved than approximation in L^2 (for analytic functions this would correspond to Mergelyan’s theorem instead of Runge’s theorem). In order to achieve this, we will assume some regularity on D and work with Sobolev norms instead.

Below, we say that D is a C^0 domain if it is locally the region above the graph of a continuous function, and we define $H^{1,p}(D) = \{u|_D : u \in W^{1,p}(\mathbb{R}^n)\}$ with the quotient norm $\|v\|_{H^{1,p}(D)} = \inf\{\|u\|_{W^{1,p}(\mathbb{R}^n)} : u|_D = v\}$. The space $H^{1,p}(D)$, defined via restriction, coincides with the standard Sobolev space $W^{1,p}(D)$ whenever D is a $W^{1,p}$ extension domain. The following version of the Runge approximation property will be relevant for us. In the rest of this section, we allow functions to be complex valued.

Proposition 3.4 *Let $1 < p < \infty$, let $\lambda > 0$, and let $D \subset \mathbb{R}^n$ be a bounded C^0 domain such that $\mathbb{R}^n \setminus \bar{D}$ is connected. Given any $v \in H^{1,p}(D)$ (possibly complex valued) with $(\Delta + \lambda^2)v = 0$ in D , there exist u_j solving $(\Delta + \lambda^2)u_j = 0$ in \mathbb{R}^n so that*

$$\|u_j - v\|_{H^{1,p}(D)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

If v is real valued, then so are u_j .

The proof of Proposition 3.3 follows rather easily:

Proof of Proposition 3.3 Since λ is not a Dirichlet eigenvalue in D , there is a real-valued solution $v = 1 + w$ of $(\Delta + \lambda^2)v = 0$ in D where $w \in H_0^1(D)$ is the unique solution of

$$(\Delta + \lambda^2)w = -\lambda^2 \quad \text{in } D.$$

We claim that $v \in W^{1,p}(D)$ for some $p > n$. Note that the $W^{1,p}(D)$ and $H^{1,p}(D)$ norms are equivalent since Lipschitz domains are $W^{1,p}$ extension domains. Now, if $v \in W^{1,p}(D)$ for some $p > n$, then by Proposition 3.4, there are global solutions u_j such that

$$\|u_j - 1\|_{C(\partial D)} \leq \|u_j - v\|_{C(\bar{D})} \leq C\|u_j - v\|_{H^{1,p}(D)} \rightarrow 0$$

where we used the Sobolev embedding. This shows the existence of a global solution that is positive on ∂D .

To prove that $v \in W^{1,p}(D)$ for some $p > n$, we note that $w \in H_0^1(D)$ solves

$$\Delta w = -\lambda^2 v \quad \text{in } D. \tag{3.1}$$

By Sobolev embedding, the right-hand side is in $L^{\frac{2n}{n-2}}(D)$ for $n \geq 3$, and in $L^r(D)$ for any $r < \infty$ for $n = 2$. In particular, the right-hand side is in $W^{-1,r}(D)$ for any $r < \infty$ for $n = 2, 3, 4$ and for $r = \frac{2n}{n-4}$ for $n \geq 5$. Since D is Lipschitz, by [29, Theorem 1.1], one has $w \in W^{1,p}(D)$ for some $p > 3$ if $n = 2, 3$. This proves the claim for Lipschitz domains in dimensions $n = 2, 3$. For C^1 domains in dimensions $n \geq 4$, using [29, Theorem 1.1], which holds with $p_0 = 1$ in C^1 domains, shows that $w \in W^{1, \frac{2n}{n-4}}(D)$. Returning to (3.1), noting that the right-hand side has more regularity, and iterating this argument shows that $v \in W^{1,p}(D)$ for all $p < \infty$ in the case of C^1 domains.

To prove Proposition 3.4, it is convenient to introduce the operator

$$P(\lambda) : C^\infty(S^{n-1}) \rightarrow C^\infty(\mathbb{R}^n), \quad P(\lambda)f(x) = \int_{S^{n-1}} e^{i\lambda x \cdot \omega} f(\omega) d\omega.$$

Functions $u = P(\lambda)f$ are called Herglotz waves, and they are particular solutions of $(\Delta + \lambda^2)u = 0$ in \mathbb{R}^n . One can think of f as a certain boundary value at infinity for u , and of $P(\lambda)$ as a Poisson integral that gives the solution of $(\Delta + \lambda^2)u = 0$ having boundary value f at infinity.

The following proof, modeled after [42, 52], will show that restrictions of Herglotz waves to D are dense in the set of all Helmholtz solutions in $H^{1,p}(D)$.

Proof of Proposition 3.4 By the Hahn–Banach theorem, it is enough to prove that any bounded linear functional ℓ on $H^{1,p}(D)$ that vanishes on $\{P(\lambda)f|_D; f \in C^\infty(S^{n-1})\}$ must also vanish on $\{v \in H^{1,p}(D); (\Delta + \lambda^2)v = 0 \text{ in } D\}$. We define a functional

$$\ell_1 : W^{1,p}(\mathbb{R}^n) \rightarrow \mathbb{C}, \quad \ell_1(u) = \ell(u|_D).$$

Then, ℓ_1 is bounded on $W^{1,p}(\mathbb{R}^n)$, and by duality

$$\ell_1(u) = (u, \mu)$$

for some $\mu \in W^{-1,p'}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and (\cdot, \cdot) is the sesquilinear distributional pairing in \mathbb{R}^n . Clearly $\mu = 0$ in $\mathbb{R}^n \setminus \bar{D}$, so μ is a compactly supported distribution. Thus, the condition $\ell(P(\lambda)f|_D) = 0$ for all $f \in C^\infty(S^{n-1})$ implies that

$$(P(\lambda)f, \mu) = 0 \quad \text{for all } f \in C^\infty(S^{n-1}). \tag{3.2}$$

Let G_λ be the outgoing fundamental solution of $\Delta + \lambda^2$, given by

$$G_\lambda(x) = c_{n,\lambda} |x|^{-\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(1)}(\lambda|x|)$$

where $H_\nu^{(1)}$ is the Hankel function (see [55, Section 1.2.3]), and let w be the distribution

$$w = G_\lambda * \mu.$$

Then, w is a distributional solution of

$$(\Delta + \lambda^2)w = \mu \text{ in } \mathbb{R}^n.$$

By elliptic regularity $w \in W_{loc}^{1,p'}(\mathbb{R}^n)$ and w is smooth outside \bar{D} , with the expression

$$w(x) = (G_\lambda(x - \cdot), \mu), \quad x \in \mathbb{R}^n \setminus \bar{D}. \tag{3.3}$$

Let $f \in C^\infty(S^{n-1})$ and $u = P(\lambda)f \in C^\infty(\mathbb{R}^n)$. By (3.2) and the fact that μ has compact support, we have

$$0 = (u, \mu) = \lim_{r \rightarrow \infty} (u, \mu)_{B_r} \tag{3.4}$$

where $(\cdot, \cdot)_{B_r}$ is the sesquilinear distributional pairing in B_r . We wish to use that $\mu = (\Delta + \lambda^2)w$. Since w is not smooth in \bar{D} , we introduce a cutoff function $\chi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \chi \leq 1$ and $\chi = 1$ near \bar{D} . Writing $u = \chi u + (1 - \chi)u$ and using that everything is smooth outside \bar{D} , we obtain from (3.4) that

$$\begin{aligned} 0 &= \lim_{r \rightarrow \infty} ((\chi u, (\Delta + \lambda^2)w)_{B_r} + ((1 - \chi)u, (\Delta + \lambda^2)w)_{B_r}) \\ &= \lim_{r \rightarrow \infty} [((\Delta + \lambda^2)(\chi u), w)_{B_r} + ((\Delta + \lambda^2)((1 - \chi)u), w)_{B_r} \\ &\quad + \int_{\partial B_r} (u \overline{\partial_\nu w} - (\partial_\nu u) \bar{w}) dS]. \end{aligned}$$

Since $(\Delta + \lambda^2)u = 0$, this reduces to

$$\lim_{r \rightarrow \infty} \int_{\partial B_r} (u \overline{\partial_\nu w} - (\partial_\nu u) \bar{w}) dS = 0. \tag{3.5}$$

Writing $x = r\theta$ where $r \geq 0$ and $\theta \in S^{n-1}$, the function $u = P(\lambda)f$ has the asymptotics

$$\begin{aligned} u(r\theta) &= c'_{n,\lambda} r^{-\frac{n-1}{2}} \left[e^{i\lambda r} f(\theta) + i^{n-1} e^{-i\lambda r} f(-\theta) \right] + O(r^{-\frac{n+1}{2}}), \\ \partial_r u(r\theta) &= c'_{n,\lambda} r^{-\frac{n-1}{2}} i\lambda \left[e^{i\lambda r} f(\theta) - i^{n-1} e^{-i\lambda r} f(-\theta) \right] + O(r^{-\frac{n+1}{2}}), \end{aligned}$$

as $r \rightarrow \infty$ (see [41, Section 1.3]). We wish to study similar asymptotics for the outgoing function w . Using (3.3) and asymptotics for the Hankel function, we see that (as in [55, Section 1.2.3])

$$\begin{aligned} w(r\theta) &= c''_{n,\lambda} r^{-\frac{n-1}{2}} e^{i\lambda r} \hat{\mu}(\lambda\theta) + O(r^{-\frac{n+1}{2}}), \\ \partial_r w(r\theta) &= c''_{n,\lambda} r^{-\frac{n-1}{2}} i\lambda e^{i\lambda r} \hat{\mu}(\lambda\theta) + O(r^{-\frac{n+1}{2}}) \end{aligned}$$

where $\hat{\mu} \in C^\infty(\mathbb{R}^n)$ is the Fourier transform of the compactly supported distribution μ . Above $c_{n,\lambda}$, $c'_{n,\lambda}$ and $c''_{n,\lambda}$ are nonzero constants.

Inserting the asymptotics for u and w into (3.5) and noting that the terms containing $f(-\theta)$ cancel yields that

$$\int_{S^{n-1}} f(\theta) \overline{\hat{\mu}(\lambda\theta)} d\theta = 0.$$

Since this is true for all $f \in C^\infty(S^{n-1})$, we must have $\hat{\mu}(\lambda\theta) = 0$. In particular, w has the asymptotics

$$w(r\theta) = O(r^{-\frac{n+1}{2}}), \quad \partial_r w(r\theta) = O(r^{-\frac{n+1}{2}}).$$

Since w is outgoing and satisfies $(\Delta + \lambda^2)w = 0$ in $\mathbb{R}^n \setminus \bar{D}$, the Rellich uniqueness theorem (see e.g., [28]) implies that $w = 0$ outside a large ball. Since $\mathbb{R}^n \setminus \bar{D}$ is connected, the unique continuation principle gives that $w = 0$ in $\mathbb{R}^n \setminus \bar{D}$.

We have now proved that the condition (3.2) implies that

$$\mu = (\Delta + \lambda^2)w \quad \text{in } \mathbb{R}^n$$

for some $w \in W^{1,p'}(\mathbb{R}^n)$ vanishing in $\mathbb{R}^n \setminus \bar{D}$. Now let $v \in H^{1,p}(D)$ be any solution of $(\Delta + \lambda^2)v = 0$ in D , and let $\tilde{v} \in W^{1,p'}(\mathbb{R}^n)$ be an extension of v . Then, we have

$$\ell(v) = \ell_1(\tilde{v}|_D) = (\tilde{v}, \mu) = (\tilde{v}, (\Delta + \lambda^2)w) = ((\Delta + \lambda^2)\tilde{v}, w).$$

Since D is a bounded C^0 domain and since $w \in W^{1,p'}(\mathbb{R}^n)$ vanishes in $\mathbb{R}^n \setminus \bar{D}$, there are $w_j \in C_c^\infty(D)$ with $w_j \rightarrow w$ in $W^{1,p'}(\mathbb{R}^n)$ (this is proved as in [40, Theorem 3.29]). It follows that

$$\ell(v) = \lim_{j \rightarrow \infty} ((\Delta + \lambda^2)\tilde{v}, w_j) = 0$$

since $(\Delta + \lambda^2)\tilde{v} = 0$ in D . This concludes the proof.

4 Free boundary methods

By arguments from Sect. 1.3, we know that $u = u_q - u_0$ satisfies the equation (see (1.11)–(1.12))

$$\Delta u = f(x)\chi_{\{u \neq 0\}} \quad \text{in } B(x^0, r), \tag{4.1}$$

where we may assume that $f(x) > 0$ in some neighborhood of $x^0 \in \partial\{u \neq 0\}$. The above equation has been treated extensively in the literature, and all regularity aspects of the problem are resolved and sorted out; see e.g., [4] and the references therein. Theorem 1.5 follows directly from [4, Theorem 1.3], and the proof of Theorem 1.6 is sketched below. Theorem 1.4 in the case where h is Dini or Lipschitz continuous follows from these results, and the higher regularity results follow from the method of [32] (see also [44, Section 6.4]).

We shall now give classical examples of singularities that can appear in the obstacle problem.

Example ([32, 47–49]) We recall from [32, page 387–390] an explicit example of cusps appearing in the free boundary. These cusps are represented by the curves

$$x_2 = \pm x_1^{\mu/2}, \quad 0 \leq x_1 \leq 1,$$

where $\mu = 4k + 1$, ($k = 1, 2, \dots$) gives nonnegative solutions and $\mu = 4k + 3$, ($k = 0, 1, \dots$) gives solutions that become negative on the negative x_1 -axis and near the origin. The solution is defined locally by

$$u(x) = x_2^2 - \frac{2}{1 + \mu/2} \rho^{1+\mu/2} \sin(1 + \mu/2)\theta + \dots, \quad x \in \Omega, |x| < \epsilon,$$

for ϵ small. Here, we have used both real and complex notation

$$x = (x_1, x_2), \quad z = \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Also the domain Ω is the image of the set

$$\{z : |z| < 1, \operatorname{Im} z > 0\}$$

under the conformal mapping $f(z) = z^2 + iz^\mu$.

Proof of Theorem 1.6 The proof of Theorem 1.6 when $h \in C^{0,1}(B(x^0, r))$ (i.e., case (1)) is somehow hidden in [15] (see their proof of Main Theorem), where it is proven that the singular set of the free boundary lies in a C^1 -manifold. In particular, this means that whenever we blow up a solution at a singular free boundary point through any sequence $u(r_j x + x^0)/r_j^2$ (here $u = u_q - u_0$ satisfies (4.1)), it will converge to a fixed polynomial $p(x)$, with the free boundary $\{p = \nabla p = 0\}$, and regardless of the sequence $\{r_j\}$. This in particular implies that the limiting free boundary lies in a plane (or lower dimensional plane) which after translation and rotation we assume it is $\{x_n = 0\}$. From here, it follows that the free boundary approaches this plane tangentially, whence the statement b) follows whenever the free boundary has a cusp at x^0 .

In case x^0 is not a cusp point, then by Theorem 1.5 in a vicinity of x^0 the free boundary is C^1 , and $u \geq 0$, and obviously $\partial_e u \geq 0$ in a smaller neighbourhood of x^0 . By results of [3] (see section 1.4.2), the free boundary is $C^{1,\alpha}$, and Theorem 1.6 is proved in case (1).

To prove Theorem 1.6 in the case (2), we work with $v = \partial_e u$, where by the assumption $e = \nabla u_0(x^0) \neq 0$. Since $u_q = u_0$ in D^c , we also have $\nabla u_q(x^0) \neq 0$. As before with $u = u_q - u_0$, we have $-(\Delta + \lambda^2)\partial_e u = \partial_e(hu_q)\chi_D = (\partial_e h u_q + h \partial_e u_q)\chi_D$ close to x^0 . Since $u_0 = u_q = 0$ on $\partial D \cap B_r(x^0)$ and $\partial_e u_q(x^0) > 0$ and $h(x^0) \neq 0$, we have that $\partial_e u$ satisfies the hypothesis of Theorem 1.6 case (1), and hence the result follows.

Acknowledgements

M. S. was supported by the Academy of Finland (Finnish Centre of Excellence in Inverse Modelling and Imaging, Grant Numbers 312121 and 309963) and by the European Research Council under Horizon 2020 (ERC CoG 770924). H. Sh. was supported in part by Swedish Research Council.

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Received: 29 June 2021 Accepted: 25 August 2021

Published online: 05 October 2021

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