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UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Academic dissertation to be publicly discussed, by permission of the Faculty of Mathematics and Science of the University of Jyväskylä, on September 16, 2021, at 12 o'clock.

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Jyväskylä, August 2021
Joonas Niinikoski

List of included articles

[A] J. Niinikoski, *Volume preserving mean curvature flows near strictly stable sets in flat torus*, J. Differ. Equ. **276** (2021), 149–186. Updated version arXiv:1907.03618.

[B] V. Julin and J. Niinikoski, *Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow*, to appear in Anal. PDE, preprint (2020).

[C] V. Julin and J. Niinikoski, *Stationary sets of the mean curvature flow with a forcing term*, Adv. Calc. Var., preprint (2020).

The author of this dissertation has actively contributed to the research of the joint articles [**B**] and [**C**].

Abstract

The main subject of this dissertation is mean curvature type of flows, in particular the volume preserving mean curvature flow. A classical flow in this context is seen as a smooth time evolution of n-dimensional sets. An important question is when a given mean curvature type of flow exists at all times, and thus does not form singularities. A singularity of a flow is a time where one cannot continue the flow, and usually the evolving set experiences topological changes. The work consists of three articles.

In the first article [A], the focus lies on a behavior of a volume preserving mean curvature flow starting nearby a so-called strictly stable set in a three- or four-dimensional flat torus. The contribution of the first article is to show that if the previous flow starts sufficiently close to the strictly stable set in the H^3-sense, then the flow exists at all times and converges, up to a small translation, to the set at an exponential rate. In particular, such a flow does not experience singularities.

The second article [B] and the third article [C] concern generalizations of mean curvature type of flows, so-called flat flows, obtained via the minimizing movement method. Advantages of such a generalization are that it is defined at all times and requires less regularity for a given initial set compared to a mean curvature type of flow. In [B], it is shown that a flat flow of volume preserving mean curvature flow, starting from a bounded set of finite perimeter, has a shape of a finite union of equisized balls with mutually disjoint interiors in the asymptotical sense. The previous result relies on a new quantitative Alexandrov's theorem, also proven in [B]. This theorem says that if a bounded C^2-regular set, with a fixed upper bound on perimeter and a fixed lower bound on volume in an n-dimensional Euclidean space, has a boundary mean curvature close to a constant value in the L^{n-1}-sense, then the set is close to a finite union of equisized balls, with mutually disjoint interiors, in the Hausdorff-sense.

In [C], it is shown that finite unions of n-dimensional tangent balls are not invariant under flat flows of any mean curvature flow with a bounded forcing. This is already proven in the two-dimensional case, so the third article generalizes this result to the higher dimensions.

Tiivistelmä

Tämän väitöksen pääaiheena ovat keskikaarevuustyyppiset virtaukset, erityisesti tilavuuden säilyttävä keskikaarevuusvirtaus. Klassinen virtaus nähdään tässä kontekstissa sileänä n-ulotteisten joukkojen aikaevoluutiona. Tärkeä kysymys on, milloin annettu keskikaarevuustyyppinen virtaus on olemassa kaikkina ajanhetkinä ja ei täten muodosta singulariteetteja. Virtauksen singulariteetti on ajanhetki, josta kyseistä virtausta ei voida jatkaa, ja tavallisesti kehittyvä joukko muuttuu topologisesti. Työ koostuu kolmesta artikkelista.

Ensimmäisessä artikkelissa [A] tarkastelun kohteena on tilavuuden säilyttävän keskikaarevuusvirtauksen käytös ns. ehdottomasti vakaan joukon lähellä kolmi- tai neliulotteisessa litteässä toruksessa. Ensimmäisen artikkelin panos on osoittaa, että jos edellinen virtaus alkaa tarpeeksi läheltä kyseistä joukkoa H^3-mielessä, niin virtaus on olemassa kaiken aikaa ja pientä siirtoa lukuunottamatta lähestyy eksponentiaalisella vauhdilla kohti samaa joukkoa. Erityisesti tällainen virtaus ei muodosta singulariteetteja.

Toinen ja kolmas artikkeli käsittelevät keskikaarevuustyyppisten virtausten yleistyksiä, ns. litteitä virtauksia, jotka saadaan liikkeiden-minimointi-menetelmällä. Tällaisen yleistyksen etuina ovat, että se on määritelty kaikkina ajanhetkinä ja vaatii vähemmän säännöllisyyttä lähtöjoukolta verrattuna keskikaarevuustyyppiseen virtaukseen. Artikkelissa [B] osoitetaan, että litteä virtaus tilavuuden säilyttävälle keskikaarevuusvirtaukselle alkaen rajoitetusta äärellisen perimetrin joukosta muistuttaa asymptoottisesti äärellisen monen samankokoisen pallon yhdistettä siten, että pallojen sisukset ovat pistevieraat. Edellinen tulos nojaa uuteen kvantitatiiviseen Alexandrovin lauseeseen, joka myöskin todistetaan artikkelissa [B]. Tämä lause sanoo, että jos C^2-säännöllisellä joukolla, jolle perimetrillä on kiinnitetty yläraja ja tilavuudelle kiinnitetty alaraja n-ulotteisessa euklidisessa avaruudessa, reunan keskikaarevuus on L^{n-1}-mielessä lähellä vakiota, niin joukko on Hausdorff-mielessä lähellä äärellisen monen samankokoisen pallon yhdistettä siten, että pallojen sisukset ovat pistevieraat.

Artikkelissa [C] osoitetaan, että äärellisen monen toisiaan sivuavan n-ulotteisen pallon yhdiste ei ole invariantti minkään pakotetun keskikaarevuusvirtauksen litteän virtauksen suhteen. Tämä on jo valmiiksi todistettu kaksiulotteisessa tapauksessa, joten kolmas artikkeli yleistää tämän tuloksen korkeampiin ulottuvuuksiin.

INTRODUCTION

This dissertation focuses on certain perturbations of *mean curvature flow* (MCF) in the n-dimensional Euclidean space $\mathbb{R}^n$ and in the *n-dimensional flat torus* $\mathbb{T}^n$. By a *smooth flow* we always mean a smooth evolution of smooth sets $t \mapsto E_t$ with $t \in [0, a)$. An initial set E_0 of a given flow is called an *initial datum* and we say that the flow *starts* from E_0. A (classical) MCF is a smooth flow, for which the motion of the boundary is described by the equation

$$V_t = -H_t, \tag{0.1}$$

where V_t is the *normal velocity* of the flow on ∂E_t at time t and H_t is the scalar *mean curvature* field on ∂E_t associated with the inside-out orientation. MCF and its perturbations are widely used to model different phenomena in material science such as an evolution of grain boundaries in a metal sheet [44]. The perturbation of mean curvature flow we are mainly interested in here is *volume preserving mean curvature flow* (VMCF), sometimes called *surface tension flow.* We say that a smooth flow with finite *perimeter* $P(E_t) = \mathcal{H}^{n-1}(\partial E_t)$ is a VMCF, if the normal velocity V_t at time t obeys the law

$$V_t = \overline{H}_t - H_t. \tag{0.2}$$

Here $\overline{H}_t = \fint_{\partial E_t} H_t \, \mathrm{d}\mathcal{H}^{n-1}$ is the integral average of H_t over ∂E_t. As the name suggests, the volume $|E_t|$ is preserved under VMCF. Both MCF and VMCF decrease perimeter and they are considered to be *gradient flows* of the perimeter functional. In fact, such flows can be seen as an evolutionary counterpart of the classical Euclidean isoperimetric problem.

In study of MCF and its perturbations, *singularities* are a great matter of interest. By a singularity, we mean a moment in time, beyond which one cannot smoothly extend a given flow. In such a situation, topological changes usually occur. Thus, it is natural to ask under which conditions there are no singularities at all. For instance, Gage [21] (in the planar case) and Huisken [27] (in the higher dimensions) prove that a VMCF with a bounded, convex and smooth initial datum does not experience singularities.

The structure of the dissertation can be divided into two separate parts. The first part concerns the stability of VMCF near stationary sets. A set is called *stationary*, if a VMCF starting from it preserves the set unchanged. These sets turn out to be bounded smooth sets with constant boundary mean curvature, such as balls in $\mathbb{R}^n$ or cylinders in $\mathbb{T}^n$. By *stability* near a stationary set E, we broadly understand that any VMCF evolution starting from a set "sufficiently" close to E and of volume $|E|$ exists at all times and stays arbitrarily close to E in some $C^{k,\alpha}$-sense. In particular, such a VMCF does not experience singularities. It follows from the article [18] by Escher-Simonett that stable and stationary sets for VMCF in $\mathbb{R}^n$ are exactly the single balls. Moreover, they prove that a VMCF with an initial datum E_0 of volume $|B|$ sufficiently close to a ball B converges to a small translate of the ball at an exponential rate. Such balls are *asymptotically* stable with respect to VMCF. The main result of [**A**] states that in the flat torus $\mathbb{T}^n$, with $n = 3, 4$, there are essentially more asymptotically stable and stationary sets besides trivial ones such as balls and cylinders.

One way to work around singularities is to introduce a notion of a *weak solution*, see [4], [10], [12], [15], [18] and [35]. The second part of the dissertation considers *flat solutions* for perturbations of MCF constructed via the *minimizing movement scheme.* These are weak solutions to (0.1) and its perturbations such as (0.2). The main advantages of flat solutions are that they exist at all times and have lower regularity requirements for initial data. Indeed, a bounded *set of finite perimeter* suffices. It is well-known that if a VMCF converges to a bounded limit set in the C^2-sense, then the limit must be bounded and have a constant boundary mean curvature. It follows from *Alexandrov's theorem* [8], that such sets in $\mathbb{R}^n$ are exactly finite unions of equisized balls, where the balls have a positive distance to each other. In the second article [**B**] we generalize this result, in a weakened form, to concern flat solutions to (0.2) in $\mathbb{R}^n$ with $n = 2, 3$. We show that a flat solution always converges asymptotically to a finite union of equisized balls with mutually disjoint interiors. A new *quantitative Alexandrov's theorem*, also proven in [**B**], is instrumental in proving the previous result. This result roughly quantifies the following fact for a given bounded C^2-regular set $E \subset \mathbb{R}^n$ with a fixed lower bound on volume and an upper bound on perimeter. If the quantity $\|H_E - \overline{H}_E\|_{L^{n-1}(\partial E)}$ tends to zero, then E becomes arbitrarily close to a tangential union of equisized balls with mutually disjoint interiors. In the last article [**C**] we extend the result proven in [20] from the planar case to the higher dimensions. The main result in [**C**] says that a flat solution to MCF with a bounded *forcing term*, see (5.6), starting from two tangential balls "welds" the balls together for a short time period. From this and [16], one may conclude that a stationary set for a MCF with a positive constant forcing term must always be a finite union of equisized balls with a positive distance to each other.

1. Notations and preliminaries

Our working space is $\mathbb{R}^n$ or the n-dimensional flat torus $\mathbb{T}^n$. We denote them by $\mathbb{K}^n$, if there is no need to make a distinction. Also, our standing assumption through the presentation is that the dimension n is at least two.

Flat torus. We consider the n-dimensional flat torus $\mathbb{T}^n$ as the quotient space $\mathbb{R}^n/\mathbb{Z}^n$. It should be noted that many authors mean by this any quotient space $\mathbb{R}^n/\,\mathbf{L}$, where $\mathbf{L}$ is a discrete subgroup of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^n$. The corresponding quotient map is denoted by q. If $x \in \mathbb{T}^n$ and $v \in \mathbb{R}^n$, then the sum $x + v$ in $\mathbb{T}^n$ is defined as $q(u + v)$, where u is any element from the lattice $q^{-1}(v)$. For a function $f : \mathbb{T}^n \to \mathbb{R}^k$ its lift is $\tilde{f} = f \circ q : \mathbb{R}^n \to \mathbb{R}^k$, which is a unique expression. On the other hand, every $\mathbb{Z}^n$-periodic function $\mathbb{R}^n \to \mathbb{R}^k$ induces a unique function $\mathbb{T}^n \to \mathbb{R}^k$ via the quotient map. Again, for a map $\phi : \mathbb{T}^n \to \mathbb{T}^n$ its lift is any function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\phi \circ q = q \circ \tilde{\phi}$. The topology and smooth structure on $\mathbb{T}^n$ are induced by q. Then $\mathbb{T}^n$ is a smooth and compact manifold (without boundary) and $q : \mathbb{R}^n \to \mathbb{T}^n$ is a smooth universal cover. The topology is metrizable with a compatible metric $d_{\mathbb{T}^n}$ given by the rule

$$d_{\mathbb{T}^n}(x, y) = \min\{|u - v| : u \in q^{-1}(x), v \in q^{-1}(y)\}.$$

The Riemannian metric we consider on $\mathbb{T}^n$ is the pullback of the Euclidean inner product $\langle \,\cdot\,,\,\cdot\,\rangle$ via the quotient map. Then q is a local isometry and hence $\mathbb{T}^n$ can be locally seen just $\mathbb{R}^n$. That is why $\mathbb{T}^n$ is called flat. In particular, the volume element in $\mathbb{T}^n$ is the Euclidean one $\mathrm{d}x$. Then for a Borel set $A \subset \mathbb{T}^n$ its n-dimensional volume $|A|$ is given by $|A| = |q^{-1}(A) \cap \mathcal{D}_n|$, where $\mathcal{D}_n = [0, 1)^n$ is the fundamental domain of q. Similarly, integration with respect to n-dimensional volume $\mathrm{d}x$ can be defined via lifts in the fundamental domain.

In the quotient topology, $f : \mathbb{T}^n \to \mathbb{R}^k$ is continuous if and only its lift is continuous and $\phi : \mathbb{T}^n \to \mathbb{T}^n$ is continuous if and only if it admits a continuous lift $\tilde{\phi}$. Such a $\tilde{\phi}$ is exactly of the form $\tilde{\phi} = L_\phi + u$, where $L_\phi \in M_n(\mathbb{Z})$ is a unique and $u : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and $\mathbb{Z}^n$-periodic function, unique modulo the $\mathbb{Z}^n$-valuable translations. Again, $f : \mathbb{T}^n \to \mathbb{R}^k$ is $C^{k,\alpha}$-regular, with $k \in \mathbb{N} \cup \{\infty\}$ and $0 \le \alpha \le 1$, exactly when its lift $\tilde{f}$ is $C^{k,\alpha}$-regular and $\phi : \mathbb{T}^n \to \mathbb{T}^n$ is $C^{k,\alpha}$-regular (diffeomorphic) exactly when its continuous lifts $\tilde{\phi}$ are $C^{k,\alpha}$-regular (diffeomorphic). Then the differentials $\mathrm{D}f$ and $\mathrm{D}\phi$ can be seen as the derivatives $\mathrm{D}\tilde{f}$ and $\mathrm{D}\tilde{\phi}$ respectively. If a diffeomorphism $\Phi : \mathbb{T}^n \to \mathbb{T}^n$ is close enough to the identity map $\mathrm{id}_{\mathbb{T}^n}$ in the sense that $\sup_{\mathbb{T}^n} d_{\mathbb{T}^n}(\Phi, \mathrm{id}_{\mathbb{T}^n})$ sufficiently small, then there is a unique diffeomorphic lift $\tilde{\Phi}$ such that $\tilde{\Phi} = \mathrm{id} + u$ and $\sup_{\mathbb{T}^n} d_{\mathbb{T}^n}(\Phi, \mathrm{id}_{\mathbb{T}^n}) = \sup_{\mathbb{R}^n} |u|$. In such a case, we set for every $l \in \mathbb{N}$ and $0 \le \alpha \le 1$ for which Φ is $C^{l,\alpha}$-regular

$$\|\Phi - \mathrm{id}_{\mathbb{T}^n}\|_{C^{l,\alpha}(\mathbb{T}^n;\mathbb{T}^n)} = \|\tilde{\Phi} - \mathrm{id}_{\mathbb{R}^n}\|_{C^{l,\alpha}(\mathbb{R}^n;\mathbb{R}^n)}.$$

Regular sets and mean curvature. For a given non-empty set A, its *signed distance function* $\bar{d}_A : \mathbb{K}^n \to [0, \infty)$ is defined by setting

$$\bar{d}_A(x) = \begin{cases} \mathrm{dist}(x, A), & x \in \mathbb{R}^n \setminus A \\ -\mathrm{dist}(x, \mathbb{R}^n \setminus A), & x \in A. \end{cases}$$

In the case $\mathbb{K}^n = \mathbb{T}^n$, we use the previous metric $d_{\mathbb{T}^n}$ to define pointwise distance to a set. Then the lift of $\bar{d}_A$ is the signed distance function of the $\mathbb{Z}^n$-periodic extension $q^{-1}(A)$ in $\mathbb{R}^n$.

We say that an non-empty open set $E \subset \mathbb{K}^n$ is $C^{k,\alpha}$-regular, if $\mathrm{int}(\overline{E}) = E$ and ∂E is a $C^{k,\alpha}$-hypersurface. In this discussion, by a $C^{k,\alpha}$-hypersurface we mean an embedded $C^{k,\alpha}$-regular submanifold (without boundary) of codimension one. We say that E is *smooth*, if $k = \infty$. Note that $E \subset \mathbb{T}^n$ is $C^{k,\alpha}$-regular exactly when its $\mathbb{Z}^n$-periodic extension $q^{-1}(E)$ is a $C^{k,\alpha}$-regular set. Thus, regular sets in $\mathbb{T}^n$ can be (canonically) seen as $\mathbb{Z}^n$-periodic regular sets in $\mathbb{R}^n$. In the case $\mathbb{K}^n = \mathbb{T}^n$, the $n-1$-dimensional volume element at $x \in \partial E$ can be seen as the volume element at $u \in \partial(q^{-1}(E))$, where $u \in q^{-1}(x)$. Thus, an integration of an integrable Borel function f on ∂E with respect to the volume element is effectively integration over the $n-1$-dimensional Hausdorff measure $\mathrm{d}\mathcal{H}^{n-1}$ restricted to ∂E and we denote it by $\int_{\partial E} f \; \mathrm{d}\mathcal{H}^{n-1}$ just like in $\mathbb{R}^n$. When there is no danger of confusion we denote by $\bar{f}$ an integral average of an integrable Borel function on ∂E.

For a $C^{k,\alpha}$-regular set we use the inside-out orientation ν_E on ∂E. Further, we identify the tangent space $T_x\partial E$ of ∂E at x as the orthogonal complement $\langle \nu_E(x)\rangle^{\perp}$. If $f \in C^1(\partial E; \mathbb{R}^k)$, its *tangential derivative* $\mathrm{D}_\tau f(x)$ at $x \in \partial E$ is given by $\mathrm{D}_\tau f(x) = \mathrm{D}f(x)(I - \nu_E(x) \otimes \nu_E(x))$, where f is any local C^1-extension of f. In the case $k = 1$, the *tangential gradient* $\nabla_\tau f(x)$ is given as the dual of $\mathrm{D}_\tau f(x)$ and if $k = n$, then the *tangential divergence* is given by $\mathrm{div}_\tau f(x) = \mathrm{tr}(\mathrm{D}_\tau f(x))$. For $\varphi \in C^2(\partial E)$ its *tangential Hessian* and *tangential Laplace* are defined as $\mathrm{D}^2_\tau \varphi(x) = \mathrm{D}_\tau \nabla_\tau \varphi(x)$ and $\Delta_\tau \varphi(x) = \mathrm{div}_\tau \nabla_\tau \varphi(x)$ respectively.

If $k \geq 2$, then ∂E admits an open tubular neighborhood $\mathcal{N} = \partial E + B(0,r)$, called *regular neighborhood*, such that $\bar{d}_E \in C^{k,\alpha}(\mathcal{N})$, every $y \in \mathcal{N}$ admits a unique distance minimizer or projection $\pi_{\partial E}(y)$ on ∂E and the decomposition $\mathrm{id}_{\mathbb{K}^n} = \pi_{\partial E} + \bar{d}_E \nu_E$ holds in $\mathcal{N}$. Moreover, $\nabla \bar{d}_E = \nu_E$ on ∂E. The *second fundamental form* $B_E(x)$ at $x \in \partial E$ is now a symmetric bilinear form $T_x\partial E \times T_x\partial E \to \mathbb{R}$ given by the rule

$$B_E(x)(u,v) = \langle u, \mathrm{D}_\tau \nu_E(x) v\rangle.$$

and the mean curvature of $H_E(x)$ is the trace of the previous operator, that is, $H_E(x) = \mathrm{div}_\tau \nu_E(x)$. Equivalently, $B_E(x)$ can be associated with the Hessian $\mathrm{D}^2 \bar{d}_E(x)$ and it holds $H_E(x) = \Delta d_E(x)$. While the scalar field H_E on ∂E depends on the choice of orientation (which is in this case the inside-out), the *mean curvature vector field* $\mathbf{H}_E = -H_E \nu_E : \partial E \to \mathbb{R}^n$ is invariant. With help of mean curvature we may write the tangential divergence formula

$$\int_{\partial E} \mathrm{div}_\tau T \; \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial E} H_E \langle T, \nu_E\rangle \; \mathrm{d}\mathcal{H}^{n-1} \tag{1.1}$$

for every compactly supported and C^1-regular vector field $T : \mathbb{K}^n \to \mathbb{R}^n$. If H_E is a constant, then E is called a *critical* set.

If ∂E is smooth, then the space of smooth vector fields $\mathcal{T}(\partial E)$ on ∂E can be identified as the collection $\{X \in C^\infty(\partial E, \mathbb{R}^n) : \langle X, \nu_E\rangle = 0\}$. Again, we consider ∂E as an embededded Riemannian manifold in $\mathbb{K}^n$ equipped with the induced metric. Keeping the previous identifications in mind, we may regard the metric tensor g on ∂E as the restriction of Euclidean inner product to $\langle \nu_E(x)\rangle^\perp$ for every $x \in \partial E$. Then for every $\varphi \in C^\infty(\partial E)$ the tangential gradient $\nabla_\tau \varphi$ corresponds to the gradient $\mathrm{grad}(\varphi)$ induced by the metric g. Further, for φ its *covariant* derivatives $\nabla^k_{\mathrm{co}} \psi$ are defined via the *Riemannian connection* which, in this case, is the *tangential connection.* Pointwise tensor norms are given as usual and further every Sobolev space $W^{k,p}(\partial E)$ is just the $L^p(\partial E)$-norm completions of the space of the k-tuples $(\varphi, \nabla_{\mathrm{co}}\varphi, \ldots, \nabla^k_{\mathrm{co}}\varphi)$, where $\varphi \in C^\infty(\partial E)$. Recall, the standard notation $H^k = W^{2,k}$.

We say that $(E_k)_{k\in\mathbb{N}}$ converges to E in the $C^{k,\alpha}$-sense ($k < \infty$) in $\mathbb{K}^n$, if there is a sequence of $C^{k,\alpha}$-diffeomorphisms $(\Phi_k)_{k\in\mathbb{N}}$ such that $\phi_k(E) = E_k$ and $\|\Phi_k - \mathrm{id}_{\mathbb{K}^n}\|_{C^{k,\alpha}(\mathbb{T}^n;\mathbb{T}^n)} \to 0$ as $k \to \infty$. Again, if E is smooth, then there is $\delta \in \mathbb{R}_+$ such that if $\Phi : \mathbb{K}^n \to \mathbb{K}^n$ is a $C^{k,\alpha}$-diffeomorphism ($k \in \mathbb{N} \cup \{\infty\}$) satisfying $\|\Phi_k - \mathrm{id}_{\mathbb{K}^n}\|_{C^1(\mathbb{T}^n;\mathbb{T}^n)} < \delta$, then there is a unique $\psi \in C^{k,\alpha}(\partial E)$ for which

$$\partial(\Phi(E)) = \{x + \psi(x)\nu_E(x) : x \in \partial E\} \subset \mathcal{N}, \tag{1.2}$$

where $\mathcal{N} = \partial E + B(0,r)$ is a fixed regular neighborhood of ∂E. In particular, it holds $\pi_{\partial E}(x + \psi(x)\nu_E(x)) = x$ and $\bar{d}_E(x + \psi(x)\nu_E(x)) = \psi(x)$. Moreover, for every $l \in \mathbb{N}$ there is a constant

$C \in \mathbb{R}_+$ such that $\|\psi\|_{C^l(\partial E)} \leq C\|\Phi - \mathrm{id}_{\mathbb{K}^n}\|_{C^l(\mathbb{T}^n;\mathbb{T}^n)}$ provided that $l \leq k$. Conversely, if $\psi \in C^{k,\alpha}(\partial E)$ and $\sup|\psi| < r$, then ψ induces an orientation preserving $C^{k,\alpha}$-diffeomorphism Φ such that $\Phi = \mathrm{id}_{\partial E} + \psi\nu_E$ on ∂E and (1.2) holds. We use the notation $E_\psi = \Phi(E)$. We call (1.2) a *graph representation* in the normal direction of ∂E and ψ the *height function* of E_ψ. The orientation ν_{E_ψ} reads as

$$\nu_{E_\psi}(\mathrm{id} + \psi\nu_E) = \frac{\nu_E - A_E(\psi)\nabla_\tau\psi}{\sqrt{1 + |A_E(\psi)\nabla_\tau\psi|^2}} \quad \text{on} \quad \partial E, \tag{1.3}$$

where $A_E(\psi) = (I + \psi B_E)^{-1}$ on ∂E. If $(\Phi_t)_{t\in I}$ is a smoothly parametrized family of smooth diffeomorphisms satisfying $\|\Phi_t - \mathrm{id}_{\mathbb{K}^n}\|_{C^1(\mathbb{T}^n;\mathbb{T}^n)} < \delta$, then the corresponding height functions of ∂E are smoothly paramterized in t.

On embedded smooth flows. Rather than seeing smooth flows as smooth evolutions of smoothly immersed manifolds in $\mathbb{K}^n$ we consider them as smooth evolution of smooth (and bounded) sets and their boundaries. One may imagine how a given initial set smoothly deformes along time while topology is preserved. Then, an obvious way to describe such an evolution is to consider smooth deformations of an initial set under a smoothly parametrized family of diffeomorphisms. By an *admissible family* $(\Phi_t)_{t\in I}$, with a non-degenerate interval I, we mean a map $\Phi \in C^\infty(\mathbb{K}^n \times I; \mathbb{K}^n)$ such that

- $\Phi_t := \Phi(\,\cdot\,, t)$ is a (smooth) diffeomorphism for every $t \in I$ and Φ_{t_0} is the identity map with some $t_0 \in I$.
- For every compact set $K \subset I$ the set of exceptional values $\{(x,t) \in \mathbb{K}^n \times K : \Phi(x,t) \neq x\}$ is pre-compact, that is, every slice Φ_t belongs to $\mathrm{Diff}_0(\mathbb{K}^n)$, the space of compactly supported diffeomomorphisms isotopic to $\mathrm{id}_{\mathbb{K}^n}$.

An admissible family Φ can be seen as a smooth path in the space $\mathrm{Diff}_0(\mathbb{K}^n)$. Note that the corresponding family of inverses $t \mapsto [\Phi(\,\cdot t)]^{-1}$ is also admissible. In the case $\mathbb{K}^n = \mathbb{T}^n$, the latter requirement is redundant. We come up with the following definition.

Definition 1.1. For a given bounded and smooth initial set $E_0 \subset \mathbb{K}^n$ and $0 < a \leq \infty$ a map $[0,a) \to \mathcal{P}(\mathbb{K}^n)$, $t \mapsto E_t$, is a smooth flow starting from E_0, also denoted by $(E_t)_{t\in[0,a)}$, if for every $t \in [0,a)$ there exist an interval $t \in I \subset [0,a)$, I open in $[0,a)$, and an admissible family $(\Phi_s)_{s\in I}$, with $\Phi_t = \mathrm{id}$, such that $\Phi_s(E_t) = E_s$ for every $s \in I$. Then we say that $(\Phi_s)_{s\in I}$ is a *local parametrization* of the flow at t. If $(\Phi_t)_{t\in[0,a)}$ is an admissible family, with $\Phi_0 = \mathrm{id}$ and $\Phi_t(E_0) = E_t$ for every $t \in [0,a)$, then we say that $(\Phi_t)_{t\in[0,a)}$ is a *global parametrization* of the flow.

It turns out that every smooth flow $(E_t)_{t\in[0,a)}$ admits a global parametrization $(\Phi_t)_{t\in[0,a)}$. A smooth flow in $\mathbb{T}^n$ can be canonically lifted to a smooth evolution of $\mathbb{Z}^n$-periodic sets in $\mathbb{R}^n$. We say that a smooth flow $(E_t)_{t\in[0,a)}$ is *autonomous*, if it admits an autonomous global parametrization $(\Phi_t)_{t\in[0,a)}$ meaning that there is a smooth vector field $X : \mathbb{K}^n \to \mathbb{R}^n$ satisfying $\partial_t\Phi_t = X \circ \Phi_t$ for every $t \in [0,a)$. Since $(\Phi_t)_{t\in[0,a)}$ is admissible, it follows that X must be compactly supported.

Usually, the parameter a is not emphasized and hence one simply denotes $(\Phi_t)_{t\geq 0}$ and $(E_t)_{t\geq 0}$. Note that we also use abbreviations $\nu_t = \nu_{E_t}$, $B_t = B_{E_t}$, $H_t = H_{E_t}$ and so forth. For each $t \in [0,a)$ we define a map $\Phi^t \in C^\infty(\mathbb{K}^n \times [0, a-t); \mathbb{K}^n)$ by setting $\Phi^t(x,s) = \Phi(\Phi_t^{-1}(x), t+s)$. Then Φ^t is an admissible family and defines a smooth flow $E^t_s := \Phi^t_s(E_t)$, $0 \leq s < a-t$, starting from E_t. This expression has the following *semi-group property*

$$E^t_s = E_{t+s} \quad \text{for every } 0 \leq t < a \text{ and for every } 0 \leq s < a - t,$$

which means that we can stop the flow $(E_t)_{t\geq 0}$ at the time t and start it again by using the local parametrization $(E^t_s)_{s\geq 0}$.

Obviously, there is no unique parametrization via admissible families of diffeomorphisms for a smooth flow and, on the other hand, we are interested in an evolution of set in whole rather than trajectories of single points. This motivates us to search an intrinsic way to describe such an evolution. A natural approach to the issue is to consider how the boundaries evolve over time. For fixed time t let us consider the behavior ∂E_s nearby ∂E_t when s is close to t. Let $(\Phi_s)_{s\in I}$ be any local parametrization of the flow at t. Since now $\|\Phi_s - \mathrm{id}\|_{C^1(\mathbb{K}^n;\mathbb{K}^n)} \to 0$ as $s \to t$ and $E_s = \Phi_s(E_t)$, then there are an interval $t \in I' \subset I$, I' open in $[0,a)$, and $\psi \in C^\infty(\partial E_t \times I')$ such that $\partial E_s \subset \mathcal{N}$ for every $s \in I'$, where $\mathcal{N}$ is a regular neighborhood of ∂E_t, and ψ provides a unique graph representation for ∂E_s in the normal direction of ∂E_t. Recall, this means $\partial E_s = \{x + \psi_s(x)\nu_t(x) : x \in \partial E_t\}$ for every $s \in I'$, where $\psi_s = \psi(\,\cdot\,, s)$, and $\pi_{\partial E}(x + \psi_s(x)\nu_t(x)) = x$ for every $x \in \partial E_t$. Therefore, the evolution of the height function ψ_s on ∂E_t purely determines the evolution of the flow nearby time t. Now $\psi_t = 0$, so at a given point $x \in \partial E_t$ the quantity $V_t(x) = \partial_s \psi_s(x)\big|_{s=t}$ tells us the rate of evolution of the boundary in the normal direction $\nu_t(x)$ at time t. That is, the *normal velocity* of the flow at the spatial-time coordinate (x,t). On the other hand, we have

$$\psi_s \circ \pi_{\partial E_t} \circ \Phi_s = \bar{d}_{E_t} \circ \Phi_s \quad \text{on} \quad \partial E_t$$

so differentiating the identity with respect to s and evaluating it at $s = t$ yields

$$V_t = \partial_s \psi_s\big|_{s=t} = \langle \partial_s \Phi_s\big|_{s=t}, \nu_t \rangle \quad \text{on } \partial E_t.$$

This expression is clearly independent of the choice of local parametrization. Thus, if $(\Phi_t)_{t\in I}$ is any local parametrization of the flow, then using the re-parametrization $(\Phi(\Phi_t^{-1}, s))_{s\in I}$ for time $t \in I$ gives us

$$V_t = \langle \partial_s \Phi_s\big|_{s=t} \circ \Phi_t^{-1}, \nu_t \rangle \quad \text{on } \partial E_t. \tag{1.4}$$

The quantity V_t tells us the speed of ∂E_t to its normal direction at time t and, further, the knowledge of a vector field $V_t \nu_t$ on ∂E_t at every time t entirely determines the evolution of the flow. For a given flow $(E_t)_{t\geq 0}$ we call V_0 the *initial normal velocity* of the flow.

Usually, a typical application, where the notion of normal velocity appears, is computing how energy integrals, associated with potentials varies, along a given flow. If $\eta \in C^1(\mathbb{K}^n \times [0,a))$ describes a (possibly time dependent) potential, then the energy $\int_{E_t} \eta(\,\cdot\,, t)\,\mathrm{d}x$ varies at the rate

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{E_t} \eta(\,\cdot\,,t)\,\mathrm{d}x = \int_{E_t} \partial_t \eta(\,\cdot\,,t)\,\mathrm{d}x + \int_{\partial E_t} \eta(\,\cdot\,,t) V_t \,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.5}$$

The special case $\eta \equiv 1$ gives us the formula of the *first variation of volume* along the flow

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = \int_{\partial E_t} V_t \,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.6}$$

Therefore, $(E_t)_{t\geq 0}$ is volume preserving exactly when V_t has a vanishing integral over ∂E_t at every time. Further, for the surface energy $\int_{\partial E_t} \eta(\,\cdot\,,t)\,\mathrm{d}\mathcal{H}^{n-1}$, with $\eta \in C^1(\mathbb{K}^n \times [0,a))$, one may compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial E_t} \eta(\,\cdot\,,t)\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial E_t} \partial_t \eta(\,\cdot\,,t) + \langle \nabla \eta(\,\cdot\,,t), \nu_t \rangle V_t + \eta(\,\cdot\,,t) H_t V_t \,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.7}$$

Again, substituting $\eta \equiv 1$ yields the formula for the *first variation of perimeter* along the flow, that is

$$\frac{\mathrm{d}}{\mathrm{d}t} P(E_t) = \int_{\partial E_t} V_t H_t \,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.8}$$

Sets of finite perimeter. In this case, we consider the setting only in $\mathbb{R}^n$ and generally refer to [36]. Recall that a measurable set $E \subset \mathbb{R}^n$ is a set of finite perimeter provided that

$$P(E) = \sup\left\{ \int_E \operatorname{div} T \,\mathrm{d}x : T \in C_0^1(\mathbb{R}^n;\mathbb{R}^n),\ |T| \leq 1 \right\} < \infty.$$

Then there exists a unique, finite and $\mathbb{R}^n$-valued Radon measure μ_E, called the *Gauss-Green measure* of E, such that $\int_E \operatorname{div} T\,\mathrm{d}x = \int_E \langle T, \mathrm{d}\mu_E \rangle$ for every $T \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$. Here $P(E)$ is the total variation of μ_E called the *perimeter* of E. *De Giorgi's structure theorem* gives us the representation $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E$, where $\partial^* E$ is the *reduced boundary* of E and $\nu_E : \partial^* E \to \partial B(0,1)$ the *measure theoretical outer unit normal.* Then $P(E) = \mathcal{H}^{n-1}(\partial^* E)$ and the divergence theorem takes the form

$$\int_E \operatorname{div} T \,\mathrm{d}x = \int_{\partial E} \langle T, \nu_E \rangle \,\mathrm{d}\mathcal{H}^{n-1}.$$

Naturally, if E is C^1-regular, then the reduced boundary agrees with the topological one and the measure theoretical outer unit normal coincides with the classical inside-out orientation. Motivated by (1.1) we say that $H_E \in L^1(\partial^* E; \mathcal{H}^{n-1})$ is a *distributional* or *weak mean curvature* of E, if for every $T \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$

$$\int_{\partial^* E} \mathrm{div}_\tau T \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* E} H_E \langle T, \nu_E \rangle \, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.9}$$

where the tangential divergence $\mathrm{div}_\tau T$ is given similarly as earlier, now in terms of the measure theoretical outer unit normal. Again, if H_E is constant, we say that E is *weakly critical.*

The notion of flows via admissible families can be generalized to the bounded set of finite perimeters. In this case, Definition 1.1 makes perfectly sense. The notion of normal velocity is defined as in (1.4), now in terms of reduced boundary and measure theoretic outer unit normal, and the equations (1.5) and (1.6) remain valid. Again, if $E \subset \mathbb{R}^n$ is a bounded set of finite perimeter and $(\Phi_t)_{t \geq 0}$ is an admissible family with $\Phi_0 = \mathrm{id}$, then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\Phi_t(E)) = \int_{\partial^* E_t} \mathrm{div}_\tau \left(\partial_t \Phi_t \big|_{t=0} \right) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{1.10}$$

2. Existence and gradient flow structure

We shortly cover familiar existence results for MCF and VMCF over a short time period in the case of a smooth and bounded initial datum. This is usually known as *short time existence*, Furthermore, we introduce formal *gradient flow structure* these flows posses and also take a quick look at stationary sets, i.e., the sets which are invariant under the flows.

Short time existence. In the case of MCF, short time existence is well-known in broader context of immersed manifolds without boundary, see for instance [30] or [37, Theorem 1.5.1] for more careful discussion. The short time existence for VMCF is also proven in the case of smooth, compact and connected hypersurfaces of $\mathbb{R}^n$ without boundary, see [18]. Naturally, the same methods can be applied in proving a short time existence for MCF/VMCF as an evolution of bounded and smooth sets in $\mathbb{K}^n$.

The general strategy is to employ graph representation to reduce an evolution of a smooth flow locally as a PDE of a height function on a fixed refence boundary. Let us consider a smooth flow $(E_t)_{t \geq 0}$ and a smooth and bounded set E in $\mathbb{K}^n$. Suppose that the set E_0 is sufficiently close to E in the C^1-sense. Then, by the earlier discussion, there is $a > 0$ such that we may write $E_t = E_{\psi_t}$ with a unique height parametrization $\psi \in C^\infty(\partial E \times [0, a))$, recall the shorthand $\psi_t = \psi(\,\cdot\,, t)$. Moreover, we may assume that ψ induces a smooth family of smooth diffeomorphism $(\tilde{\Phi}_t)_{t \in [0,a)}$ from $\mathbb{K}^n$ to $\mathbb{K}^n$ such that $E_t = \tilde{\Phi}_t(E)$, $\Phi_t = \mathrm{id} + \psi_t \nu_E$ on ∂E and $\{(x, t) \in \mathbb{K}^n \times [0, a) : \tilde{\Phi}_t(x) \neq x\}$ is bounded. Then $\Phi_t = \tilde{\Phi}_t \circ \tilde{\Phi}_0^{-1}$, $t \in [0, a)$, is a local parametrization of the flow so by substituting this into (1.4), recalling (1.3) and denoting $\psi_0 = \varphi$ we have ψ satisfying the initial value problem

$$\begin{cases} \partial_t \psi(x, t) = \sqrt{1 + |A_E(x, \psi_t) \nabla_\tau \psi_t|^2} \; V_t(x + \psi_t \nu_E(x)), \\ \psi(x, 0) = \varphi(x), \end{cases} \tag{2.1}$$

where $A_E(x, \psi_t) = (I + \psi_t B_E(x))^{-1}$. Conversely, if $\psi \in C^\infty(\partial E \times [0, a))$ satisfies $\psi_0 = \varphi$ and $\sup |\psi| < r$, where $r \in \mathbb{R}_+$ is a maximal radius such that $\partial E + B(0, r)$ is a regular neighborhood of ∂E, then ψ induces a smooth flow $(E_{\psi_t})_{t \in [0,a)}$ with a parametrization $(\Phi_t)_{t \in [0,a)}$ as before and ψ satisfies (2.1). Thus, (2.1) describes completely the dynamics of the flow, starting nearby E, as a motion of a graph surface over ∂E for a short time period. When dealing with a flow driven by curvature, i.e., the normal velocity depends on the curvature of evolving boundary, the term $V_t(x + \psi(x, t)\nu_E(x))$ contains higher order covariant derivatives of φ.

Assume that the previous flow $(E_t)_{t\geq 0}$ is a MCF. We have for sufficiently small t $\sup|\psi_t| < r$, where r is given as before, so starting from the graph representation and *local level set characterization* for mean curvature, see [1, p.10], one may compute

$$\begin{aligned} V_t(x+\psi\nu_E(x)) &= -H_{E_{\psi_t}}(x+\psi_t\nu_E(x)) \\ &= -\langle Q_E(x,\psi_t,\nabla_{\mathrm{co}}\psi_t), \nabla^2_{\mathrm{co}}\psi_t\rangle - S_E(x,\psi_t,\nabla_{\mathrm{co}}\psi_t), \end{aligned}$$

where $Q_E : \partial E \times (-r,r) \times \mathbb{R}^n \to T^2_0(\partial E)$ is a symmetric smooth section, i.e. $Q_E(x,b,z) \in T^2_0(T_x\Sigma)$ is a symmetric bilinear form for every $(x,b,z) \in \partial E \times \mathbb{R} \times \mathbb{R}^n$, with Q_E being locally elliptic and $Q_E(x,0,0) = -g_{\partial E}(x)$, and $S_E : \partial E \times (-r,r) \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $S_E(x,0,0) = H_E(x)$. This turns (2.1) into a quasilinear parabolic PDE on ∂E, which admits a unique smooth solution for a short time ε. For a detailed discussion, see [37, Appendix A]. The key idea here is to linearize the PDE locally. The complexity of the problem increases in the case of VMCF, where

$$V_t(x+\psi_t\nu_E(x)) = \overline{H}_{E_{\psi_t}} - H_{E_{\psi_t}}(x+\psi_t(x)\nu_E(x))$$

Hence, the corresponding PDE has an integral term in its principal part, which requires an extra work in [18]. Note that when $\|\psi_t\|_{C^1(\partial E)} \to 0$, then $\partial_t\psi_t$ asymptotically resembles $\Delta_\tau\psi_t - H_E$ for MCF and $\Delta_\tau\psi_t$ for VMCF. Thus, these flows are heat equation like evolutions. Recalling the semi-group property we obtain the following well-known existence result.

Theorem 2.1 (Short time existence). Let $E \subset \mathbb{K}^n$ be a smooth and bounded set. There is a unique $(E_t)_{t\in[0,T)}$ MCF (respectively VMCF) starting from E such that if $(F_t)_{t\in[0,a)}$ a MCF (resp. VMCF) starting from E, then $a \leq T$ and $F_t = E_t$ for every $t \in [0,a)$. We call T a *maximal lifetime* of the flow.

Unless otherwise stated, we mean by a MCF (resp. VMCF) $(E_t)_{t\geq 0}$ a corresponding maximal evolution and by a lifetime its maximal lifetime. These solutions are locally stable in sense of local perturbations of an initial set $E \subset \mathbb{K}^n$ meaning that a small and continuous perturbation of the initial set varies the flow continuously in a short time interval. In terms of graph surface representation, a small perturbation means a smooth initial datum φ on ∂E sufficiently close to 0 in some norm. To be more precise, we have the following for a smooth and bounded set $E \subset \mathbb{K}^n$, see [37, Theorem A.3.1.] and [18]. For $0 < \alpha < 1$ there are positive constants $\delta = \delta(E,\alpha) \in \mathbb{R}_+$ and $\varepsilon = \varepsilon(E,\alpha) \in \mathbb{R}_+$ such that if $\varphi \in C^\infty(\partial E)$, with $\|\varphi\|_{C^{1,\alpha}(\partial E)} \leq \delta$, then there is a MCF (resp. VMCF) $(E_t)_{t\geq 0}$ starting from E_φ, with a lifetime at least ε, and $(\varphi,t) \mapsto \partial E_t$ is continuous in the $C^{1,\alpha}$-sense in $(B_{C^{1,\alpha}(\partial E)}(0,\delta) \cap C^\infty(\partial E)) \times [0,\varepsilon)$.

Of course, MCF and VMCF can be treated similarly in lower regularity settings such as for C^2-sets or $C^{1,\alpha}$-sets, see for instance [18]. The parabolic nature of these flows provides instant smoothing. In certain cases, it is possible to start a MCF evolution from an unbounded set in $\mathbb{R}^n$. For instance, a MCF evolution starting from a smooth set in $\mathbb{T}^n$ corresponds a $\mathbb{Z}^n$-periodic MCF evolution in $\mathbb{R}^n$.

Gradient flow structure. One motivation for the notion of MCF can be seen coming from an attempt to decrease perimeter in a smooth and local way. Our starting point is a classical problem where we want to decrease energy in a C^1-potential field $u : \mathbb{R}^n \to \mathbb{R}$ starting from a point x by using a local minimizing strategy. This leads to the following gradient flow

$$\begin{cases} \gamma(0) = x, \\ \gamma'(t) = -\nabla u(\gamma(t)). \end{cases} \tag{2.2}$$

We would like consider a similar problem for the perimeter functional P on the set of smooth and bounded sets of $\mathbb{K}^n$. While it is possible to make a rigorous approach by introducing so called *shape spaces*, see [9], we keep our discussion at heuristic level for sake of presentation. If we consider smooth and bounded sets of $\mathbb{K}^n$, which are mutually diffeomorphich with diffeomorphisms from the class $\mathrm{Diff}_0(\mathbb{K}^n)$, as elements of an abstract configuration space, then smooth flows are a natural choice to represent admissible paths here.

Now (1.5) and (1.7) give us a vague analogy between normal velocity and "time derivative" of a path as well as between the function space $C^\infty(\partial E)$ and a "tangent space" at given element E.

Further, if we regard the L^2-inner product of $C^\infty(\partial E)$ as a "metric tensor" g_E on the tangent space we may (formally) write

$$(\mathrm{d}P)_E(V_0) = \frac{\mathrm{d}}{\mathrm{d}t} P(E_t)\Big|_{t=0} = \int_{\partial E} V_0 H_E \,\mathrm{d}\mathcal{H}^{n-1} = g_E(V_0, H_E)$$

for every smooth flow $(E_t)_{t\geq 0}$ starting from E. Now for every $\varphi \in C^\infty(\partial E)$ it is easy to construct a smooth flow starting from E, with the initial velocity φ, so we have $(\mathrm{d}P)_E(\varphi) = g_E(\varphi, H_E)$ for every $\varphi \in C^\infty(\partial E)$. From this we infer that $\nabla P(E) = H_E$ at E with respect to the "metric". Thus, a VMCF starting from E can bee seen solving the problem

$$\begin{cases} E_0 = E, \\ \frac{\mathrm{d}}{\mathrm{d}t} E_t = -\nabla P(E_t). \end{cases} \tag{2.3}$$

This heuristics motivates why MCFs are usually said to be gradient flows of perimeter.

We may also consider the same problem with a volume constraint

$$\begin{cases} E_0 = E, \\ |E_t| = |E|, \\ \frac{\mathrm{d}}{\mathrm{d}t} E_t = -\nabla P(E_t), \end{cases} \tag{2.4}$$

in a similar setting. Now, a MCF starting from E does not generally preserve volume and hence is not a solution. Let $(E_t)_{t\geq 0}$ be a volume preserving flow starting from E. Recalling that the condition $|E_t| = |E|$ for every t implies $\int_{\partial E_t} V_t \,\mathrm{d}\mathcal{H}^{n-1} = 0$ we obtain from (1.8)

$$\frac{\mathrm{d}}{\mathrm{d}t} P(E_t) = \int_{\partial E_t} V_t (H_t - \overline{H}_t) \,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.5}$$

where $\overline{H}_t = \fint_{\partial E_t} H_t \,\mathrm{d}\mathcal{H}^{n-1}$. Then we consider the bounded smooth sets diffeomorphich to each other (via the class $\mathrm{Diff}_0(\mathbb{K}^n)$), with a fixed volume, as a submanifold of the previous setting, where the smooth volume preserving flows are the admissible paths and a tangent space at each element E is identified as the space $\tilde{C}^\infty(\partial E)$. Since now for every $\varphi \in \tilde{C}^\infty(\partial E)$ there is a smooth volume preserving flow $(E_t)_{t\geq 0}$ starting from E with the initial velocity φ, then $\nabla P(E) = H_E - \overline{H}_E$ in this submanifold and a VMCF starting from E is a solution to (2.4) in this configuration. Thus, VMCFs are formally gradient flows of perimeter in the context of stationary volume.

While the previous discussion is heuristic, both MCF and VMCF do satisfy the dissipation equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(E_t) = -\int_{\partial E_t} V_t^2 \,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.6}$$

when they exist. This follows directly from (1.8) and (2.5). Note an analogy to a solution γ of (2.2) satisfying $(u \circ \gamma)' = -\langle \gamma', \gamma' \rangle$. We see later that (2.6) or rather its weak notions turn out to be useful tools when analyzing global behavior of these flows. Note that Theorem 2.1 and the stability property for a short time period can be seen saying that the problems (2.3) and (2.4) are well-posed in a local sense. By comparison, (2.2) is locally well-posed, if we require u to be $C^{1,1}_{\mathrm{loc}}$-regular. That is, for every x there is an open neighborhood U of x and $\varepsilon \in \mathbb{R}_+$ such that for every $y \in U$ there is a unique maximal solution $\theta_y : [0, t_y) \to \mathbb{R}^n$ to (2.2), with $t_y \geq \varepsilon$, and the local flow (not to be confused with Definition 1.1) $U \times [0, \varepsilon) \to \mathbb{R}^n$, $(y, t) \mapsto \theta_y(t)$, is continuous.

Stationary sets. In the context of gradient systems equilibrium points are naturally an essential topic. We say that a smooth and bounded $E \subset \mathbb{K}^n$ is stationary with respect to MCF (resp. VMCF), if a corresponding solution $(E_t)_{t\geq 0}$ starting from E is a constant solution, i.e. $E_t = E$. This is equivalent with the corresponding normal velocity V_t being identically zero at every time and hence the semi-group property and uniqueness imply that the solution has an infinite lifetime. Then E is stationary if and only if the corresponding gradient of perimeter related to the problem vanishes at E, which is analogous to the equlibrium points of (2.2) being exactly the critical points of ∇u. On the other hand, if for every admissible flow $(E_t)_{t\geq 0}$ starting from E it holds that $\frac{\mathrm{d}}{\mathrm{d}t} P(E_t)|_{t=0} = 0$ in the sense of the setting (2.3) or (2.4), then the corresponding gradient of perimeter must vanish at E. For MCF the set E is stationary if and only if its boundary has zero mean curvature, that is, ∂E is a *minimal surface* (we call it a minimal boundary). Again, for VMCF the condition is equivalent to E having a constant boundary mean curvature, i.e., E being critical. Now, a stationary set with respect to MCF is always stationary with respect to VMCF.

It is easy to see that there is no bounded set of $\mathbb{R}^n$ with a minimal boundary. Therefore, MCF does not have bounded stationary sets in $\mathbb{R}^n$ and (2.6) is always negative. Thus, it follows from the already mentioned Alexandrov's theorem [8] (see also [49]) that the only bounded and critical sets in $\mathbb{R}^n$ are the finite unions of balls of equal size with a positive distance to each other. In the flat torus $\mathbb{T}^n$ there are more (bounded) sets, with a constant boundary mean curvature, besides balls such as cylinders and *lamellae* (regions between parallel hyperplanes) to mention the most trivial structures. In particular, lamellae are the simplest stationary sets for MCF in $\mathbb{T}^n$. A suitable solid of revolution having an *unduloid* as a boundary, when $n \geq 3$, is a simple example of a non-trivial set, with a constant boundary mean curvature, in $\mathbb{T}^n$. Interesting examples of sets with a minimal boundary in $\mathbb{T}^3$ provide sets which are the $\mathbb{Z}^3$-quotients of smooth sets of $\mathbb{R}^3$ with a *triply periodic* minimal surface as boundary. For instance, the classical *Schwarz P surface* and the *lidinoid* [33] are examples of such boundaries. Overall, minimal hypersurfaces in the ambient dimension $n = 3$ are a well-studied subject [39]. Although there are quite few concrete examples of nontrivial embedded minimal hypersurfaces in the higher dimensions $n \geq 4$, we remark that a generalization of Schwarz P surface can be constructed in the dimension $n = 4$, see [13].

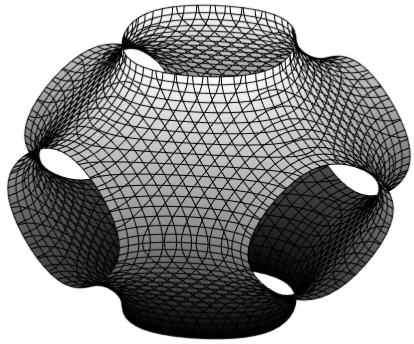

FIGURE 1. A fundamental part of Schwarz P surface

3. Global-in-time behavior and singularities

In this section, we make a short survey of the most fundamental results concerning global behavior of MCF and VMCF, mainly in the context of $\mathbb{R}^n$. In particular, we are looking for cases when such flows exist at all times. We say that a MCF or a VMCF with a finite lifetime has a singularity at the end of its lifetime. Topological changes usually take place at such a moment. As already discussed, the both flows behave stable for a short time period. Now we are interested in stability (in a global sense) and vaguely say a smooth and bounded set E to be stable with respect to MCF, if the corresponding solution, starting from any slight perturbation of E, has an infinite lifetime and stays near E. In the case of VMCF, we also assume that an initial datum is of volume $|E|$.

Mean curvature flow. Global behavior and analysis of singularities of MCF are extensively studied. Besides a smoothing effect, MCF has other properties known for general parabolic solutions such as a comparison principle, see [37, Thm 2.2.1]. This says that if $E, F \subset \mathbb{K}^n$ are bounded and smooth sets with $\operatorname{dist}(\partial E, \partial F) > 0$ and $(E_t)_{t\geq 0}$ and $(F_t)_{t\geq 0}$ are MCFs starting from E and F respectively, then the function $t \mapsto \operatorname{dist}(\partial E_t, \partial F_t)$ is non-decreasing as long as it is defined. In particular, $E \subset\subset F$ implies $E_t \subset\subset F_t$. If we purely consider an evolution of a (compact) embedded hypersurface under MCF, then embeddedness is preserved [37, Prop. 2.2.7]. Together with the comparison principle, the previous property allows us quite freely to apply many properties of MCF, stated for immersed surfaces, in the context of bounded and smooth sets.

A standard example of MCF is to consider how a ball $B_0 = B(x_0, r_0) \subset \mathbb{R}^n$ behaviors under such a motion. In this case, the corresponding MCF evolution consists of concentric balls $E_t = B(x_0, r(t))$, where the radius evolves according to $r(t) = \sqrt{r_0^2 - 2(n-1)t}$ with the lifetime $T = r_0^2/(2(n-1))$. Thus, we have that a MCF evolution starting from a ball $B(x_0, r)$ eventually collapses to x_0 just by (strictly) decreasing the radius within a finite time. Therefore, it follows from the comparison principle that for every smooth and bounded set $E \subset \mathbb{R}^n$ a maximal MCF starting from E has a finite lifetime and hence exprериences a singularity. This is in line with the observation that there is no (bounded) stationary set for MCF in $\mathbb{R}^n$.

Although a ball collapses to its center under MCF, the evolution is just a smooth evolution of concentric balls. Hence rescaling them around the center, in a way that perimeter is preserved, yields the original ball. This kind of behviour generalizes to the category of convex sets. Huisken proves in [27] that a bounded, smooth and uniformly convex set in $\mathbb{R}^n$, $n \geq 3$, shrinks under MCF smoothly to a single point within a finite time and the rescaled flow (boundary area is kept constant) converges to a ball in the C^k-sense with any $k \in \mathbb{N}$. Result by Gage-Hamilton [22] states the same result for a bounded, convex and smooth planar set. Further, Grayson in [23] generalizes [22] to cover every bounded planar set with a closed and smooth curve as a boundary. These results translates directly to the same dimensional flat torus $\mathbb{T}^n$ and correspond $\mathbb{Z}^n$-periodic MCFs in $\mathbb{R}^n$.

In the previous cases, evolving boundary asymptotically converges to a sphere until collapsing to a single point. It is also possible that something more complex happens. Indeed, there are dumbbell shaped sets in $\mathbb{R}^3$ such that a MCF starting from a such dumbbell eventually pinches the neck of the dumbbell into two cusps with their tips touching each other, see [24]. Again, an *Angenent's torus* is an example of a toroidal set in $\mathbb{R}^3$ which shrinks homothetically under MCF before collapsing to a point, see [5]. The singularities of MCF are further classified as *Type I* and *Type II* singularities, see [37, Def. 3.2.1]. Fundamental tools in analyzing singularities of MCF are *Huisken's monotonicity formula* [29] and its generalizations.

In the case of unbounded sets of $\mathbb{R}^n$, a lifetime of a corresponding MCF evolution may be infinite. For instance, Ecker-Huisken [17] prove that for a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ there is a MCF, with an infinite lifetime, starting from the subgraph of u. Moreover, if the solution does not diverge to infinity, then it must converge to a half-space. A simple example of a self-similar set under planar MCF is the epigraph of a function $(-\pi/2, \pi/2) \to \mathbb{R}$, $x \mapsto -\log(\cos(x))$. In this case, the corresponding MCF is just a vertical motion at a constant speed. This example introduced in [44] is called *Grim Reaper* by Grayson [23].

Since a MCF starting from a bounded and smooth set always experiences a singularity in $\mathbb{R}^n$, the stability does not make sense here. On the contrary, the notion becomes relevant in $\mathbb{T}^n$. For instance, by combining [17] with the comparison principle we obtain the following stability result in $\mathbb{T}^n$. If $E \subset \mathbb{T}^n$ is a lamella and $\psi_0 \in C^\infty(\partial E)$ has $\sup_{\partial E} |\psi_0|$ small enough, then the MCF starting from E_{ψ_0} has an infinite lifetime and converges to a lamella E_∞. Moreover, $E_\infty \to E$ in the Hausdorff-sense as $\sup_{\partial E} |\psi_0| \to 0$.

Volume preserving mean curvature flow. Compared to MCF, a VMCF evolution too decreases perimeter but additionally preserves volume. Thus, such an evolution cannot collapse to a single point. Again, a VMCF evolution in $\mathbb{T}^n$ corresponds a smooth $\mathbb{Z}^n$-periodic evolution in $\mathbb{R}^n$, where the integral average in (0.2) is taken over the intersection of the boundary and the fundamental domain $\mathcal{D}_n$. As we discussed in the previous section, VMCF has (bounded) stationary sets also in $\mathbb{R}^n$ and those sets are exactly a finite unions of equisized balls, where the balls have a mutually positive distance. On the other hand, if a VMCF evolution $(E_t)_{t \geq 0}$ converges to a limit set $E_\infty \subset \mathbb{K}^n$ in the C^2-sense, one may show that the flow has an infinite lifetime and E_∞ is stationary. If we replace the C^2-convergence with mere Hausdorff-convergence, then the limit set may fail to be even C^1-regular, see [20, Thm 1.4].

Although a VMCF evolution may exist at all times, there is a trade-off between the volume preserving property and non-local characteristics induced by the integral average in (0.2). This makes VMCF somewhat more rigid compared to a MCF evolution. To elaborate this, let us consider behavior of a VMCF evolution when an initial set is a finite union balls $E = \cup_{i=1}^N B(x_i, r_{i,0})$, where the balls have mutually a positive distance, that is, $|x_i - x_j| > r_{i,0} + r_{j,0}$ whenever $i \neq j$. Then the VMCF $(E_t)_{t\geq 0}$ starting from E is of the form $E_t = \cup_{i=1}^N B(x_i, r_i(t))$, where $r_i(0) = r_{i,0}$, $|x_i - x_j| > r_i + r_j$ for $i \neq j$ and r_is evolve according to the system

$$r_i' = \frac{n-1}{r_i} \frac{\sum_{j\neq i} r_j^{n-2}(r_i - r_j)}{\sum_j r_j^{n-1}}.$$

If the balls are equisized, nothing happens and E is stationary with respect to VMCF. If $N \geq 2$ and the balls are not of equal size, then a minimal radius start shrinking to zero within a finite time. Correspondingly, the maximal radius increases constantly. The evolution reaches its singularity, when the minimal radius reaches zero or possibly before that two balls with increasing radius touch each other. Besides the fact that singularities are possible for VMCF, we make the following observations from the previous example. First, VMCF enjoys no comparison principle in general. Second, neither embeddedness is necessarily preserved in the context of compact (and embedded) hypersurfaces. This means that it is possible for a VMCF evolution to drive boundary self-intersecting and after that to exist as an evolution of immersed boundary. The phenomenon may occur even when an initial boundary is connected, see [38]. Third, a stationary set may be "unstable" in small perturbations, as in the previous example, where even a slight disparity between the size of balls results the corresponding VMCF experiencing a singularity.

Naturally, this raises a question: When does a VMCF evolution not experience a singularity and have a limit? As in the case of MCF, a behavior of VMCF in the convex setting is settled. Gage proves in [21] that a VMCF starting from a bounded, convex and smooth planar set, has an infinite lifetime, preserves convexity and converges to a ball of the initial volume (area) in the C^k-sense with any $k \in \mathbb{N}$. Again, Huisken proves in [27] the same result for any VMCF starting from a bounded, smooth and uniformly convex set in $\mathbb{R}^n$ with $n \geq 3$. In the both cases, the convergence of the corresponding VMCF $(E_t)_{t\geq 0}$ has an exponential rate in the C^k-topology with any $k \in \mathbb{N}$ and, in particular, the absolute value of the normal velocity $|H_t - \overline{H}_t|$ on ∂E_t decays exponentially fast.

Escher-Simonett prove in [18] (see also [6]), using *center manifold analysis*, that a single ball is always stable in the $C^{1,\alpha}$-sense. To be more precise, suppose that a reference ball $B \subset \mathbb{R}^n$ and $0 < \alpha < 1$ are given. Then for any $\psi_0 \in C^\infty(\partial B)$ with a sufficiently small $C^{1,\alpha}(\partial B)$-norm the VMCF starting from B_{ψ_0} has an infinite lifetime and converges exponentially fast to some ball B_∞ in the C^k-sense with any $k \in \mathbb{N}$. Moreover, the limit ball B_∞ converges to the reference ball B as the $C^{1,\alpha}(\partial B)$-norm of ψ tends to zero. Note that here the initial set is not even assumed to share the same volume with B. Recalling our earlier example, we see that single balls are only stationary sets such that VMCF behaviors stable nearby them. We also note that Li gives in [34] an alternative condition for a compact and immersed hypersurface such that the VMCF starting from it converges to a sphere in $\mathbb{R}^n$. Like in the case of MCF, the previous results hold also in $\mathbb{T}^n$.

4. Perimeter minimizers and asymptotical stability of VMCF

We continue to investigate the notion of stability given heuristically in the previous section and restrict our focus on VMCF asking how it behaves near a stationary set. In other words, we want to understand how the system (2.4) behaves near an equilibrium point. For instance, a union of multiple equisized balls in $\mathbb{R}^n$ as we discussed earlier gives us an example of "unstable" stationary set with respect to VMCF. On the contrary, the main result of [18] roughly says that a single ball in $\mathbb{R}^n$ is a stable set under VMCF. Again, if a VMCF starts sufficiently close to a ball B and the flow is of volume $|B|$, then the corresponding convergence to a translate of B always happens at an exponential rate. Hence, one may regard B as an asymptotically stable set for VMCF. Thus, we want to find a reasonable condition for a stationary set $E \subset \mathbb{K}^n$ such that a VMCF starting close to E behaves similarly to [18].

For a moment, let us consider the system (2.2). If $u : \mathbb{R}^n \to \mathbb{R}$ is a C^2-regular potential and x_0 is a critical point of u with $\mathrm{D}^2 u(x_0) > 0$, i.e., the corresponding Hessian is positive-definite, then x_0 is a strict local minimum point of u and any solution of (2.2) starting sufficiently close to x_0 converges to x_0 at an exponential rate. In particular, x_0 is an asymptotically stable point of

u. Motivated by this simple example, we would like to find an analogous condition $\mathrm{D}^2P(E) > 0$ for a stationary set $E \subset \mathbb{K}^n$. The first problem is how to express a heuristic notion $\mathrm{D}^2P(E)$ in rigorous terms. If $u : \mathbb{R}^2 \to \mathbb{R}$ is a C^2-regular function and x_0 is a critical point of u, then for every autonomous and smooth path $\gamma : [0, a) \to \mathbb{R}^n$, that is to say $\gamma' = X \circ \gamma$ with some smooth vector field X, starting from x_0 it holds

$$(u \circ \gamma)''(0) = \langle \gamma'(0), \mathrm{D}^2u(x_0)\gamma'(0)\rangle. \tag{4.1}$$

If $E \subset \mathbb{K}^n$ is a stationary set, then for any autonomous, smooth and volume preserving flow $(E_t)_{t\geq 0}$ starting from E the corresponding *second variation* of the perimeter reads as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}P(E_t)\bigg|_{t=0} = \int_{\partial E} |\nabla_\tau V_0|^2 - \|B_E\|^2 V_0^2 \, \mathrm{d}\mathcal{H}^{n-1}, \tag{4.2}$$

where V_0 is the initial normal velocity of the flow on ∂E. This gives rise to a symmetric form $\mathrm{Q}P(E) : \tilde{C}^\infty(\partial E) \times \tilde{C}^\infty(\partial E) \to \mathbb{R}$ by setting

$$\mathrm{Q}P(E)[\varphi_1, \varphi_2] = \int_{\partial E} \langle \nabla_\tau \varphi_1, \nabla_\tau \varphi_1\rangle - \|B_E\|^2 \varphi_1\varphi_2 \, \mathrm{d}\mathcal{H}^{n-1}.$$

We further consider the quadratic form $\partial^2 P(E) : \tilde{C}^\infty(\partial E) \to \mathbb{R}$ associated with perimeter given by $\partial^2P(E)[\varphi] = \mathrm{Q}(E)[\varphi, \varphi]$. Comparing to (4.1), $\partial^2P(E)$ can be heuristically seen as the functional $\varphi \mapsto \langle \varphi, \mathrm{D}^2P(E)\varphi\rangle$. Again, (4.1) gives us hint how to define a positive semi-definite condition for $\mathrm{D}^2P(E)$, i.e., $\mathrm{D}^2P(E) \geq 0$. Indeed, a stationary set $E \subset \mathbb{K}^n$ is called *volume preserving stable*, or shortly *v.p. stable*, if $\partial^2P(E)[\varphi] \geq 0$ for every $\varphi \in \tilde{C}^\infty(\partial E)$. Simple examples of such sets are lamellae and single balls and cylinders in $\mathbb{T}^n$ as well as single balls in $\mathbb{R}^n$. It is easy to see that lamellae are only possible v.p. stable sets with multiple boundary components. Thus, single balls are only v.p. stable sets of $\mathbb{R}^n$ in the category bounded and smooth sets. In $\mathbb{T}^3$, there are v.p. stable sets with a triply periodic minimal surface such as the Schwarz P-surface [46] as a boundary. In fact, the v.p. stable sets are completely classified in $\mathbb{T}^3$, see [45].

While the heuristic notion $\mathrm{D}^2P(E) \geq 0$ directly translates to $\partial^2P(E)$ being non-negative, defining the condition $\mathrm{D}^2P(E) > 0$ has some obstacles, mainly due to a fact that $\partial^2P(E)$ cannot be strictly positive in $\tilde{C}^\infty(\partial E) \setminus \{0\}$. Namely, if we consider the finite dimensional subspace $T(\partial E) = \{\langle \nu_E, p\rangle : p \in \mathbb{R}^n\}$ of $\tilde{C}^\infty(\partial E)$, called the *infinitesimal translations* of ∂E then, recalling that H_E is constant it is straightforward to compute that every element $\varphi \in T(\partial E)$ satisfies

$$-\Delta_\tau \varphi = \|B_E\|^2\varphi \quad \text{on} \quad \partial E. \tag{4.3}$$

Thus, $\partial^2P(E)$ vanishes on this subspace. The space $T(\partial E)$ consists of the normal velocities of the local translates of E along a vector, that is $E_t = E + tp$ with $p \in \mathbb{R}^n$ for a short time period. Since perimeter is invariant under such operations, then (4.3) can be also seen valid via (4.2). Hence, we want to neglect these directions. Taking into account this issue the following definition arises.

Definition 4.1 (Strictly stable set). Let $E \subset \mathbb{K}^n$ be a bounded and v.p. critical set. We say that E is *v.p. strictly stable*, if there is $c \in \mathbb{R}_+$ such that

$$\partial^2P(E)[\varphi] \geq c\|\varphi\|^2_{L^2(\partial E)} \quad \text{for every} \quad \varphi \in \tilde{C}^\infty(\partial E) \cap T^{\perp, L^2}(\partial E), \tag{4.4}$$

where $T^{\perp,L^2}(\partial E)$ is the L^2-orthogonal complement of $T(\partial E)$ on ∂E.

Note that (4.4) in the definition can be relaxed to the condition $\partial^2P(E)[u] > 0$ for every $u \in \tilde{H}^1(\partial E)\setminus T^{\perp,L^2}(\partial E)$, where $\tilde{H}^1(\partial E) = \{u \in H^1(\partial E) : \int_{\partial E} u \, \mathrm{d}\mathcal{H}^{n-1} = 0\}$ is the H^1-completion of $\tilde{C}^\infty(\partial E)$ and $\partial^2P(E)$ is considered to be a functional on it, see the proof of [2, Lemma 3.6]. By the definition every v.p. strictly stable set is also v.p. stable. Conversely, using standard elliptic estimates one may argue that a v.p. stable set $E \subset \mathbb{K}^n$ is v.p. strictly stable if and only if the infinitesimal translations are the only classical solutions to (4.3) in $\tilde{C}^\infty(\partial E)$, i.e., the kernel of the operator $-(\Delta_\tau + \|B_E\|)$ on $\tilde{C}^\infty(\partial E)$ is $T(\partial E)$. This is trivially true for lamellae and well-known for single balls and cylinders. Again, this also holds true for a set having the Schwarz P-surface as boundary in $\mathbb{T}^3$, see [46] and [25, Lemma 17].

If Definition 4.1 can be seen as a formalization of the notion $D^2P(E) > 0$, then does it behave analogously to a situation $D^2u(x_0) > 0$? By the previous, we mean whether a v.p. strictly stable set is a unique (strict) local minimizer of perimeter, up to translates, when volume is unchanged. In the case of $\mathbb{R}^n$, a v.p. strictly stable set E must be a single ball which is, up to translates, a strict global minimizer of P among the set of finite perimeter of volume $|E|$. In the case of $\mathbb{T}^n$, [2] generalizes the result in [41] by showing that in $\mathbb{T}^n$ a v.p. strictly stable set E is, up to translates, an isolated local minimizer in the L^1-sense among the set of finite perimeter of the same volume. To be more precise, there are $C, \delta \in \mathbb{R}_+$ such that

$$P(F) \geq P(E) + C\alpha^2(E, F)$$

for every set of finite perimeter $F \subset \mathbb{T}^n$ satisfying $|F| = |E|$ and $\alpha(E, F) < \delta$, where the translates neglecting "distance" α is given by $\alpha(E, F) = \min_{p \in \mathbb{R}^n} |E \Delta (F + p)|$.

In this light, the main result of [18] says that up to translates the v.p. strictly stable sets in $\mathbb{R}^n$, i.e., the single balls, are asymptotically stable points for VMCF in the $C^{1,\alpha}$-sense with an exponential convergence rate. The main result of [**A**] says that the v.p. strictly stable sets in low dimensional flat tori are asymptotically stable in the H^3-sense. The result reads as follows.

Theorem 4.2 (**Main result A**)**.** Let $n = 3, 4$ and $E \subset \mathbb{T}^n$ be a v.p. strictly stable set. There exists a positive constant $\delta = \delta(E) \in \mathbb{R}_+$ such that the following hold. If a smooth set $E_0 \subset \mathbb{T}^n$ satisfies $|E_0| = |E|$ and $E_0 = E_{\psi_0}$, where $\psi_0 \in C^\infty(\partial E)$ and $\|\psi_0\|_{H^3(\partial F)} \leq \delta_0$, then the VMCF $(E_t)_{\geq 0}$ starting from E_0 has an infinite lifetime and converges to a translate $F + p$ of F exponentially fast in the $W^{2,5}(\partial(F + p))$-sense with uniform constants. Moreover, $|p| \to 0$ as $\|\psi_0\|_{H^3(\partial E)} \to 0$.

Note that the result well-known in $\mathbb{T}^2$, since only admissible sets are lamellae and single balls. As we discussed previously, there are non-trivial v.p. strictly stable sets in $\mathbb{T}^3$. The proof of Theorem 4.2 draws extensively upon the methods from [3]. This is also why the result is carried out only in low dimensions, mainly due to heavy dependency on Sobolev interpolations. Thus, an open question is whether it is possible to prove a similar result in higher dimensions $n \geq 5$. The mentioned paper [3] is interesting in its own right. Here it is shown that the v.p. strictly stable sets in $\mathbb{T}^3$ are asymptotically stable with respect to the *surface diffusion flow* and the modified *Mullins-Sekerka flow* in a similar manner as in Theorem 4.2. These flows can also be seen as gradient flows of perimeter.

5. Minimizing movement scheme and flat solutions

As we observed, MCF and VMCF evolutions have two major issues. First, an initial datum of a classical solution must have rather high regularity. The second issue is possible occurrence of singularities, although for a MCF it is possible to do *surgery* to bypass singularities (before a final termination) when an initial boundary is for instance *two-convex* [31]. We would like to find a generalized notion of a solution which is well-defined for initial sets with low regularity, such as sets of finite perimeter, and is defined at all-times. We call such generalizations weak solutions. Now our framework is purely restricted to $\mathbb{R}^n$.

Weak solutions for MCF and its perturbations have gathered lot of attention in recent years. The first step is due to Brakke in [10], where a weak solution to (0.1) is developed in the context of *varifolds*. Again, a weak solution to (0.1) can be obtained by using so called *minimial barriers* introduced by De Giorgi [15]. Since MCF satisfies a comparison principle, a *viscosity type* or *level-set* solution to (0.1) is also possible, see independent works of Evans-Spruck [19] and Chen-Giga-Goto [12]. The previous viscosity solutions are unique, exist at all times and agree with classical MCF evolutions when latter exist. However, their regularity may be very rough. For instance, a level set of such a solution may have a positive volume. This phenomenon is called *fattening*. While a level set method in the spirit of the previous works is not directly applicable for VMCF due to lack of a reasonable comparison principle, a modified notion of viscosity solution to (0.2) is possible, see [32].

There is yet another alternative which is so called minimizing movement method developed independently by Almgren-Taylor-Wang [4] and Luckhaus-Sturzenhecker [35]. This method for solids based on *discretization of time* and *selection principle* provides a viable approach from our perspective, since we are interested in an evolution of n-Hausdorff dimensional sets. In particular, this approach yields weak solutions to (0.2), as we will see later.

To motivate the minimizing movement approach, let us again consider the problem (2.2) and assume that $u : \mathbb{R}^n \to \mathbb{R}$ is C^1-regular and bounded from below. For a discrete time $h \in \mathbb{R}_+$ we define an approximative solution $\gamma_h : [0, \infty) \to \mathbb{R}^n$ by the following recursive method. We set first $\gamma_h(t) = x$ for $0 \le t < h$ and assuming γ_h is defined in $[kh, (k+1)h)$ for $k = 0, 1, 2, \dots$ we set $\gamma_h(t) = \gamma_h((k+1)h)$, for $t \in [(k+1)h, (k+2)h)$, where $\gamma_h((k+1)h)$ minimizes the functional

$$\mathbb{R}^n \to \mathbb{R}, \quad y \mapsto u(y) + \frac{1}{2h}|y - \gamma_h(kh)|^2. \tag{5.1}$$

Here the assumption $\inf u > -\infty$ is needed to find a minimizer. For any sequence $(h_i)_{i\in\mathbb{N}}$ of discrete times converging to zero, the sequence $(\gamma_{h_i})_{i\in\mathbb{N}}$ converges, up to extracting a subsequence, to a classical solution of (2.2) and the convergence is uniform on the compact intervals $[0, T]$. The method resembles the classical *Euler's method* used to approximate gradient descent (2.2), since now for every $t \ge 0$

$$\frac{\gamma_h(t+h) - \gamma_h(t)}{h} = -\nabla u(\gamma_h(t+h)). \tag{5.2}$$

This is kind of approach is used widely when one wants to generalize gradient flows to metric context, see [7] for a basic reference. Once we have understood the gradient flow structure of MCF, the minimizing movement method for MCF resembles the previous example.

5.1. **Minimizing movement method for MCF.** Now we consider a heuristic problem given by (2.3) and want to imitate the previous method. The potential u is naturally replaced with the perimeter functional P but how to take account of the distance penalization term in (5.1)? If $E \subset \mathbb{R}^n$ is a smooth and bounded set and $(E_t)_{t\ge 0}$ is a smooth flow starting from E, then for a short time period we may write $E_t = E_{\psi_t}$ with a smooth height parametrization $\psi_t = \psi(\,\cdot\,, t)$ on ∂E and for every $x \in \partial E$ we have $\bar{d}_E(x + \psi_t(x)\nu_E(x)) = \psi(x,t) - \psi(x,0)$. If $y = \psi(x,h) \in \partial E_h$ is a displacement of $x \in \partial E$ after small time period $h \in \mathbb{R}_+$, then recalling (2.1) and using uniform convergence on compact sets we may approximately write

$$\frac{y - x}{h} = V_h(y)\nu_{E_h}(y) \quad \text{and} \quad V_h(y) = \frac{\bar{d}_E(y)}{h}.$$

Again, if $(E_t)_{t\ge 0}$ is a MCF evolution, then recalling the heuristical interpretation $V_h = -\nabla P(E_t)$ and comparing to (5.2) leads one to replace the time penalization term in (5.1) with the *degenerate* L^2*-distance* $\frac{1}{h}\int_{E\Delta E_h} |\bar{d}_E|\,\mathrm{d}x$. Thus, we minimize the functional

$$\mathcal{P}_h(E)[F] = P(F) + \frac{1}{h}\int_{E\Delta F} |\bar{d}_E|\,\mathrm{d}x. \tag{5.3}$$

From now on, we assume that E is a bounded set of finite perimeter and $\mathcal{P}_h(E)$ acts on the bounded sets of finite perimeter. In general, we base our discussion on the papers [11], [20], [35] and [43]. Since modifications of E (even in a L^1-negligible set) may lead to drastic value changes of the functional $F \mapsto \int_{E\Delta F} |\bar{d}_E|\,\mathrm{d}x$, we have to use a suitable convention for the sets. For instance, one may always assume that ∂E agrees with the closure of the reduced boundary $\partial^* E$, see [36, Prop 12.19]. We denote the collection of such admissible sets by $\mathcal{A}$. Now, we define for a small fixed $h \in \mathbb{R}_+$ an approximative solution $(E_t^h)_{t\in[0,\infty)} \subset \mathcal{A}$ to (0.1), with a given initial datum $E \in \mathcal{A}$, by using a recursive procedure as in the previous example.

- We set $E_t^h = E$ for every $0 \le t < h$.
- Assuming that E_t^h is defined in the interval $[kh, (k+1)h)$ for a given $k = 0, 1, 2, \dots$ we set $E_t^h = E_{(k+1)h}^h$, for $t \in [(k+1)h, (k+2)h)$, where $E_{(k+1)h}^h$ is any minimizer of $\mathcal{P}_h(E_{kh}^h)$.

If $E_{t_0}^h$ is empty, it is natural to assume every set E_t^h to be empty for every $t \ge t_0$. Thus, we use the convention $\bar{d}_E = \infty$ in (5.3), whenever E is empty, to ensure such a behavior. Then we take a quick look at some properties of approximative sequences. First, the procedure is well-posed, i.e., there is a minimizer in $\mathcal{A}$ for each $\mathcal{P}_h(E_{kh}^h)$, although it remains an open question whether such a selection is unique. Since for each $t \ge h$ the set E_t^h is a minimizer of $\mathcal{P}_h(E_{t-h}^h)$, the set E_t^h

enjoys higher regularity. Namely, one can show that E_t^h is so called (Λ_0, r)-*minimizer*, see [36]. Hence it follows from [36, Thm 26.5 and Thm 28.1] that $\partial^* E_t^h$ is a relatively open and $C^{1,\alpha}$-regular part of ∂E, with any $0 < \alpha < 1/2$, and the dimension of the singular part $\partial E_t^h \setminus \partial^* E_t^h$ is at most $n-8$. Moreover, if E_t^h is non-empty, it has a distributional mean curvature $H_{E_t^h}$ satisfying the *Euler-Lagrange equation*

$$H_{E_t^h} = -\frac{\bar{d}_{E_{t-h}^h}}{h} \tag{5.4}$$

on $\partial^* E_t^h$ in the distributional sense. This is analogous to (5.2). Since the right hand side is Lipschitz-continuous, then by the standard elliptic estimates we obtain $C^{2,\alpha}$-regularity for $\partial^* E_t^h$. In particular, $(E_t^h)_{t \geq h}$ consists of $C^{2,\alpha}$-regular set when $n \leq 7$. Moreover, the perimeter $P(E_t^h)$ is non-increasing in time and $(E_t^h)_{t \geq 0}$ satisfies a weak notion of the dissipation equality (2.6) for MCF. To be more precise, for every $h \leq t \leq T-h$ we have the *dissipation inequality*

$$\int_t^T \int_{\partial^* E_t^h} H_{E_t^h}^2 \, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t \leq C \left(P(E_{t-h}^h) - P(E_T^h) \right) \tag{5.5}$$

with some uniform $C \in \mathbb{R}_+$ provided that h is sufficiently small.

Now, like in the motivating example, we are looking for a convergence of approximative solutions. If $((E_t^{h_i})_{t \geq 0})_{i \in \mathbb{N}}$ with $h_i \to 0$ is a sequence of approximative solutions, then there is indeed a parametrized family $(E_t)_{t \geq 0} \subset \mathcal{A}$ such that, by possibly passing to subsequence, $E_t^{h_i} \to E_t$ in the L^1-sense as $i \to \infty$. Moreover, $(E_t)_{t \geq 0}$ satisfies the following properties.

(i) Boundedness: there is a ball $B(0,R)$ containing the solution $(E_t)_{t \geq 0}$.

(ii) Uniform Hölder continuity in the L^1-sense with respect to time: there is $C \in \mathbb{R}_+$ such that $|E_t \Delta E_s| \leq C|t-s|^{\frac{1}{2}}$ for every $t, s > 0$. If E is open or closed, then the previous holds for every $t, s \geq 0$.

(iii) Perimeter $P(E_t)$ is bounded by the perimeter of the initial set E.

We call a family $(E_t)_{t \geq 0}$, obtained via the previous procedure, a *flat solution* to (0.1) (or (2.3)) starting from E. Note that it always hold $E_0 = E$. Thus, we have covered a simple minimizing movement procedure for MCF among the bounded sets of finite perimeter. Further, it can be shown that, like a MCF evolution, a flat solution to (0.1) terminates within a finite time meaning that $E_t = \emptyset$ after some time $t_0 \in \mathbb{R}_+$. Also, if a classical solution exists, then a corresponding flat solution agrees with it. However, in general, there are many open questions concerning the flat solutions. We list a couple of them.

- Is a flat solution $(E_t)_{t \geq 0}$ to (0.1) starting from $E \in \mathcal{A}$ unique? This is closely related to the selection procedure.
- Does there exist any partial regularity for $(E_t)_{t \geq 0}$? In particular, does each set E_t have a distributional mean curvature at almost every time t?

5.2. **Flat MCF with a forcing term and stationary sets.** The previous procedure can be generalized for vast amount of perturbations of MCF. In [20] authors consider flat solutions of MCF with a time-dependent forcing term. That is, a smooth flow $(E_t)_{t \geq 0}$ driven by a motion of the form

$$V_t = -H_t + f(t), \tag{5.6}$$

where $f : [0, \infty) \to \mathbb{R}$ is a smooth and bounded function called a *forcing term*. The function f represents an external time-dependent force. Note that in [35] authors already consider a similar setting where a forcing term is a spatial function. Now a flat solution to (5.6) with a given initial datum $E \in \mathcal{A}$ is essentially obtained by the same procedure as earlier. Also, smoothness of f can be relaxed to mere measurability. The major difference is that, when defining the approximating sequence, the set E_{k+1}^h in the recursive step is now taken as a minimizer of the functional

$$F \mapsto \mathcal{P}_h(E_{kh}^h)[F] - \fint_{[kh,(k+1)h]} f \, \mathrm{d}t \, |F|. \tag{5.7}$$

Note that this procedure is a generalization of the previous minimizing movement method for MCF. The approximative solutions share same regularity results and (5.4) transfroms into

$$H_{E^h_t} = -\frac{\bar{d}_{E^h_{t-h}}}{h} + \fint_{[kh,(k+1)h]} f \ \mathrm{d}t.$$

Again, a flat solution to (5.6) starting from E is obtained as earlier via approximative solutions and it satisfies the properties (i) - (iii) though in a weakened form, see [20, Prop 2.3].

In particular, a flat solution to (5.6) with a bounded and measurable forcing term f starting from $E \in \mathcal{A}$ exists at all times. Thus, one may ask how such a solution behaviors globally. In $\mathbb{R}^2$, it is proven that if the forcing term f is asymptotical to a positive constant in the L^2-sense, then a flat solution converges, up to a translation, to a finite union of equisized balls with mutually disjoint interiors, see [20, Thm 1.3]. In the previous case, the convergence means that E_t is arbitrarily Hausdorff-close to such a union of balls provided that t is big enough, and the unions of balls may be time-dependent but the number and size of balls are not. To make the setting more interesting, the generalization Alexandrov's theorem proved in [16] states that the sets of finite perimeter with a distributional constant mean curvature, i.e., weakly critical sets are exactly the finite union of equisized balls with the interiors being disjoint to each other. Then the distributional mean curvature is $(n-1)/r$, where r is the shared radius. In this light, the previous convergence result is weakly analogous to limit behavior of VMCF although it remains an open question whether there is a real convergence to a fixed union of balls. Also, a higher dimensional version of [20, Thm 1.3] is an open question.

An important special case of (5.6) is obtained, when $f = \Lambda$ with $\Lambda \in \mathbb{R}_+$. That is, we consider a motion with a constant forcing

$$V_t = -H_t + \Lambda, \tag{5.8}$$

Recalling (1.6) and (1.8), such a flow can be seen as a gradient flow of the energy $\mathcal{E}_\Lambda$ on the bounded and smooth sets of $\mathbb{R}^n$ given by $\mathcal{E}_\Lambda(E) = P(E) - \Lambda|E|$. We say that a bounded and smooth set is a critical point of the energy $\mathcal{E}_\Lambda$, if $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\Lambda(E_t)\big|_{t=0} = 0$ for every smooth flow starting from E. Compare the previous to equilibrium points of (2.3). Then the critical points of the energy are exactly the stationary sets of (5.8), i.e., the bounded and critical set with the mean curvature Λ. Again, by Alexandrov's theorem, these are exactly the finite unions of ball, where balls have radius $(n-1)/\Lambda$ and a positive distance to each other.

The energy $\mathcal{E}_\Lambda$ can be directly extended to the class $\mathcal{A}$. Then we analogously say that $E \in \mathcal{A}$ is an critical point of the energy $\mathcal{E}_\Lambda$, if for every admissible family $(\Phi_t)_{t\geq 0}$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\Lambda(\Phi_t(E))\bigg|_{t=0} = 0. \tag{5.9}$$

Using (1.9) and (2.5) this is equivant to E having Λ as a distributional mean curvature. By the earlier discussion such sets are exactly the finite union of balls, where balls have radius $(n-1)/\Lambda$ and their interiors are disjoint to each other. Now one may ask how such sets are related to the stationary sets under the flat solutions to (5.8). To be more precise, we say that set $E \in \mathcal{A}$ is stationary under the flat solutions to (5.8), if for every flat solution $(E_t)_{t\geq 0}$ to (5.8), with the initial datum E, it holds $\sup_{t\geq 0} |E \Delta E_t| = 0$, see [20, Def 3.1]. First with help of [16, Cor 2] (see [**C**, Lemma 3.6] for clarity) one may show that such stationary sets are critical in the sense of (5.9). On the other hand, it is quite straightforward to check that a union of balls, having radius $(n-1)/\Lambda$ and a positive distance to each other, i.e., the stationary sets of (5.8) in the classical sense is always stationary under the flat solutions to (5.8). This leaves us to consider the finite unions of balls of radius $(n-1)/\Lambda$, where the interiors are disjoint to each other and at least two balls are tangential. The answer is that such unions are not stationary under the flat solutions. Thus, a set $E \in \mathcal{A}$ is stationary under the flat solutions to (5.8) if and only if it is critical in the classical sense with the mean curvature Λ and the previous question is closed.

How is the negative answer concluded? We first restrict ourselves to consider a union of two tangential balls of equal radius $E = B(x_1, r) \cup B(x_2, r)$. In the dimension $n = 2$, [20, Thm 1.1] states that any flat solution $(E_t)_{t\geq 0}$ to (5.6) with a bounded and measurable forcing term f

starting from E immediately "welds" the balls together. To be more precise, there are concentric balls $B(x_1, \eta r)$ and $B(x_2, \eta r)$, $0 < \eta < 1$, such that for a short time period E_t contains a simply connected and dumbbell-shaped set which, in turn, contains the shrunk balls. Moreover, we have a lower estimate for the excess $|E_t \setminus E| \geq ct^3$. The third article [**C**] of this dissertation generalizes [20, Thm 1.1] to any dimension $n \geq 2$.

Theorem 5.1 (**Main result C**)**.** Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a union of two tangential balls $B(x_1, r)$ and $B(x_2, r)$. Let $(E_t)_{t\geq 0}$ be a flat solution to (5.6) with a measurable forcing term f bounded by $C_0 \in \mathbb{R}_+$, starting from E. There exist positive numbers δ, c_1 and c_2 depending only on n, r and C_0 such that for every $t \in (0, \delta)$ the set E_t contains a dumbbell shaped simply connected set which, in turn, contains the balls $B(x_1, r - c_1 t)$, $B(x_2, r - c_1 t)$ and $B((x_1 + x_2)/2, c_2 t)$.

Now using a weak comparison principle for approximative solutions, see [20, Prop 2.1], the same phenomenon can be seen true for any $E \in \mathcal{A}$ containing a union of two tangential balls. This closes the question concerning relation between the weakly critical (with the constant Λ) sets and the stationary sets under the flat solutions to (5.8). We remark that a similar kind of fattening phenomenon for viscosity solutions to (5.6) with $f = f(x, t)$ in $\mathbb{R}^n$ is already established in [26].

5.3. **Flat solutions for VMCF and quantitative Alexandrov's theorem.** In this last subsection, we investigate how minimizing movement scheme is carried out in the case of VMCF. While the term involving integral average in (0.2) is time-dependent, it is not a prescribed function and hence we cannot use the previous approach for (5.6). We focus on the presentation of [43]. Now for a given non-empty initial set $E \in \mathcal{A}$ and $\Lambda \in \mathbb{R}_+$ the approximative solutions to (0.2) with a sufficiently small $h \in \mathbb{R}_+$ are defined in the same way as for (0.1), except now a set $E^k_{(k+1)h}$ in the recursive step is chosen as a minimizer of the functional

$$F \mapsto \mathcal{P}_h(E^h_{kh}) + \frac{\Lambda}{\sqrt{h}}||F| - |E||,$$

where $\mathcal{P}_h(E^h_{kh})$ is defined as in (5.3). The weak volume penalization term here emulates the integral average in (0.2).

Note that in [43] authors use fixed choice $\Lambda = |E| = 1$ but clearly the same arguments hold in the general case. Again, such approximative solutions $(E^h_t)_{t\geq 0}$ to (0.2) share the same regularity results as previously and satisfy a similar Euler-Lagrange equation compared to (5.4), see [43]. Moreover, the perimeter in the family $(E^h_t)_{t\geq 0}$ is non-increasing and the dissipation inequality

$$\int_t^T \int_{\partial^* E^h_t} |\overline{H}_{E^h_t} - H_{E^h_t}|^2 \, \mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t \leq C \left(P(E^h_{t-h}) - P(E^h_T)\right) \tag{5.10}$$

holds for every $h \leq t \leq T - h$ with a uniform constant $C \in \mathbb{R}_+$. The flat solutions to (0.2) are obtained similarly as clusters of the approximative solutions. According to [43, Thm 2.2] a flat solution $(E_t)_{t\geq 0} \subset \mathcal{A}$ to (0.2) starting from $E \in \mathcal{A}$ has a constant volume $|E|$ (so it preserves volume like its classical counterpart) and satisfies the properties (ii) and (iii) listed for the flat solutions to (0.1). Again, for every $T \in \mathbb{R}_+$ the subfamily $(E_t)_{t\in[0,T]}$ is bounded. Further, if we assume that $n \leq 7$ and $P(E^{h_k}_t) \to P(E_t)$ in the distributional sense, i.e,

$$\lim_k \int_0^T P(E^{h_k}_t) \, \mathrm{d}t = \int_0^T P(E_t) \, \mathrm{d}t \quad \text{for every} \quad T \in \mathbb{R}_+, \tag{5.11}$$

then $t \mapsto P(E_t)$ is non-increasing beyond a negligible set, almost every E_t has a distributional mean curvature and $(E_t)_{t\geq 0}$ satisfies (0.2) in a distributional sense, see [43, Thm 2.3]. This of course gives rise to many new questions, not least whether the condition (5.11) can be relaxed.

Since a flat solution to (0.2) exists at all times and preserves volume, its behavior at infinity becomes a matter of interest. Recalling the asymptotical convergence result [20, Thm 1.3] in the planar case, we expect a similar kind of asymptotical behavior. Indeed, the second article [**B**] of this dissertation provides such a result for the flat solutions to (0.2) in the dimensions $n = 2, 3$.

Theorem 5.2 (**Main result B1**)**.** Assume that $E \subset \mathbb{R}^n$ with $n = 2, 3$ is in the class $\mathcal{A}$ of volume $|B_1|$ and let $(E_t)_{t\geq 0}$ be a flat solution to (0.2) starting from E. There is $N \in \mathbb{N}$ such that the following holds: for every $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for every $t \geq T_\varepsilon$ there are points $x_1, \ldots, x_N$, which may depend on time with $|x_i - x_j| \geq 2r$ for $i \neq j$, and $r = N^{-\frac{1}{n}}$ such that for $F_t = \bigcup_{i=1}^N B_r(x_i)$ it holds

$$\sup_{x \in E_t \Delta F_t} d_{\partial F_t}(x) \leq \varepsilon.$$

The assumption $|E| = |B_1|$ is purely for convenience, in the general case we have $r = (|E|/|B_1|)^{\frac{1}{n}} N^{-\frac{1}{n}}$. Again, one may ask if the flat solution convergences to a fixed union of balls. In fact, the result established in [42] states that any approximative solution $(E_t^h)_{t\geq 0}$ to (0.2) converges to a finite union of equisized balls with a positive distance to each other. Further, the convergence happens at an exponential rate in any C^k-sense. Since now there is no known control on the limit set in the parameter h and, on the other hand, a classical VMCF may converge to tangential balls, the previous convergence result is not at least directly applicable in the case of the flat solutions.

Obviously, the proof of Theorem 5.2 involves approximative solutions due to low known regularity for the flat solutions themselves. The proof relies on the dissipation inequality (5.10) and a new quantitative Alexandrov's theorem also established in the second article. With help of (5.10), one may conclude that if $(E_t^h)_{t\geq 0}$ is an approximative solution to (0.2) starting from E and $h \in \mathbb{R}_+$ is sufficiently small, then there is a sequence $(E^h_{t_l^h})_{l\in\mathbb{N}}$ such that $t_l^h \in [l^2, (l+1)^2]$ and

$$\int_{\partial^* E^h_{t_l^h}} |\overline{H}_{E^h_{t_l^h}} - H_{E^h_{t_l^h}}|^2 \, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t \leq C l^{-\frac{1}{2}}. \tag{5.12}$$

This is where the new quantitative Alexandrov's theorem from [B] comes into play.

Theorem 5.3 (**Main result B2**). Let $E \subset \mathbb{R}^n$ be a C^2-regular set such that $P(E) \leq C_0$ and $|E| \geq 1/C_0$. There are positive constants $q = q(n) \in (0,1]$, $C = C(C_0, n)$ and $\delta = \delta(C_0, n)$ such that if $\|H_E - \lambda\|_{L^{n-1}(\partial E)} \leq \delta$ for some $\lambda \in \mathbb{R}$, then $1/C \leq \lambda \leq C$ and there are points $x_1, \ldots, x_N$ with $|x_i - x_j| \geq 2R$, where $R = n/\lambda$, such that for $F = \bigcup_{i=1}^N B_R(x_i)$ it holds

$$\sup_{x \in E \Delta F} d_{\partial F}(x) \leq C \|H_E - \lambda\|^q_{L^{n-1}(\partial E)}.$$

Moreover, by denoting $\omega_n = \mathcal{H}^{n-1}(\partial B_1)$, it holds

$$\left| P(E) - N n \omega_n R^{n-1} \right| \leq C \|H_E - \lambda\|^q_{L^{n-1}(\partial E)}.$$

While there are many generalization of Alexandrov's theorem, see for instance the survey [14], the strength of Theorem 5.3 lies in the fact that it does not set any geometric requirements for a given set, such as mean convexity. In the statement, the L^{n-1}-norm is the optimal choice in order to get the first estimate works. For a finer analysis, it would be interesting to know what is optimal value for the exponent q. The proof of Theorem 5.3 is based on the diameter control by mean curvature provided by [47] and [48] and a modification of the *Montiel-Ros argument* used to prove the *Heinze-Karcher inequality*, see [40]. We also remark that in the planar case Theorem 5.3 can be proven in a strengthened form, see [20, Lemma 3.2], although this argument works only in the dimension $n = 2$.

Theorem 5.2 essentially follows from (5.12) and Theorem 5.3 via a suitable comparison argument. This also shows us the limiting factor of the proof why it is only carried out in the dimensions $n = 2, 3$. The dissipation inequality provides only the L^2-control over the quantity $\overline{H}_E - H_E$. A new approach and probably a new quantitative version of Alexandrov's theorem are needed to generalize Theorem 5.2 to higher dimensions.

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Volume preserving mean curvature flows near strictly stable sets in flat torus

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Abstract

In this paper we establish a new stability result for smooth volume preserving mean curvature flows in flat torus $\mathbb{T}^n$ in dimensions $n = 3, 4$. The result says roughly that if an initial set is near to a strictly stable set in $\mathbb{T}^n$ in H^3-sense, then the corresponding flow has infinite lifetime and converges exponentially fast to a translate of the strictly stable (critical) set in $W^{2,5}$-sense.

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Contents

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1. Introduction

A smooth evolution of sets $(E_t)_t$, that is, a smooth flow in $\mathbb{R}^n$ is a *volume preserving mean curvature flow* (VMCF), if for every time t the (outer) normal velocity of the flow on ∂E_t obeys the law

$$V_t = \bar{H}_t - H_t,$$

where H_t is the (scalar) *mean curvature* on ∂E_t and $\bar{H}_t$ its integral average over ∂E_t. As the name suggests, a VMCF preserves the volume, which is in the contrast to a classical *mean curvature flow* with a smooth and bounded initial set. Such a flow shrinks the initial volume to zero in finite time.

The short time existence for a smooth VMCF in $\mathbb{R}^n$ is well-known. For any smooth (a closed set with a smooth boundary) and compact set $E \subset \mathbb{R}^n$ with $n \geq 2$ there is a unique VMCF starting from E. However, a VMCF $(E_t)_t$ may develop *singularities* such as self-intersections of the boundary ∂E_t within a finite time, see [16]. Another type of singularities of VMCF in a free boundary setting is studied in [5], where it is shown that certain thin necks have to pinch-off under VMCF. A natural problem is to find a sufficient condition for the initial set such that the VMCF starting from the set does not form singularities and has infinite lifetime.

Several contributions concerning the previous question have been made over the years. The classical result of Huisken [14] says that for any smooth, compact and convex set $E \subset \mathbb{R}^n$ there is a unique VMCF $(E_t)_t$ starting from E such that the flow has infinite lifetime and converges exponentially fast to a closed ball of the volume $|E|$ in C^∞-topology. Second, Li [15] has formulated in $\mathbb{R}^n$ an alternative condition for a connected initial boundary based on a certain energy such that the corresponding VMCF has infinite lifetime and converges exponentially fast to a ball, when the dimension is at least three. Notice that if VMCF converges in C^2-sense, then the limit set is a finite union of balls with mutually positive distance. This follows from the *Alexandrov theorem*.

This naturally raises questions about the stability of VMCF near *stable sets*, in this case the closed Euclidean balls. Such problems are often called *stability problems*. Escher and Simonett [9] used to center manifold analysis to prove that if $E \subset \mathbb{R}^n$ is a smooth compact set and $\bar{B}(x,r)$ is a closed ball with the same volume such that ∂E is $C^{1,\alpha}$-close to $\partial B(x,r)$, then the VMCF starting from E has infinite lifetime and converges to a translate of $\bar{B}(x,r)$ exponentially fast in C^k-sense for any $k \in \mathbb{N}$.

Instead of having generic smooth sets in $\mathbb{R}^n$, we may focus on periodic smooth sets, that is, the smooth sets in $\mathbb{R}^n$ invariant under the lattice translations. This again leads us to consider the *flat torus* $\mathbb{T}^n$ in place of $\mathbb{R}^n$. One motivation for this is that there then are more different types of compact and *critical* sets than in $\mathbb{R}^n$. Also, the notion of VMCF generalizes to the flat torus and corresponds to the periodic VMCFs in $\mathbb{R}^n$. We are interested in the subclass of compact and critical sets in $\mathbb{T}^n$ called *strictly stable* sets (with respect to perimeter), see Definition 2.12. Examples of strictly stable sets (besides single balls, cylinders and strips) in $\mathbb{T}^3$ are those sets having a *Schwarz surface* as a boundary, see [17]. Acerbi, Fusco and Morini prove that strictly stable sets in $\mathbb{T}^n$ are always isolated *local perimeter minimizers* (under the notion of volume in $\mathbb{T}^n$), see [2, Theorem 1.1]. In contrast, the only smooth local perimeter minimizers in $\mathbb{R}^n$ are just the single balls. This essentially follows from [7], see also [18].

Our goal is to prove the following stability result for VMCFs near strictly stable sets in the flat torus $\mathbb{T}^n$ with $n = 3, 4$ using the notion of *graph surface representation in normal direction* and Sobolev spaces on smooth compact hypersurfaces.

Theorem *(Main result). Let $F \subset \mathbb{T}^n$, where $n = 3, 4$, be a strictly stable set and let ν_F be the unit normal of ∂F with the inside-out orientation. There exists a positive constant $\delta_0 \in \mathbb{R}_+$ depending on F such that the following hold.*

If E_0 is a smooth set in $\mathbb{T}^n$ with the same volume as F and having a boundary of the form

$$\partial E_0 = \{x + \psi_0(x)\nu_F(x) : x \in \partial F\},$$

where $\psi_0 \in C^\infty(\partial F)$ and $\|\psi_0\|_{H^3(\partial F)} \le \delta_0$, then the VMCF $(E_t)_t$ starting from E_0 has infinite lifetime and converges to a translate $F + p$ of F exponentially fast in $W^{2,5}$-sense from the point of view of the boundary $\partial(F + p)$. Moreover, $|p| \to 0$ as $\|\psi_0\|_{H^3(\partial F)} \to 0$.

Remarks. Of course, using the same arguments we obtain a similar result in $\mathbb{T}^2$. Since the convergence happens exponentially fast in time, the flow is also said to be *exponentially stable* near strictly stable set.

In terms of methods we are motivated by the paper Acerbi, Fusco, Julin and Morini [1], where they prove similar kinds of stability results for other volume preserving flows, namely the *modified Mullins-Sekerka flow* and the *surface diffusion flow* in the three dimensional flat torus $\mathbb{T}^3$. The cornerstone of our analysis (see Section 3) is to prove that H^3-closeness to a strictly stable set implies for the VMCF $(E_t)_t$ that the L^2-norm of the normal velocity over ∂E_t, that is $\|\bar{H}_t - H_t\|_{L^2(\partial E_t)}$, is decaying exponentially in time while the L^2-norm of its tangential gradient over ∂E_t is bounded in time. In proving this we are heavily dependent on Sobolev interpolation inequalities, which is the reason we have to restrict ourselves to low dimensions.

2. Preliminaries

Flat torus. Recall that for given $n \ge 2$ the (unit) flat torus $\mathbb{T}^n$ is defined as the quotient space $\mathbb{R}^n / \mathbb{Z}^n$. Here the equivalence relation is given in the obvious way: $x \sim y$ exactly when $x - y \in \mathbb{Z}^n$. Functions $f : \mathbb{T}^n \to \mathbb{R}^d$ (for $d \in \mathbb{N}$) can be (canonically) identified with the class of $\mathcal{D}_n$-periodic maps $\mathbb{R}^n \to \mathbb{R}^d$, where $\mathcal{D}_n = [0, 1[^n$ is the dyadic unit cube. Again, the functions from $\mathbb{T}^n$ to itself are naturally identified with the maps $f : \mathbb{R}^n \to \mathbb{R}^n$ for which $f - \mathrm{id}$ are $\mathcal{D}_n$-periodic. Further, the sets in $\mathbb{T}^n$ can be identified with $\mathcal{D}_n$-periodic sets in $\mathbb{R}^n$. If $x \in \mathbb{T}^n$, then for any $p \in \mathbb{R}^n$ the notation $x + p$ means the element $q(\tilde{x} + p) \in \mathbb{T}^n$, where $\tilde{x} \in q^{-1}(x)$ and $q : \mathbb{R}^n \to \mathbb{T}^n$ is the quotient map.

We consider $\mathbb{T}^n$ as a smooth and compact manifold where the smooth structure is given via the quotient map $\mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n$. The natural flat Riemannian metric on $\mathbb{T}^n$ is induced by the standard Euclidean inner product $\langle \cdot\,, \cdot \rangle$ of $\mathbb{R}^n$ via the quotient map. Indeed, one can think $\mathbb{T}^n$ is "locally" the Euclidean $\mathbb{R}^n$. The compatible distance function $\mathbb{T}^n \times \mathbb{T}^n \to [0, \infty[$ is given by

$$|x - y|_{\mathbb{T}^n} = \min\{|\tilde{x} - \tilde{y}| : \tilde{x} \in x, \tilde{y} \in y\}.$$

A function $f : \mathbb{T}^n \to \mathbb{R}^d$ is locally a C^k-map at $x \in \mathbb{T}^n$ exactly when its periodic extension $\tilde{f}$ to $\mathbb{R}^n$ is locally C^k at every representative $\tilde{x}$ of x. Thus we set the j-th derivative $\mathrm{D}^j f(x)$ at

x to be $\mathrm{D}^j \tilde{f}(\tilde{x})$ for every $1 \le j \le k$. This is done similarly in the case $f : \mathbb{T}^n \to \mathbb{T}^n$. Other familiar function classes such as the Hölder spaces $C^{k,\alpha}(\mathbb{T}^n)$ are defined by similar means. For every Borel set in A in $\mathbb{T}^n$ its s-dimensional Hausdorff measure $\mathcal{H}^s(A)$ is defined to be the corresponding Hausdorff-measure of the intersection of the periodic extension and $\mathcal{D}_n$.

Smooth hypersurfaces. A set $\Sigma \subset \mathbb{T}^n$ is a smooth hypersurface (a smooth embedded submanifold with codimension 1) exactly when its $\mathcal{D}_n$-periodic extension is a smooth hypersurface in $\mathbb{R}^n$. Other classes such as $C^{k,\alpha}$-hypersurfaces are defined similarly. From now on, we assume a given hypersurface Σ (not necessarily smooth) to be compact and connected in our discussion. The shorthand notation $|\Sigma| = \mathcal{H}^{n-1}(\Sigma)$ is used time to time without any further mention.

A function f belongs to $C^k(\Sigma; \mathbb{R}^d)$ if and only if it admits a C^k-extension to some open neighborhood of Σ. For every $x \in \Sigma$ the *geometric tangent space* $G_x\Sigma$ is defined as a unique $n-1$-dimensional subspace of $\mathbb{R}^n$ such that $x + G_x\Sigma$ is the tangent plane of Σ at x. Equivalently we can set $G_x\Sigma = \mathrm{D}\phi(\mathbb{R}^{n-1})$, where $\phi : U \to \mathbb{T}^n$ is any local parametrization of Σ at x. Moreover, the orthogonal projection from $\mathbb{R}^n$ onto $G_x\Sigma$ is denoted by $P_\Sigma(x)$. Then $P_\Sigma : \Sigma \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ is a smooth map.

For any $x \in \Sigma$, open neighborhood U of x in $\mathbb{T}^n$ and $f \in C^1(U; \mathbb{R}^k)$ the *tangential differential* $\mathrm{D}_\tau f(x) : \mathbb{R}^n \to \mathbb{R}^k$ of f with respect to Σ at x is given by

$$\mathrm{D}_\tau f(x) = \mathrm{D} f(x) P_\Sigma(x).$$

The definition does not depend on how f is extended beyond Σ. In the case $k = 1$ the dual of $\mathrm{D}_\tau f(x)$ is called the *tangential gradient* of f with respect to Σ at x denoted by $\nabla_\tau f(x)$. Then we can write $\nabla_\tau f(x) = P_\Sigma(x)\nabla f(x)$ so $\nabla_\tau f(x) \in G_x\Sigma$. Further, in the case $k = n$ the *tangential divergence* $\mathrm{div}_\tau f(x)$ of f with respect to Σ at x is defined as the trace of $\mathrm{D}_\tau f(x)$. These operations behave as their ordinary counterparts in $\mathbb{T}^n$.

Since Σ is compact and connected, then it is also orientable, i.e., it admits a smooth unit normal field $\nu : \Sigma \to \mathbb{R}^n$, where $\nu(x) \in N_x\Sigma := G_x\Sigma^\perp$ for every $x \in \Sigma$. The pair (Σ, ν) is called an *oriented hypersurface* Σ with an orientation ν. If Σ' is a C^1-hypersurface and $\Phi : \Sigma \to \Sigma'$ is a C^1 diffeomorphism, then we have the *change of variable formula*. For any $h \in L^1(\Sigma', \mathcal{H}^{n-1})$

$$\int_{\Sigma'} h \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\Sigma} (h \circ \Phi) J_\tau \Phi \,\mathrm{d}\mathcal{H}^{n-1},$$

where the *tangential Jacobian* $J_\tau\Phi$ is given by $J_\tau\Phi = |\det \mathrm{D}\tilde{\Phi}||(\mathrm{D}\tilde{\Phi})^{-\mathrm{T}}\nu|$ and $\tilde{\Phi}$ is any diffeomorphic extension of Φ. This is independent of the choice of orientation.

The *second fundamental form* $B(x)$ of Σ associated with the orientation ν at x can be seen as the linear operator from $G_x\Sigma$ to itself or equivalently $\mathbb{R}^n \to \mathbb{R}^n$ given by

$$B(x) = \mathrm{D}_\tau \tilde{\nu}(x),$$

where $\tilde{\nu}$ is any smooth extension of ν. We use both of these conventions interchangeably. If we use the latter one, then $N_x\Sigma \subset \ker B(x)$ and $\mathrm{Im}\ B(x) \subset G_x\Sigma$. The operator is symmetric and its eigenvalues and corresponding eigenspaces on $G_x\Sigma$ are called *principal values* and *principal directions*. The *mean curvature* $H(x)$ of Σ associated with the orientation at x is now defined as the trace of $B(x)$ or equivalently the sum of principal values on $G_x\Sigma$. Again, the maps B :

$\Sigma \to \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$ and $H:\Sigma\to\mathbb{R}$ are smooth. If H is constant, then Σ is said to be critical. The Frobenius norm $|B|$ on Σ does not depend on the used orientation. We define also the mean curvature vector field $\mathbf{H}:\Sigma\to\mathbb{R}^n$ by setting $\mathbf{H}=-H\nu$. Notice that $\mathbf{H}$ is independent of the choice of orientation. Since Σ is compact, then the *divergence theorem for hypersurfaces* says that for any $f\in C^1(\mathbb{T}^n,\mathbb{R}^n)$

$$\int_\Sigma \operatorname{div}_\tau f \; \mathrm{d}\mathcal{H}^{n-1} = -\int_\Sigma \langle f,\mathbf{H}\rangle \; \mathrm{d}\mathcal{H}^{n-1} = \int_\Sigma H\langle f,\nu\rangle \; \mathrm{d}\mathcal{H}^{n-1}.$$

If $f(x)\in G_x\Sigma$ for every $x\in\Sigma$, then the previous formula yields the *integration by parts formula*

$$\int_\Sigma \varphi \operatorname{div}_\tau f \; \mathrm{d}\mathcal{H}^{n-1} = -\int_\Sigma \langle \nabla_\tau\varphi, f\rangle \; \mathrm{d}\mathcal{H}^{n-1}$$

for every $\varphi\in C^1(\Sigma)$, in particular $\int_\Sigma \operatorname{div}_\tau f \; \mathrm{d}\mathcal{H}^{n-1}=0$.

While the concept of tangential derivative can be defined similarly on compact C^1-hypersurfaces, the classical notion of mean curvature is not generally possible for such surfaces due to lack of regularity. However, the following generalization is possible. We say that a C^1-hypersurface $\Gamma\subset\mathbb{T}^n$ with orientation ν has mean curvature in weak sense, if there exists a Borel function $h\in L^1(\Gamma)$ such that for every $f\in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$

$$\int_\Gamma \operatorname{div}_\tau f \; \mathrm{d}\mathcal{H}^{n-1} \int_\Gamma h\langle f,\nu\rangle \; \mathrm{d}\mathcal{H}^{n-1}.$$

In such case h is called a *weak* or *distributional mean curvature* of Γ. Moreover, if h is constant, then Γ is called *stationary*. By writing Γ locally as a graph surface in an orthonormal coordinates and applying the elliptic regularity results [4, Proposition 7.56] and [11, Theorem 9.19] we obtain the following well-known result.

Lemma 2.1. *Let $\Gamma\subset\mathbb{T}^n$ be a $C^{1,\alpha}$-hypersurface with $0<\alpha<1$. Then Σ is smooth and critical if and only if it is stationary.*

Vector fields, tensors and covariant derivatives on hypersurfaces. Since now for every $\varphi\in C^\infty(\Sigma)$ there is a smooth extension $\tilde\varphi$ to some open neighborhood of Σ and tangential differential is independent of the way an ambient function is extended beyond Σ, we may define a tangential gradient $\nabla_\tau\varphi(x)$ of φ at x by setting $\nabla_\tau\varphi(x)=\nabla_\tau\tilde\varphi(x)$. Let $T_x\Gamma$ be the *tangent space* of Γ at x. Then for every $v\in T_x\Sigma$

$$v(\varphi)=\langle z_v,\nabla_\tau\varphi(x)\rangle$$

with some unique $z_v\in G_x\Sigma$. Thus the geometric tangent space $G_x\Sigma$ can be canonically identified with the tangent space $T_x\Sigma$ and, from now on, we just use the notation $T_x\Sigma$. The tangent bundle $T\Sigma$ is then the union of ordered pair $(x,T_x\Sigma)$ equipped with the corresponding smooth structure. Further, we may canonically identify the set of smooth vector fields $\mathfrak{X}(\Sigma)$ on Σ, that is, the smooth sections $\Sigma\to T\Sigma$, with the collection

$$\mathfrak{X}(\Sigma) = \{X \in C^\infty(\Sigma; \mathbb{R}^n) : X(x) \in T_x\Sigma \text{ for every } x \in \Sigma\},$$

and for $X \in \mathfrak{X}(\Sigma)$ its action on $\varphi \in C^\infty(\Sigma)$ can be seen as $X\varphi = \langle X, \nabla_\tau \varphi\rangle$ on Σ. As usual, for $X, Y \in \mathfrak{X}(\Sigma)$ the vector field XY is determined by the rule $XY\varphi = X(Y\varphi)$.

The Riemannian metric g on Σ is the naturally induced flat metric. Keeping the previous identifications in mind this is just the restriction of the standard Euclidean inner product to the hyperspace $T_x\Sigma$. The usual flat and sharp operations induced by g for smooth vector and *covector* fields on Σ are denoted by $\flat$ and $\sharp$ correspondingly. Then for $\varphi \in C^\infty(\Sigma)$ the gradient vector field is $\operatorname{grad} \varphi = d\varphi^\sharp = \nabla_\tau \varphi$.

Slightly abusing the notations we define for a vector field $X \in \mathfrak{X}(\Sigma)$ the (geometric) tangential differential $\mathrm{D}_\tau X(x)$ at x as the linear map $\mathbb{R}^n \to \mathbb{R}^n$ (or equivalently $T_x\Sigma \to T_x\Sigma$) by setting

$$\mathrm{D}_\tau X(x) = P_\Sigma(x)\mathrm{D}_\tau \tilde{X}(x),$$

where $\tilde{X}$ is any smooth extension of X beyond Σ. The tangential divergence $\operatorname{div}_\tau X(x)$ of X at x is the trace of previous operator and $\operatorname{div}_\tau X(x) = \operatorname{div}_\tau \tilde{X}(x)$. Now the mappings $\Sigma \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, $x \mapsto \mathrm{D}_\tau X(x)$ and $\Sigma \to \mathbb{R}$, $x \mapsto \operatorname{div}_\tau X(x)$, are smooth. Again, for any $\varphi \in C^\infty(\Sigma)$ the *tangential Hessian* is given by $\mathrm{D}_\tau^2\varphi = \mathrm{D}_\tau \nabla_\tau \varphi$, which is a symmetric operator. Further, the *Laplace-Beltrami operator* or the *tangential Laplacian* Δ_τ (of Σ) acting on φ can be seen as $\Delta_\tau\varphi = \operatorname{tr} \mathrm{D}_\tau^2\varphi = \operatorname{div}_\tau \nabla_\tau \varphi$.

The compatible Riemannian connection on Σ is the *tangential connection* $\nabla^\top : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by the rule

$$\nabla_X^\top Y = (\mathrm{D}_\tau Y)X.$$

Recall that this is a symmetric connection, i.e., $\nabla_X^\top Y - \nabla_Y^\top X = [X, Y]$, where $[X, Y] = XY - YX$ is the corresponding commutator.

As usual, for every $m \in \mathbb{N} \cup \{0\}$ and $x \in \Sigma$ we denote the space of m-multilinear mappings or *m-covariant tensors* $T_x\Sigma \times \cdots \times T_x\Sigma \to \mathbb{R}$ by $T_0^m(T_x\Sigma)$. Recall the special cases $T_0^1(T_x\Sigma) = (T_x\Sigma)^*$ and $T_0^0(T_x\Sigma) = \mathbb{R}$. If $m > 0$, for given $L, G \in T_0^m(T_x\Sigma)$ the inner product is given by

$$\langle L, G\rangle = \sum_{i_1 \ldots i_m} L(v_{i_1}, \ldots, v_{i_m})G(v_{i_1}, \ldots, v_{i_m}),$$

where $v_1, \ldots, v_{n-1}$ is any orthonormal basis of $T_x\Sigma$. Thus the corresponding tensor norm for $L \in T_0^m(T_x\Sigma)$ is $|L| = \langle L, L\rangle^{\frac{1}{2}}$. If $L \in T_0^2(T_x\Sigma)$, then with help of any orthonormal basis $v_1, \ldots, v_{n-1}$ of $T_x\Sigma$ we may define the trace of L by setting

$$\operatorname{tr} L = \sum_i L(v_i, v_i).$$

The *m-covariant tensor bundle* $T_0^m(\Sigma)$ is defined by similar means as $T\Sigma$. A *covariant m-tensor field* or section $T : \Sigma \to T_0^m(\Sigma)$ is smooth exactly when $T(X_1, \ldots, X_m) \in C^\infty(\Sigma)$ for every $X_1, \ldots, X_m \in \mathfrak{X}(\Sigma)$. We denote them by $\mathcal{T}^m(\Sigma)$. Recall that $\mathcal{T}^1(\Sigma)$ is the collection of the smooth *covector* fields or 1-*forms* and $\mathcal{T}^0(\Sigma) = C^\infty(\Sigma)$.

Now for any (smooth) covariant m-tensor field $T \in \mathcal{T}^m(\Sigma)$ the covariant derivative of T, denoted by $\nabla_{\mathrm{co}}T$, is defined as the element of $\mathcal{T}^{m+1}(\Sigma)$ for which

$$\nabla_{\mathrm{co}} T(X_1, \dots, X_m, X_{m+1}) = X_{m+1} T(X_1, \dots, X_m) - \sum_{k=1}^{m} T(X_1, \dots, \nabla^{\top}_{X_{m+1}} X_k, \dots, X_m).$$

For $\varphi \in C^\infty(\Sigma)$ this simply means that $\nabla_{\mathrm{co}}\varphi = d\varphi$ and again for $T \in \mathfrak{T}^m(\Sigma)$ the $k+1$-th covariant derivative is defined recursively by setting $\nabla^{k+1}_{\mathrm{co}} T = \nabla_{\mathrm{co}}(\nabla^k_{\mathrm{co}} T)$. It is straightforward to compute that for $\varphi \in C^\infty(\Sigma)$ the *covariant Hessian* $\nabla^2_{\mathrm{co}}\varphi$ is now the symmetric 2-tensor field obtained by

$$\nabla^2_{\mathrm{co}}\varphi(X, Y) = \langle X, \mathrm{D}^2_\tau \varphi Y \rangle \ \text{ for every } \ X, Y \in \mathfrak{X}(\Sigma),$$

$\Delta_{\mathrm{co}}\varphi := \mathrm{tr}\ \nabla^2_{\mathrm{co}}\varphi = \Delta_\tau \varphi$ and $|\nabla^2_{\mathrm{co}}\varphi| = |\mathrm{D}^2_\tau \varphi|$, where $|\mathrm{D}^2_\tau\varphi|$ is the standard Frobenius norm of $\mathrm{D}^2_\tau\varphi$.

For any $T \in \mathfrak{T}^m(\Sigma)$ the C^k-norm on Σ is given in the obvious way

$$\|T\|_{C^k(\Sigma)} = \sum_{i=0}^{m} \sup_\Sigma |\nabla^i_{\mathrm{co}} T|$$

and further we set $\|X\|_{C^k(\Sigma)} = \|X^\flat\|_{C^k(\Sigma)}$ for every $X \in \mathfrak{X}(\Sigma)$.

Sobolev and Hölder spaces on hypersurfaces. Recall that the *Riemannian measure* on Σ induced by the flat metric g is the restriction of the $n-1$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ to Σ. Then the corresponding p-Lebesgue space is $L^p(\Sigma, \mathcal{H}^{n-1})$ for any $1 \le p \le \infty$. For every smooth covariant tensor field T on Σ the L^p-norm is now given by

$$\|T\|_{L^p(\Sigma)} = \left(\int_\Sigma |T|^p \ \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{1}{p}}.$$

For a $\varphi \in C^\infty(\Sigma)$ the Sobolev $W^{k,p}$-norm with $k \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$ is given as usual

$$\|\varphi\|_{W^{k,p}(\Sigma)} = \left(\sum_{j=0}^{k} \|\nabla^j_{\mathrm{co}}\varphi\|^p_{L^p(\Sigma)} \right)^{\frac{1}{p}}.$$

Here $\nabla^0_{\mathrm{co}}\varphi = \varphi$. For any $1 \le p < \infty$ the space $W^{k,p}(\Sigma)$ is now the norm completion of $C^\infty(\Sigma)$, where any of its element φ is considered as the $k+1$-tuple $(\varphi, \nabla^1_{\mathrm{co}}\varphi, \dots, \nabla^k_{\mathrm{co}}\varphi)$. Again, we use the conventional notation $H^k(\Sigma)$ for a Hilbert space $W^{k,2}(\Sigma)$, where the inner product is given in the obvious way

$$\langle \varphi_1, \varphi_2 \rangle_{H^k} = \sum_{j=0}^{k} \int_\Sigma \langle \nabla^j_{\mathrm{co}}\varphi_1, \nabla^j_{\mathrm{co}}\varphi_2 \rangle \ \mathrm{d}\mathcal{H}^{n-1}.$$

We have the standard Sobolev interpolation for L^p-norms of covariant derivatives of smooth maps, see [6, Theorem 3.70]. From now on, we denote the space of $C^\infty(\Sigma)$-maps with vanishing integrals by $\tilde{C}^\infty(\Sigma)$.

Lemma 2.2 *(Basic interpolation). Let $\Sigma \subset \mathbb{T}^n$ be a smooth hypersurface and suppose that there are $1 \le p, q < \infty$, $1 \le p \le \infty$ and integers $0 \le j < m$ such that*

$$\frac{1}{p} = \frac{j}{n-1} + \left(\frac{1}{r} - \frac{m}{n-1}\right)\alpha + \frac{1-\alpha}{q}, \tag{2.1}$$

where $\frac{j}{m} \le \alpha \le 1$ and the condition $r = \frac{n-1}{m-j} \ne 1 = \alpha$ does not hold. Then there exists a constant C depending on the previous numbers and Σ such that for every $\varphi \in \tilde{C}^\infty(\Sigma)$

$$\|\nabla_{\text{co}}^j \varphi\|_{L^p(\Sigma)} \le C\|\nabla_{\text{co}}^m \varphi\|_{L^r(\Sigma)}^\alpha \|\varphi\|_{L^q(\Sigma)}^{1-\alpha}. \tag{2.2}$$

If $j \ge 1$, the estimate holds for every $\varphi \in C^\infty(\Sigma)$.

Remark 2.3. It follows from the previous theorem that in the case $n \le 4$ there is a constant C depending on Σ such that $\|\psi\|_{C^1(\Sigma)}, \|\psi\|_{W^{2,p}(\Sigma)} \le C\|\psi\|_{H^3(\Sigma)}$ for every $1 \le p \le 6$ and $\varphi \in C^\infty(\Sigma)$.

We also recall the following interpolation inequality for covariant tensor fields, see [12, Theorem 12.1].

Lemma 2.4 *(Second order tensor interpolation). Let $\Sigma \subset \mathbb{T}^n$ be a smooth hypersurface and suppose that*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} \tag{2.3}$$

for $1 \le p, q, r \le \infty$. Then for every smooth covariant k-tensor field T on Σ it holds

$$\|\nabla_{\text{co}} T\|_{L^{2p}(\Sigma)}^2 \le (2p - 2 + n - 1)\|\nabla_{\text{co}}^2 T\|_{L^r(\Sigma)}\|T\|_{L^q(\Sigma)}. \tag{2.4}$$

Moreover, we need the following estimates, see [10, Lemma 2.3 and Remark 2.4].

Lemma 2.5. *Let $\Sigma \subset \mathbb{T}^n$ be a smooth hypersurface. There are constants C_n, depending only on n, and C_Σ, depending on Σ, such that for every $\varphi \in C^\infty(\Sigma)$*

$$\|\nabla_{\text{co}}^2 \varphi\|_{L^2(\Sigma)}^2 \le \|\Delta_{\text{co}}\varphi\|_{L^2(\Sigma)}^2 + C_n \int_\Sigma |B|^2 |\nabla_{\text{co}}\varphi|^2 \, d\mathcal{H}^{n-1} \tag{2.5}$$

$$\|\nabla_{\text{co}}^3 \varphi\|_{L^2(\Sigma)}^2 \le \|\nabla_{\text{co}}(\Delta_{\text{co}}\varphi)\|_{L^2(\Sigma)}^2 + C_\Sigma \|\varphi\|_{H^2(\Sigma)}^2. \tag{2.6}$$

For a continuous map $f : \Sigma \to \mathbb{R}$ and $0 < \alpha < 1$ the $C^\alpha(\Sigma)$-Hölder semi-norm is given by

$$[f]_{C^\alpha(\Sigma)} = \sup_{\substack{x,y \in \Sigma \\ x \ne y}} \frac{|f(x) - f(y)|}{d_g(x,y)^\alpha},$$

where d_g is the length metric induced by g. For every $\varphi \in C^\infty(\Sigma)$ we define $C^{1,\alpha}(\Sigma)$-norm by setting

$$\|f\|_{C^{1,\alpha}(\Sigma)} = \|\varphi\|_{C^1(\Sigma)} + \sup_{\substack{X \in \mathfrak{X}(\Sigma) \\ \|X\|_{C^1(\Sigma)} \leq 1}} [\nabla_{\text{co}}\varphi(X)]_{C^\alpha(\Sigma)}$$

and set the space $C^{1,\alpha}(\Sigma)$ to be the norm completion of the set of $C^\infty(\Sigma)$-maps with finite $C^{1,\alpha}(\Sigma)$-norm. Then $C^{1,\alpha}(\Sigma)$ is the space of continuous maps on Σ with $C^{1,\alpha}$-extension to some open neighborhood of Σ. Note that there is several equivalent ways to define $C^{1,\alpha}(\Sigma)$-norm.

The higher order Hölder spaces are defined similarly, but we do not need them. By using the Rellich-Kondarchov theorem, see [13, Thm 2.9] and Lemma 2.2 one obtains the following embedding result.

Lemma 2.6. *Let $\Sigma \subset \mathbb{T}^n$ be a smooth hypersurface and suppose that $k \geq 3$ is an integer and $1 < p < \infty$. If for given $0 < \alpha < 1$ the condition*

$$\alpha < 2 - \frac{n-1}{p}$$

is true, then the embedding $W^{3,p}(\Sigma) \subset C^{1,\alpha}(\Sigma)$ is compact.

In particular, Lemma 2.6 says that for $n \leq 4$ and $0 < \alpha < \frac{1}{2}$ the embedding $H^3(\Sigma) \subset C^{1,\alpha}(\Sigma)$ is compact.

Smooth sets. A closed $E \subset \mathbb{T}^n$ is called a *smooth set*, if the boundary ∂E is a smooth hypersurface. Then ∂E is always a finite disjoint union of compact and connected smooth hypersurfaces in $\mathbb{T}^n$ so the previous results can be applied on ∂E. For the boundary ∂E we always use the natural inside-out orientation denoted by ν_E. The *classical divergence theorem* takes now the following form in $\mathbb{T}^n$. For any Lipschitz function $f : \mathbb{T}^n \to \mathbb{R}^n$

$$\int_E \operatorname{div} f \, \mathrm{d}\mathcal{H}^n = \int_{\partial E} \langle f, \nu_E \rangle \, \mathrm{d}\mathcal{H}^{n-1}.$$

Further, we denote by B_E the second fundamental form on ∂E associated with ν_E and by H_E the corresponding boundary mean curvature. We use also the shorthand notation $|E|$ for the volume $\mathcal{H}^n(E)$. We recall that there exists a *regular neighborhood* of ∂E say U_E such that the *signed distance function* $\bar{d}_E : \mathbb{T}^n \to \mathbb{R}$

$$\bar{d}_E(x) = \begin{cases} \operatorname{dist}(x, \partial E), & x \in \mathbb{T}^n \setminus E \\ -\operatorname{dist}(x, \partial E), & x \in E \end{cases}$$

and the projection mapping $\pi_{\partial E}$ onto ∂E are smooth on $\bar{U}_E$ (in particular the latter is well-defined). Again, we may write $\nu_E = \nabla \bar{d}_E$ and $B_E = \mathrm{D}^2 \bar{d}_E$ on ∂E. The $C^{k,\alpha}$-sets are defined similarly.

For any C^k-sets E and E' in $\mathbb{T}^n$ the C^k-distance is given by

$$\|E, E'\|_{C^k} = \inf\{\|\Phi - \mathrm{id}\|_{C^k(\mathbb{T}^n;\mathbb{T}^n)} : \Phi \text{ is a } C^k\text{-diffeomorphism with } \Phi(E) = E'\}.$$

For a smooth set E and an open set E' we denote $E' = E_\psi$ for some $\psi \in C(\partial E)$, if we may write $\partial E'$ as a *graph of* ψ *in normal direction over* ∂E, that is,

$$\partial E' = \{x + \psi(x)\nu_E(x) : x \in \partial E\} \tag{2.7}$$

and $x + (s + \psi(x))\nu_E(x) \in \mathbb{T}^n \setminus E'$ with a small positive s for every $x \in \partial E$. Now small C^1-distance between E and a $C^{k,\alpha}$-set E' is equivalent to the graph representation $E' = E_\psi$ with a $C^{k,\alpha}(\partial E)$-map ψ having a small C^1-norm.

Lemma 2.7. *Let $E \subset \mathbb{T}^n$ be a smooth set. There exist positive constants $C \geq 1$ and δ depending on E such that the set $\{y \in \mathbb{T}^n : |\bar{d}_E(y)| \leq \delta\}$ belongs to a regular neighborhood of ∂E and the following hold for any $k \in \mathbb{N} \cup \{\infty\}$ and $0 \leq \alpha < 1$.*

(i) *For any $\psi \in C^{k,\alpha}(\partial E)$ with $\|\psi\|_{C^1(\partial E)} \leq \delta$ the set E_ψ is defined as a $C^{k,\alpha}$-set, the map $\Phi_\psi : \partial E \to \partial E_\psi$, given by*

$$\Phi_\psi(x) = x + \psi(x)\nu_E(x),$$

is $C^{k,\alpha}$-diffeomorphism and $\|E, E_\psi\|_{C^1} \leq C\|\psi\|_{C^1(\partial E)}$.

(ii) *If $E' \subset \mathbb{T}^n$ is a $C^{k,\alpha}$-set with $\|E, E'\|_{C^1} \leq \delta$, then there is a unique map $\psi \in C^{k,\alpha}(\partial E)$ for which $E' = E_\psi$ and $\|\psi\|_{C^1(\partial E)} \leq C\|E, E'\|_{C^1}$.*

When for $\psi \in C^\infty(\partial E)$ its C^1-norm is small enough, we may control B_{E_ψ} and represent H_{E_ψ} via Φ_ψ on ∂E in the following way.

Lemma 2.8. *Let $E \subset \mathbb{T}^n$ be a smooth set. There are constants $\delta = \delta(E)$ and $C = C(E)$ and smooth maps*

$$\begin{aligned} &A : \partial E \times [-\delta, \delta] \times [-\delta, \delta]^n \to T^2_0(\partial E), \\ &Z : \partial E \times [-\delta, \delta] \times [-\delta, \delta]^n \to T(\partial E) \quad \textit{and} \\ &P : \partial E \times [-\delta, \delta] \times [-\delta, \delta]^n \to \mathbb{R} \end{aligned}$$

depending on E with $A(\,\cdot\,, 0, 0) = 0$ such that the following hold. If $\psi \in C^\infty(\partial E)$ and $\|\psi\|_{C^1(\partial E)} \leq \delta$, then

$$\begin{aligned} H_{E_\psi} \circ \Phi_\psi &= -\Delta_\tau\psi + \langle A(\,\cdot\,, \psi, \nabla_\tau\psi), \nabla^2_{\mathrm{co}}\psi\rangle \\ &\quad + \nabla_{\mathrm{co}}\psi\,(Z(\,\cdot\,, \psi, \nabla_\tau\psi)) + \psi P(\,\cdot\,, \psi, \nabla_\tau\psi) + H_E \quad \textit{and} \end{aligned} \tag{2.8}$$

$$|B_{E_\psi} \circ \Phi_\psi| \leq C(1 + |\mathrm{D}^2_\tau\psi|) \tag{2.9}$$

on ∂E.

Moreover, with the same δ and C we may assume that the following holds. If $\psi \in C^\infty(\partial E)$ with $\|\psi\|_{C^1(\partial E)} \le \delta$, then for every $1 \le p < \infty$, $h \in L^p(\partial E_\psi)$ and $\varphi \in C^\infty(\partial E_\psi)$

$$C^{-1}\|h \circ \Phi_\psi\|_{L^p(\partial E)} \le \|h\|_{L^p(\partial E_\psi)} \le C\|h \circ \Phi_\psi\|_{L^p(\partial E)} \quad \text{and} \tag{2.10}$$

$$C^{-1}\|\nabla_\tau(\varphi \circ \Phi_\psi)\|_{L^p(\partial E)} \le \|\nabla_\tau \varphi\|_{L^p(\partial E_\psi)} \le C\|\nabla_\tau(\varphi \circ \Phi_\psi)\|_{L^p(\partial E)}. \tag{2.11}$$

Finally we need the following uniform estimates for low dimensions. The first one says that a Poincaré-type estimate holds with a uniform constant when slightly varying a reference boundary in C^1-sense.

Lemma 2.9. *Let $E \subset \mathbb{T}^n$, $n = 3, 4$, be a smooth set. There exist positive constants δ and C depending on E such that if $\psi \in C^\infty(\partial E)$ with $\|\psi\|_{H^3(\partial E)} \le \delta$, then for every $1 \le p \le 6$ and $\varphi \in \tilde{C}^\infty(\partial E_\psi)$*

$$\|\varphi\|_{L^p(\partial E_\psi)} \le C\|\varphi\|_{H^1(\partial E_\psi)}.$$

The second one says that we can control uniformly L^p-norm of $\nabla_\tau \varphi$ (up to $p \le 6$) by the L^2-norm of $\nabla_\tau \varphi$ and the H^1-norm of φ when slightly varying a reference boundary in H^3-sense.

Lemma 2.10. *Let $E \subset \mathbb{T}^n$, $n = 3, 4$, be a smooth set. There are constants δ and C depending on E such that if $\psi \in C^\infty(\partial E)$ and $\|\psi\|_{C^1(\partial E)} \le \delta$, then for every $1 \le p \le 6$ and $\varphi \in C^\infty(\partial E_\psi)$*

$$\|\nabla_\tau \varphi\|_{L^p(\partial E_\psi)} \le C\left(\|\Delta_\tau \varphi\|_{L^2(\partial E_\psi)} + \|\varphi\|_{H^1(\partial E_\psi)}\right).$$

Lemma 2.9 and Lemma 2.10 hold also in the case $n > 4$, if we replace the upper bound 6 with smaller number depending on n. This number converges to 2 by above as n tends to infinity. We will prove these results except Lemma 2.7 in Appendix.

Volume preserving mean curvature flow and strictly stable sets. Let us first give the formal definition of smooth flow in this setting.

Definition 2.11 *(Smooth flow).* The parametrized family $(E_t)_{t\in[0,T[}$ of smooth sets in $\mathbb{T}^n$ with $0 < T \le \infty$ is a smooth flow with an *initial set* or *initial datum* E_0, if there exists a smooth map $\Phi : \mathbb{T}^n \times [0, T[\to \mathbb{T}^n$ such that $\Phi_0 = \mathrm{id}_{\mathbb{T}^n}$, $\Phi_t := \Phi(\,\cdot, t)$ is a (smooth) diffeomorphism and $E_t = \Phi_t(E_0)$ for every $t \in [0, T[$. Again, for every $0 \le t < T$, the (outer) normal velocity of the flow on ∂E_t at the time t is defined by setting

$$V_t = \langle \partial_s \Phi_{s+t}\Big|_{s=0} \circ \Phi_t^{-1}, \nu_{E_t} \rangle.$$

The normal velocity V_0 on ∂E_0 is called *an initial velocity*. If we do not emphasize the time interval $[0, T[$, we write $(E_t)_t$ for short. Also, for the unit normal field ν_{E_t} the corresponding second fundamental form B_{E_t} and the boundary mean curvature H_{E_t} we use the shorthand notations ν_t, B_t and H_t, when there is no possibility of confusion. In the previous definition Φ is called a *smoothly parametrized family of diffeomorphisms* and we may suggestively denote it by $(\Phi_t)_{t\in[0,T[}$ or $(\Phi_t)_t$. The normal velocity V_t on ∂E_t does not depend on the choice of the parametrization Φ. Recall the *first variation of volume* under the flow

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = \int_{\partial E_t} V_t \,\mathrm{d}\mathcal{H}^{n-1} \tag{2.12}$$

and again the *first variation of perimeter*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial E_t| = \int_{\partial E_t} V_t H_t \,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.13}$$

We say that $(E_t)_t$ is *volume preserving*, if $|E_t|$ is a constant function in time. According to (2.12) this is possible exactly when V_t has a vanishing integral over ∂E_t for every t. Conversely, one may show that if E is a smooth set in $\mathbb{T}^n$ and $\varphi \in \tilde{C}^\infty(\partial E)$, there is a smooth volume preserving flow $(E_t)_t$ starting from E such that initial velocity on ∂E is φ and moreover there is a parametrization Φ of the flow autonomous with respect to time, i.e., $\partial_t \Phi = X(\Phi)$ with a $X \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$, see the proof of [2, Corollary 3.4]. Then, by using a simple approximation argument, it follows from (2.13) that a smooth set E is critical if and only if for every smooth volume preserving flow $(E_t)_t$ starting from E

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial E_t|\Big|_{t=0} = 0.$$

Further, if in the previous setting the flow admits a parametrization autonomous with respect to time, then the *second variation of perimeter* at $t = 0$ is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|\partial E_t|\Big|_{t=0} = \int_{\partial E} |\nabla_\tau V_0|^2 - |B_E|^2 V_0^2 \,\mathrm{d}\mathcal{H}^{n-1},$$

see [2, Remark 3.3]. This motivates us to define for every smooth E in $\mathbb{T}^n$ the quadratic form $\partial^2 P(E) : \tilde{H}^1(\partial E) \to \mathbb{R}$, where $\tilde{H}^1(\partial E)$ is the space of $H^1(\partial E)$-maps with vanishing integral over ∂E, by setting first for every $\varphi \in \tilde{C}^\infty(\partial E)$

$$\partial^2 P(E)[\varphi] = \int_{\partial E} |\nabla_\tau \varphi|^2 - |B_E|^2 \varphi^2 \,\mathrm{d}\mathcal{H}^{n-1}$$

and then extending $\partial^2 P(E)$ to $\tilde{H}^1(\partial E)$ in the obvious way. We say that a smooth and critical set E in $\mathbb{T}^n$ is stable, if $\partial^2 P(E)(\varphi) \geq 0$ for every $\varphi \in \tilde{H}^1(\partial E)$.

For every smooth set E in $\mathbb{T}^n$ the space of *infinitesimal translations* on ∂E is given by

$$T(\partial E) = \{\langle \nu_E, p\rangle : p \in \mathbb{R}^n\}.$$

Notice that $T(\partial E) \subset C^\infty(\tilde{\partial} E)$. These maps correspond to the initial velocities of the linear translations along a vector. It follows from [2, Theorem 3.1] that if $E_t = E + tp$, then the second variation at $t = 0$ of perimeter under this translation is $\partial^2 P(E)[\langle \nu_E, p\rangle]$. Since translations do not change perimeter, then $\partial^2 P(E)[\langle \nu_E, p\rangle] = 0$. Thus $\partial P^2(\partial E)$ is always zero on $T(\partial E)$, which is also easy to compute directly by using the definition. This leads us to define the strictly stable sets in the following way, see [2].

Definition 2.12 *(Strictly stable set).* A smooth and critical set F in $\mathbb{T}^n$ is strictly stable, if $\partial^2 P(E)[\varphi] > 0$ for every nonzero map $\varphi \in \tilde{H}^1(\partial F) \cap T^{\perp, L^2}(\partial F)$, where $T^{\perp, L^2}(\partial F)$ is the orthogonal complement of $T(\partial F)$ in $L^2(\partial F)$.

In the previous definition, we may replace the condition $\partial^2 P(E)[\varphi] > 0$ for every nonzero map $\varphi \in \tilde{H}^1(\partial F) \cap T^{\perp, L^2}(\partial F)$ with $\partial^2 P(E)[\varphi] > 0$ for every map $\varphi \in \tilde{H}^1(\partial F) \setminus T(\partial F)$ due to the fact that $\Delta_\tau \langle \nu_F, p\rangle = -|B_F|^2 \langle \nu_F, p\rangle$ for every $p \in \mathbb{R}^n$. As mentioned already in the Introduction such sets are always isolated local perimeter minimizers. For a strictly stable set F any critical set with the same volume being close enough to F in H^3-sense is a translate of F.

Lemma 2.13. *Let $F \subset \mathbb{T}^n$, $n = 3, 4$, be a strictly stable set. There exists a positive $\delta = \delta(F)$ such that if F_ψ is critical, $|F_\psi| = |F|$ and $\|\psi\|_{H^3(\partial F)} \le \delta_1$ for $\psi \in C^\infty(\partial F)$, then F_ψ is a translate of F.*

This result follows from the proof of [2, Theorem 3.9] (for clarity see the proof of [1, Proposition 2.7] although it concerns only the case $n = 3$) and Remark 2.3. Moreover, being near to a strictly stable set in H^3-sense implies that the quadratic form controls H^1-norm for the $\tilde{C}^\infty$-functions orthogonal to the infinitesimal translations in L^2-sense, see [1, Lemma 2.6].

Lemma 2.14. *Let $F \subset \mathbb{T}^n$, $n = 3, 4$, be a strictly stable set. Then there exist $\sigma_1 = \sigma_1(F)$ and $\delta = \delta(F)$ such that if $\|\psi\|_{H^3(\partial F)} \le \delta_1$ for $\psi \in C^\infty(\partial F)$, then for every $\varphi \in \tilde{C}^\infty(\partial F_\psi) \cap T^{\perp, L^2}(\partial F_\psi)$*

$$\sigma_1 \|\varphi\|^2_{H^1(\partial F_\psi)} \le \partial^2 P(F_\psi)[\varphi].$$

The authors prove the previous result in the case $n = 3$ and H^3-closeness being replaced by $W^{2,p}$-closeness with any $p > 2$, but clearly the same arguments go through for any $n \ge 2$, if we use the condition $p > \max\{n-1, 2\}$ instead of $p > 2$. Hence the result follows from Remark 2.3.

As already presented in the Introduction, a volume preserving mean curvature flow $(E_t)_t$ in $\mathbb{T}^n$ is a smooth flow obeying the rule

$$V_t = \bar{H}_t - H_t,$$

where $\bar{H}_t$ is the integral average of H_t over ∂E_t. Then by (2.12) and (2.13) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}|\partial E_t| = -\int_{E_t} (\bar{H}_t - H_t)^2 \,\mathrm{d}\mathcal{H}^{n-1} \le 0,$$

so the flow is volume preserving and decreases boundary area. The existence and uniqueness of such flow for a given smooth initial datum is well-known. This is usually known as *short time existence*. Also, the maximal lifetime of the flow is bounded by below when slightly varying the initial set in $C^{1,\alpha}$-topology with $\alpha > 0$.

Theorem 2.15 *(Short time existence). Let $E \subset \mathbb{T}^n$ be a smooth set and $0 < \alpha < 1$. There are positive constants δ and T depending on E and α such that if E_0 is a smooth set in $\mathbb{T}^n$ of the form $\partial E_0 = E_{\psi_0}$, where $\psi_0 \in C^\infty(\partial E)$ and $\|\psi_0\|_{C^{1,\alpha}(\partial E)} \le \delta$, then there exists a unique volume preserving mean curvature flow in $\mathbb{T}^n$ with the initial datum E_0 and the maximal lifetime of flow is at least T.*

The above result has been proved for bounded smooth sets in $\mathbb{R}^n$, see [9, Main Theorem] and clearly it is similar to prove it in the setting of the flat torus $\mathbb{T}^n$. From now on, when there is no danger of confusion, we denote the maximal lifetime of a given volume preserving mean curvature flow $(E_t)_t$ by T^*.

3. L^2-monotonicity

In this section we prove a monotonicity result, which is the basis of our analysis. It states that if $(E_t)_t$ is a volume preserving mean curvature flow with a smooth initial datum near enough to a strictly stable set F in H^3-sense, then it stays near F in H^3-sense for the whole lifespan of the flow, the initial velocity of flow decreases exponentially in L^2-sense and the quantity $\|\nabla_\tau H_t\|^2_{L^2(\partial E_t)} + C_0\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)}$ is decreasing in time with sufficiently large C_0. More precisely we have the following result.

Theorem 3.1 *(Monotonicity near strictly stable set). Let $F \subset \mathbb{T}^n$ $(n = 3, 4)$ be a strictly stable set. There are positive constants C_0 and ε_0 depending on F such that for every $0 < \varepsilon \le \varepsilon_0$ there is a positive $\gamma_\varepsilon < \varepsilon$ such that if $(E_t)_t$ is a volume preserving mean curvature flow starting from a smooth set $E_0 = F_{\psi_0}$, where $\psi_0 \in C^\infty(\partial F)$ and $\|\psi_0\|_{H^3(\partial F)} \le \gamma_\varepsilon$, then for every $t \in [0, T^*[$ (where $T^* > 0$ is the maximal lifetime of flow) we may write $E_t = F_{\psi_t}$ with $\psi_t \in C^\infty(\partial F)$, $\|\psi_t\|_{H^3(\partial F)} \le \varepsilon$,*

$$\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)} \le \|\psi_0\|_{H^3(\partial F)} e^{-\sigma_1 t} \quad \textit{and} \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla_\tau H_t\|^2_{L^2(\partial E_t)} + C_0\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)}\right] \le 0, \tag{3.2}$$

where σ_1 is as in Lemma 2.14.

Remark 3.2. As a byproduct of the proof of Theorem 3.1 we may replace zero with $-\frac{1}{2}\|\Delta_\tau H_t\|^2_{L^2(\partial E_t)}$ on the right-hand side of (3.2). However, we do not need that fact.

Before proving Theorem 3.1 we introduce the following useful geometric quantity employed, for instance, in [1]. For any open set $E \subset \mathbb{T}^n$ we define the map D_E from the collection of the open subsets of $\mathbb{T}^n$ to $[0, \infty[$ by setting for any open $E' \subset \mathbb{T}^n$

$$D_E(E') := \int_{E' \Delta E} \mathrm{dist}_{\partial E}\, \mathrm{d}\mathcal{H}^n = \int_{E'} \bar{d}_E\, \mathrm{d}\mathcal{H}^{n-1} - \int_{E} \bar{d}_E\, \mathrm{d}\mathcal{H}^{n-1}. \tag{3.3}$$

In case of any smooth set $E \subset \mathbb{T}^n$ this concept of "weak distance" turns out to be very useful in terms of controlling the $L^2(\partial E)$-norm of the corresponding function of given C^1-graph in normal direction over ∂E, when the corresponding $C^1(\partial F)$-norm is sufficiently small. To observe this, choose $\delta = \delta(E)$ so small that by Lemma 2.7 for any $\psi \in C^1(\partial E)$ with $\|\psi\|_{C^1(\partial E)} \le \delta$ the set E_ψ is defined as a C^1-set. Moreover, we may assume that for every $s \in [-\delta, \delta]$ the map $\Phi_s = \mathrm{id} + s\nabla \bar{d}_E$ is defined as a smooth diffeomorphism from some tubular neighborhood of ∂E to its image. With help of the coarea formula one may compute

$$D_E(E_\psi) = \int\limits_{[0,\|\psi\|_{L^\infty(\partial E)}]} s \left[\int\limits_{\{x\in\partial E:\psi(x)>s\}} J_\tau \Phi_s \, \mathrm{d}\mathcal{H}^{n-1} + \int\limits_{\{x\in\partial E:\psi(x)<-s\}} J_\tau \Phi_{-s} \, \mathrm{d}\mathcal{H}^{n-1} \right] \mathrm{d}s.$$

Since now $J_\tau \Phi_s \to 1$ uniformly on ∂E as $s \to 0$, then by decreasing δ, if necessary, it follows from the Cavalieri's principle that there exists $C \ge 1$ depending on δ such that for every $\psi \in C^1(\partial E)$ with $\|\psi\|_{C^1(\partial E)} \le \delta$

$$C^{-1}\|\psi\|^2_{L^2(\partial E)} \le D_E(E_\psi) \le C\|\psi\|^2_{L^2(\partial E)}. \tag{3.4}$$

Again, for any open subset $E \subset \mathbb{T}^n$ the map D_E behaves well under any smooth flow. If $(E_t)_t$ is a smooth flow in $\mathbb{T}^n$ with a normal velocity V_t, then $D_E(E_t)$ is differentiable in time and it is straightforward to calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} D_E(E_t) = \int\limits_{\partial E_t} \bar{d}_E V_t \, \mathrm{d}\mathcal{H}^{n-1}. \tag{3.5}$$

Now the importance of this quantity D lies on the fact, that for any smooth critical set $F \subset \mathbb{T}^n$ and $\psi \in C^\infty(\partial F)$ with sufficiently small C^1-norm, $\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F)}$ and $D_F(F_\psi)$ together control the H^3-norm of ψ.

Lemma 3.3. *Let $F \subset \mathbb{T}^n$ be a smooth critical set. There are positive constants $K = K(F) \ge 1$ and $\delta = \delta(F)$ such that whenever $\psi \in C^\infty(F)$ satisfies $\|\psi\|_{C^1(\partial F)} \le \delta$, then*

$$K^{-1}\left(\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} + \sqrt{D_F(F_\psi)}\right) \le \|\psi\|_{H^3(\partial F)} \le K\left(\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} + \sqrt{D_F(F_\psi)}\right).$$

Proof. We prove only the inequality

$$\|\psi\|_{H^3(\partial F)} \le K\left(\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} + \sqrt{D_F(F_\psi)}\right).$$

The another one is easier to show. Again, it follows from (2.11) and (3.4), that it suffices to find positive δ and K such that for every $\psi \in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial E)} \le \delta$

$$\|\psi\|_{H^3(\partial F)} \le K\left(\|\nabla_\tau\left(H_{F_\psi} \circ \Phi_\psi\right)\|_{L^2(\partial F)} + \|\psi\|_{L^2(\partial F)}\right), \tag{3.6}$$

where $\Phi_\psi : \partial F \to \partial F_\psi$ is defined as in Lemma 2.7. To this end we will prove the following auxiliary result. For suitable positive constants δ' and K' the estimate

$$\left|\nabla_\tau\left(H_{F_\psi}\circ\Phi_\psi\right)\right|^2 \geq \frac{|\nabla_{\mathrm{co}}(\Delta_{\mathrm{co}}\psi)|^2}{2} - \frac{|\nabla^3_{\mathrm{co}}\psi|^2}{4} - K'\left(|\psi|^2 + |\nabla_{\mathrm{co}}\psi|^2 + |\nabla^2_{\mathrm{co}}\psi|^2 + |\nabla^2_{\mathrm{co}}\psi|^4\right) \tag{3.7}$$

holds on ∂F for every $\psi \in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial E)} \leq \delta'$. Due to the compactness of ∂F we have only to show this holds locally. Let δ'' be the constant for F as in Lemma 2.8. Then for every $\psi \in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial F)} \leq \delta''$

$$\begin{aligned} H_{F_\psi}\circ\Phi_\psi = &-\Delta_\tau\psi + \langle A(\,\cdot\,,\psi,\nabla_\tau\psi), \nabla^2_{\mathrm{co}}\psi\rangle \\ &+ \nabla_{\mathrm{co}}\psi\,(Z(\,\cdot\,,\psi,\nabla_\tau\psi)) + \psi P(\,\cdot\,,\psi,\nabla_\tau\psi) + H_F, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &A : \partial F \ \times [-\delta'',\delta''] \times [-\delta'',\delta'']^n \to T^2_0(\partial F), \\ &Z : \partial F \ \times [-\delta'',\delta''] \times [-\delta'',\delta'']^n \to T(\partial F) \quad \text{and} \\ &P : \partial F \ \times [-\delta'',\delta''] \times [-\delta'',\delta'']^n \to \mathbb{R} \end{aligned}$$

are smooth maps depending on F and $A(\,\cdot\,,0,0) : \partial F \to T^2_0(\partial F)$ is the zero tensor field. For a fixed point $x_0 \in \partial F$ choose a local orthonormal vector frame $(E_1,\ldots,E_{n-1})$ in some open neighborhood $U_{x_0} \subset \partial F$ of x_0. Then we may write for every $\psi \in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial E)} \leq \delta''$

$$\langle A(\,\cdot\,,\psi,\nabla_\tau\psi), \nabla^2_{\mathrm{co}}\psi\rangle = \sum_{ij} a_{ij}(\,\cdot\,,\psi,\nabla_\tau\psi)\nabla^2_{\mathrm{co}}\psi(E_i,E_j) \quad \text{and} \tag{3.9}$$

$$\nabla_{\mathrm{co}}\psi\,(Z(\,\cdot\,,\psi,\nabla_\tau\psi)) = \sum_i z_i(\,\cdot\,,\psi,\nabla_\tau\psi)\nabla_{\mathrm{co}}\psi(E_i), \tag{3.10}$$

where $a_{ij}, z_i : U_{x_0} \ \times [-\delta'',\delta''] \times [-\delta'',\delta'']^n \to \mathbb{R}$ are the smooth functions given by

$$\begin{aligned} &a_{ij}(x,t,z) = A(x,t,z)(E_i(x),E_j(x)) \quad \text{and} \\ &z_i(x,t,z) = \langle E_i(x), Z(x,t,z)\rangle \end{aligned}$$

for every $(x,t,z) \in U_{x_0} \times [-\delta'',\delta''] \times [-\delta'',\delta'']^n$. Notice that then $|(a_{ij}(x,t,z)| \leq |A(x,t,z)|$ and $|z_i(x,t,z)| \leq |Z(x,t,z)|$. By taking tangential gradient over (3.8) (notice that for every $\varphi \in C^\infty(\partial F)$ we may write $\nabla_\tau\varphi = \sum_i(\nabla_{E_i}\varphi)E_i$ in U_{p_0}) and using the expressions (3.9) and (3.10) we obtain

$$\begin{aligned} \nabla_\tau(H_{F_\psi}\circ\Phi_\psi) = &-\nabla_\tau(\Delta_{\mathrm{co}}\psi) + \sum_{ijk} a_{ij}(\,\cdot\,,\psi,\nabla_\tau\psi)\nabla^3_{\mathrm{co}}\psi(E_i,E_j,E_k)E_k \\ &+ \sum_{ijk} a_{ij}(\,\cdot\,,\psi,\nabla_\tau\psi)\left(\nabla^2_{\mathrm{co}}\psi(\nabla_{E_k}E_i,E_j) + \nabla^2_{\mathrm{co}}\psi(E_i,\nabla_{E_k}E_j)\right)E_k \\ &+ \sum_{ij} \nabla^2_{\mathrm{co}}\psi(E_i,E_j)\nabla_\tau\left(a_{ij}(\,\cdot\,,\psi,\nabla_\tau\psi)\right) \end{aligned}$$

$$+\sum_{ik} z_i(\,\cdot\,,\psi,\nabla_\tau\psi)\left(\nabla^2_{\mathrm{co}}\psi(E_i,E_k)+\nabla_{\mathrm{co}}\psi(\nabla_{E_k}E_i)\right)E_k$$
$$+\sum_i \nabla_{\mathrm{co}}\psi(E_i)\nabla_\tau\left(z_i(\,\cdot\,,\psi,\nabla_\tau\psi)\right)$$
$$+P(\,\cdot\,,\psi,\nabla_\tau\psi)\nabla_\tau\psi+\psi\nabla_\tau\left(P(\,\cdot\,,\psi,\nabla_\tau\psi)\right). \tag{3.11}$$

Recall that $\Delta_{\mathrm{co}}\psi=\Delta_\tau\psi$ and H_F is a constant thus vanishing after taking gradient over (3.8). Again, for any smooth map $u: U_{x_0}\times[-\delta'',\delta'']\times[-\delta'',\delta'']^n \to \mathbb{R}$ one computes

$$\nabla_\tau\left(u(\,\cdot\,,\psi,\nabla_\tau\psi)\right)=\nabla_\tau u(\,\cdot\,,\psi,\nabla_\tau\psi)+\partial_t u(\,\cdot\,,\psi,\nabla_\tau\psi)\nabla_\tau\psi$$
$$+\mathrm{D}^2_\tau\psi\nabla_n u(\,\cdot\,,\psi,\nabla_\tau\psi)-\langle\nu_F,\nabla_n u(\,\cdot\,,\psi,\nabla_\tau\psi)\rangle B_{\partial F}\nabla_\tau\psi. \tag{3.12}$$

It follows from $A(\,\cdot\,,0,0)$ being the zero tensor field on ∂F and uniform continuity on compact sets, that there exist $0<\delta'<\delta''$ and $C\geq 1$ such that for every $(x,t,z)\in\partial F\times[-\delta',\delta']\times[-\delta',\delta']^n$

$$|A(x,t,z)|\leq\frac{1}{64(n-1)^3}\quad\text{and} \tag{3.13}$$
$$|Z(x,t,z)|,|P(x,t,z)|\leq C. \tag{3.14}$$

By shrinking U_{x_0} and increasing C, if necessary, we may assume that each $|\nabla_{E_k}E_i|$ is bounded by C in U_{x_0}. Since $|\nabla_\tau\psi|=|\nabla_{\mathrm{co}}\psi|$ and $|\mathrm{D}^2_\tau\psi|=|\nabla^2_{\mathrm{co}}\psi|$ on ∂F for every $\psi\in C^\infty(\partial F)$, then again by shrinking U_{x_0} and increasing C, if needed, it follows from (3.12) that for every $\psi\in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial F)}\leq\delta'$

$$|\nabla_\tau\left(a_{ij}(\,\cdot\,,\psi,\nabla_\tau\psi)\right)|,|\nabla_\tau\left(z_i(\,\cdot\,,\psi,\nabla_\tau\psi)\right)|,|\nabla_\tau\left(P(\,\cdot\,,\psi,\nabla_\tau\psi)\right)|\leq C(1+|\nabla^2_{\mathrm{co}}\psi|) \tag{3.15}$$

in U_{x_0}. Recalling expression (3.11) for such ψ we denote

$$a_\psi=\sum_{ijk}a_{ij}(\,\cdot\,,\psi,\nabla_\tau\psi)\nabla^3_{\mathrm{co}}\psi(E_i,E_j,E_k)E_k$$

and the sum of the lower order terms by b_ψ. Thus we may write shortly in U_{x_0}

$$\nabla_\tau(H_{F_\psi}\circ\Phi_\psi)=-\nabla_\tau(\Delta_{\mathrm{co}}\psi)+a_\psi+b_\psi.$$

Now by using (3.13) we have

$$|a_\psi|\leq\frac{|\nabla^3_{\mathrm{co}}\psi|}{64} \tag{3.16}$$

in U_p. Again, by applying the estimates (3.14) and (3.15) on b_ψ we find a positive constant C' depending on C, δ' and n such that in U_{x_0}

$$|b_\psi|\leq C'\left(|\psi|+|\nabla_{\mathrm{co}}\psi|+|\nabla^2_{\mathrm{co}}\psi|+|\nabla^2_{\mathrm{co}}\psi|^2\right). \tag{3.17}$$

Thus by using (3.16) and (3.17) as well as the Cauchy-Schwarz and Young's inequalities, we have in U_{x_0}

$$
\begin{aligned}
|\nabla_\tau(H_{F_\psi}\circ\Phi_\psi)|^2 &= |-\nabla_\tau(\Delta_{\mathrm{co}}\psi)+a_\psi+b_\psi|^2\\
&\geq |\nabla_\tau(\Delta_{\mathrm{co}}\psi)|^2-2|\nabla_\tau(\Delta_{\mathrm{co}}\psi)||a_\psi+b_\psi|\\
&\geq \frac{1}{2}|\nabla_\tau(\Delta_{\mathrm{co}}\psi)|^2-8|a_\psi+b_\psi|^2\\
&\geq \frac{1}{2}|\nabla_\tau(\Delta_{\mathrm{co}}\psi)|^2-16|a_\psi|^2-16|b_\psi|^2\\
&\geq \frac{1}{2}|\nabla_\tau(\Delta_{\mathrm{co}}\psi)|^2\\
&\quad-\frac{1}{4}|\nabla^3_{\mathrm{co}}\psi|^2-64(C')^2\left(|\psi|^2+|\nabla_{\mathrm{co}}\psi|^2+|\nabla^2_{\mathrm{co}}\psi|^2+|\nabla^2_{\mathrm{co}}\psi|^4\right)\\
&= \frac{1}{2}|\nabla_{\mathrm{co}}(\Delta_{\mathrm{co}}\psi)|^2\\
&\quad-\frac{1}{4}|\nabla^3_{\mathrm{co}}\psi|^2-64(C')^2\left(|\psi|^2+|\nabla_{\mathrm{co}}\psi|^2+|\nabla^2_{\mathrm{co}}\psi|^2+|\nabla^2_{\mathrm{co}}\psi|^4\right).
\end{aligned}
$$

This implies (3.7). By integrating the both sides of (3.7) over ∂F and applying (2.6) with the associated constant $C_{\partial F}$ we obtain

$$
\begin{aligned}
\|\nabla_\tau(H_{F_\psi}\circ\Phi_\psi)\|^2_{L^2(\partial F)} &\geq \frac{1}{2}\|\nabla_{\mathrm{co}}(\Delta_{\mathrm{co}}\psi)\|^2_{L^2(\partial F)}-\frac{1}{4}\|\nabla^3_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}\\
&\quad-K'\left(\|\psi\|^2_{H^2(\partial F)}+\|\nabla^2_{\mathrm{co}}\psi\|^4_{L^4(\partial F)}\right)\\
&\geq \frac{1}{4}\|\nabla^3_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}-(C_{\partial F}+K')\left(\|\psi\|^2_{H^2(\partial F)}+\|\nabla^2_{\mathrm{co}}\psi\|^4_{L^4(\partial F)}\right)\\
&\geq \frac{1}{16}\|\psi\|^2_{H^3(\partial F)}+\frac{3}{16}\|\nabla^3_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}\\
&\quad-\left(C_{\partial F}+K'+1\right)\left(\|\psi\|^2_{H^2(\partial F)}+\|\nabla^2_{\mathrm{co}}\psi\|^4_{L^4(\partial F)}\right).
\end{aligned}
\tag{3.18}
$$

Next we interpolate the last terms in (3.18). By using Lemma 2.2 and Lemma 2.4 (recall that $\nabla_{\mathrm{co}}(\nabla_{\mathrm{co}}\psi)=\nabla^2_{\mathrm{co}}\psi$, $\nabla^2_{\mathrm{co}}(\nabla_{\mathrm{co}}\psi)=\nabla^3_{\mathrm{co}}\psi$ and $\|\nabla_{\mathrm{co}}\psi\|_{L^\infty(\partial F}\leq\|\psi\|_{C^1(\partial F)}$) we find a positive M depending on F such that

$$
\|\nabla_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}\leq M\|\nabla^3_{\mathrm{co}}\psi\|^{\frac{2}{3}}_{L^2(\partial F)}\|\psi\|^{\frac{4}{3}}_{L^2(\partial F)},
\tag{3.19}
$$

$$
\|\nabla^2_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}\leq M\|\nabla^3_{\mathrm{co}}\psi\|^{\frac{4}{3}}_{L^2(\partial F)}\|\psi\|^{\frac{2}{3}}_{L^2(\partial F)} \quad \text{and}
\tag{3.20}
$$

$$
\|\nabla^2_{\mathrm{co}}\psi\|^4_{L^4(\partial F)}\leq M\|\nabla^3_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}\|\psi\|^2_{C^1(\partial F)}.
\tag{3.21}
$$

By applying Young's inequality on (3.19) and (3.20) we find $K''=K''(K',M,C_{\partial F})$ such that

$$\left(C_{\partial F}+K'+1\right)\|\psi\|^2_{H^2(\partial F)}\le\frac{1}{8}\|\nabla^3_{\mathrm{co}}\psi\|^2_{L^2(\partial F)}+K''\|\psi\|^2_{L^2(\partial F)}. \tag{3.22}$$

Finally by choosing $0<\delta\le\delta'$ with $\left(C_{\partial F}+K'+1\right)M\delta^2\le\frac{1}{16}$ it follows from (3.18), (3.21) and (3.22), that for every $\psi\in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial F)}\le\delta$

$$\|\nabla_\tau(H_{F_\psi}\circ\Phi_\psi)\|^2_{L^2(\partial F)}\ge\frac{1}{16}\|\psi\|^2_{H^3(\partial F)}+K''\|\psi\|^2_{L^2(\partial F)},$$

which implies (3.6). □

The previous lemma and Lemma 2.8 together yield, that for every smooth critical set $F\subset\mathbb{T}^n$ there are $\delta=\delta(F)$ and $K=K(F)$ such that for every $\psi\in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial F)}\le\delta$

$$\|\bar{H}_{F_\psi}-H_{F_\psi}\|_{H^1(\partial F_\psi)}\le K\|\psi\|_{H^3(\partial F)} \tag{3.23}$$

$$\|H_{F_\psi}\|_{H^1(\partial F_\psi)}\le K\left(\|\psi\|_{H^3(\partial F)}+|H_F|\right). \tag{3.24}$$

We will also use the following identities for the time derivatives of the L^2-norms of $\bar{H}_t-H_t$ and $\nabla_\tau H_t$. For the proof see Appendix.

Lemma 3.4. *Let $(E_t)_t$ be a volume preserving mean curvature flow in $\mathbb{T}^n$. Then for every $0\le t<T^*$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{H}_t-H_t\|^2_{L^2(\partial E_t)}=-2\partial^2P(E_t)[\bar{H}_t-H_t]+\int_{\partial E_t}H_t(\bar{H}_t-H_t)^3\,\mathrm{d}\mathcal{H}^{n-1}\quad\textit{and} \tag{3.25}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_\tau H_t\|^2_{L^2(\partial E_t)}=&-2\int_{\partial E_t}(\Delta_\tau H_t)^2-(\bar{H}_t-H_t)|B_t|^2\Delta_\tau H_t\,\mathrm{d}\mathcal{H}^{n-1}\\&-2\int_{\partial E_t}(\bar{H}_t-H_t)\langle\nabla_\tau H_t,B_t\nabla_\tau H_t\rangle\,\mathrm{d}\mathcal{H}^{n-1}\\&+\int_{\partial E_t}|\nabla_\tau H_t|^2(\bar{H}_t-H_t)H_t\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{3.26}$$

We are now ready to prove Theorem 3.1. We divide it into four steps.

Proof of Theorem 3.1. Step 1. We want first to utilize several lemmas and estimates we have gathered by controlling C^1- and $W^{2,p}$-norms for $1\le p\le 6$. Indeed, by Remark 2.3 there is a constant $K_0\in\mathbb{R}_+$ depending on F, such that for every $\psi\in C^\infty(\partial F)$ and $1\le p\le 6$

$$\|\psi\|_{C^1(\partial F)},\|\psi\|_{W^{2,p}(\partial F)}\le K_0\|\psi\|_{H^3(\partial F)}. \tag{3.27}$$

Notice that the assumption $n\le 4$ is really needed for this conclusion. Using the estimate (3.27) we find $0<\delta<1$ and $1<K<\infty$ so that for any $\psi\in C^\infty(\partial F)$ with $\|\psi\|_{H^3(\partial F)}\le\delta$ the norm $\|\psi\|_{C^1(\partial F)}$ is so small that the following four conditions hold.

(i) Lemma 2.7 is satisfied, i.e., F_ψ is a well-defined smooth set. Moreover, we may assume

$$\|E_\psi, F\|_{C^1(\mathbb{T}^n)} \le \frac{\delta_F}{2}, \tag{3.28}$$

where δ_F is as in the lemma.

(ii) The inequalities (2.10) (for $\bar{d}_F$, then $\bar{d}_F \circ \Phi_\psi = \psi$) and (3.4) are satisfied with K, i.e.,

$$K^{-1}\|\psi\|_{L^2(\partial F)} \le \|\bar{d}_F\|_{L^2(\partial F_\psi)}, \sqrt{D_F(F_\psi)} \le K\|\psi\|_{L^2(\partial F)}. \tag{3.29}$$

(iii) Lemma 3.3 and (3.23) are satisfied with K, i.e.,

$$\|\psi\|_{H^3(\partial F)} \le K\left(\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} + \sqrt{D_F(F_\psi)}\right) \tag{3.30}$$

$$\|\bar{H}_{F_\psi} - H_{F_\psi}\|_{H^1(\partial F_\psi)} \le K\|\psi\|_{H^3(\partial F)}. \tag{3.31}$$

(iv) Lemma 2.9 is satisfied for $\bar{H}_{F_\psi} - H_{F_\psi}$ and $1 \le p \le 6$ with K, i.e.,

$$\|\bar{H}_{F_\psi} - H_{F_\psi}\|_{L^p(\partial F_\psi)} \le K\|\bar{H}_{F_\psi} - H_{F_\psi}\|_{H^1(\partial F_\psi)}. \tag{3.32}$$

Moreover, we may assume δ to be so small and K to be so large that by (3.27), (2.9), Lemma 2.10 and Lemma 2.14

$$\|\nabla_\tau H_{F_\psi}\|_{L^4(\partial F_\psi)} \le K\left(\|\Delta_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} + \|\bar{H}_{F_\psi} - H_{F_\psi}\|_{H^1(\partial F_\psi)}\right), \tag{3.33}$$

$$\|H_{F_\psi}\|_{L^p(\partial F_\psi)}, \|B_{F_\psi}\|_{L^p(\partial F_\psi)} \le K \quad \text{for} \quad 1 \le p \le 6 \quad \text{and} \tag{3.34}$$

$$\sigma_1\|\bar{H}_{F_\psi} - H_{F_\psi}\|^2_{H^1(\partial F_\psi)} \le \partial^2 P(F_\psi)[\bar{H}_{F_\psi} - H_{F_\psi}]. \tag{3.35}$$

Next we fix the constants C_0 and ε_0 by setting

$$C_0 = \frac{4K^6 + 1}{\sigma_1} \tag{3.36}$$

$$\varepsilon_0 = \min\left\{\delta, \frac{\min\{\sigma_1, 1\}}{16K^6}\right\}. \tag{3.37}$$

Again, for given $0 < \varepsilon \le \varepsilon_0$ set γ_ε to be the maximal $0 < s \le \frac{\varepsilon}{2}$ satisfying

$$\frac{2}{\sigma_1}\sqrt{s} + K^2 s^2 \le \frac{\varepsilon^2}{16K^2} \quad \text{and} \quad K^2(1 + C_0)s \le \frac{\varepsilon}{4}. \tag{3.38}$$

Step 2. Suppose that $(E_t)_t$ is the volume preserving mean curvature flow with a smooth initial datum $E_0 = F_{\psi_0}$, where $\psi_0 \in C^\infty(\partial F)$ and $\|\psi_0\|_{H^3(\partial F)} \le \gamma_\varepsilon$. Since $t \mapsto \|E_t, F\|_{C^1}$ is continuous on $[0, T^*[$ (recall T^* is the maximal lifetime), then it follows from Lemma 2.7 and (3.28) that we may write $E_t = F_{\psi_t}$ for unique $\psi_t \in C^\infty(\partial F)$ with continuous $t \mapsto \|\psi_t\|_{H^3(\partial F)}$ over a short time period. Hence

$$T_\varepsilon = \sup\left\{s \in [0, T^*[\,:\, E_t = F_{\psi_t} \text{ for } \psi_t \in C^\infty(\partial F),\ \|\psi_t\|_{H^3(\partial F)} \le \varepsilon\ \forall t \in [0, s]\right\}$$

must be a positive number. The key idea is to show that the claim of theorem is satisfied for ε on the time interval $[0, T_\varepsilon[$ and by virtue of the choice of γ_ε we have in fact

$$\|\psi_t\|_{H^3(\partial F)} \le \frac{\varepsilon}{2} \quad \text{on} \quad [0, T_\varepsilon[. \tag{3.39}$$

By using a similar continuity argument as before one shows that the condition (3.39) implies $T_\varepsilon = T^*$.

Step 3. For every $t \in\ [0, T_\varepsilon[$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)} \\
&\overset{(3.25)}{=} -2\partial^2 P(E_t)[\bar{H}_t - H_t] + \int_{\partial E_t} H_t(\bar{H}_t - H_t)^3 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\overset{(3.35)}{\le} -2\sigma_1\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} + \int_{\partial E_t} H_t(\bar{H}_t - H_t)^3 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\le -2\sigma_1\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} + \|H_t\|_{L^4(\partial E_t)}\|\bar{H}_t - H_t\|^3_{L^4(\partial E_t)} \\
&\overset{(3.34)}{\le} -2\sigma_1\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} + K\|\bar{H}_t - H_t\|^3_{L^4(\partial E_t)} \\
&\overset{(3.32)}{\le} -2\sigma_1\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} + K^4\|\bar{H}_t - H_t\|^3_{H^1(\partial E_t)} \\
&\overset{(3.31)}{\le} \left(-2\sigma_1 + K^5\|\psi_t\|_{H^3(\partial F)}\right)\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} \\
&\le \left(-2\sigma_1 + K^5\varepsilon\right)\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} \\
&\overset{(3.37)}{\le} -\sigma_1\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)}.
\end{aligned} \tag{3.40}$$

Since $-\sigma_1\|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} \le -\sigma_1\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)}$, by using Grönwall's lemma we obtain

$$\begin{aligned}
\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)} &\le \|\bar{H}_0 - H_0\|^2_{L^2(\partial E_t)}e^{-\sigma_1 t} \\
&\overset{(3.31)}{\le} K^2\|\psi_0\|^2_{H^3(\partial F)}e^{-\sigma_1 t} \\
&\overset{(3.37)}{\le} \|\psi_0\|_{H^3(\partial F)}e^{-\sigma_1 t}
\end{aligned} \tag{3.41}$$

so (3.1) holds on $[0, T_\varepsilon[$. Next we estimate $D_F(E_t)$ on $[0, T_\varepsilon[$. For the time derivative

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} D_F(E_t) &\overset{(3.5)}{=} \int_{\partial E_t} \bar{d}_F(\bar{H}_t - H_t) \mathrm{d}\mathcal{H}^{n-1} \\
&\leq \|\bar{d}_F\|_{L^2(\partial E_t)} \|\bar{H}_t - H_t\|_{L^2(\partial E_t)} \\
&\overset{(3.29)}{\leq} K\|\psi_t\|_{L^2(\partial F)} \|\bar{H}_t - H_t\|_{L^2(\partial E_t)} \\
&\leq K\varepsilon \|\bar{H}_t - H_t\|_{L^2(\partial E_t)} \\
&\overset{(3.41)}{\leq} K\varepsilon \|\psi_0\|_{H^3(\partial F)}^{\frac{1}{2}} e^{-\frac{1}{2}\sigma_1 t} \\
&\overset{(3.37)}{\leq} \|\psi_0\|_{H^3(\partial F)}^{\frac{1}{2}} e^{-\frac{1}{2}\sigma_1 t}.
\end{aligned}
$$

Thus integrating over time yields

$$
\begin{aligned}
D_F(E_t) &\leq \frac{2}{\sigma_1} \|\psi_0\|_{H^3(\partial F)}^{\frac{1}{2}} \left(1 - e^{-\frac{\sigma_1}{2}}\right) + D_F(E_0) \\
&\overset{(3.29)}{\leq} \frac{2}{\sigma_1} \|\psi_0\|_{H^3(\partial F)}^{\frac{1}{2}} + K^2 \|\psi_0\|_{L^2(\partial F)}^2 \\
&\overset{(3.38)}{\leq} \frac{\varepsilon^2}{16K^2}.
\end{aligned} \tag{3.42}
$$

Step 4. In this last step we finish the proof by showing that (3.2) and (3.39) are satisfied on $[0, T_\varepsilon[$. To this end we have to estimate $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_\tau H_t\|_{L^2(\partial E_t)}^2$ on $[0, T_\varepsilon[$. Recall that by (3.26) we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_\tau H_t\|_{L^2(\partial E_t)}^2 &= -2\int_{\partial E_t} (\Delta_\tau H_t)^2 - (\bar{H}_t - H_t)|B_t|^2 \Delta_\tau H_t \, \mathrm{d}\mathcal{H}^{n-1} \\
&\quad - 2\int_{\partial E_t} (\bar{H}_t - H_t)\langle \nabla_\tau H_t, B_t \nabla_\tau H_t\rangle \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial E_t} |\nabla_\tau H_t|^2 (\bar{H}_t - H_t) H_t \, \mathrm{d}\mathcal{H}^{n-1} \\
&= T_1 + T_2 + T_3.
\end{aligned}
$$

Next we estimate the terms T_1, T_2 and T_3. First we have

$$
\begin{aligned}
T_1 &\overset{\text{Young}}{\leq} -\|\Delta_\tau H_t\|_{L^2(\partial E_t)}^2 + 4\int_{\partial E_t} |\bar{H}_t - H_t|^2 |B_t|^4 \, \mathrm{d}\mathcal{H}^{n-1} \\
&\leq -\|\Delta_\tau H_t\|_{L^2(\partial E_t)}^2 + 4\|\bar{H}_t - H_t\|_{L^6(\partial E_t)}^2 \|B_t\|_{L^6(\partial E_t)}^4 \\
&\overset{(3.34)}{\leq} -\|\Delta_\tau H_t\|_{L^2(\partial E_t)}^2 + 4K^4 \|\bar{H}_t - H_t\|_{L^6(\partial E_t)}^2 \\
&\overset{(3.32)}{\leq} -\|\Delta_\tau H_t\|_{L^2(\partial E_t)}^2 + 4K^6 \|\bar{H}_t - H_t\|_{H^1(\partial E_t)}^2.
\end{aligned} \tag{3.43}
$$

Second

$$\begin{aligned}
T_2 &\le 2 \int_{\partial E_t} |\bar{H}_t - H_t| |\nabla_\tau H_t|^2 |B_t| \, \mathrm{d}\mathcal{H}^{n-1} \\
&\le 2 \|\nabla_\tau H_t\|^2_{L^4(\partial E_t)} \|\bar{H}_t - H_t\|_{L^4(\partial E_t)} \|B_t\|_{L^4(\partial E_t)} \\
&\overset{(3.34)}{\le} 2K \|\nabla_\tau H_t\|^2_{L^4(\partial E_t)} \|\bar{H}_t - H_t\|_{L^4(\partial E_t)} \\
&\overset{(3.33)}{\le} 2K^3 \left(\|\Delta_\tau H_t\|_{L^2(\partial E_t)} + \|\bar{H}_t - H_t\|_{H^1(\partial E_t)}\right)^2 \|\bar{H}_t - H_t\|_{L^4(\partial E_t)} \\
&\overset{(3.32)}{\le} 2K^4 \|\bar{H}_t - H_t\|_{H^1(\partial E_t)} \left(\|\Delta_\tau H_t\|_{L^2(\partial E_t)} + \|\bar{H}_t - H_t\|_{H^1(\partial E_t)}\right)^2 \\
&\overset{(3.31)}{\le} 2K^5 \|\psi_t\|_{H^3(\partial F)} \left(\|\Delta_\tau H_t\|_{L^2(\partial E_t)} + \|\bar{H}_t - H_t\|_{H^1(\partial E_t)}\right)^2 \\
&\le 4K^5 \|\psi_t\|_{H^3(\partial F)} \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} + 4K^5 \|\psi_t\|_{H^3(\partial F)} \|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} \\
&\le 4K^5 \varepsilon \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} + 4K^5 \varepsilon \|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} \\
&\overset{(3.37)}{\le} \frac{1}{4} \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} + \frac{1}{2} \|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)}.
\end{aligned} \tag{3.44}$$

Finally by estimating in a similar way as above we obtain

$$T_3 \le \frac{1}{8} \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} + \frac{1}{4} \|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)}. \tag{3.45}$$

Hence (3.43), (3.44) and (3.45) together yield

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_\tau H_t\|^2_{L^2(\partial E_t)} &\le -\frac{1}{2} \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} + \left(4K^6 + 1\right) \|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)} \\
&\overset{(3.36)}{=} -\frac{1}{2} \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)} + \sigma_1 C_0 \|\bar{H}_t - H_t\|^2_{H^1(\partial E_t)}
\end{aligned}$$

on $[0, T_\varepsilon[$. Then the previous estimate with (3.40) yields that for every $t \in [0, T_\varepsilon[$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\|\nabla_\tau H_t\|^2_{L^2(\partial E_t)} + C_0 \|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)} \right] \le -\frac{1}{2} \|\Delta_\tau H_t\|^2_{L^2(\partial E_t)},$$

which implies (3.2). In particular

$$t \mapsto \|\nabla_\tau H_t\|^2_{L^2(\partial E_t)} + C_0 \|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)}$$

is decreasing map on $[0, T_\varepsilon[$ and therefore

$$\|\nabla_\tau H_t\|_{L^2(\partial E_t)} \le \|\nabla_\tau H_0\|_{L^2(\partial E_0)} + C_0 \|\bar{H}_0 - H_0\|_{L^2(\partial E_0)}. \tag{3.46}$$

Finally for every $t \in [0, T_\varepsilon[$

$$\begin{aligned}\|\psi_t\|_{H^3(\partial E_t)} &\overset{(3.30)}{\leq} K\left(\|\nabla_\tau H_t\|_{L^2(\partial E_t)} + \sqrt{D_F(E_t)}\right)\\ &\overset{(3.46)}{\leq} K\left(\|\nabla_\tau H_0\|_{L^2(\partial E_0)} + C_0\|\bar{H}_0 - H_0\|_{L^2(\partial E_0)}\right) + K\sqrt{D_F(E_t)}\\ &\overset{(3.42)}{\leq} K\,(1+C_0)\,\|\bar{H}_0 - H_0\|_{H^1(\partial E_0)} + \frac{\varepsilon}{4}\\ &\overset{(3.31)}{\leq} K^2\,(1+C_0)\,\|\psi_0\|_{H^3(\partial E_t)} + \frac{\varepsilon}{4}\\ &\overset{(3.38)}{\leq} \frac{\varepsilon}{2}. \quad \square\end{aligned}$$

4. The main result

In this section we will prove the main result. We give first the technical statement of the theorem in contrast to the heuristical one we presented in the Introduction.

Theorem 4.1 *(Main Theorem). Let $\mathbb{T}^n$ be a flat torus with $n = 3, 4$ and assume that $F \subset \mathbb{T}^n$ is a strictly stable set. There exist positive constants $\delta_0, \sigma_0 \in \mathbb{R}_+$ depending on F such that the following hold.*

If E_0 is a smooth set in $\mathbb{T}^n$ with $|E_0| = |F|$ of the form $E_0 = F_{\psi_0}$, where $\psi_0 \in C^\infty(\partial F)$ and $\|\psi_0\|_{H^3(\partial F)} \leq \delta_0$, then the volume preserving mean curvature flow $(E_t)_t$ in $\mathbb{T}^n$ with the initial datum E_0 satisfies the following conditions.

(i) *The flow has infinite lifetime.*
(ii) *There exist $p = p(F, E_0) \in \mathbb{R}^n$ and $C = C(F, E_0) \in \mathbb{R}_+$ such that the flow converges to $F + p$ exponentially fast in $W^{2,5}$-sense, that is, $E_t = (F + p)_{\varphi_t}$ for $\varphi_t \in C^\infty(\partial(F + p))$ and $\|\varphi_t\|_{W^{2,5}} \leq Ce^{-\sigma_0 t}$.*
(iii) *$|p| \to 0$ and $C \to 0$ as $\|\psi_0\|_{H^3(\partial F)} \to 0$.*

Remark 4.2. In the statement of the main theorem the $W^{2,5}$-convergence can be replaced by $W^{2,q}$-convergence, where $1 \leq q < \infty$, if $n = 3$, and $1 \leq q < 6$, if $n = 4$. In this case the proof would be similar to the original proof.

The main idea of the proof is obviously to employ the short time existence (Theorem 2.15) and the monotonicity result (Theorem 3.1).

Proof of the Main Theorem. Let $F \subset \mathbb{T}^n$ be a strictly stable set and let us fix the constants δ_0 and σ_0 for F as in the statement of the main theorem. Let $\varepsilon_0, \sigma_1 \in \mathbb{R}_+$ and $0 < \gamma_\varepsilon < \varepsilon$ (for every $0 < \varepsilon \leq \varepsilon_0$) for F be as in Theorem 3.1. Choose first a positive c so small that the following hold.

(i) The condition $\|\psi\|_{C^1(\partial F)} \leq c$ for $\psi \in C^{1,\frac{1}{4}}(\partial F)$ implies via Lemma 2.7 that the set E_ψ is defined as a $C^{1,\frac{1}{4}}$-set, the map Φ_ψ defined as in the same lemma is a $C^{1,\frac{1}{4}}$-diffeomorphism from ∂F to ∂F_ψ and

$$K^{-1}|\partial F| \leq |\partial F_\psi| \leq K|\partial F| \tag{4.1}$$

for some real number $K \geq 1$ depending on c and F.

(ii) Further, if ψ is smooth, the same condition implies via (3.24) that by increasing K, if necessary,

$$\|H_{F_\psi}\|_{H^1(F_\psi)} \leq K\left(\|\psi\|_{H^3(\partial F)} + |H_F|\right). \tag{4.2}$$

(iii) Since a translate of F satisfies Lemma 3.3 and (3.4) with the same bounds as F, then by using the previous lemmas and Lemma 2.7, possibly decreasing c and increasing K, we obtain the following. Assume $F + p = F_g$ with $g \in C^\infty(\partial F)$ and $\|g\|_{C^1(\partial F)} \leq c$. Then for every $\psi \in C^\infty(\partial F)$ with $\|\psi\|_{C^1(\partial F)} \leq c$ there is a unique $\varphi \in C^\infty(\partial(F + p))$ such that $F_\psi = (F + p)_\varphi$ and we have the following uniform estimates

$$K^{-1}\sqrt{D_{F+p}(F_\psi)} \leq \|\varphi\|_{L^2(\partial(F+p))} \leq K\sqrt{D_{F+p}(F_\psi)} \quad \text{and} \tag{4.3}$$

$$K^{-1}\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} \leq \|\varphi\|_{H^3(\partial(F+p))} \leq K\left(\|\nabla_\tau H_{F_\psi}\|_{L^2(\partial F_\psi)} + \sqrt{D_{F+p}(F_\psi)}\right). \tag{4.4}$$

(iv) Finally the condition $\|\psi\|_{C^{1,\frac{1}{4}}(\partial F)} \leq c$ for $\psi \in C^\infty(\partial F)$ means that Theorem 2.15 (in the case $\beta = \frac{1}{4}$) is satisfied for the initial set E_ψ.

Next choose $0 < \varepsilon_1 \leq \varepsilon_0$ such that the condition $\|\psi\|_{H^3(\partial F)} \leq \varepsilon_1$ for $\psi \in C^\infty(\partial F)$ means that first Lemma 2.13 is satisfied, provided that F_ψ is critical with $|F_\psi| = |F|$, and second via Lemma 2.6 $\|\psi\|_{C^1(\partial F)} \leq \|\psi\|_{C^{1,\frac{1}{4}}(\partial F)} \leq c$ (here we really need the assumption $n \leq 4$). At this point we set

$$\delta_0 = \gamma_{\varepsilon_1} \quad \text{and} \quad \sigma_0 = -\frac{\sigma_1}{2}\left(\frac{1}{3} - \frac{n-1}{10}\right). \tag{4.5}$$

Fix an arbitrary $\psi_0 \in C^\infty(\partial F)$ with $\|\psi_0\|_{H^3(\partial F)} \leq \delta_0$ and $|F_{\psi_0}| = |F|$. Then by Theorem 2.15 there exists a unique volume preserving mean curvature flow $(E_t)_t$ starting from $E_0 = F_{\psi_0}$ and the maximal lifetime T^* is bounded from below by $T = T(F, \frac{1}{4}) > 0$ as in Theorem 2.15. Again, by Theorem 3.1 we may write $E_t = F_{\psi_t}$, where $\psi_t \in C^\infty(\partial F)$ with $\|\psi_t\|_{H^3(\partial F)} \leq \varepsilon_1$, and the inequalities (3.1) and (3.2) are satisfied for every $t \in [0, T^*[$. We divide the proof into three steps, whose statements are listed below.

Step 1. The flow $(E_t)_t$ has infinite lifetime and there exists $\psi_\infty \in H^3(\partial F)$ with $\|\psi_\infty\|_{H^3(\partial F)} \leq \varepsilon$ such that $\psi_t \to \psi_\infty$ in $C^{1,\frac{1}{4}}(\partial F)$. Further, there exist a positive constant C_F independent of the choice of ψ_0 and a decreasing $\rho : [0, \delta_0] \to [0, \infty[$ with $\lim_{s\to 0+} \rho(s) = 0$ such that every $t \in [0, \infty[$

$$D_{E_\infty}(E_t) \leq C_F\|\psi_0\|_{H^3(\partial F)}e^{-\sigma_1 t} \quad \text{and} \tag{4.6}$$

$$\|\psi_\infty\|_{L^\infty(\partial F)} \leq \rho\left(\|\psi_0\|_{H^3(\partial F)}\right), \tag{4.7}$$

where $E_\infty = F_{\psi_\infty}$ is the corresponding $C^{1,\frac{1}{4}}$-limit set.

Step 2. The limit set is of the form $E_\infty = F + p$, where $p \to 0$ as $\|\psi_0\|_{H^3(\partial F)} \to 0$.
Step 3. The $W^{2,5}$-convergence of the flow: For each $t \in [0, \infty[$ there is $\varphi_t \in C^\infty(\partial E_\infty)$ with $E_t = (E_\infty)_{\varphi_t}$ and $\|\varphi_t\|_{W^{2,5}(\partial F)} \le Ce^{-\sigma_0 t}$, where $C \to 0$ as $\|\psi_0\|_{H^3(\partial F)} \to 0$.

Since $\psi_0 \in C^\infty(\partial F)$ with $\|\psi_0\|_{H^3(\partial F)} \le \delta_0$ was arbitrarily chosen, the claim of theorem follows immediately from these statements. Let us prove them in order as listed.

Proof of Step 1. Assume by contradiction $T^* < \infty$ and choose $\hat{t} \in [0, T^*[$ such that $T^* - \hat{t} < T$, where $T = T(F, \frac{1}{4})$ as in Theorem 2.15. Now $\|\psi_{\hat{t}}\|_{H^3(\partial F)} \le \varepsilon_1$ so $\|\psi_{\hat{t}}\|_{C^{1,\frac{1}{4}}(\partial F)} \le c$ and hence by Theorem 2.15 there exists a unique volume preserving mean curvature flow $(\hat{E}_t)_t$ starting from $E_{\hat{t}}$ with a maximal lifetime at least T. It follows from the uniqueness and from the semigroup property of $(E_t)_t$ that $E_t = \hat{E}_{t-\hat{t}}$ for every $t \in [\hat{t}, T^*[$. This means that the flow $(E_t)_t$ can be extended beyond T^*, which contradicts its maximality. Therefore it holds $T^* = \infty$.

Take a sequence $(t_k)_{k=1}^\infty \subset \mathbb{R}_+$ with $t_k \to \infty$. Since $\|\psi_{t_k}\|_{H^3(\partial F)} \le \varepsilon_1$ for every k and $H^3(\partial F)$ is weakly compact, then, up to a subsequence, there is a weak limit $\psi_\infty \in H^3(\partial F)$ with $\|\psi_\infty\|_{H^3(\partial F)} \le \varepsilon_1$. Further, it follows from Lemma 2.6 that the sequence converges to ψ_∞ in $C^{1,\frac{1}{4}}(\partial F)$. Now C^1-convergence implies that $\|\psi_\infty\|_{C^1(\partial F)} \le c$ so $E_\infty := F_{\psi_\infty}$ is defined as a $C^{1,\frac{1}{4}}$-set and the map Φ_{ψ_∞} is a $C^{1,\frac{1}{4}}$-diffeomorphism from ∂F to ∂E_∞.

Next we check that (4.6) holds for every t. Notice first that $|E_{t_k} \Delta E_\infty| \to 0$, which implies $D_{E_\infty}(E_{t_k}) \to 0$. For a fixed $t \in [0, \infty[$ and every $t_k > t$ we may estimate

$$
\begin{aligned}
|D_{E_\infty}(E_{t_k}) - D_{E_\infty}(E_t)| &= \left| \int_t^{t_k} \frac{\mathrm{d}}{\mathrm{d}s} D_{E_\infty}(E_s)\, \mathrm{d}s \right| \\
&\overset{(3.5)}{=} \left| \int_t^{t_k} \int_{\partial E_s} \bar{d}_{E_\infty}(\bar{H}_s - H_s)\, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}s \right| \\
&\le \int_t^{t_k} \|\bar{d}_{E_\infty}\|_{L^2(\partial E_s)} \|\bar{H}_s - H_s\|_{L^2(\partial E_s)}\, \mathrm{d}s \\
&\le \sqrt{n} \int_t^{t_k} |\partial E_s| \|\bar{H}_s - H_s\|_{L^2(\partial E_s)}\, \mathrm{d}s \\
&\overset{(4.1)}{\le} \sqrt{n} K |\partial F| \int_t^{t_k} \|\bar{H}_s - H_s\|_{L^2(\partial E_s)}\, \mathrm{d}s \\
&\overset{(3.1)}{\le} \sqrt{n} K |\partial F| \|\psi_0\|_{H^3(\partial F)} \int_t^{t_k} e^{-\sigma_1 s}\, \mathrm{d}s \\
&\le \frac{\sqrt{n} K |\partial F|}{\sigma_1} \|\psi_0\|_{H^3(\partial F)} e^{-\sigma_1 t}.
\end{aligned}
$$

Since $D_{E_\infty}(E_{t_k}) \to 0$, then the previous estimate implies (4.6) for t. By doing a similar estimate for $D_F(E_{t_k}) - D_F(E_0)$, using (4.3) for F and Lemma (4.7) and recalling that $\|\psi_{t_k}\|_{H^3(\partial F)} \le \varepsilon_1$ we find a constant $\tilde{C}_F$ not depending on the choice of ψ_0 such that

$$\|\psi_{t_k}\|_{L^\infty(\partial F)} \le \tilde{C}_F \|\psi_0\|_{H^3(\partial F)}^{\frac{1}{2}(1-\frac{n-1}{6})}.$$

Thus by passing to limit we see that (4.7) holds for ψ_∞. Finally we check that the full $C^{1,\frac{1}{4}}$-convergence in time holds. To this end, it suffices to show that every sequence $(\tilde{t}_k)_{k=1}^\infty \subset \mathbb{R}_+$ with $\tilde{t}_k \to \infty$ has a subsequence converging to ψ_∞ in $C^{1,\frac{1}{4}}(\partial F)$. Indeed, by arguing as previously in a case of such sequence $(\tilde{t}_k)_{k=1}^\infty \subset \mathbb{R}_+$, we find a subsequence converging to some limit $\tilde{\psi}_\infty \in H^3(\partial F)$ in $C^{1,\frac{1}{4}}(\partial F)$ and $F_{\tilde{\psi}_\infty}$ is defined as a $C^{1,\frac{1}{4}}$-set. We may again assume that the subsequence is the whole sequence and hence $|E_{\tilde{t}_k} \Delta F_{\tilde{\psi}_\infty}| \to 0$, which implies together (4.6) and the boundedness of $\bar{d}_{E_\infty}$ that

$$D_{E_\infty}\left(F_{\tilde{\psi}_\infty}\right) = \lim_k D_{E_\infty}\left(E_{\tilde{t}_k}\right) = 0.$$

This implies that $E_\infty = F_{\psi_\infty}$ and further $\psi_\infty = \tilde{\psi}_\infty$. Thus the first step has been concluded.

Proof of Step 2. First we show that E_∞ is a smooth and critical set. Since E_∞ is a $C^{1,\frac{1}{4}}$-set, then thanks to Lemma 2.1 it suffices to show it to be stationary. We need to find $\lambda_\infty \in \mathbb{R}$ such that for every $f \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$

$$\int_{\partial E_\infty} \operatorname{div}_\tau f \, d\mathcal{H}^{n-1} = \lambda_\infty \int_{\partial E_\infty} \langle f, \nu_\infty \rangle \, d\mathcal{H}^{n-1}, \tag{4.8}$$

where ν_∞ is the corresponding unit normal field of ∂E_∞ with inside-out orientation. Since $\psi_t \to \psi_\infty$ in $C^1(\partial F)$, then $\Phi_{\psi_t} \to \Phi_{\psi_\infty}$, $\nu_t \circ \Phi_{\psi_t} \to \nu_\infty \circ \Phi_{\psi_\infty}$ and $J_\tau \Phi_{\psi_t} \to J_\tau \Phi_{\psi_\infty}$ uniformly on ∂F. Thus by using the change of variables formula we obtain for every $f \in C^\infty(\mathbb{T}^n;\mathbb{R}^n)$

$$\int_{\partial E_t} \operatorname{div}_\tau f \, d\mathcal{H}^{n-1} \longrightarrow \int_{\partial E_\infty} \operatorname{div}_\tau f \, d\mathcal{H}^{n-1} \quad \text{and} \tag{4.9}$$

$$\int_{\partial E_t} \langle f, \nu_t \rangle \, d\mathcal{H}^{n-1} \longrightarrow \int_{\partial E_\infty} \langle f, \nu_\infty \rangle \, d\mathcal{H}^{n-1}. \tag{4.10}$$

By using (4.1), (4.2) and Hölder's inequality we see that $\bar{H}_t$ is bounded in time and hence we find a sequence $(\bar{H}_{t_k})_k$, $t_k \to \infty$, converging to some real number say λ_∞. By the divergence theorem

$$\int_{\partial E_{t_k}} \operatorname{div}_\tau f \, d\mathcal{H}^{n-1} = \int_{\partial E_{t_k}} H_{t_k} \langle f, \nu_t \rangle \, d\mathcal{H}^{n-1}$$

$$= \bar{H}_{t_k} \int_{\partial E_{t_k}} \langle f, \nu_{t_k} \rangle \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial E_{t_k}} (H_{t_k} - \bar{H}_{t_k}) \langle f, \nu_t \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

and thus by letting $t_k \to \infty$ and recalling (4.9) and (4.10) we obtain (4.8), since

$$\left| \int_{\partial E_{t_k}} (H_{t_k} - \bar{H}_{t_k}) \langle f, \nu_{t_k} \rangle \, \mathrm{d}\mathcal{H}^{n-1} \right| \leq \sup_{\mathbb{T}^n} |f| |\partial E_{t_k}|^{\frac{1}{2}} \| (H_{t_k} - \bar{H}_{t_k}) \|_{L^2(\partial E_{t_k})}$$
$$\stackrel{(4.1)}{\leq} \sup_{\mathbb{T}^n} |f| \, (K |\partial F|)^{\frac{1}{2}} \, \| (H_{t_k} - \bar{H}_{t_k}) \|_{L^2(\partial E_{t_k})}$$
$$\stackrel{(3.1)}{\leq} \sup_{\mathbb{T}^n} |f| \left(K |\partial F| \|\psi_0\|_{H^3(\partial F)} e^{-\sigma_1 t_k} \right)^{\frac{1}{2}} .$$

Thus E_∞ is a smooth and critical set and since $\|\psi_\infty\|_{H^3(\partial F)} \leq \varepsilon_1$ (recall the choice of ε_1) and $|E_\infty| = |F|$ (by (4.4)), it follows from Lemma 2.13 that $E_\infty = F + p$ for some $p \in \mathbb{R}^n$. Since now $d_H(F, E_\infty) \leq \|\psi_\infty\|_{L^\infty(\partial F)}$, then it follows from (4.7) that $d_H(F, E_\infty) \to 0$ as $\|\psi_0\|_{H^3(\partial F)}$ tends to zero. This implies that we may choose p in such a way that simultaneously $p \to 0$.

Proof of Step 3. Since now $E + p = F_\infty$ and $\|\psi_\infty\|_{C^1(\partial E)}, \|\psi_t\|_{C^1(\partial E)} \leq c$, then by (iii) we may write ∂E_t as a smooth graph in normal direction over $\partial(F + p)$, i.e., for every $t \in [0, \infty[$ there is a unique $\varphi_t \in C^\infty(\partial E_\infty)$ for which $E_t = (E_\infty)_{\varphi_t}$. Again, for every $t \in [0, \infty[$

$$\|\varphi_t\|_{L^2(\partial E_\infty)} \stackrel{(4.3)}{\leq} K \sqrt{D_{E_\infty}(E_t)} \stackrel{(4.6)}{\leq} K C_F^{\frac{1}{2}} e^{-\frac{\sigma_1}{2} t}$$

and

$$\|\varphi_t\|_{H^3(\partial E_\infty)} \stackrel{(4.4)}{\leq} K \left(\|\nabla_\tau H_t\|_{L^2(\partial E_t)} + \sqrt{D_{E_\infty}(E_t)} \right)$$
$$\stackrel{(3.2)}{\leq} K \left(\|\nabla_\tau H_0\|_{L^2(\partial E_0)} + C_0^{\frac{1}{2}} \|\bar{H}_0 - H_0\|_{L^2(\partial E_0)} + \sqrt{D_{E_\infty}(E_t)} \right)$$
$$\stackrel{(4.4)}{\leq} K \left(K \|\psi_0\|_{H^3(\partial F)} + C_0^{\frac{1}{2}} \|\bar{H}_0 - H_0\|_{L^2(\partial E_0)} + \sqrt{D_{E_\infty}(E_t)} \right)$$
$$\stackrel{(3.1)}{\leq} K \left(K \|\psi_0\|_{H^3(\partial F)} + C_0^{\frac{1}{2}} \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} + \sqrt{D_{E_\infty}(E_t)} \right)$$
$$\stackrel{(4.6)}{\leq} K \left(K \|\psi_0\|_{H^3(\partial F)} + C_0^{\frac{1}{2}} \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} + C_F^{\frac{1}{2}} \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} \right)$$
$$\leq K \left(K \varepsilon_1^{\frac{1}{2}} + C_0^{\frac{1}{2}} + C_F^{\frac{1}{2}} \right) \|\psi_0\|_{H^3(\partial F)}$$

This means that there exists a positive constant C'_F independent of the choice of ψ_0 such that $\|\varphi_t\|_{L^2(\partial E_\infty)} \leq C'_F \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} e^{-\frac{\sigma_1}{2} t}$ and $\|\varphi_t\|_{H^3(\partial E_\infty)} \leq C'_F \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)}$ for every $t \in [0, \infty[$.

Since $\partial(F+p)$ shares same interpolation bounds than ∂F, then by using the previous estimates and Lemma 2.2 in the case

$$\frac{1}{5} = \frac{j}{n-1} + \left(\frac{1}{2} - \frac{3}{n-1}\right)\alpha + \frac{1-\alpha}{2} \tag{4.11}$$

with $\alpha = \frac{j}{3} + \frac{n-1}{10}$, $j = 0, 1, 2$ and a corresponding interpolation constant (which we may assume to be the same C'_F) we obtain

$$\begin{aligned}
\|\nabla^j_{\mathrm{co}}\varphi_t\|_{L^5(\partial E_\infty)} &\le C'_F \|\nabla^3_{\mathrm{co}}\varphi_t\|^{\frac{j}{3}+\frac{n-1}{10}}_{L^2(\partial E_\infty)} \|\varphi_t\|^{\frac{3-j}{3}-\frac{n-1}{10}}_{L^2(\partial E_\infty)} \\
&\le C'_F \|\varphi_t\|^{\frac{j}{3}+\frac{n-1}{10}}_{H^3(\partial E_\infty)} \|\varphi_t\|^{\frac{3-j}{3}-\frac{n-1}{10}}_{L^2(\partial E_\infty)} \\
&\le (C'_F)^2 \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} e^{-\frac{\sigma_1}{2}\left(\frac{3-j}{3}-\frac{n-1}{10}\right)t} \\
&\le (C'_F)^2 \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} e^{-\frac{\sigma_1}{2}\left(\frac{1}{3}-\frac{n-1}{10}\right)t} \\
&\overset{(4.5)}{=} (C'_F)^2 \|\psi_0\|^{\frac{1}{2}}_{H^3(\partial F)} e^{-\sigma_0 t},
\end{aligned}$$

which implies that there exists such C as claimed. □

Let us finally recall Remark 4.2. In the last step of the previous proof we may replace 5 in the left hand side of (4.11) with any $q \ge 1$ as long as the corresponding α is strictly less than 1, because $\sigma_0 = \frac{\eta_0}{2}(1-\alpha)$ would then be strictly positive. In the case $n = 3$ by replacing 5 with any $1 \le q < \infty$ we obtain

$$\alpha = 1 - \frac{2}{3q},$$

so we see that any such q will do. Whereas in the case $n = 4$ doing so yields

$$\alpha = \frac{7}{6} - \frac{1}{q}$$

and hence q may take any values in the interval $[1, 6[$.

5. Appendix

C^1- and H^3-bounds. In this subsection we prove the estimates (2.10) and (2.11), Lemma 2.8, Lemma 2.9 and Lemma 2.10. We will use the same notation as earlier without any further mention. For sake of simplicity, we use the generic symbol C for a constant which may change line to line in the estimates.

Let us first fix a smooth set E and let U be a regular neighborhood of ∂E. Recall that $\bar{d}_E$ and $\pi_{\partial E}$ are C^k-bounded in U for every $k \in \mathbb{N}$. For every $\psi \in C^\infty(\partial E)$ we set the smooth extension $\psi_E = \psi \circ \pi_{\partial E}$. Then $\nabla \psi_E = \nabla_\tau \psi$ on ∂E and moreover the following decomposition holds on ∂E

$$\mathrm{D}^2\psi_E = -\nu_E \otimes B_E \nabla_\tau \psi - B_E \nabla_\tau \psi \otimes \nu_E + D_\tau^2 \psi. \tag{5.1}$$

For every $\psi \in C^\infty(\partial E)$ we set $\Phi_\psi : \partial E \to \mathbb{T}^n$, $\Phi_\psi(x) = x + \psi \nu_E(x)$, as in Lemma 2.7. We extend Φ_ψ to be the smooth map $U \to \mathbb{T}^n$ given by $\Phi_\psi(x) = x + \psi_E(x)\nabla \bar{d}_E(x)$. Then

$$\mathrm{D}\Phi_\psi = \begin{cases} I + \psi_E \mathrm{D}^2 \bar{d}_F + \nabla \bar{d}_F \otimes \nabla \psi_\pi & \text{in } U \text{ and} \\ I + \psi B_E + \nu_E \otimes \nabla_\tau \psi & \text{on } \partial E. \end{cases}$$

Now $\mathrm{D}\Phi_\psi \to I$ uniformly in U as $\|\psi\|_{C^1(\partial E)} \to 0$. Thus Φ_ψ is an orientation preserving diffeomorphism from U to its image and the set E_ψ is well-defined, when $\|\psi\|_{C^1(\partial E)}$ is small enough.[1] The inverse matrix on ∂E is then

$$(\mathrm{D}\Phi_\psi)^{-1} = (I - \nu_E \otimes \nabla_\tau \psi)\,(I + \psi B_E)^{-1}.$$

From now on, we assume $\|\psi\|_{C^1(\partial E)} \le \delta$ with δ small enough so that the previous hold. Further, we use the shorthand notation $A_\psi = (I + \psi B_E)^{-1}$ on ∂E. Set $u_\psi : \Phi_\psi(U) \to \mathbb{R}$, $u_\psi = \bar{d}_E \circ \Phi_\psi^{-1}$. Then $\partial E_\psi = u_\psi^{-1}(0)$ and $\nu_{E_\psi} = {}^{\nabla u_\psi}/_{|\nabla u_\psi|}$ on ∂E_ψ. Again

$$\nabla u_\psi \circ \Phi_\psi = (\mathrm{D}\Phi_\psi)^{-\mathrm{T}} \nu_E = \nu_E - A_\psi \nabla_\tau \psi \longrightarrow \nu_E$$

uniformly on ∂E as $\|\psi\|_{C^1(\partial E)} \to 0$. Thus $\nu_{E_\psi} \circ \Phi_\psi$ also converges uniformly to ν_E as ψ goes to zero in C^1-sense. The second fundamental form on ∂E_ψ can be written, with help of u_ψ, as

$$B_{E_\psi} = P_{\partial E_\psi} \mathrm{D}_\tau \left(\frac{\nabla u_\psi}{|\nabla u_\psi|} \right) = \frac{1}{|\nabla u_\psi|} (I - \nu_{E_\psi} \otimes \nu_{E_\psi}) \mathrm{D}^2 u_\psi (I - \nu_{E_\psi} \otimes \nu_{E_\psi}). \tag{5.2}$$

Omitting the details we may further compute that

$$\mathrm{D}^2 u_\psi \circ \Phi_\psi = A_\psi \left[B_E - \psi \left(\sum_{k=1}^n \langle \nu_\Gamma - A_\psi \nabla_\tau \psi, v_k \rangle \partial_{v_k} \mathrm{D}^2 \bar{d}_E \right) - \mathrm{D}^2 \psi_E \right] A_\psi \tag{5.3}$$

on ∂E, where $v_1, \dots, v_n$ is any orthonormal base of $\mathbb{R}^n$. Hence (5.1), (5.2) and (5.3) and the C^1-bound δ imply that $|\mathrm{D}^2 u_\psi \circ \Phi_\psi| \le C(1 + |\mathrm{D}_\tau^2 \psi|)$ on ∂E with some constant C and so (2.9) holds. Again, by combining the expressions (5.2) and (5.3) we may write on ∂E

$$H_{E_\psi} \circ \Phi_\psi = \mathrm{tr} \left(Q(\,\cdot\,, \psi, \nabla_\tau \psi) \left[B_E - \psi \left(\sum_{k=1}^n \langle \nu_\Gamma - A_\psi \nabla_\tau \psi, v_k \rangle \partial_{v_k} \mathrm{D}^2 \bar{d}_E \right) - \mathrm{D}^2 \psi_E \right] \right), \tag{5.4}$$

where $Q : \partial E \times [-\delta, \delta] \times [-\delta, \delta] \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ is a smooth map with $Q(\,\cdot\,, 0, 0) = P_{\partial E}$. Thus by using Taylor's expansion we may write on ∂E

$$Q(\,\cdot\,, \psi, \nabla_\tau \psi) = P_{\partial E} + \psi S(\,\cdot\,, \psi, \nabla_\tau \psi) + [\langle \nabla_\tau \psi, r_{ij}(\,\cdot\,, \psi, \nabla_\tau \psi) \rangle]_{ij} \tag{5.5}$$

[1] This implies the first part of Lemma 2.7 for smooth functions (the other cases are similar to check). The details of second part are left to the reader.

with some smooth S and r_{ij}. Thus by substituting (5.5) and (5.1) to (5.4) we obtain the expression (2.8) after regrouping the terms and Lemma 2.8 is clear.

Suppose that $h \in L^p(\partial E_\psi)$ with $1 \le p < \infty$ and $\varphi \in C^\infty(\partial E_\psi)$. By using the change of variable formula we have

$$\int_{\partial E_\psi} |h|^p \, \mathrm{d}\mathcal{H}^{n-1} = \int_E |h \circ \Phi_\psi|^p J_\tau \Phi_\psi \, \mathrm{d}\mathcal{H}^{n-1} \quad \text{and}$$

$$\begin{aligned}\int_{\partial E_\psi} |\nabla_\tau \varphi|^p \, \mathrm{d}\mathcal{H}^{n-1} &= \int_{\partial E_\psi} |P_{\partial E_\psi} \nabla \left((\varphi \circ \Phi_\psi)_E \circ \Phi_\psi^{-1} \right)|^p \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial E_\psi} |P_{\partial E_\psi} (\mathrm{D}\Phi_\psi^{-1})^{\mathrm{T}} \nabla (\varphi \circ \Phi_\psi)_E \circ \Phi_\psi^{-1}|^p \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial E} |(I - \nu_{E_\psi} \circ \Phi_\psi \otimes \nu_{E_\psi} \circ \Phi_\psi)(\mathrm{D}\Phi_\psi)^{-\mathrm{T}} \nabla_\tau (\varphi \circ \Phi_\psi)|^p J_\tau \Phi_\psi \, \mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Since now $(I - \nu_{E_\psi} \circ \Phi_\psi \otimes \nu_{E_\psi} \circ \Phi_\psi)(\mathrm{D}\Phi_\psi)^{-\mathrm{T}} \to P_{\partial E}$ and $J_\tau \Phi_\psi \to 1$ uniformly on ∂E as $\|\psi\|_{C^1(\partial E)}$ tends to zero, then by decreasing δ, if necessary, we find a uniform constant C such that

$$C^{-1} \|h \circ \Phi_\psi\|^p_{L^p(\partial E)} \le \|h\|^p_{L^p(\partial E_\psi)} \le C \|h \circ \Phi_\psi\|^p_{L^p(\partial E)} \quad \text{and}$$
$$C^{-1-p} \|\nabla_\tau (\varphi \circ \Phi_\psi)\|^p_{L^p(\partial E)} \le \|\nabla_\tau \varphi\|^p_{L^p(\partial E_\psi)} \le C^{1+p} \|\nabla_\tau (\varphi \circ \Phi_\psi)\|^p_{L^p(\partial E)},$$

whenever $\|\psi\|_{C^1(\partial E)} \le \delta$. This establishes (2.10) and (2.11).

It follows from (2.10) that for every $h \in L^6(\partial E_\psi)$ the norm $\|h\|_{L^6(\partial E_\psi)}$ controls uniformly every $\|h\|_{L^p(\partial E_\psi)}$ norm with $1 \le p \le 6$. Hence it is sufficient to check Lemma 2.9 in the case $p = 6$. Suppose that $\varphi \in \tilde{C}^\infty(\partial E_\psi)$ (so $\overline{\varphi} = \fint_{\partial E_\psi} \varphi \, \mathrm{d}\mathcal{H}^{n-1} = 0$). Now Lemma 2.2 is satisfied with $p = 6$, $r = q = 2$, $j = 0$, $m = 1$ and $n = 3, 4$. In the case $n = 3$ the interpolation exponent is $\alpha_6 = \frac{2}{3}$ whereas for $n = 4$ we have $\alpha_6 = 1$. Now we estimate

$$\begin{aligned}\|\varphi\|_{L^6(\partial E_\psi)} &\overset{(2.10)}{\le} C \|\varphi \circ \Phi_\psi\|_{L^6(\partial E)} \\ &\le C \|\varphi \circ \Phi_\psi - \overline{\varphi \circ \Phi_\psi}\|_{L^6(\partial E)} + C |\partial E|^{-\frac{5}{6}} \left| \int_{\partial E} \varphi \circ \Phi_\psi \, \mathrm{d}\mathcal{H}^{n-1} \right| \\ &= C \|\varphi \circ \Phi_\psi - \overline{\varphi \circ \Phi_\psi}\|_{L^6(\partial E)} + C |\partial E|^{-\frac{5}{6}} \left| \int_{\partial E} (1 - J_\tau \Phi_\psi) \varphi \circ \Phi_\psi \, \mathrm{d}\mathcal{H}^{n-1} \right| \\ &\le C \|\varphi \circ \Phi_\psi - \overline{\varphi \circ \Phi_\psi}\|_{L^6(\partial E)} + C \max_{\partial E} |1 - J_\tau \Phi_\psi| \|\varphi \circ \Phi_\psi\|_{L^6(\partial E)} \\ &\overset{2.2}{\le} C \|\nabla_\tau (\varphi \circ \Phi_\psi - \overline{\varphi \circ \Phi_\psi})\|^{\alpha_6}_{L^2(\partial E)} \|\varphi \circ \Phi_\psi - \overline{\varphi \circ \Phi_\psi}\|^{1-\alpha_6}_{L^2(\partial E)}\end{aligned}$$

$$
\begin{aligned}
&+ C \max_{\partial E} |1 - J_\tau \Phi_\psi| \|\varphi \circ \Phi_\psi\|_{L^6(\partial E)} \\
&\le C \|\nabla_\tau (\varphi \circ \Phi_\psi)\|_{L^2(\partial E)}^{\alpha_6} \|\varphi \circ \Phi_\psi\|_{L^2(\partial E)}^{1-\alpha_6} + C \max_{\partial E} |1 - J_\tau \Phi_\psi| \|\varphi \circ \Phi_\psi\|_{L^6(\partial E)} \\
&\overset{(2.10)}{\le} C \|\nabla_\tau \varphi\|_{L^2(\partial E_\psi)}^{\alpha_6} \|\varphi\|_{L^2(\partial E_\psi)}^{1-\alpha_6} + C \max_{\partial E} |1 - J_\tau \Phi_\psi| \|\varphi\|_{L^6(\partial E_\psi)} \\
&\le C \|\varphi\|_{H^1(\partial E_\psi)} + C \max_{\partial E} |1 - J_\tau \Phi_\psi| \|\varphi\|_{L^6(\partial E_\psi)}.
\end{aligned}
$$

Thus by decreasing δ, if necessary, we obtain

$$
\|\varphi\|_{L^6(\partial E_\psi)} \le C \|\varphi\|_{L^2(\partial E_\psi)}
$$

with some constant C and Lemma 2.9 follows. Again, it is enough to prove Lemma 2.10 for $p = 6$. For that choose an arbitrary $\varphi \in C^\infty(\partial E_\psi)$. We define the smooth extension $\varphi_{E_\psi} = \varphi \circ \pi_{\partial E_\psi}$ to some regular neighborhood of ∂E_ψ as before. A straightforward but rather long computation yields

$$
\begin{aligned}
\mathrm{D}_\tau^2 (\varphi \circ \Phi_\psi) &= P_{\partial E} (\mathrm{D}^2 \varphi_{E_\psi} \circ \Phi_\psi) \mathrm{D}\Phi_\psi P_{\partial E} + \langle \nu_E, \nabla_\tau \varphi \circ \Phi_\psi \rangle (\mathrm{D}_\tau^2 \psi - B_E) \\
&+ \psi \sum_{i=1}^n P_{\partial E} e_i \otimes P_{\partial E} \left((\partial_i \mathrm{D} \bar{d}_E)(\nabla_\tau \varphi \circ \Phi_\psi) + (\mathrm{D}\Phi_\psi)^{\mathrm{T}} \mathrm{D}^2 \varphi_{E_\psi} \partial_i \nabla \bar{d}_E \right) \\
&+ B_E (\nabla_\tau \varphi \circ \Phi_\psi) \otimes \nabla_\tau \psi + \nabla_\tau \psi \otimes B_E (\nabla_\tau \varphi \circ \Phi_\psi) \\
&+ \nabla_\tau \psi \otimes P_{\partial E} (\mathrm{D}\Phi_\psi)^{\mathrm{T}} (\mathrm{D}^2 \varphi_{E_\psi} \circ \Phi_\psi) \nu_E.
\end{aligned}
$$

Hence with help of the previous identity

$$
\begin{aligned}
|\mathrm{D}_\tau^2 (\varphi \circ \Phi_\psi)| &\le C |\mathrm{D}^2 \varphi_{E_\psi} \circ \Phi_\psi| + (C + |\mathrm{D}_\tau^2 \psi|) |\nabla_\tau \varphi \circ \Phi_\psi| \\
&\overset{(5.1)}{\le} C |\mathrm{D}_\tau^2 \varphi \circ \Phi_\psi| + C \left(1 + |B_{E_\psi} \circ \Phi_\psi| + |\mathrm{D}_\tau^2 \psi| \right) |\nabla_\tau \varphi \circ \Phi_\psi| \\
&\overset{(2.9)}{\le} C |\mathrm{D}_\tau^2 \varphi \circ \Phi_\psi| + C \left(1 + |\mathrm{D}_\tau^2 \psi| \right) |\nabla_\tau \varphi \circ \Phi_\psi|. \qquad (5.6)
\end{aligned}
$$

Again, Lemma 2.2 is satisfied for φ with $p = 6$, $r = q = 2$, $j = 1$, $m = 1$ and $n = 3, 4$, where $\alpha_6 = \frac{5}{6}$ for $n = 3$ and $\alpha_6 = 1$ for $n = 4$. Then

$$
\begin{aligned}
\|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)} &\overset{(2.11)}{\le} C \|\nabla_\tau (\varphi \circ \Phi_\psi)\|_{L^6(\partial E)} \\
&\overset{\text{Lemma 2.2}}{\le} C \|\mathrm{D}_\tau^2 (\varphi \circ \Phi_\psi)\|_{L^2(\partial E)}^{\alpha_6} \|\varphi \circ \Phi_\psi\|_{L^2(\partial E)}^{1-\alpha_6} \\
&\le C \|\mathrm{D}_\tau^2 (\varphi \circ \Phi_\psi)\|_{L^2(\partial E)} + C \|\varphi \circ \Phi_\psi\|_{L^2(\partial E)} \\
&\overset{(5.6)}{\le} C \|\mathrm{D}_\tau^2 \varphi \circ \Phi_\psi\|_{L^2(\partial E)} + C \|\nabla_\tau \varphi \circ \Phi_\psi\|_{L^2(\partial E)} + C \|\varphi \circ \Phi_\psi\|_{L^2(\partial E)}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\left(\int_{\partial E} |D^2_\tau \psi|^2 |\nabla_\tau \varphi \circ \Phi_\psi|^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \\
&\overset{(2.10)+(2.11)}{\leq} C\|\mathrm{D}^2_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} \\
&+ C\left(\int_{\partial E} |D^2_\tau \psi|^2 |\nabla_\tau \varphi \circ \Phi_\psi|^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \\
&\leq C\|\mathrm{D}^2_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi \circ \Phi_\psi\|_{L^6(\partial E)} \\
&\overset{(2.10)}{\leq} C\|\mathrm{D}^2_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)} \\
&\overset{(2.5)}{\leq} C\|\Delta_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)} \\
&+ C\left(\int_{\partial E_\psi} |B_{E_\psi}|^2 |\nabla_\tau \varphi|^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \\
&\overset{(2.10)}{\leq} C\|\Delta_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)} \\
&+ C\left(\int_{\partial E} |B_E \circ \Phi_\psi|^2 |\nabla_\tau \varphi \circ \Phi_\psi|^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \\
&\overset{(2.9)}{\leq} C\|\Delta_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)} \\
&+ C\left(\int_{\partial E} |\mathrm{D}^2_\tau \psi|^2 |\nabla_\tau \varphi \circ \Phi_\psi|^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} + C\|\nabla_\tau \varphi \circ \Phi_\psi\|_{L^2(\partial E)} \\
&\leq C\|\Delta_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)} \\
&+ C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi \circ \Phi_\psi\|_{L^6(\partial E)} + C\|\nabla_\tau \varphi \circ \Phi_\psi\|_{L^2(\partial E)} \\
&\overset{(2.11)}{\leq} C\|\Delta_\tau \varphi\|_{L^2(\partial E_\psi)} + C\|\varphi\|_{H^1(\partial E_\psi)} + C\|D^2_\tau \psi\|_{L^3(\partial E)} \|\nabla_\tau \varphi\|_{L^6(\partial E_\psi)}.
\end{aligned}
$$

Again, by Remark 2.3 $\|\psi\|_{C^1(\partial E_\psi)}, \|\mathrm{D}^2_\tau \psi\|_{L^3(\partial E)} \leq C\|\psi\|_{H^3(\partial E_\psi)}$, when n is 3 or 4, so the previous estimate implies Lemma 2.10 in the case $p = 6$.

Time derivatives. In this subsection we derive the formulas (3.25) and (3.26) of Lemma 3.4. In particular, (3.25) is probably well-known, but for sake of completeness we compute it too.

It follows from the *semi-group property* that we need to check (3.25) and (3.26) at the time $t = 0$. At first we list some facts. For that let $(E_t)_{t \in [0,T[}$ be any smooth flow in $\mathbb{T}^n$ with a corresponding smoothly parametrized family of diffeomorphism $(\Phi_t)_{t\in[0,T[}$. Set the initial velocity vector field

$$X_0 = \partial_t \Phi_t\Big|_{t=0}.$$

Then $V_0 = \langle X_0, \nu_0\rangle$ and the following hold on the initial boundary ∂E_0.

$$\frac{\partial}{\partial t}\nu_t \circ \Phi_t\Big|_{t=0} = -(\mathrm{D}_\tau X)^{\mathrm{T}}\nu_0, \tag{5.7}$$

$$\frac{\partial}{\partial t}J_\tau \Phi_t\Big|_{t=0} = \mathrm{div}_\tau X \quad \text{and} \tag{5.8}$$

$$\frac{\partial}{\partial t}H_t \circ \Phi_t\Big|_{t=0} = -\mathrm{div}_\tau\left((\mathrm{D}_\tau X)^{\mathrm{T}}\nu_0\right) - \mathrm{tr}\,(B_0 \mathrm{D}_\tau X)\,. \tag{5.9}$$

For instance, (5.7) is directly computed in [8]. There are also an open neighborhood U of $\bigcup_{t\in[0,T[}\partial E_t \times \{t\}$ and a smooth map $H : U \to \mathbb{T}^n$ such that $H(\,\cdot\,,t) = H_t$ on ∂E_t for every $t \in [0,T[$. Again, we recall that every smooth flow admits a unique *normal parametrization*, for a compact setting in $\mathbb{R}^n$ see [3, Theorem 8] (the corresponding periodic case goes similarly), so we may assume $(\Phi_t)_{t\in[0,T[}$ to be such a parametrization. That is,

$$\partial_s \Phi_{t+s}\Big|_{s=0} = (V_t \circ \Phi_t)(\nu_t \circ \Phi_t)$$

on ∂E_0 for every $t \in [0,T[$, in particular $X_0 = V_0\nu_0$ on ∂E_0. Suppose from now on that $(E_t)_t$ is a volume preserving mean curvature flow, so we may write $X_0 = (\bar{H}_0 - H_0)\nu_0$ and

$$\mathrm{D}_\tau X_0 = (\bar{H}_0 - H_0)B_0 - \nu_0 \otimes \nabla_\tau H_0 \tag{5.10}$$

on ∂E_0. Hence (5.7), (5.8) and (5.9) can be rewritten as

$$\frac{\partial}{\partial t}\nu_t \circ \Phi_t\Big|_{t=0} = \nabla_\tau H_0, \tag{5.11}$$

$$\frac{\partial}{\partial t}J_\tau \Phi_t\Big|_{t=0} = (\bar{H}_0 - H_0)H_0 \quad \text{and} \tag{5.12}$$

$$\frac{\partial}{\partial t}H_t \circ \Phi_t\Big|_{t=0} = \Delta_\tau H_0 - (\bar{H}_0 - H_0)|B_0|^2. \tag{5.13}$$

The identity (5.13) can be also obtained in a more elegant way using the results from [8]. By using the change of variables formula and integration by parts we compute first

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{H}_t - H_t\|^2_{L^2(\partial E_t)}\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial E_0}\left(\bar{H}_t \circ \Phi_t - H_t \circ \Phi_t\right)^2 J_\tau \Phi_t \,\mathrm{d}\mathcal{H}^{n-1}\Big|_{t=0} \\ &= \int_{\partial E_0} 2\left(\bar{H}_0 - H_0\right)\left(\frac{\partial}{\partial t}\bar{H}_t \circ \Phi_t\Big|_{t=0} - \frac{\partial}{\partial t}H_t \circ \Phi_t\Big|_{t=0}\right)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

$$
\begin{aligned}
&+\int_{\partial E_0} \left(\bar H_0 - H_0\right)^2 \frac{\partial}{\partial t} J_\tau \Phi_t\Big|_{t=0} \, \mathrm{d}\mathcal{H}^{n-1} \\
&= -2 \int_{\partial E_0} \frac{\partial}{\partial t} H_t \circ \Phi_t\Big|_{t=0} \left(\bar H_0 - H_0\right) \, \mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{\partial E_0} \left(\bar H_0 - H_0\right)^2 \frac{\partial}{\partial t} J_\tau \Phi_t\Big|_{t=0} \, \mathrm{d}\mathcal{H}^{n-1} \\
&\overset{(5.12)+(5.13)}{=} -2 \int_{\partial E_0} \left(\Delta_\tau H_0 - (\bar H_0 - H_0)|B_0|^2\right)\left(\bar H_0 - H_0\right) \, \mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{\partial E_0} H_0 \left(\bar H_0 - H_0\right)^3 \, \mathrm{d}\mathcal{H}^{n-1} \\
&= -2 \int_{\partial E_0} |\nabla_\tau H_0|^2 - |B_0|^2 \left(\bar H_0 - H_0\right)^2 \, \mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{\partial E_0} H_0 \left(\bar H_0 - H_0\right)^3 \, \mathrm{d}\mathcal{H}^{n-1} \\
&= -2\partial^2 P(\partial E_0)[\bar H_0 - H_0] + \int_{\partial E_0} H_0 \left(\bar H_0 - H_0\right)^3 \, \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
$$

To compute (3.26) at $t = 0$, we evaluate the term $\frac{\partial}{\partial t} (\nabla_\tau H_t \circ \Phi_t)\Big|_{t=0}$ on ∂E_0. We use the notation ∇ for the spatial gradient. Now

$$
\begin{aligned}
\frac{\partial}{\partial t} (\nabla_\tau H_t \circ \Phi_t)\Big|_{t=0} &= \frac{\partial}{\partial t} (I - \nu_t \circ \Phi_t \otimes \nu_t \circ \Phi_t)\, \nabla H(\,\cdot\,, t) \circ \Phi_t\Big|_{t=0} \\
&= -\left(\frac{\partial}{\partial t} \nu_t \circ \Phi_t\Big|_{t=0} \otimes \nu_0 + \nu_0 \otimes \frac{\partial}{\partial t} \nu_t \circ \Phi_t\Big|_{t=0}\right) \nabla H(\,\cdot\,, 0) \\
&\quad + (I - \nu_0 \otimes \nu_0) \frac{\partial}{\partial t} \nabla H(\,\cdot\,, t) \circ \Phi_t\Big|_{t=0} \\
&\overset{(5.11)}{=} -\langle \nu_0, \nabla H(\,\cdot\,, 0)\rangle \nabla_\tau H_0 - |\nabla_\tau H_0|^2 \nu_0 \\
&\quad + (I - \nu_0 \otimes \nu_0) \frac{\partial}{\partial t} \nabla H(\,\cdot\,, t) \circ \Phi_t\Big|_{t=0}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t} \nabla H(\,\cdot\,, t) \circ \Phi_t\Big|_{t=0} &= \frac{\partial}{\partial t} (\mathrm{D}\Phi_t)^{-\mathrm{T}} \nabla \left(H(\,\cdot\,, t) \circ \Phi_t\right)\Big|_{t=0} \\
&= \frac{\partial}{\partial t} (\mathrm{D}\Phi_t)^{-\mathrm{T}}\Big|_{t=0} \nabla H(\,\cdot\,, 0) + \frac{\partial}{\partial t} \nabla \left(H(\,\cdot\,, t) \circ \Phi_t\right)\Big|_{t=0}
\end{aligned}
$$

$$= -\left(\mathrm{D}\frac{\partial}{\partial t}\Phi_t\Big|_{t=0}\right)^{\mathrm{T}} \nabla H(\,\cdot\,,0) + \nabla \frac{\partial}{\partial t}\,(H(\,\cdot\,,t)\circ\Phi_t)\,\Big|_{t=0}$$
$$= -\,(\mathrm{D}X_0)^{\mathrm{T}}\,\nabla H(\,\cdot\,,0) + \nabla \frac{\partial}{\partial t}\,(H(\,\cdot\,,t)\circ\Phi_t)\,\Big|_{t=0}.$$

By combining the two previous expressions

$$\begin{aligned}\frac{\partial}{\partial t}\,(\nabla_\tau H_t\circ\Phi_t)\,\Big|_{t=0} &= -\langle \nu_0, \nabla H(\,\cdot\,,0)\rangle \nabla_\tau H_0 - |\nabla_\tau H_0|^2\nu_0\\ &\quad - (\mathrm{D}_\tau X_0)^{\mathrm{T}}\,\nabla H(\,\cdot\,,0) + \nabla_\tau \frac{\partial}{\partial t}\,(H(\,\cdot\,,t)\circ\Phi_t)\,\Big|_{t=0}\\ &\overset{(5.10)}{=} -\langle \nu_0, \nabla H(\,\cdot\,,0)\rangle \nabla_\tau H_0 - |\nabla_\tau H_0|^2\nu_0\\ &\quad - \big((\bar{H}_0 - H_0)B_0 - \nu_0\otimes\nabla_\tau H_0\big)^{\mathrm{T}}\,\nabla H(\,\cdot\,,0) + \nabla_\tau \frac{\partial}{\partial t}\,(H(\,\cdot\,,t)\circ\Phi_t)\,\Big|_{t=0}\\ &= -|\nabla_\tau H_0|^2\nu_0 - (\bar{H}_0 - H_0)B_0\nabla_\tau H_0 + \nabla_\tau \frac{\partial}{\partial t}\,(H(\,\cdot\,,t)\circ\Phi_t)\,\Big|_{t=0}.\end{aligned} \tag{5.14}$$

Thus by using the change of variables formula and integrating by parts we finally compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_\tau H_t\|^2_{L^2(\partial E_t)}\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial E_0} \langle \nabla_\tau H_t\circ\Phi_t, \nabla_\tau H_t\circ\Phi_t\rangle J_\tau\Phi_t \,\mathrm{d}\mathcal{H}^{n-1}\Big|_{t=0}\\ &= 2\int_{\partial E_0}\langle \frac{\partial}{\partial t}\nabla_\tau H_t\circ\Phi_t\Big|_{t=0}, \nabla_\tau H_0\rangle \,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad + \int_{\partial E_0} |\nabla_\tau H_t|^2 \frac{\partial}{\partial t}J_\tau\Phi_t\Big|_{t=0}\,\mathrm{d}\mathcal{H}^{n-1}\\ &\overset{(5.12)}{=} 2\int_{\partial E_0}\langle \frac{\partial}{\partial t}\nabla_\tau H_t\circ\Phi_t\Big|_{t=0}, \nabla_\tau H_0\rangle \,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad + \int_{\partial E_0} |\nabla_\tau H_t|^2(\bar{H}_0 - H_0)H_0\,\mathrm{d}\mathcal{H}^{n-1}\\ &\overset{(5.14)}{=} 2\int_{\partial E_0}\langle -(\bar{H}_0 - H_0)B_0\nabla_\tau H_0 + \nabla_\tau\frac{\partial}{\partial t}\,(H(\,\cdot\,,t)\circ\Phi_t)\,\Big|_{t=0}, \nabla_\tau H_0\rangle\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad + \int_{\partial E_0} |\nabla_\tau H_t|^2(\bar{H}_0 - H_0)H_0\,\mathrm{d}\mathcal{H}^{n-1}\\ &= -2\int_{\partial E_0} \Delta_\tau H_0\frac{\partial}{\partial t}H_t\circ\Phi_t\Big|_{t=0}\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

$$
\begin{aligned}
&-2 \int_{\partial E_0} (\bar{H}_0 - H_0)\langle \nabla_\tau H_0, B_0 \nabla_\tau H_0\rangle \, d\mathcal{H}^{n-1} \\
&+ \int_{\partial E_0} |\nabla_\tau H_t|^2 (\bar{H}_0 - H_0) H_0 \, d\mathcal{H}^{n-1} \\
\overset{(5.13)}{=} &-2 \int_{\partial E_0} (\Delta_\tau H_0)^2 - (\bar{H}_0 - H_0)|B_0|^2 \Delta_\tau H_0 \, d\mathcal{H}^{n-1} \\
&-2 \int_{\partial E_0} (\bar{H}_0 - H_0)\langle \nabla_\tau H_0, B_0 \nabla_\tau H_0\rangle \, d\mathcal{H}^{n-1} \\
&+ \int_{\partial E_0} |\nabla_\tau H_t|^2 (\bar{H}_0 - H_0) H_0 \, d\mathcal{H}^{n-1}.
\end{aligned}
$$

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Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow

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QUANTITATIVE ALEXANDROV THEOREM AND ASYMPTOTIC BEHAVIOR OF THE VOLUME PRESERVING MEAN CURVATURE FLOW

VESA JULIN AND JOONAS NIINIKOSKI

Abstract. We prove a new quantitative version of the Alexandrov theorem which states that if the mean curvature of a regular set in $\mathbb{R}^{n+1}$ is close to a constant in L^n-sense, then the set is close to a union of disjoint balls with respect to the Hausdorff distance. This result is more general than the previous quantifications of the Alexandrov theorem and using it we are able to show that in $\mathbb{R}^2$ and $\mathbb{R}^3$ a weak solution of the volume preserving mean curvature flow starting from a set of finite perimeter asymptotically convergences to a disjoint union of equisize balls, up to possible translations. Here by weak solution we mean a flat flow, obtained via the minimizing movements scheme.

Contents

1. Introduction

The main purpose of this article is to study the asymptotic behavior of the weak solution of the volume preserving mean curvature flow starting from a set of finite perimeter. In the classical setting we are given a smooth set $E_0 \subset \mathbb{R}^{n+1}$ and we let it evolve into a smooth family of sets $(E_t)_t$ according to the law, where the normal velocity V_t is proportional to the mean curvature of E_t as

$$V_t = -(H_{E_t} - \bar{H}_{E_t}) \qquad \text{on } \partial E_t, \tag{1.1}$$

where $\bar{H}_{E_t} = \fint_{\partial E_t} H_{E_t}\, d\mathcal{H}^n$. Mean curvature type of equations are important in geometry, where one usually studies the geometric properties of ∂E_t which are inherited from ∂E_0. The equation (1.1) can also be seen as a volume preserving gradient flow of the surface area. These equations arise naturally in physical models involving surface tension (see [33]).

The main issue with (1.1) is that it may develop singularities in finite time even in the plane [24, 25]. In order to pass over the singular time one may try to do a surgery procedure and restart the flow after a singular time as in [18] or to define a weak solution of (1.1), which is what we will consider here. For the mean curvature flow one may define a weak solution by using the varifold setting by Brakke [3], the level set solution developed independently by Chen-Giga-Goto [6] and Evans-Spruck [13], or by using the minimizing movements scheme developed independently by Almgren-Taylor-Wang [2] and Luckhaus-Stürzenhecker [21]. Since we want the solution of (1.1) to be a family of sets and since (1.1) does not satisfy the comparison principle, the natural choice is to define a weak solution via the minimizing movements scheme as in [2, 21]. This solution is usually called a flat flow and it is well-defined due to [29], but might not be unique.

The advantage of the flat flow is that it is defined for all times for any bounded initial set with finite perimeter and we may thus study its asymptotic behavior. Heuristically, one may guess that the flat flow converges to a critical point of the static problem, which are classified in the recent work by Delgadino-Maggi [9] as disjoint union of balls, possibly tangent to each other. The asymptotic convergence of (1.1) has been proved for initial sets with certain geometric properties such as convexity [17], nearly spherical [12] or sets which are near a stable critical set in the flat torus in low dimensions [30]. We note that in these cases the flow does not develop singularities and is thus classically well-defined for all times. The result in [19] shows that the convergence holds also for star-shaped sets, up to possible translations. For the mean curvature flow with forcing the asymptotic behavior has been studied for the level set solution in [15, 16] and for the flat flow in the plane in [14]. The result closest to ours is the recent work by Morini-Ponsiglione-Spadaro [28], where the authors prove that the discrete-in-time approximation of the flat flow of (1.1) converges exponentially fast to disjoint union balls. Here we are able to pass the time discretization to zero and characterize the limit sets for the flat flow of (1.1) in $\mathbb{R}^2$ and $\mathbb{R}^3$. The precise definition of the flat flow is given in Section 4.

Theorem 1.1. *Assume $E_0 \subset \mathbb{R}^{n+1}$, with $n \leq 2$ and $|E_0| = |B_1|$, is a bounded set of finite perimeter which is either essentially open or essentially closed and let $(E_t)_{t\geq 0}$ be a flat flow of (1.1) starting from E_0. There is $N \in \mathbb{N}$ such that the following holds: for every $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for every $t \geq T_\varepsilon$ there are points $x_1, \dots, x_N$, which may depend on time, with $|x_i - x_j| \geq 2r$ for $i \neq j$ and $r = N^{-\frac{1}{n+1}}$ such that for $F_t = \bigcup_{i=1}^N B_r(x_i)$ it holds*

$$\sup_{x \in E_t \Delta F_t} d_{\partial F_t}(x) \leq \varepsilon.$$

Here $d_{\partial F}$ denotes the distance function. To the best of our knowledge this is the first result on the characterization of the asymptotic limit of (1.1) in $\mathbb{R}^3$. The above result holds for any limit of the approximative flat flow and we do not need the additional assumption on the convergence of the perimeters as in [21, 29]. We note that the assumption on E_0 being either essentially open or closed is only needed to ensure that the flow is continuous up to time zero. It plays no role in the asymptotic analysis.

Concerning the limiting configurations, Theorem 1.1 is sharp since the flow (1.1) may converge to tangent balls as it is shown in [14]. On the other hand, we believe that one may rule out the possible translations and the flow actually convergences to a disjoint union of balls. The higher dimensional case and the possible speed of convergence are also open problems.

Quantitative Alexandrov theorem. The proof of Theorem 1.1 is based on the dissipation inequality proven in [29] and stated in Proposition 4.1. This implies that there is a sequence of times $t_j \to \infty$ such that the mean curvatures of the evolving sets E_{t_j} are asymptotically close to a constant with respect to the L^2-norm. Therefore, we need a quantified version of the Alexandrov theorem which enables us to conclude that the sets E_{t_j} are close to a disjoint union of balls.

There is a lot of recent research on generalization of the Alexandrov theorem [8, 9, 10, 11, 20, 23]. We refer the survey paper [7] for the state-of-the-art. Unfortunately, none of the available results is applicable to our problem, and we are also not able to use the characterization of the critical sets by Delgadino-Maggi [9, Corollary 2] to identify the limit set. Indeed, even if we know that the sets E_{t_j} converge to a set of finite perimeter and their mean curvatures converge to a constant, it is not clear why the limit set is a set of finite perimeter with weak mean curvature as this class of sets is not in general closed. Our main result of the paper is the following quantification of the Alexandrov theorem, which is the main technical tool in the proof of Theorem 1.1.

Theorem 1.2. *Let $E \subset \mathbb{R}^{n+1}$ be a C^2-regular set such that $P(E) \leq C_0$ and $|E| \geq 1/C_0$. There are positive constants $q = q(n) \in (0,1]$, $C = C(C_0, n)$ and $\delta = \delta(C_0, n)$ such that if $\|H_E - \lambda\|_{L^n(\partial E)} \leq \delta$ for some $\lambda \in \mathbb{R}$, then $1/C \leq \lambda \leq C$ and there are points $x_1, \dots, x_N$ with $|x_i - x_j| \geq 2R$, where $R = n/\lambda$, such that for $F = \bigcup_{i=1}^N B_R(x_i)$ it holds*

$$\sup_{x \in E \Delta F} d_{\partial F}(x) \leq C \|H_E - \lambda\|_{L^n(\partial E)}^q.$$

Moreover,

$$\Big| P(E) - N(n+1)\omega_{n+1} R^n \Big| \leq C \|H_E - \lambda\|_{L^n(\partial E)}^q.$$

The main advantage of Theorem 1.2 with respect to the previous results in the literature is that we do not assume any geometric restriction on E such as mean convexity. Moreover, we assume the mean curvature to be close to a constant only in the L^n-sense, which is exactly what we need for the asymptotic analysis in Theorem 1.1. This makes the proof challenging as we, e.g., cannot use the estimates from the Allard regularity theory [1].

Theorem 1.2 is sharp in the sense that $\|H_E - \lambda\|_{L^n(\partial E)}$ cannot be replaced by a weaker L^p-norm. This can be easily seen by considering a set which is a union of the unit ball and a ball of small radius ε far away. On the other hand, the dissipation inequality in Proposition 4.1 controls only the L^2-norm of the mean curvature, which is the reason why we cannot prove Theorem 1.1 in higher dimensions. The proof of Theorem 1.2 is done in a constructive way and we obtain an explicit bound on the exponent $q = (n+2)^{-3}$. It would be interesting to obtain the sharp one as it might be crucial in order to obtain the possible exponential convergence of (1.1) as in [28]. In the two-dimensional case the optimal power $q = 1$ is proven in [14].

Outline of the proof of Theorem 1.2. Since the proof of Theorem 1.2 is rather long, we give its outline here. As in [9], also our argument is based on the proof of the Heinze-Karcher inequality by Montiel-Ros [27], which is originally an alternative proof for [31]. In [9] Delgadino-Maggi are able to generalize the Montiel-Ros argument to sets of finite perimeter with weak distrubutional mean curvature. Here we revisit the argument by Montiel-Ros and deduce in Proposition 3.3 that for E and R as in Theorem 1.2 and for $0 < r < R$ the volume of the set $E_r = \{x \in E : \operatorname{dist}(x, \partial E) > r\}$ satisfies the estimate

$$\left| |E_r| - \frac{|E|}{R^{n+1}}(R - r)^{n+1} \right| \leq C \|H_E - \lambda\|_{L^n(\partial E)}.$$

We use this in Step 1 of the proof of Theorem 1.2 to deduce that for r close to R the set E_r is a union of finite number of components, or clusters, with positive distance to each other.

We note that the above inequality is not enough to conclude the proof as, for example, the cube $Q = (-1,1)^{n+1}$ satisfies $|Q_r| = (1-r)^{n+1}|Q|$. Therefore, we need further information from the Montiel-Ros argument and we prove in Proposition 3.3 that the Minkowski sum $E_r + B_\rho = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, E_r) < \rho\}$, with $0 < \rho < r < R$, satisfies

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1} \right| \leq \frac{C}{(R-r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)}.$$

This enables us to prove that the components of $E_r + B_\rho \subset E$, with properly chosen ρ and r, are almost spherical. In particular, if E satisfies the above estimate with $C = 0$, then it is a disjoint union of balls. This, together with the density estimate from [34], concludes the proof.

2. Notation and preliminary results

In this section we briefly introduce our notation and recall some results from differential geometry. Given a set $E \subset \mathbb{R}^{n+1}$ the distance function $d_E : \mathbb{R}^{n+1} \to [0,\infty)$ is defined, as usual, as

$$d_E(x) := \inf_{y \in E} |x - y|$$

and we denote the signed distance function by $\bar{d}_E : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$\bar{d}_E(x) := \begin{cases} -d_{\partial E}(x), & \text{for } x \in E \\ d_{\partial E}(x), & \text{for } x \in \mathbb{R}^{n+1} \setminus E. \end{cases}$$

Then clearly it holds $d_{\partial E} = |\bar{d}_E|$. We denote the ball with radius r centered at x by $B_r(x)$ and by B_r if it is centered at the origin. Given a set $E \subset \mathbb{R}^{n+1}$ we denote its ρ-enlargement by the Minkowski sum

$$E + B_\rho = \{x + y \in \mathbb{R}^{n+1} : x \in E,\ y \in B_\rho\} = \{x \in \mathbb{R}^{n+1} : d_E(x) < \rho\}.$$

For a measurable set $E \subset \mathbb{R}^{n+1}$ the shorthand notation $|E|$ denotes its Lebesgue measure and we denote the k-dimensional measure of the unit ball in $\mathbb{R}^k$ by ω_k. In some cases, we may use the shorthand notation $|E|$ more generally for a measurable set $E \subset \mathbb{R}^k$ to denote its k-dimensional Lebesgue measure but this shall be clear from context.

For a set of finite perimeter $E \subset \mathbb{R}^{n+1}$ we denote its reduced boundary by $\partial^* E$ and the perimeter by $P(E)$. Recall that $P(E) = \mathcal{H}^n(\partial^* E)$ and for regular enough set it holds $\partial^* E = \partial E$. The relative isoperimetric inequality states that for every set of finite perimeter E and for every ball $B_r(x)$ it holds

$$\mathcal{H}^n(\partial^* E \cap B_r(x))^{\frac{n+1}{n}} \geq c_n \min\left\{|E \cap B_r(x)|, |B_r(x) \setminus E|\right\},$$

for a dimensional constant. We refer to [22] for an introduction to the topic.

We define the *tangential differential* of $F \in C^1(\mathbb{R}^{n+1};\mathbb{R}^m)$ on ∂E by

$$\mathrm{D}_\tau F(x) = \mathrm{D}F(x)(I - \nu_E(x) \otimes \nu_E(x)),$$

where ν_E denotes the unit outer normal of E. For a function $f \in C^1(\mathbb{R}^{n+1};\mathbb{R})$ we denote by $\nabla_\tau f$ its tangential gradient which is a vector in $\mathbb{R}^{n+1}$. We define the tangential divergence of $F \in C^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ by $\operatorname{div}_\tau F = \operatorname{Tr}(\mathrm{D}_\tau F)$. Then the divergence theorem on manifolds generalizes to

$$\int_{\partial^* E} \operatorname{div}_\tau F \, d\mathcal{H}^n = \int_{\partial^* E} H_E \langle F, \nu_E \rangle \, d\mathcal{H}^n,$$

where $H_E \in L^1(\partial^* E)$ is the distributional mean curvature. When ∂E is smooth H_E agrees with the classical definition of the mean curvature, which for us is the sum of the principal curvatures.

We begin by recalling the well-known inequality proven first by Simon [32] in $\mathbb{R}^3$ and then by Topping [34] in the general case.

Theorem 2.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact and connected C^2-hypersurface. Then*

$$\mathrm{diam}(\Sigma) \leq C_n \int_\Sigma |H_\Sigma|^{n-1} \, \mathrm{d}\mathcal{H}^n, \tag{2.1}$$

where C_n depends only on the dimension.

We need also the Michael-Simon inequality [26].

Theorem 2.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a compact C^2-hypersurface. Then for every non-negative $\varphi \in C^1(\mathbb{R}^{n+1})$*

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\Sigma)} \leq C_n \int_\Sigma |\nabla_\tau \varphi| + \varphi |H_\Sigma| \, \mathrm{d}\mathcal{H}^n, \tag{2.2}$$

where C_n depends only on the dimension.

The following density-type estimate is essentially proven in [28, Lemma 2.1].

Proposition 2.3. *Let $E \subset \mathbb{R}^{n+1}$ be a set of finite perimeter with $P(E) > 0$ and $0 < \beta < 1$. There is a positive constant $c = c(n, \beta)$ such that*

$$r_{E,\beta} := \sup\left\{r \in \mathbb{R}_+ : \text{there is } x \in \mathbb{R}^{n+1} \text{ with } |B_r(x) \cap E| \geq \beta |B_r(x)|\right\} \geq c \frac{|E|}{P(E)}.$$

We use the previous results to prove the following lemma, which is useful when we bound the Lagrange multipliers and the number of the components of the flat flow of (1.1).

Lemma 2.4. *Let $E \subset \mathbb{R}^{n+1}$ be a bounded set of finite perimeter with a distributional mean curvature $H_E \in L^1(\partial^* E)$, $\lambda \in \mathbb{R}$ and $1 \leq C_0 < \infty$. There is a positive constant $C = C(C_0, n)$ such that the following hold.*

(i) If $P(E) \leq C_0$ and $|E| \geq 1/C_0$, then

$$1/C - C\|H_E - \lambda\|_{L^1(\partial^* E)} \leq \lambda \leq C + C\|H_E - \lambda\|_{L^1(\partial^* E)}.$$

(ii) If $P(E) \leq C_0$, $|E| \geq 1/C_0$ and E is C^2-regular, then the number of the components of E is bounded by $C(1 + \|H_E - \lambda\|^n_{L^n(\partial E)})$ and their diameters are bounded by $C(1 + \|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)})$.

Proof. Our standing assumptions throughout the proof are $P(E) \leq C_0$ and $|E| \geq 1/C_0$. The perimeter bound and the global isoperimetric inequality yield

$$|E| \leq c_n P(E)^{\frac{n+1}{n}} \leq c_n C_0^{\frac{n+1}{n}}.$$

By the assumptions on E and by the divergence theorems we compute for any vector field $F \in C^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$

$$\begin{aligned} \lambda \int_E \operatorname{div} F \, \mathrm{d}x &= \int_{\partial^* E} \lambda \langle F, \nu_E \rangle \, \mathrm{d}\mathcal{H}^n \\ &= \int_{\partial^* E} H_E \langle F, \nu_E \rangle \, \mathrm{d}\mathcal{H}^n + \int_{\partial^* E} (\lambda - H_E) \langle F, \nu_E \rangle \, \mathrm{d}\mathcal{H}^n \\ &= \int_{\partial^* E} \operatorname{div}_\tau F \, \mathrm{d}\mathcal{H}^n + \int_{\partial^* E} (\lambda - H_E) \langle F, \nu_E \rangle \, \mathrm{d}\mathcal{H}^n. \end{aligned} \tag{2.3}$$

Our goal is to construct a suitable vector field F to obtain (i) from (2.3). To this aim, we use first the relative isoperimetric inquality, Proposition 2.3 and a suitable continuity argument to find positive $r_0 = r_0(C_0, n)$, $R_0 = R_0(C_0, n)$ and r such that $r_0 \le r \le R_0$ and, by possibly translating the coordinates, $|B_r \cap E| = |B_r|/2$. Again, it follows from the relative isoperimetric inequality that $\mathcal{H}^n(\partial^* E \cap B_r) \ge c$ with some positive $c = c(C_0, n)$. Choose a decreasing C^1-function $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(t) = \begin{cases} (2r)^{-1}, & \text{for } t \le \frac{3}{2} r \\ t^{-1}, & \text{for } t \ge \frac{5}{2} r \end{cases}$$

and the conditions $f(t) \le \min\{(2r)^{-1}, t^{-1}\}$, $|f'(t)| \le (2r)^{-2}$ hold on $[\frac{3}{2}r, \frac{5}{2}r]$. We define $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting $F(x) = f(|x|)x$. Then F is a C^1-vector field with

$$\begin{aligned} \mathrm{D}F(x) &= f(|x|) I + \frac{f'(|x|)}{|x|} \, x \otimes x, && \text{for every } \ x \in \mathbb{R}^{n+1}, \\ \operatorname{div} F(x) &= (n+1) f(|x|) + f'(|x|)|x|, && \text{for every } \ x \in \mathbb{R}^{n+1} \ \text{ and} \\ \operatorname{div}_\tau F(x) &= n f(|x|) + f'(|x|) \left(|x| - \frac{\langle x, \nu_E \rangle^2}{|x|} \right), && \text{for every } \ x \in \partial^* E. \end{aligned}$$

Then $0 < \operatorname{div} F \le (n+1)(2r)^{-1}$ everywhere and $\operatorname{div} F = (n+1)(2r)^{-1}$ in B_r so by using these and the earlier observations we obtain

$$\frac{n+1}{4R_0} |B_{r_0}| \le \frac{n+1}{4r} |B_r| = \frac{n+1}{2r} |B_r \cap E| \le \int_E \operatorname{div} F \, \mathrm{d}x \le \frac{n+1}{2r} |E| \le \frac{c_n (n+1)}{2 r_0} C_0^{\frac{n+1}{n}}. \tag{2.4}$$

Again, $0 \le \operatorname{div}_\tau F \le n(2r)^{-1}$ on $\partial^* E$ and $\operatorname{div}_\tau F = n(2r)^{-1}$ on $\partial^* E \cap B_r$ and thus

$$\frac{nc}{2R_0} \le \frac{n}{2r} \mathcal{H}^n(\partial^* E \cap B_r) \le \int_{\partial^* E} \operatorname{div}_\tau F \, \mathrm{d}\mathcal{H}^n \le \frac{n P(E)}{2r} \le \frac{n C_0}{2 r_0}. \tag{2.5}$$

We use (2.3), (2.4), (2.5) and $|F| \le 1$ to obtain (i).

The claim (ii) is easy to prove in the planar case and therefore we assume that $n \ge 2$. Let $E_1, E_2, \dots, E_N$ denote the connected components of E. We apply Theorem 2.2 on ∂E_i with $\varphi = 1$ and use Hölder's inequality to obtain

$$C_n^{-1} \le \|H_{E_i}\|_{L^n(\partial E_i)} \le \|H_{E_i} - \lambda\|_{L^n(\partial E_i)} + |\lambda| P(E_i)^{\frac{1}{n}},$$

from which we conclude using (i) and Hölder's inequality

$$\begin{aligned} NC_n^{-n} &\le 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^n |\lambda|^n P(E) \\ &\le 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^{2n} C_0 C^n \left(1 + \|H_E - \lambda\|^n_{L^1(\partial E)}\right) \\ &\le 2^n \|H_E - \lambda\|^n_{L^n(\partial E)} + 2^{2n} C_0 C^n \left(1 + C_0^{n-1} \|H_E - \lambda\|^n_{L^n(\partial E)}\right). \end{aligned} \tag{2.6}$$

On the other hand, Theorem 2.1 together with (i) and Hölder's inequality implies

$$\begin{aligned} \sum_i \operatorname{diam}(E_i) &\le \sum_i C_n \int_{\partial E_i} |H_{E_i}|^{n-1} \, d\mathcal{H}^n \\ &\le \sum_i 2^{n-1} C_n \left(\int_{\partial E_i} |H_{E_i} - \lambda|^{n-1} \, d\mathcal{H}^n + |\lambda|^{n-1} P(E_i) \right) \\ &\le 2^{n-1} C_n \left(\int_{\partial E} |H_E - \lambda|^{n-1} \, d\mathcal{H}^n + P(E) |\lambda|^{n-1} \right) \\ &\le 2^{n-1} C_n \left(\|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)} + 2^{n-1} C_0 C^n (1 + \|H_E - \lambda\|^{n-1}_{L^1(\partial E)}) \right) \\ &\le 2^{n-1} C_n \left(\|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)} + 2^{n-1} C_0 C^n (1 + C_0^{n-2} \|H_E - \lambda\|^{n-1}_{L^{n-1}(\partial E)}) \right). \end{aligned} \tag{2.7}$$

Thus, by possibly increasing C, the second claim follows from (2.6) and (2.7). □

3. Quantitative Alexandrov theorem

We split the proof of Theorem 1.2 into two parts. We first revisit the Montiel-Ros argument in Proposition 3.3 where all the technical heavy lifting is done. The idea of Proposition 3.3 is to transform the (local) information of the mean curvature of E being close to a constant, into information on the ρ-enlargement of the level sets of the distance function of ∂E. We note that the statement of Proposition 3.3 is given by the sharp exponent. The proof of Theorem 1.2 is then based on purely geometric arguments.

We first state the following equivalent formulation of the theorem.

Remark 3.1. Once we prove that in Theorem 1.2 the number of component of E is bounded, the statement on the L^∞-distance is equivalent to the fact that, under the assumption $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$, there are points $x_1, \dots, x_N$ such that

$$\bigcup_{i=1}^N B_{\rho_-}(x_i) \subset E \subset \bigcup_{i=1}^N B_{\rho_+}(x_i),$$

where $\rho_- = R - C\|H_E - \lambda\|^q_{L^n(\partial E)}$, $\rho_+ = R + C\|H_E - \lambda\|^q_{L^n(\partial E)}$, $R = n/\lambda$ and the balls $B_{\rho_-}(x_1), \dots, B_{\rho_-}(x_N)$ are disjoint to each other. We leave the details to the reader.

In Theorem 1.2 we assume that the mean curvature is bounded only in the L^n-sense and thus the estimates from the Allard's regularity theory [1] are not available for us. Indeed, the L^n-boundedness of the mean curvature is not strong enough to give proper density estimates. Moreover, even in the three dimensional case $\mathbb{R}^3$ we cannot use the results from [32], because we do not have a uniform bound on the Euler characteristic of the set E. However, if we know that the mean curvature is close to a constant with respect to the L^n-norm, then the following density estimate holds. The proof is based on [34, Lemma 1.2].

Lemma 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a compact C^2-hypersurface and $\lambda \in \mathbb{R}_+$. There is a positive dimensional constant δ_n such that if $\|H_\Sigma - \lambda\|_{L^n(\Sigma)} \leq \delta_n$, then*

$$\delta_n \leq \frac{\mathcal{H}^n(B(x,r) \cap \Sigma)}{r^n}$$

for every $x \in \Sigma$ and $0 < r \leq \frac{\delta_n}{\lambda}$.

Proof. The planar case $n = 1$ is rather obvious and we leave it to the reader. Let us assume $n \geq 2$. Fix $x \in \Sigma$ and define $V : [0,\infty) \to [0,\infty)$ as $V(r) = \mathcal{H}^n(B_r(x) \cap \Sigma)$. Since V is increasing, the derivative $V'(r)$ is defined for almost every $r \in [0,\infty)$ and

$$\int_{r_1}^{r_2} V'(\rho)\, \mathrm{d}\rho \leq V(r_2) - V_2(r_1) \quad \text{whenever} \quad 0 \leq r_1 < r_2.$$

By a standard foliation argument we have that $\mathcal{H}^n(\partial B_r(x) \cap \Sigma) > 0$ at most countably many $r \in \mathbb{R}_+$. Thus $V'(r)$ is defined and $\mathcal{H}^n(\partial B_r(x) \cap \Sigma) = 0$ for almost every $r \in [0,\infty)$. Fix such r and choose $h \in \mathbb{R}_+$ for which $\mathcal{H}^n(\partial B_{r+h}(x) \cap \Sigma) = 0$. Define a cut-off function $f_h : \mathbb{R}^{n+1} \to \mathbb{R}$ by setting

$$f_h(y) = \begin{cases} 1, & y \in B_r(x) \\ 1 - \frac{|y-x|}{h}, & y \in B_{r+h}(x) \setminus B_r(x) \\ 0, & y \in \mathbb{R}^{n+1} \setminus B_{r+h}(x). \end{cases}$$

By using a suitable approximation argument combined with Theorem 2.2 we obtain

$$V(r)^{\frac{n-1}{n}} \leq C_n \left(\frac{V(r+h) - V(r)}{h} + \|f_h H_\Sigma\|_{L^1(\Sigma)} \right).$$

In turn, we may choose a sequence $(h_k)_k$ such that $h_k \to 0$ and $\mathcal{H}^n(\partial B_{r+h_k}(x) \cap \Sigma) = 0$. Then by letting $k \to \infty$ the previous estimate yields

$$\begin{aligned} V(r)^{\frac{n-1}{n}} &\leq C_n \left(V'(r) + \int_{\overline{B}_r(x) \cap \Sigma} |H_\Sigma|\, \mathrm{d}\mathcal{H}^n \right) \\ &\leq C_n \left(V'(r) + \int_{B_r(x) \cap \Sigma} |H_\Sigma|\, \mathrm{d}\mathcal{H}^n \right) \\ &\leq C_n \left(V'(r) + \int_{B_r(x) \cap \Sigma} |H_\Sigma - \lambda|\, \mathrm{d}\mathcal{H}^n + \lambda V(r) \right) \\ &\leq C_n \left(V'(r) + \|H_\Sigma - \lambda\|_{L^n(\Sigma)} V(r)^{\frac{n-1}{n}} + \lambda V(r) \right). \end{aligned}$$

Thus for almost every $r \in (0,\infty)$ it holds

$$\left(\frac{C_n^{-1} - \|H_\Sigma - \lambda\|_{L^n(\Sigma)}}{V(r)^{\frac{1}{n}}} - \lambda \right) V(r) \leq V'(r).$$

If $\|H_\Sigma - \lambda\|_{L^n(\Sigma)} \leq \delta_n$ for small δ_n then the above inequality implies

$$\frac{1}{2C_n} V(r)^{1-\frac{1}{n}} - \lambda V(r) \leq V'(r).$$

Fix $r < \delta_n/\lambda$. We assume that $V(r) \leq \delta_n r^n$, since otherwise the claim is trivially true. By the monotonicity we have $V(\rho)^{\frac{1}{n}} \leq V(r)^{\frac{1}{n}} \leq \delta_n/\lambda$ for all $0 < \rho < r$. For δ_n small enough the above inequality then yields

$$\frac{1}{4C_n} V(\rho)^{1-\frac{1}{n}} \le V'(\rho)$$

for almost every $0 < \rho < r$. The claim follows by integrating this over $(0, r)$.

□

3.1. **Montiel-Ros argument.** We recall that for $E \subset \mathbb{R}^{n+1}$ we denote

$$E_r := \{x \in E : \operatorname{dist}(x, \partial E) > r\}. \tag{3.1}$$

We use the fact that E is C^2-regular and say that $x \in \partial E$ satisfies interior ball condition with radius r, if for $y = x - r\nu_E(x)$ it holds $B_r(y) \subset E$. For $r > 0$ we define

$$\Gamma_r := \{x \in \partial E : x \text{ satisfies interior ball condition with radius } r\}. \tag{3.2}$$

Proposition 3.3. *Let $\lambda \in \mathbb{R}$ and suppose that a bounded and C^2-regular set $E \subset \mathbb{R}^{n+1}$ satisfies $P(E) \le C_0$ and $|E| \ge 1/C_0$ with $C_0 \in \mathbb{R}_+$. Then for $0 < r < R$ with $R = n/\lambda$ it holds*

$$\left| |E_r| - \frac{|E|}{R^{n+1}} (R - r)^{n+1} \right| \le C \|H_E - \lambda\|_{L^n(\partial E)}$$

and

$$\mathcal{H}^n(\partial E \setminus \Gamma_r) \le \frac{C}{(R - r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)},$$

provided that $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$, where the constants C and δ depend only on C_0 and on the dimension. Moreover, under the same assumptions, for $0 < \rho < r < R$ it holds

$$\left| |E_r + B_\rho| - \frac{|E|}{R^{n+1}} (R - (r - \rho))^{n+1} \right| \le \frac{C}{(R - r)^{n+1}} \|H_E - \lambda\|_{L^n(\partial E)}.$$

Proof. As we already mentioned the proof is based on the Montiel-Ros argument for the Heinze-Karcher inequality, which we recall shortly. To that aim we define $\zeta : \partial E \times \mathbb{R} \to \mathbb{R}^{n+1}$ as

$$\zeta(x, t) = x - t\nu_E(x).$$

We denote the principle curvatures of ∂E at x by $k_1(x), \dots k_n(x)$ and assume that they are pointwise ordered as $k_i(x) \le k_{i+1}(x)$. If we consider $\partial E \times \mathbb{R}$ as a hypersurface embedded in $\mathbb{R}^{n+2}$ then its tangential Jacobian is

$$J_\tau \zeta(x, t) = \prod_{i=1}^{n} |1 - tk_i(x)| \qquad \text{on } \partial E \times \mathbb{R}.$$

For every bounded Borel set $M \subset \partial E \times \mathbb{R}$ we have by the area formula

$$\int_{\zeta(M)} \mathcal{H}^0(\zeta^{-1}(y) \cap M) \, \mathrm{d}y = \int_M J_\tau \zeta \, \mathrm{d}\mathcal{H}^{n+1}.$$

In the proof, C denotes a positive constant which may change from line to line, depending only on C_0 and on the dimension.

Step 1: In order to utilize Lemma 2.4, we choose $\delta = \delta(C_0, n)$ to be same as in the lemma and assume $\|H_E - \lambda\|_{L^n(\partial E)} \le \delta$. Then E has N many connected components with $N \le C$. We may thus prove the claim componentwise and assume that E is connected. We denote

$$\Sigma := \{x \in \partial E : |H_E(x) - \lambda| < \lambda/2\}.$$

By Lemma 2.4 it holds $\lambda \geq 1/C$ and thus by Hölder's inequality it holds

$$\mathcal{H}^n(\partial E \setminus \Sigma) \leq \frac{2}{\lambda}\int_{\partial E} |H_E(x) - \lambda| \, d\mathcal{H}^n \leq C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3.3}$$

Moreover, we have

$$\begin{aligned}\frac{n}{n+1}\int_{\Sigma} \frac{1}{H_E} \, d\mathcal{H}^n &= \frac{n}{n+1}\int_{\Sigma} \left(\frac{1}{\lambda} + \left(\frac{1}{H_E} - \frac{1}{\lambda}\right)\right) d\mathcal{H}^n \\ &\leq \frac{nP(E)}{(n+1)\lambda} + C\|H_E(x) - \lambda\|_{L^n(\partial E)}.\end{aligned}$$

Since E is connected, Lemma 2.4 yields $\operatorname{diam}(E) \leq \tilde{R}$ with $\tilde{R} = \tilde{R}(C_0, n) \geq R$. Choose $x_0 \in E$. Then using (2.3) with $F(x) = x - x_0$ we obtain

$$nP(E) = (n+1)\lambda|E| + \int_{\partial E} (H_E - \lambda)\langle (x - x_0), \nu_E\rangle \, d\mathcal{H}^n,$$

which in turn implies

$$\big| nP(E) - (n+1)\lambda|E| \big| \leq C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3.4}$$

Hence, we deduce

$$\frac{n}{n+1}\int_{\Sigma} \frac{1}{H_E} \, d\mathcal{H}^n \leq |E| + C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3.5}$$

Next we define

$$Z = \{(x,t) \in \Sigma \times [0,\infty) : 0 \leq t \leq 1/k_n(x)\}.$$

Note that this is well-defined, since $x \in \Sigma$ implies $k_n(x) \geq \frac{H_E(x)}{n} \geq \frac{\lambda}{2n} > 0$. We also set

$$\Sigma_1' = \{x \in \partial E \setminus \Sigma : k_n(x) \leq 1/\tilde{R}\} \qquad \text{and} \qquad \Sigma_2' = \{x \in \partial E \setminus \Sigma : k_n(x) > 1/\tilde{R}\},$$

$$Z_1' = \Sigma_1' \times [0, \tilde{R}] \qquad \text{and} \qquad Z_2' = \{(x,t) \in \Sigma_2' \times [0,\infty) : 0 \leq t \leq 1/k_n(x)\}$$

and finally

$$Z' = Z_1' \cup Z_2'.$$

Then Z and Z' are disjoint and bounded Borel sets and it holds $E \subset \zeta(Z \cup Z')$. To see this fix $y \in E$ and let $x \in \partial E$ be such that $r = d_{\partial E}(y) = |x - y|$. Then we may write $y = x - r\nu_E(x)$ and by the maximum principle $k_n(x) \leq 1/r$. Since $\operatorname{diam}(E) \leq \tilde{R}$, then $r \leq \tilde{R}$ and we conclude that $(x, r) \in Z \cup Z'$ and $y = \zeta(x, r)$.

We now recall the Montiel-Ros argument. We use the fact that $E \subset \zeta(Z \cup Z')$, the area formula, the arithmetic geometric inequality and the fact that for $x \in \Sigma$ it holds $1/k_n(x) \leq$

$n/H_E(x)$ to obtain

$$\begin{aligned}|E| &\le |\zeta(Z)| + |\zeta(Z')| \le \int_{\zeta(Z)} \mathcal{H}^0(\zeta^{-1}(y)\cap Z)\,\mathrm{d}y + |\zeta(Z')|\\ &= \int_Z J_\tau\zeta \,\mathrm{d}\mathcal{H}^{n+1} + |\zeta(Z')|\\ &= \int_\Sigma \int_0^{1/k_n(x)} \prod_{i=1}^n (1 - tk_i(x))\,\mathrm{d}t\mathrm{d}\mathcal{H}^n + |\zeta(Z')|\\ &\le \int_\Sigma \int_0^{1/k_n(x)} \Big(1 - \frac{t}{n}H_E(x)\Big)^n\,\mathrm{d}t\mathrm{d}\mathcal{H}^n + |\zeta(Z')|\\ &\le \int_\Sigma \int_0^{n/H_E(x)} \Big(1 - \frac{t}{n}H_E(x)\Big)^n\,\mathrm{d}t\mathrm{d}\mathcal{H}^n + |\zeta(Z')| = \frac{n}{n+1}\int_\Sigma \frac{1}{H_E}\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')|.\end{aligned}$$

Next we quantify the previous four inequalities. To that aim we define the non-negative numbers R_1, R_2, R_3 and R_4 as

$$R_1 = |\zeta(Z)\setminus E| \tag{3.6}$$

$$R_2 = \int_{\zeta(Z)} |\mathcal{H}^0(\zeta^{-1}(y)\cap Z) - 1|\,\mathrm{d}y \tag{3.7}$$

$$R_3 = \int_\Sigma \int_0^{1/k_n(x)} \Big|\Big(1 - \frac{t}{n}H_E(x)\Big)^n - \prod_{i=1}^n (1 - tk_i(x))\Big|\,\mathrm{d}t\mathrm{d}\mathcal{H}^n \tag{3.8}$$

$$R_4 = \int_\Sigma \int_{1/k_n(x)}^{n/H_E(x)} \Big|1 - \frac{t}{n}H_E(x)\Big|^n\,\mathrm{d}t\mathrm{d}\mathcal{H}^n. \tag{3.9}$$

Then by repeating the Montiel-Ros argument we deduce that

$$|E| \le \frac{n}{n+1}\int_\Sigma \frac{1}{H_E}\,\mathrm{d}\mathcal{H}^n + |\zeta(Z')| - R_1 - R_2 - R_3 - R_4.$$

Therefore, by (3.5) it holds

$$R_1 + R_2 + R_3 + R_4 \le |\zeta(Z')| + C\|H_E - \lambda\|_{L^n(\partial E)},$$

where R_i are defined in (3.6)-(3.9).

Let us next show that

$$|\zeta(Z')| \le C\|H_E(x) - \lambda\|_{L^n(\partial E)}. \tag{3.10}$$

Indeed, by the area formula we have

$$\begin{aligned}|\zeta(Z')| &\le \int_{Z'} J_\tau\zeta\,\mathrm{d}\mathcal{H}^{n+1}\\ &= \int_{\Sigma_1'}\int_0^{\tilde R}\prod_{i=1}^n |1 - tk_i(x)|\,\mathrm{d}t\mathrm{d}\mathcal{H}^n + \int_{\Sigma_2'}\int_0^{1/k_n(x)}\prod_{i=1}^n |1 - tk_i(x)|\,\mathrm{d}t\mathrm{d}\mathcal{H}^n.\end{aligned} \tag{3.11}$$

By the definition of Σ_1' it holds $|1 - tk_i(x)| = (1 - tk_i(x))$ for every $(x,t)\in \Sigma_1'\times[0,\tilde R]$ and therefore by the arithmetic-geometric inequality we may estimate

$$\prod_{i=1}^n |1 - tk_i(x)| \le C(1 + |H_E(x)|^n) \qquad \text{for } (x,t)\in\Sigma_1'\times[0,\tilde R].$$

Similarly, we deduce that

$$\prod_{i=1}^{n} |1 - tk_i(x)| \le C(1 + t^n |H_E(x)|^n) \qquad \text{for } x \in \Sigma_2' \text{ and } 0 \le t \le 1/k_n(x).$$

On the other hand, by the definition of Σ_2' it holds $1/k_n(x) < \tilde{R}$. Therefore, by (3.11), $\lambda \le C$ and (3.3) we have

$$\begin{aligned}|\zeta(Z')| &\le C \int_{\Sigma_1' \cup \Sigma_2'} \int_0^{\tilde{R}} (1 + |H_E(x)|^n)\, \mathrm{d}t \mathrm{d}\mathcal{H}^n = C\tilde{R} \int_{\partial E \setminus \Sigma} (1 + |H_E(x)|^n)\, \mathrm{d}\mathcal{H}^n \\ &\le C \int_{\partial E \setminus \Sigma} (1 + \lambda^n + |H_E - \lambda|^n)\, \mathrm{d}\mathcal{H}^n \le C(\mathcal{H}^n(\partial E \setminus \Sigma) + \|H_E - \lambda\|^n_{L^n(\partial E)}) \\ &\le C\|H_E - \lambda\|_{L^n(\partial E)},\end{aligned}$$

when $\|H_E - \lambda\|_{L^n(\partial E)} \le 1$. Hence by decreasing δ, if needed, we have (3.11). In particular, it holds

$$R_1 + R_2 + R_3 + R_4 \le C\|H_E - \lambda\|_{L^n(\partial E)} \tag{3.12}$$

where R_i are defined in (3.6)-(3.9).

Step 2: Here we utilize the estimate (3.12) and prove the following auxiliary result. For a Borel set $\Gamma \subset \partial E$ and $0 < r < R$ it holds

$$|E \cap \zeta(Z \cap (\Gamma \times (r, R)))| \ge \frac{\mathcal{H}^n(\Gamma)}{(n+1)R^n}(R - r)^{n+1} - C\|H_E - \lambda\|_{L^n(\partial E)}. \tag{3.13}$$

We prove (3.13) by 'backtracking' the Montiel-Ros argument. By the definition of R_1, R_2, R_3, R_4 and (3.12) we may estimate

$$\begin{aligned}|E \cap \zeta(Z \cap (\Gamma \times (r, R)))| &\ge |\zeta(Z \cap (\Gamma \times (r, R)))| - R_1 \\ &\ge \int_{\zeta(Z \cap (\Gamma \times (r,R)))} \mathcal{H}^0(\zeta^{-1}(y) \cap Z \cap (\Gamma \times (r, R)))\, dy - R_1 - R_2 \\ &= \int_{\Gamma \cap \Sigma} \int_{\min\{r, 1/k_n(x)\}}^{\min\{R, 1/k_n(x)\}} \prod_{i=1}^{n} (1 - tk_i(x))\, \mathrm{d}t \mathrm{d}\mathcal{H}^n - R_1 - R_2 \\ &\ge \int_{\Gamma \cap \Sigma} \int_{\min\{r, 1/k_n(x)\}}^{\min\{R, 1/k_n(x)\}} \left(1 - \frac{t}{n} H_E(x)\right)^n \mathrm{d}t \mathrm{d}\mathcal{H}^n - R_1 - R_2 - R_3 \\ &\ge \int_{\Gamma \cap \Sigma} \int_{\min\{r, 1/k_n(x)\}}^{\min\{R, n/H_E(x)\}} \left(1 - \frac{t}{n} H_E\right)^n \mathrm{d}t \mathrm{d}\mathcal{H}^n - R_1 - R_2 - R_3 - R_4 \\ &\ge \int_{\Gamma \cap \Sigma} \int_{\min\{r, n/H_E(x)\}}^{\min\{R, n/H_E(x)\}} \left(1 - \frac{t}{n} H_E\right)^n \mathrm{d}t \mathrm{d}\mathcal{H}^n - R_1 - R_2 - R_3 - R_4.\end{aligned}$$

Recall that for $x \in \Sigma$ it holds $\lambda/2 \le H_E(x) \le 2\lambda$ and that $R = n/\lambda$. Therefore, we may estimate

$$\begin{aligned}\int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}}\left(1-\frac{t}{n}H_E\right)^n \mathrm{d}t\mathrm{d}\mathcal{H}^n &\ge \int_{\Gamma\cap\Sigma}\int_{\min\{r,n/H_E(x)\}}^{\min\{R,n/H_E(x)\}}\left(1-\frac{t}{n}\lambda\right)^n \mathrm{d}t\mathrm{d}\mathcal{H}^n - C\|H_E-\lambda\|_{L^n(\partial E)}\\ &\ge \int_{\Gamma\cap\Sigma}\int_{r}^{R}\left(1-\frac{t}{n}\lambda\right)^n \mathrm{d}t\mathrm{d}\mathcal{H}^n - C\|H_E-\lambda\|_{L^n(\partial E)}\\ &= \frac{\mathcal{H}^n(\Gamma\cap\Sigma)\,n}{(n+1)\lambda}\left(1-\frac{\lambda}{n}r\right)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}\\ &= \frac{\mathcal{H}^n(\Gamma\cap\Sigma)\,R}{(n+1)}\left(1-\frac{r}{R}\right)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}.\end{aligned}$$

Hence, we obtain (3.13) from the previous two inequalities, from (3.3) and (3.12).

Step 3: Here we finally prove the proposition. Recall the definition of E_r in (3.1). Let us first prove that

$$|E_r| \ge \frac{P(E)}{(n+1)R^n}(R-r)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)} \tag{3.14}$$

for all $0 < r < R$.

To this aim, we claim that it holds

$$E\cap\zeta(Z\cap(\Sigma\times(r,R))) \subset E_r \cup \{y\in\zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2\} \cup \zeta(Z'). \tag{3.15}$$

The point of this inclusion is that almost every point which is of the form $y = x - t\nu_E(x)$, for $x \in Z$ and $t \in (r,R)$, belongs to E_r.

To this aim let $y \in E \cap \zeta(\Sigma\times(r,R))$. Then we may write $y = x - t\nu_E(x) = \zeta(x,t)$ for some $x \in \Sigma$ and $t \in (r,R)$, with $(x,t)\in Z$. If $d_{\partial E}(y) = |y-x|$ then $y \in E_r$ because $|x-y| = t > r$. Otherwise, $d_{\partial E}(y) = |y-\tilde{x}| = \tilde{r} < t$ for $\tilde{x}\in\partial E$, so we may write $y = \tilde{x} - \tilde{r}\nu_E(x) = \zeta(\tilde{x},\tilde{r})$ and $(\tilde{x},\tilde{r}) \in Z\cup Z'$. Again, if $(\tilde{x},\tilde{r})\notin Z'$, then $(\tilde{x},\tilde{r})\in Z$ and thus $\mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2$. Hence, we have (3.15).

Recall that by the definition of R_2 and by (3.12) it holds

$$\begin{aligned}|\{y\in\zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y)\cap Z)\ge 2\}| &\le \int_{\zeta(Z)} |\mathcal{H}^0(\zeta^{-1}(y)\cap Z) - 1|\,\mathrm{d}y\\ &\le C\|H_E-\lambda\|_{L^n(\partial E)}.\end{aligned} \tag{3.16}$$

We then use (3.15), (3.16), (3.10) and (3.13) with $\Gamma = \Sigma$ to deduce

$$\begin{aligned}|E_r| &\ge |E\cap\zeta(Z\cap(\Sigma\times(r,R)))| - C\|H_E-\lambda\|_{L^n(\partial E)}\\ &\ge \frac{\mathcal{H}^n(\Sigma)}{(n+1)R^n}(R-r)^{n+1} - C\|H_E-\lambda\|_{L^n(\partial E)}.\end{aligned}$$

The inequality (3.14) then follows from (3.3).

Let us next show that for all $r\in(0,R)$ it holds

$$|E_r| \le \frac{\mathcal{H}^n(\Gamma_r)}{(n+1)R^n}(R-r)^{n+1} + C\|H_E-\lambda\|_{L^n(\partial E)}, \tag{3.17}$$

where $\Gamma_r \subset \partial E$ is defined in (3.2).

First, we show

$$|E_R| \le C\|H_E-\lambda\|_{L^n(\partial E)}. \tag{3.18}$$

This follows from an already familiar argument, so we only sketch it. It is easy to see that $E_R \subset \zeta(Z') \cup \zeta(Z \cap (\Sigma \times (R,\infty)))$. Moreover, since $\lambda/2 \leq H_E(x) \leq 2\lambda$ for $x \in \Sigma$, it holds

$$J_\tau \zeta(x,t) = \prod_{i=1}^n |1 - tk_i(x)| \leq C(1 + |H_E(x)|^n) \leq C \qquad \text{for } (x,t) \in Z \cap (\Sigma \times (R,\infty)).$$

Recall that $R = n/\lambda$. Therefore, we have

$$\begin{aligned} |\zeta(Z \cap (\Sigma \times (R,\infty))| &\leq \int_\Sigma \int_R^{\max\{n/H_E(x),R\}} J_\tau \zeta(x,t)\, dt\mathcal{H}^n \\ &\leq C \int_\Sigma \Big|\frac{n}{H_E} - R\Big|\, dt\mathcal{H}^n \leq C\|H_E - \lambda\|_{L^n(\partial E)}. \end{aligned}$$

The estimate (3.18) then follows from $|E_R| \leq |\zeta(Z \cap (\Sigma \times (R,\infty))| + |\zeta(Z')|$ and (3.10).

Note that for all $\rho \in (r,R)$ it holds $\{x \in E : d_{\partial E}(x) = \rho\} = \zeta(\Gamma_\rho, \rho)$ and $\Gamma_\rho \subset \Gamma_r$. We set $\zeta_\rho = \zeta(\cdot,\rho) : \partial E \to \mathbb{R}^{n+1}$ and thus it holds $\{x \in E : d_{\partial E}(x) = \rho\} = \zeta_\rho(\Gamma_\rho)$ and

$$J_\tau \zeta_\rho(x) = \prod_{i=1}^n |1 - \rho k_i(x)| \leq \left(1 - \frac{H_E}{n}\rho\right)^n \qquad \text{for } x \in \Gamma_\rho.$$

Therefore, by (3.18) and by co-area and area formulas we obtain

$$\begin{aligned} |E_r| &\leq |E_r| - |E_R| + C\|H_E - \lambda\|_{L^n(\partial E)} \leq \int_r^R \mathcal{H}^n(\{x \in E : d_{\partial E} = \rho\})\, d\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\ &= \int_r^R \mathcal{H}^n(\zeta_\rho(\Gamma_\rho))\, d\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\ &\leq \int_r^R \int_{\Gamma_\rho} J_\tau \zeta_\rho(x)\, d\mathcal{H}^n d\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\ &\leq \int_r^R \int_{\Gamma_\rho} \left(1 - \frac{H_E}{n}\rho\right)^n d\mathcal{H}^n d\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\ &\leq \int_r^R \mathcal{H}^n(\Gamma_\rho) \left(1 - \frac{\lambda}{n}\rho\right)^n d\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\ &\leq \mathcal{H}^n(\Gamma_r) \int_r^R \left(1 - \frac{\rho}{R}\right)^n d\rho + C\|H_E - \lambda\|_{L^n(\partial E)} \\ &= \frac{\mathcal{H}^n(\Gamma_r)}{(n+1)R^n}(R - r)^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}. \end{aligned}$$

Hence, we have (3.17).

The second claim of the proposition follows immediately from (3.14) and (3.17). These also imply

$$\left| |E_r| - \frac{P(E)}{(n+1)R^n}(R - r)^{n+1} \right| \leq C\|H_E - \lambda\|_{L^n(\partial E)}.$$

The first claim thus follows from (3.4) and $R = n/\lambda$.

For the last claim we refine the inclusion (3.15) and show that for $0 < \rho < r < R$ and $r' \in (r,R)$ it holds

$$E \cap \zeta(Z \cap (\Gamma_{r'} \times (r' - \rho, R))) \subset (E_r + B_\rho) \cup \{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2\} \cup \zeta(Z'). \tag{3.19}$$

Indeed, let $y \in E \cap \zeta(Z \cap (\Gamma_{r'} \times (r' - \rho, R)))$. Then we may write $y = x - t\nu_E(x)$ for some $x \in \Sigma \cap \Gamma_{r'}$ and $t \in (r' - \rho, R)$, with $(x,t) \in Z$. If $t \in (r', R)$ then by (3.15) it holds

$$
\begin{aligned}
y \in E \cap \zeta(Z \cap (\Sigma \times (r, R))) &\subset E_r \cup \{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2\} \cup \zeta(Z') \\
&\subset (E_r + B_\rho) \cup \{y \in \zeta(Z) : \mathcal{H}^0(\zeta^{-1}(y) \cap Z) \geq 2\} \cup \zeta(Z').
\end{aligned}
$$

Let us then assume that $t \in (r' - \rho, r']$. We write $y = x - r'\nu_E(x) + (r' - t)\nu_E(x)$. Since $x \in \Gamma_{r'}$, i.e., ∂E satisfies the interior ball condition at x with radius $r' > r$, then necessarily $x - r'\nu_E(x) \in E_r$. Therefore, since $0 \leq r' - t < \rho$, we conclude that $y \in E_r + B_\rho$ and (3.19) follows.

We use (3.10), (3.13), (3.16) and (3.19) to conclude

$$
\begin{aligned}
|E_r + B_\rho| &\geq |E \cap \zeta(Z \cap (\Gamma_{r'} \cap \times (r' - \rho, R)))| - C\|H_E - \lambda\|_{L^n(\partial E)} \\
&\geq \frac{\mathcal{H}^n(\Gamma_{r'})}{(n+1)R^n}(R - (r' - \rho))^{n+1} - C\|H_E - \lambda\|_{L^n(\partial E)}.
\end{aligned}
$$

By using the second claim of the proposition and then letting $r' \to r$ we deduce

$$
|E_r + B_\rho| \geq \frac{P(E)}{(n+1)R^n}(R - (r - \rho))^{n+1} - \frac{C}{(R-r)^{n+1}}\|H_E - \lambda\|_{L^n(\partial E)}.
$$

On the other hand, it clearly holds $E_r + B_\rho \subset E_{r-\rho}$. Then by (3.17) we have

$$
|E_r + B_\rho| \leq |E_{r-\rho}| \leq \frac{P(E)}{(n+1)R^n}(R - (r - \rho))^{n+1} + C\|H_E - \lambda\|_{L^n(\partial E)}.
$$

The last claim thus follows from the two previous inequalities and (3.4). $\square$

3.2. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. Let E, λ, C_0 be as in the formulation of Theorem 1.2. Recall that we denote $R = n/\lambda$. As before C denotes a constant which may change from line to line but always depends only on C_0 and n. Let us denote

$$
\varepsilon := \|H_E - \lambda\|_{L^n(\partial E)}.
$$

If $\varepsilon = 0$, then E is a disjoint union balls by [9]. Let us then assume that $0 < \varepsilon \leq \delta$, where δ is initially set as in Proposition 3.3. We might shrink δ several times but always in such a way that it depends only on C_0 and the dimension n. Indeed, by shrinking δ, if needed, Proposition 2.4 provides the estimates

$$
1/C \leq \lambda, R \leq C
$$

and hence the first claim of Theorem 1.2 is clear. We will use these estimates repeatedly without further mention.

By Proposition 2.4 the number of the connected components of E and their diameters are bounded by C. Thus, by applying a similar argument as in the proof of Proposition 3.3 (to obtain (3.4)) on each component and then summing these estimates up we obtain

$$
\big|nP(E) - (n+1)\lambda|E|\big| \leq C\varepsilon. \tag{3.20}
$$

By possibly shrinking δ we have $R - \delta^{\frac{1}{n+2}} \geq R/2$. Choose $r_0 = R - \varepsilon^{\frac{1}{n+2}}$. Then the volume estimates given by Proposition 3.3 read as

$$
\left| |E_r| - \frac{|E|}{R^{n+1}}(R - r)^{n+1} \right| \leq C\varepsilon \tag{3.21}
$$

for all $0 \le r < R$ and

$$\Big| |E_r + B_\rho| - \frac{|E|}{R^{n+1}}(R-(r-\rho))^{n+1}\Big| \le C\varepsilon^{\frac{1}{n+2}} \tag{3.22}$$

for all $0 \le \rho \le r \le r_0$. We remark that by (3.21) we have

$$|E_{r_0}| \ge \frac{|E|}{R^{n+1}}\varepsilon^{\frac{n+1}{n+2}} - C\varepsilon \ge \frac{1}{C}\varepsilon^{\frac{n+1}{n+2}} - C\varepsilon.$$

Hence by decreasing δ, if needed, we may assume that E_{r_0} is non-empty. This implies that $E_{r'}$ is non-empty for $r' > r_0$, when $|r' - r_0|$ is small enough. Since for any $r' > r_0$ it is geometrically clear that $\Gamma_{r'} \subset \partial E_{r_0} + \overline{B}_{r_0}$, then by using Proposition 3.3 and $r_0 = R - \varepsilon^{\frac{1}{n+2}}$ we have

$$\mathcal{H}^n(\partial E \setminus (\overline{E}_{r_0} + \overline{B}_{r_0})) \le \mathcal{H}^n(\partial E \setminus \Gamma_{r'}) \le C\frac{\varepsilon}{(r_0 - r' + \varepsilon^{\frac{1}{n+2}})^{n+1}}.$$

Thus by letting $r' \to r_0$ the previous estimate yields

$$\mathcal{H}^n(\partial E \setminus (\overline{E}_{r_0} + \overline{B}_{r_0})) \le C\varepsilon^{\frac{1}{n+2}}. \tag{3.23}$$

As previously, we divide the proof into three steps.

Step 1: Recall that $r_0 = R - \varepsilon^{\frac{1}{n+2}} \ge R/2$. We prove that there is a positive constant $d_0 = d_0(C_0, n) \le R/4$ such that if $x, y \in E_{r_0}$, then

$$\text{either} \qquad |x-y| < \varepsilon^{\frac{1}{2(n+2)}} \qquad \text{or} \qquad |x-y| \ge d_0. \tag{3.24}$$

Let us fix $x, y \in E_{r_0}$. We denote $d := |x-y|$ and the segment from x to y by $J_{xy} := \{tx + (1-t)y : t \in [0,1]\}$. We may assume that d is small, since otherwise the claim (3.24) is trivially true. To be more precise we assume

$$d \le \min\left\{\frac{R}{4}, 1\right\}. \tag{3.25}$$

Let us first show that

$$J_{xy} \subset E_{r_0 - R^{-1}d^2}. \tag{3.26}$$

Note that $r_0 - R^{-1}d^2 > 0$ by $r_0 \ge R/2$ and (3.25) and hence $E_{r_0 - R^{-1}d^2}$ is well-defined and non-empty. Choose $z \in \mathbb{R}^{n+1} \setminus E$ and $z' \in J_{xy}$ such that

$$|z - z'| = \operatorname{dist}(\mathbb{R}^{n+1} \setminus E, J_{xy}).$$

If $z' = x$ or $z' = y$, then it follows from $x, y \in E_{r_0}$ that $|z - z'| > r_0$. If not, then from the fact that z' is the closest point on J_{xy} to z, we deduce that the vector $x - z'$ is orthogonal to $z - z'$, i.e., $\langle x - z', z - z'\rangle = 0$. Note also that $\min\{|x-z'|, |y-z'|\} \le d/2$ and we may thus assume that $|x - z'| \le d/2$. Therefore, we have by Pythagorean theorem

$$|x-z|^2 = |x-z'|^2 + |z-z'|^2 \le \frac{d^2}{4} + |z-z'|^2.$$

Since $|x - z| > r_0$, the previous estimate gives us

$$|z-z'|^2 > r_0^2 - \frac{d^2}{4}.$$

We deduce from $r_0 \geq R/2$ and (3.25) that

$$\left(r_0^2 - \frac{d^2}{4}\right)^{1/2} \geq r_0 - \frac{d^2}{R}.$$

The previous two estimates yield $|z - z'| > r_0 - R^{-1}d^2$ and the claim (3.26) follows due to the choice of z and z'.

Again, we use $r_0 \geq R/2$ and (3.25) to observe

$$r_0 - (1 + R^{-1})d^2 \geq r_0 - d - R^{-1}d^2 \geq \frac{R}{2} - \frac{R}{4} - \frac{R}{16} > 0.$$

Thus $E_{r_0-(1+R^{-1})d^2}$ is well-defined and non-empty. Next, we deduce from (3.26) and $E_r + B_\rho \subset E_{r-\rho}$ that

$$J_{xy} + B_{d^2} \subset E_{r_0-R^{-1}d^2} + B_{d^2} \subset E_{r_0-(1+R^{-1})d^2}. \tag{3.27}$$

Since $J_{xy} + B_{d^2}$ contains the cylinder $J_{xy} \times B^n_{d^2}$, it is clear that

$$|J_{xy} + B_{d^2}| \geq \omega_n d^{1+2n}.$$

On the other hand, (3.21) and $\varepsilon \leq 1$ (we may assume $\delta \leq 1$) imply

$$\begin{aligned}
|E_{r_0-(1+R^{-1})d^2}| &\leq \frac{|E|}{R^{n+1}}\big(R - (r_0 - (1 + R^{-1})d^2)\big)^{n+1} + C\varepsilon \\
&= \frac{|E|}{R^{n+1}}\big(\varepsilon^{\frac{1}{n+2}} + (1 + R^{-1})d^2)\big)^{n+1} + C\varepsilon \\
&\leq \frac{|E|}{R^{n+1}}\big(\varepsilon^{\frac{1}{n+2}} + (1 + R^{-1})d^2)\big)^{n+1} + C\varepsilon^{\frac{n+1}{n+2}} \\
&\leq Cd^{2(n+1)} + C\varepsilon^{\frac{n+1}{n+2}}.
\end{aligned}$$

Then (3.27) yields

$$\omega_n d^{1+2n} \leq Cd^{2(n+1)} + C\varepsilon^{\frac{n+1}{n+2}}.$$

If $d \geq \varepsilon^{\frac{1}{2(n+2)}}$, then

$$\omega_n d^{1+2n} \leq Cd^{2(n+1)}.$$

This implies $d \geq c > 0$ for some $c = c(C_0, n)$. By recalling (3.25) the claim (3.24) follows.

Step 2: By (3.24) and possibly replacing δ with $\min\{\delta, (d_0/8)^{2(n+2)}\}$ we may divide the set E_{r_0} into N many clusters $E^1_{r_0}, \dots, E^N_{r_0}$ such that we fix a point $x_i \in E_{r_0}$ and define the corresponding cluster $E^i_{r_0}$ as

$$E^i_{r_0} = \{x \in E_{r_0} : |x - x_i| \leq d_0/8\}.$$

By (3.24) it holds $E^i_{r_0} \subset B_{\varepsilon_0}(x_i)$, where $\varepsilon_0 = \varepsilon^{\frac{1}{2(n+2)}}$, and $|x_i - x_j| \geq d_0$ for $i \neq j$. Therefore, we have for every $\rho > 0$

$$\bigcup_{i=1}^N B_\rho(x_i) \subset E_{r_0} + B_\rho \subset \bigcup_{i=1}^N B_{\rho+\varepsilon_0}(x_i). \tag{3.28}$$

Since $r_0 \geq R/2 > R/4 \geq d_0$ and $|x_i - x_j| \geq d_0$ for $i \neq j$, then the balls $B_\rho(x_1), \dots, B_\rho(x_N)$ with $\rho = d_0/4$ are disjoint and contained in E, which, in turn, implies there is an upper bound $N_0 = N_0(C_0, n) \in \mathbb{N}$ for the number of clusters N.

Next we improve the lower bound $|x_i - x_j| \geq d_0$ and prove that there is a positive constant $C_1 = C_1(C_0, n)$ such that

$$|x_i - x_j| \geq 2R - 2C_1 \varepsilon^{\frac{1}{(n+2)^2}} \qquad \text{for all pairs } i \neq j. \tag{3.29}$$

As a byproduct we prove the last statement of the theorem, i.e., we show

$$\Big| P(E) - N(n+1)\omega_{n+1} R^n \Big| \leq C\varepsilon^{\frac{1}{2(n+2)}}. \tag{3.30}$$

Recall that the balls $B_{d_0/4}(x_1), \dots, B_{d_0/4}(x_N)$ are disjoint. Therefore, using $N \leq N_0$ and (3.28) with $\rho = d_0/4$ we deduce

$$\Big| |E_{r_0} + B_{d_0/4}| - N\omega_{n+1} \left(\frac{d_0}{4}\right)^{n+1} \Big| \leq C\varepsilon_0 = C\varepsilon^{\frac{1}{2(n+2)}}.$$

On the other hand, we have $d_0/4 \leq R/16 < R/2 \leq r_0$ so we may use (3.22) to obtain

$$\Big| |E_{r_0} + B_{d_0/4}| - \frac{|E|}{R^{n+1}} \left(\frac{d_0}{4} + \varepsilon^{\frac{1}{n+2}}\right)^{n+1} \Big| \leq C\varepsilon^{\frac{1}{n+2}}.$$

These two estimates and $\varepsilon \leq 1$ imply

$$\Big| |E| - N\omega_{n+1} R^{n+1} \Big| \leq C\varepsilon^{\frac{1}{2(n+2)}}. \tag{3.31}$$

Thus, (3.20), $R = n/\lambda$ and (3.31) yield (3.30).

To obtain (3.29), let us assume that there is $0 < h < R/2$ such that $|x_i - x_j| < 2R - 2h$ for some $i \neq j$. This implies that the balls $B_R(x_i)$ and $B_R(x_j)$ intersect each other such that a set enclosed by a spherical cap of height h is included in their intersection. As the volume enclosed by the spherical cap of height h has a lower bound $c_n R^{n+1} h^{\frac{n+2}{2}}$, with some dimensional constant c_n, then there is $c = c(C_0, n)$ such that

$$\Big| B_R(x_i) \cap B_R(x_j) \Big| \geq c h^{\frac{n+2}{2}}.$$

We use the previous estimate as well as (3.22), (3.28), (3.31), $\varepsilon \leq 1$ and $N \leq N_0$ to estimate

$$\begin{aligned}
N\omega_{n+1} R^{n+1} &\leq |E| + C\varepsilon_0 \\
&\leq |E_{r_0} + B_{r_0}| + C\varepsilon_0 + C\varepsilon^{\frac{1}{n+2}} \\
&\leq \Big| \bigcup_{i=1}^N B_{R+\varepsilon_0}(x_i) \Big| + C\varepsilon_0 + C\varepsilon^{\frac{1}{n+2}} \\
&\leq \Big| \bigcup_{i=1}^N B_R(x_i) \Big| + N\omega_{n+1}((R+\varepsilon_0)^{n+1} - R^{n+1}) + C\varepsilon_0 + C\varepsilon^{\frac{1}{n+2}} \\
&\leq N\omega_{n+1} R^{n+1} - \Big| B_R(x_i) \cap B_R(x_j) \Big| + C\varepsilon_0 + C\varepsilon^{\frac{1}{n+2}} \\
&\leq N\omega_{n+1} R^{n+1} - c h^{\frac{n+2}{2}} + C\varepsilon_0 + C\varepsilon^{\frac{1}{n+2}} \\
&= N\omega_{n+1} R^{n+1} - c h^{\frac{n+2}{2}} + C\varepsilon^{\frac{1}{2(n+2)}} + C\varepsilon^{\frac{1}{n+2}} \\
&\leq N\omega_{n+1} R^{n+1} - c h^{\frac{n+2}{2}} + C\varepsilon^{\frac{1}{2(n+2)}}.
\end{aligned}$$

Thus $h^{\frac{n+2}{2}} \leq C\varepsilon^{\frac{1}{2(n+2)}}$ and (3.29) follows.

Step 3: Let C_1 be as in (3.29). By decreasing δ, if needed, we may assume

$$0 < R - C_1\varepsilon^{\frac{1}{(n+2)^2}} < R - \varepsilon^{\frac{1}{n+2}} = r_0.$$

Then we have by (3.28) and (3.29) that the balls $B_\rho(x_1), \dots, B_\rho(x_N)$, with $\rho = R - C_1\varepsilon^{\frac{1}{(n+2)^2}}$, are disjoint and

$$\bigcup_{i=1}^{N} B_\rho(x_i) \subset E_{r_0} + B_\rho \subset E_{r_0-\rho} \subset E. \tag{3.32}$$

This, $\varepsilon \le 1$, $N \le N_0$ and (3.31) imply

$$\left| E \setminus \bigcup_{i=1}^{N} B_\rho(x_i) \right| \le C\varepsilon^{\frac{1}{(n+2)^2}}. \tag{3.33}$$

Set $\varepsilon_1 = \varepsilon^{\frac{1}{(n+2)^3}}$. We prove

$$E \subset \bigcup_{i=1}^{N} B_\eta(x_i) \tag{3.34}$$

for $\eta = R + C_2\varepsilon_1$ with some positive $C_2 = C_2(n, C_0)$. By decreasing δ, if necessary, we deduce from (3.33) that

$$|B_{\varepsilon_1}| > \left| E \setminus \bigcup_{i=1}^{N} B_\rho(x_i) \right|.$$

Thus, if $x \in E_{\varepsilon_1}$, then $B_{\varepsilon_1}(x) \cap \bigcup_{i=1}^{N} B_\rho(x_i)$ must be non-empty. This implies

$$E_{\varepsilon_1} \subset \bigcup_{i=1}^{N} B_{\rho+\varepsilon_1}(x_i). \tag{3.35}$$

Assume that for $x \in \partial E$ it holds

$$d_x := \operatorname{dist}\left(x, \overline{E}_{r_0} + \overline{B}_{r_0}\right) > 0.$$

Then by (3.23)

$$\mathcal{H}^n(\partial E \cap B(x, d_x)) \le C\varepsilon^{\frac{1}{n+2}}.$$

Let $\delta_n \in \mathbb{R}_+$ be as in Lemma 3.2 and set $r_x = \min\{d_x, \delta_n/\lambda\}$. Again, by possibly decreasing δ so that $\delta \le \delta_n$, Lemma 3.2 yields

$$\delta_n r_x^n \le \mathcal{H}^n\left(\partial E \cap B_{r_x}(x)\right).$$

By combining the two previous estimates we have

$$\min\left\{d_x, \frac{\delta_n}{\lambda}\right\} \le C\varepsilon^{\frac{1}{n(n+2)}}.$$

Since $\delta_n/\lambda \ge \delta_n/C$, then by decreasing δ, if necessary, the previous estimate implies $r_x = d_x$ and further gives us

$$d_x \le C\varepsilon^{\frac{1}{n(n+2)}} \le C\varepsilon^{\frac{1}{(n+2)^2}}. \tag{3.36}$$

On the other hand, by (3.28)

$$\overline{E}_{r_0} + \overline{B}_{r_0} \subset E_{r_0} + B_R \subset \bigcup_{i=1}^{N} B_{R+\varepsilon_0}(x_i), \tag{3.37}$$

where $\varepsilon_0 = \varepsilon^{\frac{1}{2(n+2)}} \le \varepsilon^{\frac{1}{(n+2)^2}}$. Thus, (3.36) and (3.37) imply

$$\partial E \subset \bigcup_{i=1}^{N} B_{\tilde{\eta}}(x_i)$$

with $\tilde{\eta} = R + C\varepsilon^{\frac{1}{(n+2)^2}}$. By combining this observation with (3.35) we obtain (3.34).

Finally, by decreasing δ one more time, if necessary, (3.30), (3.32) and (3.34) yield

$$\bigcup_{i=1}^{N} B_{\rho_-}(x_i) \subset E \subset \bigcup_{i=1}^{N} B_{\rho_+}(x_i),$$

where $\rho_- = R - C\varepsilon^{\frac{1}{(n+2)^3}}$, $\rho_+ = R + C\varepsilon^{\frac{1}{(n+2)^3}}$, the balls $B_{\rho_-}(x_1), \dots, B_{\rho_-}(x_N)$ are mutually disjoint, for N it holds

$$\Big| P(E) - N(n+1)\omega_{n+1}R^n \Big| \le C\varepsilon^{\frac{1}{(n+2)^3}}$$

and $C = C(C_0, n) \in \mathbb{R}_+$. The claim of Theorem 1.2 then follows by Remark 3.1. □

4. Asymptotic behavior of the volume preserving mean curvature flow

In this section we first define the flat flow and recall some of its basic properties. We do this in the general dimensional case $\mathbb{R}^{n+1}$ and resctrict ourself to the case $n \le 2$ only in the proof of Theorem 1.1. We begin by defining the flat flow of (1.1).

Assume that $E_0 \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter with the volume of the unit ball $|E_0| = \omega_{n+1}$. For a given $h \in \mathbb{R}_+$ we construct a sequence of sets $(E_k^h)_{k=1}^{\infty}$ by iterative minimizing procedure called minimizing movements, where initially $E_0^h = E_0$ and E_{k+1}^h is a minimizer of the following problem

$$\mathcal{F}_h(E, E_k) = P(E) + \frac{1}{h}\int_E \bar{d}_{E_k}\, dx + \frac{1}{\sqrt{h}}\Big||E| - \omega_{n+1}\Big|. \tag{4.1}$$

Recall, that $\bar{d}_{E_k}$ is the signed distance function from E_k. We then define the approximative flat flow $(E_t^h)_{t\ge 0}$ by

$$E_t^h = E_k^h, \qquad \text{for } (k-1)h \le t < kh. \tag{4.2}$$

By [29] we know that there is a subsequence of the approximative flat flow which converges

$$(E_t^{h_l})_{t\ge 0} \to (E_t)_{t\ge 0},$$

where for every $t > 0$ the set E_t is a set of finite perimeter with $|E_t| = \omega_{n+1}$. Any such limit is called a flat flow of (1.1). It follows from [29] that when $n \le 6$ and if the perimeters of E_t^h converge, i.e., $\lim_{h\to 0} P(E_t^h) = P(E_t)$ for every $t > 0$, then the flat flow is a weak solution of the volume preserving mean curvature flow. It is not known if the flat flow coincide with the classical solution of (1.1) when the latter is well defined and smooth, but the result in [5] seems to indicate this (see also [4]).

4.1. Preliminary results.

Let us take more rigorous approach to the concepts heuristically introduced above. We base this mainly on [29], where the only difference is that the volume constraint has a different value. Obviously, this does not affect the arguments.

First, we take a closer look at the functional $\mathcal{F}_h$ given by (4.1). If $E, F \subset \mathbb{R}^{n+1}$ are bounded sets of finite perimeter, then it is easy to see that modifications of E in a set of measure zero do not affect the value $\mathcal{F}_h(E, F)$ whereas such modifications of F may lead drastic changes of the of $\mathcal{F}_h(E, F)$. To eliminate this issue, we use a convention that a topological boundary of a set of finite perimeter is always the support of the corresponding Gauss-Green measure. Thus, we consider $\mathcal{F}_h$ as a functional $X_{n+1} \times \{A \in X_{n+1} : A \neq \emptyset\} \to \mathbb{R}$, where

$$X_{n+1} = \{E \subset \mathbb{R}^{n+1} : E \text{ is a bounded set of finite perimeter with } \partial E = \operatorname{spt} \mu_E\}.$$

We remark that if E_0 is essentially open or closed and $E_0 \in X_{n+1}$, then we may assume it to be open or closed respectively.

For a non-empty $F \in X_{n+1}$ there is always a minimizer E of the functional $\mathcal{F}_h(\,\cdot\,, F)$ in the class X_{n+1} satisfying the *discrete dissipation inequality*

$$P(E) + \frac{1}{h}\int_{E\Delta F} d_{\partial F}\, \mathrm{d}x + \frac{1}{\sqrt{h}}\,\big||E| - \omega_{n+1}\big| \leq P(F) + \frac{1}{\sqrt{h}}\,\big||F| - \omega_{n+1}\big|, \tag{4.3}$$

see [29, Lemma 3.1]. Moreover, there is a dimensional constant C_n such that

$$\sup_{E\Delta F} d_{\partial F} \leq C_n\sqrt{h}, \tag{4.4}$$

see [29, Proposition 3.2]. The minimizer E is always a (Λ, r_0) *-minimizer* in any open neighborhood of E with suitable $\Lambda, r_0 \in \mathbb{R}_+$ satisfying $\Lambda r_0 \leq 1$. Thus, by the standard regularity theory [22, Thm 26.5 and Thm 28.1] $\partial^* E$ is relatively open in ∂E and $C^{1,\alpha}$-regular with any $0 < \alpha < 1/2$ and the Hausdorff dimension of the singular part $\partial E \smallsetminus \partial^* E$ is at most $n-7$. These imply that E can always be chosen as an open set. On the other hand, if E is non-empty, it has a Lipschitz-continuous distributional mean curvature H_E satisfying the Euler-Lagrange equation

$$\frac{\bar{d}_F}{h} = -H_E + \lambda_E, \tag{4.5}$$

where the *Lagrange multiplier* can be written in the case $|E| \neq \omega_{n+1}$ as

$$\lambda_E = \frac{1}{\sqrt{h}}\operatorname{sgn}\left(\omega_{n+1} - |E|\right), \tag{4.6}$$

see [29, Lemma 3.7]. Thus, by using standard elliptic estimates one can show that $\partial^* E$ is in fact $C^{2,\alpha}$-regular and (4.5) holds in the classical sense on $\partial^* E$. In particular, E is $C^{2,\alpha}$-regular when $n \leq 6$. Moreover, if $x \in \partial E$ satisfies exterior or interior ball condition with any r, then it must belong to the reduced boundary of E. This is well-known and follows essentially from [9, Lemma 3].

Let us turn our focus back on flat flows. Let $E_0 \in X_{n+1}$ be a set with volume ω_{n+1} and $0 < h < (\omega_{n+1}/P(E_0))^2$. Then we find a minimizer $E_1^h \in X_{n+1}$ for $\mathcal{F}_h(\,\cdot\,, E_0)$ and by (4.3) we have $\big||E_1^h| - \omega_{n+1}\big| \leq \sqrt{h}P(E_0)$ implying, via the condition $h < (\omega_n/P(E_0))^2$, that E_1^h is non-empty. Again, we find a minimizer $E_2^h \in X_{n+1}$ for $\mathcal{F}_h(\,\cdot\,, E_1)$ and using (4.3) twice we obtain $\big||E_2^h| - \omega_{n+1}\big| \leq \sqrt{h}P(E_0)$ and thus E_2^h is also non-empty. By continuing the procedure we find non-empty sets $E_0^h, E_1^h, E_2^h, \ldots \in X_{n+1}$ as mentioned earlier, i.e., $E_0^h = E_0$ and E_k^h is a minimizer of $\mathcal{F}_h(\,\cdot\,, E_{k-1})$ for every $k \in \mathbb{N}$. Thus, we may define an approximate flat flow $(E_t^h)_{t\geq 0}$, with the initial set E_0, defined by (4.2). Further a flat flow as a limit is defined as before. By iterating (4.3) we obtain

$$P(E_{kh}^h) + \frac{1}{h}\sum_{j=1}^{k}\int_{E_{jh}^h \Delta E_{(j-1)h}^h} d_{\partial E_{(j-1)h}^h}\, \mathrm{d}x + \frac{1}{\sqrt{h}}\big||E_{kh}^h| - \omega_{n+1}\big| \leq P(E_0) \quad \text{for every } k \in \mathbb{N}. \tag{4.7}$$

By the earlier discussion we may assume that E_t^h, for every $t \geq h$, is an open set and ∂E_t^h is C^2-regular up to the singular part $\partial E_t^h \setminus \partial^* E_t^h$ with Hausdorff dimesion at most $n-7$. We use the shorthand notation λ_t^h for the corresponding Lagrange multiplier.

Next, we list some basic properties of the approximative flat flow.

Proposition 4.1. *Let $(E_t^h)_{t\geq 0}$ be an approximative flat flow starting from $E_0 \in X_{n+1}$ with volume ω_{n+1} and $P(E_0) \leq C_0$. There is a positive constant $C = C(C_0, n)$ such that the following holds for every $0 < h < (\omega_n / P(E_0))^2$:*

(i) For every s, t with $h \leq s \leq t-h$ it holds $|E_s^h \Delta E_t^h| \leq C\sqrt{t-s}$.

(ii) Suppose that for given $T_1 \geq 0$ it holds $|E_{T_1}^h| = \omega_{n+1}$. Then $P(E_{T_1}^h) \geq P(E_t^h)$ for every $t \geq T_1$ and

$$\int_{T_1+h}^{T_2} \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \, d\mathcal{H}^n \mathrm{d}t \leq C(P(E_{T_1}^h) - P(E_{T_2}^h))$$

for every $T_2 \geq T_1 + h$. Moreover, for every $h \leq T_1 < T_2$ it holds

$$\int_{T_1}^{T_2} \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \, d\mathcal{H}^n \mathrm{d}t \leq C P(E_0).$$

(iii) For every $T > 0$ there is $R = R(E_0, T)$ such that $E_t^h \subset B_R$ for all $0 \leq t \leq T$.

(iv) If $(h_k)_k$ is a sequence of positive numbers converging to zero, then up to a subsequence there exist approximative flat flows $((E_t^{h_k})_{t\geq 0})_k$ which converges to a flat flow $(E_t)_{t \geq 0}$, where $E_t \in X_{n+1}$, in L^1-sense in space and pointwise time, i.e., for every $t \geq 0$ it holds

$$\lim_{h_k \to 0} |E_t^{h_k} \Delta E_t| = 0.$$

The limit flow also satisfies $|E_s \Delta E_t| \leq C\sqrt{t-s}$ for every $0 < s < t$ and $|E_t| = \omega_{n+1}$ for every $t \geq 0$.

(v) If E_0 is either open or closed, then the sequence in (iv) converges to $(E_t)_{t\geq 0}$ in L^1 in space and compactly uniformly in time, i.e., for a fixed T it holds

$$\lim_{h_k \to 0} \sup_{t \in [0,T]} |E_t^{h_k} \Delta E_t| = 0.$$

Moreover, $|E_s \Delta E_t| \leq C\sqrt{t-s}$ for every $0 \leq s < t$.

Proof. The claims (i) - (iv) are essentially proved in [29], see the proofs of Proposition 3.5, Lemma 3.6 and Theorem 2.2.

To prove (v), we first show that

$$|E_h^h \Delta E_0| \to 0 \quad \text{as} \quad h \to 0$$

which immediately implies via (iv) that $|E_0 \Delta E_t| \leq C\sqrt{t}$ for every $t \geq 0$ and hence the second claim of (v) holds. Then the compactly uniform convergence in time is a rather direct consequence of this and (i).

To this aim, let $(h_k)_k$ be an arbitrary sequence of positive numbers converging to zero. By (iii) and by the standard compactness property of sets of finite perimeter there is a bounded set of finite perimeter E_∞ such that, up to extracting a subsequence, $E_{h_k}^{h_k} \to E_\infty$ in L^1 -sense. In particular, by (4.7) we have $|E_\infty| = \omega_{n+1} = |E_0|$. Again, by using $|E_{h_k}^{h_k} \Delta E_\infty| \to 0$ and (4.4) we have

$$|E_\infty \setminus \{y \in \mathbb{R}^n : \bar{d}_{E_0}(y) \leq j^{-1}\}| = 0 \quad \text{and} \quad |\{y \in \mathbb{R}^n : \bar{d}_{E_0}(y) \leq -j^{-1}\} \setminus E_\infty| = 0$$

for every $j \in \mathbb{N}$. Thus, by letting $j \to \infty$ we obtain $|E_\infty \setminus \bar{E}_0| = 0$ and $|\mathrm{int}(E_0) \setminus E_\infty| = 0$. Since E_0 is open or closed, this means either $|E_\infty \setminus E_0| = 0$ or $|E_0 \setminus E_\infty| = 0$. But now $|E_\infty| = |E_0|$ so the previous yields $|E_\infty \Delta E_0| = 0$. Thus, $|E_{h_k}^{h_k} \setminus E_0| \to 0$ up to a subsequence and since $(h_k)_k$ was arbitrarily chosen, then it holds $|E_h^h \Delta E_0| \to 0$. □

We note that the claim (v) does not hold for every bounded set of finite perimeter E_0. As an example one may construct a wild set of finite perimeter E_0 such that $|E_h^h \Delta E_0| \geq c_0 > 0$ for all $h > 0$

By [29, Corollary 3.10] it holds, for a fixed time $T \geq h$, that the integral $\int_h^T |\lambda_t^h|^2 dt$ is uniformly bounded in h and hence, via (4.6), it holds $\big|\{t \in (h, T) : |E_t^h| \neq \omega_{n+1}\}\big| \leq Ch$, where C depends also on T. We may improve this by using Lemma 2.4.

Proposition 4.2. *Let $C_0 > 0$ and $E_0 \in X_{n+1}$ be a set of finite perimeter with volume ω_{n+1} and $P(E_0) \leq C_0$. There are positive constants $C = C(C_0, n)$ and $h_0 = h_0(C_0, n)$ such that if $h \leq h_0$ and $(E_t^h)_{t\geq 0}$ is an approximative flat flow starting from E_0, then for every $h \leq T_1 \leq T_2$*

$$\int_{T_1}^{T_2} |\lambda_t^h|^2 dt \leq C(T_2 - T_1 + 1) \quad \text{and}$$
$$\big|\{t \in (T_1, T_2) : |E_t^h| \neq \omega_{n+1}\}\big| \leq Ch(T_2 - T_1 + 1).$$

Proof. By (4.7) we may choose $h_0 = h_0(C_0, n)$ such that $|E_t^h| \geq \frac{\omega_{n+1}}{2}$ whenever $h \leq h_0$. We may also assume $C_0 > 2\omega_{n+1}$ so $|E_t^h| \geq 1/C_0$ for $h \leq h_0$. Thus, by Lemma 2.4 and $P(E_t^h) \leq C_0$ we find a positive $C = C(C_0, n)$ such that for every $t \geq h$ and $h \leq h_0$ it holds

$$|\lambda_t^h|^2 \leq C\left(1 + \int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \mathrm{d}\mathcal{H}^n\right).$$

Therefore,

$$\int_{T_1}^{T_2} |\lambda_t^h|^2 dt \leq C(T_2 - T_1) + C\int_{T_1}^{T_2}\int_{\partial^* E_t^h} (H_{E_t^h} - \lambda_t^h)^2 \mathrm{d}\mathcal{H}^n \mathrm{d}t.$$

By Lemma 4.1 (ii) we obtain the first inequality. The first inequality implies, via (4.6), the second inequality with the same constant C. □

We need also the following comparison result for the proof.

Lemma 4.3. *Let $1 \leq C_0 < \infty$. Assume $E_0 \in X_{n+1}$ is a set of finite perimeter with volume ω_{n+1} and $P(E_0) \leq C_0$, and let $F = \bigcup_{i=1}^N B_r(x_i)$ with $|x_i - x_j| \geq 2r$ and $1/C_0 \leq r \leq C_0$. There is a positive constant $\varepsilon_0 = \varepsilon_0(C_0, n)$ such that if $(E_t^h)_{t\geq 0}$ is an approximative flat flow starting from E_0 and*

$$\sup_{x \in E_{t_0}^h \Delta F} d_{\partial F}(x) \leq \varepsilon \quad \text{with} \quad \varepsilon \leq \varepsilon_0$$

for $t_0 \geq 0$, then it holds

$$\sup_{x \in E_t^h \Delta F} d_{\partial F}(x) \leq C\varepsilon^{\frac{1}{9}} \qquad \text{for all } t_0 < t < t_0 + \sqrt{\varepsilon}$$

provided that $h \leq \min\{\sqrt{\varepsilon}, h_0\}$, where $h_0 = h_0(C_0, n)$ is as in Proposition 4.2.

Proof. Our standing assumptions are $h \le \min\{\sqrt{\varepsilon}, h_0\}$ and $\varepsilon \le \min\{1/(2C_0), 1\}$. As usual, C denotes a positive constant which may change from line to line but depends only on the parameters C_0 and n.

Without loss of generality we may assume $t_0 = 0$. Fix an arbitrary $x_i \in \{x_1, \dots, x_N\}$. Up to translating the coordinates we may assume that $x_i = 0$. We set for every $k = 0, 1, 2, \dots$

$$\rho_k = \inf\{|x| : x \in \mathbb{R}^{n+1} \setminus E^h_{kh}\} \quad \text{and} \quad r_k = \min\{r, \rho_0, \dots, \rho_k\}.$$

We claim that it holds

$$r^2_{k+1} - r^2_k \ge -C_1(1 + |\lambda^h_{(k+1)h}|)h, \tag{4.8}$$

with some positive constant $C_1 = C_1(C_0, n)$. First, if $r_{k+1} = r_k$, the claim (4.8) is trivially true. Thus, we may assume $r_{k+1} < r_k$ which implies $\rho_{k+1} = r_{k+1} < r_k \le \rho_k$. Then $\rho_k > 0$ which, in turn, means

$$\rho_k = \min_{\partial E^h_{kh}} |x|.$$

Since $E^h_{(k+1)h}$ is bounded and open, there is a point $x \in \mathbb{R}^{n+1} \setminus E^h_{(k+1)h}$ with $\rho_{k+1} = |x|$. Let x' be a closest point to x on ∂E^h_{kh}. Then

$$r_{k+1} + |\bar{d}_{E^h_{kh}}(x)| = |x| + |\bar{d}_{E^h_{kh}}(x)| \ge |x'| \ge \rho_k \ge r_k.$$

The condition $|x| < \rho_k$ means $x \in E^h_{kh}$ so the previous estimate yields

$$r_{k+1} - r_k \ge \bar{d}_{E^h_{kh}}(x). \tag{4.9}$$

Again, $x \in E^h_{kh} \setminus E^h_{(k+1)h}$ so by Lemma 4.4 $|\bar{d}_{E^h_{kh}}(x)| \le C_n\sqrt{h}$ and hence

$$r_{k+1} - r_k \ge -C_n\sqrt{h}. \tag{4.10}$$

We split the argument into two cases. First, if $r_{k+1} < C_n\sqrt{h}$, then by (4.10) we have $r_k < 2C_n\sqrt{h}$. Therefore, using (4.10) we obtain

$$r^2_{k+1} - r^2_k \ge -C_n(r_{k+1} + r_k)\sqrt{h} \ge -3C_n^2 h. \tag{4.11}$$

If $r_{k+1} \ge C_n\sqrt{h}$, then by (4.10) $r_k \le 2r_{k+1}$. Since $r_{k+1} > 0$, then it holds $x \in \partial E^h_{(k+1)h}$ and $E^h_{(k+1)h}$ satisfies interior ball condition of radius r_{k+1} at x. Thus, by discussion in Section 2 x belongs to the reduced boundary of $E^h_{(k+1)h}$ and therefore by the maximum principle it holds $H_{E^h_{(k+1)h}}(x) \le \frac{n}{r_{k+1}}$. Again, by the previous estimate, (4.9), the Euler-Lagrange equation (4.5) and $r_{k+1} \le C_0$ we obtain

$$\frac{r_{k+1} - r_k}{h} \ge \frac{\bar{d}_{E^h_{kh}}(x)}{h} \ge -\frac{n}{r_{k+1}} - |\lambda^h_{(k+1)h}| \ge -\frac{1}{r_{k+1}}\left(n + C_0|\lambda^h_{(k+1)h}|\right).$$

Therefore,

$$\frac{r^2_{k+1} - r^2_k}{h} \ge -\left(1 + \frac{r_k}{r_{k+1}}\right)\left(n + C_0|\lambda^h_{(k+1)h}|\right) \ge -3\left(n + C_0|\lambda^h_{(k+1)h}|\right). \tag{4.12}$$

Thus, (4.11) and (4.12) yield the claim (4.8) in the case $r_{k+1} < r_k$.

We iterate (4.8) up to $K \in \mathbb{N}$, which is chosen so that $Kh \in (\sqrt{\varepsilon}, 2\sqrt{\varepsilon})$ (recall $h < \sqrt{\varepsilon}$), and use Proposition 4.2 to obtain

$$
\begin{aligned}
r_K^2 - r_0^2 &\geq -C_1 \sum_{k=0}^{K-1} (1 + |\lambda_{(k+1)h}^h|)h \\
&= -C_1 Kh - C_1 \int_h^{(K+1)h} |\lambda_t^h| \, dt \\
&\geq -2C_1\sqrt{\varepsilon} - C_1 \int_h^{3\sqrt{\varepsilon}} |\lambda_t^h| \, dt \\
&\geq -2C_1\sqrt{\varepsilon} - \int_h^{3\sqrt{\varepsilon}} \varepsilon^{-\frac{1}{4}} + \varepsilon^{\frac{1}{4}} |\lambda_t^h|^2 \, dt \\
&\geq -C\varepsilon^{\frac{1}{4}} \left(1 + \int_h^{3\sqrt{\varepsilon}} |\lambda_t^h|^2 \, dt \right) \geq -C\varepsilon^{\frac{1}{4}}.
\end{aligned}
\tag{4.13}
$$

By the assumption $\sup_{x \in E_0 \Delta F} d_{\partial F}(x) \leq \varepsilon$ we have $r - \varepsilon \leq r_0$. Thus we divide $r_K^2 - r_0^2$ by $r_K + r_0$ and use $r_0 \geq r - \varepsilon \geq r/2 \geq 1/(2C_0)$ as well as (4.13) to find a positive constant $C_2 = C_2(C_0, n)$ such that $r_K \geq r - C_2 \varepsilon^{\frac{1}{4}}$. This means that

$$
\inf_{\mathbb{R}^{n+1} \setminus E_t^h} \bar{d}_{B_r(x_i)} \geq -C_2 \varepsilon^{\frac{1}{4}} \qquad \text{for all } t < \sqrt{\varepsilon}
$$

and again due to the arbitrariness of $x_i \in \{x_1, \dots, x_N\}$

$$
\inf_{\mathbb{R}^{n+1} \setminus E_t^h} \bar{d}_F \geq -C_2 \varepsilon^{\frac{1}{4}} \qquad \text{for all } t < \sqrt{\varepsilon}.
$$

To conclude the proof, we show that there is a positive constant $\varepsilon_1 = \varepsilon_1(C_0, n)$ such that

$$
\sup_{E_t^h} \bar{d}_F \leq 2\varepsilon^{\frac{1}{9}} \qquad \text{for all } t < \sqrt{\varepsilon} \tag{4.14}
$$

provided that $\varepsilon \leq \varepsilon_1$. To this aim we choose an arbitrary $x_0 \in \mathbb{R}^{n+1} \setminus \bar{F}$ with $\bar{d}_F(x_0) \geq 2\varepsilon^{\frac{1}{9}}$. We set for every $k = 0, 1, 2, \dots$

$$
\rho_k = \inf_{x \in E_{kh}^h} |x - x_0| \quad \text{and} \quad r_k = \min\{2\varepsilon^{\frac{1}{9}}, \rho_1, \dots, \rho_k\}.
$$

In particular, $r_k \leq 2C_0^{\frac{1}{9}}$. A slight modification of the procedure we used to obtain (4.13) yields

$$
r_K^2 - r_0^2 \geq -C\varepsilon^{\frac{1}{4}},
$$

where K is the same as earlier. Again, the conditions $\sup_{x \in E_0 \Delta F} d_{\partial F}(x) \leq \varepsilon$ and $\varepsilon \leq 1$ imply $r_0 \geq 2\varepsilon^{\frac{1}{9}} - \varepsilon \geq \varepsilon^{\frac{1}{9}}$. Thus

$$
r_K - r_0 \geq -C \frac{\varepsilon^{\frac{1}{4}}}{r_0} \geq -C\varepsilon^{\frac{5}{36}} = -C\varepsilon^{\frac{1}{36}} \varepsilon^{\frac{1}{9}}
$$

and thus $r_K \geq (1 - C\varepsilon^{\frac{1}{36}})\varepsilon^{\frac{1}{9}} > \frac{1}{2}\varepsilon^{\frac{1}{9}}$, when ε is small enough. Since x_0, with $d_F(x_0) \geq 2\varepsilon^{\frac{1}{9}}$, was arbitrarily chosen we deduce that

$$
E_{kh}^h \subset \{x \in \mathbb{R}^{n+1} : d_F(x) \leq 2\varepsilon^{\frac{1}{9}}\} \qquad \text{for all } k = 0, \dots, K.
$$

The claim (4.14) then follows from the choice of K.

□

4.2. **Proof of Theorem 1.1.** The proof of Theorem 1.1 is based on Theorem 1.2. We first use it together with the dissipation inequality in Proposition 4.1 (ii) to deduce that there exists a sequence of times $t_j \to \infty$ such that the sets E_{t_j} are close to a disjoint union of balls. Since perimeter of the approximative flat flow is essentially decreasing then the number of the balls is also monotone. In particular, we deduce that after some time, the sets E_{t_j} are close to a fixed number, say N, of balls. We use the second statement of Theorem 1.2 to deduce that the perimeters of E_{t_j} converges to the perimeter of N many balls with volume ω_{n+1} and thus the right-hand-side of the dissipation inequality converges to zero. This allows us to improve our estimate and use Theorem 1.2 again to deduce that the flat flow E_t is close to a disjoint union of N many balls for all large t except a set of times with small measure. The statement then finally follows from Lemma 4.3.

Proof of Theorem 1.1. Assume that the initial set $E_0 \in X_{n+1}$ has the volume of the unit ball $|E_0| = \omega_{n+1}$, fix a positive C_0 with $C_0 \geq \max\{1, P(E_0)\}$ and assume $h < (C_0/\omega_{n+1})^2$. Let $(E_t)_{t\geq 0}$ be a flat flow starting from E_0 and let $(E_t^{h_l})_{t\geq 0}$ be an approximative flat flow which by Proposition 4.1 converges to $(E_t)_{t\geq 0}$ locally uniformly in L^1. We simplify the notation and denote the converging subsequence again by h. Since we are now in the dimensions 2 and 3 ($n = 1, 2$), the sets E_t^h are C^2-regular.

Step 1: Let us denote

$$\Sigma_h := \{t \in (0, \infty) : |E_t^h| \neq \omega_{n+1}\} \tag{4.15}$$

By (4.7) and Proposition 4.2 we find a constant $h_0 = h_0(C_0, n) < 1$ such that $|E_t^h| \geq 1/C_0$ for every $t \geq 0$ and

$$|(T_1, T_2) \cap \Sigma_h| \leq \frac{1}{3}(T_2 - T_1)$$

for every $T_1 \geq 1$ and $T_2 \geq T_1 + 1$ provided that $h \leq h_0$. On the other hand, by Proposition 4.1 (ii) we have for every $h \leq h_0$ and $l \in \mathbb{N}$

$$I_{l,h} := \fint_{l^2}^{(l+1)^2} \|H_{E_t^h} - \lambda_t^h\|_{L^2(\partial E_t^h)}^2 \, dt \leq \frac{C}{l}.$$

By Chebysev's inequality

$$\left|\{t \in (l^2, (l+1)^2) : \|H_{E_t^h} - \lambda_t^h\|_{L^2(\partial E_t^h)}^2 \geq 3I_{l,h}\}\right| \leq \frac{1}{3}((l+1)^2 - l^2).$$

Therefore, by choosing $T_1 = l^2$ and $T_2 = (l+1)^2$ we deduce that the set

$$\left\{t \in (T_1, T_2) : |E_t^h| = \omega_{n+1}, \|H_{E_t^h} - \lambda_t^h\|_{L^2(\partial E_t^h)}^2 < 3I_{l,h}\right\}$$

is non-empty. Thus, if $h \leq h_0$, then there is a sequence of times $(T_l^h)_l$, with $l^2 \leq T_l^h \leq (l+1)^2$, such that the corresponding sets satisfy $|E_{T_l^h}^h| = \omega_{n+1}$ and

$$\|H_{E_{T_l^h}^h} - \lambda_{T_l^h}^h\|_{L^2(\partial E_{T_l^h}^h)} \leq C l^{-\frac{1}{2}}. \tag{4.16}$$

By slight abuse of the notation we set $E_l^h := E_{T_l^h}^h$ and $\lambda_{l,h} := \lambda_{T_l^h}^h$ for $h \leq h_0$. Since the sets E_l^h are C^2-regular and bounded, then thanks to $P(E_0) \leq C_0$, $|E_l^h| \geq 1/C_0$, (4.16) and Theorem

1.2 we find $l_0 = l_0(C_0, n)$ such that for every $l \ge l_0$ we have $1/C \le \lambda_{l,h} \le C$,

$$|P(E_l^h) - N_l^h(n+1)\omega_{n+1}(r_l^h)^n| \le Cl^{-\frac{q}{2}} \qquad \text{and} \qquad \sup_{E_l^h \Delta F_l^h} d_{\partial F_l^h} \le Cl^{-\frac{q}{2}}, \tag{4.17}$$

where $r_l^h = n/\lambda_{l,h}$ and F_l^h is a union of N_l^h-many pairwise disjoint (open) balls of radius $r_{l,h}$. Since $1/C \le \lambda_{l,h} \le C$, then also $1/C \le r_{l,h} \le C$, which together with the perimeter estimate $P(E_l^h) \le P(E_0) \le C_0$ implies that there is $N_0 = N_0(C_0, n) \in \mathbb{N}$ such that $N_l^h \le N_0$. Further the distance estimate in (4.17), together with $1/C \le r_{l,h} \le C$ and $N_l^h \le N_0$, yields

$$|E_l^h \Delta F_l^h| \le Cl^{-\frac{q}{2}}.$$

Since $|E_l^h| = \omega_{n+1}$, then the estimate above implies $|(r_{l,h})^{n+1} N_l^h - 1| \le Cl^{-\frac{q}{2}}$ and further $|(r_{l,h})^n (N_l^h)^{\frac{n}{n+1}} - 1| \le Cl^{-\frac{q}{2}}$. This inequality, the perimeter estimate in (4.17) and $N_l^h \le N_0$ imply

$$|P(E_l^h) - (n+1)\omega_{n+1}(N_l^h)^{\frac{1}{n+1}}| \le Cl^{-\frac{q}{2}}. \tag{4.18}$$

Since by Proposition (4.1) (ii) $(P(E_l^h))_{l\ge l_0}$ is non-increasing, then (4.18) implies that there is a positive integer $l_1 = l_1(C_0, n) \ge l_0$ for which $(N_l^h)_{l\ge l_1}$ is non-increasing for all $h \le h_0$.

Step 2: For $l \ge l_1$ and $h \le h_0$ the sets E_l^h are thus close to N_l^h many balls. We claim that there are $N \in \mathbb{N}$ and $l_2 \ge l_1$ such that for every integer $L \ge l_2$ it holds

$$N_l^h = N \qquad \text{for all } l_2 \le l \le L \tag{4.19}$$

provided that h is small enough.

By using a standard diagonal argument and possibly passing to a subsequence we find a sequence of positive integers $(N_l)_{l\ge l_1}$, with $N_l \le N_0$, such that $N_l^h \to N_l$ for every $l \ge l_1$. Since $(N_l^h)_{l\ge l_1}$ is non-increasing, then $(N_l)_{l\ge l_1}$ is non-increasing too and hence there are $N, l_2 \in \mathbb{N}$, $l_2 \ge l_1$, such that $N_l = N$ for every $l \ge l_2$. Hence, we have (4.19) by the convergence of N_l^h to N_l.

We obtain from (4.18) and (4.19) that

$$|P(E_l^h) - (n+1)\omega_{n+1}(N)^{\frac{1}{n+1}}| \le Cl^{-\frac{q}{2}} \tag{4.20}$$

for $l_2 \le l \le L$, provided that h is small enough. Therefore, it follows from Proposition 4.1 (ii) that

$$\int_{T_l^h + h}^{T_L^h} \|H_{E_t^h} - \lambda_t^h\|_{L^2(\partial E_t^h)}^2 \, \mathrm{d}t \le Cl^{-\frac{q}{2}}.$$

Since $h \le 1$, and $L > 1$ was arbitrary, the above yields

$$\sup_{T \ge (l+2)^2} \left[\limsup_{h\to 0} \int_{(l+2)^2}^{T} \|H_{E_t^h} - \lambda_t^h\|_{L^2(\partial E_t^h)}^2 \, \mathrm{d}t \right] \le Cl^{-\frac{q}{2}} \tag{4.21}$$

for every $l \ge l_2$.

Step 3: Let us fix small δ, which choice will be clear later. Then it follows from (4.21), (4.20) and the fact $t \mapsto P(E_t^h)$ is non-increasing in Σ_h that there is T_δ such that for every $T \ge T_\delta + 1$ there is $h_{\delta,T}$ such that

$$\int_{T_\delta}^{T} \|H_{E_t^h} - \lambda_t^h\|_{L^2(\partial E_t^h)}^2 \, \mathrm{d}t \le \delta \tag{4.22}$$

for all $h \le h_{\delta,T}$ and

$$|P(E_t^h) - (n+1)\omega_{n+1}N^{\frac{1}{n+1}}| \le \delta \tag{4.23}$$

for all $t \in (T_\delta, T) \smallsetminus \Sigma_h$. On the other hand, by Proposition 4.2 and by decreasing $h_{\delta,T}$ if necessary we deduce that

$$|\Sigma_h \cap (T_\delta, T)| \le \delta \qquad \text{for all } h \le h_{\delta,T}. \tag{4.24}$$

Let $\varepsilon > 0$ and let us fix $t \ge T_\delta + 1$. (The time $T_\delta + 1$ will be T_ε in the claim.) We claim that, when δ is chosen small enough, it holds

$$\sup_{E_t^h \Delta F_t^h} d_{\partial F_t^h} \le \varepsilon, \tag{4.25}$$

for $h \le h_{\delta,T}$, where F_t^h is a union of N-many pairwise disjoint (open) balls of radius $r = N^{-\frac{1}{n+1}}$ with volume ω_{n+1}.

Fix $T \ge t+1$. Then it follows from (4.22) that

$$\int_{t-\delta^{1/4}}^t \|H_{E_\tau^h} - \lambda_\tau^h\|_{L^2(\partial E_\tau^h)}^2 \, d\tau \le \delta$$

and from (4.23) and (4.24) that

$$|P(E_\tau^h) - (n+1)\omega_{n+1}N^{\frac{1}{n+1}}| \le \delta \qquad \text{for all } \tau \in (t-\delta^{1/4}, t) \smallsetminus \Sigma_h$$

and $|\Sigma_h \cap (t-\delta^{1/4}, t)| \le \delta$. Using these estimates we deduce that there is $t_0 \in (t-\delta^{1/4}, t)$ such that $|E_{t_0}^h| = \omega_{n+1}$,

$$\left|P(E_{t_0}^h) - (n+1)\omega_{n+1}N^{\frac{1}{n+1}}\right| \le \delta \tag{4.26}$$

and

$$\|H_{E_{t_0}^h} - \lambda_{t_0}^h\|_{L^2(\partial E_{t_0}^h)} \le \delta^{1/4}.$$

Theorem 1.2 implies that

$$\sup_{E_{t_0}^h \Delta F_{t_0}^h} d_{\partial F_{t_0}^h} \le C\delta^{q/4},$$

for all $h \le h_{\delta,T}$, where $F_{t_0}^h$ is a union of $N_{t_0,h}$-many pairwise disjoint (open) balls of radius $r_{t_0,h}$ with volume ω_{n+1} and

$$\left|P(E_{t_0}^h) - N_{t_0,h}(n+1)\omega_{n+1}r_{t_0,h}^n\right| \le C\delta^{q/4}.$$

Since $1/C \le r_{t_0,h} \le C$, then, as in Step 1, we deduce from the previous two estimates above that $|E_{t_0}^h \Delta F_{t_0}^h| \le C\delta^{q/4}$. Then by (4.26) and $|F_{t_0}^h| = \omega_{n+1}$ we further conclude that $N_{t_0,h} = N$, i.e., $F_{t_0}^h$ is a union of N-many pairwise disjoint (open) balls with volume ω_{n+1} and radius $r = N^{-\frac{1}{n+1}}$.

By Lemma 4.3 it holds

$$\sup_{E_\tau^h \Delta F_{t_0}^h} d_{\partial F_{t_0}^h} \le C\delta^{\frac{q}{36}} \qquad \text{for all } t_0 < \tau < t_0 + \delta^{\frac{q}{8}}$$

and $h \le h_{\delta,T}$. In particular, since $\delta^{\frac{q}{8}} > \delta^{\frac{1}{4}}$ the above inequality holds for t. This proves (4.25) by choosing $F_t^h = F_{t_0}^h$ and δ small enough. The claim follows by letting $h \to 0$. Note that by Proposition 4.1 (iii) there is $R > 0$ such that $F_t^h \subset B_R$ for all $h \le h_{\delta,T}$. Therefore, by passing to another subsequence if necessary, we have that $F_t^h \to F_t$, where F_t is a union of N-many pairwise disjoint (open) balls with volume ω_{n+1} and by (4.25) it holds

$$\sup_{E_t \Delta F_t} d_{\partial F_t} \leq \varepsilon.$$

□

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Stationary sets of the mean curvature flow with a forcing term

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STATIONARY SETS OF THE MEAN CURVATURE FLOW WITH A FORCING TERM

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Abstract. We consider the flat flow solution to the mean curvature equation with forcing in $\mathbb{R}^n$. Our main result states that tangential balls in $\mathbb{R}^n$ under a flat flow with a bounded forcing term will experience fattening, which generalizes the result in [11] from the planar case to higher dimensions. Then, as in the planar case, we characterize stationary sets in $\mathbb{R}^n$ for a constant forcing term as finite unions of equisize balls with mutually positive distance.

1. Introduction

In this article, we consider the mean curvature flow (MCF) with a bounded forcing term for compact embedded hypersurfaces. By definition, this is a family of embedded surfaces $(\Sigma_t)_{t\in[0,\infty)}$ in $\mathbb{R}^n$, with initial set Σ_0, and which moves according to the law

$$V_t = -H_{\Sigma_t} + f(t), \tag{1.1}$$

where V_t is the normal velocity, H_{Σ_t} the mean curvature and f a bounded measurable function. It is well known that the flow may develop singularities for a smooth initial set when $n \geq 3$ [12] and even in the plane when $f \neq 0$ [4]. In order to define the flow over the singular times and in order to define it for rough initial sets, one may define a weak solution by using either the level set formulation [7, 10], the flat flow via the minimizing movement scheme [1, 14] or Brakke's varifold formulation [5]. The main issue is that there is no unique way to define the weak solution, and the previous methods may give rise to a different solution. The level-set approach provides a unique *function* which is a solution of the corresponding partial differential equation in the viscosity sense, but its level sets may have positive volume. We call this phenomenon *fattening*. De Giorgi's minimal and maximal barriers provide essentially the same solution as the level-set approach, and in this context the fattening means that the minimal and the maximal solution do not agree. The fattening may occur instantaneously if the initial set is not regular [4, 10] or after a finite time for regular initial sets [4]. In this work, we consider the flat flow of (1.1), which is a solution obtained via the minimizing movement scheme as in [1, 14]. The flat flow can be defined for rough embedded initial hypersurfaces which are boundaries of sets of finite perimeter. Therefore, it is more natural in this context to define the flow for sets rather than surfaces. If the initial set is smooth, then the flat flow agrees with the classical solution for a short time interval, but in case of fattening it is not clear if it is uniquely defined.

Here we study the fattening for the flat flow of (1.1) in the specific case when the initial set is a union of two tangent balls. It is well known that in this case the level-set solution produces instantaneously fattening [4, 13]. We also mention the work [9], where Dirr, Luckhaus and Novaga study the same setting but add randomness to the flow. For a general introduction to the topic, we refer to [3]. In our main theorem, we generalize the result in [11] from the plane to $\mathbb{R}^n$ and prove that the flat flow instantaneously connects the two tangent balls with a thin neck which continues to grow at least for a short period of time.

Theorem 1.1. *Let $E_0 \subset \mathbb{R}^n$, $n \geq 2$, be a union of two tangential balls $B(x_1, r)$ and $B(x_2, r)$. Let $(E_t)_t$ be a flat flow with a forcing term f, which is bounded by $C_0 \in \mathbb{R}_+$, starting from E_0. There exist positive numbers δ, c_1 and c_2 depending only on n, r and C_0 such that for every $t \in (0, \delta)$ the set E_t contains a dumbbell-shaped simply connected set which again contains the balls $B(x_1, r - c_1 t)$, $B(x_2, r - c_1 t)$ and $B((x_1 + x_2)/2, c_2 t)$.*

We note that the above result immediately generalizes to the case when the two balls do not have the same radii. This follows from Theorem 1.1 and a standard comparison argument (see Proposition 3.2).

Theorem 1.1 implies that a union of tangent balls cannot be a stationary set of the flow (1.1). Therefore, we may use the characterization of critical points of the isoperimetric problem from [8] to characterize all stationary points of the flow (1.1).

Theorem 1.2. *A bounded set of finite perimeter $E_0 \subset \mathbb{R}^n$, with $n \geq 2$, is a stationary set of the flow (1.1) (see definition 3.5) with a positive constant forcing Λ exactly when it is a finite union of balls of radius $r = (n-1)/\Lambda$ with mutually positive distance.*

Let us finally mention a few words about the proof of Theorem 1.1. We begin the proof as in the planar case [11] by showing that any discrete approximation of the flat flow creates at the first step a neck which connects the two balls. After this, we need to show that this neck is growing until the time δ. In the planar case, it is enough to construct a single barrier set to show that the neck is growing (see [11, Proof of Theorem 1.1]). In the higher-dimensional case, we need to construct a family of comparison sets which, together with a delicate comparison argument, implies that the neck is growing. The novelty of the proof is the construction of this discrete barrier flow. A similar idea is used in [13] in the context of level set solutions. The main difference is that in our case the flow is defined via time discretization.

2. Notation and preliminary results

Let us introduce some basic concepts and notation. First, our standing assumption throughout the paper is that the dimension n is at least two, and for $x \in \mathbb{R}^n$ we use the decomposition $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$. For a given set $E \subset \mathbb{R}^n$, the distance function $d_E : \mathbb{R}^n \to \mathbb{R}$ is given by $d_E(x) = \inf_{y \in E} |x - y|$, and further the signed distance function $\bar{d}_E : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\bar{d}_E(x) = \begin{cases} -d_E(x), & x \in E \\ d_E(x), & x \in \mathbb{R}^n \setminus E. \end{cases}$$

For the empty set, we use the convention that its signed distance function is ∞ everywhere. If $E \subset \mathbb{R}^n$ is Lebesgue-measurable, then we will denote its n-dimensional Lebesgue-measure by $|E|$.

For a set of finite perimeter $E \subset \mathbb{R}^n$, the term $\partial^* E$ denotes its reduced boundary as usual. Recall that then $\overline{\partial^* E}$ is the support of the corresponding Gauss-Green measure and the perimeter of E is given by $P(E) = \mathcal{H}^{n-1}(\partial^* E)$. If E is C^1-regular, then we have $\partial^* E = \partial E$. Moreover, we may always assume $\partial E = \overline{\partial^* E}$. The measure theoretic outer unit normal is defined in $\partial^* E$ and we denote it by ν_E. If E is a C^1-set, then ν_E agrees with the classical outer unit normal of E. Again, for every C^1-vector field $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ the tangential differential at x is defined by

$$\mathrm{D}_\tau \Psi(x) = \mathrm{D}\Psi(x)(I - \nu_E(x) \otimes \nu_E(x))$$

and the tangential divergence by $\mathrm{div}_\tau\, \Psi = \mathrm{Tr}(\mathrm{D}_\tau \Psi(x))$.

For an orientable C^2 -hypersurface $\Sigma \subset \mathbb{R}^n$, with orientation $\nu_\Sigma : \Sigma \to \partial B(0,1)$, the corresponding mean curvature $H_\Sigma(x)$ at $x \in \Sigma$ is defined as the sum of the principal curvatures $k_1(x), \dots, k_{n-1}(x)$. If $E \subset \mathbb{R}^n$ is a C^2 -set, then $H_E(x)$ for $x \in \partial E$ denotes $H_{\partial E}(x)$, with the orientation ν_E, and we have the classical (surface) divergence theorem

$$\int_{\partial E} \mathrm{div}_\tau\, \Psi \; \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial E} H_E \langle \Psi, \nu_E \rangle \; \mathrm{d}\mathcal{H}^{n-1}$$

for every $\Psi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$. In general, we say that a set of finite perimeter $E \subset \mathbb{R}^n$ has a *distributional mean curvature* $H_E \in L^1(\partial^* E)$ if for every $\Psi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$ it holds

$$\int_{\partial^* E} \mathrm{div}_\tau\, \Psi \; \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* E} H_E \langle \Psi, \nu_E \rangle \; \mathrm{d}\mathcal{H}^{n-1}. \tag{2.1}$$

Note that for C^2-regular sets the distributional mean curvature agrees with the classical mean curvature. Finally, we say that a set of finite perimeter $E \subset \mathbb{R}^n$ is *critical* if it has a constant distributional mean curvature. By [8, Theorem 1], we know that the critical sets are characterized as finite unions of balls with equal radius and mutually disjoint interiors. As a consequence, we have the following convergence result; see [8, Corollary 2].

Theorem 2.1. *Let $(E_i)_{i=1}^\infty$ be a sequence of sets of finite perimeter in $\mathbb{R}^n$ with distributional mean curvature H_{E_i}, let $E \subset \mathbb{R}^n$ be a set of finite perimeter with a positive volume, and let Λ be a positive constant such that $|E \Delta E_i| \to 0$, $P(E_i) \to P(E)$ and $H_{E_i} \to \Lambda$ in the distributional sense, i.e., for every $\Psi \in C_0^1(\mathbb{R}^n; \mathbb{R}^n)$ it holds*

$$\lim_{i \to \infty} \int_{\partial^* E_i} \mathrm{div}_\tau\, \Psi - \Lambda \langle \Psi, \nu_{E_i} \rangle \; \mathrm{d}\mathcal{H}^{n-1} = 0.$$

Then E is a finite union of balls with the equal radius $r = (n-1)/\Lambda$ and the balls have mutually disjoint interiors.

We will use *solids of revolution*, which are obtained by rotating a non-negative function around the x_1-axis in $\mathbb{R}^n$. If g is a non-negative function defined on an interval $[a, b]$, then we will denote by $\mathbf{C}(g, [a, b])$ the solid of revolution

$$\mathbf{C}(g, [a, b]) \coloneqq \{x \in \mathbb{R}^n : x_1 \in [a, b],\ x' \in \bar{B}^{n-1}(0, g(x_1))\}.$$

Again, by the *heads* of $\mathbf{C}(g, [a, b])$ we mean the vertical part of the boundary

$$\{x \in \mathbb{R}^n : x_1 \in \{a, b\},\ x' \in \bar{B}^{n-1}(0, g(x_1))\}.$$

In the special case of a cylinder, symmetric to the hyperplane $\{x_1 = 0\}$, i.e., $g \equiv R > 0$ and $b = -a$, we simply denote $\mathbf{C}(R, a) = \mathbf{C}(R, [-a, a])$. In the case where g is continuous on $[a, b]$ and vanishes at the endpoints, we make the following technical observation.

Remark 2.2. Suppose that $g \in C([a, b])$ is non-negative with $g(a) = 0 = g(b)$ and set $E = \mathbf{C}(g, [a, b])$. Then, for every $x_1 \in \mathbb{R}$, the set $\bar{d}_E(x_1, \,\cdot\,) : \mathbb{R}^{n-1} \to \mathbb{R}$ is a radially symmetric function strictly increasing in radius.

If $g \in C([a,b]) \cap C^2((a,b))$ and g is strictly positive, then for the surface of revolution

$$\Gamma = \{(x_1, x') \in \mathbb{R}^n : x_1 \in (a,b), x' \in \partial B^{n-1}(0, g(x_1))\}$$

with the inside-out orientation of $\mathbf{C}(g,[a,b])$ one computes

$$H_\Gamma(x) = -\frac{g''(x_1)}{(1+g'(x_1)^2)^{\frac{3}{2}}} + \frac{1}{(1+g'(x_1)^2)^{\frac{1}{2}}}\frac{n-2}{g(x_1)} \tag{2.2}$$

for every $x \in \Gamma$.

A solid of revolution $\mathbf{C}(g,[a,b])$ is an example of a *Schwarz symmetric* set. Recall that for every measurable set $E \subset \mathbb{R}^n$ its Schwarz symmetrization, or $(n-1)$-dimensional Steiner symmetrization, with respect to a direction $e \in \partial B(0,1)$ is a measurable set E_e^* such that for every $t \in \mathbb{R}$ the section $\{z \in \langle e \rangle^\perp : te + z \in E_e^*\}$ is an open $(n-1)$-dimensional ball centered at the origin and it holds

$$\mathcal{H}^{n-1}(\{z \in \langle e \rangle^\perp : te + z \in E\}) = \mathcal{H}^{n-1}(\{z \in \langle e \rangle^\perp : te + z \in E_e^*\}).$$

Note that $|E_e^*| = |E|$ and if E is a set of finite perimeter, then E_e^* is also a set of finite perimeter and $P(E_e^*) \le P(E)$ (see[2]). A set E is Schwarz symmetric with respect to e if $E_e^* = E$ holds, up to a set of measure zero.

3. Flat flows with forcing and stationary sets

Let us first heuristically explain how a flat flow with a forcing term is obtained via the minimizing movement scheme. Let $C_0 \in \mathbb{R}_+$ be a fixed constant and let $f : [0,\infty) \to \mathbb{R}$ be a measurable function satisfying the condition

$$\sup_{t \ge 0} |f(t)| \le C_0. \tag{3.1}$$

The function f will act as a time dependent forcing term in the dynamics. Now if E_0 is a bounded set of finite perimeter, then we define for every $0 < h \le 1$ a sequence of bounded sets of finite perimeter $(E^{h,k})_{k=0}^\infty$, a so-called *approximative sequence*, inductively by setting first $E^{h,0} = E_0$ and for $k = 0,1,2,\dots$ we set $E^{h,k+1}$ to be a minimizer of the functional

$$F \mapsto P(F) + \frac{1}{h}\int_F \bar{d}_{E^{h,k}} \, dx - \bar{f}(h,k)|F|, \tag{3.2}$$

where $\bar{f}(h,k) = \fint_{kh}^{(k+1)h} f(t)\, dt$. Then we define an *approximate flat flow* $(E_t^h)_{t\ge 0}$ by setting

$$E_t^h = E^{h,k} \quad \text{for} \quad kh \le t < (k+1)h. \tag{3.3}$$

If there is a subsequence $(h_k)_{k\in\mathbb{N}}$ with $h_k \to 0$ and a family of bounded sets of finite perimeter $(E_t)_{t\ge 0}$ such that $E^{h_k} \to E_t$ for every $t \ge 0$ in the L^1-sense, then we call $(E_t)_{t\ge 0}$ a *flat flow* with forcing f starting from E_0. An existence of such a cluster point is always guaranteed; see, for instance, [11, Proposition 2.3].

Let us next make the above argument more precise by using the results in [11, 16]. We note that in [16] Mugnai, Seis and Spadaro consider flat flow for volume preserving mean curvature flow, but the arguments will remain valid in our setting. Our first observation is that the functional in (3.2) may change its values if we perturb the set $E^{h,k}$ by a set of measure zero due to the distance function. In order to use the notion of distance function consistently, we define the class

$$X_n = \{E \subset \mathbb{R}^n : E \text{ is a bounded set of finite perimeter with } \partial E = \overline{\partial^* E}\}.$$

Recall that every (essentially) bounded set of finite perimeter has an L^1-equivalent set from X_n. For given $0 < h \le 1$ and $\Lambda \in [-C_0, C_0]$, we define the functional

$$\mathcal{F}_{h,\Lambda} : X_n \times X_n \to \mathbb{R} \cup \{\infty\}$$

by setting

$$\mathcal{F}_{h,\Lambda}(F, E) = P(F) + \frac{1}{h}\int_F \bar{d}_E \,\mathrm{d}x - \Lambda|F|. \tag{3.4}$$

For every $E \in X_n$, the functional $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ admits a minimizer $E_{\min} \in X_n$; see [16, proof of Lemma 3.1]. If E is empty, then $\bar{d}_E = \infty$, and hence necessarily $E_{\min}$ must be empty too. Minimizers have the following distance property; see the proof of [14, Lemma 2.1] (or [16, Proposition 3.2]): there is a positive constant $\gamma = \gamma(n, C_0)$ such that for every $E \in X_n$ and every minimizer $E_{\min} \in X_n$ of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ it holds

$$|\bar{d}_E| \le \gamma h^{\frac{1}{2}} \quad \text{in} \quad E \Delta E_{\min}. \tag{3.5}$$

Now, (3.5) has the following consequence.

Remark 3.1. Suppose that $E_1, E_2, \dots, E_k \in X_n$ have a mutually positive distance of at least d. Then there is a positive $h_d = h_d(n, C_0, d) \le 1$ such that for any $h \le h_d$ it holds that any minimizer of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, \bigcup_i E_i)$ must be a union of minimizers of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E_i)$.

In general, the uniqueness of a minimizer of $\mathcal{F}_{h,\Lambda}(\cdot, E)$ is not known. However, the following weak comparison principle holds; see [6, proof of Lemma 7.2].

Proposition 3.2. *Let $E, E' \in X_n$ and $\Lambda, \Lambda' \in [-C_0, C_0]$, with $\Lambda > \Lambda'$.*

(i) If $E' \subset\subset E$ and $E_{\min}, E'_{\min} \in X_n$ are minimizers of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ and $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E')$ respectively, then $|E'_{\min} \setminus E_{\min}| = 0$.

(ii) If $E' \subset E$ and $E_{\min}, E'_{\min} \in X_n$ are minimizers of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ and $\mathcal{F}_{h,\Lambda'}(\,\cdot\,, E')$ respectively, then $|E'_{\min} \setminus E_{\min}| = 0$.

Concerning the regularity of a minimizer $E_{\min}$ of (3.4), it is not difficult to see that it is a (Λ_0, r_0)-perimeter minimizer (using the notation from [15]) with suitable $\Lambda_0, r_0 \in \mathbb{R}_+$ satisfying $\Lambda_0 r_0 \le 1$. Then it follows [15, Theorem 26.5 and Theorem 28.1] that $\partial^* E_{\min}$ is relatively open in $\partial E_{\min}$, a $C^{1,\alpha}$-regular hypersurface for every $0 < \alpha < 1/2$ and the (closed) singular part $\partial E \setminus \partial^* E$ has Hausdorff-dimension at most $n - 8$. In particular, from now on *we will use the convention that the minimizers are always open sets.*

Moreover, by considering local variations $(\Phi_t)_t$ of the form $\Phi_t = \mathrm{id} + t\Psi$, with $\Psi \in C^1_0(\mathbb{R}^n, \mathbb{R}^n)$, and differentiating $t \mapsto \mathcal{F}_{h,\Lambda}(\Phi_t(E_{\min}), E)$ at zero, we see that $E_{\min}$ has distributional mean curvature $H_{E_{\min}}$ which satisfies the Euler-Lagrange equation in the distributional sense

$$\frac{\bar{d}_E}{h} = -H_{E_{\min}} + \Lambda \quad \text{on} \quad \partial^* E_{\min}. \tag{3.6}$$

Since $H_{E_{\min}}$ is Lipschitz continuous on $\partial^* E$, then, by standard elliptic estimates, $\partial^* E_{\min}$ is $C^{2,\alpha}$-regular and (3.6) holds in the classical sense on the reduced boundary. In particular, $E_{\min}$ is a $C^{2,\alpha}$-set when $n \le 7$. Finally, we note that if ∂E satisfies an exterior or interior ball condition at x, then x must belong to the regular part $\partial^* E_{\min}$. This follows essentially from [8, Lemma 3].

The next proposition states the somewhat obvious fact that for a ball $E = B(x, r)$ any non-empty minimizer of $\mathcal{F}_{h,\Lambda}(\cdot, E)$ must be a concentric ball.

Proposition 3.3. *For a ball $E = B(x,r)$, every minimizer (3.4) must be an open concentric ball or the empty set. There is a positive constant $h_0 = h_0(n, C_0) \le 1$ such that if $h \le h_0$, then every ball $B(x,r)$, with $r \ge (n-1)/C_0$, has a concentric ball $B(x, r_{\min})$ as a unique minimizer of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ and it holds*

$$r_{\min} - r = \left[\Lambda - \frac{n-1}{r} + \mathcal{O}(h)\right] h. \tag{3.7}$$

In the case $\Lambda = (n-1)/r$, the error term $\mathcal{O}(h)$ vanishes and hence $r_{\min} = r$.

Proof. The first claim is easy to see by using the isoperimetric inequality and the fact that for a given non-zero volume V an open ball of the volume V, centered at x, is a unique minimizer of the energy $\int_F \bar{d}_{B(x,r)}\, \mathrm{d}y$ among the open sets F of the volume V. Again, by using (3.5), we see that if h is sufficiently small compared to the radius, then every minimizer must be non-empty and hence a concentric ball. Thus, the uniqueness and (3.7) follow from (3.5) and the Euler-Lagrange equation (3.6). □

Let us denote the Schwarz symmetrization of E with respect to the x_1-axis simply by E^*. As we mentioned above, Schwarz symmetrization decreases the perimeter and preserves the volume. Moreover, for a smooth set in the case of equality $P(E^*) = P(E)$, it holds that every vertical slice $E_{x_1} = \{x' \in \mathbb{R}^{n-1} : (x_1, x') \in E\}$ is an $(n-1)$-dimensional ball [2]. We also notice that if the set E is Schwarz symmetric with respect to the x_1-axis, then Schwarz symmetrization also decreases the dissipation term of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ defined in (3.4). This follows rather directly from Fubini's theorem. For a suitable solid of revolution E around the x_1-axis, there is invariance of minimizers under the symmetrization.

Proposition 3.4. *If $E = \mathbf{C}(g, [a,b])$, with a non-negative and continuous g attaining the zero value at the endpoints, then every (open) minimizer F of $\mathcal{F}_{h,\Lambda}(\,\cdot\,, E)$ defined in (3.4) is Schwarz symmetric with respect to the x_1-axis.*

Proof. Let F be a such a minimizer. We may assume F to be non-empty. Now $P(F^*) \le P(F)$ and $|F^*| = |F|$. By Remark 2.2, every section $\bar{d}_E(x_1, \,\cdot\,)$ is radially symmetric and strictly increasing in radius, which implies via Fubini's theorem that the $(n-1)$-dimensional Lebesgue measure of the symmetric difference $|(F^*)_{x_1} \Delta F_{x_1}|_{n-1}$ of the vertical slices $(F^*)_{x_1}$ and F_{x_1} is zero for almost every x_1, since otherwise it would hold

$$\int_{F^*} \bar{d}_E \, \mathrm{d}x < \int_F \bar{d}_E \, \mathrm{d}x$$

and hence $\mathcal{F}_{h,\Lambda}(F^*, E) < \mathcal{F}_{h,\Lambda}(F, E)$ contradicting the minimality of F. Since F is open, every vertical slice $F_{x_1} \subset \mathbb{R}^{n-1}$ is open too, and thus the previous observation guarantees that $(F^*)_{x_1} = F_{x_1}$ for almost every x_1. Thus, the openness of F implies that the equality holds for every x_1.

□

After this discussion, we are convinced that an approximative sequence $(E^{h,k})_{k=0}^{\infty}$, starting from E_0, where for every $k = 1, 2\ldots$, the set $E^{h,k+1}$ is defined as a minimizer of the functional $\mathcal{F}_{h,\bar{f}(h,k)}(\,\cdot\,, E^{h,k})$ defined in (3.4), is well-defined. Further, we may define the approximative flat flow $(E_t^h)_{t\ge 0}$ as in (3.3). We have for every $t \ge h$ that the set E_t^h is open and C^2-regular up to a singular part $\partial E_t^h \setminus \partial^* E_t^h$ of Hausdorff-dimesion at most $n-8$. Moreover, E_t^h, with $t \ge h$, has a distributional mean curvature $H_{E_h^t}$ which satisfies the Euler-Lagrange equation (3.6), with $\Lambda = \bar{f}(h, \lfloor t/h \rfloor - 1)$, in a weak sense and on $\partial^* E_t^h$ in the classical sense. For more properties of the approximative flat flows, when the forcing term satisfies (3.1), such as local Hölder continuity of $(t,s) \mapsto |E_t^h \Delta E_s^h|$ and perimeter control we refer to [11, Proposition 2.3].

Next, we define *stationary sets* of (1.1) with constant forcing term by using flat flows as in [11, Definition 3.1].

Definition 3.5. A non-empty set $E_0 \in X_n$ is a stationary set of (1.1) for a constant forcing term $f \equiv \Lambda > 0$ if for any flat flow, starting from E_0 it holds

$$\sup_{0 \le t \le T} |E_t \Delta E_0| = 0$$

for every $T > 0$.

By using Remark 3.1 and Proposition 3.3, one may conclude the obvious direction of Theorem 1.2, that is, a finite union of equisize balls with a mutually positive distance is a stationary set for the constant forcing term $\Lambda = (n-1)/r$, where r is the radius of the balls. In turn, the following lemma states that the converse is almost true, that is, a stationary set is also critical, i.e., a finite union of balls with equal radius and mutually disjoint interiors.

Lemma 3.6. *Every stationary set $E_0 \subset \mathbb{R}^n$ for a positive constant forcing term Λ, is a finite union of balls of radius $r = (n-1)/\Lambda$ with mutually disjoint interiors.*

Proof. The lemma is already established in the two-dimensional case in [11, Lemma 3.4]. Again, the proof of the general case is analogous to the proof of [11, Lemma 3.4] with the only essential change being that we use Theorem 2.1 instead of [11, Lemma 3.2]. Therefore, we only sketch the proof. Besides Theorem 2.1, we also use some basic properties of approximate flat flows proven in [11].

We begin by fixing times $0 < T_1 < T_2$. Then, by Definition 3.5 and [11, Proposition 2.3], we have a decreasing sequence $(h_i)_{i=1}^{\infty}$, with $0 < h_i < 1$ and h_i converging to zero, such that the approximate flat flows $(E_t^{h_i})_{t \ge 0}$, with constant forcing $f \equiv \Lambda$ and starting from E_0, satisfy

$$\lim_{i \to \infty} \sup_{t \in [T_1, T_2]} |E_0 \Delta E_t^{h_i}| = 0. \tag{3.8}$$

Moreover, since the forcing term is constant, it follows from the argument in the proof of [11, Proposition 2.4] that there is $C \in \mathbb{R}_+$, independent of h, such that for every $t \in [T_1, T_2]$ with $t > h$ it holds

$$\int_h^t \int_{\partial^* E_s^h} |H_{E_s^h} - \Lambda|^2 \, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}s \le C \left[(P(E_0) - P(E_t^h)) + (|E_t^h| - |E_0|) \right]. \tag{3.9}$$

Now, (3.8) and (3.9) imply $\limsup_{i \to \infty} P(E_t^{h_i}) \le P(E_0)$, for every $t \in [T_1, T_2]$. On the other hand, by [11, Proposition 2.3], there is a radius $R > 0$, independent of i, such that $E_t^{h_i} \subset B(0,R)$ for every $t \in [0, T_2]$. Then, by the lower semi-continuity of the perimeter and by the previous estimate, we have $P(E_t^{h_i}) \to P(E_0)$ for every $t \in [T_1, T_2]$. Thus, (3.9) yields

$$\lim_{i \to \infty} \int_{T_1}^{T_2} \int_{\partial^* E_t^{h_i}} |H_{E_t^{h_i}} - \Lambda|^2 \, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t = 0$$

and further, by the mean value theorem, we find times $t_i \in (T_1, T_2)$ such that

$$\lim_{i \to \infty} \int_{\partial^* E_{t_i}^{h_i}} |H_{E_{t_i}^{h_i}} - \Lambda|^2 \, \mathrm{d}\mathcal{H}^{n-1} = 0.$$

Since $P(E_{t_i}^{h_i})$ are uniformly bounded, we deduce by the previous estimate that $H_{E_{t_i}^{h_i}} \to \Lambda$ in the distributional sense. We also have $|E_0 \Delta E_{t_i}^{h_i}| \to 0$ and $P(E_{t_i}^{h_i}) \to P(E_0)$, so the claim follows from Theorem 2.1. □

Now, the non-trivial direction of Theorem 1.2 is a rather straightforward consequence of Lemma 3.6 and Theorem 1.1, since the latter guarantees that a critical set having two tangential balls cannot be stationary. The reasoning is exactly the same as in the planar case, but for the sake of completeness we sketch the argument here. To this end, let E_0 be a finite union of balls with equal radius r containing a union of two tangential balls, say

$$E_0' = B(x_1, r) \cup B(x_2, r).$$

Let $(E_t)_{t\geq 0}$ be any flat flow with a bounded forcing f starting from E_0. By applying the second claim of Proposition 3.2, we find a a flat flow $(E_t')_{t\geq 0}$ with the forcing $f - 1$ starting from E_0' such that $|E_t' \setminus E_t| = 0$ for every $t \geq 0$. By Theorem 1.1, we have $|B((x_1 + x_2)/2, ct) \setminus E_t'| = 0$, and thus $|B((x_1 + x_2)/2, ct) \setminus E_t| = 0$ for some $c \in \mathbb{R}_+$ and for all small $t > 0$. On the other hand, it clearly holds $|B((x_1 + x_2)/2, ct) \setminus E_0| > 0$. Therefore, we deduce that $|E_0 \Delta E_t| > 0$ for all small $t > 0$. Thus, E_0 cannot be stationary.

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Let E_0 be a union of two tangential balls of radius r. We may assume that $E_0 = B(-re_1, r) \cup B(re_1, r)$. Recall that for $0 < h \leq 1$ an approximative sequence $(E^{h,i})_{i=0}^{\infty}$ is defined recursively by setting first $E^{h,0} = E_0$ and for each $i = 0, 1, 2, \ldots$ the set $E^{h,i+1}$ is chosen to be a minimizer of $\mathcal{F}_{h,\bar{f}(h,i)}(\,\cdot\,, E^{h,i})$ defined in (3.4), where $\bar{f}(h,i) = \fint_{ih}^{(i+1)h} f(t)\,dt$. Recall also that $f : [0,\infty) \to \mathbb{R}$ is measurable and satisfies (3.1) (and hence $|\bar{f}(h,i)| \leq C_0$). Now each $E^{h,i}$, with $i \geq 1$, satisfies the Euler-Lagrange equation (3.6) with the constant $\Lambda = \bar{f}(h, i-1)$. Again, the corresponding approximative flat flow $(E^h)_{t\geq 0}$ is given by (3.3).

Our aim is to show that for a time interval $(0, \delta]$, with δ small enough, we may construct barrier sets $G^{h,i} \subset E^{h,i}$ for $i = 1, \ldots, \lfloor \delta/h \rfloor + 1$ such that for every $h \leq t \leq \delta$ the barrier $G^{h,\lfloor t/h \rfloor}$ contains a simply connected set A_t defined by

$$A_t = \mathbf{C}(c_1 t, r) \cup B(-re_1, r - c_2 t) \cup B(re_1, r - c_2 t), \tag{4.1}$$

with some $c_1, c_2 \in \mathbb{R}_+$, depending only on n, r and C_0, provided that h is small enough. Now, if $(E_t)_{t\geq 0}$ is any cluster flow, then $A_t \subset E_t^h$ implies $|A_t \setminus E_t| = 0$ for every $t \in (0, \delta)$. Further, since $E_t \in X_n$, this means $\mathrm{int}(A_t) \subset E_t$. The rest of the claim follows trivially from this.

We first note that it is easy to see that the balls $B(\pm re_1, r - c_2 t)$ are contained in E_t. Indeed, by possibly replacing C_0 with $\max\{C_0, 4(n-1)/r\}$, we may assume that $r/4 \geq (n-1)/C_0$. Then, by (i) of Proposition 3.2 and Proposition 3.3, we find $\eta = \eta(n, r/2, C_0) \in \mathbb{R}_+$ and $0 < h_0 = h_0(n, r/2, C_0) \leq 1$ such that for every $0 < h \leq h_0$ the following implication holds

$$B(x, \tilde{r}) \subset E^{h,i} \text{ with } \tilde{r} \geq r/2 \implies \bar{B}(x, \tilde{r} - \eta h) \subset E^{h,i+1}. \tag{4.2}$$

We split the proof into three steps.

Step 1: First, we prove that there is a positive $\alpha = \alpha(n, r, C_0)$, such that the set $E^{h,1}$ contains the cylinder $\mathbf{C}(\alpha h^{\frac{1}{4}}, \alpha h^{\frac{1}{2}})$ provided that h is sufficiently small.

To this end, let $\tau > 0$ be a small number, which we will fix later. We use C and c for positive constants which may change from line to line but always depend only on n, r and C_0. We also use a further shorthand notation $\mathbf{C}_{h,\tau}$ for the cylinder $\mathbf{C}(\tau h^{\frac{1}{4}}, \tau h^{\frac{1}{2}})$

By (4.2), the balls $\bar{B}(\pm re_1, r-\eta h)$ are contained in $E^{h,1}$ provided that h is small enough. Again, assuming $\tau \le r/2$ and h to be sufficiently small, we have

$$(r-\eta h)^2 - (r-\tau h^{\frac12})^2 = (2r-\eta h-\tau h^{\frac12})(\tau-\eta h^{\frac12})h^{\frac12}$$
$$> \frac{\tau r}{2}h^{\frac12} \ge \tau^2 h^{\frac12}.$$

Therefore, the heads of the cylinder $\mathbf{C}_{h,\tau}$, which are the vertical parts of the boundary, are contained in $B(\pm re_1, r-\eta h)$ and therefore, in turn, in the set $E^{h,1}$. Since $\partial E^{h,1}$ is C^2 (possibly up to a closed singular part of Hausdorff-dimension at most $n-8$), by a foliation and continuity argument, we may assume that $\mathcal{H}^{n-1}(\partial\mathbf{C}_{h,\tau}\cap\partial E^{h,1}) = 0$. Otherwise, we would choose $\tau/2 \le \tilde\tau < \tau$ such that the heads of the cylinder $\mathbf{C}_{h,\tilde\tau}$ are contained in $B(\pm re_1, r-\eta h)$ and $\mathcal{H}^{n-1}(\partial\mathbf{C}_{h,\tilde\tau}\cap\partial E^{h,1}) = 0$. This implies

$$P(\mathbf{C}_{h,\tau}\cup E^{h,1}) = \mathcal{H}^{n-1}(\partial\mathbf{C}_{h,\tau}\setminus E^{h,1}) + \mathcal{H}^{n-1}(\partial E^{h,1}\setminus\mathbf{C}_{h,\tau}). \tag{4.3}$$

Again, we have the following estimates

$$\mathcal{H}^{n-1}(\partial\mathbf{C}_{h,\tau}\setminus E^{h,1}) \le C\tau^{n-1}h^{\frac{n}{4}}, \tag{4.4}$$

$$\bar{d}_{E_0} \le C\tau^2 h^{\frac12} \quad \text{in} \quad \mathbf{C}_{h,\tau} \quad \text{and} \tag{4.5}$$

$$|\mathbf{C}_{h,\tau}| \le C\tau^n h^{\frac{n+1}{4}}. \tag{4.6}$$

We show that $\mathbf{C}(\tau h^{\frac14}/2, \tau h^{\frac12}) \subset E^{h,1}$ which implies the claim of Step 1 by choosing $\alpha = \tau/2$. Suppose by contradiction that this does not hold. We first notice that $E_0 = \mathbf{C}(g, [-2r, 2r])$ with a continuous g having the zero value at the endpoints, and therefore by Proposition 3.4 it holds $E^{h,1} = (E^{h,1})^*$, i.e., $E^{h,1}$ is Schwarz symmetric with respect to the x_1-axis. Using this and the fact that the heads of $\mathbf{C}_{h,\tau}$ are in $E^{h,1}$, we conclude

$$\mathcal{H}^{n-1}(\partial E^{h,1}\cap\mathbf{C}_{h,\tau}) \ge c\tau^{n-1}h^{\frac{n-1}{4}}.$$

We use the set $\mathbf{C}_{h,\tau}\cup E^{h,1}$ as a competitor in the energy $\mathcal{F}_{h,\bar f(h,0)}(\,\cdot\,, E_0)$. By using the previous estimate as well as (4.3), (4.4), (4.5) and (4.6) and assuming h to be small enough, we estimate

$$\begin{aligned}
\mathcal{F}_{h,\bar f(h,0)}(\mathbf{C}_{h,\tau}\cup E^{h,1}, E_0) &= \mathcal{F}_{h,\bar f(h,0)}(E^{h,1}, E_0) + \mathcal{H}^{n-1}(\partial\mathbf{C}_{h,\tau}\setminus E^{h,1}) - \mathcal{H}^{n-1}(\partial E^{h,1}\cap\mathbf{C}_{h,\tau})\\
&\quad + \frac1h\int_{\mathbf{C}_{h,\tau}\setminus E^{h,1}} \bar d_{E_0}\,\mathrm{d}x - \bar f(h,0)|\mathbf{C}_{h,\tau}\setminus E^{h,1}|\\
&\le \mathcal{F}_{h,\bar f(h,0)}(E^{h,1}, E_0) + C\tau^{n-1}h^{\frac n4} - c\tau^{n-1}h^{\frac{n-1}{4}}\\
&\quad + C(\tau^2 h^{-\frac12}+1)|\mathbf{C}_{h,\tau}|\\
&\le \mathcal{F}_{h,\bar f(h,0)}(E^{h,1}, E_0) + C\tau^{n-1}h^{\frac n4} - c\tau^{n-1}h^{\frac{n-1}{4}} + C\tau^{n+2}h^{\frac{n-1}{4}}\\
&\le \mathcal{F}_{h,\bar f(h,0)}(E^{h,1}, E_0) + \frac c2\tau^{n-1}h^{\frac{n-1}{4}} - c\tau^{n-1}h^{\frac{n-1}{4}} + C\tau^{n+2}h^{\frac{n-1}{4}}\\
&= \mathcal{F}_{h,\bar f(h,0)}(E^{h,1}, E_0) + \tau^{n-1}h^{\frac{n-1}{4}}\left(C\tau^2 - \frac c2\right).
\end{aligned}$$

Thus by choosing $\tau < \sqrt{c/(2C)}$, we have $\mathcal{F}_{h,\bar f(h,0)}(\mathbf{C}_{h,\tau}\cup E^{h,1}, E_0) < \mathcal{F}_{h,\bar f(h,0)}(E^{h,1}, E_0)$ which contradicts the minimality of the set $E^{h,1}$. Hence, we have $\mathbf{C}(\alpha h^{\frac14}, \alpha h^{\frac12}) \subset E^{h,1}$ for $\alpha = \tau/2$.

Step 2: We proceed by constructing a candidate family for the barrier sets $G^{h,i}$ for every $i = 1, \dots, \lfloor \delta/h \rfloor + 1$ and small δ, which satisfy for every $h \le t \le \delta$ the condition $A_t \subset G^{h, \lfloor t/h \rfloor}$, where A_t is defined in (4.1). To be more precise, we will define positive numbers $d_{h,i}$, $l_{h,i}$ and $r_{h,i}$ (such that $l_{h,i}$ increases and $r_{h,i}$ decreases linearly in discrete time and $r_{h,1} \to r$ as $h \to 0$) and suitable convex and positive functions $\varphi_{h,i} : [-d_{h,i}, d_{h,i}] \to \mathbb{R}$ with $l_{h,i}/2 \le \varphi_{h,i} \le l_{h,i}$. Then we define the barrier sets $G^{h,i}$, see Figure 4, as the union

$$G^{h,i} = \mathbf{C}(\varphi_{h,i}, [-d_{h,i}, d_{h,i}]) \cup \bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}).$$

Here it follows from the selection of the parameters and the functions that the heads of the

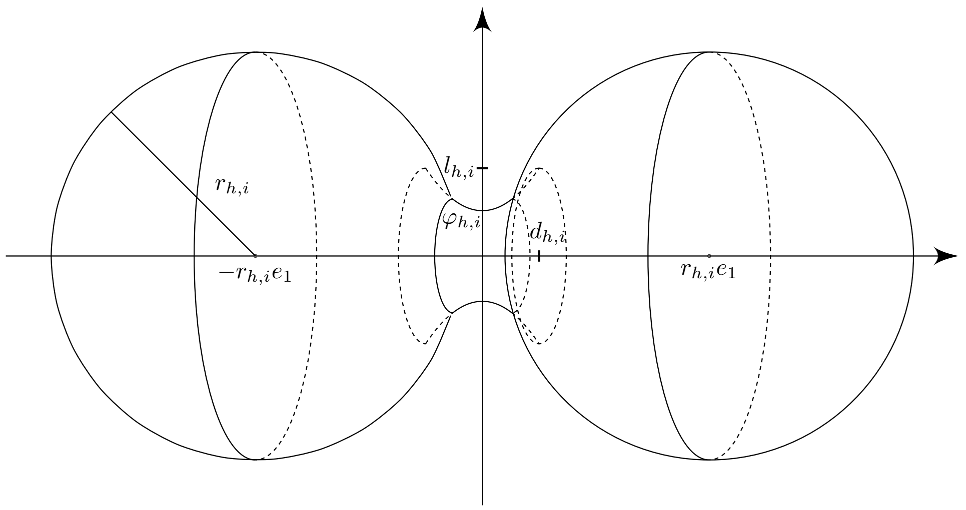

FIGURE 4.1. A visualization of the barrier set $G^{h,i}$.

neck $\mathbf{C}(\varphi_{h,i}, [-d_{h,i}, d_{h,i}])$ are contained in the balls $B(\pm re_1, r_{h,i})$ so $G^{h,i}$ will contain a simply connected set

$$\mathbf{C}(l_{h,i}/2, r) \cup \bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}).$$

and hence the behavior of $l_{h,i}$ and $r_{h,i}$ yields the condition $A_t \subset G^{h, \lfloor t/h \rfloor}$.

To this end, let $0 < \delta < 1$ be a sufficiently small number which will ultimately depend only on n, r and C_0. Note that it holds $hi \le 2\delta$ for all $i = 1, \dots, \lfloor \delta/h \rfloor + 1$ when h is small. We begin by setting $r_{h,i} = r - \eta h i$. Now $r_{h,i} \ge r - 2\eta\delta$, so by assuming δ to be small enough, we have $r_{h,i} \ge r/2$. Hence, thanks to (4.2)

$$\bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}) \subset E^{h,i}. \tag{4.7}$$

Again, set $\Lambda_0 = \max\{4\eta^2, 2^9(n-2)^2, 1\}$ and for each $i = 1, \dots, \lfloor \delta/h \rfloor + 1$ define

$$l_{h,i} = \Lambda_0 h(i-1) + \alpha h^{\frac{1}{4}} \quad \text{and} \quad d_{h,i} = 2\eta h(i-1) + \alpha h^{\frac{1}{2}}. \tag{4.8}$$

It follows from the choice of Λ_0 that for δ small enough ($\delta \le \Lambda_0^{-2}$) it holds

$$\Lambda_0^{\frac{1}{2}} d_{h,i} \le l_{h,i}. \tag{4.9}$$

Moreover, $l_{h,i} \le \Lambda_0 \delta + \alpha \delta^{\frac{1}{4}}$, so by decreasing δ we may assume that $d_{h,i}$ and $l_{h,i}$ are as small as we need. Note that by Step 1 we have

$$\mathbf{C}(l_{h,1}, d_{h,1}) \subset E^{h,1}. \tag{4.10}$$

Further, by replacing α with $\min\{\alpha, r/4\}$, if necessary, we have

$$\begin{aligned} r_{h,i}^2 - (r - d_{h,i})^2 &= (r_{h,i} + r - d_{h,i})(r_{h,i} - r + d_{h,i}) \\ &= (2r - \eta h(3i-2) - \alpha h^{\frac{1}{2}})(\eta h(i-2) + \alpha h^{\frac{1}{2}}) \\ &\geq r(\eta h(i-2) + \alpha h^{\frac{1}{2}}) \\ &= r\eta h(i-1) + r(\alpha - \eta h^{\frac{1}{2}})h^{\frac{1}{2}} \\ &\geq r\eta h(i-1) + \frac{\alpha r}{2} h^{\frac{1}{2}} \\ &\geq 2\delta \Lambda_0^2 h(i-1) + 2\alpha^2 h^{\frac{1}{2}} \\ &\geq 2\Lambda_0^2 h^2 (i-1)^2 + 2\alpha^2 h^{\frac{1}{2}} \\ &\geq \left(\Lambda_0 h(i-1) + \alpha h^{\frac{1}{4}}\right)^2 = l_{h,i}^2, \end{aligned}$$

when δ is small. Therefore, by the Pythagorean theorem

$$\{(\pm d_{h,i}, x') \in \mathbb{R}^n : x' \in \bar{B}^{n-1}(0, l_{h,i})\} \subset \bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}), \tag{4.11}$$

i.e., the heads of the cylinder $\mathbf{C}(l_{h,i}, d_{h,i})$ are contained in the balls $\bar{B}(\pm re_1, r_{h,i})$.

We define for each $i = 1, \dots, \lfloor \delta/h \rfloor + 1$ a convex function $\varphi_{h,i} : [-d_{h,i+1}, d_{h,i+1}] \to \mathbb{R}$ by setting

$$\varphi_{h,i}(t) = \frac{a_{h,i}}{2}(t^2 - d_{h,i}^2) + l_{h,i},$$

where $a_{h,i} = \Lambda_0^{\frac{1}{2}}/l_{h,i}$. Note that by (4.9) we have $a_{h,i} d_{h,i} \leq 1$, and further $a_{h,i} d_{h,i}^2 \leq l_{h,i}$. Thus, $\varphi_{h,i}$ is 1-Lipschitz and

$$\varphi_{h,i} \geq \varphi_{h,i}(0) = l_{h,i} - \frac{1}{2} a_{h,i} d_{h,i}^2 \geq \frac{l_{h,i}}{2}. \tag{4.12}$$

Recall that we set $G^{h,i}$ as the union

$$G^{h,i} = \mathbf{C}(\varphi_{h,i}, [-d_{h,i}, d_{h,i}]) \cup \bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}).$$

By (4.11) and (4.12), the barrier $G^{h,i}$ contains the simply-connected set

$$\mathbf{C}(l_{h,i}/2, r) \cup B(-re_1, r_{h,i}) \cup B(re_1, r_{h,i}).$$

Thus recalling $l_{h,i} = \Lambda_0 h(i-1) + \alpha h^{\frac{1}{4}}$ and $r_{h,i} = r - \eta h i$, we find $c_1, c_2 \in \mathbb{R}_+$, depending only on n, r and C_0, such that for every $h \leq t \leq \delta$ the barrier $G^{h,\lfloor t/h \rfloor}$ contains the set A_t defined in (4.1) with the constants c_1 and c_2.

Step 3: We finish the proof by showing that each barrier $G^{h,i}$ constructed in the previous step is actually contained in $E^{h,i}$. First, we conclude from (4.8), (4.9) and $\Lambda_0 \geq 2\eta$ that, when δ is small enough, for every $i = 2, \dots, \lfloor \delta/h \rfloor + 1$ it holds

$$|\varphi_{h,i} - \varphi_{h,i-1}| \leq 2\Lambda_0 h \quad \text{on} \quad [-d_{h,i-1}, d_{h,i-1}]. \tag{4.13}$$

Further, using the facts that $\varphi_{h,i}$ is 1-Lipschitz and $r_{h,i-1} \geq r_{h,i}$, we obtain

$$G^{h,i} \subset \{x \in \mathbb{R}^n : \bar{d}_{G^{h,i-1}}(x) \leq 4\Lambda_0 h\}. \tag{4.14}$$

By (4.7) and (4.10), we have $G^{h,1} \subset E^{h,1}$. Thus, we argue by induction. Assume that for $i = 2, \dots, \lfloor \delta/h \rfloor + 1$ it holds $G^{h,i-1} \subset E^{h,i-1}$. By (4.8) and (4.12), we have for small δ

$$\varphi_{h,i} - 2\Lambda_0 h \geq \frac{l_{h,i}}{2} - 2\Lambda_0 h \geq \frac{\alpha}{2} h^{\frac{1}{4}} - 2\Lambda_0 h > 0$$

and hence the set $\mathbf{C}(\varphi_{h,i} - 2\Lambda_0 h, [-d_{h,i}, d_{h,i}])$ is well-defined. Again, by (4.13) and $\varphi_{h,i} - 2\Lambda_0 h \leq l_{h,i-1}$, it holds

$$\mathbf{C}(\varphi_{h,i} - \Lambda_0 h, [-d_{h,i}, d_{h,i}]) \subset G^{h,i-1}. \tag{4.15}$$

Next, we define an auxiliary set $\tilde{G}^{h,i} \subset G^{h,i}$ by

$$\tilde{G}^{h,i} = \mathbf{C}(\varphi_{h,i} - 2\Lambda_0 h, [-d_{h,i}, d_{h,i}]) \cup \bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}).$$

Then, by the induction assumption (4.15) and $r_{h,i-1} \geq r_{h,i}$, we have $\tilde{G}^{h,i} \subset E^{h,i-1}$. Therefore, by (3.5),

$$\{x \in \tilde{G}^{h,i} : \mathrm{dist}(x, \partial \tilde{G}^{h,i}) > \gamma h^{\frac{1}{2}}\} \subset E^{h,i}. \tag{4.16}$$

Since the function $t \mapsto \max\{s : \{t\} \times \bar{B}^{n-1}(0,s) \subset \tilde{G}^{h,i}\}$ is increasing in $[0, r]$, decreasing in $[-r, 0]$ and $\tilde{G}^{h,i}$ is a solid of revolution, then for any

$$x \in \mathbf{C}(\varphi_{h,i} - 2\Lambda_0 h, [-d_{h,i}, d_{h,i}]) \setminus (\bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i}))$$

the closest point $y \in \partial \tilde{G}^{h,i}$, i.e. $\mathrm{dist}(x, \partial \tilde{G}^{h,i}) = |x - y|$, must lie on

$$\partial \mathbf{C}(\varphi_{h,i} - 2\Lambda_0 h, [-d_{h,i}, d_{h,i}]) \setminus \left(\bar{B}(-re_1, r_{h,i}) \cup \bar{B}(re_1, r_{h,i})\right).$$

Let us write $x = (x_1, x')$ and $y = (y_1, y')$. Since $\varphi_{h,i}$ is a 1-Lipschitz function, we have

$$\begin{aligned} |\varphi_{h,i}(x_1) - |x'|| &\leq |\varphi_{h,i}(x_1) - \varphi_{h,i}(y_1)| + |\varphi_{h,i}(y_1) - |x'|| \\ &\leq |x_1 - y_1| + ||y'| - |x'|| \\ &\leq 2|x - y| = 2\mathrm{dist}(x, \partial \tilde{G}^{h,i}). \end{aligned} \tag{4.17}$$

We have $\gamma h^{\frac{1}{2}} > 2\Lambda_0 h$ and $\alpha h^{\frac{1}{4}} > 12\gamma h^{\frac{1}{2}}$, provided that δ is small. Thus by (4.8) and (4.12),

$$\frac{l_{h,i}}{4} \leq \varphi_{h,i} - 3\gamma h^{\frac{1}{2}} < \varphi_{h,i} - 2\Lambda_0 h - 2\gamma h^{\frac{1}{2}} \quad \text{on} \quad [-d_{h,i}, d_{h,i}]. \tag{4.18}$$

Therefore, it follows from (4.7), (4.16), (4.17) and (4.18) that the set $\mathbf{C}(\varphi_{h,i} - 3\gamma h^{\frac{1}{2}}, [-d_{h,i}, d_{h,i}])$ is well-defined and contained in $E^{h,i}$.

We argue by contradiction and assume that $G^{h,i}$ is not contained in $E^{h,i}$. Since

$$\mathbf{C}(\varphi_{h,i} - 3\gamma h^{\frac{1}{2}}, [-d_{h,i}, d_{h,i}]) \subset E^{h,i},$$

we may lift up the graph of $\varphi_{h,i} - 3\gamma h^{\frac{1}{2}}$ until it touches the boundary $\partial E^{h,i}$. To be more precise, by a continuity argument and (4.7), there is $0 < \tau < 3\gamma h^{\frac{1}{2}}$ such that

$$\mathbf{C}(\varphi_{h,i} - \tau, [-d_{h,i}, d_{h,i}]) \subset E^{h,i}$$

and there is a point $z \in \Gamma \cap \partial E^{h,i}$, where

$$\Gamma = \{(x_1, x') \in \mathbb{R}^n : x_1 \in (-d_{h,i}, d_{h,i}), x' \in \partial B^{n-1}(0, \varphi_{h,i}(x_1) - \tau)\}.$$

In particular, the boundary $\partial E^{h,i}$ satisfies the interior ball condition at z, and thus z belongs to the regular part of $\partial E^{h,i}$. Hence, by the comparison principle, we have $H_{E^{h,i}}(z) \le H_\Gamma(z)$, where H_Γ is chosen to be compatible with the inside-out orientation of

$$\mathbf{C}(\varphi_{h,i} - \tau, [-d_{h,i}, d_{h,i}]).$$

Recalling (2.2), (4.18), $a_{h,i} d_{h,i} \le 1$ and the choice of Λ_0, we estimate

$$\begin{aligned} H_\Gamma(z) &= -\frac{\varphi''_{h,i}(z_1)}{(1+(\varphi'_{h,i}(z_1))^2)^{\frac{3}{2}}} + \frac{1}{(1+(\varphi'_{h,i}(z_1))^2)^{\frac{1}{2}}} \frac{(n-2)}{\varphi_{h,i}(z_1) - \tau} \\ &= -\frac{a_{h,i}}{(1+(a_{h,i} z_1)^2)^{\frac{3}{2}}} + \frac{1}{(1+(a_{h,i} z_1)^2)^{\frac{1}{2}}} \frac{(n-2)}{\varphi_{h,i}(z_1) - \tau} \\ &\le -\frac{a_{h,i}}{2^{\frac{3}{2}}} + \frac{4(n-2)}{l_{h,i}} \\ &= \frac{2^{\frac{7}{2}}(n-2) - \Lambda_0^{\frac{1}{2}}}{2^{\frac{3}{2}} l_{h,i}} \le -\frac{\Lambda_0^{\frac{1}{2}}}{2^{\frac{5}{2}} l_{h,i}}. \end{aligned}$$

Thus, by choosing δ to be small enough, we have $H_\Gamma(z) \le -(5\Lambda_0 + C_0)$. Then the Euler-Lagrange equation (3.6) for $E^{h,i}$ and $H_{E^{h,i}}(z) \le H_\Gamma(z)$ yields

$$\bar{d}_{G^{h,i-1}}(z) = -H_{E^{h,i}}(z) h + \bar{f}(h, i-1) h \ge 5\Lambda_0 h.$$

However, by the construction we have $z \in G^{h,i}$, and thus the above contradicts (4.14). Hence, we have $G^{h,i} \subset E^{h,i}$ for every $i = 1, \dots, \lfloor \delta/h \rfloor + 1$. □

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