Simon Bortz, Olli Tapiola

ε-approximability of harmonic functions in L^p implies uniform rectifiability

Proceedings of the American Mathematical Society

DOI: 10.1090/proc/14394

Accepted Manuscript

ε-APPROXIMABILITY OF HARMONIC FUNCTIONS IN L^p IMPLIES UNIFORM RECTIFIABILITY

SIMON BORTZ AND OLLI TAPIOLA

ABSTRACT. Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is an open set satisfying the corkscrew condition with an n-dimensional ADR boundary, $\partial\Omega$. In this note, we show that if harmonic functions are ε-approximable in L^p for any $p > n/(n-1)$, then $\partial\Omega$ is uniformly rectifiable. Combining our results with those in [HT] gives us a new characterization of uniform rectifiability which complements the recent results in [HMM], [GMT] and [AGMT].

1. INTRODUCTION

Carleson measures of the type $|\nabla u(Y)|\,dY$ are very useful with various questions related to harmonic analysis, PDE and geometric measure theory (see e.g. [FS, Gar, KKiPT]) but unfortunately, whether the distributional gradient of a given function defines a Carleson measure is a highly non-trivial question. Even strong analytic properties or controlled oscillation are not enough to guarantee a positive answer since there exist examples of bounded harmonic functions and BMO functions that do not have this property [Gar]. The purpose of *ε-approximability* is to find a suitable substitute when the given function fails this property: instead of working with the original function u, we find another function φ such that it approximates u well in L^∞ sense and $|\nabla\varphi(Y)|\,dY$ is a Carleson measure. To be more precise:

Definition 1.1. We say that a function u in an open set $\Omega \subset \mathbb{R}^{n+1}$ with non-empty boundary $\partial\Omega$ is *ε-approximable* if there exists a constant C_ε and a function $\varphi = \varphi^\varepsilon \in BV_{\text{loc}}(\Omega)$ satisfying

$$\|u-\varphi\|_{L^\infty(\Omega)} < \varepsilon \qquad \text{and} \qquad \sup_{x\in\partial\Omega, r>0} \frac{1}{r^n} \iint_{B(x,r)\cap\Omega} |\nabla\varphi(Y)|\,dY \leq C_\varepsilon.$$

As usual, if φ is not differentiable, $\iint_{B(x,r)\cap\Omega} |\nabla\varphi|\,dY$ means the *total variation of φ in $B(x,r)\cap\Omega$*,

$$\iint_{B(x,r)\cap\Omega} |\nabla\varphi|\,dY := \sup_{\substack{\vec{\Psi}\in C_0^1(B(x,r)\cap\Omega) \\ \|\vec{\Psi}\|_{L^\infty(B(x,r)\cap\Omega)\leq 1}}} \iint_{B(x,r)\cap\Omega} \varphi \operatorname{div}\vec{\Psi}\,dY,$$

and we denote $\varphi \in BV(\Omega)$ if the total variation over Ω is finite and $\varphi \in BV_{\text{loc}}(\Omega)$ if the total variation over any open relatively compact set $\Omega' \subset \Omega$ is finite. For the

The first author was supported by the NSF INSPIRE Award DMS-1344235. The second author was supported by Emil Aaltosen Säätiö through Foundations' Post Doc Pool grant. Both authors would like to thank Steve Hofmann for his mentorship and advice during the preparation of this article.

basic theory of measures of the type $|\nabla\varphi(Y)|\,dY$ for functions $\varphi \in BV_{\text{loc}}$, see e.g. [EG, Section 5].

The notion of ε-approximability was first introduced by Varopoulos who proved that every harmonic function in $\mathbb{R}^{n+1}_+$ is ε_0-approximable for some $\varepsilon_0 \in (0,1)$ [Var]. He used the property to prove the so called Varopoulos extension theorem which gives an alternative characterization for BMO functions. In his work, it was not necessary to have the approximability property for all $\varepsilon \in (0,1)$ but in later developments a sharper version of the property has been crucial. Garnett [Gar] extended Varopoulos's result for all $\varepsilon \in (0,1)$ and his result in turn was generalized for bounded Lipschitz domains by Dahlberg [Dah]. An L^p version was recently introduced by Hytönen and Rosén [HR, Theorem 1.3] who showed that any weak solution to certain elliptic partial differential equations in $\mathbb{R}^{n+1}_+$ are ε-approximable in L^p for every $\varepsilon \in (0,1)$ and every $p \in (1,\infty)$. The definition of this L^p version is included in Theorem 1.3.

The notion of ε-approximability has been used to e.g. explore the absolute continuity properties of elliptic measures [KKoPT, HKMP15] and, more recently, study the connections between the properties of solutions to elliptic PDE and rougher boundary geometries for which the absolute continuity of harmonic measure with respect to the surface measure may fail. In particular, a recent fundamental question at the interface of harmonic analysis and geometric measure theory has been the following: what PDE properties serve to characterize uniform rectifiability (UR) (Definition 2.3) of the boundary of an open set with an Ahlfors-David regular (ADR) (Definition 2.1) boundary? Uniform rectifiability is a quantitative form of rectifiability that was introduced by David and Semmes [DS1, DS2] in the early 1990's. They showed that it is both sufficient and necessary for the L^2-boundedness of certain classes of singular integral operators. Although UR is not strong enough to imply the weak-A_∞ condition for harmonic measure (see the well-known counterexample by Bishop and Jones [BJ]), the property has numerous different characterizations with respect to other geometric conditions, harmonic analysis results and properties of solutions to elliptic PDE. Among the more recent results is the following theorem:

Theorem 1.2 ([GMT, HMM])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain satisfying the interior corkscrew condition with n-dimensional ADR boundary, $\partial\Omega$. Then $\partial\Omega$ is uniformly rectifiable if and only if bounded harmonic functions on Ω are ε-approximable.*

A remarkable thing about this theorem is that it does not have any assumptions on the connectivity of Ω or $\partial\Omega$. For a long time, giving up connectivity assumptions was a serious obstacle in the field. The theorem was generalized for a class of elliptic operators in [AGMT].

The purpose of this note is to answer a question posed in [HT]: If $E \subset \mathbb{R}^{n+1}$ is an n-ADR set, does ε-approximability of harmonic functions in L^p for some fixed p in $\mathbb{R}^{n+1} \setminus E$ imply uniform rectifiability of E? We answer this question affirmatively with the following theorem.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-dimensional ADR boundary such that Ω satisfies the corkscrew condition. Let $p > n/(n-1)$ and suppose that every harmonic function is* ε-approximable in L^p *for every $\varepsilon \in (0,1)$: there exist* *constants C_ε and C_p and a function $\varphi = \varphi^\varepsilon \in BV_{loc}(\Omega)$ such that*

$$\begin{cases} \|N_*(u-\varphi)\|_{L^p(\partial\Omega,\sigma)} \le \varepsilon C_p \|N_* u\|_{L^p(\partial\Omega,\sigma)}, \\ \|C(\nabla\varphi)\|_{L^p(\partial\Omega,\sigma)} \le C_\varepsilon C_p \|N_* u\|_{L^p(\partial\Omega,\sigma)}. \end{cases}$$

Then $\partial\Omega$ is uniformly rectifiable. Here $N_* u$ is the non-tangential maximal function (see Definition 2.7) and

$$C(\nabla\varphi)(x) := \sup_{r>0} \frac{1}{r^n} \iint_{B(x,r)\cap\Omega} |\nabla\varphi|\, dY.$$

We note that in the proof we only need the approximability property for *some* sufficiently small $\varepsilon_1 \in (0,1)$ depending on p and the structural constants. The restriction $p > n/(n-1)$ is due to the proof of Lemma 3.3; a suitable version of the Serrin-Weinberger theory [SW] would allow one to reach $p = n/(n-1)$ (see [HR, Proposition 5.1]).

Together with the main result in [HT], this theorem gives a new characterization for uniform rectifiability. Our proof draws strongly on the ideas of [GMT] and [HR]. In particular, we will follow a central idea in [GMT] and prove the following result:

Theorem 1.4. *Suppose that the hypotheses of Theorem 1.3 hold. Then the harmonic measure ω admits a Corona decomposition (see Definition 4.2).*

By [GMT, Proposition 5.1], Theorem 1.4 is enough to imply Theorem 1.3.

Combining our main result with results in [HMM, HMM2, GMT] and [HT] gives us the following characterization result:

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set with n-dimensional ADR boundary such that Ω satisfies the corkscrew condition. Then the following conditions are equivalent.*

(a) $\partial\Omega$ is uniformly rectifiable.
(b) Every bounded harmonic function u satisfies the following Carleson measure estimate:

$$\sup_{x\in\partial\Omega, r>0} \frac{1}{r^n} \iint_{B(x,r)\setminus\partial\Omega} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial\Omega)\, dY \lesssim \|u\|^2_{L^\infty(\Omega)}.$$

(c) For any fixed $p \in [2,\infty)$, every function $u \in C_0(\overline{\Omega})$ that is harmonic in Ω satisfies the following $S \lesssim N$ estimate:

$$\|Su\|_{L^p(\partial\Omega)} \le C_p \|N_* u\|_{L^p(\partial\Omega)}.$$

(d) Every bounded harmonic function on Ω is ε-approximable for all $\varepsilon > 0$.
(e) For a fixed $p > n/(n-1)$, every harmonic function on Ω is ε-approximable in L^p for all $\varepsilon > 0$.
(f) For every fixed $p > 1$, every harmonic function on Ω is ε-approximable in L^p for all $\varepsilon > 0$.

We note that the results in [HMM, HMM2] and [HT] are stated for the complement of an ADR set and in that setting the interior corkscrew condition is automatically satisfied. However, like it is noted in [GMT], it is straightforward to show that the result holds in the current context, too.

Finally, we remark that the conditions (a), (b) and (d) remain equivalent for divergence form elliptic operators $L = -\operatorname{div} A\nabla$ whose coefficients satisfy a Carleson measure condition as well as the point-wise gradient bound [AGMT]. Here – and this is important – one must assume that conditions (b) and (d) are satisfied for every (bounded) solution to $Lu = 0$ *and* every (bounded) solution to $L^* v = 0$, where $L^* = -\operatorname{div} A^* \nabla$. For the sake of brevity and notational convenience, we only treat the case of the Laplacian in this note, but all of the ingredients of the proof of Theorem 1.3 work directly for the class of operators defined [AGMT] once we replace the tools from [GMT] with the analogous tools from [AGMT] and the rest of the modifications are straightforward. We leave the details to the interested reader, but let us indicate the necessary changes. Instead of using the point-wise values of the fundamental solution in Lemma 3.1, use the point-wise bounds for the fundamental solution of any elliptic operator with real coefficients. This is a classical result and was established in [LSW] (symmetric coefficients) and [GW] (in general), see also [Hof-Kim, Theorem 3.1]. Next, one must replace Lemma 4.3 by [AGMT, Lemma 3.3]. The remainder of the proof is identical, except we replace the passage from Theorem 1.4 to Theorem 1.3, [GMT, Proposition 5.1], by [AGMT, Theorem 1.1]. Since the techniques used in [HT] rely on the same tools as [HMM] and some additional dyadic analysis, almost the same proofs work for the class of elliptic operators similarly as in the generalization from [HMM] to the operators considered in [AGMT]. Therefore, the equivalence of (a), (b), (d), (e) and (f) in Theorem 1.5 continues to hold in the context of operators whose coefficients satisfy the conditions in [AGMT], provided that we replace "harmonic function" with "solution to the elliptic equation and solution to the adjoint elliptic equation."

2. Preliminaries and Definitions

Definition 2.1 (ADR (Ahlfors-David regular) sets)**.** We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is *ADR* if it is closed, and if there is some uniform constant C such that

$$\frac{1}{C}\, r^n \leq \sigma\big(\Delta(x,r)\big) \leq C\, r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{2.2}$$

where $\operatorname{diam}(E)$ may be infinite. Here, $\Delta(x,r) := E \cap B(x,r)$ is the "surface ball" of radius r, and $\sigma := H^n|_E$ is the "surface measure" on E, where H^n denotes n-dimensional Hausdorff measure.

Definition 2.3 (UR (uniformly rectifiable) sets)**.** Following [DS1, DS2], we say that an ADR set $E \subset \mathbb{R}^{n+1}$ is *UR* if it contains "big pieces of Lipschitz images" (BPLI) of $\mathbb{R}^n$: there exist constants $\theta, M' > 0$ such that for every $x \in E$ and $r \in (0, \operatorname{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r} \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz norm no larger that M', such that

$$H^n(E \cap B(x,r) \cap \rho(\{y \in \mathbb{R}^n : |y| < r\})) \geq \theta r^n.$$

Definition 2.4 (Corkscrew Condition)**.** We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *corkscrew condition* if there exists a constant M such that for every $x \in \partial\Omega$ and $r \in (0, \operatorname{diam}(\partial\Omega))$ there exists a point $Y \in \Omega$ such that $B(Y, \frac{r}{M}) \subset B(x,r) \cap \Omega$. If Y is as above, we say Y is a corkscrew point relative to x at scale r.

Our standing assumption will be that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n-dimensional ADR boundary $\partial\Omega$ and that Ω satisfies the corkscrew condition with constant M. Let us fix some notation:

- We will use lowercase letters $x, y, z \ldots$ to denote points on $\partial\Omega$ and capital letters $X, Y, Z, \ldots$ to denote generic points in $\mathbb{R}^{n+1}$.
- We use the standard notation $B(X, r)$ for usual Euclidean balls in $\mathbb{R}^{n+1}$ and $\Delta(x, r) := B(x, r) \cap \partial\Omega$ for surface balls. As usual, we use the notation $\kappa B(X, r) := B(X, \kappa r)$ and $\kappa\Delta(x, r) := \Delta(x, \kappa r)$ for the dilations of the balls.
- We set $\delta(X) := \operatorname{dist}(X, \partial\Omega)$ for every $X \in \Omega$.
- We let $\omega = \omega^X$ to denote the harmonic measure for Ω.
- We denote $\alpha \lesssim \beta$ if there exists a structural constant c (i.e. a constant depending only on dimension, ADR constant, UR constants and corkscrew constant) such that $\alpha \leq c\beta$. If $\alpha \lesssim \beta \lesssim \alpha$, we denote $\alpha \approx \beta$.

We will use actively well-known dyadic techniques on $\partial\Omega$. We note that in our context there is no canonical choice for the dyadic system but it is not necessary to know the exact structure of the "cubes".

Lemma 2.5 (Dyadic systems for ADR sets)**.** *Suppose that $E \subset \mathbb{R}^{n+1}$ is closed n-dimensional ADR set. Then there exist constants $a_0 > 0$ and $a_1 < \infty$, depending only on dimension and the ADR constant, such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(1) $E = \cup_j Q_j^k$ for each $k \in \mathbb{Z}$ and the union is disjoint.
(2) If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.
(3) For each (j, k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.
(4) For each (j, k) there exists a point $x_j^k \in Q_j^k$ such that

$$\Delta(x_j^k, a_0 2^{-k}) \subset Q_j^k \subset \Delta(x_j^k, a_1 2^{-k}) := B(x_j^k, a_1 2^{-k}) \cap E.$$

In the literature, there exist numerous proofs for this result with many additional properties (see e.g. [DS1, DS2, Chr, Hyt-Kar, HMMM]) but we listed only the ones that we need. Let us fix some notation related to the dyadic system.

- If the set E is bounded, then we simply have $Q_j^k = E$ for all sufficiently small $k \in \mathbb{Z}$. Because of this, we denote $\mathbb{D} := \bigcup_k \mathbb{D}_k$, where the union runs over the k such that $2^{-k} \lesssim \operatorname{diam}(E)$.
- For each $Q := Q_j^k \in \mathbb{D}$ we denote

$$x_Q := x_j^k, \qquad B_Q := B(x_Q, a_1 2^{-k}), \qquad \Delta_Q := B_Q \cap E.$$

  We shall refer to the point x_Q as the "center" of Q.
- For each $Q := Q_j^k \in \mathbb{D}$, we set $\ell(Q) := 2^{-k}$. We refer to this quantity as the "side length" of Q. Since we assume that $\ell(Q) \lesssim \operatorname{diam}(E)$ for every $Q \in \mathbb{D}$, we have $\ell(Q) \approx \operatorname{diam}(Q)$ and $\ell(Q)^n \approx \sigma(Q)$.
- For each $R \in \mathbb{D}$, we let $\mathbb{D}_R \subset \mathbb{D}$ be the collection of the subcubes of R.

Next, we define "dyadic" Whitney regions similar to those in [HM]: for each $Q \in \mathbb{D}$ we choose a bounded number of usual Whitney cubes $I_Q^i \subset \Omega$ such that $\bigcup_i I_Q^i$ acts as a substitute for a region of the type $P \times (\ell(P)/2, \ell(P)]$, the standard Whitney region in the upper half-space. Our Whitney regions are not disjoint but they have a bounded overlap property which is enough for us. In [HM, HMM, HT], it was crucial to fatten the cubes I_Q^i to ensure that most of the regions break into components with strong geometric properties (see particularly [HMM, Lemma 3.24]). In our situation, we do not have to fatten the cubes but we note that we have to make sure that the Whitney regions are large enough for the non-tangential maximal function we use (see Definition 2.7) to be equivalent in L^p sense with the non-tangential maximal function used in [HT]. See [HT, Lemma 1.23] and its proof for more details about this and corresponding Fefferman-Stein [FS] type arguments.

Let us be more precise. Let $\eta \ll 1 \ll K$ be parameters depending on n, M and the ADR constant whose values we will determine later in the proofs. Suppose that $\mathcal{W} := \{I\}_I$ is a Whitney decomposition of Ω, that is, $\{I\}_I$ is a collection of closed $(n+1)$-dimensional Euclidean cubes whose interiors are disjoint such that $\bigcup_I I = \mathbb{R}^{n+1} \setminus \partial\Omega$ and

$$4\,\mathrm{diam}(I) \le \mathrm{dist}(4I, \partial\Omega) \le \mathrm{dist}(I, \partial\Omega) \le 40\,\mathrm{diam}(I)\,, \qquad \forall\, I \in \mathcal{W}$$

and

$$1/4\,\mathrm{diam}(I_1) \le \mathrm{diam}(I_2) \le 4\,\mathrm{diam}(I_1)$$

whenever $I_1 \cap I_2 \neq \emptyset$. For every $Q \in \mathbb{D}(\partial\Omega)$ we set

$$W_Q(\eta, K) = \{I \in \mathcal{W} : \eta^{1/4}\ell(Q) \le \ell(I) \le K^{1/2}\ell(Q), \mathrm{dist}(I, Q) \le K^{1/2}\ell(Q)\}.$$

and

$$U_Q(\eta, K) = \bigcup_{I \in W_Q(\eta,K)} I.$$

We call $U_Q(\eta, K)$ the Whitney region relative to $Q \in \mathbb{D}$.

Remark 2.6. We put some initial restrictions on the parameters η and K to ensure that $U_Q(\eta, K)$ is non-empty for every $Q \in \mathbb{D}$. We notice that by the corkscrew condition for every $Q \in \mathbb{D}$ there exists a corkscrew point X_Q such that $|X_Q - x_Q| < \ell(Q)$ and $\mathrm{dist}(X_Q, \partial\Omega) > M^{-1}\ell(Q)$. It follows that $X_Q \in I$ for some $I \in \mathcal{W}$ with

$$\mathrm{dist}(I, Q) \approx \ell(Q).$$

Choosing $\eta \ll 1 \ll K$ depending on the corkscrew condition and n we obtain that $U_Q(\eta, K) \neq \emptyset$. We later impose further assumptions on η, K, but these will continue to only depend on n, the ADR constant and the corkscrew condition. We will drop the η, K from $U_Q(\eta, K)$ for notational convenience.

Finally, let us define cones and the non-tangential maximal operator. We first denote the usual cone of aperture $\alpha > 1$ at $x \in \partial\Omega$ by $\widetilde{\Gamma}_\alpha(x) := \{Y \in \Omega : |x - Y| < \alpha \cdot \delta(Y)\}$. We notice that if $\partial\Omega$ is bounded, then we have $\mathbb{R}^{n+1} \setminus B(x, R) \subset \widetilde{\Gamma}_\alpha(x)$ for large enough R and every $x \in \partial\Omega$. Thus, since we only construct the regions U_Q for such Q that $\ell(Q) \lesssim \mathrm{diam}(\partial\Omega)$, we set

$$\Gamma(x) := \begin{cases} \bigcup_{Q \in \mathbb{D}, Q \ni x} U_Q, & \text{if } \mathrm{diam}(\partial\Omega) = \infty \\ \bigcup_{Q \in \mathbb{D}, Q \ni x} U_Q \cup (\mathbb{R}^{n+1} \setminus B(x, R_0)), & \text{if } \mathrm{diam}(\partial\Omega) < \infty \end{cases},$$

for a fixed R_0 large enough. It is straightforward to verify that by choosing the constants η and K in a suitable way, there exist $\alpha_1 > \alpha_0 > 1$ such that $\widetilde{\Gamma}_{\alpha_0}(x) \subset \Gamma(x) \subset \widetilde{\Gamma}_{\alpha_1}(x)$ for every $x \in \partial\Omega$.

Definition 2.7 (Non-tangential maximal function)**.** For any function $g : \Omega \to \mathbb{R}$ we define the non-tangential maximal function $N_* g : \partial\Omega \to \mathbb{R}$ by

$$N_* g(y) := \sup_{X \in \Gamma(y)} |g(X)|.$$

3. Two lemmas

In order to present the proof of Theorem 1.4 we prove two simple lemmas.

Lemma 3.1. *Let $Q \in \mathbb{D}$. Suppose that f is a Borel function such that $|f| \leq 1_Q$ and set $u(X) = \int_{\partial\Omega} f(y)\, d\omega^X(y)$ for every $X \in \Omega$. Then*

$$|u(X)| \lesssim 1_{4B_Q}(X) + 1_{(4B_Q)^c}(X) \left(\frac{\ell(Q)}{|X - x_Q|} \right)^{n-1},$$

where the implicit constants depend on n and the ADR constant.

Proof. Let us set

$$H(X) := \frac{1}{\ell(Q)} \int_{B_Q} \mathcal{E}(X, y)\, d\sigma(y),$$

where

$$\mathcal{E}(X, Y) := \frac{c_n}{|X - Y|^{n-1}}$$

is the fundamental solution to the Laplacian in $\mathbb{R}^{n+1}$. By the ADR condition and the local σ-integrability of $\mathcal{E}$, we know that H is bounded in $\mathbb{R}^{n+1}$ and we have

$$H(y) \gtrsim 1, \quad \forall y \in Q. \tag{3.2}$$

We also notice that

$$0 \leq H(X) \lesssim \left(\frac{\ell(Q)}{|X - x_Q|} \right)^{n-1}, \quad \forall X \in (4B_Q)^c,$$

where the implicit constants above depend on n and the ADR constant. It is straightforward to show that H is harmonic in Ω and continuous in $\overline{\Omega}$. Thus, we have

$$H(X) = \int_{\partial\Omega} H(y)\, d\omega^X(y).$$

In particular, since $|f| \leq 1_Q$, we obtain

$$|u(X)| \leq \int_Q |f(y)|\, d\omega^X(y) \overset{(3.2)}{\lesssim} H(X) \lesssim 1_{4B_Q}(X) + 1_{(4B_Q)^c}(X) \left(\frac{\ell(Q)}{|X - x_Q|} \right)^{n-1}$$

by the boundedness of H. □

This lemma readily yields a bound also for the non-tangential maximal operator acting on functions of the same type:

Lemma 3.3. *Let $Q \in \mathbb{D}$. Suppose that f is a Borel function satisfying $|f| \le 1_Q$ an set $u(X) = \int_{\partial\Omega} f(y)\, d\omega^X(y)$ for every $X \in \Omega$. Then for all $p > n/(n-1)$ we have*

$$\|N_* u\|_{L^p(\partial\Omega,\sigma)} \le C_1 \sigma(Q)^{1/p},$$

where C_1 depends on n, the ADR constant, η, K and p.

Proof. Let $y \in \partial\Omega$ and suppose that $X \in \Gamma(y)$. Then by the definition of $\Gamma(y)$ we have that

$$|X - y| \approx_{\eta,K} \delta(X) \le |X - x_Q|.$$

It follows that

$$|y - x_Q| \le |X - y| + |X - x_Q| \lesssim |X - x_Q|.$$

Thus, Lemma 3.1 yields

$$\begin{aligned} |u(X)| &\lesssim 1_{4B_Q}(X) + 1_{(4B_Q)^c}(X) \left(\frac{\ell(Q)}{|X - x_Q|} \right)^{n-1} \\ &\lesssim 1_{4B_Q}(X) + 1_{(4B_Q)^c}(X) \left(\frac{\ell(Q)}{|y - x_Q|} \right)^{n-1}, \quad \forall X \in \Gamma(y). \end{aligned}$$

Let us set $A_k := 2^{k+1}\Delta_Q \setminus 2^k \Delta_Q$ for every $k \ge 2$. By the ADR property, we have

$$\begin{aligned} \int_{\partial\Omega} N_* u^p \, d\sigma &\le \int_{4\Delta_Q} N_* u^p \, d\sigma + \sum_{k=2}^{\infty} \int_{A_k} N_* u^p \, d\sigma \\ &\lesssim \sigma(4\Delta_Q) + \sum_{k=2}^{\infty} \int_{A_k} \left(\frac{\ell(Q)}{|y - x_Q|} \right)^{p(n-1)} d\sigma \\ &\lesssim \sigma(Q) + \sum_{k=2}^{\infty} \int_{A_k} \frac{1}{2^{k(n-1)p}} \, d\sigma \\ &\lesssim \sigma(Q) + \sigma(Q) \sum_{k=2}^{\infty} 2^{kn - k(n-1)p} \lesssim \sigma(Q), \end{aligned}$$

since $p > n/(n-1)$. □

4. Corona decomposition for harmonic measure and the proof of Theorem 1.4

In this section, we prove Theorem 1.4 which, as we pointed out earlier, is enough to imply Theorem 1.3. Before the proof, we recall some definitions and results from [GMT].

Definition 4.1 ([DS2])**.** Let $\mathbf{S} \subset \mathbb{D}$. We say that $\mathbf{S}$ is *coherent* if the following conditions hold:

(a) $\mathbf{S}$ contains a unique maximal element $Q(\mathbf{S})$ which contains all other elements of $\mathbf{S}$ as subsets.
(b) If Q belongs to $\mathbf{S}$ and $Q \subset Q' \subset Q(\mathbf{S})$ for any $Q' \in \mathbb{D}$, then $Q' \in \mathbf{S}$.
(c) Given a cube $Q_j^k \in \mathbf{S}$, either all of its children (i.e. the cubes $P \in \mathbb{D}_{k+1}$ such that $P \subset Q_j^k$) belong to $\mathbf{S}$, or none of them do.

Definition 4.2 (Corona decomposition for harmonic measure [GMT]). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the corkscrew condition with n-dimensional ADR boundary, and let ω be the harmonic measure for Ω. We say that ω *admits a corona decomposition* if $\mathbb{D}$ can be decomposed into disjoint coherent subcollections $\mathbf{S}$ such that the following two conditions hold.

(1) The maximal cubes, $Q(\mathbf{S})$, satisfy a Carleson packing condition

$$\sum_{Q(\mathbf{S})\subset R} \sigma(Q(\mathbf{S})) \leq C\sigma(R), \quad \forall R \in \mathbb{D}(\partial\Omega).$$

(2) For each $Q(\mathbf{S})$ there exists $P_{Q(\mathbf{S})} \in \Omega$ such that

$$c^{-1}\ell(Q(\mathbf{S}))\operatorname{dist}(P_{Q(\mathbf{S})}, Q(\mathbf{S})) \leq \operatorname{dist}(P_{Q(\mathbf{S})}, \partial\Omega) \leq c\ell(Q(\mathbf{S})),$$

$$\omega^{P_{Q(\mathbf{S})}}(3R) \approx \frac{\sigma(R)}{\sigma(Q(\mathbf{S}))} \quad \forall R \in \mathbf{S},$$

where the implicit constants above and c are uniform in $\mathbf{S}$ and R.

Let us fix the value of $\varepsilon \ll 1$ later. For each cube in $\mathbb{D}$, let $P_Q \in B_Q$ be a corkscrew point at scale $\varepsilon\ell(Q)$ relative to x_Q. Then we have

$$\frac{\varepsilon}{M}\ell(Q) \leq \delta(P_Q) \leq \varepsilon(\ell(Q)),$$

Let $y_Q \in \partial\Omega$ be the touching point for the point P_Q, that is, $|y_Q - P_Q| = \delta(P_Q)$. By choosing $\varepsilon \ll 1$ to be small enough, we may assume that

$$B(y_Q, |y_Q - P_Q|) \subset B(x_Q, \tfrac{3}{4}a_0\ell(Q))^{1}.$$

Next, for some parameter τ to be chosen later depending on M, dimension and the ADR constant, we set

$$\begin{aligned} S_Q &:= y_Q + 2\tau(P_Q - y_Q), \\ V_Q &:= B(P_Q, (1-\tau)\delta(P_Q)), \\ V_Q^1 &:= B(P_Q, (1/2)\delta(P_Q)), \\ V_Q^2 &:= B(S_Q, \tau\delta(P_Q)). \end{aligned}$$

Notice that $V_Q^1, V_Q^2 \subset V_Q$. We then have:

Lemma 4.3 ([GMT, Lemma 3.3, proof of Lemma 3.7]). *Suppose that $\tau \ll 1$ and $\varepsilon \ll \tau$ are chosen appropriately depending on M, dimension and the ADR constant. Then if $E_Q \subset Q \in \mathbb{D}$ is such that*

$$\omega^{P_Q}(E_Q) \geq (1-\varepsilon)\omega^{p_Q}(Q),$$

there exists a non-negative harmonic function u_Q on Ω and a Borel function f_Q with

$$u_Q(X) = \int_{\partial\Omega} f_Q \, d\omega^X, \quad 0 \leq f_Q \leq 1_{E_Q}$$

satisfying

$$|m_{V_Q^1} u_Q - m_{V_Q^2} u_Q| \geq c_1, \tag{4.4}$$

where c_1 depends only on M, n and the ADR constant and $m_{B_i}u = \fint_{B_i} u\, dX$.

[1] Note that $B(x_Q, \frac{3}{4}a_0\ell(Q))$ is exactly "B_Q" from [GMT]. See Lemma 2.5 (4)

Definition 4.5 (High and low density cubes [GMT])**.** Let $0 < \delta \ll 1$ be a fixed constant. For a cube $R \in \mathbb{D}$ we say that a subcube $Q \in \mathbb{D}_R$ is a *low density cube* and write $Q \in \mathrm{LD}(R)$ if Q is a maximal cube (with resepect to containment) satisfying

$$\frac{\omega^{P_R}(Q)}{\sigma(Q)} \le \delta \frac{\omega^{P_R}(R)}{\sigma(R)}.$$

For any cube $R \in \mathbb{D}$, we denote $\mathrm{LD}^0(R) = \{R\}$ and define $\mathrm{LD}^k(R)$, $k \ge 1$, inductively by

$$\mathrm{LD}^k(R) = \bigcup_{Q \in \mathrm{LD}^{k-1}(R)} \mathrm{LD}(Q).$$

In the proof of the corona decomposition for harmonic measure in [GMT], ε-approximability is used only to prove a packing condition for the low density cubes [GMT, Lemma 3.7]. Thus, we have:

Lemma 4.6. *Suppose that for any $m \ge 1$ and $R \in \mathbb{D}$ we have*

$$\sum_{k=0}^{m} \sum_{Q \in LD^k(R)} \sigma(Q) \le C\sigma(R), \tag{4.7}$$

where C is independent of m and R. Then ω admits a corona decomposition.

We now prove Theorem 1.4.

Proof of Theorem 1.4. Let us start by fixing η and K depending on τ and ε so that

$$V_Q \subset U_Q(\eta, K). \tag{4.8}$$

This can be done since every point $Y \in V_Q$ has the property that

$$\delta(Y) \approx_{\varepsilon,\tau} \ell(Q).$$

As ε and τ depend only on M, n and the ADR constant so do η and K. We also note that by the construction of the regions $U_Q(\eta, K)$ we have

$$\sum_{Q \in \mathbb{D}} 1_{V_Q}(X) \lesssim 1, \quad \forall X \in \Omega, \tag{4.9}$$

where the implicit constant depends on η, K, M, n and the ADR constant.

By Lemma 4.6, to prove the theorem it is enough enough for us to show that (4.7) holds. In order to do this, we set

$$\mathcal{A}(Q, m) := \frac{1}{\sigma(Q)} \sum_{Q' \in \cup_{k=0}^{m} \mathrm{LD}^k(Q)} \sigma(Q')$$

for every cube $Q \in \mathbb{D}$ and every $m \ge 1$. Then (4.7) is equivalent to the statement

$$\mathcal{A}(m) := \sup_{Q \in \mathbb{D}} \mathcal{A}(Q, m) \le C, \tag{4.10}$$

where C is independent of m.

Let us fix $R \in \mathbb{D}$ and set $\mathcal{F}_{1,m} := \bigcup_{k=1}^{m} \mathrm{LD}^k(R)$. For $Q \in \mathcal{F}_{1,m}$ we set $L_Q := \cup_{Q' \in \mathrm{LD}(Q)} Q'$ and

$$E_Q := Q \setminus L_Q.$$

The sets E_Q, $Q \in \mathcal{F}_{1,m}$, are pairwise disjoint by definition. Moreover, by the definition of $\mathrm{LD}(Q)$, we have

$$\omega^{P_Q}(L_Q) \leq \sum_{Q' \in \mathrm{LD}(Q)} \omega^{P_Q}(Q') \leq \delta \sum_{Q' \in \mathrm{LD}(Q)} \frac{\sigma(Q')}{\sigma(Q)} \omega^{P_Q}(Q) \leq \delta \omega^{P_Q}(Q)$$

and hence

$$\omega^{P_Q}(E_Q) \geq (1-\delta)\omega^{P_Q}(Q) \geq (1-\varepsilon)\omega^{P_Q}(Q) \tag{4.11}$$

as long as we choose $\delta \leq \varepsilon$. Then we may apply Lemma 4.3 to obtain a collection of functions $\{u_Q\}_{Q \in \mathcal{F}_{1,m}}$ such that

$$u_Q(X) = \int_{\partial\Omega} f_Q \, d\omega^X, \quad 0 \leq f_Q \leq 1_{E_Q}$$

for some Borel function f_Q and

$$|m_{V_Q^1} u_Q - m_{V_Q^2} u_Q| \geq c_1.$$

Let $\varepsilon_1 > 0$ to be chosen. Let Ξ denote the collection of sequences $\{a = (a_Q) : Q \in \mathcal{F}_{1,m}, a_Q \in \{-1,+1\}\}$ and let λ be a probability measure on Ξ which assigns equal probability to -1 and $+1$. For every $a \in \Xi$ we set

$$u_a(X) = \sum_{Q \in \mathcal{F}_{1,m}} a_Q u_Q(X).$$

Note that since f_Q are Borel functions with disjoint supports, $\sum_{Q \in \mathcal{F}_{1,m}} a_Q f_Q$ is a Borel function and clearly

$$|u_a(X)| \leq \int_{\partial\Omega} \sum_{Q \in \mathcal{F}_{1,m}} |a_Q| f_Q \, d\omega^X \leq \sum_{Q \in \mathcal{F}_{1,m}} \omega^X(E_Q) \leq 1, \quad \forall X \in \Omega.$$

We now apply the ε-approximability in L^p property (see Theorem 1.3) with "ε" = ε_1: for each $a \in \Xi$ there exists $\varphi_a \in BV_{\mathrm{loc}}$ such that

$$\begin{cases} \|N_*(u_a - \varphi_a)\|_{L^p(\partial\Omega,\sigma)} \leq \varepsilon_1 C_p \|N_* u_a\|_{L^p(\partial\Omega,\sigma)} \\ \|C(\nabla\varphi_a)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\varepsilon_1} C_p \|N_* u_a\|_{L^p(\partial\Omega,\sigma)}. \end{cases}$$

By Lemma 3.3 we also have

$$\|N_*(u_a - \varphi_a)\|_{L^p(\partial\Omega,\sigma)} \leq \varepsilon_1 \|N_* u_a\|_{L^p(\partial\Omega,\sigma)} \leq C_1 \varepsilon_1 \sigma(R)^{1/p} \tag{4.12}$$

and

$$\|C(\nabla\varphi_a)\|_{L^p(\partial\Omega,\sigma)} \leq C_{\varepsilon_1} C_1 \sigma(R)^{1/p}, \tag{4.13}$$

where C_1 depends on n, the ADR constant, η, K and p. By Chebyshev's inequality and (4.12), for each $a \in \Xi$ we have

$$\sigma(\{x \in R : N_*(u_a - \varphi_a)(x) > C_2 \varepsilon_1\}) \leq \frac{C_1^p}{C_2^p} \sigma(R).$$

We will fix the exact value of $\gamma \in (0,1)$ later but regardless of the exact value, we may choose C_2 so that $C_1^p / C_2^p < \gamma/2$. Let us set $\varepsilon_0 := C_2 \varepsilon_1$. It follows that for each $a \in \Xi$ there exists a set $F(R,a) \subset R$ such that $\sigma(F(R,a)) > (1-\gamma)\sigma(R)$ and for all $y \in F(R,a)$

$$|u_a(X) - \varphi_a(X)| \leq \varepsilon_0, \quad \forall X \in \Gamma(y).$$

Now for each $a \in \Xi$ we let $\mathcal{F}_{2,m}(a)$ be the collection of cubes $Q \in \mathcal{F}_{1,m}$ such that $Q \cap F(R, a) = \emptyset$. We then let $\widetilde{\mathcal{F}}_{2,m}(a)$ be the collection of maximal cubes in $\mathcal{F}_{2,m}(a)$ with respect to inclusion and $\mathcal{F}_{3,m}(a) = \mathcal{F}_{1,m} \setminus \mathcal{F}_{2,m}(a)$. By maximality,

$$\sum_{Q^* \in \widetilde{\mathcal{F}}_{2,m}(a)} \sigma(Q^*) \leq \gamma \sigma(R). \tag{4.14}$$

Suppose that $Q \in \mathcal{F}_{3,m}(a)$. Then there exists $y \in F(R, a) \cap Q$. It follows that $U_Q \subset \Gamma(y)$ and hence

$$|u_a(X) - \varphi_a(X)| \leq \varepsilon_0, \quad \forall X \in U_Q.$$

In particular, by (4.8) we have for all $Q \in \mathcal{F}_{3,m}(a)$

$$|u_a(X) - \varphi_a(X)| \leq \varepsilon_0, \quad \forall X \in V_Q. \tag{4.15}$$

Now, using (4.4) and Khintchine's inequality we obtain for every $Q \in \mathcal{F}_{1,m}$

$$\begin{aligned} c_1 &\leq |m_{V_Q^1} u_Q - m_{V_Q^2} u_Q| \\ &\leq \left(\sum_{Q' \in \mathcal{F}_{1,m}} |m_{V_Q^1} u_{Q'} - m_{V_Q^2} u_{Q'}|^2 \right)^{1/2} \\ &\leq \frac{1}{c_3} \int_\Xi \left| \sum_{Q' \in \mathcal{F}_{1,m}} a_{Q'} (m_{V_Q^1} u_{Q'} - m_{V_Q^2} u_{Q'}) \right| d\lambda(a) \\ &= \frac{1}{c_3} \int_\Xi |m_{V_Q^1} u_a - m_{V_Q^2} u_a| \, d\lambda(a), \end{aligned} \tag{4.16}$$

where c_3 is a universal constant. Then it follows that integrating over V_Q (note that $\operatorname{diam}(V_Q) \approx \ell(Q)$) and summing in $Q \in \mathcal{F}_{1,m}$ we have

$$\begin{aligned} \sum_{Q \in \mathcal{F}_{1,m}} \sigma(Q) &\lesssim \sum_{Q \in \mathcal{F}_{1,m}} \ell(Q)^n \\ &\overset{(4.16)}{\lesssim} \sum_{Q \in \mathcal{F}_{1,m}} \int_{V_Q} \int_\Xi \frac{1}{\ell(Q)} |m_{V_Q^1} u_a - m_{V_Q^2} u_a| \, d\lambda(a) \, dX \\ &\lesssim \int_\Xi \sum_{Q \in \mathcal{F}_{3,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q^1} u_a - m_{V_Q^2} u_a| \, dX \, d\lambda(a) \\ &\quad + \int_\Xi \sum_{Q \in \mathcal{F}_{2,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q^1} u_a - m_{V_Q^2} u_a| \, dX \, d\lambda(a) \\ &\lesssim \int_\Xi \sum_{Q \in \mathcal{F}_{3,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q^1} u_a - m_{V_Q^2} u_a| \, dX \, d\lambda(a) \\ &\quad + \int_\Xi \sum_{Q \in \mathcal{F}_{2,m}(a)} \sigma(Q) \, d\lambda(a), \end{aligned} \tag{4.17}$$

where the implicit constants depend on M, n and the ADR constant and we used that $\|u\|_\infty \leq 1$ in the last inequality. We have by (4.14) and the definitions of $\widetilde{\mathcal{F}}_{2,m}(a)$

and $\mathcal{A}(m)$ that

$$\begin{aligned}\frac{1}{\sigma(R)}\sum_{Q\in\mathcal{F}_{2,m}(a)}\sigma(Q) &= \frac{1}{\sigma(R)}\sum_{Q^*\in\widetilde{\mathcal{F}}_{2,m}(a)}\ \sum_{Q\in\bigcup_{k=0}^{m-1}\mathrm{LD}^k(Q^*)}\sigma(Q)\\ &\le \frac{1}{\sigma(R)}\sum_{Q^*\in\widetilde{\mathcal{F}}_{2,m}(a)}\sigma(Q^*)\mathcal{A}(m)\le\gamma\mathcal{A}(m).\end{aligned}\tag{4.18}$$

Set $B_R^{**} := B(x_R, 5a_1\ell(R))$ and $\Delta_R^{**} = B_R^{**}\cap\Omega$ and note that $V_Q \subset B_R^{**}$ for all $Q \in \mathbb{D}(R)$. We immediately have that

$$\begin{aligned}\int_{B_R^{**}\cap\Omega}|\nabla\varphi(X)|\,dX &\lesssim \ell(R)^n \fint_{\Delta_R^{**}} C(\nabla\varphi)(y)\,d\sigma(y)\\ &\approx \sigma(R)\fint_{\Delta_R^{**}} C(\nabla\varphi)(y)\,d\sigma(y).\end{aligned}$$

Using (4.15), (4.9), the Poincaré inequality[2] and (4.13) we obtain

(4.19)

$$\begin{aligned}&\int_\Xi\sum_{Q\in\mathcal{F}_{3,m}(a)}\int_{V_Q}\frac{1}{\ell(Q)}|m_{V_Q^1}u_a - m_{V_Q^2}u_a|\,dX\,d\lambda(a)\\ &\quad\lesssim \int_\Xi\sum_{Q\in\mathcal{F}_{3,m}(a)}\int_{V_Q}\frac{1}{\ell(Q)}|m_{V_Q^1}\varphi_a - m_{V_Q^2}\varphi_a|\,dX\,d\lambda(a) + \varepsilon_0\int_\Xi\sum_{Q\in\mathcal{F}_{3,m}(a)}\sigma(Q)\\ &\quad\lesssim \int_\Xi\sum_{Q\in\mathcal{F}_{1,m}}\int_{V_Q}|\nabla\varphi_a|\,dX\,d\lambda + \varepsilon_0\sum_{Q\in\mathcal{F}_{1,m}}\sigma(Q)\\ &\quad\lesssim \int_\Xi\int_{B_R^{**}\cap\Omega}|\nabla\varphi_a|\,dX\,d\lambda + \varepsilon_0\sum_{Q\in\mathcal{F}_{1,m}}\sigma(Q)\\ &\quad\lesssim \int_\Xi\fint_{\Delta^{**}} C(\nabla\varphi)(y)\,d\sigma(y)\,d\lambda + \varepsilon_0\sum_{Q\in\mathcal{F}_{1,m}}\sigma(Q)\\ &\quad\lesssim \int_\Xi\sigma(R)\left(\fint_{\Delta^{**}}(C(\nabla\varphi_a)(y))^p\,d\sigma(y)\right)^{1/p}d\lambda + \varepsilon_0\sum_{Q\in\mathcal{F}_{1,m}}\sigma(Q)\\ &\quad\lesssim C_{\varepsilon_1}C_1\sigma(R) + \varepsilon_0\sum_{Q\in\mathcal{F}_{1,m}}\sigma(Q),\end{aligned}$$

where the *implicit* constant depends only on M, n and the ADR constant. Dividing (4.17) by $\sigma(R)$ and using (4.18) and (4.19) we have shown

$$\mathcal{A}(R,m)\lesssim C_{\varepsilon_1}C_1 + \varepsilon_0\mathcal{A}(R,m) + \gamma\mathcal{A}(m),$$

where the implicit constant depends only on M, n and the ADR constant. Taking the supremum over $R \in \mathbb{D}$ and recalling that $\varepsilon_0 = C_2\varepsilon_1$ we have

$$\mathcal{A}(m)\lesssim C_{\varepsilon_1}C_1 + C_2\varepsilon_1\mathcal{A}(m) + \gamma\mathcal{A}(m).$$

[2] See e.g. the proof of [Z, Theorem 5.11.1], for the case of BV functions.

We recall the order we have chosen the constants and first choose $\gamma \ll 1$; this choice dictates C_2 (C_2 depends on M, p, γ, n, the ADR constant and C_1). Finally, we choose $\varepsilon_1 \ll 1$ depending on C_2, M, n and the ADR constant. Thus,

$$\mathcal{A}(m) \le C_{M,n,ADR} C_{\varepsilon_1} C_1,$$

which is the claimed inequality (4.10). Here we have used the fact that the quantity $\mathcal{A}(m)$ is finite (to be more precise, we have $\mathcal{A}(m) \le m + 1 < \infty$), which allows us to choose γ and ε_1 to be so small that $\mathcal{A}(m) - c(C_2\varepsilon_1 + \gamma)\mathcal{A}(m) > 0$ for a structural constant c which was implicit in the estimates. This concludes the proof. □

References

[AGMT] J. Azzam, J. Garnett, M. Mourgoglou and X. Tolsa, Uniform rectifiability from Carleson measure estimates and ε-approximability of bounded harmonic functions. *Duke Math. J.* 167 (2018), no. 8, 1473-1524. 1, 2, 4

[BJ] C. J. Bishop and P. W. Jones, Harmonic measure and arclength. *Ann. of Math. (2)* **132** (1990), no. 3, p. 511–547. 2

[Chr] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, **LX/LXI** (1990), 601–628. 5

[Dah] Björn E. J. Dahlberg, Approximation of harmonic functions. *Ann. Inst. Fourier (Grenoble)*, 30(2):vi, 97–107, 1980. 2

[DS1] G. David and S. Semmes, *Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs*, *Asterisque* **193** (1991). 2, 4, 5

[DS2] G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, volume 38 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1993. 2, 4, 5, 8

[EG] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, Stud. Adv. Math. CRC Press, New York, 1992. 2

[FS] C. Fefferman and E. M. Stein, Hp spaces of several variables. *Acta Math.* 129 (1972), no. 3-4, 137-193. 1, 6

[Gar] J. Garnett, *Bounded Analytic Functions*, Academic Press, San Diego, 1981. 1, 2

[GMT] John Garnett, Mihalis Mourgoglou, and Xavier Tolsa. Uniform rectifiability from carleson measure estimates and ε-approximability of bounded harmonic functions. Preprint 2016. *arXiv:1611.00264*. 1, 2, 3, 4, 8, 9, 10

[GW] M. Grter, K.-O. Widman, The Green function for uniformly elliptic equations. *Manuscripta Math.* **37** (1982), no. 3, 303?342. 4

[HKMP15] Steve Hofmann, Carlos Kenig, Svitlana Mayboroda, and Jill Pipher. Square function/non- tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators. *J. Amer. Math. Soc.*, 28(2):483–529, 2015. 2

[HM] S. Hofmann and J.M. Martell, Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in Lp. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(3):577–654, 2014. 6

[HMM] S. Hofmann, J.M. Martell, S. Mayboroda, Uniform rectifiablity, Carleson measure estimates, and approximation of harmonic functions. *Duke Math. J.* 165 (2016), no.12, 2331-2389. 1, 2, 3, 4, 6

[HMM2] S. Hofmann, J.M. Martell, S. Mayboroda, Uniform rectifiablity, Carleson measure estimates, and approximation of harmonic functions II. *unpublished*. 3

[HMMM] S. Hofmann, D. Mitrea, M. Mitrea, A. Morris, L^p-Square Function Estimates on Spaces of Homogeneous Type and on Uniformly Rectifiable Sets. *Mem. Amer. Math. Soc.* 245 (2017), no. 1159, v+108. 5

[HT] S. Hofmann and O. Tapiola, Uniform rectifiability and ε-approximability of harmonic functions in L^p. Preprint 2017. *arXiv:1710.05528*. 1, 2, 3, 4, 6

[Hof-Kim] S. Hofmann and S. Kim, The Green function estimates for strongly elliptic systems of second order. *Manuscripta Math.* 124 (2007), no. 2, 139-172. 4

[Hyt-Kar] T. Hytönen and A. Kairema, Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, 126(1):1–33, 2012. 5

[HR] T. Hytönen and A. Rosén, Bounded variation approximation of L_p dyadic martingales and solutions to elliptic equations. Preprint 2016. *arXiv:1405.2153 [v3]*. 2, 3

[KKiPT] C. Kenig, B. Kirchheim, J. Pipher and T. Toro, Square functions and the $A_{-}\infty$ property of elliptic measures. *J. Geom. Anal.*, 26(3):2383–2410, 2016. 1

[KKoPT] C. Kenig, H. Koch, J. Pipher and T. Toro, A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations. *Adv. Math.*, 153(2):231–298, 2000. 2

[LSW] W. Littman, G. Stampacchia, H. F. Weinberger, Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa* (3) **17** 1963 43-77. 4

[SW] J. Serrin and H.F. Weinberger, Isolated singularities of solutions of linear elliptic equations. *Amer. J. Math.* 88 1966 258-272. 3

[Var] N. Th. Varopoulos, A remark on functions of bounded mean oscillation and bounded harmonic functions. *Pacific J. Math.* **74**, 1978, 257-259. 2

[Z] W. P. Ziemer Weakly differentiable functions. Sobolev spaces and functions of bounded variation. Graduate Texts in Mathematics, 120. *Springer-Verlag*, New York, 1989. 13

Simon Bortz, School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: bortz010@umn.edu

Olli Tapiola, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: olli.m.tapiola@gmail.com