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Coexistence of hidden attractors and multistability in counterexamples to the Kalman conjecture

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Abstract: The Aizerman and Kalman conjectures played an important role in the theory of global stability for control systems and set two directions for its further development – the search and formulation of sufficient stability conditions, as well as the construction of counterexamples for these conjectures. From the computational perspective the latter problem is nontrivial, since the oscillations in counterexamples are hidden, i.e. their basin of attraction does not intersect with a small neighborhood of an equilibrium. Numerical calculation of initial data of such oscillations for their visualization is a challenging problem. Up to now all known counterexamples to the Kalman conjecture were constructed in such a way that one locally stable limit cycle (hidden oscillation) co-exists with a locally stable equilibrium. In this paper we demonstrate a multistable configuration of three co-existing hidden oscillations (limit cycles) and a locally stable equilibrium in the phase space of the fourth-order system, which provides a new class of counterexamples to the Kalman conjecture.

Keywords: global stability, hidden attractors, multistability, Kalman conjecture, periodic oscillations

1. INTRODUCTION

The necessity to study stability and limit dynamical regimes (attractors) arises in classical theoretical and applied problems. One of the first such problems is related to the design of automatic control systems, which ensure the transition of the controlled object to the operating regime and its stability with respect to external disturbances. The first dynamical models of control systems were constructed in a way that the operating regime corresponded to the unique globally stable equilibrium state. After that models with oscillating operating regimes (periodic attractors) and chaotic regimes (chaotic attractors) were obtained. Later on, multistable models with different co-existing regimes (attractors) were discovered. Control of system states and their transfer into the basin of attraction of a desired attractor is the subject for study of the oscillation control theory (see e.g. [Fradkov and Pogromsky, 1998, Fradkov and Evans, 2005]). One of the first theoretical problems on multistability is the second part of the famous Hilbert's 16th problem on the number and mutual disposition of coexisting periodic attractors in two-dimensional polynomial systems. For chaotic multidimensional dynamical systems a similar problem on the number and mutual disposition of chaotic attractors and, in particular, their dependence on the degree of polynomials in the model is discussed in [Leonov and Kuznetsov, 2015, Kuznetsov et al., 2018].

For nonlinear systems with a unique equilibrium and bounded solutions, the question arose: how to find a class of systems for which the condition for the absence of the possibility for birth of self-excited oscillations implies the absence of hidden oscillations [1] and the global stability of the equilibrium. This problem has its origins in the Watt governor stability studies. In 1877, I.A. Vyshnegradsky [Vyshnegradsky, 1877] for the closed dynamic model "machine + governor" studied an approximate linear mathematical model without dry friction and proposed the stability conditions of the desired operating regime corresponding to the equilibrium state (trivial attractor). However, the question about a rigorous proof of the Vyshnegradsky problem on the validity of the linearization procedure for a system by discarding dry friction remained open. In 1885, M.H. Léauté showed [Léauté, 1885] the

[1] In 2009, G.A. Leonov and N.V. Kuznetsov proposed the classification of oscillations as being hidden or self-excited and laid the foundations of the *theory of hidden oscillations*, which reflects the modern stage of development of the A.A. Andronov's theory of oscillations. Self-excited oscillations can be visualized numerically by a trajectory starting from a point in a neighborhood of an unstable equilibrium. In contrast, the basin of attraction for a hidden oscillation is not connected with equilibria and, thus is necessary to develop a special analytical-numerical methods to find initial points for their visualization. The current progress in the development of theory of hidden oscillations was recently presented at a plenary lecture at the 5th IFAC Conference on Analysis and Control of Chaotic Systems (see `https://chaos2018.dc.wtb.tue.nl`).

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possibility of the appearance of limit periodic oscillations in dynamical models of control systems with dry friction. After that, publications appeared (see e.g. [Zhukovsky, 1909, p. 6]), which criticized Vyshnegradsky approach and questioned his conclusions. In response to this criticism, A.A. Andronov and A.G. Maier [Andronov and Maier, 1944] provided a rigorous global analysis of the nonlinear model of the Watt governor with dry friction and proved the sufficiency of the Vyshnegradsky conditions for the absence of limit oscillations and global stability of the operating regime [2] (i.e. the existence of a rest segment that attracts trajectories from any initial data). Further development and generalization of the results by Vyshnegradsky, Andronov and Maier were done by G.A. Leonov in [Leonov, 1971] (see also survey [Leonov et al., 2017]).

In 1949, inspirited by the discussion of the work [Andronov and Maier, 1944] at the Andronov's scientific seminar in the Institute of Automation and Remote Control (USSR Academy of Sciences, Moscow) [Bissell, 1998], M.A. Aizerman formulated a new problem. His question was whether the sufficient conditions of global stability of a class of nonlinear Lurie systems with a unique equilibrium coincide with the necessary stability conditions when the smooth nonlinearity belongs to the sector of linear stability [Aizerman, 1949]. Independently, a similar conjecture was later advanced by R.E. Kalman in 1957, with the additional requirement that the derivative of nonlinearity belong to the linear stability sector [Kalman, 1957]: *"If $\varphi(\sigma)$ in Fig. 1 is replaced by constants k corresponding to all possible values of $\varphi'(\sigma)$, and it is found that the closed-loop system is stable for all such k, then it is intuitively clear that the system must be monostable; i.e. all transient solutions will converge to a unique, stable critical point."*

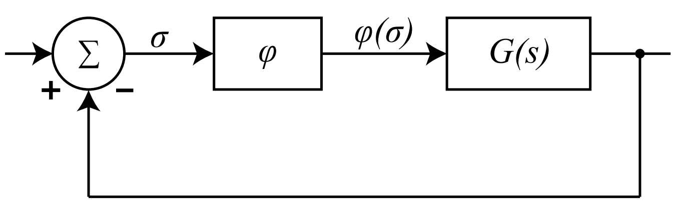


Fig. 1. Nonlinear control system. $G(s)$ is a linear transfer function, $\varphi(\sigma)$ is a single-valued smooth function [Kalman, 1957].

Kalman's statement can be reformulated in the following:

Conjecture 1. (The Kalman Conjecture). Consider the following control system in the Lurie form

$$\dot{\mathrm{x}} = A\mathrm{x} + b\varphi(\sigma), \quad \sigma = c^*\mathrm{x}, \tag{1}$$

where A is a constant $n\times n$ matrix, b and c are constant n-dimensional columns, with all values being, sign $*$ denotes the transpose, and φ is a smooth scalar function with $\varphi(0) = 0$, satisfying the condition

$$k_1 < \varphi'(\sigma) < k_2, \quad \sigma \in (-\infty, +\infty), \tag{2}$$

where k_1 is a number or $-\infty$, and k_2 is a number or $+\infty$. If the linear system $\dot{\mathrm{x}} = A\mathrm{x} + kbc^*\mathrm{x}$, with $k \in (k_1, k_2)$ is asymptotically stable, then system (1) is stable in large (i.e. a zero solution of system (1) is asymptotically stable and any solution tends to zero as $t \to +\infty$).

[2] This result was specially remarked when in 1946 A.A. Andronov was elected to the Academy of Sciences of the USSR where he became the first academician in control theory.

The Aizerman and Kalman conjectures played an important role in the theory of global stability for control systems and set two directions for its further development – the search and formulation of sufficient stability conditions (see pioneering works [Popov, 1961, Kalman, 1963, Gelig et al., 1978]), as well as the construction of counterexamples for these conjectures. From the computational perspective, the latter problem is nontrivial, since the oscillations in counterexamples are hidden, i.e. their basin of attraction does not intersect with small neighborhood of an equilibrium. Numerical calculation of initial data of such oscillations for their visualization is a challenging problem. Up to now all known counterexamples to the Kalman conjecture were constructed in such a way that one locally stable limit cycle (hidden oscillation) co-exists with a locally stable equilibrium. In this paper we demonstrate a multistable configuration of three co-existing hidden oscillations (limit cycles) and a locally stable equilibrium in the phase space of the fourth-order system, which provides a new class of counterexamples to the Kalman conjecture.

2. PREVIOUS COUNTEREXAMPLES TO KALMAN CONJECTURE

First known attempt to construct counterexamples to the Kalman conjecture was made by R.E. Fitts [Fitts, 1966], who experimentally studied a fourth-order system with a cubic nonlinearity. As a result, Fitts experimentally observed a periodic solution of considered system. Later on, N.E. Barabanov [Barabanov, 1988] claimed that some Fitts' results are not true and suggested to use discontinuous nonlinearity `sign`$(\cdot)$ to derive counterexamples analytically. His work also raised critical discussions in [Bernat and Llibre, 1996, Meisters, 1996, Glutsyuk, 1998]. In particular, Bernat and Llibre [1996] pointed out the necessity to rigorously analyze non-local bifurcations while smoothing discontinuous nonlinearities. They suggested to start the procedure for constructing counterexamples with a piecewise linear nonlinearity `sat`$(\cdot)$. In [Bragin et al., 2010, 2011, Leonov and Kuznetsov, 2011], it was introduced an effective approach for construction of counterexamples to the Kalman conjecture relying on an analytical-numerical search for periodic solutions by applying harmonic balance and numerical continuation methods and using smooth nonlinearity `tanh`$(\cdot)$. For discrete-time systems Heath et al. [2015] demonstrated that Kalman conjecture is false even for second-order systems using counterexamples with stable periodic solutions [3]. Also construction of counterexamples to the Kalman conjecture is discussed in [Burkin and Khien, 2014].

3. COEXISTING LIMIT CYCLES

To construct numerically a new counterexample to the Kalman conjecture providing three co-existing limit cycles we combined Fitts' linear system, Barabanov's idea

[3] Remark that the difference between the dimensions of the phase spaces of a discrete-time system and a continuous-time system defined by autonomous ODE, for which the Kalman conjecture is not true, is equal to 2. This value coincides with the difference between the dimensions of the spaces in which chaos can occur (for discrete-time systems the dimension is equal to 1, for continuous-time systems – 3).

of considering $\mathtt{sign}(\cdot)$, and the idea from [Leonov and Kuznetsov, 2011] to use numerical continuation procedure while passing from $\mathtt{sign}(\cdot)$ to $\mathtt{tanh}(\cdot)$.

Consider the control system in the Lurie form (1) with

$$A=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix},\ b=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\ c=\begin{pmatrix}0\\0\\-1\\0\end{pmatrix},\quad (3)$$

and $a_0=(m_1^2+\beta^2)(m_2^2+\beta^2)$, $a_1=2\beta(m_1^2+m_2^2+2\beta^2)$, $a_2=m_1^2+m_2^2+6\beta^2$, $a_3=4\beta$, $m_1=0.9, m_2=1.1, \beta=0.03$, $\varphi(\sigma)=\mathtt{tanh}(\sigma/\varepsilon)$, $\varepsilon=0.01$. The linear part of system (1) is defined by the transfer function

$$W(p)=\frac{p^2}{((p+\beta)^2+m_1^2)\,((p+\beta)^2+m_2^2)}.\quad (4)$$

Initial data for visualization of periodic oscillations were obtained using Andronov point mapping method [Andronov et al., 1966] [4] for system (1), (3) with non-linearity $\varphi(\sigma)=\mathtt{sign}(\sigma)$ and numerical continuation method [5] for smoothing the discontinuous nonlinearity (see e.g. [Leonov and Kuznetsov, 2013, Leonov et al., 2017]). Corresponding initial points for each stable limit cycle are presented below in Table 1. In system (1), (3) with the smooth nonlinearity $\varphi(\sigma)=\mathtt{tanh}(\sigma/\varepsilon)$, $\varepsilon=0.01$ for obtained initial points the trajectories were numerically integrated, which after the transient process allows us to visualize three hidden periodic attractors (see Fig. 2 and Table 2 with initial data). For each periodic attractor, an additional analysis of the local basin of attraction was carried out by choosing a grid of points in the vicinity of the periodic attractor and checking the attraction of all the trajectories with initial data from these points to the periodic attractor.

Table 1. Initial data for modeling of the three periodic attractors for system (1), (3) with nonlinearity $\varphi(\sigma)=\mathtt{sign}(\sigma)$.

	1st and 2nd	3rd
x_1	±0.62520516260693109	−2.113517446278802
x_2	±3.73240970726506105	0.664336179538623
x_3	0	0.891912878629890
x_4	∓3.47541697286971208	0.278600965570120

Table 2. Initial data for modeling of the three periodic attractors for system (1), (3) with nonlinearity $\varphi(\sigma)=\mathtt{tanh}(\sigma/\varepsilon)$, $\varepsilon=0.01$.

	1st and 2nd	3rd
x_1	±0.625216695745867	−2.11395731851229
x_2	±3.73239217905780	0.663680374961913
x_3	0	0.891701229667371
x_4	∓3.47341560599714	0.279201499188914

[4] Other methods for searching periodic oscillations of dynamical models with $\mathtt{sign}(\cdot)$ nonlinearity can be found e.g. in [Tsypkin, 1984, Boiko, 2008].

[5] The idea is to consider system (1), (3) with the nonlinearity $\varphi(\sigma)=\mathtt{sign}(\sigma)+\mu(\mathtt{tanh}(\sigma/\varepsilon)-\mathtt{sign}(\sigma))$, $\mu\in[0,1]$ and to switch from the system with nonlinearity $\mathtt{sign}(\cdot)$ to the system with a smooth nonlinearity $\mathtt{tanh}(\cdot)$ by varying the parameter μ from 0 to 1 with some small step. During the switching on each next step, the initial point for a trajectory to be integrated is chosen as the last point of the trajectory integrated on the previous step.

3.1 Sector of linear stability

It can be seen that the eigenvalues of the Jacobi matrix at the zero equilibrium are

$$-\beta\pm m_1 i,\qquad -\beta\pm m_2 i,$$

and, thus this equilibrium is locally stable.

Consider the matrix

$$A+kbc^*=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_0 & -a_1 & -a_2-k & -a_3 \end{pmatrix}.\quad (5)$$

Characteristic polynomial of the matrix (5) is

$$\lambda^4+a_3\lambda^3+(a_2+k)\lambda^2+a_1\lambda+a_0.\quad (6)$$

Using Routh-Hurwitz criterion it is possible to show that for each $\beta>0$ the linear system $\dot{\mathrm{x}}=A\mathrm{x}+k\,b\,c^*\mathrm{x}$, given by matrices (3), is globally asymptotically stable for

$$k\in\big(-4\beta^2-\tfrac{(m_1^2-m_2^2)^2}{2(2\beta^2+m_1^2+m_2^2)},+\infty\big).$$

All the roots of the characteristic polynomial (6) have negative real parts, iff all the leading principal minors

$$\Delta_1=a_3=4\beta,\qquad \Delta_2=a_3(a_2+k)-a_1,$$
$$\Delta_3=a_1a_3k-a_1^2+a_1a_2a_3-a_0a_3^2,\qquad \Delta_4=a_0\Delta_3$$

of the Hurwitz matrix

$$\begin{pmatrix} a_3 & a_1 & 0 & 0\\ 1 & a_2+k & a_0 & 0\\ 0 & a_3 & a_1 & 1\\ 0 & 1 & a_2+k & a_0 \end{pmatrix}$$

are positive. This implies the inequality $k>\frac{a_0a_3^2+a_1^2-a_1a_2a_3}{a_1a_3}$, which defines a sector of linear stability.

3.2 Describing function method and Popov criterion

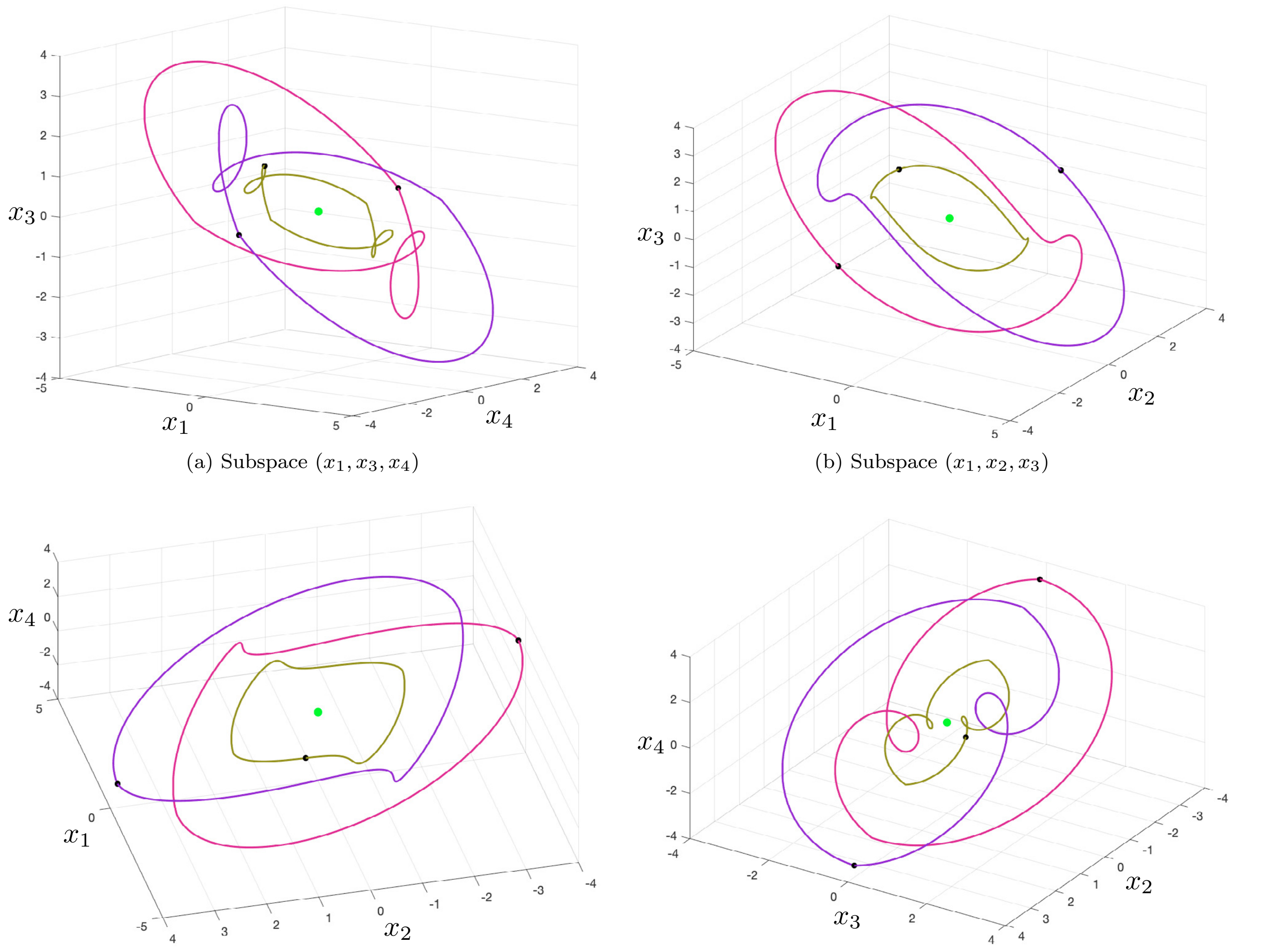
Let us show that the application of the classical describing function method [6] and Popov method to system (1), (3) demonstrates the necessity of their further development to be able to obtain the necessary and sufficient conditions for the birth of oscillations and stability.

Suppose system (1), (3) has periodic solution with amplitude a and frequency ω_0. Hence, according to the harmonic balance method, frequency of this solution can be found from the following equality $\operatorname{Im}W(i\omega_0)=0$ and, therefore, $\omega_0=\sqrt{\beta^2+\frac{m_1^2+m_2^2}{2}}>0$. Also, from the equality $\operatorname{Re}W(i\omega_0)=0$ we can get a coefficient of harmonic linearization

$$k_{\mathrm{hl}}=-\frac{1}{\operatorname{Re}W(i\omega_0)}=-\left(4\beta^2+\frac{(m_1^2-m_2^2)^2}{2(2\beta^2+m_1^2+m_2^2)}\right)<0.$$

The describing function is defined as follows:

[6] Describing function method belongs to the approximate methods of analysis of control systems and there exist various examples of systems for which it leads to incorrect results in both prediction of stability (see e.g. [Bragin et al., 2011, Leonov and Kuznetsov, 2013]) and prediction of the existence of oscillations (see e.g. [Leonov and Kuznetsov, 2018a,b]).



(a) Subspace (x_1, x_3, x_4) (b) Subspace (x_1, x_2, x_3)

(c) Subspace (x_1, x_2, x_4) (d) Subspace (x_2, x_3, x_4)

Fig. 2. Co-existence of a stable equilibrium (light green) and three hidden limit cycles, two large symmetric ones (red and purple), and a small one (dark green), in the phase space of system (1), (3) with $\varphi(\sigma) = \mathtt{tanh}(\sigma/\varepsilon)$, $\varepsilon = 0.01$.

$$\Phi(a) = \int\limits_0^{\frac{2\pi}{\omega_0}} \tanh(\cos(\omega_0 t)a)\cos(\omega_0 t)dt - \\ -ak_{\text{hl}} \int\limits_0^{\frac{2\pi}{\omega_0}} (\cos(\omega_0 t))^2 dt \geq -\frac{\pi a k_{\text{hl}}}{\omega_0}. \tag{7}$$

If $a \neq 0$, then $\Phi(a) > 0$ and there is no such a that $\Phi(a) = 0$. Therefore, there are no periodic solutions in the system (1) according to the describing function method.

Consider the Popov criterion on the absolute stability (see e.g. [Popov, 1961, p. 961],[Yakubovich et al., 2004, p. 79]) for system (1), (3) and non-linearity $\varphi(\sigma) = \mathtt{tanh}(\sigma)$. First two conditions of the Popov criterion, i.e. asymptotic stability of the linear part and $0 \leq \frac{\tanh(\sigma)}{\sigma} \leq \infty, \sigma \neq 0, \tanh(0) = 0$, are satisfied. The third condition of the Popov criterion has the following form:

$$\operatorname{Re}[(1 + i\omega\vartheta)W(i\omega)] = \operatorname{Re} W(i\omega) - \omega\vartheta \operatorname{Im} W(i\omega) \geq 0 \Leftrightarrow \\ -\omega^2(\omega^4 - \omega^2 a_2 + a_0) \geq 2\vartheta\omega^4\beta\left(-2\omega^2 + \frac{a_1}{2\beta}\right).$$

If $\omega = 0$, then this inequality holds. Else, if $\omega \neq 0$, then this condition takes the form:

$$(4\vartheta\beta - 1)\omega^4 - (\vartheta a_1 - a_2)\omega^2 - a_0 \geq 0. \tag{8}$$

Note that since $a_0 > 0$, then for each $\vartheta \geq 0$ there exists small enough $\omega > 0$ such that (8) is not true. Therefore, the conditions of the criterion are not satisfied.

4. CONCLUSION

Thus, the results obtained here show the limits of applicability of existing analytical methods and demonstrate the difficulty of identifying classes of systems for which it is possible to match the necessary and sufficient conditions for global stability.

In the general case, when considering various nonlinearities, it is possible to synthesize systems with a large number of coexisting attractors (equilibria, limit cycles, chaotic attractors), see e.g. [Wang and Chen, 2013, Zhang and Chen, 2017, Stankevich et al., 2017, Kuznetsov et al., 2017, Chen et al., 2017]. However, in these examples the nonlinearities were non-scalar, or the derivatives of the nonlinearities changed their signs. Therefore, these nonlinearities did not satisfy the conditions of Kalman conjecture. In this article, we demonstrate new counterexample to the Kalman conjecture with three co-existing stable limit cycles. The mutual disposition of co-existing attractors in counterexamples to the Kalman conjecture (depending on the dimension of the system) and possibility of managing the number of attractors (e.g. finding the maximum possible number of attractors) are open problems for the further study.

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