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On lower-bound estimates of the Lyapunov dimension and topological entropy for the Rossler systems

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Abstract: In this paper, on the example of the Rössler systems, the application of the Pyragas time-delay feedback control technique for verification of Eden's conjecture on the maximum of local Lyapunov dimension, and for the estimation of the topological entropy is demonstrated. To this end, numerical experiments on computation of finite-time local Lyapunov dimensions and finite-time topological entropy on a Rössler attractor and embedded unstable periodic orbits are performed. The problem of reliable numerical computation of the mentioned dimension-like characteristics along the trajectories over large time intervals is discussed.

Keywords: chaos, hidden and self-excited attractors, Lyapunov dimension, Lyapunov exponents, unstable periodic orbit, time-delay feedback control

1. INTRODUCTION

Consider the following Rössler systems [Rössler, 1976a,b, 1979]:

$$\dot{x} = -y - z, \quad \dot{y} = x + a_1 y, \quad \dot{z} = b_1 - c_1 z + xz, \tag{1}$$

$$\dot{x} = -y - z, \quad \dot{y} = x + a_2 y, \quad \dot{z} = b_2 x - c_2 z + xz, \tag{2}$$

$$\dot{x} = -y - z, \quad \dot{y} = x, \quad \dot{z} = -b_3 z + a_3(y - y^2), \tag{3}$$

with arbitrary real parameters $a_{1,2,3},\ b_{1,2,3},\ c_{1,2} \in \mathbb{R}$. System (1) if $c_1^2 \geq 4a_1b_1$ has the following equilibria:

$$O_1^{\pm} = (a_1p^{\pm}, -p^{\pm}, p^{\pm}), \quad \text{where} \quad p^{\pm} = \frac{c_1 \pm \sqrt{c_1^2 - 4a_1b_1}}{2a_1}, \tag{4}$$

which coincide for $c_1^2 = 4a_1b_1$. It is known [Barrio et al., 2011] that the equilibrium O_1^+ is always unstable and O_1^- is linearly stable iff the parameters a_1, c_1 belongs to the region $\{(a_1, c_1) \mid a_1 \leq 1, c_1 > 2a_1\}$ or $\{(a_1, c_1) \mid a_1 \in (1, \sqrt{2}), c_1 \in (2a_1, \frac{2a_1}{a_1^2 - 1})\}$, and the parameter b_1 satisfies the inequalities: $b_1^*(a, c) \leq b_1 \leq \frac{c_1^2}{4a_1}$, where

$$b_1^*(a_1, c_1) = \frac{a_1}{2(a_1^2+1)^2}\Big[2 - a_1^4 + c_1a_1^3 + 2a_1^2 - c_1a_1 + c_1^2$$
$$+ (c_1 - a_1)\sqrt{a_1^6 - 4a_1^4 + 2c_1a_1^3 - 4a_1^2 + c_1^2}\Big]. \tag{5}$$

As it was mentioned in [Algaba et al., 2015], in the region of parameters where the equilibria $O_1^{\pm}$ exist system (1) is equivalent to system (2) with respect to the following linear change of variables:

$$x \to x + a_1p^+, \quad y \to y - p^+, \quad z \to z + p^+, \tag{6}$$

and parameters $a_2 = a_1$, $b_2 = -p^+$, $c_2 = c_1 + a_1p^+$.

System (3) for arbitrary real parameters $a_3, b_3 \in \mathbb{R}$ ($a_3 \neq 0$), has the following equilibria:

$$O_3^+ = (0, 0, 0), \quad O_3^- = \big(0, \tfrac{a_3+b_3}{a_3}, -\tfrac{a_3+b_3}{a_3}\big). \tag{7}$$

Stability analysis of $O_3^{\pm}$ shows that if $0 < -a_3 < b_3$, then O_3^+ is locally stale, and if $0 < b_3 < -a_3 < 2b_3$, then O_3^- is locally stable.

For some values of parameters systems (1),(2),(3) exhibit chaotic behavior. To get a visualization of chaotic attractor one needs to choose an initial point in the basin of attraction of the attractor and observe how the trajectory, starting from this initial point, after a transient process visualizes the attractor. An attractor is called a *self-excited attractor* if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise, it is called a *hidden attractor* [Leonov et al., 2011, Leonov and Kuznetsov, 2013, Leonov et al., 2015, Kuznetsov, 2016a]. It was discovered numerically by Rössler that in the phase space of system (1) with parameters $a_1 = 0.2$, $b_1 = 0.2$, $c_1 = 5.7$ and system (3) with parameters $a_3 = 0.386$, $b_3 = 0.2$ there exist *chaotic attractors* of different shapes, which are self-excited with respect to both equilibria $O_{1,3}^{\pm}$, respectively.

One of the building blocks of chaotic attractor are embedded unstable periodic orbits (UPOs) (see e.g. [Afraımovic et al., 1977, Auerbach et al., 1987, Cvitanović, 1991]). A proof of the existence of UPOs for discrete and continuous-time dynamical systems can be done, for example, using a special analytical-numerical technique [Galias, 1999, 2006b, Galias and Tucker, 2008, Barrio et al., 2015].

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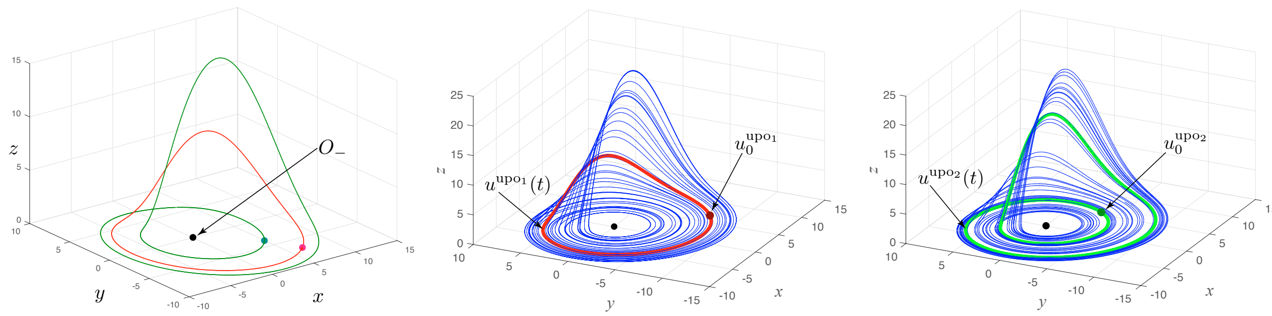


(a) (b) (c)

Fig. 1. The UPO u^{upo_1} with period $\tau_1 = 5.8811$ (red) and the UPO u^{upo_2} with period $\tau_2 = 11.7586$ (green) in system (1) with parameters $a_1 = 0.2$, $b_1 = 0.2$, $c_1 = 5.7$, stabilized using TDFC method.

Since for system (1) (and, also, (2)) corresponding results about the existence of low-periodic UPOs were obtained in [Galias, 2006a], further in this paper we will restrict ourselves to the case of system (1).

One of the effective methods[1] for the numerical visualization of the UPOs is the *time-delay feedback control* (TDFC) approach, suggested by K. Pyragas [Pyragas, 1992]. The main idea behind the Pyragas' approach is to stabilize UPOs by constructing a control force proportional to the difference between the current state of the system and an earlier state of the system (delayed by some time interval).

From the mathematical perspective, the Pyragas method is as follows. Let $u^{\text{upo}}(t)$ be an UPO with period $\tau > 0$, $u^{\text{upo}}(t-\tau) = u^{\text{upo}}(t)$, satisfying a differential equation $\dot{u} = f(u)$. To compute the UPO, we add the TDFC:

$$\dot{u} = f(u) + KB\,C^*\big[u(t-T) - u(t)\big], \tag{8}$$

where B, C are column vectors, $K > 0$ is a feedback gain and operator $*$ denotes transposition. If $T = \tau$, then $KB\,C^*\big[u(t-T) - u(t)\big] = 0$ along the UPO, and the periodic solution of system (8) coincides with the periodic solution of system the $\dot{u} = f(u)$. There exist various modifications of the TDFC that allow to calculate the values of feedback gain and period adaptively, within the stabilization procedure [Chen and Yu, 1999, Cruz-Villar, 2007, Lin et al., 2010, Lehnert et al., 2011, Pyragas and Pyragas, 2011, 2013]. Remark that TDFC has some limitations (see e.g. [Hooton and Amann, 2012, Kuznetsov et al., 2015]) which nevertheless do not occur in the Rössler system.

For system (1) with parameters $a_1 = 0.2$, $b_1 = 0.2$, $c_1 = 5.7$, we solved time-delayed differential equation (8) with $B^* = C^* = (0,1,0)$ and for different values of feedback gain K stabilized two UPOs, $u^{\text{upo}_{1,2}}$, by the trajectories on the time interval $t \in [0, 1000]$ with initial data $u_0 = (1,1,1)$ on $[0, -\tau_{1,2}]$. Using the feedback gain $K = 0.3$ the trajectory tends to the UPO u^{upo_1} with period $\tau_1 = 5.8811$[2], using the feedback gain $K = 0.2$ the trajectory tends to the UPO u^{upo_2} with period $\tau_2 = 11.7586$ (see Fig. 1), wherein $u^{\text{upo}_{1,2}}$ are also the solutions of the initial system (1) (i.e. a system without the control). Here the values of period $\tau_{1,2}$ can be found e.g. using the algorithm by Lin et al. [2010].

Then for the initial points u_0^{upo}, chosen on the UPO $u_1^{\text{upo}} = \big\{u^{\text{upo}_1}(t),\, t \in [0,\tau_1]\big\}$, either $u_2^{\text{upo}} = \big\{u^{\text{upo}_2}(t),\, t \in [0,\tau_2]\big\}$, we numerically compute the trajectory $\tilde{u}(t, u_0^{\text{upo}})$ of system (8) without the stabilization (i.e. with $K = 0$) on sufficiently large time interval $[0, T = 500]$ (see Fig. 1b,1c). One can see that on the initial small time interval $[0, T_1 \approx 60]$, even without the control, the obtained trajectory $\tilde{u}(t, u_0^{\text{upo}})$ traces approximately the "true" periodic orbit $u^{\text{upo}}(t, u_0^{\text{upo}})$. But for $t > T_1$ without control the trajectory $\tilde{u}(t, u_0^{\text{upo}})$ diverge from u^{upo} and wind on the attractor $\mathcal{A}$[3].

For systems (1),(2),(3) it was numerically verified [Kuznetsov et al., 2014] the conjecture stating that the maximum of the Lyapunov dimension on attractor is reached at an equilibrium (a special case of the Eden's conjecture [Eden, 1989] which claims that *for an arbitrary attractor its maximum of the Lyapunov dimension is reached at an equilibrium or on a unstable periodic orbit.* In general, a *conjecture on the Lyapunov dimension of self-excited attractor* [Kuznetsov, 2016b, Kuznetsov et al., 2018] is that for a typical system the Lyapunov dimension of a self-excited attractor does not exceed the Lyapunov dimension of one of unstable equilibria, the unstable manifold of which intersects with the basin of attraction and visualize the attractor. In this article we are going to verify numerically the Eden's conjecture for system (1)

[1] The first known approach to control chaos and detect the UPOs was the so-called OGY method suggested by Ott, Grebogi, Yorke [Ott et al., 1990]. The idea behind the method was to use a small time-dependent perturbation in the form of feedback to an accessible system parameter.

[2] Periodic orbits in a continuous-time system often regarded as period-k orbits in accordance with the smallest positive number k of distinct points $\{u_j = \Pi^j(u_0) \mid j = 0, \dots, k-1\}$ with $u_0 = \Pi^k(u_0)$ on a certain Poincaré map Π.

[3] Rigorous analysis of the time interval choices for reliable numerical computation of trajectories for the Lorenz system leads to the following results [Kehlet and Logg, 2013, Liao and Wang, 2014, Kehlet and Logg, 2017]. The time interval for reliable computation with 16 significant digits and error 10^{-4} is estimated as $[0, 36]$, with error 10^{-8} is estimated as $[0, 26]$, and reliable computation for a longer time interval, e.g. $[0, 10000]$ in [Liao and Wang, 2014], is a challenging task that requires significant increase of the precision of the floating-point representation and the use of supercomputers. Analytical aspects of this problem are concerned with the so-called shadowing theory (see e.g. [Pilyugin, 2011]) which for some classes of systems can guarantee the existence of a "true" trajectory in the vicinity of its approximation.

(and (2)) using the procedures for estimation of finite-time Lyapunov dimension.

2. LYAPUNOV DIMENSION, EDEN CONJECTURE AND TOPOLOGICAL ENTROPY

Below we follow the concept of the *finite-time Lyapunov dimension*, which is convenient for carrying out numerical experiments with finite time. The *finite-time local Lyapunov dimension* [Kuznetsov, 2016b, Kuznetsov et al., 2018] can be defined via an analog of the *Kaplan-Yorke formula* with respect to the set of finite-time Lyapunov exponents:

$$\dim_{\mathrm{L}}(t,u) = d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(t,u)\}_{i=1}^{3}) = j(t,u) + \frac{\mathrm{LE}_1(t,u)+\cdots+\mathrm{LE}_{j(t,u)}(t,u)}{|\mathrm{LE}_{j(t,u)+1}(t,u)|}, \quad (9)$$

where $j(t,u) = \max\{m : \sum_{i=1}^{m} \mathrm{LE}_i(t,u) \geq 0\}$. Then the *finite-time Lyapunov dimension* (of dynamical system generated by (1) on compact invariant set $\mathcal{A}$) is defined as

$$\dim_{\mathrm{L}}(t,\mathcal{A}) = \sup_{u\in\mathcal{A}} \dim_{\mathrm{L}}(t,u). \quad (10)$$

The *Douady–Oesterlé theorem* [Douady and Oesterle, 1980] implies that for any fixed $t > 0$ the finite-time Lyapunov dimension, defined by (10), is an upper estimate of the Hausdorff dimension: $\dim_{\mathrm{H}} \mathcal{A} \leq \dim_{\mathrm{L}}(t,\mathcal{A})$. The best estimation is called the *Lyapunov dimension* [Kuznetsov, 2016b]:

$$\dim_{\mathrm{L}} \mathcal{A} = \inf_{t>0} \sup_{u\in\mathcal{A}} \dim_{\mathrm{L}}(t,u) = \liminf_{t\to+\infty} \sup_{u\in\mathcal{A}} \dim_{\mathrm{L}}(t,u).$$

The notion of Lyapunov dimension is closely connected (see e.g. [Boichenko et al., 2005]) with another important characteristic of dynamical and control systems which is called *topological entropy*. Topological entropy was proposed in [Adler et al., 1965] as an analog of Kolmogorov-Sinai entropy that can be introduced without consideration of invariant measures. It plays an important role in the development of large-scale control systems in which the control tasks are distributed among numerous processors via a communication network [Matveev and Pogromsky, 2017]. As the size of such systems grows, limitations caused by the network finite capacity becomes irremovable via the design of control algorithms. Here, topological entropy comes to the fore, since according to the fundamental data rate theorem [Nair et al., 2007] the rate at which the channel is capable of reliable data communication should exceed the topological entropy of the open-loop system.

It is known that Rössler system (1) is non-dissipative in the sense that it does not possess a bounded convex absorbing set [Leonov and Reitmann, 1986] containing a global attractor. Nevertheless, for parameters $a_1 = 0.2$, $b_1 = 0.2$, $c_1 = 5.7$ in numerical experiments it is possible to observe an invariant attracting set $\mathcal{A}$ which could be localized numerically within a cuboid $\mathcal{C}$ (Fig. 2). On the set $\mathcal{A}$ we define a dynamical system $\{\varphi^t\}_{t\geq 0}$, generated by equations (1). Here $\varphi^t\big((x_0,y_0,z_0)\big)$ is a solution of (1) with the initial data (x_0,y_0,z_0).

In Fig. 2 is shown the grid of points $\mathcal{C}_{\mathrm{grid}}$ filling the attractor: the grid of points fills cuboid $\mathcal{C} = [-9.5,12] \times [-11,8] \times [0,23]$ with the distance between points equals to 0.5. The time interval is $[0, T = 500]$, $k = 5000$, $\tau = 0.1$, and the integration method is MATLAB ode45 with predefined parameters. The infimum on the time interval is computed at the points $\{t_k\}_1^N$ with time step $\tau = t_{i+1} - t_i = 0.5$. Note that if for a certain time $t = t_k$ the computed trajectory is out of the cuboid, the corresponding value of finite-time local Lyapunov dimension is not taken into account in the computation of maximum of the finite-time local Lyapunov dimension (e.g. there are trajectories with initial data in cuboid, which tend to infinity).

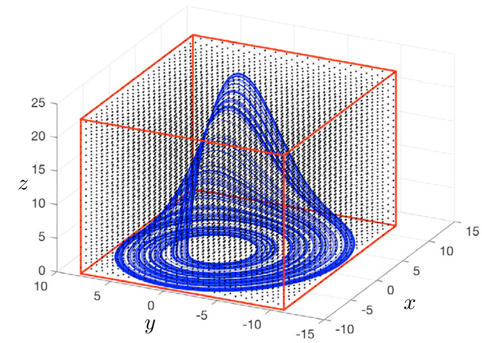


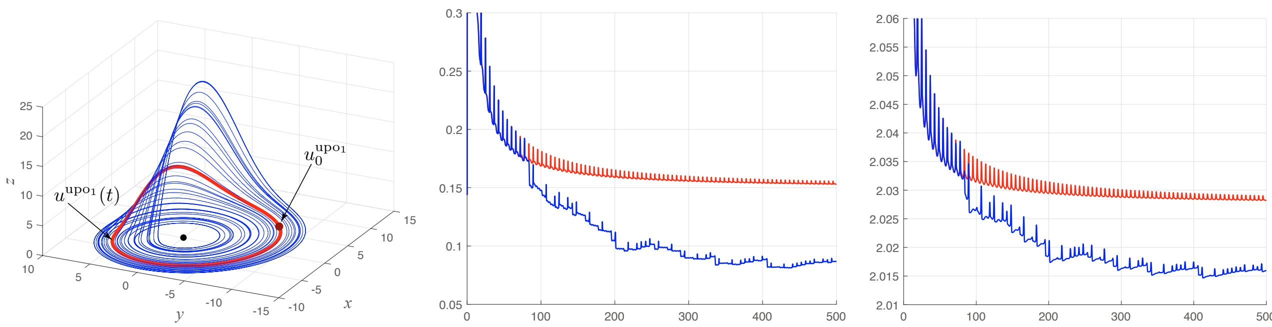
Fig. 2. Localization of the chaotic attractor of system (1) with parameters $a_1 = 0.2$, $b_1 = 0.2$, $c_1 = 5.7$ by the cuboid $\mathcal{C} = [-9.5,12] \times [-11,8] \times [0,23]$ (red) and the corresponding grid of points $\mathcal{C}_{\mathrm{grid}}$ (black).

For the considered set of parameters we use MATLAB realization [4] of *the adaptive algorithm of finite-time Lyapunov dimension and Lyapunov exponents computation* [Kuznetsov et al., 2018] and obtain the following values:

(i) maximum of the finite-time local Lyapunov dimensions at the points of grid, $\max_{u\in\mathcal{C}_{\mathrm{grid}}} \dim_{\mathrm{L}}(t,u)$, at the time points $t = t_k = 0.1\,k$ $(k = 1,..,5000)$;
(ii) finite-time Lyapunov dimensions $\dim_{\mathrm{L}}(500,\cdot)$ for the stabilized UPOs with periods $\tau_1 = 5.8811$ and $\tau_2 = 11.7586$;
(iii) approximate value of topological entropy which is defined by the maximum of the finite-time local topological entropies around the points of grid: $\max_{u\in\mathcal{C}_{\mathrm{grid}}} H_{\mathrm{loc}}(500,u)$;
(iv) the values of finite-time local local topological entropy on the two UPOs: $H_{\mathrm{loc}}\big(500, u^{\mathrm{upo}_{1,2}}\big)$.

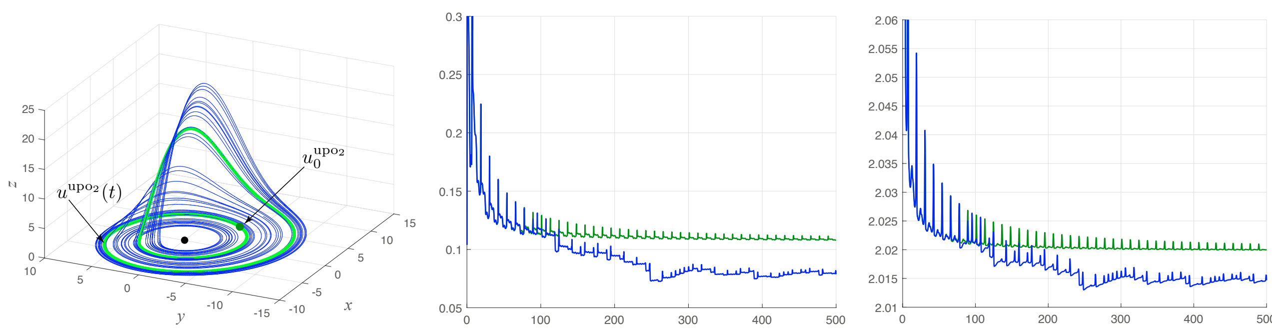
For numerical computation of finite-time Lyapunov dimension for the UPO $u^{\mathrm{upo}} = \big\{u^{\mathrm{upo}}(t), t\in[0,\tau]\big\}$ we choose an initial point $u_0^{\mathrm{upo}} \in u^{\mathrm{upo}}$ and apply adaptive algorithm togehter with Pyragas control to keep the computation along the $u^{\mathrm{upo}}(t)$. Starting from the same point u_0^{upo} on UPO we also compute finite-time Lyapunov dimension along the trajectory but without stabilization (i.e. when $K = 0$ in (8)) which stops to trace the UPO and starts to wind on the chaotic attractor. The corresponding results obtained for two stabilized UPOs, $u^{\mathrm{upo}_1} = \big\{u^{\mathrm{upo}_1}(t), t\in[0,\tau_1]\big\}$ and $u^{\mathrm{upo}_2} = \big\{u^{\mathrm{upo}_2}(t), t\in[0,\tau_2]\big\}$, with periods $\tau_1 = 5.8811$ and $\tau_2 = 11.7586$, respectively, are presented in Fig. 3 and Fig. 4. The results are given in Table 1.

[4] Various realizations of algorithms that compute finite-time Lyapunov exponents and Lyapunov dimension for discontinuous, piecewise continuous and fractional order systems can be found e.g. in [Danca, 2015, 2018, Danca and Kuznetsov, 2018, Danca et al., 2018].



(a) (b) (c)

Fig. 3. Numerical computation of $\mathrm{LE}_1(t, u_0^{\mathrm{upo}_1})$ and $\dim_{\mathrm{L}}(t, u_0^{\mathrm{upo}_1}) = d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(t, u_0^{\mathrm{upo}_1})\}_1^3)$ for the time interval $t \in [0, 500]$ along the period-1 UPOs $u^{\mathrm{upo}_1}(t)$ (red) and along the trajectory integrated without stabilization (blue). Each trajectory starts from the point $u_0^{\mathrm{upo}_1} = (6.491, -7.0078, 0.1155)$ (dark red).



(a) (b) (c)

Fig. 4. Numerical computation of $\mathrm{LE}_1(t, u_0^{\mathrm{upo}_2})$ and $\dim_{\mathrm{L}}(t, u_0^{\mathrm{upo}_2}) = d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(t, u_0^{\mathrm{upo}_2})\}_1^3)$ for the time interval $t \in [0, 500]$ along the period-2 UPOs $u^{\mathrm{upo}_2}(t)$ (green) and along the trajectory integrated without stabilization (blue). Each trajectory starts from the point $u_0^{\mathrm{upo}_2} = (5.3914, -3.2889, 0.1099)$ (dark green).

Table 1. Approximation of the finite-time Lyapunov dimensions and entropy.

	$t = 60$	$t = 500$
$\max_{u \in \mathcal{C}_{\mathrm{grid}}} \dim_{\mathrm{L}}(t, u)$	2.0209	2.0160
$\dim_{\mathrm{L}}(t, u_0^{\mathrm{upo}_2})$	2.0227	2.0200
$\dim_{\mathrm{L}}(t, u_0^{\mathrm{upo}_1})$	2.0406	2.0283
$\max_{u \in \mathcal{C}_{\mathrm{grid}}} H_{\mathrm{loc}}(t, u)$	0.1754	0.1250
$H_{\mathrm{loc}}(t, u^{\mathrm{upo}_2})$	0.1756	0.1571
$H_{\mathrm{loc}}(t, u^{\mathrm{upo}_1})$	0.3126	0.2219

The comparison of the obtained values of $\mathrm{LE}_1(t, u_0^{\mathrm{upo}})$ and $\dim_{\mathrm{L}}(t, u_0^{\mathrm{upo}}) = d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(t, u_0^{\mathrm{upo}})\}_1^3)$ computed along the stabilized UPO and the trajectory without stabilization gives us the following results [5]. On the initial part of the time interval, one can indicate the coincidence of these values with a sufficiently high accuracy. For the period-1 UPO and for the unstabilized trajectory the largest Lyapunov exponents $\mathrm{LE}_1(t, u_0^{\mathrm{upo}_1})$ coincide up to the 5th decimal place inclusive on the interval $[0, T_{\mathrm{match}}^1 \approx 5\tau_1]$, up to the 4th decimal place inclusive on the interval $[0, T_{\mathrm{match}}^2 \approx 9\tau_1]$, up to the 3th decimal place inclusive on the interval $[0, T_{\mathrm{match}}^3 \approx 12\tau_1]$. After $t > T_{\mathrm{match}}^3$ the difference in values becomes significant and the corresponding graphics diverge in such a way that the part of the graph corresponding to the unstabilized trajectory is lower than the part of the graph corresponding to the UPO (see Fig. 3b). The same situation is observed for the period-2 UPO, but the obtained time intervals are smaller: the corresponding values of the largest Lyapunov exponents $\mathrm{LE}_1(t, u_0^{\mathrm{upo}_2})$ coincide up to the 5th decimal place inclusive on the interval $[0, T_{\mathrm{match}}^1 \approx 1.1\tau_2]$, up to the 4th decimal place inclusive on the interval $[0, T_{\mathrm{match}}^2 \approx 3.6\tau_2]$, up to the 3th decimal place inclusive on the interval $[0, T_{\mathrm{match}}^3 \approx 5.6\tau_2]$.

[5] The idea to compare finite-time Lyapunov dimensions of trajectories computed along UPOs with and without Pyragas control was discussed in [Kuznetsov and Mokaev, 2019].

For the considered values of parameters $a_1 = 0.2$, $b_1 = 0.2$, $c_1 = 5.7$ the Jacobian at the equilibria $O_1^{\pm}$ has the following simple eigenvalues

$$\lambda_1(O_1^+) = 0.1930, \qquad \lambda_{2,3}(O_1^+) = -4.5 \cdot 10^{-6} \pm 5.428\mathrm{i},$$
$$\lambda_{1,2}(O_1^-) = 0.0970 \pm 0.9952\mathrm{i}, \qquad \lambda_3(O_1^-) = -5.6870.$$

Therefore, we get

$$\begin{aligned} \dim_{\mathrm{L}} O_1^+ &= d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{Re}\lambda_i(O_1^+)\}_{i=1}^3) = 3, \\ \dim_{\mathrm{L}} O_1^- &= d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{Re}\lambda_i(O_1^-)\}_{i=1}^3) = 2.0341. \end{aligned} \tag{11}$$

Let us mention here previous results on dimension of Rössler attractor. In [Peitgen et al., 2004, p. 644] it is stated that the fractal dimension is between 2.01 and 2.02, in [Fuchs, 2013, p. 80] it is stated that the correlation dimension is equal to 2.01. In literature we found the following values for the Lyapunov dimension: 2.014 [Froehling et al., 1981], 2.01 [Sano and Sawada, 1985], 2.0132 [Sprott, 2003, 2007, Fuchs, 2013], and 2.09635 [Awrejcewicz et al., 2018].

Using the formula of the local topological entropy of a system around an equilibrium point: $H_{\text{loc}}(u_{\text{eq}}) = \frac{1}{\ln 2}\sum_{j=1}^{3}\max\{\text{Re}[\lambda_j(u_{\text{eq}})], 0\}$, we get the the following values of the local topological entropy of system (1) around $O_1^{\pm}$: $H_{\text{loc}}(O_1^+) = 0.2784,\ H_{\text{loc}}(O_1^-) = 0.2799$.

3. CONCLUSION

In this note we have confirmed the Eden conjecture for the Rössler system (1) [6] and obtained the following relations between the Lyapunov dimensions: $3 = \dim_{\text{L}} O_1^+ > 2.0341 = \dim_{\text{L}} O_1^- > 2.0283 = \dim_{\text{L}}(500, u^{\text{upo}_1}) > 2.02 = \dim_{\text{L}}(500, u^{\text{upo}_2}) > 2.0160 = \max_{u\in\mathcal{C}_{\text{grid}}} \dim_{\text{L}}(500, u)$ and various values of topological entropy: $0.2799 = H_{\text{loc}}(O_1^-) > 0.2784 = H_{\text{loc}}(O_1^+) >$
$0.2219 = H_{\text{loc}}(500, u^{\text{upo}_1}) > 0.1571 = H_{\text{loc}}(500, u^{\text{upo}_2}) >$
$0.1250 = \max_{u\in\mathcal{C}_{\text{grid}}} H_{\text{loc}}(500, u)$.

Then for any invariant set or attractor contaning period-1 UPO: $\mathcal{A} \supset u^{\text{upo}_1}$, we have the following lower-bound estimates for the Lyapunov dimension

$$\dim_{\text{L}} \mathcal{A} \geq 2.0283 \approx \dim_{\text{L}}(u^{\text{upo}_1}),$$

and the topological entropy:

$$H(\mathcal{A}) \geq 0.2219 \approx H_{\text{loc}}(u^{\text{upo}_1}).$$

The above numerical values are close to the exact values computed via multipliers of $u^{\text{upo}_{1,2}}$. For the upper-bound estimates one can use special analytical methods (see, e.g. [Leonov, 1991, Boichenko and Leonov, 1998, Kuznetsov, 2016b]).

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[6] For system (3) with $a_3 = 0.386$, $b_3 = 0.2$ for period-1 and period-2 UPOs $u^{\text{upo}_{1,2}}$ corresponding local Lyapunov dimensions are: $\dim_{\text{L}} u^{\text{upo}_1} = 2.316$, $\dim_{\text{L}} u^{\text{upo}_2} = 2.241$, and local topological entropies are: $H_{\text{loc}}(u^{\text{upo}_1}) = 0.134$, $H_{\text{loc}}(u^{\text{upo}_2}) = 0.092$. Similar results can be obtained for the Lorenz system.

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