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Pauls rectifiable and purely Pauls unrectifiable smooth hypersurfaces

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ABSTRACT

This paper is related to the problem of finding a good notion of rectifiability in sub-Riemannian geometry. In particular, we study which kind of results can be expected for smooth hypersurfaces in Carnot groups. Our main contribution will be a consequence of the following result: there exists a C^∞-hypersurface S without characteristic points that has uncountably many pairwise non-isomorphic tangent groups on every positive-measure subset. The example is found in a Carnot group of topological dimension 8, it has Hausdorff dimension 12 and so we use on it the Hausdorff measure $\mathcal{H}^{12}$. As a consequence, we show that any Lipschitz map defined on a subset of a Carnot group of Hausdorff dimension 12, with values in S, has negligible image with respect to the Hausdorff measure $\mathcal{H}^{12}$. In particular, we deduce that S cannot be Lipschitz parametrizable by countably many maps each defined on some subset of some Carnot group of Hausdorff dimension 12. As main consequence we have that a notion of rectifiability proposed by S. Pauls is not equivalent to one proposed by B. Franchi, R. Serapioni and F. Serra Cassano, at least for arbitrary Carnot groups. In addition, we show that, given a subset U of a homogeneous subgroup of Hausdorff dimension 12 of a Carnot group, every bi-Lipschitz map $f : U \to S$ satisfies $\mathcal{H}^{12}(f(U)) = 0$. Finally, we prove that such an example does not exist in Heisenberg groups: we prove that all C^∞-hypersurfaces in $\mathbb{H}^n$ with $n \geq 2$ are countably $\mathbb{H}^{n-1} \times \mathbb{R}$-rectifiable according to Pauls' definition, even with bi-Lipschitz maps.

1. Introduction

State of the art: Measure-theoretic notions of rectifiability in Carnot groups have been deeply studied in the last 20 years. At the end of the 90s it was understood by the work of Ambrosio and Kirchheim [1] that

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the classical notion of k-rectifiability in metric spaces was not the correct one in this setting. Recall that a set is k-rectifiable if it is covered by a countable union of Lipschitz images of subsets of $\mathbb{R}^k$, up to a set that is negligible with respect to the Hausdorff measure $\mathcal{H}^k$. Indeed, in [1] (see also more general results in [39]) the authors showed that the first Heisenberg group $\mathbb{H}^1$ is purely k-unrectifiable, which is stronger than being not k-rectifiable, for $k \geq 2$. Then, in [21], in the setting of Heisenberg groups $\mathbb{H}^n$, Franchi, Serapioni and Serra Cassano proposed a definition, named C^1_{H}-hypersurface, that was meant to mimic the notion of regular hypersurface in the Euclidean setting. Indeed, a C^1_{H}-hypersurface is locally the zero-level set of a C^1_{H}-function with non-vanishing intrinsic gradient. In the same paper the authors proposed a definition of rectifiability for codimension-one sets in the setting of Heisenberg groups: a set in $\mathbb{H}^n$ is *codimension-one C^1_{H}-rectifiable* if it can be covered by countably many C^1_{H}-hypersurfaces up to a $\mathcal{H}^{2n+1}$-negligible set. With this definition, they proved that the reduced boundary of a set of finite perimeter is codimension-one C^1_{H}-rectifiable thus showing that this could be a good notion of rectifiability at least for Heisenberg groups. The same definitions and results were soon generalized to Carnot groups of step 2 in [19].

After the mentioned works, the class of C^1_{H}-submanifolds has been intensively studied, also in general codimensions and beyond the Heisenberg setting. For example, in [22] the authors provide an implicit function theorem for C^1_{H}-functions in the setting of Carnot groups (see also [10]), showing also that a C^1_{H}-hypersurface is locally the boundary of a set of finite perimeter. In [24] a systematic study of low dimensional and low codimensional C^1_{H}-submanifolds in $\mathbb{H}^n$ has been performed and a general definition of *k-dimensional C^1_{H}-rectifiability* has been given.

At the same time the problem of the regularity of the parametrization of C^1_{H}-submanifolds was widely studied: in [23] the authors proposed the definitions of intrinsic Lipschitz function and intrinsic differentiable function in the setting of Heisenberg groups. In that reference the authors proved that a C^1_{H}-submanifold is locally the graph of an intrinsic Lipschitz function — see also [16, Theorem A.5]. Thus, still in [23], the authors proposed, in the setting of Heisenberg groups and for any dimension k, a definition of rectifiability *a priori* more general than k-dimensional C^1_{H}-rectifiability, by using coverings with graphs of intrinsic Lipschitz functions. Later, in [25], the authors generalized the notion of intrinsic Lipschitz function and intrinsic differentiable function in arbitrary Carnot groups. They also showed that a Rademacher-type theorem for intrinsic Lipschitz functions defined on codimension-one subgroups holds in $\mathbb{H}^n$. As a consequence, they proved the equivalence between the two codimension-one definitions of rectifiability, i.e., the codimension-one C^1_{H}-rectifiability and the one with covering by means of graphs of intrinsic Lipschitz functions. The Rademacher-type theorem and the equivalence of the two notions of codimension-one rectifiability have been extended to a larger class of Carnot groups in [18], but not yet to all Carnot groups.

A comprehensive presentation of intrinsic Lipschitz functions is contained in [20]. We point out that earlier studies of this notion were also contained in [2] in the setting of Heisenberg groups $\mathbb{H}^n$. In this reference it is showed that a C^1_{H}-hypersurface is locally the graph of a uniformly intrinsic differentiable function that solves a *Burger-type* equation. Generalizations of this result are contained in [3] in the setting of C^1_{H}-submanifolds in $\mathbb{H}^n$, in [14] for C^1_{H}-hypersurfaces in Carnot groups of step 2 and in [13] for C^1_{H}-submanifolds in Carnot groups of step 2. Further studies of metric properties of intrinsic Lipschitz graphs are contained also in [11].

At the beginning of 2000 Pauls proposed a different notion of rectifiability in [47, Definition 4.1]. According to his definition, given $\mathbb{G}$ a Carnot group of Hausdorff dimension Q, a subset E of another Carnot group is $\mathbb{G}$-rectifiable if it can be covered $\mathcal{H}^Q$-a.e. by countably many Lipschitz images of subsets of $\mathbb{G}$. The relation between the two notions of differentiability – namely Pauls' one and the one(s) by Franchi, Serapioni and Serra Cassano – is far from being well understood. Notice that in [12, Definition 3] the authors propose another definition of rectifiability in which they allow $\mathbb{G}$ of the previous definition to be a homogeneous subgroup of a Carnot group.

The query that has been left open is whether a k-dimensional C^1_{H}-submanifold in a Carnot group is Lipschitz (or better bi-Lipschitz) parametrizable by subsets of k-dimensional homogeneous subgroups of a Carnot group. One positive result in this direction has been obtained in [12] in which the authors proved that any C^1-hypersurface in $\mathbb{H}^1$ is N-rectifiable, where N is a vertical plane in $\mathbb{H}^1$ and the maps used for the parametrization are even defined on open sets. Then this result was improved by Bigolin and Vittone in [6] showing that any non-characteristic point of a C^1-hypersurface in $\mathbb{H}^1$ admits a neighbourhood U and a bi-Lipschitz chart between an open subset of N and U. In [6] the authors also provided a partial negative answer to the query: they showed the existence of a C^1_{H}-hypersurface in $\mathbb{H}^1$ that has a point with no bi-Lipschitz map from an open subset of N and any of its neighbourhoods.

As far as we know, apart from these results, there are no general positive answers in the direction of parametrizing an arbitrary C^1_{H}-hypersurface in a Carnot group — either with Lipschitz or bi-Lipschitz maps defined on measurable subsets of Carnot groups with the same dimension of the hypersurface. However recently, in a slightly different direction, Le Donne and Young in [37] proved that a sub-Riemannian manifold with constant Gromov–Hausdorff tangents $\mathbb{G}$, is countably $\mathbb{G}$-rectifiable, where $\mathbb{G}$ is a Carnot group. This result gives a possible way to show that smooth hypersurfaces in Carnot groups – sufficiently smooth in order to carry a sub-Riemannian structure – are $\mathbb{G}$-rectifiable for some $\mathbb{G}$. This is exactly what we do in the second part of this paper with smooth non-characteristic hypersurfaces in $\mathbb{H}^n$ with $n \geq 2$. We add that, using some ideas coming from the theory of quantitative differentiability, Orponen recently showed in [45] that any $C^{1,\alpha}_{\mathrm{H}}$-hypersurface with $\alpha > 0$ in $\mathbb{H}^1$ is Lipschitz parametrizable with subsets of a vertical plane N.

We point out that very recently A. Merlo started the study of rectifiability of measures in Carnot groups, from a Geometric Measure Theoretic point of view. In particular he proposed the definition of $\mathcal{P}$-rectifiable and $\mathcal{P}^*$-rectifiable measures. The former are Radon measures with strictly positive lower density and finite upper density such that at almost every point the blow-ups are flat, **but supported on the same homogeneous subgroup**. The latter are Radon measures with the same assumptions on the density, such that at almost every point the blow-ups are flat. In [43] the author obtains a Marstrand–Mattila type theorem in Carnot groups. The author proves that $\mathcal{P}^*$-rectifiable measures of codimension one are supported on a countable union of C^1_{H}-hypersurfaces. Moreover, with this result, and the one obtained in [42], the author concludes Preiss' theorem in $\mathbb{H}^n$. We warmly thank A. Merlo for having shared with us a manuscript of [43].

Results: The paper is essentially divided into two parts: in the first one we provide a *negative* result and in the second one we provide a *positive* result.

In the first part we show the following (see Corollary 5.9):

Theorem 1.1. *There exist a Carnot group $\mathbb{G}$ and a C^∞ non-characteristic hypersurface $S \subseteq \mathbb{G}$ that is not Pauls Carnot rectifiable, see Definition 3.5.*

Let us remark that the hypersurface S we construct to show Theorem 1.1 is actually **algebraic**, thus analytic, and then much more than C^∞. See Remark 5.3.

Pauls Carnot rectifiability is just a generalization of Pauls rectifiability defined in [47, Definition 4.1] in which we allow countably many Carnot groups, see Definition 3.5. Our result shows that even very regular objects, such as smooth non-characteristic hypersurfaces, which for sure are rectifiable according to Franchi, Serapioni and Serra Cassano, are not Pauls Carnot rectifiable.

Later on in the paper, we show that such an example does not exist in the setting of $\mathbb{H}^n$ with $n \geq 2$ (see Theorem 6.15 and Remark 6.16 for a more exhaustive statement):

Theorem 1.2. *Let S be a C^∞-hypersurface in the nth Heisenberg group $\mathbb{H}^n$ with $n \geq 2$. Then S is $\mathbb{H}^{n-1} \times \mathbb{R}$-rectifiable according to Pauls' definition of rectifiability [12, Definition 3], even with bi-Lipschitz maps.*

Comments and ideas of the proofs: To prove Theorem 1.1, whose proof is in Section 5, we will show the existence of a C^∞-hypersurface S - of Hausdorff dimension 12 in a Carnot group of topological dimension

8 - that cannot be $\mathcal{H}^{12}$-a.e. covered by countably many Lipschitz images of subsets of Carnot groups of Hausdorff dimension 12. Notice that we also allow the Carnot groups to vary, thus using a more general definition of rectifiability with respect to the one given in [47, Definition 4.1].

We will actually show a more general property for S: for every Carnot group $\mathbb{G}$ of Hausdorff dimension 12, every Lipschitz map $f : U \subseteq \mathbb{G} \to S$ satisfies $\mathcal{H}^{12}(f(U)) = 0$ (see Theorem 5.7). We will call this property *purely Pauls Carnot unrectifiability* (Definition 3.5), which implies that S is not Pauls Carnot rectifiable, see Remark 3.8. The key property for the proof of the previous result is that every $\mathcal{H}^{12}$-positive subset of S has uncountably many points with pairwise non-isomorphic Carnot groups as tangents, see the statement and the proof of Theorems 5.4, and 5.12.

The idea to build such a hypersurface is the following: at first, in Proposition 4.5, we show the existence of a Carnot algebra $\mathfrak{g}$ of dimension 8 that has uncountably many pairwise non-isomorphic Carnot subalgebras of dimension 7. This is done by exploiting the existence of an uncountable family $\mathcal{F}$ of Carnot algebras of dimension 7 that are known to be pairwise non-isomorphic, see [27]. Notice that 7 is the minimal dimension for which this fact holds. Indeed, there are, up to isomorphisms, only finitely many Carnot algebras of dimension ≤ 6, see again [27]. Then we construct examples of smooth non-characteristic hypersurfaces S in the Carnot group whose Lie algebra is $\mathfrak{g}$, with the property that the tangent spaces of S form an uncountable subfamily of $\mathcal{F}$. With a particular choice of S, see Remark 5.3, we show that every $\mathcal{H}^{12}$-positive subset of S has uncountably many points with pairwise non-isomorphic Carnot groups as tangents.

Having in our hands the pathological example S, we prove our main result, see Theorem 5.7. We do it via a blow up analysis and using the area formula for Lipschitz maps between Carnot groups proved by Magnani in [38].

We point out that we also construct, in every Carnot group $\mathbb{G}$, a smooth non-characteristic hypersurface that has every subgroup of codimension-one of $\mathbb{G}$ as tangent, see Lemma 2.22.

We also prove a variant of Theorem 1.1. Namely, we show in Corollary 5.5 that our example S is not *bi-Lipschitz homogeneous rectifiable* (Definition 3.2). More precisely, it is impossible to $\mathcal{H}^{12}$-a.e. cover S by countably many bi-Lipschitz images of subsets of metric spaces of Hausdorff dimension 12 that have bi-Lipschitz equivalent tangents. Actually, again, we prove more: we show that S is *purely bi-Lipschitz homogeneous unrectifiable* according to Definition 3.2, after having provided a general criterion for *purely bi-Lipschitz homogeneous unrectifiability* (Lemma 3.4).

Notice that, from this last result, it follows that S is not rectifiable according to the countable bi-Lipschitz variant of the definition given in [12, Definition 3], that is, the one that allows the parametrizing spaces to be homogeneous subgroups of Carnot groups, see also Remark 3.3. Nevertheless, we are still not able to prove that our counterexample is not rectifiable according to [12, Definition 3], see Remark 5.6.

We remark here that, from how we are going to construct the example S, it follows that any tangent to S is a Carnot group. Consequently, together with the previously discussed results, we immediately deduce that S is also an example of metric space that cannot be Lipschitz parametrized by countably many of its tangents, see Remark 5.11.

In Remark 5.14 we observe that S has a structure of sub-Riemannian manifold and if we consider on it the sub-Riemannian distance it is still purely bi-Lipschitz homogeneous unrectifiable.

We remark that the notions of rectifiability that we study are just particular cases of a very general definition. Given a metric space (X, d), we introduce the notion of $(\mathcal{F}, \mu)$-rectifiability, where $\mathcal{F}$ is a family of metric spaces and μ is an outer measure on X.

Definition 1.3 (*$(\mathcal{F}, \mu)$-rectifiability*)**.** Given a family $\mathcal{F}$ of metric spaces we say that a metric space (X, d), with an outer measure μ on it, is *$(\mathcal{F}, \mu)$-rectifiable* if there exist countably many bi-Lipschitz embeddings $f_i : U_i \subseteq (X_i, d_i) \to (X, d)$ where $(X_i, d_i) \in \mathcal{F}$, $i \in \mathbb{N}$, and

$$\mu\left(X \setminus \bigcup_{i\in\mathbb{N}} f_i(U_i)\right) = 0.$$

We say that a metric space (X,d) is *purely* $(\mathcal{F},\mu)$*-unrectifiable* if for every $(X',d')\in\mathcal{F}$ and every bi-Lipschitz embedding $f:U\subseteq(X',d')\to(X,d)$ it holds

$$\mu(f(U))=0.$$

To prove Theorem 1.2, we will use [49, Theorem 1.1], [11, Proposition 3.8], and [37, Theorem 2]. The proof is contained in Section 6.

The idea is the following: first we show that every smooth non-characteristic hypersurface S in $\mathbb{H}^n$, with $n\geq 2$, carries a structure of polarized manifold (Proposition 6.14). Indeed, we show that the intersection of the horizontal bundle of $\mathbb{H}^n$ with the tangent bundle of S is a step-2 bracket generating distribution (Proposition 6.11). This was already known from [49, Theorem 1.1], but we give a different proof based on simple explicit computations.

Before going on, let us notice that Proposition 6.14 is very likely to hold for $C^{1,1}$ non-characteristic hypersurfaces. The reason for which we stated it in the C^∞-category is merely technical. Indeed, Proposition 6.11 is stated for C^2 non-characteristic hypersurfaces, but its proof can be adapted to work in the $C^{1,1}$ case. Moreover, in the proof of Proposition 6.14, we use the fundamental results in [44], and [5] (see also [30]), which require C^∞-regularity, but can be very likely adapted to $C^{1,1}$-regularity in our case. The serious difficult point seems to pass from this $C^{1,1}$-regularity to C^1_{H}, that would probably require a completely different argument.

In order to conclude the proof we show that every sub-Riemannian structure on the polarized manifolds S gives rise to a distance that is locally bi-Lipschitz equivalent to the distance on S seen as subset of $\mathbb{H}^n$ (Proposition 6.12). We will call these distances the *intrinsic distance* and the *induced distance*, respectively. The equivalence is due to the general fact that in $\mathbb{H}^n$, with $n\geq 2$, the intrinsic distance and the induced distance on the graph of an intrinsic Lipschitz function are equivalent (Proposition 6.10). This tells us also that in Proposition 6.12, we are merely using the fact that S is locally the graph of an intrinsic Lipschitz function.

The proof of Proposition 6.10 result was suggested to us by Fässler and Orponen, and it is reminiscent of the result already known from [11, Proposition 3.8]. Eventually we use the fundamental tool [37, Theorem 2] and the key fact that the tangents to the hypersurface are all isomorphic to $\mathbb{H}^{n-1}\times\mathbb{R}$ (Lemma 6.13). With these three steps we conclude the proof of Theorem 1.2.

Structure of the paper: The structure of the paper is the following: in Section 2 we collect general definitions and tools that are useful for our aims. We there collect general definitions about Carnot groups and metric measure spaces; we recall the area formula for Lipschitz maps between Carnot groups; we revise some basic definitions and statements about C^1_{H}-hypersurfaces, showing in particular that a C^1_{H}-hypersurface has Hausdorff dimension $(Q-1)$ – Q being the Hausdorff dimension of the group in which it lives – and $\mathcal{H}^{Q-1}$ is a locally doubling measure on it (Proposition 2.14). We also stress that the tangent group to a C^1_{H}-hypersurface is the Hausdorff tangent, which is a fact due to [31, Theorem 3.1.1] (see Proposition 2.15).

In Section 3 we give different notions of rectifiability, such as bi-Lipschitz homogeneous rectifiability (Definition 3.2) and Pauls Carnot rectifiability (Definition 3.5), the latter one being more general than Pauls rectifiability [47, Definition 4.1]. Namely, a metric space of Hausdorff dimension k is *Pauls Carnot rectifiable* if it is $\mathcal{H}^k$-a.e. covered by countably many Lipschitz images of subsets of Carnot groups of Hausdorff dimension k; a metric space of Hausdorff dimension k is *bi-Lipschitz homogeneous rectifiable* if it is $\mathcal{H}^k$-a.e. covered by countably many bi-Lipschitz images of subsets of metric spaces of Hausdorff dimension k that have bi-Lipschitz equivalent tangents. We also give the notions of purely bi-Lipschitz homogeneous unrectifiability (Definition 3.2) and purely Pauls Carnot unrectifiability (Definition 3.5), which are stronger version (see Remarks 3.1 and 3.8) of not being bi-Lipschitz homogeneous rectifiable and not being Pauls Carnot rectifiable, respectively. In Lemma 3.4 we provide a criterion for a metric space to be purely bi-Lipschitz homogeneous unrectifiable. In Remark 3.3 we discuss some definitions of rectifiability that are less general than Definition 3.2, while in Remark 3.7 we briefly discuss some Lipschitz counterpart to Definition 3.2.

In Section 4 we construct a Carnot algebra of dimension 8 that has uncountably many pairwise non-isomorphic Carnot subalgebras of dimension 7. The construction is done in Proposition 4.5.

In Sections 5 and 6 we show the main theorems we discussed above. Namely, we first prove the main result Theorem 1.1, see Corollary 5.9, and the variant we discussed above, see Corollary 5.5. Secondly we prove Theorem 1.2, see Theorem 6.15.

2. Preliminaries

2.1. Some standard definitions

For definitions and theory about Carnot groups one can see [7,34] and [19, Section 2]. We recall here some basic facts and terminology.

A Carnot group is a simply connected nilpotent Lie group whose Lie algebra is stratified and generated by the first stratum. If $\mathbb{G}$ is a Carnot group and $\mathfrak{g}$ is its Lie algebra we thus have

$$\mathfrak{g} = V_1 \oplus \ldots \oplus V_s,$$

with $V_{i+1} = [V_1, V_i]$ for every $1 \leq i \leq s-1$, $V_s \neq \{0\}$ and $[V_1, V_s] = \{0\}$. The number s is called *step* of the group $\mathbb{G}$. The dimension of the first stratum V_1 is denoted by m and the dimension of $\mathfrak{g}$ by n. The identity element of $\mathbb{G}$ is denoted by e.

For a Carnot group the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism. Thus by means of this map, after a choice of a basis of $\mathfrak{g}$, we can identify $\mathbb{G}$ with $\mathbb{R}^n$ with an operation $\cdot$ that can be explicitly written by making use of the Baker–Campbell–Hausdorff formula. We will use exponential coordinates

$$(x_1, \ldots, x_n) \mapsto \exp(x_1 X_1 + \cdots + x_n X_n),$$

where $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$ adapted to the stratification. Then $\{X_1, \ldots, X_m\}$ is a basis of V_1. With a little abuse of notation we will indicate with $X_i \in \mathfrak{g}$ both a tangent vector at the identity element of $\mathbb{G}$ and the left-invariant vector field on $\mathbb{G}$ that agrees with it at the identity. We call $\{X_1, \ldots, X_m\}$ a *basis of the horizontal space* V_1.

On the Lie algebra $\mathfrak{g}$ we have a family of linear maps δ_λ that act as

$$\delta_\lambda(v_i) = \lambda^i v_i, \qquad \text{if} \quad v_i \in V_i.$$

With an abuse of notation, we denote by δ_λ the group endomorphism on $\mathbb{G}$ with differential δ_λ. Namely, by means of the exponential map we have $\delta_\lambda := \exp \circ \delta_\lambda \circ \exp^{-1}$ on $\mathbb{G}$ as well. We call a homomorphism $\varphi : \mathbb{G} \to \mathbb{H}$ between two Carnot groups a *Carnot homomorphism* if

$$\varphi \circ \delta_\lambda = \delta_\lambda \circ \varphi, \qquad \forall \lambda > 0.$$

Remark 2.1. Every Carnot homomorphism induces a linear map $\varphi_* : \mathfrak{g} \to \mathfrak{h}$, which is a Lie algebra homomorphism, such that $\varphi_* \circ \delta_\lambda = \delta_\lambda \circ \varphi_*$. From this property it easily follows that for every $1 \leq i \leq s$ we get $\varphi_*(V_i) \subseteq V_i^{\mathfrak{h}}$, where V_i and $V_i^{\mathfrak{h}}$ are the ith strata of $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

A *left-invariant homogeneous distance* $d : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_{\geq 0}$ - sometimes we call it $d_{\mathbb{G}}$ - is a distance on $\mathbb{G}$ satisfying

$$d(hg_1, hg_2) = d(g_1, g_2), \qquad d(\delta_\lambda g_1, \delta_\lambda g_2) = \lambda d(g_1, g_2), \qquad \forall h, g_1, g_2 \in \mathbb{G}, \qquad \forall \lambda > 0. \tag{2.1}$$

It follows from [34, Proposition 3.5] that such a distance is continuous with respect to the manifold topology on $\mathbb{G}$. Given d a left-invariant homogeneous distance, we can define a homogeneous norm associated to d as

$$\|g\|_d := d(e, g). \tag{2.2}$$

There is a distinguished class of left-invariant homogeneous distances, known as *Carnot-Carathéodory distances*. If we fix a norm $\|\cdot\|$ on the first stratum V_1 of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, we can extend it left-invariantly to the horizontal bundle

$$\mathbb{V}_1(x) := (L_x)_* V_1,$$

for $x \in \mathbb{G}$, where L_x is the left translation by x. We say that an absolutely continuous curve $\gamma : [0,1] \to \mathbb{G}$ is *horizontal* if

$$\gamma'(t) \in \mathbb{V}_1(\gamma(t)), \qquad \text{for a.e.} \quad t \in [0,1].$$

We define

$$d_{cc}^{\|\cdot\|}(x,y) := \inf \left\{ \int_I \|\gamma'(t)\| : \quad \gamma(0) = x, \quad \gamma(1) = y, \quad \gamma \quad \text{horizontal} \right\}.$$

The Chow–Rashevsky theorem states that this distance is finite. It is clearly homogeneous and left-invariant.

It is clear that any two homogeneous left-invariant distances d_1 and d_2 are equivalent: we write $d_1 \sim d_2$ and we mean that there exists $C \geq 1$ such that $\frac{1}{C} d_1 \leq d_2 \leq C d_1$. Then, from now on, we do not specify what homogeneous left-invariant distance we are choosing as we prove results that are true up to bi-Lipschitz equivalence.

We remind that the Hausdorff dimension of a Carnot group $\mathbb{G}$ with respect to any left-invariant homogeneous distance is

$$Q := \sum_{i=1}^{s} i \dim V_i,$$

which we also call *homogeneous dimension*.

We recall now some definitions about metric spaces and metric measure spaces. Given a metric space (X,d), we indicate the open ball in the metric d centred at x of radius r with $B^d(x,r)$. When the distance is clear we just write $B(x,r)$. The closed ball is indicated with $\overline{B}^d(x,r)$. We say that (X, d_X) is *bi-Lipschitz equivalent* to (Y, d_Y) if there exists a bijective map $f : X \to Y$ such that

$$\frac{1}{C} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2).$$

Given a separable metric space (X,d) and μ a Borel regular measure on X that is finite on bounded sets, we say that μ is locally doubling if for each $a \in X$ there exists $R_a > 0$ and $C_a > 0$ such that

$$0 < \mu(B^d(x,2r)) \leq C_a \mu(B^d(x,r)) < +\infty, \qquad \forall x \in B^d(a, R_a), \qquad \forall 0 \leq r \leq R_a.$$

In this case we say that (X,d,μ) is a locally doubling metric measure space.

For a locally compact locally doubling metric measure space, as a consequence of the Gromov compactness theorem, we can say that for every $x \in X$, the set of Gromov–Hausdorff tangents $\mathrm{Tan}(X,d,x)$ is nonempty. Indeed, we can say that for every sequence of positive numbers $\lambda_i \to 0$, up to subsequences it holds

$$(X, \lambda_i^{-1} d, x) \to (X_\infty, d_\infty, x_\infty)$$

in the pointed Gromov–Hausdorff convergence. For general definitions and theory about (pointed) Gromov–Hausdorff convergence one can see [50, Chapter 27] and [8, Chapters 7, 8].

We indicate with $\mathcal{H}_d^k$ the Hausdorff measure of dimension k associated to d and with $\mathcal{S}_d^k$ the *spherical* Hausdorff measure of dimension k associated to d. When the distance will be clear, we will omit the subscript d in the previously defined measures. We denote with $\dim_H X$ the Hausdorff dimension of the metric space X. For these general definitions see [17, 2.10.2]. We indicate with d_H the Hausdorff distance between sets, see [17, 2.10.21].

2.2. Area formula for Lipschitz functions between Carnot groups

We shall recall the area formula for Lipschitz maps in Carnot groups which is due to Magnani. First we recall Rademacher theorem in this setting, which is due to Pansu. The following statement is in [38, Theorem 3.9].

Theorem 2.2 (*Pansu, Magnani*). *Let $\mathbb{G}$ and $\mathbb{H}$ be two Carnot groups. Let us call Q the homogeneous dimension of $\mathbb{G}$. Then any Lipschitz map $f : A \subseteq (\mathbb{G}, d_{\mathbb{G}}) \to (\mathbb{H}, d_{\mathbb{H}})$, where A is a measurable set, is differentiable $\mathcal{H}^Q$-a.e., i.e., there exists, at $\mathcal{H}^Q$-a.e. point x of A, a Carnot homomorphism $Df_x : \mathbb{G} \to \mathbb{H}$ such that*

$$\lim_{y\in A, y\to x} \frac{d_{\mathbb{H}}(f(x)^{-1}f(y), Df_x(x^{-1}y))}{d_{\mathbb{G}}(x,y)} = 0. \tag{2.3}$$

Remark 2.3. We discuss here how Df_x is defined. From [38, Step 1 and Step 2 of Theorem 3.9], and [38, Equation (3) in Step 1 of Theorem 3.9], it follows that

$$Df_x(z) := \lim_{x\delta_t z\in A, t\to 0} \delta_{1/t}\left(f(x)^{-1}f(x\delta_t z)\right)$$

does exist for every x in a $\mathcal{H}^Q$-full measure set $A_\omega \subseteq A$, and every z in a countable dense subset of $\mathbb{G}$. Then, from [38, Step 2 of Theorem 3.9], for $x \in A_\omega$, the map Df_x can be extended to all $z \in \mathbb{G}$, by density.

Definition 2.4 (*Jacobian of a Lipschitz Map*)**.** Given any Lipschitz map $f : A \subseteq (\mathbb{G}, d_{\mathbb{G}}) \to (\mathbb{H}, d_{\mathbb{H}})$ we can define the *Jacobian*

$$J_Q(Df_x) := \frac{\mathcal{H}^Q(Df_x(B(0,1)))}{\mathcal{H}^Q(B(0,1))},$$

at any differentiability point x of f.

The following result is proved in [38, Theorem 4.4].

Theorem 2.5 (*Magnani*). *Given any Lipschitz map $f : A \subseteq (\mathbb{G}, d_{\mathbb{G}}) \to (\mathbb{H}, d_{\mathbb{H}})$, where A is a measurable set, we have*

$$\int_A J_Q(Df_x)\,\mathrm{d}\,\mathcal{H}^Q(x) = \int_{\mathbb{H}} \sharp(f^{-1}(y)\cap A)\,\mathrm{d}\,\mathcal{H}^Q(y).$$

2.3. Parametrizations of a C^1_{H}-hypersurface and its Hausdorff dimension

We give here some definitions about hypersurfaces in Carnot groups. One of our references is the summary given in [14, Section 2] and references therein, such as [18,20,21].

Definition 2.6 (C^1_{H}*-hypersurfaces*)**.** In a Carnot group $\mathbb{G}$, with $\{X_1, \dots, X_m\}$ as basis of horizontal space, a subset S is a C^1_{H}-*hypersurface* if for all $p \in S$ there exists a neighbourhood U of p and a continuous function $f : U \to \mathbb{R}$ with $X_1 f, \dots, X_m f$ continuous in U, such that

$$S \cap U = \{x \in U : f(x) = 0\}, \tag{2.4}$$

and $(X_1 f, \dots, X_m f)$ does not vanish on U.

Definition 2.7 (*Characteristic Points*)**.** Let $\mathbb{G}$ be a Carnot group. Given S a C^1-hypersurface in $\mathbb{G} \equiv \mathbb{R}^n$, we say that $x \in S$ is a *characteristic point* for S if

$$\mathbb{V}_1(x) \subseteq T_x S, \tag{2.5}$$

where $\mathbb{V}_1(x)$ is the horizontal bundle at x and T_xS is the "Euclidean tangent" of S, i.e., the tangent space of S seen as submanifold of $\mathbb{G} \equiv \mathbb{R}^n$. We shall use the term "Euclidean" in contrast with the intrinsic sub-Riemannian one.

We will say that a C^1-hypersurface S is *non-characteristic* if it does not have characteristic points as in (2.5).

Remark 2.8. We identify $\mathbb{G}$ with $\mathbb{R}^n$ by means of exponential coordinates and we call m the dimension of the first stratum of the Lie algebra. If we take $f \in C^1(\mathbb{G})$ we will denote with $\nabla f|_x$ the *full gradient* of f at x, i.e., the vector $\sum_{i=1}^n (\partial_{x_i} f)(x)\partial_{x_i}|_x$, and with $\nabla_{\mathbb{H}} f|_x$ the *horizontal gradient* of f at x, i.e., the vector $\sum_{i=1}^m (X_i f)(x) X_i|_x$.

If S is a C^1-hypersurface in $\mathbb{G}$, for every point $p \in S$ there exist an open neighbourhood U_p of it and $f \in C^1(U_p)$ such that

$$S \cap U_p = \{x \in U_p : f(x) = 0\}, \tag{2.6}$$

with $\nabla f \neq 0$ on $S \cap U_p$. The Euclidean tangent space of S at an arbitrary point $x \in S \cap U_p$ is

$$T_xS := \{v : \langle v, \nabla f|_x \rangle_x = 0\}, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle_x$ is the usual inner product, i.e., $\langle \partial_{x_i}|_x, \partial_{x_j}|_x \rangle_x = \delta_{ij}$, and $v = \sum_{i=1}^n v_i \partial_{x_i}|_x$. Then $x \in U_p$ is a characteristic point (2.5) if and only if (see (2.7)) it holds that $X_i f(x) = 0$ for all $i = 1, \ldots, m$.

Thus a C^1-hypersurface S in $\mathbb{G}$ with non-characteristic points is C^1_{H}, because we have the representation in (2.6) with $(X_1 f, \ldots, X_m f) \neq 0$ on U_p.

For C^1_{H}-hypersurfaces we have a notion of tangent group [19, page 14]. In the following definition we are identifying the group $\mathbb{G}$ with $\mathbb{R}^n$ by means of the exponential coordinates and we denote by $\{X_1, \ldots, X_m\}$ a basis of the horizontal space.

Definition 2.9 (*Tangent Group to a C^1_{H}-hypersurface*)**.** Given S a C^1_{H}-hypersurface, a point $p \in S$ and a representative f around p as in (2.4), we can define the *tangent group*, or the *intrinsic tangent* of S at p, as

$$T_p^{\mathrm{I}}S := \left\{ v \in \mathbb{G} \equiv \mathbb{R}^n : \sum_{i=1}^m v_i X_i f(p) = 0 \right\}.$$

Remark 2.10. The tangent group defined in Definition 2.9 is a subgroup of $\mathbb{G}$ and does not depend on the representative f given in Definition 2.6 [19, page 14]. Indeed, we have, see Section 2.4 for details, that

$$T_p^{\mathrm{I}}S = \lim_{\lambda \to 0} \delta_{\lambda^{-1}}(p^{-1} \cdot S).$$

We deal now with the problem of the parametrization of a C^1_{H}-hypersurface. We say that a subgroup $\mathbb{W}$ of $\mathbb{G}$ is a *homogeneous subgroup* if

$$\delta_\lambda w \in \mathbb{W}, \qquad \forall w \in \mathbb{W}, \qquad \forall \lambda > 0.$$

We first introduce the notion of intrinsic Lipschitz function. See [18, Section 2].

Definition 2.11. Set $\mathbb{G}$ a Carnot group with identity e. If $\mathbb{W}$ and $\mathbb{H}$ are homogeneous subgroups of $\mathbb{G}$ such that $\mathbb{W} \cap \mathbb{H} = \{e\}$ and

$$\mathbb{G} = \mathbb{W} \cdot \mathbb{H},$$

we say that $\mathbb{W}$ and $\mathbb{H}$ are *complementary subgroups* in $\mathbb{G}$. We write $p = p_{\mathbb{W}} \cdot p_{\mathbb{H}}$, denoting with $p_{\mathbb{W}}$ and $p_{\mathbb{H}}$ the projections on the two subgroups.

If $\mathbb{W}$ and $\mathbb{H}$ are complementary subgroups in a Carnot group $\mathbb{G}$, then the *cone* $C_{\mathbb{W},\mathbb{H}}(q,\alpha)$ of base $\mathbb{W}$ and axis $\mathbb{H}$, centred at q and of opening $\alpha \geq 0$, is defined as

$$C_{\mathbb{W},\mathbb{H}}(q,\alpha) := q \cdot \{p \in \mathbb{G} : d(p_{\mathbb{W}}, e) \leq \alpha d(p_{\mathbb{H}}, e)\}.$$

Given $f : \Omega \subseteq \mathbb{W} \to \mathbb{H}$, we define the *graph* of f as

$$\mathrm{graph}(f) := \{w \cdot f(w) : w \in \Omega\} \subseteq \mathbb{G}.$$

Definition 2.12. Let us assume $\mathbb{W}$ and $\mathbb{H}$ are complementary subgroups in a Carnot group $\mathbb{G}$. Given $f : \Omega \subseteq \mathbb{W} \to \mathbb{H}$, with Ω open, we say that f is *L-intrinsic Lipschitz* in Ω, with $L > 0$, if

$$C_{\mathbb{W},\mathbb{H}}\left(p, \frac{1}{L}\right) \cap \mathrm{graph}(f) = \{p\}, \qquad \text{for all } p \in \mathrm{graph}(f),$$

where $C_{\mathbb{W},\mathbb{H}}\left(p, \frac{1}{L}\right)$ is defined in Definition 2.11.

Remark 2.13. It is not always true that an intrinsic Lipschitz function is Lipschitz in exponential coordinates. Nevertheless an arbitrary intrinsic Lipschitz function is locally Hölder continuous, see [18, Proposition 2.3.6].

Now we recall that the Hausdorff dimension of a C^1_{H}-hypersurface in a Carnot group of Hausdorff dimension Q is $Q-1$. This comes from an implicit function theorem (see e.g., [16, Theorem A.5], and [41, Theorem 1.4]), that allows to locally write the hypersurface as the graph of an intrinsic Lipschitz function. Thus, an estimate on the Hausdorff measure of the graph of an intrinsic Lipschitz function, see [18, Theorem 2.3.7], concludes the following proposition.

Proposition 2.14. *Let $\mathbb{G}$ be a Carnot group of Hausdorff dimension Q. Let S be a C^1_{H}-hypersurface of $\mathbb{G}$. Then the Hausdorff dimension of S is $Q-1$ and $\mathcal{H}^{Q-1}$ restricted to S is a locally doubling measure on S.*

2.4. The tangent group of a C^1_{H}-hypersurface as the Hausdorff tangent

In this subsection we remind that the tangent group $T^{\mathrm{I}}_p S$ at $p \in S$ of an arbitrary C^1_{H}-hypersurface (see Definition 2.9) is the Hausdorff tangent at p to S. This follows from [31, Theorem 3.1.1] and the identification between the kernel of the differential of a C^1_{H}-function f with the tangent group of the surface defined by f, see [19, Proposition 2.11].

The result that we state here is simplified for our aims. For the general statement we refer to [31, Theorem 3.1.1].

Proposition 2.15 (*Kozhevnikov*)**.** *Let S be a C^1_{H}-hypersurface in a Carnot group $\mathbb{G}$. Let us consider $p \in S$ and a function f whose 0-level set coincides locally with S, as in (2.4). Then we have that there exists $\beta : (0,+\infty) \to (0,+\infty)$, with $\beta(r) \to 0^+$ if $r \to 0^+$, such that for all $r > 0$*

$$d_H\left(B(p,r) \cap S, B(p,r) \cap (p \cdot T^{\mathrm{I}}_p S)\right) \leq \beta(r) r,$$

where $T^{\mathrm{I}}_p S$ is defined in Definition 2.9.

Remark 2.16. From the result in Proposition 2.15 we eventually get also that if we consider the metric space (S, d), the Gromov–Hausdorff tangent at any point $p \in S$ is (isometric to) $(T^{\mathrm{I}}_p S, d)$.

We give here part of the statement of [31, Theorem 3.3.1], because it is useful for our aims. For the complete theorem one can see the reference.

Definition 2.17 (*$\mathrm{Tan}^+_{\mathbb{G}}(S,a)$*)**.** Given a subset S of a Carnot group $\mathbb{G}$ and $a \in S$, we say that $v \in \mathbb{G}$ is an element of $\mathrm{Tan}^+_{\mathbb{G}}(S,a)$ if there exist a positive sequence $\{t_m\}$ with $t_m \to 0$ and a sequence $\{a_m\}$ of elements of S such that

$$\lim_{m\to+\infty} d(a_m, a) = 0, \qquad \lim_{m\to+\infty} \delta_{1/t_m}(a^{-1}\cdot a_m) = v.$$

Lemma 2.18 (*Kozhevnikov*)**.** *Let S be a closed set of a Carnot group $\mathbb{G}$, and let $a \in S$. If there exists a closed homogeneous set W such that*

$$r^{-1} d_H(B(a,r)\cap S, B(a,r)\cap(a\cdot W)) \to 0, \qquad \textit{when} \quad r \to 0,$$

then

$$\mathrm{Tan}^+_{\mathbb{G}}(S,a) \subseteq W.$$

2.5. Vertical surfaces

Now we give the definition of *vertical surface*. Loosely speaking, a vertical surface in a Carnot group $\mathbb{G}$ is a C^1-surface that depends only on the horizontal coordinates.

Definition 2.19. Let $\mathbb{G}$ be a Carnot group identified with $\mathbb{R}^n$ by means of exponential coordinates. Let m be the dimension of the first stratum of the Lie algebra. A *vertical surface* V is

$$V := \{x \in \Omega \times \mathbb{R}^{n-m} : f(x_1,\dots,x_m) = 0\},$$

where $f : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, with Ω open, is a C^1 function with $\nabla f \neq 0$ on the set $\{\omega \in \Omega : f(\omega_1,\dots,\omega_m) = 0\}$.

Moreover, if f is linear we say that V is a *vertical subgroup of codimension one*.

Remark 2.20. An arbitrary vertical surface as in Definition 2.19 is a C^1-hypersurface with no characteristic points, i.e., points that satisfy (2.5). This is due to the fact that, if $1 \le i \le m$, in exponential coordinates we have

$$X_i = \partial_{x_i} + r_i(x),$$

where $r_i(x)$ is a polynomial combination of $\partial_{x_{i+1}},\dots,\partial_{x_n}$, see [19, Proposition 2.2], and then, for all $x \in \omega$,

$$X_i f(x) = \partial_{x_i} f(x),$$

as f depends only on the first m variables. Thus, from Remark 2.8, a vertical surface is also a C^1_{H}-hypersurface.

Remark 2.21. Every tangent group, as defined in Definition 2.9, is a vertical subgroup of codimension one.

Lemma 2.22. *Given a Carnot group $\mathbb{G}$, there exists a vertical surface V such that for every vertical subgroup $\mathbb{W}$ of codimension one in $\mathbb{G}$ there exists $p \in V$ such that $T^{\mathrm{I}}_p V = \mathbb{W}$.*

Proof. Let us consider

$$V := \left\{ x \in \mathbb{G} \equiv \mathbb{R}^n : \sum_{i=1}^m x_i^2 = 1 \right\}.$$

At an arbitrary point $p = (x_1, \dots, x_m, x_{m+1} \dots, x_n)$,we have that, by Definition 2.9 and Remark 2.20,

$$T_p^{\mathrm{I}} V = \left\{ v \in \mathbb{G} \equiv \mathbb{R}^n : \sum_{i=1}^{m} v_i x_i = 0 \right\},$$

and then, as any linear function $f : \mathbb{R}^m \to \mathbb{R}$ can be written as $f(v) = \sum_{i=1}^{m} v_i x_i$ for a vector $(x_1, \dots, x_m)$ of norm 1, we get the desired conclusion. $\square$

3. Notions of rectifiability

In this section we introduce several kinds of notions of rectifiability, which are particular cases of Definition 1.3. In Definition 3.2, we specialize the notion of $(\mathcal{F}, \mu)$-rectifiability by taking $\mathcal{F}$ as the class of metric spaces that are locally compact, locally doubling and with bi-Lipschitz equivalent tangents. In Remark 3.3 we discuss further specializations of this notion.

Then we give the notion of Pauls Carnot rectifiability in Definition 3.5, generalizing the definition given in [47, Definition 4.1]. In Remark 3.7 we briefly discuss some Lipschitz variants of $(\mathcal{F}, \mu)$-rectifiability, for specific families $\mathcal{F}$.

We stress here that from now on **every metric space (X, d) will be separable**. We also remark that if (X, d) is locally complete we can equivalently ask each set U_i to be closed in Definition 1.3. Indeed, in this case every bi-Lipschitz map $f_i : U_i \to (X, d)$ extends, locally in the closure $\overline{U_i}$, to a bi-Lipschitz map. We will freely use this observation throughout the paper.

Remark 3.1. Having a look at Definition 1.3, assuming we have $\mu(X) > 0$, which will be always in our case, we see that one necessary condition for the $(\mathcal{F}, \mu)$-rectifiability of (X, d) is the existence of at least one bi-Lipschitz map $f : U \subseteq (X', d') \to (X, d)$, where $(X', d') \in \mathcal{F}$ and $\mu(f(U)) > 0$. So if a metric space (X, d), with an outer measure μ on it such that $\mu(X) > 0$, is $(\mathcal{F}, \mu)$-purely unrectifiable then it cannot be $(\mathcal{F}, \mu)$-rectifiable.

Specializing the family $\mathcal{F}$ and μ in Definition 1.3, we can give the following definitions.

Definition 3.2 (*bi-Lipschitz Homogeneous Rectifiability*)**.** Let (X, d) be a metric space of Hausdorff dimension k. Set $\mathcal{T}_k := \{(X_i, d_i)\}_{i \in I}$ to be the family of all the metric spaces (X_i, d_i) such that:

- $(X_i, d_i, \mathcal{H}^k)$ is a locally compact locally doubling metric measure space, with $k = \dim_H X_i$;
- any two tangent spaces, at any two points of X_i, are bi-Lipschitz equivalent.

We say that (X, d) is *bi-Lipschitz homogeneous rectifiable* if it is $(\mathcal{T}_k, \mathcal{H}^k)$-rectifiable according to Definition 1.3. We say that (X, d) is *purely bi-Lipschitz homogeneous unrectifiable* if it is purely $(\mathcal{T}_k, \mathcal{H}^k)$-unrectifiable according to Definition 1.3.

Remark 3.3. The family $\mathcal{T}_k$ defined in Definition 3.2 is very rich. For example it contains all homogeneous Lie groups $\mathbb{G}$ equipped with a left-invariant homogeneous distance $d_{\mathbb{G}}$, with Hausdorff dimension k. Indeed, by homogeneity, every tangent space at any point of such a group $\mathbb{G}$ is isometric to $(\mathbb{G}, d_{\mathbb{G}})$ and this metric space is locally compact and k-Ahlfors-regular [35, Theorem 4.4, (iii)], and then $\mathcal{H}^k$ is a doubling measure on it. We remark here that the larger class of self-similar metric Lie groups of Hausdorff dimension k, whose definition is in [35], is still a subclass of $\mathcal{T}_k$. Going beyond Lie groups, we remark that in $\mathcal{T}_k$ one has all those Carnot-Carathéodory spaces whose nilpotentization is constantly equal to a fixed Carnot group of homogeneous dimension k. This last statement is a consequence of Mitchell's theorem (see [44] and [5]) and the bi-Lipschitz equivalence of left-invariant homogeneous distances on Carnot groups.

In the very rich class of homogeneous Lie groups we distinguish homogeneous subgroups of Hausdorff dimension k of arbitrary Carnot groups, with the restricted distance, and obviously also Carnot groups of Hausdorff dimension k. We can then give different notions of rectifiability for each of these subfamilies of $\mathcal{T}_k$.

Notice that if we take the subfamily of $\mathcal{T}_k$ made of arbitrary homogeneous subgroups, of dimension k, of Carnot groups, we obtain a notion of rectifiability that is a variation of [12, Definition 3] where we now allow countably many homogeneous subgroups but we require bi-Lipschitz maps. Similarly, if we only consider Carnot groups, we obtain a similar variation of [47, Definition 4.1].

We give next a criterion for purely bi-Lipschitz homogeneous unrectifiability.

Lemma 3.4. *Let $(X, d, \mathcal{H}^k)$ be a locally compact locally doubling metric measure space, with $k = \dim_H X$. If every $\mathcal{H}^k$-positive measure subset of X contains two points that have two tangent spaces that are not bi-Lipschitz equivalent, then (X, d) is purely bi-Lipschitz homogeneous unrectifiable (according to Definition 3.2).*

Proof. We prove that there is no bi-Lipschitz map $f : U \subseteq (X', d') \to (X, d)$, where $\mathcal{H}^k(f(U)) > 0$ and $(X', d') \in \mathcal{T}_k$. As (X, d) is locally compact, we can restrict ourselves to consider U closed.

If there exists such a map, first of all notice that $\mathcal{H}^k(U) > 0$ because f is bi-Lipschitz. Now we can restrict ourselves to the points of density of U with respect to $\mathcal{H}^k$, say W, and W is a set of full $\mathcal{H}^k$-measure in U [48, Corollary 2.13, Theorem 3.1 and Remark 3.3]. Then, by the fact that f is bi-Lipschitz, the set $f(W)$ has full $\mathcal{H}^k$-measure in $f(U)$. The set Z of points in $f(W)$ of density of $f(U)$ with respect to $\mathcal{H}^k$, is still a set of full $\mathcal{H}^k$-measure in $f(U)$ because it is the intersection of two sets of full $\mathcal{H}^k$-measure in $f(U)$. Then it holds $\mathcal{H}^k(Z) > 0$ since $\mathcal{H}^k(f(U)) > 0$.

By hypothesis there exist two points $x, y \in W$ and $p = f(x), q = f(y) \in Z$ with two non-bi-Lipschitz tangent spaces T_p and T_q. Because of the fact that we are dealing with 1-density points, we can say that $\mathrm{Tan}\,(U, x, d') = \mathrm{Tan}\,(X', x, d')$ and $\mathrm{Tan}(f(U), p, d) = \mathrm{Tan}(X, p, d)$ and the same holds with y and q, see [32, Proposition 3.1]. Passing to the tangents in p and x we get, as in [37, Section 5.2], some induced bi-Lipschitz map between T_p and one element of $\mathrm{Tan}\,(X', x, d')$. In the same way we get a bi-Lipschitz map between T_q and one element of $\mathrm{Tan}\,(X', y, d')$. By hypothesis each element of $\mathrm{Tan}\,(X', x, d')$ is bi-Lipschitz equivalent to each element of $\mathrm{Tan}\,(X', y, d')$, so that at the end T_p is bi-Lipschitz equivalent to T_q, which is a contradiction. □

Let us point out that in Definition 1.3 we require the parametrizing maps to be bi-Lipschitz while for the classical definitions of rectifiability one may just ask for the map to be Lipschitz. We next give the Lipschitz counterpart to Definition 1.3 for the family of Carnot groups.

Definition 3.5 (*Pauls Carnot Rectifiability*)**.** Let (X, d) be a metric space of Hausdorff dimension k. We say that (X, d) is *Pauls Carnot rectifiable* if there exist countably many Carnot groups $\mathbb{G}_i$ of Hausdorff dimension k and Lipschitz maps $f_i : U_i \subseteq (\mathbb{G}_i, d_i) \to (X, d)$ such that

$$\mathcal{H}^k\left(X \setminus \bigcup_{i\in\mathbb{N}} f_i(U_i)\right) = 0.$$

We say that (X, d) is *purely Pauls Carnot unrectifiable* if for every Carnot group $\mathbb{G}$ of Hausdorff dimension k, every Lipschitz map $f : U \subseteq (\mathbb{G}, d_{\mathbb{G}}) \to (X, d)$ satisfies

$$\mathcal{H}^k(f(U)) = 0.$$

Remark 3.6. The definition given in Definition 3.5 is a generalization of [47, Definition 4.1] where it was considered only one Carnot group for the parametrization of X. The definition of *purely* $\mathbb{G}$*-unrectifiability*, with one Carnot group $\mathbb{G}$, was already given in [39, Definition 3.1]. That is, given a Carnot group $\mathbb{G}$ of Hausdorff dimension k, we say that a metric space (X, d) is *purely* $\mathbb{G}$*-unrectifiable* if every Lipschitz map $f : U \subseteq \mathbb{G} \to X$ satisfies $\mathcal{H}^k(f(U)) = 0$.

Remark 3.7. In this paper we will not focus on the Lipschitz counterpart to Definition 3.2. Restricting to the subfamily of $\mathcal{T}_k$ made of homogeneous subgroups of Carnot groups, such Lipschitz counterpart would lead to a variant of [12, Definition 3] allowing countably many possibly different subgroups. We think there are pathological examples and more easy-to-ask questions that we are not able to answer up to now.

For example Peano's curve tells that the Euclidean plane $\mathbb{R}^2$ can be Lipschitz rectified with $\left(\mathbb{R}, \|\cdot\|^{1/2}\right)$. Notice that $\left(\mathbb{R}, \|\cdot\|^{1/2}\right)$ is the vertical line in the Heisenberg group.[2]

Forcing the topological dimension to be the same, we also wonder, for example, whether there exists a Lipschitz map

$$f : U \subseteq \left(\mathbb{R}^3, \|\cdot\|^{3/4}\right) \to \mathbb{H}^1,$$

with $\mathcal{H}^4(f(U)) > 0$.

Remark 3.8. As in Remark 3.1, if (X, d) has Hausdorff dimension k and $\mathcal{H}^k(X) > 0$, it holds that if (X, d) is purely Pauls Carnot unrectifiable then it is not Pauls Carnot rectifiable.

4. A Carnot algebra with uncountably many non-isomorphic Carnot subalgebras

In this section we prove that there exists a Carnot algebra $\mathfrak{g}$ of dimension 8 that has uncountably many pairwise non-isomorphic Carnot subalgebras of dimension 7. The Lie algebra $\mathfrak{g}$ is constructed in Definition 4.3 and in Proposition 4.5 we prove the result.

Definition 4.1. Given $\mu \in \mathbb{R}$, we call $\mathfrak{g}_\mu$ the Carnot algebra of step 3 and dimension 7 given by

$$\mathfrak{g}_\mu := V_\mu^1 \oplus V_\mu^2 \oplus V_\mu^3,$$

where

$$V_\mu^1 := \operatorname{span}\{X_1, X_2, X_3\}, \quad V_\mu^2 := \operatorname{span}\{X_4, X_5, X_6\}, \quad V_\mu^3 := \operatorname{span}\{X_7\},$$

with the following relations

$$\begin{aligned} &[X_1, X_2] = X_4, && [X_1, X_3] = -X_6, && [X_2, X_3] = X_5; \\ &[X_1, X_5] = -X_7, && [X_2, X_6] = \mu X_7, && [X_3, X_4] = (1-\mu) X_7, \end{aligned} \tag{4.1}$$

where all the other commutators between two vectors of the basis $\{X_1, \ldots, X_7\}$ that are not listed above are zero.

Remark 4.2. The family $\{\mathfrak{g}_\mu\}_{\mu \in \mathbb{R}}$ in Definition 4.1 consists of uncountably many pairwise non-isomorphic Carnot algebras, which are called of type 147E, see [27]. Indeed, if $\mu_1, \mu_2 \notin \{0, 1\}$, the Lie algebra $\mathfrak{g}_{\mu_1}$ is isomorphic to $\mathfrak{g}_{\mu_2}$ if and only if $I(\mu_1) = I(\mu_2)$, where

$$I(\mu) := \frac{(1 - \mu + \mu^2)^3}{\mu^2(\mu - 1)^2}.$$

[2] We have evidence that every Carnot group of Hausdorff dimension Q that admits lattices can be Lipschitz rectified with $\left(\mathbb{R}, \|\cdot\|^{1/Q}\right)$, which is a subgroup of every Carnot group of step Q.

Our plan is to next add a direction X_0 in the first stratum of a specific Carnot algebra given by Definition 4.1, namely the one with $\mu = 0$. We show the existence of uncountably many pairwise non-isomorphic Carnot subalgebras of dimension 7 in this new Carnot algebra of dimension 8.

Definition 4.3. We call $\mathfrak{g}$ the Carnot algebra of step 3 and dimension 8 given by

$$\mathfrak{g} \coloneqq V^1 \oplus V^2 \oplus V^3,$$

where

$$V^1 \coloneqq \mathrm{span}\{X_0, X_1, X_2, X_3\}, \quad V^2 \coloneqq \mathrm{span}\{X_4, X_5, X_6\}, \quad V^3 \coloneqq \mathrm{span}\{X_7\},$$

with the following bracket relations

$$\begin{aligned} &[X_1, X_2] = X_4, \quad [X_1, X_3] = -X_6, \quad [X_1, X_0] = -X_4, \quad [X_2, X_3] = X_5; \\ &[X_1, X_5] = -X_7, \quad [X_3, X_4] = X_7, \quad [X_0, X_6] = X_7, \end{aligned} \tag{4.2}$$

and all the other commutators between two elements of the basis $\{X_0, X_1, \ldots, X_7\}$ that are not listed above are 0.

Remark 4.4. Let us show that the one defined in Definition 4.3 is a Lie algebra. It suffices to verify Jacobi identity on triples of pairwise different vectors of the basis. Since the step of the stratification is equal to 3, it suffices to show the Jacobi identity on vectors in the first stratum V^1. Then, as we are extending $\mathfrak{g}_0$ in Definition 4.1, we just have to check the Jacobi identity on the triples $\{X_1, X_2, X_0\}$, $\{X_2, X_3, X_0\}$ and $\{X_1, X_3, X_0\}$. A simple computation yields

$$\begin{aligned} &[X_1, [X_2, X_0]] + [X_2, [X_0, X_1]] + [X_0, [X_1, X_2]] = 0 + [X_2, X_4] + [X_0, X_4] = 0, \\ &[X_2, [X_3, X_0]] + [X_3, [X_0, X_2]] + [X_0, [X_2, X_3]] = 0 + 0 + [X_0, X_5] = 0, \\ &[X_1, [X_3, X_0]] + [X_3, [X_0, X_1]] + [X_0, [X_1, X_3]] = 0 + [X_3, X_4] - [X_0, X_6] = X_7 - X_7 = 0, \end{aligned} \tag{4.3}$$

which is what we want.

Now we are ready for the main proposition of this section.

Proposition 4.5. *If $\mathfrak{g}$ is the Carnot algebra of dimension 8 and step 3 in Definition 4.3, then there exist uncountably many Carnot subalgebras of dimension 7 of $\mathfrak{g}$ that are pairwise non-isomorphic.*

Proof. We present explicitly an uncountable family of Carnot subalgebras of dimension 7 of $\mathfrak{g}$, indexed by $\lambda \in \mathbb{R}$, which are isomorphic to $\mathfrak{g}_\lambda$ in Definition 4.1 if $\lambda \neq 1$. Then by Remark 4.2 we get the conclusion.

Given $\lambda \in \mathbb{R}$, with $\lambda \neq 1$, let us define the following vector in $V^1 \subseteq \mathfrak{g}$,

$$Y_2 \coloneqq X_2 + \lambda X_0. \tag{4.4}$$

Then $\{X_1, Y_2, X_3\}$ are linearly independent vectors of V^1. By explicit computations, using the relations in (4.2), we have

$$\begin{aligned} &[X_1, Y_2] = (1 - \lambda) X_4 \eqqcolon Y_4, \\ &[X_1, X_3] = -X_6, \\ &[Y_2, X_3] = X_5; \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&[X_1, Y_4] = 0,\\
&[X_1, X_5] = -X_7,\\
&[X_1, X_6] = 0,\\
&[Y_2, Y_4] = 0,\\
&[Y_2, X_5] = 0,\\
&[Y_2, X_6] = \lambda X_7,\\
&[X_3, Y_4] = (1-\lambda) X_7,\\
&[X_3, X_5] = 0,\\
&[X_3, X_6] = 0,
\end{aligned}
\tag{4.6}
$$

and all the other commutators between two elements of the linearly independent vectors $\{X_1, Y_2, X_3, Y_4, X_5, X_6, X_7\}$, that are not listed above, vanish. Then in view of (4.5) and (4.6), if $\lambda \neq 1$, the subspace $W^1 := \text{span}\{X_1, Y_2, X_3\}$ generates a Carnot subalgebra of step 3 and dimension 7 in $\mathfrak{g}$, which is isomorphic to $\mathfrak{g}_\lambda$ in Definition 4.1. ☐

5. Main results

In this section we construct the example that satisfies Theorem 1.1 and we prove the properties discussed in the introduction. We build the hypersurface S in the Carnot group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is as in Definition 4.3.

First of all let us identify $\mathbb{G}$ with $\mathbb{R}^8$ by using exponential coordinates and the ordered basis $(X_0, X_1, \ldots, X_7)$

$$
\begin{aligned}
&x = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \to\\
&\quad \to \exp\left(x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6 + x_7X_7\right).
\end{aligned}
\tag{5.1}
$$

In these coordinates we can express the left-invariant vector fields $X_0(x), X_1(x), X_2(x), X_3(x)$ that extend X_0, X_1, X_2, X_3, in this way, see [19, Proposition 2.2]:

$$
\begin{aligned}
X_0(x) &= \partial_{x_0} + r_0(x),\\
X_1(x) &= \partial_{x_1} + r_1(x),\\
X_2(x) &= \partial_{x_2} + r_2(x),\\
X_3(x) &= \partial_{x_3} + r_3(x),
\end{aligned}
\tag{5.2}
$$

where $r_0(x), r_1(x), r_2(x), r_3(x)$ are combinations, with polynomial coefficients of the coordinates, of $\partial_{x_4}, \partial_{x_5}, \partial_{x_6}, \partial_{x_7}$. Now we are ready to state and prove one of the main results of this article.

Proposition 5.1. *There exist a Carnot group $\mathbb{G}$ and a C^∞ non-characteristic hypersurface $S \subseteq \mathbb{G}$ with uncountably many pairwise non-isomorphic tangent groups.*

Proof. Let us consider the Carnot algebra $\mathfrak{g}$ in Definition 4.3 and $\mathbb{G} := \exp \mathfrak{g}$ identified with $\mathbb{R}^8$ by means of the exponential coordinates in (5.1).

Let us consider the family of vertical subgroups of codimension one in $\mathbb{G}$ given by

$$
\mathbb{G}_\lambda := \{v \in \mathbb{G} \equiv \mathbb{R}^8 : \lambda v_2 - v_0 = 0\}.
$$

The Lie algebra of $\mathbb{G}_\lambda$ is isomorphic to the algebra $\mathfrak{g}_\lambda$ if $\lambda \neq 1$ according to the proof of Proposition 4.5. Then the family $\{\mathbb{G}_\lambda\}_{\lambda \in \mathbb{R}}$ contains uncountably many non-isomorphic Carnot groups and the conclusion follows taking the vertical surface in $\mathbb{G}$ given by Lemma 2.22 that is smooth and non-characteristic due to Remark 2.20. ☐

Remark 5.2. In particular, every S as in Proposition 5.1 is not bi-Lipschitz equivalent to an open set in a Carnot group. This follows from a blow-up argument and Pansu's differentiability theorem [46]. The argument will be made clear in the proof of the forthcoming Theorem 5.4. We stress that even for some sub-Riemannian manifolds the constancy of the tangent may not give bi-Lipschitz local equivalence with the tangent Carnot group, see [36].

Remark 5.3. We give another example of smooth non-characteristic hypersurface satisfying Proposition 5.1. The particular form of this example will help us in showing the forthcoming Theorem 5.4 and Corollary 5.9. Let us consider again the Carnot algebra $\mathfrak{g}$ in Definition 4.3 and $\mathbb{G} := \exp \mathfrak{g}$ identified with $\mathbb{R}^8$ by means of the exponential coordinates in (5.1). Let us consider the vertical surface

$$S = \left\{ x \in \mathbb{G} \equiv \mathbb{R}^8 : f(x) := \frac{1}{3}x_2^3 + x_0 = 0 \right\}. \tag{5.3}$$

By Remark 2.20 this is a smooth non-characteristic hypersurface. From easy computations due to the particular form of X_i's in (5.2) and from the definition of tangent group in Definition 2.9, it follows that

$$\mathrm{Lie}(T_x^{\mathrm{I}}S) = \mathrm{span}\{X_1, X_2 - x_2^2 X_0, X_3, X_4, X_5, X_6, X_7\},$$

and then $\mathrm{Lie}(T_x^{\mathrm{I}}S)$ is isomorphic to the Carnot algebra generated by W^1 defined at the end of the proof of Proposition 4.5, where the λ there is now equal to $-x_2^2$. Then, the Lie algebra $\mathrm{Lie}(T_x^{\mathrm{I}}S)$ is isomorphic to $\mathfrak{g}_{-x_2^2}$ defined in Definition 4.1. Because of the fact that given any $\lambda \le 0$ there is always a point in S satisfying $\lambda = -x_2^2$, Remark 4.2 grants us that the family $\{\mathrm{Lie}(T_x^{\mathrm{I}}S)\}_{x\in S}$ contains uncountably many pairwise non-isomorphic Carnot algebras and then the family $\{T_x^{\mathrm{I}}S\}_{x\in S}$ contains uncountably many pairwise non-isomorphic Carnot groups.

Theorem 5.4. *There exist a Carnot group $\mathbb{G}$ and a C^∞ non-characteristic hypersurface $S \subseteq \mathbb{G}$, of Hausdorff dimension 12, such that on every $\mathcal{H}^{12}$-positive measure subset of it there are two points with non-isomorphic tangents. Moreover, the set S is purely bi-Lipschitz homogeneous unrectifiable according to Definition 3.2.*

Proof. Let us consider $\mathfrak{g}$ the Carnot algebra in Definition 4.3. Let us identify $\mathbb{G} := \exp \mathfrak{g}$ with $\mathbb{R}^8$ by means of the exponential coordinates in (5.1) and let us fix a left-invariant homogeneous distance d on $\mathbb{G}$. Let us consider S as in Remark 5.3. By Proposition 2.14, the definition of S, and the fact that the Hausdorff dimension of $\mathbb{G}$ is 13, we get that the Hausdorff dimension of S is 12 and $(S, d, \mathcal{H}^{12})$ is a locally compact locally doubling metric measure space. Notice also that by Proposition 2.15 (see also Remark 2.16) we get that the Gromov–Hausdorff tangent at each point $x \in S$ is unique and isometric to $(T_x^{\mathrm{I}}S, d)$. Notice that for all $x \in S$, the space $T_x^{\mathrm{I}}S$ is a Carnot group, as it is shown in the construction of S given in Remark 5.3.

We claim that

$$\mathcal{H}^{12}(S \cap \{x_2 = \lambda\}) = 0, \qquad \forall \lambda \in \mathbb{R}. \tag{5.4}$$

Indeed, we know that $S \cap \{x_2 = \lambda\}$ is the intersection of two C^1_{H}-hypersurfaces. Moreover the tangent subgroup to $S \cap \{x_2 = \lambda\}$ at an arbitrary point x is

$$\mathbb{W} := \{v \in \mathbb{G} \equiv \mathbb{R}^8 : x_2^2 v_2 + v_0 = 0\} \cap \{v \in \mathbb{G} \equiv \mathbb{R}^8 : v_2 = 0\} = \{v \in \mathbb{G} \equiv \mathbb{R}^8 : v_0 = v_2 = 0\}.$$

Since $\mathbb{W}$ is complemented by the horizontal subgroup $\mathbb{H} := \{\exp(tX_0 + sX_2) : t, s \in \mathbb{R}\}$, we can apply [16, Theorem A.5] to get that $S \cap \{x_2 = \lambda\}$ is locally the graph of an intrinsic Lipschitz function defined on $\mathbb{W}$ with values in $\mathbb{H}$. Notice that $\mathbb{H}$ is a subgroup because $[X_0, X_2] = 0$. By using [35, Theorem 4.4, (iii)] we get that $\mathbb{W}$ has Hausdorff dimension 11 with respect to the distance d and then by the estimate on the Hausdorff measure in [18, Theorem 2.3.7] we get (5.4).

Now we claim that each subset U of S that satisfies $\mathcal{H}^{12}(U) > 0$ has at least two points with two non-bi-Lipschitz Gromov–Hausdorff tangents. Indeed, Eq. (5.4) tells us that for each $U \subseteq S$ with $\mathcal{H}^{12}(U) > 0$, the coordinate function x_2 takes on U uncountably many values. This, according to the fact that $T_x^{\mathrm{I}} S$ is a Carnot group isomorphic to the one with Lie algebra $\mathfrak{g}_{-x_2^2}$ (see Remark 5.3), immediately tells that there are in U at least two points with two non-isomorphic (because of Remark 4.2) Carnot groups as tangent. By Pansu's theorem [46], two non-isomorphic Carnot groups cannot be bi-Lipschitz equivalent, so the claim follows. Now the proof is completed by using the criterion shown in Lemma 3.4. $\square$

From Remark 3.1 we have the following consequence to Theorem 5.4.

Corollary 5.5. *There exist a Carnot group $\mathbb{G}$ and a C^∞ non-characteristic hypersurface $S \subseteq \mathbb{G}$ that is not bi-Lipschitz homogeneous rectifiable according to Definition 3.2*

Remark 5.6. Notice that from Corollary 5.5 it follows that S is not rectifiable according to the countable bi-Lipschitz variant of [12, Definition 3], see Remark 3.3 for details. We notice here that we still are not able to prove that our counterexample is not rectifiable according to [12, Definition 3], see Remark 3.7 for further discussions. Nevertheless, in the forthcoming Theorem 5.7 we show that S is not rectifiable according to [47, Definition 4.1].

Theorem 5.7. *There exist a Carnot group $\mathbb{G}$ and a C^∞ non-characteristic hypersurface $S \subseteq \mathbb{G}$ that is purely Pauls Carnot unrectifiable according to Definition 3.5.*

Proof. Let us take S and $\mathbb{G}$ as in Remark 5.3. Let us fix on $\mathbb{G}$ a homogeneous left-invariant distance d. Then from Proposition 2.14 we get that the Hausdorff dimension of S is 12, because the Hausdorff dimension of $\mathbb{G}$ is 13. We will show there is no Lipschitz map $f : U \subseteq \hat{\mathbb{G}} \to (S, d)$, with $\hat{\mathbb{G}}$ a Carnot group, $\dim_H \hat{\mathbb{G}} = 12$ and $\mathcal{H}^{12}(f(U)) > 0$.

Suppose by contradiction there is such a map. We can assume U closed, because S is complete. By composing the map f with the inclusion $i : S \hookrightarrow \mathbb{G}$ we get a Lipschitz map $\tilde{f} : U \subseteq \hat{\mathbb{G}} \to \mathbb{G}$. We will make use of results and notation in Section 2.

Let us call $U_{\mathrm{ND}} \subseteq U$ the set of points where $\tilde{f}$ is non-differentiable, $U_{\mathrm{I}} \subseteq U$ the set of differentiability points x of $\tilde{f}$ for which $D\tilde{f}_x : \hat{\mathbb{G}} \to \mathbb{G}$ is injective and $U_{\mathrm{NI}} \subseteq U$ the set of differentiability points x of $\tilde{f}$ for which $D\tilde{f}_x$ is not injective. We thus have $U = U_{\mathrm{ND}} \sqcup U_{\mathrm{I}} \sqcup U_{\mathrm{NI}}$, $\tilde{f}(U) = \tilde{f}(U_{\mathrm{ND}}) \cup \tilde{f}(U_{\mathrm{I}}) \cup \tilde{f}(U_{\mathrm{NI}})$ and we know, from Rademacher theorem (Theorem 2.2) and the fact that $\tilde{f}$ is Lipschitz, that $\mathcal{H}^{12}(\tilde{f}(U_{\mathrm{ND}})) = \mathcal{H}^{12}(U_{\mathrm{ND}}) = 0$.

We claim that $\mathcal{H}^{12}(\tilde{f}(U_{\mathrm{I}})) > 0$. Indeed, the Hausdorff dimension of $\hat{\mathbb{G}}$ is 12. Thus, for $x \in U_{\mathrm{NI}}$, we get that $D\tilde{f}_x(\hat{\mathbb{G}})$ is a homogeneous subgroup of $\mathbb{G}$ of Hausdorff dimension at most 11, see Lemma 5.8. Then

$$J_{12}(D\tilde{f}_x) = \frac{\mathcal{H}^{12}(D\tilde{f}_x(B(0,1)))}{\mathcal{H}^{12}(B(0,1))} = 0,$$

and from Theorem 2.5 applied to $\tilde{f} : U_{\mathrm{NI}} \to \mathbb{G}$ we get $\mathcal{H}^{12}(\tilde{f}(U_{\mathrm{NI}})) = 0$. Now we conclude the proof of the claim:

$$\begin{aligned}\mathcal{H}^{12}(\tilde{f}(U_{\mathrm{I}})) &= \mathcal{H}^{12}(\tilde{f}(U_{\mathrm{I}})) + \mathcal{H}^{12}(\tilde{f}(U_{\mathrm{NI}})) + \mathcal{H}^{12}(\tilde{f}(U_{\mathrm{ND}})) \\ &\geq \mathcal{H}^{12}(\tilde{f}(U)) > 0.\end{aligned}$$

For every point z in U_{I} there exists an injective Carnot homomorphism $D\tilde{f}_z : \hat{\mathbb{G}} \to \mathbb{G}$. For how it is constructed the differential $D\tilde{f}_z$ (see Remark 2.3) we know that for ω in a dense subset Ω of $\hat{\mathbb{G}}$ we have

$$D\tilde{f}_z(\omega) = \lim_{z\delta_t\omega \in U, t \to 0} \delta_{1/t}\left(\tilde{f}(z)^{-1}\tilde{f}(z\delta_t\omega)\right).$$

From $\tilde{f}(z\delta_t\omega) \in S$ and $\tilde{f}(z\delta_t\omega) \to \tilde{f}(z)$ we thus get that $D\tilde{f}_z(\omega) \in \mathrm{Tan}^+_{\mathbb{G}}(S, \tilde{f}(z))$ (see Definition 2.17). Then thanks to Lemma 2.18, applied with $W = T^{\mathrm{I}}_{\tilde{f}(z)}S$ in view of Proposition 2.15, we get that $D\tilde{f}_z(\omega)$ takes values in $T^{\mathrm{I}}_{\tilde{f}(z)}S$ for $\omega \in \Omega$. Now taking into account that $D\tilde{f}_z$ is defined on all of $\hat{\mathbb{G}}$ by density (see Remark 2.3) and considering that $T^{\mathrm{I}}_{\tilde{f}(z)}S$ is closed, we get that $D\tilde{f}_z$ takes values in $T^{\mathrm{I}}_{\tilde{f}(z)}S$, which is a Carnot subgroup of $\mathbb{G}$ of Hausdorff dimension 12, thanks to the explicit expression of the tangent in Remark 5.3 and [35, Theorem 4.4, (iii)]. Thus as $\hat{\mathbb{G}}$ has Hausdorff dimension 12 itself and $D\tilde{f}_z$ is injective, we get that $D\tilde{f}_z$ is an isomorphism and so $\hat{\mathbb{G}}$ is isomorphic to $T^{\mathrm{I}}_{\tilde{f}(z)}S$ for every $z \in U_{\mathrm{I}}$.

In order to conclude, we notice that in the proof of Theorem 5.4 we showed that on every $\mathcal{H}^{12}$-positive measure subset of S there are at least two non-isomorphic tangent spaces, so that, because $\mathcal{H}^{12}(\tilde{f}(U_I)) > 0$ holds, we should have at least two non-isomorphic tangent spaces on $\tilde{f}(U_I)$. But we proved that all of them are isomorphic to $\hat{\mathbb{G}}$, thus we get a contradiction. □

Lemma 5.8. *Let $\varphi : \mathbb{G} \to \mathbb{H}$ be a Carnot homomorphism between two Carnot groups. If φ is not injective then it holds*

$$\dim_H \mathbb{G} \geq \dim_H \varphi(\mathbb{G}) + 1.$$

Proof. By definition of Carnot homomorphism we get that $\mathrm{Ker}\varphi$ is a homogeneous subgroup of $\mathbb{G}$ and $\varphi(\mathbb{G})$ is a homogeneous subgroup of $\mathbb{H}$. If an element g is in V_i for some i, we say that i is the degree of g and write $\deg g = i$. We take $\{e_1, \dots, e_l, e_{l+1}, \dots, e_n\} \subseteq \cup_{i=1}^s V_i$ a basis of $\mathfrak{g}$, such that $\{e_1, \dots, e_l\}$ is a basis of $\mathrm{Ker}\varphi$. Then $\{\varphi_*(e_{l+1}), \dots, \varphi_*(e_n)\}$ is a basis of the Lie algebra of $\varphi(\mathbb{G})$. By the fact that φ_* preserves the stratification (see Remark 2.1), we get

$$\deg \varphi_*(e_i) = \deg e_i$$

for each $l + 1 \leq i \leq n$. Then by [35, Theorem 4.4, (iii)] and the previous equation we get

$$\dim_H \varphi(\mathbb{G}) = \sum_{i=l+1}^{n} \deg \varphi_*(e_i) = \sum_{i=l+1}^{n} \deg e_i < \sum_{i=1}^{n} \deg e_i = \dim_H \mathbb{G}.$$

where we used in the strict inequality that $l > 0$ being φ not injective. □

From Remark 3.8 we have this consequence to Theorem 5.7.

Corollary 5.9. *There exist a Carnot group $\mathbb{G}$ and a C^∞ non-characteristic hypersurface $S \subseteq \mathbb{G}$ that is not Pauls Carnot rectifiable according to Definition 3.5.*

Remark 5.10. The examples of Corollary 5.9 are actually C^1_{H}-hypersurfaces because they are smooth and non-characteristic, see Remark 2.8. Thus they are rectifiable in the sense of Franchi, Serapioni and Serra Cassano but we proved they are not in the sense of [47, Definition 4.1]. Indeed, the definition of Pauls Carnot rectifiability is a generalization of [47, Definition 4.1], see Remark 3.6.

Remark 5.11. We notice that every tangent group to S as in the proof of Theorem 5.7 is a Carnot group. So S is an example of a smooth non-characteristic hypersurface in a Carnot group that cannot be Lipschitz parametrizable by countably many subsets of its tangents.

We state here as a theorem something we already proved in Theorem 5.4.

Theorem 5.12. *There exists a locally compact locally doubling metric measure space $(X, d, \mathcal{H}^k)$, where $k := \dim_H X$, that satisfies the following two properties:*

1. *For each $x \in X$, there exists (up to isometry) only one element in* $\mathrm{Tan}(X, d, x)$ *and it is a Carnot group;*
2. *For each $U \subseteq X$ with $\mathcal{H}^k(U) > 0$ there exists an uncountable family $\{x_i\}_{i\in I} \subseteq U$ of points such that the tangent spaces at these points are pairwise non-bi-Lipschitz equivalent.*

Proof. The example and the proof are exactly the same as in the proof of Theorem 5.4. □

Remark 5.13. Another example (a sub-Riemannian manifold) that satisfies Theorem 5.12 was presented in [37, Proposition 16].

Remark 5.14 (*S has a Structure of Sub-Riemannian Manifold*)**.** For an introduction to sub-Riemannian manifolds, also in the Finsler case, one can see [33, Chapter 2]. We show that the example in Remark 5.3 has also the structure of an equiregular sub-Riemannian manifold. Indeed, the set S is a smooth hypersurface of $\mathbb{R}^8$ and we can consider the distribution

$$\mathcal{D}_x := \mathrm{span}\{X_1(x), X_2(x) - x_2^2 X_0(x), X_3(x)\} \subseteq T_x S,$$

where $X_i(x)$ are defined in Eq. (5.2) and $T_x S$ is the Euclidean tangent of S. We equip the distribution $\mathcal{D}$ with some smooth scalar product $\mathbf{g}$. By the computations made in Proposition 4.5 and by considering (5.2), we get

$$\begin{aligned}
\mathcal{D}_x^2 &:= \mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x \\
&= \mathrm{span}\{X_1(x), X_2(x) - x_2^2 X_0(x), X_3(x), X_4(x), X_5(x), X_6(x)\}, \\
\mathcal{D}_x^3 &:= \mathcal{D}_x^2 + [\mathcal{D}, \mathcal{D}^2]_x \\
&= \mathrm{span}\{X_1(x), X_2(x) - x_2^2 X_0(x), X_3(x), X_4(x), X_5(x), X_6(x), X_7(x)\} \\
&= T_x S.
\end{aligned} \tag{5.5}$$

Then we get that, for all $x \in S$, $\dim \mathcal{D}_x = 3$, $\dim \mathcal{D}_x^2 = 6$ and $\dim \mathcal{D}_x^3 = 7$. Thus we define the sub-Riemannian distance d_{sR} associated to the sub-Riemannian structure $(S, \mathcal{D}, \mathbf{g})$. We get from the results in [5] (see also [30, Theorem 2.5], [30, page 25]) and the explicit expressions in (5.5), that the tangent space at $x \in S$ to (S, d_{sR}) is isometric to the Carnot group $\mathbb{G}_{-x_2^2}$ with Lie algebra $\mathfrak{g}_{-x_2^2}$, defined in Definition 4.1, equipped with the Carnot-Carathéodory distance induced by the left-invariant scalar product that, on the first stratum of the Lie algebra, coincides with $\mathbf{g}_x$. Also from [44, Theorem 2] (see also [26, Theorem 3.1]) we get that the Hausdorff dimension of (S, d_{sR}) is 12.

Then a slightly change of the proof of Theorem 5.4 - namely at the end of the proof we need to use the results about the Hausdorff dimension of smooth submanifolds in a sub-Riemannian manifold in [28, 0.6.B] (see also [26, Theorem 5.3]) to obtain $\mathcal{H}^{12}(S \cap \{x_2 = \lambda\}) = 0$ - gives that (S, d_{sR}) is purely bi-Lipschitz homogeneous unrectifiable. Also, reasoning exactly as in the proof of Theorem 5.4, we have that $(S, d_{sR}, \mathcal{H}^{12})$ satisfies Theorem 5.12.

6. Pauls rectifiability of C^∞-hypersurfaces in heisenberg groups

In this section we prove that C^∞-hypersurfaces in the nth Heisenberg group, $\mathbb{H}^n$ with $n \geq 2$, are rectifiable according to [12, Definition 3], see also Theorem 6.15 and Remark 6.16. We start with a Lipschitz-type estimate, which more generally holds for Carnot groups of step 2, see Proposition 6.6.

We remark here that, in order to prove the main result of this section, we will use Proposition 6.6 only for Heisenberg groups, but we prove it in the general case of Carnot groups of step 2. We point out that the result in Proposition 6.10 requires this Lipschitz-type estimate, plus a connectibility argument that makes that proof work only for $\mathbb{H}^n$, with $n \geq 2$. For arbitrary Carnot groups of step 2, the intrinsic distance d^Γ in the statement of Proposition 6.10 might not even be finite.

Then in Section 6.3 we show the equivalence of intrinsic distance and induced distance for intrinsic Lipschitz graphs in Heisenberg groups, which has been suggested to us by Fässler and Orponen, adapting an argument of [11]. After that, in Section 6.4, we prove that C^∞ non-characteristic hypersurfaces in $\mathbb{H}^n$, with $n \geq 2$, carry a sub-Riemannian structure, see Proposition 6.11 and [49, Theorem 1.1]. Therefore, we show that the sub-Riemannian distance is locally equivalent to the induced distance, see Proposition 6.12. By means of [37, Theorem 2] we are able to conclude the result: see Theorem 6.15.

6.1. Carnot groups of step 2

In this subsection we recall the geometry of step 2 groups in exponential coordinates. We stress here that sometimes we use Einstein notation: we do not use the symbol Σ and we remind that, in this case, we are tacitly taking the sum over the repeated indices. Every Carnot group of step 2 arises as follows.

Let $(B^1_{jl}), \dots, (B^n_{jl})$ be n linearly independent skew-symmetric $m \times m$ matrices with $j, l = 1, \dots, m$. Consider the Carnot group $(\mathbb{R}^m \times \mathbb{R}^n, \cdot, \delta_\lambda)$ where the operation is

$$\begin{aligned}(x_1, \dots, x_m, y_1, \dots, y_n) \cdot (\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n) :=\\ \left(x_1 + \tilde{x}_1, \dots, x_m + \tilde{x}_m, y_1 + \tilde{y}_1 + \frac{1}{2} B^1_{jl} \tilde{x}_j x_l, \dots, y_n + \tilde{y}_n + \frac{1}{2} B^n_{jl} \tilde{x}_j x_l\right),\end{aligned} \tag{6.1}$$

and the dilations are

$$\delta_\lambda(x_1, \dots, x_m, y_1, \dots, y_n) := (\lambda x_1, \dots, \lambda x_m, \lambda^2 y_1, \dots, \lambda^2 y_n), \tag{6.2}$$

for every $\lambda > 0$.

In the Carnot groups defined above we call X_j, with $j = 1, \dots, m$, the left-invariant vector fields that agree with ∂_{x_j} at the origin. We call Y_k, with $k = 1, \dots, n$, the left-invariant vector fields that agree with ∂_{y_k} at the origin. It holds

$$\begin{aligned} X_j &= \partial_{x_j} + \frac{1}{2} B^k_{jl} x_l \partial_{y_k}, \\ Y_k &= \partial_{y_k}. \end{aligned} \tag{6.3}$$

We shall consider the two following homogeneous subgroups

$$\mathbb{L} := \{(x_1, 0, \dots, 0)\}, \qquad \mathbb{W} := \{(0, x_2, \dots, x_m, y_1, \dots, y_n)\}. \tag{6.4}$$

In what follows $\tilde{\varphi} : \mathbb{W} \to \mathbb{L}$ will be an intrinsic L-Lipschitz function (Definition 2.12) and $\varphi : \mathbb{R}^{m+n-1} \to \mathbb{R}$ is defined according to

$$\tilde{\varphi}(0, x_2, \dots, x_m, y_1, \dots, y_n) = (\varphi(x_2, \dots, x_m, y_1, \dots, y_n), 0, \dots, 0). \tag{6.5}$$

The following vector fields on $\mathbb{W}$ are strictly related to the intrinsic gradient of a function, see [14, Section 5].

Definition 6.1. Given $\tilde{\varphi}$ and φ as in (6.5) we define, for $j = 2, \dots, m$, the vector fields on $\mathbb{W}$ at $\bar{x} := (0, x_2, \dots, x_m, y_1, \dots, y_n)$ as

$$D^\varphi_j|_{\bar{x}} := X_j|_{\bar{x}} + \varphi(x_2, \dots, x_m, y_1, \dots, y_n) B^k_{j1} Y_k|_{\bar{x}}. \tag{6.6}$$

Definition 6.2. We will say that an absolutely continuous curve $\tilde{\gamma} : I \to \mathbb{W}$ is *horizontal for the family of vector fields* $\{D^\varphi_j\}_{j=2,\dots,m}$, if there exist $(a_2(t), \dots, a_m(t)) \in L^1(I; \mathbb{R}^{m-1})$ such that

$$\tilde{\gamma}'(t) = a_j(t) D^\varphi_j|_{\tilde{\gamma}(t)}, \qquad \text{for a.e.} \quad t \in I. \tag{6.7}$$

Then the φ-*length* of $\tilde{\gamma}$ is defined as

$$\ell_\varphi(\tilde{\gamma}) := \int_I \sqrt{a_2(s)^2 + \dots + a_m(s)^2}\, \mathrm{d}\, s. \tag{6.8}$$

Remark 6.3. Notice that, due to the specific form of X_j, Y_k and D_j^φ in (6.3) and (6.6) respectively, if

$$\tilde{\gamma}(t) := (0, x_2(t), \dots, x_m(t), y_1(t), \dots y_n(t)), \tag{6.9}$$

then

$$\ell_\varphi(\tilde{\gamma}) = \int_I \sqrt{x_2'(s)^2 + \cdots + x_m'(s)^2}\,\mathrm{d}\,s. \tag{6.10}$$

Remark 6.4. Using the notation in Section 6.1, the Heisenberg group $\mathbb{H}^{\bar{n}}$ is obtained when $m = 2\bar{n}$, $n = 1$ and $B_{ij}^1 = 1$ if and only if $i = j + \bar{n}$, otherwise it is zero.

6.2. Length comparison for Carnot groups of step 2

In this subsection we will show that for Carnot groups of step 2, the length of the curve $\tilde{\gamma} \cdot \tilde{\varphi}(\tilde{\gamma})$ measured with a left-invariant homogeneous distance d in the group $\mathbb{G}$ is controlled from above by $\ell_\varphi(\tilde{\gamma})$ up to a multiplicative constant.

Remark 6.5. For the general theory of sub-Riemannian manifolds, including the Finsler case, one can check [33, Chapter 2]. We recall that in $\mathbb{G}$ we have two interpretations for the length of an absolutely continuous curve. Indeed, as in any Carnot-Carathéodory space, if the distance d on $\mathbb{G}$ is induced by a norm $\|\cdot\|$ on the horizontal bundle $\mathbb{V}_1$ of $\mathbb{G}$, the length of a continuous curve $\gamma : I \to \mathbb{G}$ equals the following values

$$\mathrm{length}(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(s_{i-1}), \gamma(s_i)) \right\} = \int_I \|\gamma'(t)\|\,\mathrm{d}\,t, \tag{6.11}$$

where the sup is over the partitions $\sqcup_{i=0}^{n} [s_i, s_{i+1}]$ of I.

The proof of the forthcoming proposition was pointed out to us by Fässler and Orponen in the Heisenberg group and it is substantially contained in [11, Proposition 3.8]. We present here a general proof for step 2 groups.

Proposition 6.6. *Let $\mathbb{G}$ be a step 2 Carnot group with the choice of coordinates as in Section 6.1 and $\mathbb{W}$ and $\mathbb{L}$ as in (6.4). Let $\tilde{\varphi} : \mathbb{W} \to \mathbb{L}$ be intrinsic L-Lipschitz. Set $\varphi, D_j^\varphi, \ell_\varphi$ as in (6.5), (6.6), and (6.8), respectively. If $\tilde{\gamma} : I \to \mathbb{W}$ is horizontal with respect to $\{D_j^\varphi\}_{j=2,\dots,m}$, then*

$$\mathrm{length}(\tilde{\gamma} \cdot \tilde{\varphi}(\tilde{\gamma})) \le C \cdot \ell_\varphi(\tilde{\gamma}),$$

where $C = C(\mathbb{G}, L)$.

Moreover, if the norm of the controls $a_j(t)$ of $\tilde{\gamma}$ as in Definition 6.2 are bounded by K, the projection on the first component of the curve $s \mapsto \tilde{\varphi}(\tilde{\gamma}(s))$ is L'-Lipschitz, with $L' = L'(L, K, \mathbb{G})$.

Proof. Set $\gamma : I \to \mathbb{R}^{m+n-1}$ be the curve

$$\gamma(t) := (x_2(t), \dots, x_m(t), y_1(t), \dots, y_n(t)),$$

where we use the notation (6.9). By the fact that $\tilde{\gamma}$ is horizontal with respect to $\{D_j^\varphi\}_{j=2,\dots,m}$ we get by easy computations that for each $k = 1, \dots, n$

$$y_k'(t) = x_j'(t) \left(\frac{1}{2} B_{jl}^k x_l(t) + \varphi(\gamma(t)) B_{j1}^k \right), \qquad \text{for a.e.} \quad t \in I, \tag{6.12}$$

where we sum over j and l from 2 to m. Now we consider the curve $\tilde{\gamma}$ between two intermediary times $t < t_1$ and we claim that

$$\sum_{i=2}^{m} |x_i(t_1) - x_i(t)| \leq C_1 \ell_\varphi \left(\tilde{\gamma}|_{[t,t_1]} \right), \tag{6.13}$$

with $C_1 = C_1(m)$. Indeed, this is a consequence of the fundamental theorem of calculus, Cauchy–Schwarz and (6.10).

Set $\Phi(\tilde{\gamma}(t)) := \tilde{\gamma}(t) \cdot \tilde{\varphi}(\tilde{\gamma}(t))$. By the definition of length it suffices to show that for all $[t, t_1] \subseteq I$ there exists a constant $C = C(L, \mathbb{G})$ such that

$$d(\Phi(\tilde{\gamma}(t)), \Phi(\tilde{\gamma}(t_1))) \leq C \cdot \ell_\varphi(\tilde{\gamma}|_{[t,t_1]}). \tag{6.14}$$

By the fact that $\tilde{\varphi}$ is intrinsic Lipschitz and [18, Proposition 2.3.4] one has that, setting $\|\cdot\|$ the homogeneous norm on $\mathbb{G}$ associated to d (see (2.2)), there exists a constant $C_0 = C_0(L)$ such that

$$d(\Phi(\tilde{\gamma}(t)), \Phi(\tilde{\gamma}(t_1))) \leq C_0 \left\| \pi_{\mathbb{W}} \left(\Phi(\tilde{\gamma}(t))^{-1} \cdot \Phi(\tilde{\gamma}(t_1)) \right) \right\|. \tag{6.15}$$

Then we leave to the reader to verify the algebraic equality, which depends on the fact that $\mathbb{W}$ is a normal subgroup,

$$\pi_{\mathbb{W}} \left(\Phi(\tilde{\gamma}(t))^{-1} \cdot \Phi(\tilde{\gamma}(t_1)) \right) = \tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(t_1) \cdot \tilde{\varphi}(\tilde{\gamma}(t)). \tag{6.16}$$

By exploiting the formula for the group law, it holds

$$\begin{aligned} &\tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(t_1) \cdot \tilde{\varphi}(\tilde{\gamma}(t)) = \\ &= (0, x_2(t_1) - x_2(t), \ldots, x_m(t_1) - x_m(t), \sigma_1(t_1, t), \ldots, \sigma_n(t_1, t)), \end{aligned} \tag{6.17}$$

where for each $k = 1, \ldots, n$ we have

$$\sigma_k(t_1, t) := y_k(t_1) - y_k(t) + B^k_{1j} \varphi(\gamma(t))(x_j(t_1) - x_j(t)) - \frac{1}{2} B^k_{jl} x_j(t_1) x_l(t), \tag{6.18}$$

where the sums on indices j and l run from 2 to m.

Then by (6.17) and the fact that $\|\cdot\|$ is equivalent to any other homogeneous norm on $\mathbb{G}$, we have that

$$\|\tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(t_1) \cdot \tilde{\varphi}(\tilde{\gamma}(t))\| \sim \sum_{i=2}^{m} |x_i(t_1) - x_i(t)| + \sum_{k=1}^{n} \sqrt{|\sigma_k(t_1, t)|}. \tag{6.19}$$

Using (6.12) and that $B^k_{jl} x_j(t) x_l(t) = 0$ by skew-symmetry of B^k we can rewrite $\sigma_k(t_1, t)$ as follows

$$\begin{aligned} \sigma_k(t_1, t) &= \left(\int_t^{t_1} y_k'(\xi) \, \mathrm{d}\,\xi \right) + B^k_{1j} \varphi(\gamma(t))(x_j(t_1) - x_j(t)) - \frac{1}{2} B^k_{jl} x_j(t_1) x_l(t) \\ &= \int_t^{t_1} \left(x_j'(\xi) B^k_{j1} \left(\varphi(\gamma(\xi)) - \varphi(\gamma(t)) \right) + \frac{1}{2} x_j'(\xi) B^k_{jl} (x_l(\xi) - x_l(t)) \right) \mathrm{d}\,\xi. \end{aligned} \tag{6.20}$$

Set

$$f(t_1, t) := \sup_{\xi \in [t, t_1]} |\varphi(\gamma(\xi) - \varphi(\gamma(t)))|.$$

It follows from (6.20), (6.13) and Cauchy–Schwarz inequality that

$$|\sigma_k(t_1, t)| \leq C_2 \left(\ell_\varphi(\tilde{\gamma}|_{[t,t_1]}) + f(t_1, t) \right) \ell_\varphi(\tilde{\gamma}|_{[t,t_1]}), \tag{6.21}$$

where $C_2 = C_2(m, B)$ and $B := \max |B^k_{jl}|$. Now for each $\xi \in [t, t_1]$ we get by the fact φ is intrinsic Lipschitz, (6.16), (6.19), (6.13) and (6.21) with ξ instead of t_1, that

$$\begin{aligned} |\varphi(\gamma(\xi)) - \varphi(\gamma(t))| &\leq L \left\| \pi_{\mathbb{W}} \left(\Phi(\tilde{\gamma}(t))^{-1} \cdot \Phi(\tilde{\gamma}(\xi)) \right) \right\| \\ &= \|\tilde{\varphi}(\tilde{\gamma}(t))^{-1} \cdot \tilde{\gamma}(t)^{-1} \cdot \tilde{\gamma}(\xi) \cdot \tilde{\varphi}(\tilde{\gamma}(t))\| \\ &\sim \sum_{i=1}^{m} |x_i(\xi) - x_i(t)| + \sum_{k=1}^{n} \sqrt{|\sigma_k(\xi, t)|} \\ &\leq C_3 \left(\ell_\varphi(\tilde{\gamma}|_{[t,\xi]}) + \sqrt{f(\xi, t)} \sqrt{\ell_\varphi(\tilde{\gamma}|_{[t,\xi]})} \right), \end{aligned} \tag{6.22}$$

where $C_3 = C_3(m, B, L)$. Now passing to the supremum as $\xi \in [t, t_1]$ in both sides of (6.22) we get

$$f(t_1, t) \leq C_3 \left(\ell_\varphi(\tilde{\gamma}|_{[t,t_1]}) + \sqrt{f(t_1, t)} \sqrt{\ell_\varphi(\tilde{\gamma}|_{[t,t_1]})} \right),$$

from which there exists $C_4 = C_4(m, B, L)$ such that

$$f(t_1, t) \leq C_4 \ell_\varphi \left(\tilde{\gamma}|_{[t,t_1]} \right). \tag{6.23}$$

Finally by chaining (6.15), (6.16), (6.19), (6.13), (6.21), and (6.23), we get (6.14) which was what we wanted. For the second part of the lemma we just chain (6.22) and (6.23) with ξ instead of t_1, and use the fact that $\ell_\varphi(\tilde{\gamma}|_{[t,\xi]})$ is bounded from above by $C(K, m)|\xi - t|$ by the definition of ℓ_φ in (6.8). $\square$

Remark 6.7. The second part of Proposition 6.6 recovers also the statement of [15, Proposition 3.6]. Notice that results that are similar to Proposition 6.6 have been proved by Kozhevnikov. In particular, in [31, Proposition 4.2.16] it is proved, in the general setting of Carnot groups, a characterization of intrinsic Lipschitz graphs by means of metric properties of integral curves, on $\mathbb{W}$, of some operators $\{D^\varphi_j\}$. We point out that these operators D^φ_j are the push-forward of the vector fields X_j by means of $\pi_{\mathbb{W}}$, the projection on $\mathbb{W}$, restricted to $\mathbb{W} \cdot \tilde{\varphi}(\mathbb{W})$, see [31, Definition 4.1.11].

In particular, the implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ in [31, Proposition 4.2.16] show the statement of Proposition 6.6 but just for horizontal curves $\tilde{\gamma} : I \to \mathbb{W}$ with constant controls $a_j(t) \equiv a_j$ in (6.7). A similar result was already known from [10, Theorem 1.2].

6.3. Equivalence of intrinsic distance and induced distance on intrinsic Lipschitz graphs in the heisenberg groups

Definition 6.8. Given $\tilde{\varphi}$ and φ as in (6.5) we define the intrinsic length distance on the graph $\Gamma := \mathrm{graph}(\tilde{\varphi}) = \{w \cdot \tilde{\varphi}(w) : w \in \mathbb{W}\} \subseteq \mathbb{G}$ as follows

$$d^\Gamma(x, y) := \inf\{\mathrm{length}(\gamma) | \gamma : [0, 1] \to \Gamma, \gamma(0) = x, \gamma(1) = y, \quad \gamma \quad \text{horizontal}\}. \tag{6.24}$$

Remark 6.9. Up to a globally bi-Lipschitz change of distance, we can suppose to work with a left-invariant homogeneous distance d on $\mathbb{G}$ coming from a scalar product $\mathbf{g}$ on the horizontal bundle $\mathbb{V}_1$. Notice that if Γ is a smooth submanifold of $\mathbb{G}$ and the horizontal bundle $\mathbb{V}_1$ intersects the tangent bundle of Γ in a bracket generating distribution, then the distance $d^\Gamma(x, y)$ is exactly the sub-Riemannian distance, let us call it $d_{\mathrm{int}}(x, y)$, associated to the sub-Riemannian structure $\left(\Gamma, \mathbb{V}_1 \cap T\Gamma, \mathbf{g}|_{(\mathbb{V}_1 \cap T\Gamma) \times (\mathbb{V}_1 \cap T\Gamma)}\right)$.

We now stress that in the specific case of the Heisenberg groups the induced distance d is bi-Lipschitz equivalent to d^Γ defined in (6.24). The proof was suggested to us by Fässler and Orponen.

Proposition 6.10. *With the same assumptions and notation as in Proposition 6.6, if $\mathbb{G} = \mathbb{H}^n$ with $n \geq 2$, then*

$$d(x, y) \sim d^{\Gamma}(x, y), \qquad \forall x, y \in \Gamma,$$

where d^{Γ} is defined in (6.24) and d is the induced distance, restriction of the one in $\mathbb{H}^n$.

Proof. First of all notice that, using the notation in (6.6) and taking into account Remark 6.4, if $\mathbb{G} = \mathbb{H}^n$, then $D^{\varphi}_j|_{\bar{x}} = X_j|_{\bar{x}}$ for all $j = 2, \ldots, 2n$ and $j \neq n+1$, while $D^{\varphi}_{n+1}|_{\bar{x}} = X_{n+1}|_{\bar{x}} + \varphi(x_2, \ldots, x_{2n}, y_1)Y_1|_{\bar{x}}$. We also have $[X_j, X_{n+j}] = Y_1$ for every $j = 1, \ldots, n$ and all the other commutators are zero. By definition of d^{Γ} (6.24), exploiting the definition of length (6.11) and the triangle inequality, we get

$$d(x, y) \leq d^{\Gamma}(x, y), \qquad \forall x, y \in \Gamma.$$

Now we want to prove the opposite inequality up to a multiplicative constant. First of all, by a left translation, we can assume $x = 0$ and $y = w \cdot \tilde{\varphi}(w)$ for $w \in \mathbb{W}$. It holds that

$$d(x, y) = d(0, y) = \|w \cdot \tilde{\varphi}(w)\|_d \geq C_0\|w\|_d, \tag{6.25}$$

where $\|\cdot\|_d$ is the homogeneous norm associated to d and $C_0 = C_0(\mathbb{W}, \mathbb{H})$, see [18, Proposition 2.2.2]. From now on, in this proof, we will set $\|\cdot\|_d := \|\cdot\|$.

We claim that we can conclude if we show that for each $w \in \mathbb{W}$ there exists $\tilde{\gamma} \subseteq \mathbb{W}$, connecting 0 to w, horizontal for $\{D^{\varphi}_j\}_{j=2,\ldots,2n+1}$, such that

$$\ell_{\varphi}(\tilde{\gamma}) \leq C_1\|w\|, \tag{6.26}$$

for some constant C_1 independent on w. Indeed, if (6.26) holds, then from the first part of Proposition 6.6 and (6.25) we get that, setting $\Phi(\tilde{\gamma}) := \tilde{\gamma} \cdot \tilde{\varphi}(\tilde{\gamma})$,

$$\text{length}(\Phi(\tilde{\gamma})) \leq C_2 d(x, y),$$

where C_2 is a constant independent on w, and $\Phi(\tilde{\gamma})$ is a curve contained in Γ connecting $x = 0$ to $y = w\cdot\tilde{\varphi}(w)$. Since the length of $\Phi(\tilde{\gamma})$ is finite, we get that it is a horizontal curve [33, Theorem 2.4.5] and then we get

$$d^{\Gamma}(x, y) \leq C_2 d(x, y),$$

that finishes the proof.

Now we show the existence of $\tilde{\gamma}$, with the required properties, such that (6.26) holds. We concatenate two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, horizontal for $\{D^{\varphi}_j\}_{j=2,\ldots,2n+1}$, to reach $w := (0, x_2, \ldots, x_{2n}, y_1)$ from 0. Due to the fact that φ is continuous, because of Remark 2.13, Peano's theorem [29, Theorem 1.1] ensures that there exists a local solution to the continuous ODE

$$\begin{cases} \tau'(s) &= \varphi(0, \ldots, 0, s, 0, \ldots, 0, \tau(s)), \\ \tau(0) &= 0, \end{cases} \tag{6.27}$$

where s in the $(n+1)$th coordinate. Set

$$\tilde{\gamma}_1(s) := (0, \ldots, 0, s, 0, \ldots, 0, \tau(s)),$$

the curve with values in $\mathbb{W} \subseteq \mathbb{H}^n$, with s in the $(n+1)$th coordinate. By (6.27) it holds, whenever $\tilde{\gamma}_1(s)$ is defined,

$$\tilde{\gamma}_1'(s) = D^{\varphi}_{n+1}|_{\tilde{\gamma}_1(s)}. \tag{6.28}$$

We show that $\tau(s)$ is defined globally on $\mathbb{R}$, arguing as in [9, (4.1) and after]. Indeed, whenever $\tau(s)$ exists,

$$\tau(s) = \int_0^s \tau'(\xi)\,\mathrm{d}\,\xi = \int_0^s \tilde{\varphi}(\tilde{\gamma}_1(\xi))\,\mathrm{d}\,\xi. \tag{6.29}$$

Notice that here there is a little abuse of notation: by $\tilde{\varphi}(\tilde{\gamma}_1(\xi))$, that a priori has values in $\mathbb{L}$, we mean the projection of it on the first coordinate in $\mathbb{H}^n$. By (6.28) and the second part of Proposition 6.6 we have that $s \mapsto \tilde{\varphi}(\tilde{\gamma}_1(s))$ is L'-Lipschitz, with $L' = L'(L)$. Then, by (6.29) and the fact that $\varphi(0) = 0$, because we are assuming $x = 0$, we have

$$|\tau(s)| \leq \frac{1}{2} L' s^2. \tag{6.30}$$

Thus, as any solution to (6.27) escapes every compact set [29, Theorem 2.1], we get from (6.30) that $\tau(s)$ is globally defined. Then $\tau(s)$ is defined up to $s = x_{n+1}$ and by the previous argument

$$|\tau(x_{n+1})| \leq \frac{1}{2} L' x_{n+1}^2. \tag{6.31}$$

We notice that we can identify the arbitrary point $(x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}, y_1)$ with a point in $\mathbb{H}^{n-1}$. Thus we can connect the point $(0, \dots, 0, \tau(x_{n+1}))$, where we just removed the first and the $(n+1)$th coordinate from $\tilde{\gamma}_1(x_{n+1})$, to the point $(x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}, y_1)$, by using a horizontal geodesic in $\mathbb{H}^{n-1}$ with respect to the Carnot-Carathéodory distance $d_{\mathbf{g}}$ induced, on $\mathbb{H}^{n-1}$, by the scalar product $\mathbf{g}$ that makes $X_2, \dots, X_n, X_{n+2}, \dots, X_{2n}$ orthonormal. We set $\tilde{\gamma}_2 : I \to \mathbb{W}$ to be the lifting of this horizontal geodesic in $\mathbb{H}^n$, where the $(n+1)$th coordinate of $\tilde{\gamma}_2$ is constantly equal to x_{n+1}. We notice that it is horizontal with respect to the family $\{D_j^{\varphi}\}_{j=2,\dots,n,n+2,\dots,2n}$, because $D_j^{\varphi} = X_j$ for $j = 2, \dots, 2n$ and $j \neq n+1$. Then we have

$$\begin{aligned} \ell_{\varphi}(\tilde{\gamma}_2) &= d_{\mathbf{g}}((0, \dots, 0, \tau(x_{n+1})), (x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}, y_1)) \\ &\leq C_3 \left(|y_1 - \tau(x_{n+1})|^{1/2} + \sum_{i=2, i \neq n+1}^{2n} |x_i| \right) \\ &\leq C_4 \left(|y_1|^{1/2} + \sum_{i=2}^{2n} |x_i| \right) \leq C_5 \|w\|, \end{aligned} \tag{6.32}$$

where the first equality follows by the definition of ℓ_{φ} (6.8) and the fact that $\tilde{\gamma}_2$, restricted to the copy of $\mathbb{H}^{n-1}$ made of points with zero in the first coordinate and x_{n+1} in the $(n+1)$th coordinate, is a $d_{\mathbf{g}}$-geodesic; the second is true because any two homogeneous norms are equivalent, the third one is true because of (6.31), and the last one again by the fact that any two homogeneous norms are equivalent. Now we have

$$\ell_{\varphi}\left(\tilde{\gamma}_1|_{[0, x_{n+1}]}\right) = |x_{n+1}| \leq |y_1|^{1/2} + \sum_{i=2}^{2n} |x_i| \leq C_6 \|w\|, \tag{6.33}$$

where the first equality is true by the definition of ℓ_{φ} and (6.28) and the third is true again because of the equivalence of homogeneous norms. Now if we set $\tilde{\gamma} := \tilde{\gamma}_1|_{[0,x_{n+1}]} \star \tilde{\gamma}_2$ the concatenation of the two curves, we get that $\tilde{\gamma}$ is horizontal and connects 0 to w. Summing (6.32) and (6.33) we get (6.26) with $C_1 := C_5 + C_6$, which was what was left to prove. □

6.4. Sub-Riemannian structure of a C^{∞} non-characteristic hypersurface in the heisenberg groups

6.4.1. The restriction of the horizontal bundle is bracket generating

Now we are going to prove that for non-characteristic C^2-hypersurfaces (see Definition 2.7) in $\mathbb{H}^n$, with $n \geq 2$, the intersection between the horizontal bundle of $\mathbb{H}^n$ and the tangent bundle of S is bracket generating. This result was already known and it is a consequence of a more general one [49, Theorem 1.1]. Nevertheless we give here a simple proof by making explicit computations.

Proposition 6.11. *Consider in $\mathbb{H}^n$, with $n \geq 2$, a C^2-hypersurface S. If S has no characteristic points, then the bundle*

$$x \mapsto \mathcal{D}_x := \mathbb{V}_1(x) \cap T_x S, \tag{6.34}$$

gives a step-2 bracket generating distribution on the hypersurface S.

Proof. We refer, for the notation, to Section 6.1. In particular, for the Heisenberg groups $\mathbb{H}^n$, see Remark 6.4. We need to prove that

$$\forall x \in S, \qquad \mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x = T_x S.$$

Let us give the proof first for $n = 2$. We work locally around $x \in S$ so that we can assume that there exists $f \in C^2(\mathbb{H}^n)$ such that

$$S = \{x \in \mathbb{H}^n : f(x) = 0\}.$$

We define locally the vector fields

$$\begin{aligned} Z_1 &:= -(X_2 f)X_1 + (X_1 f)X_2 - (X_4 f)X_3 + (X_3 f)X_4, \\ Z_2 &:= -(X_3 f)X_1 + (X_4 f)X_2 + (X_1 f)X_3 - (X_2 f)X_4, \\ Z_3 &:= -(X_4 f)X_1 - (X_3 f)X_2 + (X_2 f)X_3 + (X_1 f)X_4. \end{aligned}$$

We have that for each $x \in S$, the linear space $\mathcal{D}_x$ is a three-dimensional subspace of $T_x S$, because x is a non-characteristic point (Definition 2.7). Then, because $Z_1|_x$, $Z_2|_x$ and $Z_3|_x$ are linearly independent and are in $\mathcal{D}_x$, we have

$$\mathcal{D}_x = \operatorname{span}\{Z_1|_x, Z_2|_x, Z_3|_x\}, \qquad \forall x \in S. \tag{6.35}$$

Now by doing the computations exploiting the definition of Z_i, and using that $[X_1, X_3] = [X_2, X_4] = Y_1$, we can show that

$$\begin{aligned} [Z_1, Z_2] &= \star_1 - 2(X_1 f X_2 f + X_3 f X_4 f)Y_1, \\ [Z_1, Z_3] &= \star_2 + ((X_1 f)^2 + (X_3 f)^2 - (X_2 f)^2 - (X_4 f)^2)Y_1, \\ [Z_2, Z_3] &= \star_3 + 2(X_1 f X_4 f - X_2 f X_3 f)Y_1, \end{aligned}$$

where $\star_1, \star_2, \star_3$ are some combinations of X_1, X_2, X_3, X_4 with function coefficients.

It is easy to check that it is not possible to have, at some point $x \in S$,

$$\begin{aligned} &X_1 f(x) X_2 f(x) + X_3 f(x) X_4 f(x) = 0, \\ &(X_1 f)^2(x) + (X_3 f)^2(x) - (X_2 f)^2(x) - (X_4 f)^2(x) = 0, \\ &X_1 f(x) X_4 f(x) - X_2 f(x) X_3 f(x) = 0, \end{aligned}$$

because otherwise $X_1 f(x) = X_2 f(x) = X_3 f(x) = X_4 f(x) = 0$, which is impossible because there are no characteristic points. Then, for every $x \in S$, at least one among $[Z_1, Z_2]|_x$, $[Z_1, Z_3]|_x$, and $[Z_2, Z_3]|_x$ has a component along $Y_1|_x$. Thus, as $X_1|_x, X_2|_x, X_3|_x, X_4|_x$ and $Y_1|_x$ are linearly independent, this means that there exists at least one among $[Z_1, Z_2]|_x$, $[Z_1, Z_3]|_x$, and $[Z_2, Z_3]|_x$ which is not in $\mathcal{D}_x$. Then as $[\mathcal{D}, \mathcal{D}]_x \subseteq T_x S$, and it holds that there exists an element in $[\mathcal{D}, \mathcal{D}]_x$ which is not in $\mathcal{D}_x$, we get the conclusion.

If $n > 2$ we can argue exactly in the same way. Indeed, because there are no characteristic points, for every $x \in S$ there exists i with $1 \leq i \leq 2n$ such that $X_i f(x) \neq 0$ and one runs the same computations substituting X_1, X_2, X_3, X_4 with $X_i, X_j, X_{i+n}, X_{j+n}$ with $j \neq i$. ☐

6.4.2. Local equivalence of the sub-Riemannian distance and the induced distance

Let S be a smooth non-characteristic hypersurface in the Heisenberg group $\mathbb{H}^n$, $n \geq 2$. From Proposition 6.11 we have a bracket generating distribution $\mathcal{D}$ in the Euclidean tangent bundle TS of S. Hence S has the structure of sub-Riemannian manifold: we fix a scalar product $\mathbf{g}$ on $\mathbb{V}_1$, the horizontal bundle of $\mathbb{H}^n$, which induces a scalar product on $\mathcal{D}$. This scalar product defines a sub-Riemannian distance on S by taking the infimum of the length – measured with the norm $\|\cdot\|_{\mathbf{g}}$ associated to $\mathbf{g}$ - of all the horizontal – according to $\mathcal{D}$ - curves in S. We will call this distance d_{int}, the *intrinsic distance* on S. We can also equip S with the restriction of the distance of $\mathbb{H}^n$, which we will call induced distance and with a little abuse of notation we denote it by d.

Proposition 6.12. *Let $(\mathbb{H}^n, d)$, with $n \geq 2$, be the Heisenberg group equipped with the sub-Riemannian distance coming from a scalar product on the horizontal distribution. Let S be a C^∞ non-characteristic hypersurface in $\mathbb{H}^n$. For each $p \in S$ there exists an open neighbourhood U_p of p such that*

$$d(x, y) \sim d_{\text{int}}(x, y) \qquad \forall x, y \in U_p.$$

Proof. By Remark 2.8, S is a $C^1_{\mathbb{H}}$-hypersurface. Then, by the implicit function theorem, we get that locally around $p \in S$ the hypersurface S is the graph Γ of a globally defined intrinsic Lipschitz function on the tangent group $\mathbb{W} := T^{\mathrm{I}}_x S$. By changing coordinates if necessary (see also Lemma 6.13), we can assume $\mathbb{W}$ as in (6.4). Then by Proposition 6.10 we get that $d^\Gamma \sim d$ and from Remark 6.9 we get that, in a neighbourhood of p, $d_{\text{int}} = d^\Gamma$, so that we get the result. □

6.4.3. Tangents of C^∞ non-characteristic hypersurfaces

Now we know that a C^∞ non-characteristic hypersurface in the Heisenberg groups $\mathbb{H}^n$, $n \geq 2$, is a sub-Riemannian manifold. With the aim of using the rectifiability result from [37, Theorem 2], we calculate the possible tangents of S. We recall this well-known lemma, see for example [24, Lemma 3.26].

Lemma 6.13. *Every vertical subgroup of codimension one in $\mathbb{H}^n$, $n \geq 2$, is isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$, which is a Carnot group.*

Proposition 6.14. *Let S be a C^∞-hypersurface in $\mathbb{H}^n$, $n \geq 2$, with no characteristic points. Let $\mathcal{D}$ be as in (6.34) and $\boldsymbol{g}$ be a scalar product on the horizontal bundle $\mathbb{V}_1$ of $\mathbb{H}^n$.*

Then the triple $(S, \mathcal{D}, \boldsymbol{g}|_{\mathcal{D}\times\mathcal{D}})$ is an equiregular sub-Riemannian manifold with Hausdorff dimension $2n+1$. At each point $x \in S$ we have that the Gromov–Hausdorff tangent is unique and it is isometric the Carnot group $\mathbb{H}^{n-1} \times \mathbb{R}$ endowed with some Carnot-Carathéodory distance.

Proof. Because of the fact that S is non-characteristic it follows that $\mathcal{D}_x$ has dimension $2n-1$ at each point $x \in S$. Also it is a direct consequence of Proposition 6.11 that, for each $x \in S$, the linear space $\mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x$ has dimension $2n$. Then $(S, \mathcal{D}, \mathbf{g}|_{\mathcal{D}\times\mathcal{D}})$ is an equiregular sub-Riemannian manifold with weights $(2n-1, 1)$. Then the Hausdorff dimension of S with respect to the sub-Riemannian distance d_{int} is $2n+1$, since d_{int} is equivalent to d, see Proposition 6.12.

By [5] (see also [30, Theorem 2.5], [30, page 25]) it follows, as we are in the equiregular case, that the Gromov–Hausdorff tangent at any point $x \in S$ is isometric to the Carnot group, endowed with some Carnot distance, that has Lie algebra

$$\mathbb{V}_x := \mathcal{D}_x \oplus ((\mathcal{D}_x + [\mathcal{D}, \mathcal{D}]_x)/\mathcal{D}_x),$$

with the bracket operation inherited by the brackets in the Heisenberg group. Then $\mathbb{V}_x$ is isomorphic to a vertical subgroup of $\mathbb{H}^n$ of codimension one and thus it is isomorphic to $\mathbb{H}^{n-1} \times \mathbb{R}$ by Lemma 6.13. □

6.4.4. Carnot-rectifiability of C^∞-hypersurfaces

We conclude with the main result of this section.

Theorem 6.15. *Let $(\mathbb{H}^n, d)$, with $n \geq 2$, be the nth Heisenberg group equipped with a left-invariant homogeneous distance d. If S is a C^∞-hypersurface in $\mathbb{H}^n$, then the metric space (S, d) has Hausdorff dimension $2n+1$ and it is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}^{2n+1})$-rectifiable according to Definition 1.3.*

Proof. The fact that (S, d) has Hausdorff dimension $2n+1$ follows from Proposition 2.14. Let us assume first that S has no characteristic points. In this case it directly follows from [37, Theorem 2] and Proposition 6.14

that the metric space (S, d_{int}) has Hausdorff dimension $2n+1$ and it is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}^{2n+1}_{d_{\mathrm{int}}})$-rectifiable according to Definition 1.3. Then by Proposition 6.12 we obtain that (S, d) is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}^{2n+1}_{d})$-rectifiable.

In the general case, calling Σ_S the set of characteristic points, we know that $\mathcal{H}^{2n+1}(\Sigma_S) = 0$ by [4, Theorem 1.1] (see also [40, Theorem 2.16]). Moreover if $x \in S$ is a non-characteristic point, there exists U_x open subset of S containing x such that U_x is a smooth non-characteristic hypersurface. Then we can use the previous argument to conclude that (U_x, d) is $(\{\mathbb{H}^{n-1} \times \mathbb{R}\}, \mathcal{H}^{2n+1})$-rectifiable and by covering $S \setminus \Sigma_S$ with countably many U_x's we get the conclusion. □

Remark 6.16. By Theorem 6.15 it follows that any smooth hypersurface S is $\mathbb{H}^{n-1} \times \mathbb{R}$-rectifiable according to the bi-Lipschitz variant of Pauls' definition [12, Definition 3], see Remark 3.3 for more details about this definition.

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