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Assouad type dimensions in geometric analysis

Juha Lehrbäck

Abstract We consider applications of the dual pair of the (upper) Assouad dimension and the lower (Assouad) dimension in analysis. We relate these notions to other dimensional conditions such as a Hausdorff content density condition and an integrability condition for the distance function. The latter condition leads to a characterization of the Muckenhoupt A_p properties of distance functions in terms of the (upper) Assouad dimension. It is also possible to give natural formulations for the validity of Hardy–Sobolev inequalities using these dual Assouad dimensions, and this helps to understand the previously observed dual nature of certain cases of these inequalities.

Key words: Assouad dimension, Lower dimension, Aikawa condition, Muckenhoupt weight, Hardy–Sobolev inequality
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1 Introduction

Mathematicians working in fractal geometry and related fields are well aware of the fact that there can not be a unique definition for the concept of dimension of a set, since different problems require different ways to deal with dimensional information. In fact, what sometimes may seem like a negligible nuance in the definition might actually lead to interesting discoveries concerning the fine structure of sets. On the flip side, the multitude of the notions of dimension may easily create confusion, and thus it is important to be able to justify the existence of all these concepts via natural applications.

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The purpose of this article is to describe some recent observations concerning the applications of the dual pair of the upper and lower Assouad dimension, often simply called the Assouad dimension and the lower dimension, respectively. These notions provide geometric information which is relevant not only in fractal geometry, but also for instance in harmonic analysis, potential theory, and partial differential equations. One manifestation of these connections can be seen via the validity of the so-called Hardy–Sobolev inequalities. Our aim is not so much in presenting any novelties on the level of the details or techniques, but rather in trying to illustrate how a new point of view in terms of dimensional conditions may offer clarity and reveal connections between known results. On the other hand, we do give proofs for some basic results, hoping that these will help the reader to gain familiarity with the relevant concepts.

We begin in Section 2 by recalling the definitions of the upper and lower Assouad dimension and relating them to the more familiar Hausdorff dimension. In particular, we explain the connection between the lower Assouad dimension and a Hausdorff content density condition. In Section 3 we study integrability conditions for distance functions $w(x) = d(x, E)^{-\alpha}$, where $E \subset \mathbb{R}^n$ and (usually) $0 < \alpha < n$. Such conditions, originally introduced by Aikawa, can be used to characterize the upper Assouad dimension, see Theorem 3.5. Next, in Section 4, we ask when a distance function w as above belongs to the important class of Muckenhoupt A_p weights. As it turns out, the answer can be given in terms of the upper Assouad dimension, using the integrability conditions from Section 3 as a helpful stepping stone. Finally, Section 5 completes the circle by showing how both upper and lower Assouad dimension play an important role when examining the validity of the Hardy–Sobolev inequalities in an open set $\Omega \subset \mathbb{R}^n$. In particular, a previously observed duality between certain cases of such inequalities becomes more transparent and natural when the conditions are formulated in terms of suitable dimensions.

Much of the theory presented in this survey can be extended to more general metric spaces satisfying standard structural assumptions. We give some comments and remarks related to such extensions, but for simplicity we focus on the case of the n-dimensional Euclidean space $\mathbb{R}^n$.

Notation

The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\},$$

and $\overline{B}(x, r)$ is the corresponding closed ball. When $A \subset \mathbb{R}^n$, we write $\operatorname{diam}(A)$ for the diameter of A, and $d(x, A)$ denotes the distance from a point $x \in \mathbb{R}^n$ to the set A. The complement of A is $A^c = \mathbb{R}^n \setminus A$. If A is (Lebesgue) measurable, then the Lebesgue measure of A is denoted by $|A|$. If $0 < |A| < \infty$ and $f \in L^1(A)$, then the mean value integral of f over A is

$$\fint_A f(x)\,dx = \frac{1}{|A|}\int_A f(x)\,dx.$$

As usual, C denotes a constant whose exact value may change at each occurrence.

For simplicity, we use the following versions of Hausdorff contents and measures. It is easy to see that these are comparable to the more standard definitions in e.g. [9, 30].

Definition 1.1. Let $E \subset \mathbb{R}^n$ and $\lambda \geq 0$. For $0 < \delta \leq \infty$, the λ-dimensional Hausdorff δ-content of E is

$$\mathcal{H}^\lambda_\delta(E) = \inf\Bigg\{\sum_{i=1}^\infty r_i^\lambda : E \subset \bigcup_{i=1}^\infty B(x_i, r_i),\ 0 < r_i \leq \delta\Bigg\}.$$

(In the case $\lambda = 0$ we allow also finite summations.) Then the (spherical) λ-dimensional Hausdorff measure of E is

$$\mathcal{H}^\lambda(E) = \lim_{\delta\to 0_+} \mathcal{H}^\lambda_\delta(E) = \sup_{\delta>0} \mathcal{H}^\lambda_\delta(E),$$

and the Hausdorff dimension of E is defined as

$$\dim_{\mathrm{H}}(E) = \inf\big\{\lambda \geq 0 : \mathcal{H}^\lambda(E) = 0\big\} = \inf\big\{\lambda \geq 0 : \mathcal{H}^\lambda_\infty(E) = 0\big\}.$$

2 Assouad type dimensions

The definitions of the Assouad type dimensions of a set $E \subset \mathbb{R}^n$ are based on simple and natural local covering properties of E: we consider pieces $E \cap \bar{B}(x, R)$, with $x \in E$ and $0 < R < \operatorname{diam}(E)$, and ask how many balls of radius $0 < r < R$ are needed at most (upper Assouad), or respectively at least (lower Assouad), to cover such pieces. Thus these concepts reveal the most "extreme" local behavior of sets, whereas other notions of dimension usually tell more about the "average" properties of sets.

When $A \subset \mathbb{R}^n$ is a bounded set and $r > 0$, we let $N(A, r)$ denote the minimal number of open balls of radius r that are needed to cover the set A.

Definition 2.1. Let $E \subset \mathbb{R}^n$. The upper Assouad dimension $\overline{\dim}_{\mathrm{A}}(E)$ is the infimum of $\lambda \geq 0$ for which there exists a constant C such that

$$N\big(E \cap \bar{B}(x, R), r\big) \leq C\Big(\frac{r}{R}\Big)^{-\lambda} = C\Big(\frac{R}{r}\Big)^{\lambda} \tag{2.1}$$

for every $x \in E$ and $0 < r < R < \operatorname{diam}(E)$.

In particular, the estimate in (2.1) holds whenever $\lambda > \overline{\dim}_{\mathrm{A}}(E)$, and possibly also when $\lambda = \overline{\dim}_{\mathrm{A}}(E)$. If $E \subset E'$, then clearly $\overline{\dim}_{\mathrm{A}}(E) \leq \overline{\dim}_{\mathrm{A}}(E')$. It is also easy to see that $0 \leq \overline{\dim}_{\mathrm{A}}(E) \leq n$ for every $E \subset \mathbb{R}^n$.

In the literature, the upper Assouad dimension is often called the Assouad dimension and denoted by $\dim_{\mathrm{A}}(E)$. This concept was used by Assouad in connection with the bi-Lipschitz embedding problem between metric and Euclidean spaces, see e.g. [4]. A nice account on the basic properties and history of the Assouad dimension is given in [29]. See also the survey by Fraser [11] in this same volume (and the references therein) for recent fractal geometric applications of the (upper) Assouad dimension and its generalizations.

We illustrate the definition by proving the fact that the Hausdorff dimension always gives a lower bound for the upper Assouad dimension.

Lemma 2.2. *Let* $E \subset \mathbb{R}^n$. *Then* $\dim_{\mathrm{H}}(E) \le \overline{\dim}_{\mathrm{A}}(E)$.

Proof. By the countable stability of the Hausdorff dimension it suffices to show that

$$\dim_{\mathrm{H}}\big(E \cap \bar{B}(x,R)\big) \le \overline{\dim}_{\mathrm{A}}(E)$$

for every $x \in E$ and $R > 0$. Let $s > \overline{\dim}_{\mathrm{A}}(E)$, choose λ satisfying $\overline{\dim}_{\mathrm{A}}(E) < \lambda < s$, and fix $x \in E$ and $R > 0$. Then $E \cap \bar{B}(x,R)$ can be covered by

$$N \le C\Big(\frac{R}{r}\Big)^{\lambda}$$

balls of radius r, for every $0 < r < R$. Thus, by the definition of Hausdorff content,

$$\mathcal{H}^s_r\big(E \cap \bar{B}(x,R)\big) \le N r^s \le C_1 R^{\lambda} r^{s-\lambda}.$$

Letting $r \to 0$ gives $\mathcal{H}^s\big(E \cap \bar{B}(x,R)\big) = 0$, and we conclude that $\dim_{\mathrm{H}}\big(E \cap \bar{B}(x,R)\big) \le \overline{\dim}_{\mathrm{A}}(E)$. □

Definition 2.3. Let $E \subset \mathbb{R}^n$. The lower Assouad dimension $\underline{\dim}_{\mathrm{A}}(E)$ is the supremum of $\lambda \ge 0$ for which there exists a constant C such that

$$N\big(E \cap \bar{B}(x,R), r\big) \ge C\Big(\frac{r}{R}\Big)^{-\lambda} = C\Big(\frac{R}{r}\Big)^{\lambda} \tag{2.2}$$

for every $x \in E$ and $0 < r < R < \operatorname{diam}(E)$.

In particular, the estimate in (2.2) holds whenever $0 \le \lambda < \underline{\dim}_{\mathrm{A}}(E)$, and possibly also when $\lambda = \underline{\dim}_{\mathrm{A}}(E)$. In the case $E = \{x_0\}$, $x_0 \in \mathbb{R}^n$, we remove the requirement $R < \operatorname{diam}(E)$ from the definition and hence $\underline{\dim}_{\mathrm{A}}(\{x_0\}) = 0$. It is easy to verify that $0 \le \underline{\dim}_{\mathrm{A}}(E) \le \overline{\dim}_{\mathrm{A}}(E) \le n$ for every $E \subset \mathbb{R}^n$. However, it should be noted that, unlike (most) other natural concepts of dimension, the lower Assouad dimension is not monotone. For instance, $\underline{\dim}_{\mathrm{A}}(\{0\} \cup [1,2]) = 0$, due to the isolated point 0, but for the subset $[1,2]$ we have $\underline{\dim}_{\mathrm{A}}([1,2]) = 1$.

The lower Assouad dimension is often called the lower dimension and denoted by $\dim_L(E)$. Thus the pair of Assouad-type dimensions can be referred to as the (upper) Assouad dimension $\overline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{A}}(E)$ and the lower (Assouad) dimension $\underline{\dim}_{\mathrm{A}}(E) = \dim_L(E)$. Also other names, such as (uniform) metric dimension and minimal dimensional number, respectively, have been used. An early reference concerning the lower (Assouad) dimension is [21], and more recently some basic properties of this dimension have been discussed e.g. in [10] and [18].

Remark 2.4. It should be noted that in the literature there are some slight differences in the definitions of the upper and lower Assouad dimensions. In particular, sometimes the covering inequalities in (2.1) and (2.2) are required to hold only for $0 < r < R \le R_0$, for some fixed $R_0 < \infty$. This change may affect the dimensions of unbounded sets. Notice also that in (2.1) we may omit the upper bound $R < \operatorname{diam}(E)$ without altering the value of the upper Assouad dimension. On the other hand, if we omit this upper bound in (2.2), then all bounded sets would have lower Assouad dimension equal to zero, which is perhaps not so desirable.

Recall that a closed set $E \subset \mathbb{R}^n$ is called (Ahlfors–David) λ-regular, or a λ-set, for $0 \le \lambda \le n$, if there is a constant $C \ge 1$ such that

$$C^{-1} r^\lambda \le \mathcal{H}^\lambda\big(E \cap \bar{B}(x,r)\big) \le C r^\lambda \tag{2.3}$$

for every $x \in E$ and $0 < r < \operatorname{diam}(E)$; for $\lambda = 0$ the upper bound $r < \operatorname{diam}(E)$ is omitted.

Examples of λ-regular sets include subspaces of $\mathbb{R}^n$ and self-similar fractals satisfying the open set condition. It is not hard to see that for a λ-regular set $E \subset \mathbb{R}^n$ the upper and lower Assouad dimensions agree. More precisely, if $E \subset \mathbb{R}^n$ is λ-regular then

$$\overline{\dim}_{\mathrm{A}}(E) = \underline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{H}}(E) = \lambda.$$

In order to examine the relation between the lower Assouad dimension and the Hausdorff dimension for more general sets, we consider the following density condition for Hausdorff contents.

Definition 2.5. Let $0 \le \lambda \le n$. We say that a set $E \subset \mathbb{R}^n$ satisfies the λ-Hausdorff content density condition if there exists a constant C such that

$$\mathcal{H}^\lambda_\infty\big(E \cap \bar{B}(x,R)\big) \ge C R^\lambda \tag{2.4}$$

for every $x \in E$ and $0 < R < \operatorname{diam}(E)$.

Sometimes the upper bound $R < \operatorname{diam}(E)$ is omitted in Definition 2.5, but then a bounded set can not satisfy this condition for any $\lambda > 0$.

The λ-Hausdorff content density condition holds for a set $E \subset \mathbb{R}^n$ if and only if there is a constant C such that if $\{B(x_i, r_i) : i \in \mathbb{N}\}$ is a cover of $E \cap \bar{B}(x,R)$, for $x \in E$ and $0 < R < \operatorname{diam}(E)$, then

$$\sum_{i=1}^{\infty} r_i^\lambda \ge C R^\lambda. \tag{2.5}$$

If we only use balls $B(x_i, r)$ having a fixed radius $0 < r < R$, then (2.5) reads as

$$\sum_{i=1}^{N} r^{\lambda} \geq C R^{\lambda}, \quad \text{or equivalently, } N \geq C\Big(\frac{R}{r}\Big)^{\lambda}, \tag{2.6}$$

which is exactly (2.2) for $E \cap \bar{B}(x, R)$.

Condition (2.6) might seem *a priori* much weaker than (2.5). However, when required to hold uniformly for every $x \in E$ and $0 < R < \operatorname{diam}(E)$, these conditions are almost equivalent for closed sets. That is, the estimate in (2.7), for covers using balls of fixed radii r, yields a corresponding estimate (2.8) for covers where balls of all radii are allowed. The price to pay is a small drop in the dimensional parameter λ.

Lemma 2.6. *Let $E \subset \mathbb{R}^n$ be a closed set. Assume that there exist $0 < \lambda_0 \leq n$ and a constant C_1 such that*

$$N\big(E \cap \bar{B}(x, R), r\big) \geq C_1\Big(\frac{R}{r}\Big)^{\lambda_0} \tag{2.7}$$

for every $x \in E$ and $0 < r < R < \operatorname{diam}(E)$. Then, for every $0 < \lambda < \lambda_0$, there exists a constant C such that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap \bar{B}(x, R)\big) \geq C R^{\lambda} \tag{2.8}$$

for every $x \in E$ and $0 < R < \operatorname{diam}(E)$.

The proof of Lemma 2.6 requires a bit work. Roughly speaking, the idea is to construct a Cantor-type set $F \subset E \cap \bar{B}(x, R)$ by using (2.7) iteratively, and then deduce (2.8) with the help of the equally distributed probability measure μ on F. We omit the details, which are similar to those in [17, Theorem 3.1] and [23, Lemma 4.1].

Lemma 2.6 has several important consequences. The following theorem shows that the lower Assouad dimension of closed sets can be characterized using the Hausdorff content density condition.

Theorem 2.7. *Let $E \subset \mathbb{R}^n$ be a closed set and assume that $0 \leq \lambda < \underline{\dim}_{\mathrm{A}}(E)$. Then E satisfies the λ-Hausdorff content density condition. Moreover, $\underline{\dim}_{\mathrm{A}}(E)$ is the supremum of the exponents $\lambda \geq 0$ for which E satisfies the λ-Hausdorff content density condition.*

Proof. Choose λ_0 satisfying $0 \leq \lambda < \lambda_0 < \underline{\dim}_{\mathrm{A}}(E)$. The definition of the lower Assouad dimension implies that (2.7) holds with a constant C_1 for every $x \in E$ and $0 < r < R < \operatorname{diam}(E)$. Thus we obtain from Lemma 2.6 that

$$\mathcal{H}^{\lambda}_{\infty}\big(E \cap \bar{B}(x, R)\big) \geq C R^{\lambda}$$

for every $x \in E$ and $0 < R < \operatorname{diam}(E)$; that is, E satisfies the λ-Hausdorff content density condition.

Assume then that E satisfies the λ-Hausdorff content density condition. Fix $x \in E$ and $0 < r < R < \operatorname{diam}(E)$, and let $\{B(x_i, r) : i = 1, \dots, N\}$ be a cover of $E \cap \bar{B}(x, R)$. Then

$$R^{\lambda} \leq C \mathcal{H}^{\lambda}_{\infty}\big(E \cap \bar{B}(x, R)\big) \leq C \sum_{i=1}^{N} r^{\lambda} = C N r^{\lambda},$$

and so $N \geq C\big(\frac{R}{r}\big)^{\lambda}$. Since this holds for all such covers, we have

$$N\big(E \cap \bar{B}(x,R), r\big) \geq C\Big(\frac{R}{r}\Big)^{\lambda}.$$

Thus $\underline{\dim}_{\mathrm{A}}(E) \geq \lambda$, and the proof is complete. □

Theorem 2.7 yields a comparison between the Hausdorff dimension and the lower Assouad dimension of a closed set. Such a comparison was first obtained in [21].

Corollary 2.8. *Let $E \subset \mathbb{R}^n$ be a closed set. Then*

$$\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}\big(E \cap \bar{B}(x,r)\big) \leq \dim_{\mathrm{H}}(E)$$

for every $x \in E$ and $r > 0$.

Proof. The second inequality follows from the monotonicity of the Hausdorff dimension. For the first inequality we may clearly assume that $\underline{\dim}_{\mathrm{A}}(E) > 0$ and $0 < r < \operatorname{diam}(E)$. Fix $0 \leq \lambda < \underline{\dim}_{\mathrm{A}}(E)$. By Theorem 2.7, we then have $\mathcal{H}^{\lambda}_{\infty}\big(E \cap \bar{B}(x,r)\big) > 0$. Hence $\lambda \leq \dim_{\mathrm{H}}\big(E \cap \bar{B}(x,r)\big)$, and the claim follows. □

The assumption that E is closed is necessary in Corollary 2.8. Indeed, it is easy to see that $\underline{\dim}_{\mathrm{A}}(\bar{E}) = \underline{\dim}_{\mathrm{A}}(E)$ for all $E \subset \mathbb{R}^n$, and hence for instance

$$\underline{\dim}_{\mathrm{A}}(\mathbb{Q}^n) = \underline{\dim}_{\mathrm{A}}(\mathbb{R}^n) = n \not\leq 0 = \dim_{\mathrm{H}}\big(\mathbb{Q}^n \cap B(x,r)\big)$$

for every $x \in \mathbb{Q}^n$ and $r > 0$.

For comparison, we recall also the definitions of the Minkowski (or box-counting) dimensions of bounded sets. As before, we let $N(E,r)$ be the minimal number of open balls of radius r that are needed to cover the bounded set $E \subset \mathbb{R}^n$. Then the upper Minkowski dimension of E, $\overline{\dim}_{\mathrm{M}}(E)$, can be defined as the infimum of all $\lambda \geq 0$ for which there exists a constant C such that $N(E,r) \leq Cr^{-\lambda}$ for every $0 < r < \operatorname{diam}(E)$. Correspondingly, the lower Minkowski dimension of E, $\underline{\dim}_{\mathrm{M}}(E)$, is the supremum of all $\lambda \geq 0$ for which there exists a constant C such that $N(E,r) \geq Cr^{-\lambda}$ for every $0 < r < \operatorname{diam}(E)$.

It follows easily from these definitions that

$$\underline{\dim}_{\mathrm{A}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E)$$

for all bounded sets $E \subset \mathbb{R}^n$. Moreover, if $E \subset \mathbb{R}^n$ is compact, then

$$\underline{\dim}_{\mathrm{A}}(E) \leq \dim_{\mathrm{H}}(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{A}}(E).$$

A typical example with strict inequalities is the set $E = \{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$, for which $\underline{\dim}_{\mathrm{A}}(E) = \dim_{\mathrm{H}}(E) = 0$, $\underline{\dim}_{\mathrm{M}}(E) = \overline{\dim}_{\mathrm{M}}(E) = \frac{1}{2}$, and $\overline{\dim}_{\mathrm{A}}(E) = 1$.

3 The Aikawa condition

The following integrability condition for the distance function creates a natural link between the (upper) Assouad dimension and the Muckenhoupt A_p properties of distance weights, see Section 4. This condition was introduced and used by Aikawa in connection with the so-called quasiadditivity property of Riesz capacities in [1], see also [2, Part II, Section 7]. In [20] and [22] this condition was applied in the context of Hardy inequalities.

Definition 3.1. Let $E \subset \mathbb{R}^n$ be a non-empty set. We say that E satisfies the Aikawa condition for $\alpha \in \mathbb{R}$, if there exists a constant C (depending on α) such that

$$\int_{B(x,r)} d(y,E)^{-\alpha}\, dy \le C r^{n-\alpha} \tag{3.9}$$

or, equivalently,

$$\fint_{B(x,r)} d(y,E)^{-\alpha}\, dy \le C r^{-\alpha} \tag{3.10}$$

for every $x \in E$ and $r > 0$. Here we use the convention that $0^0 = 1$, and if $\alpha > 0$ then we also require that $|\overline{E}| = 0$.

We let $\mathcal{A}(E)$ denote the set of all $\alpha \in \mathbb{R}$ for which E satisfies the Aikawa condition.

It is easy to see that a non-empty set $E \subset \mathbb{R}^n$ satisfies the Aikawa condition for all $\alpha \le 0$. On the other hand, if $\alpha \ge n$, then

$$\int_{B(x,r)} d(y,E)^{-\alpha}\, dy \ge \int_{B(x,r)} |y-x|^{-\alpha}\, dy = \infty$$

for every $x \in E$ and $r > 0$, and thus E does not satisfy the Aikawa condition for any $\alpha \ge n$. Hence we may restrict our attention to the range $0 < \alpha < n$ in the Aikawa condition.

We now begin to examine the close connections between the upper Assouad dimension and the Aikawa condition.

Lemma 3.2. *Let $E \subset \mathbb{R}^n$. If $\alpha \in \mathcal{A}(E)$, then $\overline{\dim}_{\mathrm{A}}(E) \le n - \alpha$.*

Proof. If $\alpha \le 0$, then the claim is clear since $\overline{\dim}_{\mathrm{A}}(E) \le n$. Hence we may assume that $0 < \alpha < n$. Fix $x \in E$ and $0 < r < R$, and write $F = E \cap \overline{B}(x,R)$. By the existence of maximal packings there are pairwise disjoint open balls $B(x_i, \frac{r}{2})$, $i = 1, \ldots, N$, with $x_i \in F$, such that $F \subset \bigcup_{i=1}^N B(x_i, r)$.

Let F_r be the r-neighborhood of F, that is,

$$F_r = \{y \in \mathbb{R}^n : d(y,F) < r\} \subset B(x,2R).$$

Using the pairwise disjointness of the balls $B(x_i, \frac{r}{2}) \subset F_r$, the fact that $d(y,E) \le d(y,F) < r$ for all $y \in F_r$, and the assumed Aikawa condition (3.9), we obtain

$$NCr^n \le \sum_{i=1}^{N} \left|B(x_i, \tfrac{r}{2})\right| \le |F_r| \le r^\alpha \int_{F_r} d(y,E)^{-\alpha}\, dy$$
$$\le r^\alpha \int_{B(x,2R)} d(y,E)^{-\alpha}\, dy \le r^\alpha C R^{n-\alpha} = Cr^n \Big(\frac{R}{r}\Big)^{n-\alpha}.$$

Thus

$$N\big(E \cap \overline{B}(x,R), r\big) = N(F,r) \le N \le C\Big(\frac{R}{r}\Big)^{n-\alpha},$$

and the claim $\overline{\dim}_{\mathrm{A}}(E) \le n-\alpha$ follows since $n-\alpha > 0$. □

For the converse direction we need to assume a strict upper bound for the dimension. See, however, also Theorem 3.5 below concerning the strict inequality in the previous Lemma 3.2.

Lemma 3.3. *Let $E \subset \mathbb{R}^n$ be a non-empty set. If $\alpha \in \mathbb{R}$ and $\overline{\dim}_{\mathrm{A}}(E) < n-\alpha$, then $\alpha \in \mathcal{A}(E)$.*

Proof. Again, the claim is clear if $\alpha \le 0$, and so we may assume that $\alpha > 0$. Choose $\overline{\dim}_{\mathrm{A}}(E) < \lambda < n-\alpha$, and let $x \in E$ and $r > 0$. Define

$$F_j = \big\{ y \in B(x,r) : d(y,E) < 2^{-j+1} r \big\} \quad \text{and} \quad A_j = F_j \setminus F_{j+1},$$

for $j \in \mathbb{N}$. Since $\lambda > \overline{\dim}_{\mathrm{A}}(E)$, there is a constant C_1 such that the set $E \cap \overline{B}(x,2r)$ can be covered by $N_j \le C_1 2^{j\lambda}$ balls of radius $2^{1-j} r$, for every $j \in \mathbb{N}$. It follows that each F_j can be covered by at most N_j balls of radius $2^{2-j} r$. If B_i^j, $i = 1, \ldots, N_j$, are such balls, then

$$|F_j| \le \sum_{i=1}^{N_j} \left|B_i^j\right| \le N_j C (2^{2-j} r)^n \le C (2^{-j})^{n-\lambda} r^n. \tag{3.11}$$

Since $\overline{E} \cap B(x,r) \subset F_j$ for all $j \in \mathbb{N}$ and $\lambda < n-\alpha < n$, by letting $j \to \infty$ we see in particular that $|\overline{E} \cap B(x,r)| = 0$. Here $r > 0$ is arbitrary, and thus $|\overline{E}| = 0$.

If $y \in A_j$, then $2^{-j} r \le d(y,E) < 2^{-j+1} r$. In addition, $A_j \subset F_j$ for all $j \in \mathbb{N}$ and the sets A_j cover $B(x,r)$ up to the set $\overline{E} \cap B(x,r)$, which has measure zero. By using estimate (3.11) we obtain

$$\int_{B(x,r)} d(y,E)^{-\alpha}\, dy \le C \sum_{j=1}^{\infty} \int_{A_j} d(y,E)^{-\alpha}\, dy \le C \sum_{j=1}^{\infty} |F_j| (2^{-j} r)^{-\alpha}$$
$$\le C r^{n-\alpha} \sum_{j=1}^{\infty} (2^{-j})^{n-\lambda-\alpha} \le C r^{n-\alpha},$$

where the geometric series converges since $\lambda < n-\alpha$. This together with the fact $|\overline{E}| = 0$ shows that $\alpha \in \mathcal{A}(E)$. □

In order to combine the two lemmas above into a characterization, we need the following improvement property for the Aikawa condition, observed in [20]. It is easy to see that the Aikawa condition, for $0 < \alpha < n$, implies a reverse Hölder inequality, see (3.12) below. After that we can apply a suitable version of the so-called Gehring lemma, see [13, Lemma 3], which is a deep result concerning the improvement of reverse Hölder inequalities. This leads to the Aikawa condition for an exponent larger than α. (Notice that conversely it is easy to see that the Aikawa condition, for $0 < \alpha < n$, implies Aikawa conditions for all exponents smaller than α.)

Theorem 3.4. *Let $E \subset \mathbb{R}^n$ and $0 < \alpha < n$. If $\alpha \in \mathcal{A}(E)$, then there exists $\alpha < \alpha' < n$ such that $\alpha' \in \mathcal{A}(E)$.*

Proof. Fix a ball $B(x,r) \subset \mathbb{R}^n$ and assume first that $B(x,2r) \cap E \neq \emptyset$. Then $d(y,E) \le 3r$ for every $y \in B(x,r)$, and thus the assumed Aikawa condition (3.10) implies

$$\fint_{B(x,r)} d(y,E)^{-\alpha}\,dy \le Cr^{-\alpha} = C\big(r^{-\frac{\alpha}{2}}\big)^2 \le C\bigg(\fint_{B(x,r)} d(y,E)^{-\frac{\alpha}{2}}\,dy\bigg)^2.$$

It is easy to see that the same conclusion holds also in the case $B(x,2r) \cap E = \emptyset$. Writing $f(y) = d(y,E)^{-\frac{\alpha}{2}}$, we obtain the reverse Hölder inequality

$$\bigg(\fint_{B(x,r)} f(y)^2\,dy\bigg)^{\frac{1}{2}} \le C \fint_{B(x,r)} f(y)\,dy, \tag{3.12}$$

for every ball $B(x,r) \subset \mathbb{R}^n$.

By the Gehring lemma, there exists $p > 2$ such that

$$\bigg(\fint_{B(x,r)} f(y)^p\,dy\bigg)^{\frac{1}{p}} \le C \fint_{B(x,r)} f(y)\,dy \le C\bigg(\fint_{B(x,r)} f(y)^2\,dy\bigg)^{\frac{1}{2}},$$

for every ball $B(x,r) \subset \mathbb{R}^n$, where the second inequality is just the usual Hölder's inequality. Choose $\alpha' = \frac{p}{2}\alpha > \alpha$. Then the estimate above and the assumed Aikawa condition give

$$\bigg(\fint_{B(x,r)} d(y,E)^{-\alpha'}\,dy\bigg)^{\frac{\alpha}{2\alpha'}} \le C\bigg(\fint_{B(x,r)} d(y,E)^{-\alpha}\,dy\bigg)^{\frac{1}{2}} \le Cr^{-\frac{\alpha}{2}},$$

for every $x \in E$ and $r > 0$, and this implies the Aikawa condition for $\alpha' > \alpha$. □

We are now prepared to characterize the upper Assouad dimension in terms of the Aikawa condition. This result is essentially from [26], where corresponding characterizations were obtained also in more general metric spaces.

Theorem 3.5. *Let $E \subset \mathbb{R}^n$ be a non-empty set and let $\alpha > 0$. Then $\alpha \in \mathcal{A}(E)$ if and only if $\overline{\dim}_{\mathrm{A}}(E) < n - \alpha$.*

Proof. If $\overline{\dim}_A(E) < n - \alpha$, then $\alpha \in \mathcal{A}(E)$ by Lemma 3.3.

Assume then that $0 < \alpha \in \mathcal{A}(E)$. Since $\alpha < n$, by Theorem 3.4 there is $\alpha' > \alpha$ such that also $\alpha' \in \mathcal{A}(E)$. Thus Lemma 3.2 yields $\overline{\dim}_A(E) \le n - \alpha' < n - \alpha$, as desired. □

Notice that the assumption $\alpha > 0$ in Theorem 3.5 is essential: if $E \subset \mathbb{R}^n$ and $\overline{\dim}_A(E) = n$, then $0 \in \mathcal{A}(E)$, but $\overline{\dim}_A(E) \not< n - 0$.

4 Muckenhoupt weights

A measurable function $w\colon \mathbb{R}^n \to \mathbb{R}$ is called a weight in $\mathbb{R}^n$ if $w(x) > 0$ for almost every $x \in \mathbb{R}^n$ and $\int_B w(x)\,dx < \infty$ for all balls $B \subset \mathbb{R}^n$. When w is a weight in $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is a measurable set, we write

$$w(E) = \int_E w(x)\,dx.$$

The following classes of Muckenhoupt weights are important tools for instance in harmonic analysis; we refer to [12, Chapter IV] for a thorough discussion. Muckenhoupt weighted $\mathbb{R}^n$ is also an example of a metric space with a doubling measure and supporting a p-Poincaré inequality, which are the standard assumptions in analysis on metric spaces; see for instance [6, 14] and the references therein for more information.

Definition 4.1. Let w be a weight in $\mathbb{R}^n$. We say that w belongs to the Muckenhoupt class

(a) A_p, for $1 < p < \infty$, if there is a constant C such that

$$\left(\fint_B w(x)\,dx\right)\left(\fint_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le C \tag{4.13}$$

for every ball $B \subset \mathbb{R}^n$.

(b) A_1, if there is a constant C such that

$$\left(\fint_B w(x)\,dx\right)\operatorname*{ess\,sup}_{x\in B}\frac{1}{w(x)} \le C, \tag{4.14}$$

for every ball $B \subset \mathbb{R}^n$.

(c) A_∞, if there are constants $C, \delta > 0$ such that

$$\frac{w(E)}{w(B)} \le C\left(\frac{|E|}{|B|}\right)^{\delta}$$

whenever $B \subset \mathbb{R}^n$ is a ball and $E \subset B$ is a measurable set.

It is easy to verify directly from the A_p condition (4.13) that if $1 < p < \infty$ and w is a weight in $\mathbb{R}^n$, then

$$w \in A_p \quad \text{if and only if} \quad w^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}}. \tag{4.15}$$

Moreover, an application of Hölder's inequality shows that if $1 \le p < q < \infty$, then $A_p \subset A_q$.

The class A_∞ can be characterized as the union of all A_p, for $1 \le p < \infty$, that is,

$$A_\infty = \bigcup_{1 \le p < \infty} A_p. \tag{4.16}$$

Neither of the inclusions in (4.16) is trivial. The main tool for establishing both of them is a reverse Hölder inequality, but we omit the details; see e.g. [12, Chapter IV, Section 2]. We do not really need the class A_∞ below, since all statements "$w \in A_\infty$" could be replaced by the statement "$w \in A_p$ for some $1 \le p < \infty$".

Example 4.2. Consider the weight $w(y) = |y|^{-\alpha}$ for every $y \in \mathbb{R}^n \setminus \{0\}$. It is straightforward to verify by direct computations that $w \in A_1$ if and only if $0 \le \alpha < n$, and $w \in A_p$, for $1 < p < \infty$, if and only if $(1-p)n < \alpha < n$.

Our main interest in this section is in the generalizations of Example 4.2 to more general distance functions, that is, for weights of the type $w(y) = d(y, E)^{-\alpha}$, with $E \subset \mathbb{R}^n$ satisfying $|\overline{E}| = 0$. The Aikawa condition is tailor-made for the study of this problem; see [1, 2], in particular [2, p. 151].

Theorem 4.3. *Let $E \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, and define $w(y) = d(y, E)^{-\alpha}$ for every $y \in \mathbb{R}^n$. Then the following assertions hold.*

1. *If $0 \le \alpha \in \mathcal{A}(E)$, then $w \in A_p$ for every $1 \le p \le \infty$.*
2. *If $\alpha < 0$ and $1 < p < \infty$ are such that $\frac{-\alpha}{p-1} \in \mathcal{A}(E)$, then $w \in A_p$.*

Proof. Consider first part 1. If $\alpha = 0$, then $w(y) = 1$ for every $y \in \mathbb{R}^n$, and it follows that $w \in A_p$ for every $1 \le p \le \infty$. Assume then that $0 < \alpha < n$ and that (3.9) holds with a constant C_1, that is,

$$\int_{B(x,r)} w(y)\, dy \le C_1 r^{n-\alpha} < \infty$$

for every $x \in E$ and $r > 0$. This implies that w is locally integrable. Since $\alpha \in \mathcal{A}(E)$ and $\alpha > 0$, we have $|\overline{E}| = 0$. Therefore $w(x) > 0$ for almost every $x \in \mathbb{R}^n$, and thus w is a weight.

Since $A_1 \subset A_p$ for every $p \ge 1$, it suffices to show that $w \in A_1$. Fix a ball $B(x,r) \subset \mathbb{R}^n$ and assume first that $B(x, 2r) \cap E \neq \emptyset$. Then $B(x,r) \subset B(z, 3r)$, for some $z \in E$, and so the assumed Aikawa condition (3.10) implies

$$⨍_{B(x,r)} w(y)\, dy \le C ⨍_{B(z,3r)} d(y, E)^{-\alpha}\, dy \le C(3r)^{-\alpha} = Cr^{-\alpha}.$$

On the other hand, if $y \in B(x,r) \setminus \overline{E}$, then

$$\frac{1}{w(y)} = d(y,E)^\alpha \le d(y,z)^\alpha \le (3r)^\alpha = Cr^\alpha,$$

since $\alpha > 0$. By combining the estimates above and recalling that $|\overline{E}| = 0$, we obtain

$$\left(\fint_{B(x,r)} w(y)\,dy\right) \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{w(y)} \le C.$$

This shows that the A_1 condition (4.14) holds for the ball $B(x,r)$ if $B(x,2r) \cap E \neq \emptyset$.

Assume then that $B(x,2r) \cap E = \emptyset$. In this case

$$\tfrac{1}{2} d(y,E) \le d(x,E) \le 2d(y,E)$$

for every $y \in B(x,r)$, and thus

$$\left(\fint_{B(x,r)} w(y)\,dy\right) \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{w(y)} \le Cd(x,E)^{-\alpha} d(x,E)^\alpha \le C.$$

Hence (4.14) holds also in the case $B(x,2r) \cap E = \emptyset$, and the proof of part 1 is complete.

In part 2, we let

$$\sigma(y) = w(y)^{-\frac{1}{p-1}} = d(y,E)^{\frac{\alpha}{p-1}}$$

for every $y \in \mathbb{R}^n$. By part 1 we have $\sigma \in A_1 \subset A_{\frac{p}{p-1}}$, and the claim $w \in A_p$ follows from the duality property (4.15) of A_p weights. □

There is also a partial converse of Theorem 4.3, see Theorem 4.5 below. We recall that a set $E \subset \mathbb{R}^n$ is porous, if there exists a constant C such that for every $x \in \mathbb{R}^n$ and $r > 0$ there exists $z \in \mathbb{R}^n$ such that $B(z, Cr) \subset B(x,r) \setminus E$. Porosity can also be characterized using the upper Assouad dimension:

Theorem 4.4. *A set $E \subset \mathbb{R}^n$ is porous if and only if $\overline{\dim}_{\mathrm{A}}(E) < n$.*

For the proof of Theorem 4.4, see for instance [29, Theorem 5.2]. Note that by Theorem 3.5 the conditions in Theorem 4.4 hold if and only if there is $\alpha > 0$ such that $\alpha \in \mathcal{A}(E)$.

Theorem 4.5. *Assume that $E \subset \mathbb{R}^n$ is a non-empty porous set. Let $\alpha \in \mathbb{R}$ and define $w(y) = d(y,E)^{-\alpha}$ for every $y \in \mathbb{R}^n$. Then the following assertions hold.*

1. *If $\alpha \ge 0$, $1 \le p < \infty$, and $w \in A_p$, then $\alpha \in \mathcal{A}(E)$.*
2. *If $\alpha < 0$, $1 < p < \infty$, and $w \in A_p$, then $\frac{-\alpha}{p-1} \in \mathcal{A}(E)$.*

Proof. In part 1 we may assume that $p > 1$. Let $B_0 = B(x,r)$ be a ball. Since E is porous, there is $z \in B_0$ such that $B(z, Cr) \subset B(x,r) \setminus E$. Then $d(y,E) \ge \frac{C}{2} r$ for every $y \in B = B(z, \frac{C}{2} r)$, and since the measures of B_0 and B are comparable, we obtain

$$\left(\fint_{B_0} w(y)^{-\frac{1}{p-1}}\,dy\right)^{p-1} \ge C\left(\fint_{B} w(y)^{-\frac{1}{p-1}}\,dy\right)^{p-1}$$
$$\ge C\left(\fint_{B} r^{\frac{\alpha}{p-1}}\,dy\right)^{p-1} \ge Cr^{\alpha}.$$

On the other hand, the A_p condition (4.13) for $w \in A_p$ gives

$$\left(\fint_{B_0} w(y)\,dy\right)\left(\fint_{B_0} w(y)^{-\frac{1}{p-1}}\,dy\right)^{p-1} \le C.$$

By combining the two estimates above we obtain

$$\fint_{B_0} d(y,E)^{-\alpha}\,dy = \fint_{B_0} w(y)\,dy \le C\left(\fint_{B_0} w(y)^{-\frac{1}{p-1}}\,dy\right)^{1-p} \le Cr^{-\alpha},$$

and thus $\alpha \in \mathcal{A}(E)$.

Then we consider part 2. If $w \in A_p$, for $1 < p < \infty$, we have by the A_p duality in (4.15) that

$$d(\cdot,E)^{-(\frac{-\alpha}{p-1})} = d(\cdot,E)^{\frac{\alpha}{p-1}} = w^{-\frac{1}{p-1}} \in A_{\frac{p}{p-1}}.$$

Hence the claim follows from part 1. □

For porous sets we now have a complete characterization of the A_p properties of the distance functions.

Theorem 4.6. *Let $1 < p < \infty$ and assume that $E \subset \mathbb{R}^n$ is a non-empty porous set. Let $\alpha \in \mathbb{R}$ and define $w(y) = d(y,E)^{-\alpha}$ for every $y \in \mathbb{R}^n$. Then the following assertions hold.*

1. $w \in A_1$ if and only if $0 \le \alpha < n - \overline{\dim}_{\mathrm{A}}(E)$.
2. $w \in A_p$ if and only if

$$(1-p)\big(n - \overline{\dim}_{\mathrm{A}}(E)\big) < \alpha < n - \overline{\dim}_{\mathrm{A}}(E). \tag{4.17}$$

Proof. Since E is porous, $\overline{\dim}_{\mathrm{A}}(E) < n$ by Theorem 4.4.

We consider first part 2. If $0 \le \alpha < n - \overline{\dim}_{\mathrm{A}}(E)$, we have $\alpha \in \mathcal{A}(E)$ by Lemma 3.3 and thus part 1 of Theorem 4.3 implies $w \in A_p$. On the other hand, if

$$(1-p)\big(n - \overline{\dim}_{\mathrm{A}}(E)\big) < \alpha < 0,$$

then

$$0 < \frac{-\alpha}{p-1} < n - \overline{\dim}_{\mathrm{A}}(E).$$

From Lemma 3.3 we obtain $\frac{-\alpha}{p-1} \in \mathcal{A}(E)$ and hence $w \in A_p$ by part 2 of Theorem 4.3.

Conversely, assume that $w \in A_p$. If $\alpha > 0$, part 1 of Theorem 4.5 implies $\alpha \in \mathcal{A}(E)$, and so $\alpha < n - \overline{\dim}_{\mathrm{A}}(E)$ by Theorem 3.5. If $\alpha = 0$, then (4.17) holds since $\overline{\dim}_{\mathrm{A}}(E) < n$ by porosity. Finally, if $\alpha < 0$, then $\frac{-\alpha}{p-1} \in \mathcal{A}(E)$ by part 2 of Theorem 4.5. Theorem 3.5 gives

$$0 < \frac{-\alpha}{p-1} < n - \overline{\dim}_{\mathrm{A}}(E),$$

showing that (4.17) holds also in this case. The proof of part 2 is complete.

Consider then part 1. If $0 \leq \alpha < n - \overline{\dim}_{\mathrm{A}}(E)$, the claim $w \in A_1$ follows from Lemma 3.3 and part 1 of Theorem 4.3 just as in part 2. Conversely, if $w \in A_1$ and $\alpha > 0$, then $\alpha < n - \overline{\dim}_{\mathrm{A}}(E)$ by part 1 of Theorem 4.5 and Theorem 3.5. If $\alpha = 0$, then $0 \leq \alpha < n - \overline{\dim}_{\mathrm{A}}(E)$ holds since $\overline{\dim}_{\mathrm{A}}(E) < n$ by porosity. Finally, it is easy to see that $\alpha \geq 0$ is necessary in part 1, and this completes the proof. □

Remark 4.7. The fact that the A_p properties of the weights $w(y) = d(y, E)^{-\alpha}$ depend on the dimension(s) of $E \subset \mathbb{R}^n$ has certainly been part of the mathematical folklore, at least for suitably nice sets E. Aikawa [1, 2] mentions explicitly the connections between the Aikawa condition and A_p weights. Horiuchi [15, 16] used a different dimensional condition, called property $P(s)$, in the study of A_p properties of distance weights and in particular distance weighted Sobolev-type embeddings. It was shown in [27, Theorem 3.4] that also this property $P(s)$ can be characterized using the upper Assouad dimension. A sufficient condition in the spirit of Theorem 4.3 was given in [7, Lemma 3.3] for subsets of λ-regular sets of $\mathbb{R}^n$.

Theorem 4.6 was formulated in [8], where corresponding results were also obtained in metric spaces in terms of the so-called lower Assouad codimension. Metric space results of this type were considered in [3], as well, but using a completely different approach and under the stronger assumption that both the space X and the set $E \subset X$ satisfy Ahlfors–David regularity conditions; see [3, Theorems 6 and 7].

5 Hardy–Sobolev inequalities

Hardy–Sobolev inequalities interpolate between the Sobolev inequality and the p-Hardy inequality. Indeed, for $q = p^* = \frac{np}{n-p}$ inequality (5.18) is the Sobolev inequality, while for $q = p$ we recover the p-Hardy inequality.

Definition 5.1. Let $1 < p \leq q \leq \frac{np}{n-p} < \infty$ and let $\Omega \subsetneq \mathbb{R}^n$ be an open set. We say that the (q, p)-Hardy–Sobolev inequality holds in Ω if there is a constant C such that

$$\left(\int_\Omega |u(x)|^q d(x, \Omega^c)^{\frac{q}{p}(n-p)-n}\, dx \right)^{\frac{1}{q}} \leq C \left(\int_\Omega |\nabla u(x)|^p\, dx \right)^{\frac{1}{p}} \tag{5.18}$$

for every $u \in C_0^\infty(\Omega)$.

We also consider weighted versions of these inequalities and say that the (q, p, β)-Hardy–Sobolev inequality holds in Ω, for $\beta \in \mathbb{R}$, if there is a constant C such that

$$\left(\int_\Omega |u(x)|^q d(x,\Omega^c)^{\frac{q}{p}(n-p+\beta)-n}\,dx\right)^{\frac{1}{q}} \le C\left(\int_\Omega |\nabla u(x)|^p d(x,\Omega^c)^\beta\,dx\right)^{\frac{1}{p}} \tag{5.19}$$

for every $u \in C_0^\infty(\Omega)$.

For $q = p$, the inequality in (5.19) is called the (p,β)-Hardy inequality.

In this final section we formulate (without proofs) sufficient and necessary conditions for Hardy–Sobolev inequalities in $\Omega \subset \mathbb{R}^n$, given in terms of the upper and lower Assouad dimensions (and also other dimensions) of the complement $\Omega^c = \mathbb{R}^n \setminus \Omega$. It has been understood already for a long time that sufficient conditions for these inequalities naturally split into two cases: either the complement Ω^c has to be sufficiently "thick" or sufficiently "thin". The thickness has been formulated, for instance, using capacity density or Hausdorff content density, and thinness using the Aikawa condition. With Assouad dimensions this duality becomes more transparent: thickness means that Ω^c has large lower Assouad dimension, while thinness means that Ω^c has small upper Assouad dimension.

It can also be shown that suitable combinations of such thick and thin parts give sufficient conditions for Hardy–Sobolev inequalities, as well, but these cases will not be discussed in this work; see e.g. [25, Section 7] for details.

In the case of thin complements, the Hardy–Sobolev inequalities can be obtained by using the following general two weight embedding result together with the Aikawa condition and the knowledge about the A_p properties of the distance functions.

Theorem 5.2. *Let $1 < p \le q < \infty$ and let (w,v) be a pair of weights such that $w \in A_\infty$ and $\sigma = v^{-\frac{1}{p-1}} \in A_\infty$. Assume that there exists a constant C_1 such that*

$$\left(\frac{1}{|B|^{1-\frac{1}{n}}}\right)^p w(B)^{\frac{p}{q}}\sigma(B)^{p-1} \le C_1 \tag{5.20}$$

for all balls $B \subset \mathbb{R}^n$. Then there exists a constant C such that

$$\left(\int_{\mathbb{R}^n} |u(x)|^q w(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p v(x)\,dx\right)^{\frac{1}{p}}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$.

Theorem 5.2 can be proved using the mapping properties of Riesz potentials and maximal operators. Muckenhoupt and Wheeden [31, Theorem 1] gave a single weight control for the Riesz potential I_1 in terms of a fractional maximal operator, and Pérez [32, Theorem 1.1] proved a two weight L^p–L^q control for such maximal operators under the assumption in (5.20). The claim of Theorem 5.2 then follows from the the pointwise estimate $|u(x)| \le CI_1|\nabla u|(x)$ for the Riesz potential and the boundedness properties of the maximal operator. See also [33] and [8] for discussion and generalizations of these results to metric spaces.

From Theorem 5.2 we obtain the following weighted global Hardy–Sobolev inequalities where the integrals can be taken over the whole $\mathbb{R}^n$. This is possible since $\overline{\dim}_{\mathrm{A}}(E) < n$ by the assumptions, and consequently $|E| = 0$.

Theorem 5.3. *Let $E \subset \mathbb{R}^n$ be a non-empty closed set and assume that*

$$1 < p \le q \le \frac{np}{n-p} < \infty$$

and

$$\overline{\dim}_A(E) < \min\left\{\frac{q}{p}(n-p+\beta), n - \frac{\beta}{p-1}\right\}.$$

Then the inequality

$$\left(\int_{\mathbb{R}^n} |u(x)|^q d(x,E)^{\frac{q}{p}(n-p+\beta)-n}\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p d(x,E)^\beta\,dx\right)^{\frac{1}{p}} \tag{5.21}$$

holds for every $u \in C_0^\infty(\mathbb{R}^n)$.

In particular, if $E = \Omega^c$ satisfies the assumptions in Theorem 5.3, then the (q,p,β)-Hardy inequality holds in Ω. The dimensional condition in Theorem 5.3, together with Theorem 4.3, implies that the weights in (5.21) satisfy the A_∞ conditions in Theorem 5.2, and then (5.20) for these weights can be checked with the help of the Aikawa condition; see [8, Section 4] for the computations (in the metric setting).

Actually, by the results of Horiuchi [15] (see also [16] and [27, Section 5]) the bound $\overline{\dim}_A(E) < n - \frac{\beta}{p-1}$ can be removed if $\overline{\dim}_A(E) < n-1$, while by [27, Example 4.4] this bound is really needed when $\overline{\dim}_A(E) \ge n-1$ and also sharp at least when $\overline{\dim}_A(E) = n-1$. The proofs in [15] for the case $\overline{\dim}_A(E) < n-1$ however require a completely different approach based on relative isoperimetric inequalities.

On the other hand, it is not hard to show that for $\beta \ge 0$ the bound $\overline{\dim}_A(E) < \frac{q}{p}(n-p+\beta)$ is also necessary for the global Hardy–Sobolev inequality to hold with respect to E (see [27, Theorem 6.1]). Thus we have the following characterization in the case $\beta = 0$.

Theorem 5.4. *Let $1 < p \le q < \frac{np}{n-p} < \infty$ and assume that $E \subset \mathbb{R}^n$ is a non-empty closed set. Then there exists a constant C such that*

$$\left(\int_{\mathbb{R}^n} |u(x)|^q d(x,E)^{\frac{q}{p}(n-p)-n}\,dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\right)^{\frac{1}{p}}, \tag{5.22}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$, if and only if

$$\overline{\dim}_A(E) < \frac{q}{p}(n-p).$$

Under some additional conditions the bound $\overline{\dim}_A(E) < \frac{q}{p}(n-p+\beta)$ is necessary also for $\beta < 0$, see [27, Theorem 6.2] and compare also to Theorem 5.7 below.

We now turn to the case of thick complements. A well-known sufficient condition for the unweighted p-Hardy inequality in $\Omega \subset \mathbb{R}^n$ is the uniform p-fatness of the complement Ω^c, see e.g. [28, 34]. Uniform fatness is a density condition for the variational p-capacity, but in fact Ω^c is uniformly p-fat if and only if Ω^c is unbounded and satisfies the λ-Hausdorff density condition in Definition 2.5 for some $\lambda > n - p$; see [19, Section 2.4] for a discussion.

The Hausdorff content density condition is a natural assumption also for weighted Hardy inequalities, but for $\beta \geq p - 1$ an additional accessibility condition for the boundary $\partial\Omega$ is needed. This leads to the following theorem. We omit the details and refer to [19] and [24] for the definitions and proofs; see also [5] for recent progress concerning such accessibility conditions.

Theorem 5.5. *Let $1 < p < \infty$, $\lambda \geq 0$, and $\beta \in \mathbb{R}$ be such that $\lambda > n - p + \beta$. Assume that $\Omega \subset \mathbb{R}^n$ is an open set such that Ω^c is unbounded and satisfies the λ-Hausdorff content density condition. Moreover, if $\beta \geq p-1$, we assume an additional accessibility condition for $\partial\Omega$. Then the (p, β)-Hardy inequality holds in Ω.*

Combining this with Theorem 2.7 and an interpolation result in [27, Theorem 2.1], we obtain the corresponding Hardy–Sobolev inequalities under an assumption for the lower Assouad dimension of the complement.

Theorem 5.6. *Let $1 < p \leq q \leq \frac{np}{n-p} < \infty$ and $\beta \in \mathbb{R}$ and assume that $\Omega \subset \mathbb{R}^n$ is an open set such that Ω^c is unbounded and $\underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p + \beta$. Moreover, if $\beta \geq p - 1$, we assume an additional accessibility condition for $\partial\Omega$. Then the (q, p, β)-Hardy–Sobolev inequality holds in Ω.*

Proof. Let $\lambda \geq 0$ be such that $\underline{\dim}_{\mathrm{A}}(\Omega^c) > \lambda > n - p + \beta$. By Theorem 2.7 the complement Ω^c satisfies the λ-Hausdorff content density condition (2.4) and thus the (p, β)-Hardy inequality holds in Ω by Theorem 5.5. The (q, p, β)-Hardy–Sobolev inequality then follows from [27, Theorem 2.1]. $\square$

We have seen in Theorems 5.3 and 5.6 that the "dual" conditions

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \frac{q}{p}(n - p + \beta) \quad \text{and} \quad \underline{\dim}_{\mathrm{A}}(\Omega^c) > n - p + \beta,$$

possibly together with some additional requirements, are sufficient for the (q, p, β)-Hardy–Sobolev inequality in $\Omega \subset \mathbb{R}^n$. As was already mentioned, also suitable combinations of these conditions suffice for Hardy–Sobolev inequalities, and this rules out the possibility that the conditions above could characterize the validity of Hardy–Sobolev inequalities. Nevertheless, these conditions are not that far from being also necessary, and at least the dimensional bounds $\frac{q}{p}(n-p+\beta)$ and $n-p+\beta$ are optimal. This can be seen from the following result, which is [27, Theorem 4.6]. Interestingly, also the Hausdorff dimension and the (lower) Minkowski dimension are needed here, and they can not be changed to $\underline{\dim}_{\mathrm{A}}(\Omega^c)$ in the respective bounds. However, in the case $q = p$ the inequalities in these dimensional lower bounds can be made strict, see [22]. From this it follows that if

$$\dim_{\mathrm{H}}(\Omega^c) \leq n - p + \beta \leq \overline{\dim}_{\mathrm{A}}(\Omega^c),$$

then the (p, β)-Hardy inequality can not hold in Ω.

Theorem 5.7. *Let $1 < p \leq q < \frac{np}{n-p} < \infty$ and $\beta \in \mathbb{R}$, and assume that the (q, p, β)-Hardy–Sobolev inequality* (5.19) *holds in an open set $\Omega \subset \mathbb{R}^n$.*

1. If $\beta \geq 0$ and $\frac{q}{p}(n - p + \beta) \neq n$, then either

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \frac{q}{p}(n - p + \beta) \quad or \quad \dim_{\mathrm{H}}(\Omega^c) \geq n - p + \beta.$$

2. If $\beta < 0$ and Ω^c is compact and porous, then either

$$\overline{\dim}_{\mathrm{A}}(\Omega^c) < \frac{q}{p}(n - p + \beta) \quad or \quad \underline{\dim}_{\mathrm{M}}(\Omega^c) \geq n - p + \beta.$$

Examples in [27] show that for $\beta < 0$ the compactness assumption can not be completely omitted. However, compactness can be relaxed to the condition that $x \mapsto d(x, \Omega^c)^\beta$ is locally integrable, which in turn holds, for instance, if we assume that $\overline{\dim}_{\mathrm{M}}(\Omega^c \cap B) < n + \beta$ for all balls B centered at Ω^c. It is not known if the porosity assumption is necessary or if the lower Minkowski dimension (instead of the Hausdorff dimension) is really needed in the case $\beta < 0$.

In conclusion, the moral of this final section is not so much in the actual formulations of all these conditions for Hardy–Sobolev inequalities, but rather in the fact that all five notions of dimensions mentioned in this article (Hausdorff, upper and lower Assouad, and upper and lower Minkowski) have made an appearance. Moreover, in the light of examples at least three of these (Hausdorff, upper and lower Assouad) are certainly needed in order to state the optimal conditions for the validity of Hardy–Sobolev inequalities in a somewhat uniform and condensed way.

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