The Black-Scholes Model and Risk-Sensitive Asset Management

Santeri Helin

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University of Jyvaskyla
Department of Mathematics and Statistics
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1. Tiivistelmä

Optiohinnoittelun teoria on keskeisessä osassa tutkielmaamme ja tavoitteenamme on saada optiohinnoittelun teoriaa käyttäen teoreettinen estimaatti option reilusta hinnoittelusta. Tätä option reilua hintaa sijoittajat voivat käyttää myöhemmin salkkujensa arvon maksimointiin. Yksi kuuluisimmista malleista optioiden hinnoittelussa on Black-Scholes-malli.

Black-Scholes-malli on keskeisessä roolissa modernissa finanssiteoriassa ja on käytössä myös tällä hetkellä. Mallin käyttämisessä yksi suurimmista eduista on, että malli riippuu ainoastaan yhdestä ei havaittavissa olevasta parametrista σ nimeltään volatiliteetti. Tämä huomataan tutkielmassa johdettaessa Black-Scholes-yhtälöä. Tämän volatiliteetin johtamiseen on olemassa myös keinoja, mutta emme keskity niihin tutkielman aikana.

Oletamme tutkielman aikana, että volatiliteetti pysyy vakiona, jotta laskut voitaisiin tehdä. Tämä ei kuitenkaan vastaa oikeaa tilannetta sijoittamisessa, sillä volatiliteetti voi vaihdella ajan kuluessa. Black-Scholes-yhtälöä johdettaessa oletamme myös, että sijoittaessa ei ilmaannu veroja tai rahansiirron aikana tulevia kustannuksia. Lisäksi tutkimme Black-Scholes-mallissa ainoastaan Euroopan optioita, koska kyseisessä mallissa optiot voidaan suorittaa ainoastaan niiden ennalta säädellyn viimeisen käyttöpäivän ajanhetkellä.

Tutkimuksemme koostuu kahdesta päätavoitteesta. Näistä ensimmäinen on Euroopan put ja call optioiden reilun hinnan määrittäminen, jolla tarkoitetaan, että kenenkään ei tulisi saada riskitöntä voittoa. Tämän tavoitteen suorittamista varten käytämme Black-Scholes-mallia. Aloitamme mallin esittelyllä kappaleessa 5 ja jatkamme tästä esittelemällä todennäköisyysmitan vaihtamisen kappaleessa 6. Kolmannessa kappaleessa on esitelty tärkeimmät stokastiikan perustyökalut laskemista varten. Koska stokastinen integrointi on tärkeässä roolissa tutkielmassamme, esittelemme myös yhden kuuluisimmista stokastisista integraaleista nimeltä Itô integraali. Stokastinen integrointi ja Itôn lause esitellään neljännessä kappaleessa. Kappaleessa 7 käytämme aiemmin esittelemiämme teorioita, kuten todennäköisyysmitan vaihtoa ja stokastista laskentaa, Black-Scholes-yhtälön ratkaisemiseen.

Kuten ensimmäisessä päätavoitteessa, oletamme myös toisessa päätavoitteessamme, että mahdollisia veroja tai rahansiirron kustannuksia ei ole. Toisen päätavoitteen tarkoituksena on mallintaa optimaalista investoimista. Tässä meillä on käytössä yleisempi malli, joka koostuu monesta erilaisesta riskialttiista komponentista ja riskittömästä sijoittajan omaisuudesta pankkitilillä. Valitsemme sopivan rahastohallinnon ja yritämme löytää sille optimaalisen strategian h^* maksimoimalla valitun apuväline funktion. Apuväline funktioita (utility function) on valittavana monenlaisia ja siten yhtä oikeaa valintaa ei voi määritellä. Tutkimusta tehdessä valitsemme usein funktion, jota on matemaattisesti helppo käsitellä ja jolla on mielekkäitä matemaattisia ominaisuuksia.

2. Introduction

Option pricing theory is a concept where we aim to value an option theoretically by using variables such as stock price, exercise price, volatility, interest rate and expiration date. By using option pricing theory we can obtain the theoretical estimation of an options fair value which can be used later by, for example, traders to maximize profits. One commonly used model in option pricing that we are going to introduce is called the Black-Scholes model.

The Black-Scholes model has a big role in the modern financial theory and is still widely used today. This model was first developed in 1973 by Fischer Black, Robert Merton and Myron Scholes. Due to its success the creators of the model Robert Merton and Myron Scholes were even given the Nobel price award. Fisher Black were also in close collaboration with Robert Merton and Myron Scholes but since he died before the Noble price was granted he did not have enough time to get the reward. One of the main features of the Black-Scholes model is that the pricing formula depends only on one non-observable parameter σ, the so called volatility. The volatility can be evaluated for example by using the historical method or the implied method. This is one of the main reasons behind the success of the Black-Scholes formula.

The focus of the thesis is the modelling of the two basic activities on a financial market. The first one we discuss is the option pricing and the second one is the optimal investment. The prices of both of these activities on certain underlyings are modelled by the same processes, exponential diffusion processes, and both actions can be performed on the same underlyings at the same time.

To be more precise we have two main objectives to accomplish. The first one is to determine a fair price for the European call and put options, which is done by using the Black-Scholes model. We start by introducing our model in chapter 5 and then continue by introducing the change of measure technique in the chapter 6. The basic tools needed for the computations are in the third chapter. Since stochastic integration is used in our theorems we also introduce one of the most popular stochastic integrals, the Itô integral, ensuring the foundation for our theorems and main results. Stochastic integration and Itô's formula will be introduced in the fourth chapter. In chapter 7 we finally show that how one can derive the Black-Scholes formula by using the change of measure technique and stochastic calculus.

In the final part our second objective is to find the most suitable strategy for a given utility function. Like in the Black-Scholes model we also need to assume that there are no transaction costs or taxes but in this case we can have many possible solutions depending on the utility function. We consider the Risk-sensitive asset management criterion in the special case, where asset and factor risks are not correlated. Here our main objective is to maximise the expected log return of the portfolio by using the risk-sensitive asset management criterion. This criterion is known for giving penalty for high variance, negative skewness and high kurtosis while rewarding positive skewness (see [**2**] Chapter 2.2).

Choosing logarithm of the portfolio value as a reward function provides us with a setting where the calculations can be carried out. This leads to a risk-sensitive asset management criterion, which is a great choice when managing portfolio value. For example this criterion works well with Markowitz' mean-variance analysis. and is consistent with utility theory (see [**2**] Chapter 2.2). We can also show that the risk-sensitive asset management criterion is a log coherent optimization criterion meaning that is satisfies the four axioms that we are going to introduce in the chapter 8. The Appendix part discusses existence and uniqueness of solutions for the SDEs we use in the Risk-sensitive asset management part.

3. Basic tools from probability theory

3.1. σ-algebra. The σ-algebra is a basic tool in probability theory since it serves, for example, as domain of definition of a probability measure.

DEFINITION 3.1. Let Ω be a non-empty set. A system $\mathcal{F}$ of subsets $A \subseteq \Omega$ is called σ-algebra on Ω if the following is satisfied:

- $\emptyset, \Omega \in \mathcal{F}$,
- if $A \in \mathcal{F}$ then also $A^c \in \mathcal{F}$,
- if $A_1, A_2, \dots \in \mathcal{F}$ then we also have that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

3.2. Filtration. The investor can not predict the future which means that he does not know at time 0 what is going to happen to the values S_t. When $s \in]0, T]$ and at time $t > 0$ he knows all the values S_s when $s \in [0, t]$ but does not know the values when $s \in]t, T]$. For modeling this situation we use a filtration.

DEFINITION 3.2. Let I be an index set. A filtration is a family of σ-algebras $(\mathcal{F}_t)_{t \in I}$ satisfying the following property:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \,\text{for all}\, 0 \leq s \leq t \in I$$

DEFINITION 3.3. We define a natural filtration of a family of random variables $(X_t)_{t \geq 0}$ on $\{\Omega, \mathcal{F}\}$ by setting:

$$\mathcal{F}_t^X = \sigma\{X_u, u \in [0, t]\}, \quad t \geq 0,$$

i.e. $\mathcal{F}_t^X$ is the smallest σ-algebra such that all X_u, $u \in [0, t]$, are measurable.

3.3. Brownian motion. The Brownian motion is a particularly important example of a stochastic process and it can be seen as a core of our financial model. It is used to model random phenomena in finance.

A Brownian motion is a real-valued continuous stochastic process $(X_t)_{t \geq 0}$ with independent and stationary increments.

DEFINITION 3.4. A family of random variables $(X_t)_{t \geq 0}$ is called an $(\mathcal{F}_t)_{t \geq 0}$- Brownian motion if the following is satisfied:

- X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.
- Continuity: $\mathbb{P}$ almost surely the map $t \to X_t(\omega) : [0, \infty) \to \mathbb{R}$ is continuous.
- Independent increments: If $s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.
- Stationary increments: If $s \leq t$, $X_t - X_s$ and $X_{t-s} - X_0$ have the same probability law.

Notice that this definition induces the distribution of the process $(X_t)_{t \geq 0}$.

REMARK 3.5. A Brownian motion $(X_t)_{t \geq 0}$ is called standard if

$$X_0 = 0, \qquad \mathbb{E}(X_t) = 0, \qquad \mathbb{E}(X_t^2) = t.$$

From now on we will assume that the Brownian motion $(X_t)_{t \geq 0}$ we use is standard if nothing else is mentioned. The random variable X_t is normally distributed:

$$\mathbb{P}(X_t \leq x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{z^2}{2t}} \, dz.$$

3.4. Conditional expectation.

Definition 3.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra in $\mathcal{F}$. Assume a $\mathcal{F}$-measurable random variable $X : \Omega \to \mathbb{R}$ such that $\mathbb{E}|X| < \infty$. Then a random variable $Y : \Omega \to \mathbb{R}$ is called a conditional expectation of X given $\mathcal{G}$ if

(1) Y is $\mathcal{G}$-measurable
(2) $\mathbb{E}(Y 1\!\!1_G) = \mathbb{E}(X 1\!\!1_G)$ for all $G \in \mathcal{G}$.

Then we have that $\mathbb{E}[X|\mathcal{G}] = Y$.

Remark 3.7. If $\mathbb{E}|X| < \infty$ then $\mathbb{E}[X|\mathcal{G}]$ always exists and is a.s. unique. More about this can be found in [**10**] Theorem 10.1.1.

3.5. Martingales and Doob's inequality.

Definition 3.8. Let $(M_t)_{0\le t\le T}$ be $(\mathcal{F}_t)_{0\le t\le T}$-adapted and such that $\mathbb{E}|M_t| < \infty$ for all $t \in [0, T]$.

M is called **martingale** provided that for all s and t such that $0 \le s \le t \le T$ one has

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad a.s.$$

Definition 3.9. According to [**9**] Definition 1.1.18 a **local martingale** is a process such that there exists an increasing sequence $(T_n)_n$ of stopping times satisfying: $\lim_{n\to\infty} T_n = \infty$ a.s. and every stopped process $X^{T_n} = (X_{t\wedge T_n})_{t\ge 0}$ is an uniformly integrable martingale.

Proposition 3.10. *Let $M = (M_t)_{0\le t\le T}$ be a right- continuous martingale. Then one has, for $\lambda, t \ge 0$ and $p \in (1, \infty)$, that*

$$\mathbb{E}\left(\sup_{t\le T} |M_t|^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_T|^p.$$

This inequality is known as the Doob's inequality and the proof can be seen in [**7**] Proposition 3.1.16.

4. Stochastic integration

Before starting to think about the Black-Scholes model we will briefly introduce stochastic integration and some pivotal tools in stochastic calculus like Ito's formula.

We use the index t to indicate time.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we will use is equipped with the filtration $(\mathcal{F}_t)_{t\ge 0}$, which satisfies the usual conditions. The usual conditions are the following:

- $(\Omega, \mathcal{F}, \mathbb{P})$ is complete
- $A \in \mathcal{F}_t$ for all sets $A \in \mathcal{F}$ with the property $\mathbb{P}(A) = 0$
- the filtration $(\mathcal{F}_t)_{t\ge 0}$ is right continuous.

We are interested in progressively measurable processes $(H_t)_{t\ge 0}$.

Definition 4.1. A process $H : [0, \infty] \times \Omega \to \mathbb{R}$ is called progressively measurable with respect to a filtration $(\mathcal{F}_t)_{t\ge 0}$ if the preimage $\{(t, \omega) \in [0, s] \times \Omega : H(t, \omega) \in B\}$ belongs to $\mathcal{B}([0, s]) \otimes \mathcal{F}_s$ for all $B \in \mathcal{B}(\mathbb{R})$ and for all $s \in [0, \infty]$.

The important thing to notice is that we do not require the preimage to be only in $\mathcal{B}([0,s]) \otimes \mathcal{F}$ but in the smaller σ-algebra $\mathcal{B}([0,s]) \otimes \mathcal{F}_s$.

The value of a portfolio in discrete time can be calculated by taking a sum of the differences of stock prices multiplied by the trading strategy. For initial wealth V_0 and a self-financing strategy $\phi = (H_t)_{0 \leq t \leq T}$ we have that the value of a portfolio is

$$V_0 + \sum_{j=1}^{t} H_j(\widetilde{S}_j - \widetilde{S}_{j-1}),$$

where $\widetilde{S}_t$ is the discounted stock price at time t . Naturally when modelling stock prices in continuous time we are interested in integrals of the form $\int H_t \, d\widetilde{S}_t$. The problem is that the processes modelling stock prices are usually functions of one or multiple Brownian motions hence we can not use the Stieltjes integral for our calculations because the Brownian motion a.s does not have paths of finite variation. Moreover, we know from the Paley, Wiener-Zygmund Theorem that a standard Brownian motion is nowhere differentiable. See for example [**6**][Theorem 10.3]. That means we do not have the equality $\int H(t) \, dB_t = \int H(t) B_t' \, dt$. Our goal is to define a new integral with respect to a Brownian motion, the Itô-integral.

4.1. Construction of the Itô-integral for simple processes. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_{t \geq 0}))$ satisfying the usual conditions. We approach in the same way as when defining the Riemann integral which means that we start from simple processes and then generalize our integral to progressively measurable processes.

Definition 4.2. A process $(H_t)_{t \leq T}$ is called simple if it can be written in the following form:

$$H_t(\omega) = \sum_{i=1}^{n} \phi_i(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(t).$$

Here $0 \leq t_0 < t_1 < \cdots < t_n = T$ and ϕ_i is an $\mathcal{F}_{t_{i-1}}$-measurable random variable and satisfies

$$\max_i \sup_\omega |\phi_i(\omega)| \leq c,$$

where $c > 0$ is a constant. We denote the space of simple functions by $\mathcal{H}_0$.

By using the above definition we construct the stochastic integral as a continuous process $(I(H)_t)_{0 \leq t \leq T}$ defined for any $t \in]t_k, t_{k+1}]$ as

$$I(H)_t = \sum_{1 \leq i \leq k} \phi_i (B_{t_i} - B_{t_{i-1}}) + \phi_{k+1}(B_t - B_{t_k}). \tag{4.1}$$

We can write this integral as a sum going from 1 to n by using minimum of t_i and t as

$$I(H)_t = \sum_{1 \leq i \leq n} \phi_i (B_{t_i \wedge t} - B_{t_{i-1} \wedge t}). \tag{4.2}$$

This can be seen by considering $t \in]t_k, t_{k+1}]$ as above in three cases. When $i \le k$ we have the expression $\sum_{1\le i\le k} \phi_i(B_{t_i} - B_{t_{i-1}})$ and when $i = k+1$ the above definition has the term $\phi_{k+1}(B_t - B_{t_k})$. The final case is when $i > k+1$ and now we always get zero since $t_i \wedge t = t_{i-1} \wedge t = t$. From the expression (4.2) one can see the continuity of $t \to I(H)_t$. The continuity comes from the continuity of the Brownian motion. We will write $\int_0^t H_s \, dB_s$ for $I(H)_t$.

4.2. Properties of the Itô-integral for simple processes.

Proposition 4.3. *The Itô integral for simple processes is linear. This means that for constants α and β we have that $I(\alpha H + \beta K)_t = \alpha I(K)_t + \beta I(H)_t$, where H and K are processes in $\mathcal{H}_0$.*

Proof. Let $H_t(\omega) = \sum_{i=1}^n \phi_i(\omega)\mathbb{1}_{(t_{i-1},t_i]}(t)$ and $K_t(\omega) = \sum_{j=1}^m \psi_j(\omega)\mathbb{1}_{(s_{j-1},s_j]}(t)$ be simple processes. Because $H_t(\omega)$ and $K_t(\omega)$ may be constant on different intervals we define a new partition $0 = u_0 < u_1 < \cdots < u_N = T$, where $H_t(\omega)$ and $K_t(\omega)$ both are constant. Since now we have more time points than before the former numbering does not fit and hence we use new functions $\widehat{\phi}$ and $\widehat{\psi}$. We have $H_u(\omega) = \sum_{k=1}^N \widehat{\phi}_k(\omega)\mathbb{1}_{(u_{i-1},u_i]}(u)$ and $K_u(\omega) = \sum_{k=1}^N \widehat{\psi}_k(\omega)\mathbb{1}_{(u_{i-1},u_i]}(u)$, where $N \ge n, m$. The new representations are consistent with the old representations as the $(u_i)_{i=0}^N$ partition is a finer partition than the $(s_j)_{j=0}^m$ and $(t_i)_{i=0}^n$ partitions.

Then let α and β be constants. We get for any $u \in]u_N, u_{N+1}]$

$$\begin{aligned} I(\alpha H + \beta K)_u(\omega) = &\sum_{k=1}^N (\alpha\widehat{\phi}_k(\omega) + \beta\widehat{\psi}_k(\omega))(B_{u_k}(\omega) - B_{u_{k-1}}(\omega)) \\ &+ (\alpha\widehat{\phi}_{N+1}(\omega) + \beta\widehat{\psi}_{N+1}(\omega))(B_u(\omega) - B_{u_N}(\omega)). \end{aligned}$$

After rearranging terms we get

$$\begin{aligned} I(\alpha H + \beta K)_u(\omega) &= \sum_{k=1}^N \alpha\widehat{\phi}_k(\omega)(B_{u_k}(\omega) - B_{u_{k-1}}(\omega)) + \alpha\widehat{\phi}_{N+1}(\omega)(B_u(\omega) - B_{u_N}(\omega)) \\ &\quad + \sum_{k=1}^N \beta\widehat{\psi}_k(\omega)(B_{u_k}(\omega) - B_{u_{k-1}}(\omega)) + \beta\widehat{\psi}_{N+1}(\omega)(B_u(\omega) - B_{u_N}(\omega)) \\ &= \alpha\sum_{k=1}^N \widehat{\phi}_k(\omega)(B_{u_k}(\omega) - B_{u_{k-1}}(\omega)) + \widehat{\phi}_{N+1}(\omega)(B_u(\omega) - B_{u_N}(\omega)) \\ &\quad + \beta\sum_{k=1}^N \widehat{\psi}_k(\omega)(B_{u_k}(\omega) - B_{u_{k-1}}(\omega)) + \widehat{\psi}_{N+1}(\omega)(B_u(\omega) - B_{u_N}(\omega)) \end{aligned}$$

which is by definition $\alpha I(H)_u + \beta I(K)_u$. □

Proposition 4.4. *If the process $(H_t)_{0\le t\le T}$ is defined like in Definition 2.1. we have that $\left(\int_0^t H_s \, dB_s\right)_{0\le t\le T}$ is a continuous $(\mathcal{F}_t)_{0\le t\le T}$-martingale.*

PROOF. Since the continuity of $I(H)_t$ is clear it is enough to show the three conditions of a martingale. For our proof we use the expression $I(H)_t = \sum_{1\le i\le k} \phi_i(B_{t_i} - B_{t_{i-1}}) + \phi_{k+1}(B_t - B_{t_k})$ for $t \in [t_k, t_{k+1}]$. The measurability condition holds because each term in the expression is $\mathcal{F}_t$-measurable for all $t \ge 0$ and a sum of measurable terms is measurable.

For the integrability condition we show that

$$\mathbb{E}|I(H)_t| = \mathbb{E}\left| \sum_{1\le i\le k} \phi_i(B_{t_i} - B_{t_{i-1}}) + \phi_{k+1}(B_t - B_{t_k})\right| < \infty.$$

By using the triangle inequality we put the absolute value inside the sum. Then we use the upper bound c for the random variables ϕ_i and pull c out of the expectation. We get

$$\mathbb{E}|I(H)_t| \le c \sum_{1\le i\le k} \mathbb{E}\big|B_{t_i} - B_{t_{i-1}}\big| + c\mathbb{E}\big|B_t - B_{t_k}\big|.$$

Now using Hölder's inequality the terms $\mathbb{E}|B_{t_i} - B_{t_{i-1}}|$ can be estimated in the following way.

$$\mathbb{E}|B_{t_i} - B_{t_{i-1}}| \le \left(\mathbb{E}|B_{t_i} - B_{t_{i-1}}|^2\right)^{\frac{1}{2}}.$$

Stationary increments of a Brownian motion now gives us

$$\mathbb{E}|B_{t_i} - B_{t_{i-1}}|^2 = t_i - t_{i-1}.$$

Consequently we have a finite sum whose each term is also finite, therefore integrability holds.

Finally we are going to check the martingale property which is that for any $s \le t$ on the interval $[0, T]$ we have

$$\mathbb{E}\left[I(H)_t|\mathcal{F}_s\right] = I(H)_s \quad a.s.$$

For our convenience let us use the expression

$$M_{t_k} = \sum_{1\le i\le k} \phi_i(B_{t_i} - B_{t_{i-1}}). \tag{4.3}$$

By using this expression the relation (4.1) gets the form

$$I(H)_t = M_{t_k} + \phi_{k+1}(B_t - B_{t_k}).$$

Since $(B_t)_{t\ge0}$ is a Brownian motion we have that $(B_t)_{t\ge0}$ is a $(\mathcal{F}_t)_{t\ge0}$-martingale. In view of expression in (4.3) and recalling that $B_0 = 0$ we get terms $M_{t_1} = \phi_1 B_{t_1}$, $M_{t_2} = \phi_1 B_{t_1} + \phi_2(B_{t_2} - B_{t_1}), \dots,$ $M_{t_k} = \phi_1 B_{t_1} + \cdots + \phi_k(B_{t_k} - B_{t_{k-1}})$. Since the sequence M_{t_i} is a combination of $\mathcal{F}_{t_{i-1}}$- measurable random variables $\phi_1 \dots \phi_i$ and $B_{t_1} \dots B_{t_i}$ it is adapted.

By linearity we get

$$\begin{aligned}\mathbb{E}[M_{t_k}|\mathcal{F}_s] &= \mathbb{E}\left[\sum_{1\le i\le k}\phi_i(B_{t_i}-B_{t_{i-1}})|\mathcal{F}_s\right]\\ &= \sum_{1\le i\le k}\mathbb{E}[\phi_i(B_{t_i}-B_{t_{i-1}})|\mathcal{F}_s],\end{aligned}$$

hence we can calculate the conditional expectations term by term.

Let m be such that $s\in]t_m,t_{m+1}]$ and assume that $m+1\le k$. Let us consider an arbitrary interval $(t_{i-1},t_i]$. There are three possible cases for s:

(1) $s\le t_{i-1}$
(2) $t_i<s$
(3) $t_{i-1}<s\le t_i$.

In the first case in order to pull ϕ_i out, we can use the tower property for $\mathcal{F}_{t_{i-1}}\supseteq\mathcal{F}_s$ and get

$$\begin{aligned}\mathbb{E}[\phi_i(B_{t_i}-B_{t_{i-1}})|\mathcal{F}_s] &= \mathbb{E}\left[\mathbb{E}[\phi_i(B_{t_i}-B_{t_{i-1}})|\mathcal{F}_{t_{i-1}}]|\mathcal{F}_s\right]\\ &= \mathbb{E}\phi_i\left[\mathbb{E}[B_{t_i}-B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]|\mathcal{F}_s\right]\\ &= \mathbb{E}\phi_i\left[\mathbb{E}[B_{t_{i-1}}-B_{t_{i-1}}]|\mathcal{F}_s\right]\\ &= 0.\end{aligned}$$

Next we consider the second case $t_i<s$. In this case we have that t_i is smaller than t_m for all $1\le i\le m$ in the sum, hence every term is $\mathcal{F}_s$-measurable and we get

$$\mathbb{E}[\phi_i(B_{t_i}-B_{t_{i-1}})|\mathcal{F}_s]=\phi_i(B_{t_i}-B_{t_{i-1}}).$$

Finally we have the third case $t_{i-1}<s\le t_i$. Here we get

$$\begin{aligned}\mathbb{E}[\phi_i(B_{t_i}-B_{t_{i-1}})|\mathcal{F}_s] &= \phi_{m+1}(\mathbb{E}[B_{t_i}|\mathcal{F}_s]-\mathbb{E}[B_{t_{i-1}}|\mathcal{F}_s])\\ &= \phi_{m+1}(B_s-B_{t_m}).\end{aligned}$$

Now we have that for all s and m such that $s\in]t_m,t_{m+1}]$ and $m+1<k$

$$\mathbb{E}[M_{t_k}|\mathcal{F}_s]=\sum_{1\le i\le m}\phi_i(B_{t_i}-B_{t_{i-1}})+\phi_{m+1}(B_s-B_{t_m}).$$

For $s\in]t_m,t_{m+1}]$ with $m+1\le k$ we also have

$$\mathbb{E}[\phi_{k+1}(B_t-B_{t_k})|\mathcal{F}_s]=0$$

like in case (1) above. Hence $\mathbb{E}\left[I(H)_t|\mathcal{F}_s\right]=I(H)_s$. If $s\in]t_k,t]$, then also it holds $\mathbb{E}\left[I(H)_t|\mathcal{F}_s\right]=I(H)_s$ by the arguments above. As a result we have shown that $\mathbb{E}\left[I(H)_t|\mathcal{F}_s\right]=I(H)_s$ and the other conditions for a martingale which means that $(I(H)_t)_{0\le t\le T}$ is a continuous $(\mathcal{F}_t)_{0\le t\le T}$-martingale. □

The next property is unique for stochastic integrals and is called Itô isometry. We first define and prove Itô isometry for simple processes.

Proposition 4.5. *For a simple process $(H_t)_{t\geq 0}$ we have that*

(1) $\mathbb{E}\left(\left(\int_0^t H_s\,dB_s\right)^2\right) = \mathbb{E}\left(\int_0^t H_s^2\,ds\right)$

(2) $\mathbb{E}\left(\sup_{t\leq T}\left|\int_0^t H_s\,dB_s\right|^2\right) \leq 4\mathbb{E}\left(\int_0^T |H_s|^2\,ds\right)$.

Proof. To prove (1) we use the notation M_{t_k} from the previous proof and have

$$\begin{aligned}\mathbb{E}(M_{t_k}^2) &= \mathbb{E}\left(\left(\sum_{1\leq i\leq k}\phi_i(B_{t_i}-B_{t_{i-1}})\right)^2\right)\\ &= \mathbb{E}\left(\sum_{1\leq i\leq k}\sum_{1\leq j\leq k}\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})\right).\end{aligned}$$

In this expression when $i=j$ we sum the terms $\phi_i^2(B_{t_i}-B_{t_{i-1}})^2$ and when $i\neq j$ we sum terms that have the form $\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})$. By using the linearity of the expectation we can move the expectation inside the sum and calculate the expectation of every term one by one. When $i<j$ we use the tower property of the conditional expectation and get

$$\mathbb{E}\left[\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})\right] = \mathbb{E}\left[\mathbb{E}[\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}]\right].$$

Since we have $i<j$ the term $\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})$ is $\mathcal{F}_{t_{j-1}}$-measurable and we can pull it out. We get

$$\mathbb{E}\left[\mathbb{E}[\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}]\right] = \mathbb{E}\left[\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})\mathbb{E}[(B_{t_j}-B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}]\right].$$

Since we know that $(B_t)_{0\leq t\leq T}$ is a Brownian motion we can use the same procedure as in the proof of Proposition 4.4 and obtain $\mathbb{E}[(B_{t_j}-B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}]=0$. This means that $\mathbb{E}\left[\phi_i\phi_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})\right]=0$. Then if $i=j$ we have,

$$\begin{aligned}&\mathbb{E}\left[\phi_i^2(B_{t_i}-B_{t_{i-1}})^2\right].\\ &=\mathbb{E}\left[\mathbb{E}[\phi_i^2(B_{t_i}-B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]\right]\\ &=\mathbb{E}\left[\phi_i^2\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]\right].\end{aligned}$$

Since $(B_{t_i}-B_{t_{i-1}})^2$ is independent of $\mathcal{F}_{t_{i-1}}$ hence we have that

$$\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}] = \mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2] = t_i - t_{i-1}.$$

The last equality comes from the fact that the standard Brownian motion has stationary increments, mean zero and variance $\mathrm{var}(B_t)=t$.

Finally we combine all the steps we made and we see that

$$\mathbb{E}M_{t_k}^2 = \mathbb{E}\left(\left(\sum_{1\le i\le k}\phi_i(B_{t_i}-B_{t_{i-1}})\right)^2\right) = \mathbb{E}\left(\sum_{1\le i\le k}\phi_i^2\mathbb{E}((B_{t_i}-B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}})\right)$$
$$= \mathbb{E}\left(\sum_{1\le i\le k}\phi_i^2(t_i-t_{i-1})\right).$$

Then we take the expectation of the square of the integral $I(H)_t$ and get

$$\mathbb{E}(I(H)_t^2) = \mathbb{E}M_{t_k}^2 + 2\mathbb{E}\left(M_{t_k}\phi_{k+1}(B_t-B_{t_k})\right) + \mathbb{E}(\phi_{k+1}^2(B_t-B_{t_k})^2)$$
$$= \mathbb{E}M_{t_k}^2 + \phi_{k+1}^2(t-t_k).$$

The term $2\mathbb{E}\left(M_{t_k}\phi_{k+1}(B_t-B_{t_k})\right)$ is zero by the tower property and the last term can be treated in the same way as in the $\mathbb{E}M_{t_k}^2$ calculation. The equality

$$\mathbb{E}(I(H)_t^2) = \mathbb{E}\left(\sum_{1\le i\le k}\phi_i^2(t_i-t_{i-1})\right) + \phi_{k+1}^2(t-t_k) = \mathbb{E}\left(\int_0^t H_s^2\,ds\right)$$

can be seen by the following way. We define

$$H_s(\omega) = \sum_{i=1}^n \phi_i(\omega)\mathbb{1}_{(t_{i-1},t_i]}(s)$$

and have $H_s(\omega)^2 = \sum_{i=1}^n \phi_i(\omega)^2\mathbb{1}_{(t_{i-1},t_i]}(s)$. The integral of $H_s(\omega)^2$ can be calculated simply by multiplying the value of the function $\phi_i^2(\omega)$ by the length of the interval $]t_{i-1},t_i]$, hence we get

$$\int_0^t H_s^2\,ds = \int_0^t\sum_{i=1}^n \phi_i(\omega)^2\mathbb{1}_{(t_{i-1},t_i]}(s)\,ds = \sum_{i=1}^k \phi_i^2(t_i-t_{i-1}) + \phi_{k+1}^2(t-t_k).$$

To prove (2) we use Doob's inequality applied to the continuous martingale $(I(H)_t)_{0\le t\le T}$ and get

$$\mathbb{E}\left(\sup_{t\le T}|I(H)_t|^2\right) \le 4\mathbb{E}|I(H)_T|^2 = 4\mathbb{E}\left|\int_0^T H_s\,dB_s\right|^2.$$

To finish our proof we use Itô's isometry and get

$$\mathbb{E}\left(\sup_{t\le T}|I(H)_t|^2\right) \le 4\mathbb{E}\left(\int_0^T |H_s|^2\,ds\right).$$

□

4.3. Extension of the Itô-integral to a class of square integrable processes. Now since we have defined a stochastic integral for simple processes our next goal is to extend this definition for a larger class of progressively measurable processes $\mathcal{H}$

$$\mathcal{H} = \left\{ (H_t)_{0\le t\le T} : (\mathcal{F}_t)_{t\ge 0} - \text{progressively measurable}, \ \mathbb{E}\left(\int_0^T H_s^2\, ds \right) < +\infty \right\}.$$

PROPOSITION 4.6. *Let $(B_t)_{t\ge0}$ be an $(\mathcal{F}_t)$-Brownian motion. There exists a unique linear mapping J from $\mathcal{H}$ to the space of continuous $(\mathcal{F}_t)_{t\in[0,T]}$-martingales defined on the interval $[0,T]$ such that if $(H_t)_{0\le t\le T}$ is a simple process then $\mathbb{P}$ almost surely for any $0\le t\le T$ it holds $J(H)_t = I(H)_t$ and if $t\le T$, $\mathbb{E}(J(H)_t^2) = \mathbb{E}(\int_0^t H_s^2\, ds)$.*

LEMMA 4.7. *If $(H_s)_{s\le T}$ belongs to $\mathcal{H}$, then there exists a sequence $(H_s^n)_{s\le T}$ of simple processes such that*

$$\lim_{n\to+\infty} \mathbb{E}\left(\int_0^T |H_s - H_s^n|^2\, ds \right) = 0 \tag{4.4}$$

A proof of this lemma can be found in [**5**] Problem 2.5.

Let us have $H\in\mathcal{H}$ and a sequence of simple processes $(H^n)_{n=1}^{\infty}$ converging to H like in Lemma 4.7. Proposition 4.5 (2) gives us the following result:

$$\mathbb{E}\left(\sup_{t\le T} |I(H^{n+p})_t - I(H^n)_t|^2 \right) \le 4\mathbb{E}|I(H^{n+p})_T - I(H^n)_T|^2 \tag{4.5}$$

$$= 4\mathbb{E}\left(\int_0^T |H_s^{n+p} - H_s^n|^2\, ds \right). \tag{4.6}$$

Therefore because we have (4.4) we get that there exists a subsequence $(H^{n_k})_{k=0}^{\infty}$ with $H^{n_0}\equiv 0$ such that

$$\mathbb{E}\left(\sup_{t\le T} |I(H^{n_{k+1}})_t - I(H^{n_k})_t|^2 \right) \le \frac{1}{2^k},$$

which gives us also that

$$\sum_{k=0}^{\infty} \left(\mathbb{E} \sup_{t\le T} |I(H^{n_{k+1}})_t - I(H^{n_k})_t|^2 \right)^{\frac12} < \infty.$$

The almost sure convergence of the series

$$\sum_{k=0}^{\infty} \sup_{t\le T} |I(H^{n_{k+1}})_t - I(H^{n_k})_t|$$

can be seen by using Hölder's inequality: Let $a_k := \sup_{t\le T} |I(H^{n_{k+1}})_t - I(H^{n_k})_t|^2$. We get

$$\mathbb{E}\sum_{k=0}^{\infty} a_k^{\frac12} = \sum_{k=0}^{\infty} \mathbb{E}a_k^{\frac12} \le \sum_{k=0}^{\infty} (\mathbb{E}a_k)^{\frac12} \le \sum_{k=0}^{\infty} \left(\frac{1}{2^k}\right)^{\frac12} < \infty,$$

where expectation can be moved inside the series because of Fubini's theorem. The inequality $\mathbb{E}a_k^{\frac12} \le (\mathbb{E}a_k)^{\frac12}$ follows from Hölder's inequality. Since we have an expression that has a finite expectation we also can conclude that the probability that $\sum_{k=0}^{\infty} a_k^{\frac12} < \infty$ is one, hence

$$\mathbb{P}\left(\sum_{n=0}^{\infty}\sup_{t\le T}|I(H^{n_{k+1}})_t - I(H^{n_k})_t| < \infty\right) = 1.$$

Thus the series whose general term is $I(H^{n_{k+1}})_t - I(H^{n_k})_t$ is uniformly convergent. We set $\Omega_T := \{\omega \in \Omega : \sum_{n=0}^{\infty}\sup_{t\le T}|I(H^{n_{k+1}})_t - I(H^{n_k})_t| < \infty\}$ and define

$$J(H)_t(\omega) := \begin{cases} \sum_{k=0}^{\infty}[I(H^{n_{k+1}})_t - I(H^{n_k})_t] & : \omega \in \Omega_T \\ 0 & : \omega \notin \Omega_T. \end{cases}$$

Now we see that the process $(J(H)_t)_{0\le t\le T}$ is path-wise continuous. This follows from the fact that the general term $I(H^{n_{k+1}})_t - I(H^{n_k})_t$ in the sum is continuous and the partial sums are uniformly convergent (converging in a supremum norm) which gives us that the limit, which is $J(H)_t(\omega)$, is continuous. The integral for a process H in $\mathcal{H}$ we denote also by $\int_0^t H_s\, dB_s = J(H)_t$.

We can also prove that the process $(J(H)_t)_{0\le t\le T}$ is a martingale in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. We can see that $(I(H^n)_t)_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. First we use the fact that the Itô integral is linear for simple processes and get

$$\mathbb{E}|I(H^n)_t - I(H^m)_t|^2 = \mathbb{E}|I(H^n - H^m)_t|^2.$$

Then Itô isometry for simple processes (Proposition 4.5 (1)) gives us

$$\mathbb{E}|I(H^n - H^m)_t|^2 = \mathbb{E}\left(\int_0^t (H_s^n - H_s^m)\, dB_s\right)^2 = \mathbb{E}\int_0^t (H_s^n - H_s^m)^2\, ds.$$

From Lemma 4.7 we have that $\mathbb{E}\int_0^t (H_s^n - H_s^m)^2\, ds \to 0$. This means that

$$\mathbb{E}|I(H^n)_t - I(H^m)_t|^2 < \epsilon \quad \text{for all} \quad n, m \ge N(\epsilon). \tag{4.7}$$

Now because the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is closed we know that there exists a unique limit $X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$L^2 - \lim_n I(H^n)_t = X_t. \tag{4.8}$$

Now we have that

$$X_t = \sum_{n=0}^{\infty}[I(H^{n_{k+1}})_t - I(H^{n_k})_t] = J(H)_t \quad a.s. \tag{4.9}$$

because from the $L^2(\Omega, \mathcal{F}, \mathbb{P})$-convergence of $I(H^n)_t$ to X_t and the a.s. convergence to $J(H)_t$ follows also the convergence in probability. Both X_t and $J(H)_t$ are limits of the same sequence and hence X_t and $J(H)_t$ must be equal.

Because the $I(H^n)_t$ is adapted and it converges to $J(H)_t$ in L^2 also the limit has to be adapted. Integrability is also straightforward since it follows from L^2 convergence. It is sufficient to prove the martingale inequality. Since we know that the processes $(I(H^n)_t)_{0\le t\le T}$ are martingales we prove that the limit $(J(H)_t)_{0\le t\le T}$ is also a martingale.

From (4.8) and (4.9) we can conclude that for any $u \in [0,T]$

$$\mathbb{E}|J(H)_u - I(H^n)_u|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

For $G \in \mathcal{F}_s$ we get by using Hölder's inequality that

$$\left|\mathbb{E}\left(J(H)_u 1\!\!1_G\right) - \mathbb{E}\left(I(H^n)_u 1\!\!1_G\right)\right| \le \left(\mathbb{E}|J(H)_u - I(H^n)_u|^2\right)^{\frac{1}{2}} \left(\mathbb{E}1\!\!1_G\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.10}$$

Since $(I(H^n)_t)_{0\le t\le T}$ is a martingale we have for $u = t$ that

$$\begin{aligned}\mathbb{E}\left(J(H)_t 1\!\!1_G\right) &= \lim_{n\to\infty} \mathbb{E}\left(I(H^n)_t 1\!\!1_G\right)\\ &= \lim_{n\to\infty} \mathbb{E}\left(I(H^n)_s 1\!\!1_G\right)\\ &= \mathbb{E}\left(J(H)_s 1\!\!1_G\right),\end{aligned}$$

where in the last step we use (4.10) with $u = s$.

Proposition 4.8. *For a process $(H_t)_{0\le t\le T}$ that belongs to $\mathcal{H}$ we have:*

(1) $\mathbb{E}\left(\sup_{t\le T}|J(H)_t|^2\right) \le 4\mathbb{E}\left(\int_0^T H_s^2\,ds\right)$

(2) If $t \le T$, $\mathbb{E}(J(H)_t^2) = \mathbb{E}\left(\int_0^t |H_s|^2\,ds\right)$

(3) $\int_0^\tau H_s\,dB_s = \int_0^T 1\!\!1_{s\le\tau} H_s\,dB_s$ *a.s* *for any $(\mathcal{F}_t)_{t\ge 0}$-stopping time τ.*

Proof. We show (1): from Proposition 4.5 (2) we know that

$$\mathbb{E}\left(\sup_{t\le T}|I(H^n)_t|^2\right) \le 4\mathbb{E}\left(\int_0^T |H_s^n|^2\,ds\right).$$

Then by using (4.8) and (4.9) we get by taking the limit $n \to \infty$ that

$$\mathbb{E}\left(\sup_{t\le T}|J(H)_t|^2\right) \le 4\mathbb{E}\left(\int_0^T H_s^2\,ds\right).$$

We show (2): from Proposition 4.5 (1) it follows that

$$\mathbb{E}(I(H^n)_t^2) = \mathbb{E}\left(\left(\int_0^t H_s^n\,dB_s\right)^2\right) = \mathbb{E}\left(\int_0^t |H_s^n|^2\,ds\right)$$

then by taking the limit $n \to \infty$ we get

$$\mathbb{E}(J(H)_t^2) = \mathbb{E}\left(\int_0^t |H_s|^2\,ds\right).$$

The proof of assertion (3) can be found in [**1**] Proposition 3.4.5. □

4.4. Extension from $\mathcal{H}$ to $\bar{\mathcal{H}}$. Because of the problems we face in modelling we usually do not have the condition $\mathbb{E}\left(\int_0^T H_s^2\,ds\right) < +\infty$, we define processes that only satisfy the weaker integrability condition $\int_0^T H_s^2\,ds < +\infty$ a.s. That is why we define a new set of processes $\bar{\mathcal{H}}$ in the following way:

$$\bar{\mathcal{H}} = \left\{ (H_t)_{0\leq s\leq T},\ (\mathcal{F}_t)_{t\geq 0} - \text{progressively measurable}, \int_0^T H_s^2\,ds < +\infty \quad \text{a.s.} \right\}.$$

Next we define an extension of the stochastic integral from $\mathcal{H}$ to $\bar{\mathcal{H}}$ with the following properties:

PROPOSITION 4.9. *There exists a unique linear mapping $\bar{J}$ from $\bar{\mathcal{H}}$ into the vector space of continuous processes defined on $[0,T]$, such that:*

(1) Extension property: If $(H_t)_{0\leq t\leq T}$ is a simple process, then $\mathbb{P}$ almost surely for any $0 \leq t \leq T$ it holds $\bar{J}(H)_t = I(H)_t$
(2) Continuity property: If $(H^n)_{n\geq 0}$ is a sequence of processes defined in $\bar{\mathcal{H}}$ such that $\lim_{n\to\infty}\int_0^T (H_s^n)^2\,ds = 0$ almost surely we also have that $\sup_{t\leq T}|\bar{J}(H^n)_t|$ converges to 0 in probability.
For $\bar{J}(H)_t$ we use the same notation as for $J(H)_t$ and we write $\bar{J}(H)_t =: \int_0^t H_s\,dB_s$.

The proof can be found in [**1**] Proposition 3.4.6.

REMARK 4.10. In this case the process $\left(\int_o^t H_s\,dB_s\right)_{0\leq t\leq T}$ is a local martingale.

PROPOSITION 4.11. *The Itô integral is linear, which means that for constants α and β we have that $\bar{J}(\alpha X + \beta Y)_t = \alpha\bar{J}(X)_t + \beta\bar{J}(Y)_t$, where X and Y are processes in $\bar{\mathcal{H}}$.*

PROOF. First let us consider two processes X and Y from $\bar{\mathcal{H}}$ and define

$$T_n = \inf\left\{0 \leq s \leq T, \int_0^s X_u^2\,du \geq n\right\} \quad \text{and} \quad \widehat{T}_n = \inf\left\{0 \leq s \leq T, \int_0^s Y_u^2\,du \geq n\right\}.$$

Then we define two sequences X_t^n and Y_t^n such that $X_s^n = X_s \mathbb{1}_{\{s\leq T_n\}}$ and $Y_s^n = Y_s \mathbb{1}_{\{s\leq \widehat{T}_n\}}$. These two sequences X_t^n and Y_t^n are defined such that $\int_0^T |X_s^n - X_s|^2\,ds$ and $\int_0^T |Y_s^n - Y_s|^2\,ds$ converge to 0 in probability (see [**1**] Proposition 3.4.6). By using Proposition 4.9 (2) (continuity of $\bar{J}$) we can take the limit in the equality $\bar{J}(\alpha X^n + \beta Y^n)_t = \alpha\bar{J}(X^n)_t + \beta\bar{J}(Y^n)_t$ and get the desired result. $\square$

To sum up let us have a stochastic process $(H_t)_{0\leq t\leq T}$ and a $(\mathcal{F}_t)$-Brownian motion $(B_t)_{t\geq 0}$. The stochastic integral $(\int_0^T H_s\,dB_s)_{0\leq t\leq T}$ can be defined if we have the condition $\int_0^T H_s^2\,ds < \infty$ a.s. and the $(\mathcal{F}_t)_{t\geq 0}$-progessive measurability of the process $(H_t)_{0\leq t\leq T}$.

4.5. Itô's Formula. From calculus we know that, if $f \in C^1(\mathbb{R})$ and $-\infty < x < y < \infty$ there is a fundamental formula such that

$$f(y) = f(x) + \int_x^y f'(u)\, du.$$

Our goal is to derive a variant of this formula for Itô integrals.

DEFINITION 4.12. A continuous and adapted process $(X)_{0 \le t \le T}$, $X_t : \Omega \to \mathbb{R}$ is called Itô-process provided that there is $L \in \bar{\mathcal{H}}$ and a progressively measurable process $(a_t)_{0 \le t \le T}$ such that

$$\int_0^t |a_u(\omega)|\, du < \infty \quad for \quad 0 \le t \le T, \quad \text{a.s.}$$

for all $0 \le t \le T$ and $\omega \in \Omega$, and $x_0 \in \mathbb{R}$ such that

$$X_t(\omega) = x_0 + \left(\int_0^t L_u\, dB_u \right)(\omega) + \int_0^t a_u(\omega)\, du \quad \text{for} \quad 0 \le t \le T, \quad \text{a.s.}$$

PROPOSITION 4.13. *Let $(X_t)_{0 \le t \le T}$ be an Itô process like in Definition 4.12 and let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that all the partial derivatives $\partial f / \partial t$, $\partial f / \partial x$, and $\partial^2 f / \partial x^2$ exist on $(0, \infty) \times \mathbb{R}$ and can be continuously extended to $[0, \infty) \times \mathbb{R}$ and are continuous. Then one has that*

$$\begin{aligned} f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial u}(u, X_u)\, du + \int_0^t \frac{\partial f}{\partial x}(u, X_u) L_u\, dB_u \\ + \int_0^t \frac{\partial f}{\partial x}(u, X_u) a_u\, du + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, X_u) L_u^2\, du. \end{aligned}$$

The above formula we call Itô's formula.

The proof for Itô's formula in a simple case can be found in [**7**] Chapter 3.3, and for the general case see [**12**] Theorem 4.4.

5. Description of the Black-Scholes model

5.1. Interest rate process. The interest rate process can be derived by first considering the interval $[0, T]$ where time goes from 0 to T. Then we divide this interval into n parts so the size of each subinterval becomes $\frac{T}{n}$. This means that we consider subintervals $I_j = [j\frac{T}{n}, (j+1)\frac{T}{n}]$. Then suppose that trading occurs only at time points $t_j = j\frac{T}{n}$, $j = 0, \dots, n-1$ with equal distance. We fix r as the riskless constant interest rate over each interval I_j and invest 1 euro at time 0, which we will get back at maturity T. Our process is now given by

$$B_{t_j}^n = (1 + r_n)^j,\ j = 0, \dots, n,$$

where r_n is the interest rate in the time interval I_j.

In the continuous time model we assume that we can trade in any momentum of time hence the price process B_t is given by

$$B_t = e^{rt}. \tag{5.1}$$

The above interest rate is called instantaneous interest rate. If we put B_0 amount of money in the bank then after time t we have $B_0 e^{rt} = B_t$ amount of money. From this we also have that for B_0 the amount of money that is kept follows the equation $B_0 = B_t e^{-rt}$. This sort of pricing is called the discounted price of the fixed deposit at time t. If B_t is the amount we should get at time t the intuition of the discounted price is that it tells what amount we should deposit now. So if B_t is the amount of money we require $B_t e^{-rt}$ is the amount we should invest.

We also find out that when solving the equation

$$(1 + r_n)^n = e^{rT}$$

we get that $r_n = e^{\frac{rT}{n}} - 1$. Then by choosing r_n this way and using the Taylor expansion for r_n we notice that the terms $\frac{rT}{n}$ and $e^{\frac{rT}{n}} - 1$ are approximately the same. From the definition of the function e^x we know see that the equality $B_T = e^{rT}$ holds when $B^n_{t_n} = B_T$ also limit wise since $\lim_{n\to\infty}(1 + \frac{rT}{n})^n = e^{rT}$ by definition.

5.2. The behaviour of price processes. For price processes in the Black Scholes model we use continuous-time processes with a riskless asset and one risky asset. Our riskless asset can be for example a bank account with S^0_t amount of money at time t and a risky asset for example a stock with price S_t at time t. We set $S^0_0 = 1$ and $S^0_t = e^{rt}$ for $r \geq 0$ and $t \geq 0$ like in equation (5.1). For this we have that S^0_t follows the following ordinary differential equation

$$dS^0_t = rS^0_t dt. \tag{5.2}$$

It is very easy to see that $S^0_t = e^{rt}$ is a solution to the equation (5.2) for $S^0_0 = 1$. For describing the behavior of the risky asset we use the geometric Brownian motion which has the following stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dB_t). \tag{5.3}$$

Here μ and σ are constants and $(B_t)_{t\geq 0}$ is a standard Brownian motion. The part $S_t\mu dt$ is a drift term and σ is a variance term which represents the volatility of the stock price. We use the above model on the interval $[0, T]$ where T is the maturity of the stock and hence the selling time.

To solve the equation (5.3) we first introduce the stochastic exponential

$$\mathcal{E}(B)_t = e^{B_t - \frac{t}{2}}, \quad t \in [0, T].$$

We will apply Itô's formula for the function $f(t, x) = e^{x - \frac{t}{2}}$ and we let $X_t = B_t$, where X_t is an Itô process. We have that $f \in C^{1,2}([0, T] \times \mathbb{R})$. For the partial derivatives we get $\frac{\partial f}{\partial u}(u, x) = -\frac{1}{2} f(u, x)$ and $\frac{\partial f}{\partial x}(u, x) = \frac{\partial^2 f}{\partial x^2}(u, x) = f(u, x)$. From Itô's formula we get

$$\begin{aligned} f(t, B_t) = e^{B_t - \frac{t}{2}} &= 1 + \int_0^t -\frac{1}{2} e^{B_u - \frac{u}{2}} du + \int_0^t e^{B_u - \frac{u}{2}} dB_u + \frac{1}{2}\int_0^t e^{B_u - \frac{u}{2}} du \\ &= 1 + \int_0^t e^{B_u - \frac{u}{2}} dB_u. \end{aligned}$$

From this we immediately get that $s_0\mathcal{E}(B)_t$ solves

$$s_0 + \int_0^t s_0 e^{B_u - \frac{u}{2}} dB_u = s_0 + \int_0^t s_0 \mathcal{E}(B_u)_u dB_u.$$

Likewise, for a constant σ, the process $\mathcal{E}(\sigma B)_t$ satisfies the equation

$$Y_t = 1 + \sigma \int_0^t Y_u dB_u.$$

This means that $s_0\mathcal{E}(\sigma B)_t = s_0 e^{\sigma B_t - \frac{\sigma^2 t}{2}}$ solves

$$Y_t = s_0 + \sigma \int_0^t Y_u dB_u. \tag{5.4}$$

Let $Y_t = e^{-\mu t} S_t$, where we assume that $(S_t)_{t\geq 0}$ solves (5.3). Then by Itô's formula applied to $f(t,x) = e^{-\mu t}x$,

$$\begin{aligned} Y_t = f(t, S_t) &= s_0 + \int_0^t -\mu Y_u \, du + \int_0^t e^{-\mu u}\sigma S_u dB_u + \int_0^t e^{-\mu u}\mu S_u du + \frac{1}{2}\int_0^t 0 \, du \\ &= s_0 + \int_0^t \sigma Y_u dB_u + \int_0^t -\mu Y_u + \mu Y_u \, du \\ &= s_0 + \int_0^t \sigma Y_u dB_u. \end{aligned}$$

From this by using (5.4) it follows that $Y_t = s_0 e^{\sigma B_t - \frac{\sigma^2 t}{2}}$ which means that

$$S_t = s_0 e^{\mu t - \frac{\sigma^2 t}{2} + \sigma B_t}. \tag{5.5}$$

Here the initial value s_0 is the price observed at time 0. From (5.5) we see that the process $(S_t)_{t\geq 0}$ is log-normally distributed (exponential function form). This means that $(S_t)_{t\geq 0}$ can not have negative values thus it fits well for price modelling. We can also see that $(S_t)_{t\geq 0}$ is a solution of (5.5) if and only if the process $(\log(S_t))_{t\geq 0}$ is a Brownian motion with drift. The process $(S_t)_{t\geq 0}$ has the following properties:

- for all $\omega \in \Omega$, $t \mapsto S_t(\omega) : [0,T] \to \mathbb{R}$ is a continuous function
- Independent relative increments: if $u \leq t$, the relative increment $\frac{S_t - S_u}{S_u}$ is independent of the σ-algebra $\sigma(S_v, v \leq u)$. This follows from $\frac{S_t - S_u}{S_u} = \frac{S_t}{S_u} - 1 = e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t - \mu u + \frac{\sigma^2}{2}u - \sigma B_u} - 1 = e^{\mu(t-u) - \frac{\sigma^2}{2}(t-u) + \sigma(B_t - B_u)} - 1$ and because $B_t - B_u$ is independent from B_v for all $v \leq t$, we have that $\frac{S_t - S_u}{S_u}$ is independent from S_v for all $v \leq u$.
- stationary relative increments: if $u \leq t$ then $\frac{S_t - S_u}{S_u}$ has the same law as $\frac{S_{t-u} - S_0}{S_0}$, hence both are log normally distributed.

5.3. Self-financing strategies. We define a trading strategy as a $\mathbb{R}^2$-valued process $\phi = (\phi_t)_{0\leq t\leq T} = (H_t^0, H_t)_{0\leq t\leq T}$ which is progressively measurable with respect to the augmented natural filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ of the Brownian motion. We call H_t^0 a quantity of riskless asset and H_t a quantity of a risky asset at time t. We define the value of a portfolio by using the inner product of the $\mathbb{R}^2$ valued vectors ϕ_t and (S_t^0, S_t). The value of the portfolio at time t is given by the equation

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t. \tag{5.6}$$

Next we formulate the self-financing condition in the continuous time case by setting

$$dV_t(\phi) = H_t^0 dS_t^0 + H_t dS_t.$$

We assume that

$$\int_0^T |H_t^0|\,dt < +\infty \qquad \text{and} \int_0^T H_t^2\,dt < +\infty \qquad \text{a.s.}$$

This implies that the integrals

$$\int_0^T H_t^0\,dS_t^0 = \int_0^T H_t^0 r e^r\,dt$$

and

$$\int_0^T H_t\,dS_t = \int_0^T H_t S_t(\mu dt + \sigma dB_t) = \int_0^T \mu H_t S_t dt + \int_0^T \sigma H_t S_t dB_t$$

are well defined because the map $t \to S_t$ is continuous and thus bounded on $[0, T]$ almost surely.

Here we used (5.2) and (5.3) respectively.

Definition 5.1. (a) A **self-financing trading strategy** consists of two progressively measurable processes $(H_t^0)_{0\le t\le T}$ and $(H_t)_{0\le t\le T}$ satisfying:

(1) $\int_0^T |H_t^0|\,dt + \int_0^T H_t^2\,dt < +\infty$ a.s.

(2) $H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0\,dS_u^0 + \int_0^t H_u\,dS_u$ a.s. for all $t \in [0, T]$.

(b) We define the discounted price process $\widetilde{S}$ by setting $\widetilde{S} = e^{-rt} S_t$.

Proposition 5.2. *Let $\phi = (H_t^0, H_t)_{0\le t\le T}$ be an $\mathbb{R}^2$-valued adapted process which satisfies condition (1) in Definition 5.1. If we set $V_t(\phi) = H_t^0 S_t^0 + H_t S_t$ and $\widetilde{V}_t = e^{-rt} V_t$ then ϕ defines a self-financing process if and only if*

$$\widetilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_u d\widetilde{S}_u \qquad a.s. \tag{5.7}$$

for all $t \in [0, T]$.

Proof. Let us consider the self-financing strategy ϕ. Since from Definition 5.1 it follows that $dV_t(\phi) = H_t^0 dS_t^0 + H_t dS_t$ is an Itô-process we can use Itô's formula. We differentiate $\widetilde{V}_t(\phi)$ by using Itô's formula. Since $\widetilde{V}_t = e^{-rt} V_t(\phi)$ we use the function $f(t, x) = e^{-rt} x$ and get

$$\widetilde{V}_t(\phi) = f(t, V_t(\phi)) = V_0(\phi) + \int_0^t (-r)\widetilde{V}_s(\phi)\,ds + \int_0^t e^{-rs}\,dV_s(\phi).$$

Now we see that

$$d\widetilde{V}_t(\phi) = -r\widetilde{V}_t(\phi)dt + e^{-rt} dV_t(\phi).$$

Then we use $S_t^0 = e^{rt}$ and $dV_t(\phi) = H_t^0 dS_t^0 + H_t dS_t$ to obtain

$$\begin{aligned} d\widetilde{V}_t(\phi) &= -re^{-rt}(H_t^0 e^{rt} + H_t S_t)dt + e^{-rt}(H_t^0 d(e^{rt}) + H_t dS_t) \\ &= -re^{-rt} H_t^0 e^{rt} dt - re^{-rt} H_t S_t dt + e^{-rt} H_t^0 r e^{rt} dt + e^{-rt} H_t dS_t \\ &= -rH_t^0 dt - re^{-rt} H_t S_t dt + rH_t^0 dt + e^{-rt} H_t dS_t \\ &= H_t(-re^{-rt} S_t dt + e^{-rt} dS_t) \\ &= H_t d\widetilde{S}_t. \end{aligned}$$

This yields the equation (5.7) because $H_t d\widetilde{S}_t$ is the differential form of the equation (5.7). $\square$

6. Change of probability and representation of martingales

Next we introduce a method to remove the drift of a Brownian motion and change processes into martingales by changing the probability measure. Our goal is to get a probability $\mathbb{Q}$ under which $\widetilde{S}_t$ is a martingale. The measure $\mathbb{Q}$ can be found by using Girsanov's theorem. Finding an equivalent martingale measure $\mathbb{Q}$ is important for achieving a fair game trading environment.

6.1. Equivalent probabilities. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Another probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called absolutely continuous relative to $\mathbb{P}$ if for every $A \in \mathcal{F}$ when $\mathbb{P}(A) = 0$ holds then we always have that also $\mathbb{Q}(A) = 0$.

THEOREM 6.1. *The probability measure $\mathbb{Q}$ is absolutely continuous relative to $\mathbb{P}$ if and only if there exists a non-negative random variable Z on $(\Omega, \mathcal{F})$ such that for all $A \in \mathcal{F}$*

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$

The random variable Z is called density of $\mathbb{Q}$ relative to $\mathbb{P}$ and sometimes denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

More information about this theorem can be found in [**1**] Theorem 4.2.1. The implication $\Rightarrow$ is in fact the Radon-Nikodym theorem. The probability measures are called *equivalent* if each one is absolutely continuous relative to the other. This means that if $\mathbb{P}(A) = 0$ then $\mathbb{Q}(A) = 0$ and if $\mathbb{Q}(A) = 0$ then also $\mathbb{P}(A) = 0$ for all $A \in \mathcal{F}$.

LEMMA 6.2. *If $\mathbb{Q}$ is absolutely continuous relative to $\mathbb{P}$ with density Z then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\mathbb{P}(Z > 0) = 1$.*

PROOF. First we assume that $\mathbb{P}(Z > 0) = 1$ and show that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. If this is not true then there exists a set $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ and $\mathbb{Q}(A) = 0$. But since $\mathbb{Q}(A) = \int_A Z \, d\mathbb{P} = \int_\Omega \mathbb{1}_A Z \, d\mathbb{P}$ we have that $\mathbb{Q}(A) = 0$ only if $\mathbb{1}_A Z = 0$ $\mathbb{P}$-a.s. This is impossible because $\mathbb{P}(Z > 0) = 1$ and $\mathbb{P}(A) > 0$.

For the other direction we first assume that for any $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ we also have $\mathbb{Q}(A) > 0$. Let us also assume that $\mathbb{P}(Z > 0) < 1$. Then we define a set $B = \{\omega \in \Omega : Z(\omega) = 0\}$ and we get $\mathbb{P}(B) = \mathbb{P}(Z = 0) = 1 - \mathbb{P}(Z > 0) > 0$. But for $\mathbb{Q}(B)$ we get that $\mathbb{Q}(B) = \int_\Omega \mathbb{1}_B Z \, d\mathbb{P} = 0$ which is against our assumption $\mathbb{P}(B) > 0 \Rightarrow \mathbb{Q}(B) > 0$. $\square$

6.2. Girsanov's theorem. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a probability space with the augmented natural filtration of a standard Brownian motion on the time interval $[0,T]$.

THEOREM 6.3. *Let $(\theta_t)_{0\le t\le T}$ be a progressively measurable process satisfying*

$$\int_0^T \theta_s^2\, ds < \infty$$

almost surely and such that the process $(L_t)_{0\le t\le T}$ defined by

$$L_t = \exp\left(-\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds\right), \quad t \in [0,T]$$

is a martingale. Then with respect to $\mathbb{Q}$, given by

$$\mathbb{Q}(A) = \int_A L_t\, d\mathbb{P}, \quad A \in \mathcal{F},$$

the process $(W_t)_{0\le t\le T}$ defined by

$$W_t = B_t + \int_0^t \theta_s\, ds, \quad 0 \le t \le T$$

is a standard Brownian motion.

Proof for Girsanov's theorem can be found in [**1**] Theorem 4.2.2.

REMARK 6.4. In applications it is often difficult to check whether $(L_t)_{0\le t\le T}$ is a martingale. As a sufficient condition for showing that $(L_t)_{0\le t\le T}$ is a martingale we can use the Novikov condition

$$\mathbb{E}e^{\frac{1}{2}\int_0^T \theta_t^2\, dt} < \infty.$$

6.3. Representation of Brownian martingales. If the filtration of the corresponding martingale is generated by a Brownian motion we have a Brownian martingale. For obtaining the fair price of the option we will use the so-called martingale representation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(B_t)_{0\le t\le T}$ be a standard Brownian motion with augmented natural filtration $(F_t)_{0\le t\le T}$. We know that if the process $(H_t)_{0\le t\le T}$ is progressively measurable, and such that $\mathbb{E}\left(\int_0^T H_t^2\, dt\right) < \infty$, then the process $\left(\int_0^t H_s dB_s\right)_{0\le t\le T}$ is a square integrable martingale which is null at 0. Next we will show that any Brownian martingale can be represented in terms of a stochastic integral.

THEOREM 6.5. *Let $(\mathcal{F}_t)_{0\le t\le T}$ be the augmented natural filtration of a Brownian motion and $(M_t)_{0\le t\le T}$ be a square-integrable martingale. There exists a progressively measurable process $(H_t)_{0\le t\le T}$ such that $\mathbb{E}\left(\int_0^T H_s^2\, ds\right) < \infty$, and*

$$M_t = M_0 + \int_0^t H_s\, dB_s \quad \textit{almost surely for all} \quad t \in [0,T]. \tag{6.1}$$

More about this theorem can be found in [**1**] Theorem 4.2.4. This representation only applies to the martingales relative to the natural filtration of the Brownian motion. The martingales we are interested in are square integrable and they are martingales with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$ and therefore we always have the representation (6.1).

Remark 6.6. From this theorem we have a representation

$$U = \mathbb{E}(U) + \int_0^T H_s \, dB_s \quad \text{a.s.}$$

for every $\mathcal{F}_T$-measurable square-integrable random variable U, where $(H_t)_{0\le t\le T}$ is a progressively measurable process such that $\mathbb{E}\left(\int_0^T H_t^2 \, ds\right) < \infty$.

Proof. Let U be an $\mathcal{F}_T$-measurable random variable satisfying $\mathbb{E}U^2 < \infty$. Let us consider the martingale given by $N_t := \mathbb{E}(U|\mathcal{F}_t)$. Firstly $(N_t)_{t\ge 0}$ is a martingale because it is $\mathcal{F}_t$-adapted by construction and $\mathbb{E}|N_t| < \infty$ for all $t \ge 0$. The relation $\mathbb{E}|N_t| < \infty$ holds because by Hölder's inequality we can see that $\mathbb{E}|N_t| \le (\mathbb{E}N_t^2)^{\frac{1}{2}}$ and

$$\mathbb{E}N_t^2 = \mathbb{E}(\mathbb{E}[U|\mathcal{F}_t])^2 \le \mathbb{E}(\mathbb{E}[U^2|\mathcal{F}_t]) = \mathbb{E}U^2 < \infty.$$

Here we used Jensen's inequality for conditional expectation and the tower property. Jensen's inequality for conditional expectation can be found in [**8**] Theorem 4 (iii).

To show the martingale property we use the tower property again and get

$$\mathbb{E}[N_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[U|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[U|\mathcal{F}_s] = N_s \qquad \text{when } s \le t.$$

This shows that $(N_t)_{t\ge 0}$ is a martingale. Finally the representation

$$U = \mathbb{E}(U) + \int_0^T H_s \, dB_s$$

comes from the fact that $N_T = U = N_0 + \int_0^T H_s \, dB_s$, and since $\mathbb{E}(U) = N_0$ we have that

$$U = \mathbb{E}(U) + \int_0^T H_s \, dB_s.$$

□

7. Pricing and hedging of options in the Black-Scholes model

7.1. A probability under which $\widetilde{S}_t$ is a martingale. We will now consider the description of the Black-Scholes model and use Theorem 3.1 in order to switch the probability measure from $\mathbb{P}$ to $\mathbb{Q}$, where the discounted share price $\widetilde{S}_t = e^{-rt}S_t$, $t \in [0, T]$, is a martingale with respect to the probability $\mathbb{Q}$. First we will show that such $\mathbb{Q}$ exists. By using Ito's product rule and the stochastic differential equation (5.3), we have

$$
\begin{aligned}
d\widetilde{S}_t = d(e^{-rt}S_t) &= -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= -re^{-rt}S_t dt + e^{-rt}(S_t(\mu dt + \sigma dB_t)) \\
&= -re^{-rt}S_t dt + e^{-rt}S_t\mu dt + e^{-rt}S_t\sigma dB_t \\
&= -r\widetilde{S}_t dt + \widetilde{S}_t\mu dt + \widetilde{S}_t\sigma dB_t \\
&= \widetilde{S}_t((\mu - r)dt + \sigma dB_t).
\end{aligned}
$$

If we then set $W_t = B_t + \frac{(\mu-r)t}{\sigma}$, we have $dB_t = dW_t - \frac{\mu-r}{\sigma}dt$, and by substituting this into $\widetilde{S}_t((\mu - r)dt + \sigma dB_t)$ we will get

$$
d\widetilde{S}_t = \widetilde{S}_t\sigma dW_t. \tag{7.1}
$$

From Girsanov's theorem by setting $\theta_t = \frac{(\mu-r)}{\sigma}$ we have that there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $(W_t)_{0\leq t\leq T} = B_t + \frac{(\mu-r)t}{\sigma}$ is a standard Brownian motion. We also see from (7.1) that $(\widetilde{S}_t)$ should be a martingale (provided that the integrand is square integrable) with respect to the probability $\mathbb{Q}$ because there is no drift.

The differential equation (7.1) with initial condition $\widetilde{S}_0$ can be solved by using a stochastic exponential multiplied by a constant σ, which is

$$
\mathcal{E}(\sigma W_t) = \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right).
$$

This we have shown with Ito's formula in chapter 3.2. From this we have that

$$
\widetilde{S}_t = \widetilde{S}_0 \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right).
$$

7.2. Pricing. In this section, we will focus on European options. Our goal is by using theorems from the previous sections to obtain a fair price, which we call a premium, of the European call-option and put-option. This means that in a fair market model it should not be possible to make a riskless profit, and the odds for gaining should be the same for the stock trader and the writer. For example if the premium is too cheap the trader's risk of losing money is slim to none and if the price is too high no one wants to start trading. This means that for instance free premium gives the trader a possibility to make riskless profit because even if he loses ($S_T < K$) he can decide to not buy the stock with strike price K and hence loses no money (loss=premium=0).

The characteristics of a European call option is that it can only be exercised at maturity time T. This means that the trader can not decide the time when to buy a stock and hence he can only decide wether buy the stock at time T or not. European option trading starts when someone buys a European call option with the price called premium. After the premium is paid the holder of a call option can decide wether to buy or not to buy shares with the price decided beforehand called the strike price K. Assume we have a European call-option with strike price $K > 0$ at time $T > 0$.

We define the payoff function of a European call-option at time T as a non-negative $\mathcal{F}_T$-measurable random variable that can be written as

$$f(S_T) = (S_T - K)^+,$$

where S_T is the price of a stock at maturity T. We take the maximum of $S_T - K$ and zero, because it is assumed that the holder will not exercise the option in case $S_T < K$, so that the value is then zero. Hence the function describes the possible gain. For example if $S_T > K$ the option holder buys a share for the price K and if he sells it immediately he makes profit

$$S_T - K - \text{premium}$$

and if $S_T \leq K$ the option holder does not buy and loses the premium amount of money that he paid before starting to trade.

In the case of a put the option holder is allowed to decide wether sell or not to sell the asset at time T with the strike price K. This means that, opposite to the call option, the stock holder buys a put option if he expects the price of the underlying to fall. In the case of a put we define a European option to be $f(S_T) = (K - S_T)^+$.

We introduce now admissible strategies which we define in the following manner:

Definition 7.1. A strategy $\phi = (H_t^0, H_t)_{0\leq t\leq T}$ is called admissible if it is self-financing and if the discounted value of the corresponding portfolio $\widetilde{V}_t(\phi) = H_t^0 + H_t\widetilde{S}_t$ is non-negative for all $t \in [0,T]$. We also assume that $\sup_{t\in[0,T]} \widetilde{V}_t$ is square integrable for all $t \in [0,T]$ with respect to $\mathbb{Q}$.

Any option considered here is a non-negative $\mathcal{F}_T$-measurable random variable. An option h is called replicable if its payoff at maturity time T is the same as the final value of an admissible strategy ϕ. This means that we have $h = V_T(\phi)$. In the case of a call for the option $h = (S_T - K)^+$ to be replicable it is useful that it is square integrable with respect to $\mathbb{Q}$. The random variable $h = (S_T - K)^+$ is square integrable because $\mathbb{E}_{\mathbb{Q}}(S_T)^2 < \infty$. In the case of a put, the random variable h is also bounded because the maximum of a function $f(S_T) = (K - S_T)^+$ is the strike price K.

Next we are going to consider a formula for calculating the value of any replicating portfolio. First we notice that from Remark 6.6 any option which is defined by a non-negative $\mathcal{F}_T$-measurable random variable h, that is square integrable under the probability $\mathbb{Q}$, is replicable. Therefore we always assume that $h \in L^2(\mathbb{Q})$. The fact that h is replicable follows from Remark 6.6 because

$$h = \mathbb{E}h + \int_0^T H_s\, dB_s = \mathbb{E}h + \int_0^T \frac{H_s}{\widetilde{S}_s\sigma}\, d\widetilde{S}_s,$$

since $d\widetilde{S}_s = \widetilde{S}_s\sigma dB_s$ by (7.1). Now we can define a trading strategy $\psi = (\widehat{H}_t^0, \widehat{H}_t)_{0\leq t\leq T}$ by setting $\widehat{H}_t = \frac{H_s}{\widetilde{S}_s\sigma}$ and $\widehat{H}_t^0 = \widetilde{V}_t - \widetilde{H}_t\widetilde{S}_t$. Then we have $h = \widetilde{V}_T(\psi)$.

Theorem 7.2. *The value at time t of any replicating portfolio is given by*

$$V_t = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h|\mathcal{F}_t).$$

Now we have another way to define the option value at time t which is the expression $\mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h|\mathcal{F}_t)$.

PROOF. First we assume that there is an admissible strategy (H^0, H) replicating the option. We use equation (5.6) to define our portfolio's value at time t which is

$$V_t = H_t^0 S_t^0 + H_t S_t,$$

and the discounted process is also defined like before by setting $\widetilde{V}_t = V_t e^{-rt}$. This gives us the equation

$$\widetilde{V}_t = H_t^0 + H_t \widetilde{S}_t.$$

Since the strategy is self-financing, we get from Proposition 3.2 and equation (7.1)

$$\begin{aligned}\widetilde{V}_t &= V_0 + \int_0^t H_u \, d\widetilde{S}_u \\ &= V_0 + \int_0^t H_u \sigma \widetilde{S}_u \, dW_u.\end{aligned}$$

By definition of admissible strategies we have to verify that $\sup_{t\in[0,T]} \widetilde{V}_t$ is square integrable with respect to the probability $\mathbb{Q}$. We can also see from the upper equality that $(\widetilde{V}_t)$ is a stochastic integral relative to (W_t). So we show that $(\widetilde{V}_t)$ is a square integrable martingale with respect to $\mathbb{Q}$, which means that we have to show that $\mathbb{E}_{\mathbb{Q}}\left(\int_0^t H_u^2 \sigma^2 \widetilde{S}_u^2 \, du\right) < \infty$.

Let us define $\widehat{H}_u := H_u \sigma \widetilde{S}_u$. Then we have

$$\int_0^t H_u \sigma \widetilde{S}_u \, dW_u = \int_0^t \widehat{H}_u \, dW_u.$$

Since $(H_t^0, H_t)_{t\geq 0}$ is a self-financing $(\mathcal{F}_t)_{t\geq 0}$ progressively measurable strategy, Definition 5.1 gives us that

$$\int_0^T H_t^2 \, dt < \infty. \quad \text{a.s.}$$

This also gives us that

$$\int_0^T \widehat{H}_t^2 \, dt = \int_0^T (H_u \sigma \widetilde{S}_u)^2 \, du < \infty \quad \text{a.s.}$$

since

$$\begin{aligned}\int_0^T (H_u \sigma \widetilde{S}_u)^2 \, du = \int_0^T H_u^2 \sigma^2 \widetilde{S}_u^2 \, du &\leq \sigma^2 \int_0^T H_u^2 \sup_{u\in[0,T]} \widetilde{S}_u^2 \, du \\ &= \sigma^2 \sup_{u\in[0,T]} \widetilde{S}_u^2 \int_0^T H_u^2 \, du < \infty.\end{aligned}$$

Here we used that $\sup_{u\in[0,T]} \widetilde{S}_u^2 < \infty$ a.s. because $(\widetilde{S}_u)_{u\in[0,T]}$ is a.s. continuous on $[0, T]$.

Next we introduce the sequence of stopping times $\tau_n = \inf\{t > 0 : \int_0^t \widehat{H}_u^2 \, du = n\}$ for $n = 1, 2, \ldots$. Now we have that $\int_0^{T\wedge\tau_n} \widehat{H}_u \, dW_u = \int_0^T \widehat{H}_u \mathbb{1}_{\{u\leq\tau_n\}} dW_u$ and

$\mathbb{E}_{\mathbb{Q}}\left(\int_0^{\tau_n} \widehat{H}_u^2 \, du\right) = n$. Then we consider the limit where n approaches infinity and by Itô-isometry we get

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \left(\int_0^{T\wedge\tau_n} \widehat{H}_u \, dW_u\right)^2 &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \left(\int_0^{T\wedge\tau_n} \widehat{H}_u^2 \, du\right) \\
&= \mathbb{E}_{\mathbb{Q}} \left(\lim_{n\to\infty} \int_0^{T\wedge\tau_n} \widehat{H}_u^2 \, du\right) \\
&= \mathbb{E}_{\mathbb{Q}} \int_0^{T} \widehat{H}_u^2 \, du.
\end{aligned}$$

We continue by using the fact that $\mathbb{E}_{\mathbb{Q}}\left(\sup_{t\in[0,T]} \widetilde{V}_t\right)^2 < \infty$. We have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} \int_0^{T} \widehat{H}_u^2 \, du = \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \left(\int_0^{T\wedge\tau_n} \widehat{H}_u \, dW_u\right)^2 &= \limsup_n \mathbb{E}_{\mathbb{Q}} \left(\int_0^{T\wedge\tau_n} \widehat{H}_u \, dW_u\right)^2 \\
\leq \mathbb{E}_{\mathbb{Q}} \left(\limsup_n \left(\int_0^{T\wedge\tau_n} \widehat{H}_u \, dW_u\right)^2\right) \\
\leq \mathbb{E}_{\mathbb{Q}} \left(\sup_{0\leq t\leq T} \left(\int_0^{t} \widehat{H}_u \, dW_u\right)^2\right).
\end{aligned}$$

If a sequence has a limit it coincides with its limit superior. Hence the lemma of Fatou can be applied.

Since $\widetilde{V}_t = V_0 + \int_0^t H_u \sigma \widetilde{S}_u \, dW_u$ we also have that

$$\int_0^t H_u \sigma \widetilde{S}_u \, dW_u = \left| \int_0^t H_u \sigma \widetilde{S}_u \, dW_u + V_0 - V_0 \right| \leq |\widetilde{V}_t| + |V_0|.$$

Taking square and supremum of both sides we get

$$\sup_{0\leq t\leq T} \left| \int_0^t H_u \sigma \widetilde{S}_u \, dW_u \right|^2 \leq 2 \left(\sup_{0\leq t\leq T} |\widetilde{V}_t|\right)^2 + 2|V_0|^2$$

which proves that $\sup_{0\leq t\leq T} |\int_0^t H_u \sigma \widetilde{S}_u \, dW_u|^2$ is square integrable since

$$\mathbb{E}_{\mathbb{Q}} \sup_{t\in[0,T]} |\widetilde{V}_t|^2 < \infty$$

by assumption. The process $(\widetilde{V}_t)_{0\leq t\leq T}$ is also martingale because $(H_u)_{0\leq u\leq T}$ is progressively measurable with respect to the given filtration. This gives us the equation

$$\widetilde{V}_t = \mathbb{E}_{\mathbb{Q}}(\widetilde{V}_T | \mathcal{F}_t).$$

By hypothesis $V_T = h$ and its discounted value at time t is given by $\widetilde{V}_t = V_t e^{-rt}$ we can write the upper equation like in Theorem 7.2.

$$V_t = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)} h | \mathcal{F}_t). \tag{7.2}$$

□

Now we have proven that if an admissible portfolio (H^0, H) replicates the option defined by random variable h we can get its value from equation (7.2).

THEOREM 7.3. *There exists processes H^0 and H defining an admissible strategy, such that*

$$H_t^0 S_t^0 + H_t S_t = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)} h | \mathcal{F}_t)$$

under the probability $\mathbb{Q}$ with $h \in L^2(\mathbb{Q})$.

This shows that the option is indeed replicable.

PROOF. We have that under the probability $\mathbb{Q}$, the process $(M_t)_{0 \le t \le T}$ given by $M_t := \mathbb{E}_{\mathbb{Q}}(e^{-rT} h | \mathcal{F}_t)$ is a square-integrable martingale. The filtration $(\mathcal{F}_t)_{0 \le t \le T}$, which is the augmented natural filtration of the Brownian motion $(B_t)_{0 \le t \le T}$, is also the augmented natural filtration of $(W_t)_{0 \le t \le T}$. Since $(M_t)_{0 \le t \le T}$ is a square-integrable martingale and adapted with respect to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$, from the representation of Brownian martingales (Theorem 6.5), there exists a progressively measurable process $(K_t)_{0 \le t \le T}$ such that $\mathbb{E}\left(\int_0^T K_u^2 \, du\right) < \infty$ and

$$\forall t \in [0, T] \qquad M_t = M_0 + \int_0^t K_u \, dW_u \qquad \text{a.s.}$$

Then the strategy $\phi = (H_t^0, H_t)_{0 \le t \le T}$ can be chosen by setting $H_t = \frac{K_t}{\sigma \widetilde{S}_t}$ and $H_t^0 = M_t - H_t \widetilde{S}_t$ (compare with the equations $\widetilde{V}_t = H_t^0 + H_t \widetilde{S}_t$ and $\widetilde{V}_t = V_0 + \int_0^t H_u \sigma \widetilde{S}_u \, dW_u$). The strategy ϕ is then, by Proposition 5.2 and equality (7.1), a self-financing strategy and its value at time t is given by

$$V_t(\phi) = e^{rt} M_t = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} h | \mathcal{F}_t\right).$$

From this expression we can see that $V_t(\phi)$ is a non-negative random variable, with $\sup_{0 \le t \le T} V_t(\phi)$ square-integrable under $\mathbb{Q}$ and $V_T(\phi) = h$. Non-negativity follows from the fact that h is a non-negative $\mathcal{F}_T$-masurable random variable. We have found an admissible strategy replicating h.

□

REMARK 7.4. Let us try to express the option value V_t at time t by using a function of t and S_t. This can be done when the random variable h can be written as $h = f(S_T)$ (as a function of the asset price at time T). By using $h = f(S_T)$ we have

$$V_t = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(S_T) | \mathcal{F}_t\right).$$

By using the formula $\widetilde{S}_T = \widetilde{S}_0 \exp\left(\sigma W_T - \frac{\sigma^2}{2} T\right)$ we have that

$$S_T = \widetilde{S}_0 \exp\left(\sigma W_T - \frac{\sigma^2}{2} T\right) e^{rT}$$

and by writing $S_T = \frac{S_T}{S_t} S_t$ we can get the expression $W_T - W_t$ to show up:

$$\begin{aligned}
V_t &= \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(S_T)|\mathcal{F}_t\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(S_t e^{\sigma W_T - \frac{\sigma^2 T}{2} - (\sigma W_t - \frac{\sigma^2 t}{2})} e^{r(T-t)})|\mathcal{F}_t\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(S_t e^{r(T-t)} e^{\sigma(W_T - W_t) - (\frac{\sigma^2}{2})(T-t)})|\mathcal{F}_t\right).
\end{aligned}$$

The random variable S_t is $\mathcal{F}_t$-measurable and, under $\mathbb{Q}$, $W_T - W_t$ is independent of $\mathcal{F}_t$. From Proposition A.2.5 [**1**], we can compute

$$\mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(S_t e^{r(T-t)} e^{\sigma(W_T - W_t) - (\frac{\sigma^2}{2})(T-t)})|\mathcal{F}_t\right)$$

as if S_t was a constant. We write

$$V_t = F(t, S_t),$$

where

$$F(t,x) = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(x e^{r(T-t)} e^{\sigma(W_T - W_t) - (\frac{\sigma^2}{2})(T-t)})\right).$$

This can also be written in the following form by calculating the product of the two exponential functions inside the function f. We get

$$F(t,x) = \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} f(x e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)})\right). \tag{7.3}$$

From the fact that $W_T - W_t$ is a zero-mean normal random variable with $T - t$ variance under $\mathbb{Q}$ we can calculate the expected value by using the density of normal distribution and writing $W_T - W_t = Y\sqrt{T-t}$, where Y is standard Gaussian random variable. We have

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f(x e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma y \sqrt{T-t}}) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

Next we calculate the function $F(t,x)$ explicitly for calls and puts. We use the definition of f we introduced in the subsection Pricing 7.2 for calculation which is $f(x) = (x - K)_+$ in the case of the call. From equality (7.3) we have,

$$\begin{aligned}
F(t,x) &= \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} (x e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)} - K)_+\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(x e^{\sigma(W_T - W_t) - \frac{\sigma^2 (T-t)}{2}} - K e^{-r(T-t)}\right)_+ \\
&= \mathbb{E}\left(x e^{\sigma\sqrt{\theta} g - \frac{\sigma^2 \theta}{2}} - K e^{-r\theta}\right)_+.
\end{aligned}$$

Here g is a standard Gaussian random variable and $\theta = T - t$.

Let us check when the expression $x e^{\sigma\sqrt{\theta} g - \frac{\sigma^2 \theta}{2}} - K e^{-r\theta}$ is positive because otherwise we get zero. We have that

$$x e^{\sigma\sqrt{\theta} g - \frac{\sigma^2 \theta}{2}} - K e^{-r\theta} \geq 0$$

and in order to remove the exponential functions we take logarithm of both sides of the above inequality after adding $Ke^{-r\theta}$ on both sides and get

$$\log(x) + \sigma\sqrt{\theta}g - \frac{\sigma^2\theta}{2} \geq \log(K) - r\theta.$$

We also want to separate our Gaussian random variable from the expression and after that we divide the inequality by $\sigma\sqrt{\theta}$. We get

$$g + \frac{\log(\frac{x}{K}) + (r - \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} \geq 0.$$

Let us set

$$d_1 = \frac{\log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} \qquad \text{and} \qquad d_2 = \frac{\log(\frac{x}{K}) + (r - \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} = d_1 - \sigma\sqrt{\theta}.$$

We can now compute the expected value for $xe^{\sigma\sqrt{\theta}g - \frac{\sigma^2\theta}{2}} - Ke^{-r\theta}$ using notations above, because we know when the expression is positive (otherwise we get zero). We get

$$\begin{aligned}
F(t,x) &= \mathbb{E}\left((xe^{\sigma\sqrt{\theta}g - \frac{\sigma^2\theta}{2}} - Ke^{-r\theta})1\!\!1_{\{g+d_2 \geq 0\}}\right) \\
&= \int_{-d_2}^{\infty} (xe^{\sigma\sqrt{\theta}y - \frac{\sigma^2\theta}{2}} - Ke^{-r\theta})\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy \\
&= \int_{-\infty}^{d_2} (xe^{-\sigma\sqrt{\theta}y - \frac{\sigma^2\theta}{2}} - Ke^{-r\theta})\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy.
\end{aligned}$$

Notice that in the last equality we have term $-\sigma\sqrt{\theta}y$ because of the transformation $y \to -y$. We continue writing the above expression as difference of two integrals:

$$F(t,x) = x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\sigma\sqrt{\theta}y - \frac{\sigma^2\theta}{2}} e^{-\frac{y^2}{2}}\, dy - Ke^{-r\theta}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\, dy.$$

In the first integral using the change of variable $z = y + \sigma\sqrt{\theta}$ the upper bound changes to $d_2 + \sigma\sqrt{\theta} = d_1$ and we obtain

$$\begin{aligned}
x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\sigma\sqrt{\theta}y - \frac{\sigma^2\theta}{2}} e^{-\frac{y^2}{2}}\, dy &= x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\sigma\sqrt{\theta}(z-\sigma\sqrt{\theta}) - \frac{\sigma^2\theta}{2}} e^{-\frac{(z-\sigma\sqrt{\theta})^2}{2}}\, dz \\
&= x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\sigma\sqrt{\theta}z + \sigma^2\theta - \sigma^2\theta/2 + \sigma\sqrt{\theta}z - \sigma^2\theta/2} e^{-\frac{z^2}{2}}\, dz \\
&= x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{z^2}{2}}\, dz.
\end{aligned}$$

Finally using the notation $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}\, dx$ for the expression

$$x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{z^2}{2}}\,dz - Ke^{-r\theta}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{y^2}{2}}\,dy,$$

we obtain the Black-Scholes formula for call options:

$$F(t,x) = xN(d_1) - Ke^{-r\theta}N(d_2).$$

8. Risk-Sensitive Asset Management

In this section we have a different problem and objective than in the first part of the thesis. Instead of having only one risky asset we can have many different risky assets and we also introduce a new concept for defining the trading strategy by taking a proportion of investor's whole wealth and different risky assets. Before our main goal was to calculate the fair price for the European call and put option but now we will choose a certain asset management and aim to find the optimal control h^*. There are many different utility functions that can be used to model the investor's wealth with a given risk, hence the decision maker can choose a value function that will suit for his needs. Here the value function we are going to use is called risk-sensitive asset management criterion. Comparing with the first section where there was no point of view about the most suitable value function, here there are many possibilities to choose.

We are going to present the risk-sensitive asset management model in a diffusion setting and solve the risk-sensitive asset management problem when asset and factor risks are uncorrelated, which means that $\Lambda\Sigma^T = 0$.

8.1. Financial Market. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$ be the underlying probability space. Our Brownian motion $B(t)$ is defined like in Definition 3.4 but this time we expand our calculations to $\mathbb{R}^N$ and define an $\mathbb{R}^N$-valued $(\mathcal{F}_t)_{0\le t\le T}$-Brownian motion with independent components $B_k(t)$, $k = 1, \dots, N$.

So far we have only considered a market consisting of a single risky asset whose price S_t satisfies the following SDE

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = s_0. \tag{8.1}$$

This time we have m-risky assets whose price $S_i(t)$, $i = 1, \dots, m$ is modelled as a diffusion process satisfying the SDE:

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dB_k(t), \quad S_i(0) = s_i, \quad i = 1, \dots, m.$$

We see that the drift of $S_i(t)$ is more complicated than in the equation (8.1). The difference is that the drift term is now an affine function of a n-dimensional Gaussian diffusion process $X = (X(t))_{t\ge 0}$ satisfying the SDE

$$dX(t) = (b + BX(t))dt + \Lambda dB(t), \quad X(0) = x. \tag{8.2}$$

We interpret the factor $X(t)$ as an exogenous macroeconomic, microeconomic or statistical process driving asset returns. Exogenous simply means that $X(t)$ affects a model without being affected by it. The process X can be used for example to model interest rates, inflation or a stochastic risk premium.

The value of a unit deposit at time 0 in the money market asset we denote by $S_0(0)$. The dynamics of S_0 is:

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A_0^T X(t))dt, \quad S_0(0) = 1.$$

From this expression we can conclude that the value S_0 in the money market is no longer risk-free, but subject to the randomness of the factor process X globally.

We set $N = m + n$ and assume that the market parameters a_0, a, b, A_0, A, B, $\Sigma := [\sigma_{ij}]$, $i = 1, \dots, m$, $j = 1, \dots, N$, $\Lambda := [\Lambda_{ij}]$, $i = 1, \dots, n$, $j = 1, \dots, N$ used before are constant vectors and matrices of appropriate dimensions which will be explained in detail below. We also assume that the matrix $\Sigma\Sigma^T$ is positive definite which means that the covariance matrix $\Sigma\Sigma^T$ has a full rank m. The product $\Sigma\Sigma^T$ is a $m \times m$ matrix because Σ is a $m \times N$ matrix and hence its transpose is a $N \times m$ matrix. Multiplying a $m \times N$ matrix with a $N \times m$ matrix we get a $m \times m$ matrix. The implication is that we can not replicate m assets by only using the portfolio for $m - 1$ assets. For example if the rank would be $m - 1$ we do not have all the share prices for our trading.

We let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security and factor processes by time t.

8.2. Investment portfolio. An investor's general objective is to start with the capital v_0 at time 0 and maximize later the portfolio value at a fixed time $T > 0$ with no intermediate consumption or external income. We define an $\mathbb{R}^m$-valued stochastic process h in order to share or allocate the total portfolio among assets. We define the ith component of the process h in the following way:

$$h_i(t) := \frac{H_i(t)S_i(t)}{V_t},$$

where $S_i(t)$ is the price of the ith component, $H_i(t)$ is the quantity of $S_i(t)$ and V_t is the total portfolio value. We see that $h_i(t)$ denotes the proportion of total portfolio value invested in the ith risky security at time t, where $i = 1, \dots, m$. For example if $h_i > 0$ an investor owns the asset, in other words, he has paid money to buy the asset. The case $h_i < 0$ means that the investor tries to sell the asset that he does not own and he benefits if the price of the asset falls. The case $h_i > 0$ is also called **long** asset and the case $h_i < 0$ **short** asset. If a proportion $h_i(t) > 1$ the investor has leverage for example by borrowing cash.

We define asset $h_0(t)$ to be the balance of the total portfolio that is not allocated to a risky asset. We can express the proportion of total portfolio value invested in the money market instrument by using the budget equation:

$$h_0(t) = 1 - \sum_{i=1}^{m} h_i(t) = 1 - h^T(t)\mathbf{1},$$

where 1 is the proportion relative to whole wealth, and then we take away everything that is invested in the risky assets, which is $\sum_{i=1}^{m} h_i(t)$. We also write $h = (h_1, \dots, h_m)$ and $\mathbf{1} \in \mathbb{R}^m$ denotes an m-element column vector with all elements set to 1. We invest the excess cash $h_0(t) > 0$ to the money market instrument while $h_0(t) < 0$ means that we have a cash requirement met by borrowing from the money market.

In order to define the Girsanov exponential in this context we introduce the multidimensional version of Girsanov's theorem.

THEOREM 8.1. *Let T be a fixed positive time, and let $\Theta(t) = (\Theta_1(t), \dots, \Theta_N(t))$ be an N-dimensional progressively measurable process with respect to $(\mathcal{G}_t)_{t\geq 0}$. We define*

$$Z(t) = \exp\Big\{ -\int_0^t \Theta(u)\, dB(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2\, du \Big\},$$

$$W(t) = B(t) + \int_0^t \Theta(u)\, du,$$

and assume that

$$\mathbb{E}\int_0^T \|\Theta(u)\|^2 Z^2(u)\, du < \infty.$$

We set $Z = Z(T)$. Then $\mathbb{E}Z = 1$, and under the probability measure $\mathbb{Q}$ given by

$$\mathbb{Q}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \, \textit{for all} \, A \in \mathcal{G},$$

the process $W(t)$ is an N-dimensional Brownian motion.

It is easy to see that (Z_t) is a martingale: Itô's formula helps us to see that (Z_t) is a local martingale (see [**5**], page 213) since it satisfies

$$Z(t) = Z(0) - \int_0^t \Theta Z(u) dB(u)$$

and by the L^2-assumption the Itô integral is a martingale.

The proof for Theorem 8.1 is similar to the one dimensional case but instead of one dimensional Levy's theorem we use N-dimensional Levy's theorem. The proof can be found in [**11**] Theorem 5.4.1.

Then we define the class of admissible strategies $\mathcal{A}$ in a more general way than in Definition 7.1. Because of the dependence on the factor process $X(t)$ we do not immediately have that $h(t)$ is bounded by a constant and hence we need almost sure square integrability of $h(t)$.

DEFINITION 8.2. The class of admissible strategies $\mathcal{A}$ consists of all control processes h such that

(1) $h(t)$ is progressively measurable with respect to $(\mathcal{G}_t)_{t\geq 0}$ and is right continuous with left limits.
(2) $\mathbb{P}\left(\int_0^t |h(s)|^2\, ds < \infty\right) = 1$ a.s., for all $t > 0$.
(3) The Girsanov exponential χ_t^h given by

$$\chi_t^h := \exp\left\{ -\theta \int_0^t h(s)^T \Sigma\, dW_s - \frac{1}{2}\theta^2 \int_0^t h(s)^T \Sigma\Sigma^T h(s)\, ds \right\},$$

is an exponential martingale.

LEMMA 8.3. *For the Girsanov exponential $Z(t)$ in Theorem 8.1 we also have the following result:*

$$\mathbb{E}[Z(T)] = 1 \quad \textit{if and only if} \quad (Z(t))_{t\in[0,T]} \quad \textit{is a martingale.}$$

PROOF. First assume that $(Z(t))_{t\in[0,T]}$ is a martingale. Then from martingale property it follows that

$$\mathbb{E}[Z(T)] = \mathbb{E}[Z(0)] = 1.$$

Then assume that $\mathbb{E}[Z(T)] = 1$. If we write $Z(t)$ with Itô's formula we see that we have only a stochastic integral and a constant left meaning that we have a local martingale since stochastic integrals are local martingales in general. We need to show that $(Z(t))_{t\in[0,T]}$ is a true martingale.

By using [**15**] Lemma 2 we know that a non-negative local supermartingale X such that X_0 is integrable is a supermartingale. Since a local martingale is also a local supermartingale and $Z(t)$ is non-negative (exponential function) we can apply this theorem (also $Z(0) = 1$ is integrable).

Since we have that $(Z(t))_{t\in[0,T]}$ is a supermartingale the supermartingale condition gives us

$$\mathbb{E}[\mathbb{1}_A Z(s)] \geq \mathbb{E}[\mathbb{1}_A Z(T)]. \tag{8.3}$$

Since this holds for any $A \in \mathcal{F}_s$ it also holds for the complement A^C and hence our assumption $\mathbb{E}[Z(T)] = 1 = \mathbb{E}[Z(0)]$ implies that (8.3) is in fact

$$\mathbb{E}[\mathbb{1}_A Z(s)] = \mathbb{E}[\mathbb{1}_A Z(T)].$$

This is because (8.3) must hold for A and A^C, but summing those up would get $1 = \mathbb{E}[Z(0)] \geq \mathbb{E}[Z(s)] \geq \mathbb{E}[Z(T)] = 1$. This implies that '$\geq$' is in fact '$=$', which means that we have a martingale condition. ☐

We can now use the Theorem 8.1 with $\Theta = h^T(s)\Sigma$ and $\Theta^2 = h^T(s)\Sigma\Sigma^T h(s)$.

8.3. Formulating the portfolio dynamics. Let us first formulate the value of the investor's portfolio in the 1-dimensional case in order to understand the general model more easily. We assume that investment strategies are self-financing meaning that there are no external inputs to or outputs from the portfolio. We also assume that there is no market friction which means that when buying or selling assets there are no transaction costs, so that the modelled price does not change when the transaction is executed. We can interpret the definition of h in the same way also in the single asset case. The adapted process $h(t)$ describes the proportion of total portfolio value at time t invested in the risky asset but now it has only one dimension. From this it follows that the the proportion invested in the cash that is not allocated to the risky asset is $(1 - h(t))$.

At time t the total portfolio value is $V(t)$ and the money value invested in stocks is $h(t)V(t)$. From the money value we can get the units of stocks by dividing $h(t)V(t)$ by $S(t)$. Then we use the equation (8.1) and get the value of a holding over a time interval dt which is

$$\frac{h(t)V(t)}{S(t)} dS(t) = h(t)V(t)(\mu dt + \sigma dB(t)). \tag{8.4}$$

If we denote the interest rate by r we can write down the cash amount that is generated in the money market account in time dt and we get $(1 - h(t))V(t)r\,dt$. The total appreciation of the portfolio value $dV(t)$ is the sum of the money market account earnings $(1 - h(t))V(t)r\,dt$ and the part invested in stocks which is the equation (8.4). We get

$$\begin{aligned} dV(t) &= (1 - h(t))V(t)r\,dt + h(t)V(t)(\mu dt + \sigma dB(t) \\ &= V(t)rdt - h(t)V(t)rdt + h(t)V(t)\mu dt + V(t)h(t)\sigma dB(t) \\ &= V(t)((\mu - r)h(t) + r)dt + V(t)h(t)\sigma dB(t). \end{aligned}$$

Now it is time to formulate our SDE in the multidimensional case while taking the budget equation in to consideration. As an investment strategy we take $h \in \mathcal{A}$ and assume that the wealth $V(t)$ in response to h satisfies the following geometric diffusion SDE:

$$\frac{dV(t)}{V(t)} = (a_0 + A_0^T X(t))dt + h^T(t)(a - a_0\mathbf{1} + (A - \mathbf{1}A_0^T)X(t))dt + h^T(t)\Sigma dB_t, \tag{8.5}$$

with initial capital $V(0) = v_0$. Here we see that our new interest rate is $a_0 + A_0^T X(t)$ instead of r and our μ is $a + AX(t)$. This means that the part

$$a - a_0\mathbf{1} + (A - \mathbf{1}A_0^T)X(t)$$

plays the same role as $\mu - r$ in the 1-dimensional case.

Since we know that all the summands in the geometric diffusion SDE must have the same dimensions we can figure out the structure of the terms. The transposes in the multidimensional version are needed to do the multiplications correctly since our $h(t)$ is now m-dimensional. The process $X(t)$ has n-dimensions since Λ is a $n \times N$ matrix which is multiplied by N-dimensional Brownian motion resulting in n-dimensional column vector and the other terms in $dX(t) = (b + BX(t))dt + \Lambda dB(t)$ also must have the same dimensions in order to define the summation.

Starting from the term $a_0 + A_0^T X(t)$ in equation (8.5) we see that a_0 is a 1-dimensional constant and A_0^T is a n-dimensional row vector which is multiplied with n-dimensional column vector $X(t)$ resulting in 1-dimensional term hence the sum of a_0 and $A_0^T X(t)$ is well defined. The term $h^T(t)(a - a_0\mathbf{1} + (A - \mathbf{1}A_0^T)X(t))$ simplifies also into a 1-dimensional object since a, $a_0\mathbf{1}$ and $(A - \mathbf{1}A_0^T)X(t)$ are column vectors with m-elements hence when we multiply m-dimensional row vector $h^T(t)$ with those terms we get 1-dimensional objects. Since a_0 is just a market parameter that is a constant it is already 1-dimensional and hence the term $a_0\mathbf{1}$ is just a column vector with every element a_0 (m-elements). To see that $(A - \mathbf{1}A_0^T)X(t)$ is an m-dimensional column vector we look at the two terms $\mathbf{1}A_0^T X(t)$ and $AX(t)$ individually. From earlier we know that $A_0^T X(t)$ is 1-dimensional hence when multiplied with m-element column vector $\mathbf{1}$ we are left with an m-element column vector. In the term $AX(t)$ we have $m \times n$ matrix A and when we multiply it with $n \times 1$ column vector $X(t)$ we are left with $m \times 1$ column vector. The final term $h^T(t)\Sigma dB_t$ is also 1-dimensional since the term ΣdB_t is a m-element column vector ($m \times N$ matrix multiplied with $N \times 1$ Brownian motion) and when $h^T(t)$ is multiplied with ΣdB_t we get a scalar product which gives a 1-dimensional object (m-row vector multiplied with m-column vector).

For our convenience we define $\tilde{a} := a - a_0\mathbf{1}$ and $\tilde{A} := A - \mathbf{1}A_0^T$. Now we can express the portfolio dynamics in the following way:

$$\frac{dV(t)}{V(t)} = (a_0 + A_0^T X(t))dt + h^T(t)(\tilde{a} + \tilde{A}X(t))dt + h^T(t)\Sigma dB_t,\ V(0) = v_0. \tag{8.6}$$

This SDE can be solved in the similar way as in section 5.2 but this time we have more dimensions and more complicated terms. The existence and uniqueness of the solution for the SDE (8.6) can be found in the Appendix part. For comparison we can use the solution of the one dimensional case for the stock price S_t to derive expression for V_t. The price S_t has the following form:

$$S_t = s_0 e^{\mu t - \frac{\sigma^2 t}{2} + \sigma B_t} = s_0 e^{\int_0^t (\mu - \frac{\sigma^2}{2})\, ds + \int_0^t \sigma\, dB_s}.$$

Simply by replacing μ with $(a_0 + A_0^T X(s) + h^T(s)(\tilde{a} + \tilde{A}X(s))$, σ^2 with $h^T(s)\Sigma\Sigma^T h(s)$ and σ with $h^T(s)\Sigma$ we get the expression for V_t:

$$\begin{aligned} V_t =& v_0 \exp\bigg\{ \int_0^t ((a_0 + A_0^T X(s) + h^T(s)(\tilde{a} + \tilde{A}X(s)) - \frac{1}{2} h^T(s)\Sigma\Sigma^T h(s))\, ds \\ &+ \int_0^t h^T(s)\Sigma\, dB_s \bigg\}. \end{aligned}$$

8.4. Risk-sensitive asset management criterion. In risk-sensitive control the aim is to optimize the following criterion:

$$J_{RS}^\theta = -\frac{1}{\theta} \ln \mathbb{E}\big[e^{-\theta F(T,x,h)}\big],$$

where $T > 0$ is the fixed time horizon and x is the initial value of the factor process $X(t)$. The function F can be chosen by the the decision maker in such a way he thinks suits best to get what he wants. The cost or reward function F is used to model the reward and the parameter $\theta \in]-1, 0[\cup]0, \infty)$ describes the decision maker's degree of risk aversion.

For the reward function $F(T, x, h)$ we choose the logarithm of the portfolio value $V(T)$ like Bielecki and Pliska proposed in [**14**]. This leads to the risk-sensitive asset management criterion:

$$J(x, h) := -\frac{1}{\theta} \ln \mathbb{E}\big[e^{-\theta \ln V(T)}\big] = -\frac{1}{\theta} \ln \mathbb{E}[V(T)^{-\theta}]. \tag{8.7}$$

In the case where θ is positive our aim is to maximise the expectation of the risk-adjusted log return of the investor's portfolio $\ln \mathbb{E}[V(T)]$. For example when θ is positive in order to make $V(T)$ big we choose such h that the expression (8.7) is large. In the other words our task is to find such h that we get the maximum of the risk-sensitive asset management criterion (8.7).

Since the initial capital v_0 in the equation

$$\begin{aligned} V_t =& v_0 \exp\bigg\{ \int_0^t ((a_0 + A_0^T X(s) + h^T(s)(\tilde{a} + \tilde{A}X(s)) - \frac{1}{2} h^T(s)\Sigma\Sigma^T h(s))\, ds \\ &+ \int_0^t h^T(s)\Sigma\, dB_s \bigg\} \end{aligned}$$

plays no role in the optimization we can set it to any positive value we want, for example, we set $v_0 = 1$. We can interpret this decision by saying that we are using the investor's initial capital as the unit of account.

For our convenience we use the following short notation for the risk-sensitive asset management criterion:

$$\delta_\theta(\ln V) = -\frac{1}{\theta}\ln \mathbb{E}[e^{-\theta \ln V}].$$

Our function δ_θ satisfies the following axioms (see [**2**] Axioms 2.3-2.7).

Axiom 1. *If V_1 and V_2 are non-negatively correlated then:*

$$\delta_\theta(\ln V_1 + \ln V_2) \geq \delta_\theta(\ln V_1) + \delta_\theta(\ln V_2).$$

If V_1 and V_2 are non-positively correlated then:

$$\delta_\sigma(\ln V_1 + \ln V_2) \leq \delta_\theta(\ln V_1) + \delta_\theta(\ln V_2).$$

Axiom 2. *Logarithmic homogeneity.*

$$\delta_\theta(\lambda \ln V) = \lambda \delta_{\lambda\theta}(\ln V).$$

Axiom 3. *Monotonicity.*

$$\delta_\theta(\ln V_1) \geq \delta_\theta(\ln V_2) \quad \textit{if and only if} \quad V_1 \geq V_2.$$

Axiom 4. *Logarithmic risk-free condition.*

$$\delta_\theta(\ln V + kr) = kr + \delta_\theta(\ln V),$$

where r is a constant risk-free rate and k is a constant.

Among these axioms the most important ones are homogeneity and logarithmic super/sub-additivity. These axioms are also equivalent to the logarithmic convexity and concavity axioms.

Axiom 5. *Logarithmic convexity. Take $0 \leq \lambda \leq 1$. If V_1 and V_2 are non-negatively correlated then:*

$$\delta_\theta(\lambda \ln V_1 + (1-\lambda)\ln V_2) \geq \lambda \delta_{\lambda\theta}(\ln V_1) + (1-\lambda)\delta_{\lambda\theta}(\ln V_2).$$

If V_1 and V_2 are non-positively correlated then:

$$\delta_\theta(\lambda \ln V_1 + (1-\lambda)\ln V_2) \leq \lambda \delta_{(1-\lambda)\theta}(\ln V_1) + (1-\lambda)\delta_{(1-\lambda)\theta}(\ln V_2).$$

8.5. Solving the risk-sensitive asset management problem when asset and factor risks are uncorrelated. In this section we solve a special case of the general problem by using the measure change technique. The key assumption which simplifies the problem is that our asset and factor risks are uncorrelated, meaning that $\Lambda\Sigma^T = 0$. Firstly we are going to use the expression for the total portfolio value $V(t)$, which was the solution for our SDE in the appendix part. By setting $v_0 = 1$, like we discussed in the risk sensitive asset management criterion part, we have the following equality:

$$V(t) = \exp\left\{ \int_0^t (a_0 + A_0^T X(s)) + h^T(s)(\tilde{a} + \tilde{A}X(s)) - \frac{1}{2}h^T(s)\Sigma\Sigma^T h(s))\, ds + \int_0^t h^T(s)\Sigma\, dB(s) \right\}.$$

Our goal is to get an expression of $V(t)$, where the Girsanov exponential χ_t^h is clearly visible. Then we can get rid of the χ_t^h term by using a new measure $\mathbb{P}^h$. We can get this expression by using the trick where we express the power function as $V(t)^{-\theta} = \exp(-\theta \ln V(t))$ like in (8.7). Then we should get the following equation:

$$e^{-\theta \ln V(t)} = \exp\left\{\theta \int_0^t g(X(s), h(s); \theta)\, ds\right\}\chi_t^h, \tag{8.8}$$

where

$$g(x, h; \theta) = \frac{1}{2}(\theta + 1)h^T \Sigma \Sigma^T h - h^T(\tilde{a} + \tilde{A}x) - a_0 - A_0^T x$$

and

$$\chi_t^h := \exp\left\{-\theta \int_0^t h(s)^T \Sigma\, dB(s) - \frac{1}{2}\theta^2 \int_0^t h(s)^T \Sigma \Sigma^T h(s)\, ds\right\}, \quad t \in [0, T].$$

This can be seen by doing the following calculations: Since

$$\begin{aligned} e^{-\theta \ln V(t)} = \exp\bigg\{ &-\theta \int_0^t a_0 + A_0^T X(s) + h^T(s)(\tilde{a} + \tilde{A}X(s)) - \frac{1}{2}h^T(s)\Sigma\Sigma^T h(s)\, ds \\ &+ \int_0^t h^T(s)\Sigma\, dB(s)\bigg\} \end{aligned}$$

and

$$\begin{aligned} &\exp\left\{\theta \int_0^t g(X(s), h(s); \theta)\, ds\right\}\chi_t^h \\ &= \exp\left\{\theta \int_0^t \frac{1}{2}(\theta+1)h^T(s)\Sigma\Sigma^T h(s) - h^T(s)(\tilde{a} + \tilde{A}X(s)) - a_0 - A_0^T X(s)\, ds\right\} \\ &\quad \times \exp\left\{-\theta \int_0^t h^T(s)\Sigma\, dB_s - \frac{1}{2}\theta^2 \int_0^t h^T(s)\Sigma\Sigma^T h(s)\, ds\right\} \\ &= \exp\bigg\{ \int_0^t \frac{1}{2}\theta^2 h^T(s)\Sigma\Sigma^T h(s) + \frac{1}{2}\theta h^T(s)\Sigma\Sigma^T h(s) - \theta h^T(s)(\tilde{a} + \tilde{A}X(s)) - \theta a_0 - \theta A_0^T X(s) \\ &\quad - \frac{1}{2}\theta^2 h^T(s)\Sigma\Sigma^T h(s)\, ds - \theta \int_0^t h^T(s)\Sigma\, dB(s)\bigg\} \\ &= \exp\bigg\{ -\theta \int_0^t a_0 + A_0^T X(s) + h^T(s)(\tilde{a} + \tilde{A}X(s)) - \frac{1}{2}h^T(s)\Sigma\Sigma^T h(s)\, ds \\ &\quad - \theta \int_0^t h^T(s)\Sigma\, dB(s)\bigg\}. \end{aligned}$$

Then we define the new measure $\mathbb{P}^h$ we mentioned in the beginning of the chapter such that

$$\frac{d\mathbb{P}^h}{d\mathbb{P}} = \chi^h(T).$$

Because $h \in \mathcal{A}$ we know that $\chi^h(T)$ is a martingale and that

$$B_t^h = B_t + \theta \int_0^t \Sigma^T h(s)\, ds, \quad t \in [0, T]$$

is a standard Brownian motion under $\mathbb{P}^h$. From our assumption $\Lambda\Sigma^T = 0$ it follows that $X(t)$ satisfies the uncontrolled SDE:

$$dX(t) = (b + BX(t))\, dt + \Lambda dB(t).$$

This is true because we can replace $dB(t)$ with $dB^h(t)$ since

$$\Lambda B^h(t) = \Lambda B(t) + \theta \int_0^t \Lambda \Sigma^T h(s)\, ds = \Lambda B(t).$$

Then we use the new expression obtained in (8.8) in the risk-sensitive asset management criterion $J(x,h)$ in (8.7) and get the auxiliary criterion function I under the measure $\mathbb{P}^h$. Under the measure $\mathbb{P}^h$ the term χ_t^h disappears and we get

$$I(x,h) = -\frac{1}{\theta} \ln \mathbb{E}^h \left[\exp \left\{ \theta \int_0^T g(X_s, h(s); \theta)\, ds \right\} \right],$$

where $\mathbb{E}^h$ is the expectation w.r.t. $\mathbb{P}^h$.

Since $X(t)$ is independent of h under the measure $\mathbb{P}^h$ our problem namely maximizing $-\frac{1}{\theta} \ln \mathbb{E}[V(T)^{-\theta}]$, can be solved by maximizing the auxiliary criterion function $I(x,h)$. Since the function $g(x,h;\theta)$ has the same structure as a multidimensional quadratic function where the sign of the leading coefficient $\frac{1}{2}(\theta+1)\Sigma\Sigma^T$ is positive we have a 'parabola' that opens upward. this means that the criterion $I(x,h)$ reaches its maximum when the function $g(x,h;\theta)$ reaches its minimum. Our task is to minimize the function $g(x,h;\theta)$, and to do that we first solve the minimum in the 1-dimensional case and then generalize the obtained solution.

In the 1-dimensional case $g(x,h;\theta)$ gets the following form:

$$\frac{1}{2}(\theta+1)h^2\Sigma^2 - h(\tilde{a} + \tilde{A}x) - a_0 - A_0^T x.$$

The derivative with respect to h gives $(\theta+1)h\Sigma^2 - \tilde{a} - \tilde{A}x$ which is zero when $h = \frac{1}{(\theta+1)\Sigma^2}(\tilde{a} + \tilde{A}x)$. In order to solve h^* we follow the same steps as in the 1-dimensional case but we take into account that our h is now a column vector with m-elements ($h^T = (h_1, \ldots, h_m)$). First we compute the partial derivatives with respect to each component h_i, where $i = 1, \ldots, m$. In order to make the computations more clear we substitute $x = X(t)$ in the function $g(x,h,\theta)$, where $X(t)$ is $n \times 1$ column vector. We notice that each partial derivative with respect to each component has a similar form which means that for every $k \in 1, \dots, m$ we get

$$
\begin{aligned}
\frac{\partial}{\partial h_k} g(X(t), h, \theta) &= \frac{\partial}{\partial h_k}\left(\frac{1}{2}(\theta+1)h^T\Sigma\Sigma^T h - h^T(\tilde{a} + \tilde{A}X(t)) - a_0 - A_0^T X(t)\right) \\
&= \frac{1}{2}(\theta+1)\frac{\partial}{\partial h_k}\left(h^T\Sigma\Sigma^T h\right) - \frac{\partial}{\partial h_k}\left(h^T\tilde{a}\right) - \frac{\partial}{\partial h_k}\left(h^T\tilde{A}X(t)\right) \\
&= \frac{1}{2}(\theta+1)\frac{\partial}{\partial h_k}\sum_{i,j=1}^{m} h_i(\Sigma\Sigma^T)_{ij}h_j - \frac{\partial}{\partial h_k}(h_1\tilde{a}_1 + \cdots + h_m\tilde{a}_m) \\
&\quad - \frac{\partial}{\partial h_k}\sum_{i=1}^{m}\sum_{j=1}^{n} h_i\tilde{A}_{ij}X_j(t). \\
&= \frac{1}{2}(\theta+1)\left(\frac{\partial}{\partial h_k}\sum_{j=1}^{m} h_k(\Sigma\Sigma^T)_{kj}h_j + \frac{\partial}{\partial h_k}\sum_{i\neq k, i=1}^{m} h_i(\Sigma\Sigma^T)_{ik}h_k\right) \\
&\quad - \tilde{a}_k - \sum_{j=1}^{n}\tilde{A}_{kj}X_j(t) \\
&= \frac{1}{2}(\theta+1)\left(2h_k(\Sigma\Sigma^T)_{kk} + \sum_{j\neq k, j=1}^{m}(\Sigma\Sigma^T)_{kj}h_j + \sum_{i\neq k, i=1}^{m} h_i(\Sigma\Sigma^T)_{ik}\right) \\
&\quad - \tilde{a}_k - \sum_{j=1}^{n}\tilde{A}_{kj}X_j(t) \\
&= \frac{1}{2}(\theta+1)\left(\sum_{j=1}^{m}(\Sigma\Sigma^T)_{kj}h_j + \sum_{i=1}^{m} h_i(\Sigma\Sigma^T)_{ik}\right) - \tilde{a}_k - \sum_{j=1}^{n}\tilde{A}_{kj}X_j(t).
\end{aligned}
$$

We notice that the two sums inside the brackets can be written in the following way:

$$
\sum_{j=1}^{m}(\Sigma\Sigma^T)_{kj}h_j + \sum_{i=1}^{m} h_i(\Sigma\Sigma^T)_{ik} = 2\sum_{j=1}^{m}(\Sigma\Sigma^T)_{kj}h_j,
$$

By substituting $2\sum_{j=1}^{m}(\Sigma\Sigma^T)_{kj}h_j$ to the equation

$$
\frac{\partial}{\partial h_k} g(X(t), h, \theta) = \frac{1}{2}(\theta+1)\left(\sum_{j=1}^{m}(\Sigma\Sigma^T)_{kj}h_j + \sum_{i=1}^{m} h_i(\Sigma\Sigma^T)_{ik}\right) - \tilde{a}_k - \sum_{j=1}^{n}\tilde{A}_{kj}X_j(t)
$$

we get for $k = 1, \dots, m$ the following expression:

$$
\left(\frac{\partial}{\partial h_k} g(X(t), h, \theta)\right)_{k=1}^{m} = (\theta+1)(\Sigma\Sigma^T)h - \tilde{a} - \tilde{A}X(t).
$$

The optimal control h^* can now be obtained by solving $\left(\frac{\partial}{\partial h_k} g(X(t), h, \theta)\right)_{k=1}^{m} = 0$ with respect to h. This leads to the formula

$$
h^* = \frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a} + \tilde{A}X(t)),
$$

where $(\Sigma\Sigma^T)^{-1}$ is a $m \times m$ matrix, $\tilde{a}$ is a $m \times 1$ vector, $\tilde{A}$ is a $m \times n$ matrix and $X(t)$ is a $n \times 1$ vector. This solution exists because by assumption $\Sigma\Sigma^T$ is positive-definite and hence the inverse matrix $(\Sigma\Sigma^T)^{-1}$ exists.

We check that the investment strategy $h^*(t) = \frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a} + \tilde{A}X(t))$ belongs to $\mathcal{A}$. Firstly $h^*(t)$ satisfies (1) in Definition 8.2 since $h^*(t)$ depends only on constants θ, Σ, $\tilde{a}$, $\tilde{A}$ and a continuous adapted Gaussian process $X(t)$, which satisfies the SDE (7.2). From this it follows that $h^*(t)$ is continuous and progressively measurable with respect to $(\mathcal{G}_t)_{t\geq 0}$. For the assertion (2) we need to show that

$$\mathbb{P}\left(\int_0^t \left|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a} + \tilde{A}X(s))\right|^2 ds < \infty\right) = 1.$$

To complete our calculations we first define a matrix norm called 'vector induced matrix norm' which has the properties we need (see [**16**] Definition 2.3.1).

Definition 8.4. $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a matrix semi-norm if the following three properties hold:

(1) $f(A) \geq 0$, $\quad A \in \mathbb{R}^{m\times n}$,
(2) $f(A+B) \leq f(A) + f(B)$, $\quad A, B \in \mathbb{R}^{m\times n}$,
(3) $f(\alpha A) = |\alpha| f(A)$, $\quad \alpha \in \mathbb{R}, A \in \mathbb{R}^{m\times n}$.

The vector induced matrix norm (also called operator norm) has the following definition:

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|},$$

where A is a matrix and x is a vector.

This norm can be constructed by using many different vector norms, for example the euclidean norm, and it also satisfies the following inequalities:

(1) $\|Ax\| \leq \|A\|\|x\|$
(2) $\|AB\| \leq \|A\|\|B\|$,

where A, B are matrices and x is a vector. We continue to show the assertion (2) in Definition 8.2. We get

$$\begin{aligned}
&\int_0^t \left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a} + \tilde{A}X(t))\right\|^2 ds \\
&\leq \int_0^t \left(2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\tilde{a}\right\|^2 + 2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\tilde{A}X(s))\right\|^2\right) ds \\
&\leq 2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\tilde{a}\right\|^2 t + 2\int_0^t \left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\right\|^2 \|\tilde{A}\|^2 \|X(s))\|^2\, ds.
\end{aligned}$$

We continue by taking an euclidean norm of the process $X(t)$ and move all the constants out of the integral. In order to move $X(t)$ out of the integral we can estimate it by taking the supremum on the compact interval $[0,t]$ and get

$$
\begin{aligned}
&2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\tilde{a}\right\|^2 t+2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\right\|^2\|\tilde{A}\|^2\int_0^t\sum_{k=1}^n\sup_{0\le u\le t}|X_k(u))|^2\,ds\\
&=2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\tilde{a}\right\|^2 t+2\left\|\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}\right\|^2\|\tilde{A}\|^2\sum_{k=1}^n\sup_{0\le u\le t}|X_k(u))|^2 t,
\end{aligned}
$$

where every $X_k(t)$ is path-wise continuous so that supremum on compact interval is finite and then we sum up n-finite values and multiply by t, which gives us a finite expression. The final assertion (3) can also be shown but we do not do it here since it would exceed the scope of this thesis.

By substituting h^* into the function $g(x,h;\theta)$ we get

$$
\begin{aligned}
g(x,h^*;\theta)&=\frac{1}{2}(\theta+1)\left(\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)\right)^T\Sigma\Sigma^T\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)\\
&\quad-\left(\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)\right)^T(\tilde{a}+\tilde{A}x)-a_0-A_0^Tx\\
&=\frac{1}{2}(\theta+1)\frac{1}{\theta+1}(\tilde{a}^T+x^T\tilde{A}^T)(\Sigma\Sigma^T)^{-1}\Sigma\Sigma^T\frac{1}{\theta+1}(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)\\
&\quad-\frac{1}{\theta+1}(\tilde{a}^T+x^T\tilde{A}^T)(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)-a_0-A_0^Tx\\
&=\frac{1}{2}\frac{1}{\theta+1}(\tilde{a}^T+x^T\tilde{A}^T)(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)\\
&\quad-\frac{1}{\theta+1}(\tilde{a}^T+x^T\tilde{A}^T)(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)-a_0-A_0^Tx\\
&=-\frac{1}{2}\frac{1}{\theta+1}(\tilde{a}^T+x^T\tilde{A}^T)(\Sigma\Sigma^T)^{-1}(\tilde{a}+\tilde{A}x)-a_0-A_0^Tx\\
&=-\frac{1}{2}\frac{1}{\theta+1}\big(x^T\tilde{A}^T(\Sigma\Sigma^T)^{-1}\tilde{A}x+\tilde{a}^T(\Sigma\Sigma^T)^{-1}\tilde{A}x+x^T\tilde{A}^T(\Sigma\Sigma^T)^{-1}\tilde{a}\\
&\quad+\tilde{a}^T(\Sigma\Sigma^T)^{-1}\tilde{a}\big)-a_0-A_0^Tx\\
&=-\frac{1}{2}\frac{1}{\theta+1}x^T\tilde{A}^T(\Sigma\Sigma^T)^{-1}\tilde{A}x-\frac{1}{\theta+1}\tilde{a}^T(\Sigma\Sigma^T)^{-1}\tilde{A}x\\
&\quad-\frac{1}{2}\frac{1}{\theta+1}\tilde{a}^T(\Sigma\Sigma^T)^{-1}\tilde{a}-a_0-A_0^Tx.
\end{aligned}
$$

In the second last equality $\tilde{a}^T(\Sigma\Sigma^T)^{-1}\tilde{A}x=x^T\tilde{A}^T(\Sigma\Sigma^T)^{-1}\tilde{a}$, because $x^T\tilde{A}^T(\Sigma\Sigma^T)^{-1}\tilde{a}$ is 1-dimensional object hence taking transpose does not change it (x^T is $1\times n$, $\tilde{A}^T$ is $n\times m$, $(\Sigma\Sigma^T)^{-1}$ is $m\times m$ and $\tilde{a}$ is $m\times 1$).

The value function $\Phi(t,x)$ for the criterion $I(x,h;T,\theta)$ is the maximum value that can be obtained by using the process h^*. Hence we get a solution for our problem by substituting h^* into $I(x, h; T, \theta)$, which gives us the following expression:

$$\begin{aligned}
\Phi(t, x) &= \sup_{h \in \mathcal{A}} I(x, h; T, \theta) \\
&= -\frac{1}{\theta} \ln \mathbb{E}^h \left[\exp \left\{ \theta \int_0^T g(X(s), h^*(s); \theta)\, ds \right\} \right] \\
&= -\frac{1}{\theta} \ln \mathbb{E}^h \Bigg[\exp \Bigg\{ -\theta \int_0^T \Bigg(\frac{1}{2} \frac{1}{\theta + 1} X^T(s) \tilde{A}^T (\Sigma\Sigma^T)^{-1} \tilde{A} X(s) \\
&\quad + \frac{1}{1+\theta} \tilde{a}^T (\Sigma\Sigma^T)^{-1} \tilde{A} X(s) \\
&\quad + \frac{1}{2} \frac{1}{\theta + 1} \tilde{a}^T (\Sigma\Sigma^T)^{-1} \tilde{a} + a_0 + A_0^T X(s) \Bigg) ds \Bigg\} \Bigg].
\end{aligned}$$

The problem in this expression is that we still have to evaluate integrals with respect to the stochastic processes $X(t)$. In order to get a fully satisfactory expression without integrals w.r.t. stochastic processes $X(t)$ one can proceed like in [**2**] Theorem 2.8 and Corollary 2.9. It is shown that the solution can be obtained by solving a Hamilton-Jacobi-Bellman PDE and then as a direct consequence we get the following solution for the case $\Lambda\Sigma^T = 0$:

Corollary 8.5. *Suppose $\Lambda\Sigma^T = 0$. Then the investment strategy $h^*(t)$ defined by*

$$h^* = \frac{1}{\theta + 1} (\Sigma\Sigma^T)^{-1} (\tilde{a} + \tilde{A} X(t))$$

belongs to $\mathcal{A}$ and is optimal in $\mathcal{A}$. Its value, defined by $\Phi(t, x) = \sup_{\mathcal{A}} I(x, h; T, \theta)$, is

$$\Phi(0, x) = \frac{1}{2} x^T Q(0) x + q^T(0) x + k(0),$$

where q, Q and k respectively solve

$$\dot{Q}(t) - Q(t)\Lambda\Lambda^T Q(t) + B^T Q(t) + Q(t) B + \frac{1}{\theta + 1} \tilde{A}^T (\Sigma\Sigma^T)^{-1} \tilde{A} = 0$$

for $t \in [0, T]$, with terminal condition $Q(T) = 0$

$$\dot{q}(t) + (B^T - Q(t)\Lambda\Lambda^T) q(t) + Q(t) b + A_0 + \frac{1}{\theta + 1} \tilde{A}^T (\Sigma\Sigma^T)^{-1} \tilde{a} = 0$$

with terminal condition $q(T) = 0$, and

$$k(s) = \int_s^T l(t)\, dt$$

for $0 \leq s \leq T$, where

$$l(t) = \frac{1}{2} tr(\Lambda\Lambda^T Q(t)) - \frac{\theta}{2} q^T(t) \Lambda\Lambda^T q(t) + b^T q(t) + \frac{1}{2} \frac{1}{\theta + 1} \tilde{a}^T (\Sigma\Sigma^T)^{-1} \tilde{a} + a_0.$$

9. Appendix

9.1. Solving the geometric diffusion SDE. We show that the process

$$\begin{aligned} V_t =& v_0 \exp\left\{ \int_0^t ((a_0 + A_0^T X(s) + h^T(s)(\tilde{a} + \tilde{A}X(s)) - \frac{1}{2} h^T(s)\Sigma\Sigma^T h(s))\, ds \right. \\ &\left. + \int_0^t h^T(s)\Sigma\, dB_s \right\}. \end{aligned}$$

solves the equation (8.6) by using an n-dimensional Itô formula from [**3**] (Theorem 5.4.1).

Proof. First, for convenience we define $a(t) := (a_0 + A_0^T X(t) + h^T(t)(\tilde{a} + \tilde{A}X(t))$ and $b(t) := \Sigma^T h(t)$. We see that $b^T b$ and $b^T dB(t)$ are scalar products hence

$$h^T(t)\Sigma\Sigma^T h(t) = \sum_{k=1}^n b_k^2(t) \quad \text{and} \quad h^T(t)\Sigma dB(t) = \sum_{k=1}^n b_k(t) dB_k(t). \tag{9.1}$$

By using the new notation we have

$$V_t = v_0 \exp\left\{ \int_0^t a(s)\, ds - \frac{1}{2}\sum_{k=1}^n \int_0^t b_k^2\, ds + \sum_{k=1}^n \int_0^t b_k dB_k(s) \right\}.$$

We use the n-dimensional Itô formula for $V_t = f(t, Y(t)) = f(Y(t))$ with the function

$$f(x) = v_0 e^x.$$

Since we do not have a time variable t in our case the Itô formula simplifies to

$$V_t = v_0 + \int_0^t \frac{\partial f}{\partial x}(Y(s))\, dY(s) + \frac{1}{2}\int_0^t \frac{\partial f}{\partial x^2}(Y(s))\, d\langle M, M\rangle_s.$$

Because we have an exponential function in Itô's formula we get

$$\begin{aligned} V_t &= v_0 + \int_0^t V_s a(s)\, ds + \sum_{k=1}^n \int_0^t V_s b_k\, dB_k(s) + \frac{1}{2}\int_0^t V_s \sum_{k=1}^n b_k^2\, ds - \frac{1}{2}\int_0^t V_s \sum_{k=1}^n b_k^2\, ds \\ &= v_0 + \int_0^t V_s a(s)\, ds + \sum_{k=1}^n \int_0^t V_s b_k\, dB_k(s), \end{aligned}$$

where we have that $\langle M, M\rangle_t = \sum_{k=1}^n \int_0^t b_k^2\, ds$. Now we can see that Itô's formula gave us the equation

$$V_t = v_0 + \int_0^t V_s a(s)\, ds + \sum_{k=1}^n \int_0^t V_s b_k\, dB_k(s). \tag{9.2}$$

We write the equation (9.2) by using the differential form and divide it by V_t and get

$$\frac{dV_t}{V_t} = a(t)dt + \sum_{k=1}^n b_k dB_k(t).$$

Then by using the previous notations we see that we have equation (8.6). Uniqueness of this solution is proved below. □

Finally we are going to verify $\langle M, M\rangle_t = \sum_{k=1}^n \int_0^t b_k^2\, ds$. According to the Definition 5.3.1 in [**3**] the quadratic variation $\langle M\rangle_t$ is defined such that $M_t^2 - \langle M\rangle_t$ is a local martingale, where M_t is a continuous local martingale. We have that $M_t = \sum_{k=1}^n \int_0^t b_k\, dB_k(s)$ and for convenience we write

$$\int_0^t b_k\, dB_k(s) = X_k(t).$$

Then we have that $M_t^2 = \big(\sum_{k=1}^n X_k(t)\big)^2$. Using Itô's formula with a function $f(x) = x^2$ gives us

$$\begin{aligned} M_t^2 &= M_0^2 + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial x_k} dX_k(s) + \frac{1}{2}\sum_{m,l=1}^n \int_0^t \frac{\partial^2 f}{\partial x_m \partial x_l} d\langle X_m, X_l\rangle_s \\ &= M_0^2 + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial x_k} dX_k(s) + \sum_{m,l=1}^n \langle X_m, X_l\rangle_t, \end{aligned}$$

where

$$\langle X_m, X_l\rangle_t = \begin{cases} \int_0^t b_m^2\, ds & : m = l \\ 0 & : m \neq l. \end{cases}$$

When $m \neq l$ we see that the product $((X_m(t)X_l(t))_{t\geq 0}$ is already a martingale so nothing needs to be substracted. From the expression of M_t^2 we can see that in order to make M_t^2 a martingale we need to substract the term $\sum_{m=1}^n \int_0^t b_m^2\, ds$ because $M_0^2 = 0$ and $\sum_{k=1}^n \int_0^t \frac{\partial f}{\partial x_k} dX_k(s)$ is already a local martingale since it is a stochastic integral. Hence we have $\langle M, M\rangle_t = \sum_{k=1}^n \int_0^t b_k^2\, ds$.

9.2. Uniqueness of the solution. We still need to prove that the solution we obtained in subsection 9.1 is unique. The classical criteria for the uniqueness of the SDE

$$S(t) = s_0 + \int_0^t \alpha(s, S(s))\, ds + \int_0^t \beta(s, S(s))\, dB(s) \tag{9.3}$$

are that coefficients α and β should be Lipschitz-continuous, in space, uniformly in time and of linear growth, i.e.

$$|\alpha(\omega, t, x) - \alpha(\omega, t, y)| \leq L|x - y|,$$

and

$$|\alpha(\omega, t, x)| \leq K(1 + |x|), \text{ and the same for } \beta.$$

However, we are not in this setting. The coefficient α in (9.2) is given by

$$\alpha(\omega, t, V(t)) := V(t)((a_0 + A_0^T X(t)) + h^T(t)(\tilde{a} + \tilde{A}X(t))).$$

For the coefficient β in the stochastic integral we choose for the $b_k(t)$ defined in (9.1) its left-continuous version.

Define $N > 0$

$$\tau_N = \inf\{t > 0 : |X(t)| \geq N \text{ or } \|b(t)\| \geq N\},$$

where $\| \ \|$ is the euclidean norm. Then we obtain an SDE with standard assumptions on the coefficients and

$$\hat{S} := (S(t \wedge \tau_N))_{t \in [0,T]}$$

has a unique solution of (9.3) by [**3**] (Theorem 6.2.1) .

In order to show that we also have a unique solution without stopping we show that when $N \to \infty$ the stopped solution is the same as the original one. Assume now that S and $\hat{S}$ are two solutions of (9.3). Then

$$\mathbb{P}(S(t) = \hat{S}(t), t \in [0,T]) = \lim_{N \to \infty} \mathbb{P}(S(t \wedge \tau_N) = \hat{S}(t \wedge \tau_N), t \in [0,T]) = 1$$

since $\mathbb{P}(\tau_N \leq T) \to 0$ as $N \to \infty$. This is true because the larger N is the later we are exceeding the bound N. In other words the probability that we are going to hit N is going to zero as N goes to infinity. Since $\hat{S}(t)$ is a unique solution anyway the fact that $S(t) = \hat{S}_t$ a.s. shows that also $S(t)$ is a unique solution.

Resources

[1] D. Lamberton and B. Lapeyre: *Introduction to Stochastic Calculus Applied to Finance.* Chapman & Hall, 1996.

[2] M.H.A.Davis and S. Lleo *Advanced Series on Statistical Science and Applied Probability-Vol.19, Risk-Sensitive Investment Management.* World Scientific Publishing Co.Pte.Ltd, 2015.

[3] G. Kallianpur and P. Sundar *Stochastic Analysis and Diffusion Processes.* Oxford University Press, 2014.

[4] N.H.Bingham and R. Kiesel *Risk-Neutral Valuation Pricing and Hedging of Financial Derivatives.* Springer, 2013.

[5] I. Karatzas and S.E. Shreve *Brownian Motion and Stochastic Calculus.* Springer, 2014.

[6] R. L. Schilling and L. Partzsch *An Introduction to Stochastic Processes.* De Gruyter, 2012.

[7] S. Geiss
Stochastic Differential Equations. May 8 2019. http://users.jyu.fi/ geiss/lectures/stochasticsde.pdf

[8] M.M.Rao *Conditional measures and Applications.* CRC Press, 2005.

[9] J.L. Prigent *Weak Convergence of Financial Markets.* Springer, 2013.

[10] R.M.Dudley *Real Analysis and Probability.* Cambridge University Press, 2004.

[11] S. E. Shreve *Stochastic Calculus for Finance II:Continuous-Time Models.* Springer, 2004.

[12] R.S. Liptser and A.N.Shiryayev *Statistics of Random Processes I General Theory.* Springer-Verlag New York Inc. 1977

[13] *https://www.investopedia.com/terms/b/blackscholes.asp 7.11.2020 6pm*

[14] T. Bielecki and S. Pliska *Economic properties of the risk sensitive criterion for portfolio management, The Review of Accounting and Finance 2.* Emerald Publishing Limited, 2003.

[15] G. Lowther *https://almostsuremath.com/2009/12/24/local-martingales/ 30.11.2020 3.45pm*

[16] G. H. Golub and C. F. Van Loan *Matrix Computation 4th Edition.* The Johns Hopkins University Press, 2013