Author(s): Xu, Haiqing

Title: Weighted estimates for diffeomorphic extensions of homeomorphisms

Year: 2020

Version: Accepted version (Final draft)

Please cite the original version:

Xu, H. (2020). Weighted estimates for diffeomorphic extensions of homeomorphisms. Rendiconti Lincei: Matematica e Applicazioni, 31(1), 151-189. https://doi.org/10.4171/RLM/884

Weighted estimates for diffeomorphic extensions of homeomorphisms

Haiqing Xu[1,2]

[1]Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland
[2]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, P. R. China
E-mail address: hqxu@mail.ustc.edu.cn

Abstract

Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. Then there is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ. We study the relationship between weighted integrability of the derivatives of h and double integrals of φ and of φ^{-1}.

Keywords: Poisson extension, diffeomorphism, internal chord-arc domain.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Suppose that φ is a homeomorphism from the unit circle $\mathbb{S}^1$ onto $\partial\Omega$. Then, by Radó [13], Kneser [7], Choquet [3] and Lewy [10], the complex-valued Poisson extension h of φ is a diffeomorphism from $\mathbb{D}$ onto Ω. We are interested in the integrability degrees of the derivatives of h. In 2007, G. C. Verchota [14] proved that the derivatives of h may fail to be square integrable but that they are necessarily p-integrable over $\mathbb{D}$ for all $p < 2$. In 2009, T. Iwaniec, G. J. Martin and C. Sbordone improved on [5] by showing that the derivatives belong to weak-L^2 with sharp estimates. Actually

$$\int_{\mathbb{D}} |Dh(z)|^2\, dz \approx \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\varphi(\xi)-\varphi(\eta)|^2}{|\xi-\eta|^2}\, |d\xi|\, |d\eta|, \tag{1.1}$$

since harmonic functions minimize the L^2-energy and the right-hand side of (1.1) is the trace norm of $\dot{W}^{1,2}(\mathbb{D})$. In [1], it was further shown that if additionally $\partial\Omega$ is a C^1-regular Jordan curve then

$$\int_{\mathbb{D}} |Dh(z)|^2\, dz < \infty \Leftrightarrow \int_{\partial\Omega}\int_{\partial\Omega} |\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||\, |d\xi|\, |d\eta| < \infty. \tag{1.2}$$

All the above results require the target domain to be convex.

If Ω is a bounded non-convex Jordan domain, then there exists a homeomorphism $\varphi : \mathbb{S}^1 \to \partial\Omega$ for which the harmonic extension fails to map $\mathbb{D}$ homeomorphically onto Ω, see [3, 7]. Hence we cannot use the harmonic extension to produce a diffeomorphic extension. Nevertheless, (weighted) analogs of the results as (1.2) for diffeomorphic extensions in the case of an internal chord-arc Jordan domain exist, see [9]. For the definition of (internal) chord-arc domains, we refer to Definition 2.1. Notice that each bounded convex Jordan domain is a chord-arc domain. In this paper, we generalize the results in [9] to the weighted L^p-setting.

Let Ω be an internal chord-arc Jordan domain with the internal distance λ_Ω. Assume that $h : \mathbb{D} \to \Omega$ is a diffeomorphism and $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a homeomorphism. Set $\delta(z) = 1 - |z|$. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define

$$I_1(p, \alpha, \lambda, h) = \int_{\mathbb{D}} |Dh(z)|^p \delta^\alpha(z) \log^\lambda(2\delta^{-1}(z))\, dz,$$

$$I_2(p, \alpha, \lambda, h) = \int_{\mathbb{D}} |Dh(z)|^p \log^\lambda(e + |Dh(z)|)\delta^\alpha(z)\, dz,$$

$$\mathcal{U}(p, \alpha, \lambda, \varphi) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{\lambda_\Omega^p(\varphi(\xi), \varphi(\eta))}{|\xi - \eta|^{p-\alpha}} \log^\lambda\big(e + \frac{\lambda_\Omega(\varphi(\xi), \varphi(\eta))}{|\xi - \eta|}\big)\, |d\eta|\, |d\xi|,$$

$$\mathcal{A}_{p,\alpha,\lambda}(t) = \int_1^t -x^{1+\alpha-p} \log_2^\lambda(x^{-1})\, dx \qquad \forall t \geq 0,$$

$$\mathcal{V}(p, \alpha, \lambda, \varphi) = \int_{\partial\Omega} \Big(\int_{\partial\Omega} \mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|)\, |d\eta|\Big)^{p-1} |d\xi|.$$

Our main result is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. There is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which, for any $p > 1$, we have that*

(1) if either $\alpha \in (p - 2, +\infty)$ and $\lambda \in \mathbb{R}$ or $\alpha = p - 2$ and $\lambda \in (-\infty, -1)$,

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are finite.

(2) if either $\alpha \in (-1, p-2)$ *and* $\lambda \in \mathbb{R}$ *or* $\alpha = p-2$ *and* $\lambda \in [-1, +\infty)$,

both $I_1(p, \alpha, \lambda, h)$ *and* $I_2(p, \alpha, \lambda, h)$ *are comparable to* $\mathcal{U}(p, \alpha, \lambda, \varphi)$.

Moreover whenever $p \in (1, 2]$

both $I_1(p, \alpha, \lambda, h)$ *and* $I_2(p, \alpha, \lambda, h)$ *dominate* $\mathcal{V}(p, \alpha, \lambda, \varphi)$,

while

$\mathcal{V}(p, \alpha, \lambda, \varphi)$ *controls both* $I_1(p, \alpha, \lambda, h)$ *and* $I_2(p, \alpha, \lambda, h)$

for all $p \in [2, +\infty)$. *Furthermore both* $I_1(p, \alpha, \lambda, h)$ *and* $I_2(p, \alpha, \lambda, h)$ *are in general comparable to* $\mathcal{V}(p, \alpha, \lambda, \varphi)$ *only for* $p = 2$.

For any $p > 1$, *there is no diffeomorphic extension* $h : \mathbb{D} \to \Omega$ *of* φ *for which* $I_1(p, \alpha, \lambda, h) < +\infty$ *for either* $\alpha \in (-\infty, -1)$ *and* $\lambda \in \mathbb{R}$ *or* $\alpha = -1$ *and* $\lambda \in [-1, +\infty)$; *or for which* $I_2(p, \alpha, \lambda, h) < +\infty$ *for some* $\alpha \in (-\infty, -1]$ *and* $\lambda \in \mathbb{R}$.

Motivated by (1.2), one could hope to use $\mathcal{V}(p, \alpha, \lambda, \varphi)$ to control both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$. Example 4.2 together with Example 4.3 shows that $\mathcal{V}(p, \alpha, \lambda, \varphi)$ is comparable to $I_1(p, \alpha, \lambda, h)$ or $I_2(p, \alpha, \lambda, h)$ only when $p = 2$. Theorem 1.1 does not cover the case where $p > 1$, $\alpha = -1$ and $\lambda \in (-\infty, -1)$. We will return to this case in a future paper.

The structure of this paper is the following. In the next section, we give some preliminaries. Section 3 is the proof of Theorem 1.1. The final section contains several examples related to Theorem 1.1 (2).

2 Preliminaries

By $s \gg 1$ and $t \ll 1$ we mean that s is sufficiently large and t is sufficiently small, respectively. By $f \lesssim g$ we mean that there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for every x. If $f \lesssim g$ and $g \lesssim f$ we may denote $f \approx g$. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of all positive integers and the set of all real numbers. Let $\mathcal{L}^2$ (respectively $\mathcal{L}^1$) be the 2-dimensional (1-dimensional) Lebesgue measure. For sets $E \in \mathbb{R}^2$ and $F \in \mathbb{R}^2$, let $\operatorname{diam}(E)$ be the diameter of E, and $\operatorname{dist}(E, F)$ be the Euclidean distance between E and F. Let $B(p, r)$ be the disk with center P and radius r.

Definition 2.1. A Jordan domain $\Omega \subset \mathbb{R}^2$ is an **internal chord-arc Jordan domain** if $\partial\Omega$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \partial\Omega$,

$$\ell(w_1, w_2) \leq C\lambda_\Omega(w_1, w_2), \tag{2.1}$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial\Omega$ joining w_1 to w_2, and $\lambda_\Omega(w_1, w_2)$ is the **internal distance** between w_1, w_2, which is defined as

$$\lambda_\Omega(w_1, w_2) = \inf_{\alpha} \ell(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha \subset \Omega$ joining w_1 and w_2; if there is no rectifiable curve joining w_1 and w_2, we set $\lambda_\Omega(w_1, w_2) = \infty$; cf. [12, Section 3.1] or [2, Section 2].

If (2.1) holds for the Euclidean distance instead of the internal distance, we call Ω be a **chord-arc domain**. Naturally, every chord-arc Jordan domain is an internal chord-arc domain, but there are internal chord-arc domains that fail to be chord-arc; e.g. the standard cardioid domain

$$\Delta = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 < 0\}.$$

2.1 Dyadic decomposition

Given $j \in \mathbb{N}$ and $k = 1, ..., 2^j$, let

$$I_{j,k} = [2\pi(k-1)2^{-j}, 2\pi k 2^{-j}],\ \Gamma_{j,k} = \{e^{i\theta} : \theta \in I_{j,k}\}. \tag{2.2}$$

Then $\{I_{j,k}\}$ is a dyadic decomposition of $[0, 2\pi]$ and $\{\Gamma_{j,k}\}$ is a dyadic decomposition of $\mathbb{S}^1$. We call $\Gamma_{j,k}$ a j-level dyadic arc. Moreover we have that

$$\ell(\Gamma_{j,k}) \approx 2^{-j} \qquad \forall j \in \mathbb{N} \text{ and } k = 1, ..., 2^j. \tag{2.3}$$

Based on (2.2), there is a decomposition of the unit disk $\mathbb{D}$ given by $\{Q_{j,k} : j \in \mathbb{N} \text{ and } k = 1, ..., 2^j\}$, where

$$Q_{j,k} = \left\{re^{i\theta} : 1 - 2^{1-j} \le r \le 1 - 2^{-j} \text{ and } \theta \in I_{j,k}\right\}. \tag{2.4}$$

By (2.3) it follows that

$$\mathcal{L}^2(Q_{j,k}) \approx 2^{-2j} \approx \ell(\Gamma_{j,k})^2 \qquad \forall j \in \mathbb{N} \text{ and } k = 1, ..., 2^j. \tag{2.5}$$

Moreover there is a uniform constant $C > 0$ such that for any $Q_{j,k}$ there is a disk $B_{j,k}$ satisfying

$$B_{j,k} \subset Q_{j,k} \subset CB_{j,k}. \tag{2.6}$$

2.2 A_p weights

Definition 2.2. For a given $p \in (1, +\infty)$, a locally integrable function $w : \mathbb{R}^2 \to [0, +\infty)$ is an A_p weight if there is a constant $C > 0$ such that for any disk $B \subset \mathbb{R}^2$ we have that

$$\frac{1}{\mathcal{L}^2(B)} \int_B w(x)\, dx \le C\Big(\frac{1}{\mathcal{L}^2(B)} \int_B w(x)^{\frac{1}{1-p}}\, dx\Big)^{1-p}.$$

Next, w is an A_1 weight if there is a constant $C > 0$ such that

$$\frac{1}{\mathcal{L}^2(B)} \int_B w(z)\, dz \le C w(x)$$

for each disk $B \subset \mathbb{R}^2$ and all $x \in B$.

For more information on A_p weights, we recommend [4, 6, 11]. Let $\delta(x) = \operatorname{dist}(\mathbb{S}^1, x)$. Given $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, we define

$$w_{\alpha,\lambda}(x) = \begin{cases} \delta(x)^\alpha \log^\lambda (2\delta^{-1}(x)) & 0 \le |x| \le 2, \\ \log^\lambda(2) & |x| \ge 2. \end{cases} \tag{2.7}$$

It is well known that $w_{\alpha,0}$ belongs to A_p. We now generalize this to all $\lambda \in \mathbb{R}$.

Proposition 2.3. *Let $p \ge 1$ and $w_{\alpha,\lambda}$ be as in* (2.7). *Then $w_{\alpha,\lambda}$ is an A_p weight for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. The idea of proof is to use the Jones factorization of A_p weights (see [6]), i.e. we should prove $w_{\alpha,\lambda} = w_1 w_2^{1-p}$ for two A_1 weights w_1 and w_2.

We first consider the case $\lambda \ge 0$. For a given $\alpha \in (-1, p-1)$, there uniquely exist $a_1 \in (0, 1)$ and $a_2 \in (0, 1)$ such that $\alpha = a_1(-1) + a_2(p-1)$. Set $\alpha_1 = -a_1$, $\alpha_2 = -a_2$, $\lambda_1 = p\lambda$ and $\lambda_2 = \lambda$. We define

$$w_1(x) = \begin{cases} \delta(x)^{\alpha_1} \log^{\lambda_1} (2\delta^{-1}(x)) & 0 \le |x| \le 2, \\ \log^{\lambda_1}(2) & |x| \ge 2, \end{cases} \tag{2.8}$$

and

$$w_2(x) = \begin{cases} \delta(x)^{\alpha_2} \log^{\lambda_2} (2\delta^{-1}(x)) & 0 \le |x| \le 2, \\ \log^{\lambda_2}(2) & |x| \ge 2. \end{cases} \tag{2.9}$$

We next prove that w_1 is an A_1 weight, i.e.

$$\fint_B w_1(x)dx \lesssim \inf_{x \in B} w_1(x) \tag{2.10}$$

for every disk $B \subset \mathbb{R}^2$. Let $d_B = \operatorname{dist}(B, \mathbb{S}^1)$.

Case 1: $d_B \geq \operatorname{diam}(B)/2$. We have that

$$d_B \leq \delta(x) \leq 3d_B \qquad \forall x \in B. \tag{2.11}$$

If $1 \leq d_B$, then $\delta(x) \geq 1$ for all $x \in B$. Therefore $w_1(x) = \log^{\lambda_1}(2)$ whenever $x \in B$. Of course (2.10) holds now. If $3d_B \leq 1$, then $w_1(x) = \delta(x)^{\alpha_1} \log^{\lambda_1}(2\delta^{-1}(x))$ for all $x \in B$. By (2.11) it hence follows that $w_1(x) \approx d_B^{\alpha_1} \log^{\lambda_1}\left(2d_B^{-1}\right)$ whenever $x \in B$. Therefore (2.10) holds. If $d_B < 1 < 3d_B$, let $B_1 = \{x \in B : d_B < \delta(x) < 1\}$ and $B_2 = \{x \in B : 1 \leq \delta(x) < 3d_B\}$. Then $B = B_1 \cup B_2$ and

$$w_1(x) = \log^{\lambda_1}(2) \qquad \text{whenever } x \in B_2. \tag{2.12}$$

Since

$$\left[t^{\alpha_1} \log^{\lambda_1}\left(2t^{-1}\right)\right]' = t^{\alpha_1 - 1} \log^{\lambda_1}(2t^{-1}) \left(\alpha_1 - \frac{\lambda_1}{\log(2t^{-1})}\right) < 0, \tag{2.13}$$

for all $t \in (0, 1]$, we have that

$$w_1(x) \leq d_B^{\alpha_1} \log^{\lambda_1}(2d_B^{-1}) \leq \frac{\log^{\lambda_1}(6)}{3^{\alpha_1}} \qquad \forall x \in B_1. \tag{2.14}$$

Combining (2.12) and (2.14) implies that

$$\begin{aligned}
\fint_B w_1(x)dx &= \frac{1}{\mathcal{L}^2(B)} \left(\int_{B_1} w_1 + \int_{B_2} w_1\right) \\
&\leq \frac{1}{\mathcal{L}^2(B)} \left(\mathcal{L}^2(B_1) \frac{\log^{\lambda_1}(6)}{3^{\alpha_1}} + \mathcal{L}^2(B_2) \log^{\lambda_1}(2)\right) \\
&\lesssim \log^{\lambda_1}(2) = \inf_{x \in B} w_1(x).
\end{aligned}$$

Case 2: $d_B < \operatorname{diam}(B)/2$ and $\operatorname{diam}(B) \leq 2/3$. Pick $x' \in \partial B$ and $x_0 \in \mathbb{S}^1$ such that $\operatorname{dist}(B, \mathbb{S}^1) = |x' - x_0|$. Let $r_B = 3\operatorname{diam}(B)/2$. Since

$$|x - x_0| \leq |x - x'| + |x' - x_0| \leq r_B$$

for all $x \in B$, we have $B \subset B(x_0, r_B)$. Let $E = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \mathbb{S}^1) < r_B\}$. Then $B(x_0, r_B) \subset E$. Since $\mathcal{L}^2(B(x_0, r_B)) = \pi r_B^2$ and $\mathcal{L}^2(E) = 4\pi r_B$, the maximal number of pairwise disjoint open disks $B(x, r_B)$ with $x \in \mathbb{S}^1$ is less than $4r_B^{-1}$. We have that

$$\begin{aligned}
\frac{1}{\mathcal{L}^2(B)} \int_B w_1(x)\, dx &\leq \frac{1}{\mathcal{L}^2(B)} \int_{B(x_0, r_B)} w_1(x)\, dx \\
&\lesssim \frac{r_B}{\mathcal{L}^2(B)} \int_E w_1(x)\, dx \approx \frac{1}{r_B} \int_0^{r_B} t^{\alpha_1} \log^{\lambda_1}\left(2t^{-1}\right)\, dt.
\end{aligned} \tag{2.15}$$

Notice that

$$(2.16)\qquad \left[t^{\alpha_1+1}\log^{\lambda_1}\left(2t^{-1}\right)\right]' = t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\left(\alpha_1+1-\frac{\lambda_1}{\log(2t^{-1})}\right) \qquad t>0.$$

Since $\lim_{t\to 0^+}\alpha_1+1-\frac{\lambda_1}{\log(2t^{-1})}=\alpha_1+1$ and $\alpha_1+1-\frac{\lambda_1}{\log(2t^{-1})}$ is decreasing with respect to $t>0$, there exists $\epsilon\in(0,1)$ determined by α_1 and λ_1 such that $\alpha_1+1-\frac{\lambda_1}{\log(2\epsilon^{-1})}\geq(\alpha_1+1)/2$. We then obtain from (2.16) that

$$\left[t^{\alpha_1+1}\log^{\lambda_1}\left(2t^{-1}\right)\right]' \geq \frac{\alpha_1+1}{2}t^{\alpha_1}\log^{\lambda_1}(2t^{-1})$$

for all $t\in[0,\epsilon r_B]$. Therefore

$$(2.17)\qquad \int_0^{\epsilon r_B} t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\,dt = \frac{2(\epsilon r_B)^{\alpha_1+1}}{\alpha_1+1}\log^{\lambda_1}(2(\epsilon r_B)^{-1}) \lesssim r_B^{\alpha_1+1}\log^{\lambda_1}\left(2r_B^{-1}\right)$$

Moreover by (2.13) we have that

$$(2.18)\qquad \int_{\epsilon r_B}^{r_B} t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\,dt \leq (r_B-\epsilon r_B)(\epsilon r_B)^{\alpha_1}\log^{\lambda_1}(2(\epsilon r_B)^{-1}) \lesssim r_B^{\alpha_1+1}\log^{\lambda_1}(2r_B^{-1}).$$

Combining (2.15), (2.17) with (2.18) implies that

$$\frac{1}{|B|}\int_B w_1(x)\,dx \lesssim r_B^{\alpha_1}\log^{\lambda_1}\left(2r_B^{-1}\right).$$

Together with

$$r_B^{\alpha_1}\log^{\lambda_1}\left(2r_B^{-1}\right) = \inf_{t\in[0,r_B]} t^{\alpha_1}\log^{\lambda_1}\left(2t^{-1}\right) = \inf_{x\in E} w_1(x) \leq \inf_{x\in B} w_1(x),$$

we hence obtain (2.10).

Case 3: $d_B<\mathrm{diam}(B)/2$ and $\mathrm{diam}(B)>2/3$. Let x' and x_0 be as in Case 2. Then $|x'|=1+\mathrm{dist}(x',\mathbb{S}^1)\leq 1+\mathrm{diam}(B)2^{-1}$. Together with the fact that $|x-x'|\leq\mathrm{diam}(B)$ for all $x\in B$, we have $B\subset B(0,1+r_B)$. Moreover by (2.17) and (2.18), we obtain that

$$\int_{B(0,2)} w_1(x)\,dx = \int_{B(0,1)} + \int_{B(0,2)\setminus B(0,1)} = 4\pi\int_0^1 t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\,dt \approx 1.$$

Therefore

$$\begin{aligned}\frac{1}{\mathcal{L}^2(B)}\int_B w_1(x)\,dx \lesssim& \frac{1}{r_B^2}\left(\int_{B(0,2)} + \int_{B(0,1+r_B)\setminus B(0,2)}\right)\\ \lesssim& \frac{1}{r_B^2}\left(\mathcal{L}^2(B(0,2)) + \log^{\lambda_1}(2)\mathcal{L}^2(B(0,1+r_B)\setminus B(0,2))\right)\\ (2.19)\qquad \lesssim& 1.\end{aligned}$$

Moreover by the monotonicity of $t^{\alpha_1}\log^{\lambda_1}(2t^{-1})$ on $(0,+\infty)$, we have that

$$\log^{\lambda_1}(2) = \inf_{t\in[0,1+r_B]} t^{\alpha_1}\log^{\lambda_1}\left(2t^{-1}\right) = \inf_{x\in B(0,1+r_B)} w_1(x) \le \inf_{x\in B} w_1(x). \tag{2.20}$$

By combining (2.19) with (2.20), we obtain (2.10).

By the analogous arguments as for (2.10), we obtain $w_2 \in A_1$. Therefore the Jones factorization theorem implies that $w_{\alpha,\lambda} \in A_p$ for all $\alpha \in (-1, p-1)$ and $\lambda \ge 0$.

When $\lambda < 0$, define w_1 and w_2 as in (2.8) and (2.9) with $\lambda_1 = -\lambda$, $\lambda_2 = 2\lambda(1-p)^{-1}$ and both α_1 and α_2 invariant. By the same arguments as for the case $\lambda \ge 0$, we obtain that $w_{\alpha,\lambda} \in A_p$ whenever $\alpha \in (-1, p-1)$ and $\lambda < 0$. □

2.3 A class of functions

We define the Hardy-Littlewood maximal function for a Lebesgue measurable function f in $\mathbb{R}^2$ as

$$M_f(x) = \sup_{x\in B} \fint_B |f(z)|dz = \sup_{x\in B}\frac{1}{|B|}\int_B |f(z)|\,dz$$

where the supremum is taken over all disks $B \subset \mathbb{R}^2$ containing x. Let $p \in (1,\infty)$ and w be a weight. It is well-known that

$$\int_{\mathbb{R}^2} |M_f(x)|^p w(x)\,dx \lesssim \int_{\mathbb{R}^2} |f(x)|^p w(x)\,dx$$

if and only if w is an A_p weight. We generalize this to weighted Orlicz spaces. We begin with some definitions.

Definition 2.4. Let $\mathcal{F}$ be the collection of $\Phi : [0,\infty) \to [0,\infty)$, which is increasing and satisfies $\lim_{t\to 0}\Phi(t) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. We say that $\Phi \in \mathcal{F}$ is the Young function, if Φ is convex on $[0,\infty)$ and $\lim_{t\to 0}\Phi(t)/t = \lim_{t\to\infty} t/\Phi(t) = 0$.

Definition 2.5. We say that a function $\Phi : [0,+\infty) \to [0,+\infty)$ satisfies the Δ_2-condition if there is a constant $C > 0$ such that

$$\Phi(2t) \le C\Phi(t)$$

for all $t \in [0,+\infty)$.

Let $\Phi \in \mathcal{F}$ satisfying the Δ_2-condition. Put

$$h_\Phi(s) = \sup_{t>0}\frac{\Phi(st)}{\Phi(t)} \qquad s > 0.$$

We define the lower index of Φ by

$$i(\Phi) = \lim_{s\to 0} \frac{\log h_\Phi(s)}{\log s} = \sup_{0<s<1} \frac{\log h_\Phi(s)}{\log s}.$$

The quantity $i(\Phi)$ is well defined, see [8]. The following lemma is from [8, Theorem 2.1.1].

Lemma 2.6. *Let Φ be a Young function satisfying the Δ_2-condition and w be a weight on $\mathbb{R}^2$. Then the following conditions are equivalent:*

1. $\int_{\mathbb{R}^2} \Phi(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Phi(|f(x)|)w(x)\,dx$,
2. $w \in A_{i(\Phi)}$.

We next consider a special class of Young functions. Given $p > 1$ and $\lambda \in \mathbb{R}$, we set

$$\Phi_{p,\lambda}(t) = t^p \log^\lambda(e+t) \qquad \text{for } t \in [0, +\infty). \tag{2.21}$$

Proposition 2.7. *Let $\Phi_{p,\lambda}$ be as in (2.21) with $p > 1$ and $\lambda \ge 0$. Then $\Phi_{p,\lambda}$ is a Young function and satisfies the Δ_2-condition on $[0,\infty)$. Moreover $i(\Phi_{p,\lambda}) = p$.*

Proof. Simple calculations show that

$$\Phi'_{p,\lambda}(t) = \left(p\log(e+t) + \lambda\frac{t}{e+t}\right) t^{p-1}\log^{\lambda-1}(e+t) \tag{2.22}$$

and

$$\begin{aligned}\Phi''_{p,\lambda}(t) = &\left(p(p-1)\log^2(e+t) + \lambda e\frac{t}{(e+t)^2}\log(e+t) + R_{p,\lambda}(t)\right)\\ &\times t^{p-2}\log^{\lambda-2}(e+t)\end{aligned} \tag{2.23}$$

where $R_{p,\lambda}(t) = \lambda(\lambda-1)(t(e+t)^{-1})^2 + \lambda(2p-1)t(e+t)^{-1}\log(e+t)$. Since $p\log(e+t) + \lambda t(e+t)^{-1} > 0$ for all $t \in (0,+\infty)$, it follows from (2.22) that $\Phi'_{p,\lambda}(t) > 0$ for all $t > 0$. Therefore $\Phi_{p,\lambda}$ is strictly increasing on $[0,\infty)$. If $\lambda \ge 1$, we have that

$$R_{p,\lambda}(t) \ge 0 \qquad \text{for all } t \ge 0. \tag{2.24}$$

Whenever $0 \le \lambda < 1$, since $t/(e+t) < 1$ and $\log(e+t) \ge 1$ for all $t \ge 0$ we have that

$$\begin{aligned}R_{p,\lambda}(t) =&\frac{t}{e+t}\left(\lambda(\lambda-1)\frac{t}{e+t} + \lambda(2p-1)\log(e+t)\right)\\ \ge&\frac{t}{e+t}\big(\lambda(\lambda-1) + \lambda(2p-1)\big) = \frac{t}{e+t}\left(\lambda^2 + 2\lambda(p-1)\right) \ge 0\end{aligned} \tag{2.25}$$

for all $t \geq 0$. By (2.23), (2.24) and (2.25), we have that $\Phi''_{p,\lambda}(t) \geq 0$ for all $t \geq 0$. Therefore $\Phi_{p,\lambda}$ is convex on $[0, +\infty)$. Hence $\Phi_{p,\lambda}$ is a Young function whenever $p > 1$ and $\lambda \geq 0$.

Since both t^p and $\log^\lambda(e+t)$ satisfy the Δ_2-condition on $[0, +\infty)$, $\Phi_{p,\lambda}$ satisfies Δ_2-condition also. Since $h_{\Phi_{p,\lambda}}(s) = s^p$ whenever $s \in (0, 1)$, we have $i(\Phi_{p,\lambda}) = p$. □

Remark 2.8. For $p > 1$, let $\Phi_{p,\lambda}$ be as in (2.21) with $\lambda \geq 0$. Assume that w is an A_p weight. Given a Lebesgue measurable function f, by Lemma 2.6 and Proposition 2.7 we have that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(|f(x)|)w(x)\,dx.$$

Let $\Phi_{p,\lambda}$ be as in (2.21) with $p > 1$ and $\lambda < 0$. By (2.22) and (2.23), we have that both monotonicity and convexity of $\Phi_{p,\lambda}$ may fail whenever $t \ll 1$, but still hold for all $t \gg 1$. We modify $\Phi_{p,\lambda}$ in a neighborhood of the origin such that it satisfies all conclusions in Proposition 2.7.

Since $2^{-1}(p+1)\log(e+t) \leq p\log(e+t) + \lambda t(e+t)^{-1}$ whenever $t \gg 1$, by (2.22) there is a constant $t_2 \gg 1$ such that

$$\frac{(p+1)\Phi_{p,\lambda}(t)}{2t} = \frac{p+1}{2} t^{p-1} \log^\lambda(e+t) \leq \Phi'_{p,\lambda}(t) \tag{2.26}$$

for all $t \geq t_2$. Without loss of generality, we assume that $\Phi_{p,\lambda}$ is strictly increasing and convex on $[t_2, \infty)$. Since $pt_2 t^{p-1} + t^p \leq (p+1)t_2 t^{p-1} \leq t_2^p \log^\lambda(e+t_2)$ for any $t \ll 1$, we have that

$$pt^{p-1}(t_2 - t) \leq \Phi_{p,\lambda}(t_2) - t^p \tag{2.27}$$

for all $t \ll 1$. Moreover when $t \leq t_2(p-1)/(p+1)$, we have that

$$\frac{\Phi_{p,\lambda}(t_2) - t^p}{t_2 - t} \leq \frac{\Phi_{p,\lambda}(t_2)}{t_2 - t} \leq \frac{(p+1)\Phi_{p,\lambda}(t_2)}{2t_2}. \tag{2.28}$$

Therefore by (2.26), (2.27) and (2.28), there exists a constant $t_1 \ll 1$ such that

$$pt_1^{p-1} \leq \frac{\Phi_{p,\lambda}(t_2) - t_1^p}{t_2 - t_1} \leq \frac{(p+1)\Phi_{p,\lambda}(t_2)}{2t_2} \leq \Phi'_{p,\lambda}(t_2).$$

Let $k = (\Phi_{p,\lambda}(t_2) - t_1^p)/(t_2 - t_1)$. Given $p > 1$ and $\lambda < 0$, we define

$$\Psi_{p,\lambda}(t) = \begin{cases} t^p & 0 \leq t < t_1, \\ k(t - t_1) + t_1^p & t_1 \leq t < t_2, \\ \Phi_{p,\lambda}(t) & t_2 \leq t. \end{cases} \tag{2.29}$$

Proposition 2.9. *The function $\Psi_{p,\lambda}$ is a Young function and satisfies the Δ_2-condition on $[0,\infty)$. Moreover $i(\Psi_{p,\lambda}) = p$.*

Proof. It is easy to see that $\Psi_{p,\lambda}$ is strictly increasing, continuous and convex on $[0,+\infty)$. Hence $\Psi_{p,\lambda}$ is a Young function. To prove the Δ_2-condition, it suffices to check that

$$\Psi_{p,\lambda}(2t) \le C\Psi_{p,\lambda}(t) \tag{2.30}$$

for all $t \in [0,+\infty)$. In fact, (2.30) is trivial if either $t \ge t_2$ or $2t < t_1$. Whenever $t \in [t_1/2, t_2]$, by the monotonicity of $\Psi_{p,\lambda}$ we have that

$$\frac{\Psi_{p,\lambda}(2t)}{\Psi_{p,\lambda}(2t_2)} \le 1 \le \frac{\Psi_{p,\lambda}(t)}{\Psi_{p,\lambda}(t_1/2)}.$$

Let $s \ll 1$. Without loss of generality, we assume $s \le t_1/t_2$. In order to prove $i(\Psi_{p,\lambda}) = p$, we first estimate $h_{\Psi_{p,\lambda}}(s)$. By (2.29), we have that

$$\frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \begin{cases} s^p & \forall t \in (0,t_1), \\ \frac{(st)^p}{k(t-t_1)+t_1^p} \approx s^p & \forall t \in [t_1,t_2). \end{cases} \tag{2.31}$$

Moreover we obtain that

$$\frac{s^p}{\log^\lambda(e+t_2)} \le \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{(st)^p}{\Phi_{p,\lambda}(t)} \le \frac{s^p}{\log^\lambda(e+t_1 s^{-1})} \tag{2.32}$$

for all $t \in [t_2, t_1/s)$ and

$$\frac{t_1^p s^p}{t_2^p \log^\lambda(e+t_2 s^{-1})} \le \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{k(st-t_1)+t_1^p}{\Phi_{p,\lambda}(t)} \le \frac{\Phi_{p,\lambda}(t_2)s^p}{t_1^p \log^\lambda(e+t_1 s^{-1})} \tag{2.33}$$

for all $t \in [t_1/s, t_2/s)$. Assume $t \in [t_2/s, +\infty)$. It follows that

$$\frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{\Phi_{p,\lambda}(st)}{\Phi_{p,\lambda}(t)} = s^p \left(\frac{\log(e+st)}{\log(e+t)}\right)^\lambda. \tag{2.34}$$

By the monotonicity of function $(\log(s) + \log(e+\cdot))\log^{-1}(e+\cdot)$, we have that

$$\frac{\log(e+st)}{\log(e+t)} \ge \frac{\log(s)+\log(e+t)}{\log(e+t)} \ge \frac{\log(s)+\log(e+t_2 s^{-1})}{\log(e+t_2 s^{-1})} \ge \frac{\log(t_2)}{\log(e+t_2 s^{-1})}$$

for all $t \ge t_2/s$. Hence we derive from (2.34) that

$$s^p \le \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} \le \log^\lambda(t_2)\frac{s^p}{\log^\lambda(e+t_2 s^{-1})}. \tag{2.35}$$

Combining (2.31), (2.32), (2.33) with (2.35) implies that

$$s^p \lesssim h_{\Psi_{p,\lambda}}(s) \lesssim s^p \log^{-\lambda}(s^{-1}) \tag{2.36}$$

whenever $s \ll 1$. By (2.36), we therefore have that $i(\Psi_{p,\lambda}) = p$. □

Remark 2.10. For $p > 1$ and $\lambda < 0$, let $\Psi_{p,\lambda}$ be as in (2.29) and $\Phi_{p,\lambda}$ be as in (2.21). Analogously to Remark 2.8, we have that

$$\int_{\mathbb{R}^2} \Psi_{p,\lambda}(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Psi_{p,\lambda}(|f(x)|)w(x)\,dx. \tag{2.37}$$

Since $\lim_{t\to 0+} \Psi_{p,\lambda}(t)/\Phi_{p,\lambda}(t) = 1$, it follows that

$$\Psi_{p,\lambda}(t) \approx \Phi_{p,\lambda}(t) \tag{2.38}$$

whenever $t \in [0, +\infty)$. Hence we derive from (2.37) that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(|f(x)|)w(x)\,dx.$$

3 Proof of Theorem 1.1

We begin by proving the following special case of Theorem 1.1.

Theorem 3.1. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the harmonic extension of φ. For any $p > 1$, we have that*

(1) if either $\alpha \in (p-2, +\infty)$ and $\lambda \in \mathbb{R}$, or $\alpha = p-2$ and $\lambda \in (-\infty, -1)$,

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are finite.

(2) if either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$, or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$, then

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are comparable to $\mathcal{U}(p, \alpha, \lambda, \varphi)$.

Moreover whenever $p \in (1, 2]$

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ dominate $\mathcal{V}(p, \alpha, \lambda, \varphi)$,

while

$\mathcal{V}(p, \alpha, \lambda, \varphi)$ controls both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$

for all $p \in [2, +\infty)$. Furthermore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are in general comparable to $\mathcal{V}(p, \alpha, \lambda, \varphi)$ only for $p = 2$.

(3) if either $\alpha \in (-\infty, -1)$ and $\lambda \in \mathbb{R}$, or $\alpha = -1$ and $\lambda \in [-1, +\infty)$, we have that $I_1(p, \alpha, \lambda, h) = \infty$. While $I_2(p, \alpha, \lambda, h) = \infty$ for all $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.

Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda$$

and

$$\mathcal{E}_2(p, \alpha, \lambda, \varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \Phi_{p,\lambda}\Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big) \ell(\Gamma_{j,k})^{2+\alpha}$$

where $\Phi_{p,\lambda}(t)$ is from (2.21).

Lemma 3.2. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. For any $p > 1, \alpha \in (-1, +\infty)$ and every $\lambda \in \mathbb{R}$, the dyadic energies $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ and $\mathcal{E}_2(p, \alpha, \lambda, \varphi)$ are equivalent.*

Proof. We first consider the case $\lambda \geq 0$. Let $\Phi_{p,\lambda}$ be as in (2.21). Since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ and $\ell(\Gamma_{j,k}) \approx 2^{-j}$ for all $j \in \mathbb{N}$ and $k \in \{1, ..., 2^j\}$, by the monotonicity and Δ_2-property of the standard logarithm we have that

$$\Phi_{p,\lambda}\Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big) \lesssim \Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big)^p \log^\lambda \big(e + 2\pi \cdot 2^j\big) \lesssim \Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big)^p j^\lambda.$$

Hence

$$\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi). \tag{3.1}$$

Given $p > 1$ and $\alpha \in (-1, +\infty)$, there is $\beta \in (0, 1)$ such that $\alpha > (1-\beta)p - 1 > -1$. Define

$$\chi_{j,k} = \begin{cases} 1 & \text{if } \ell(\varphi(\Gamma_{j,k})) \geq 2^{-j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

We decompose $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ as

$$\begin{aligned} \mathcal{E}_1(p, \alpha, \lambda, \varphi) =& \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \chi_{j,k} \\ &+ \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda (1 - \chi_{j,k}) \\ =:& \mathcal{E}_1'(p, \alpha, \lambda, \varphi) + \mathcal{E}_1''(p, \alpha, \lambda, \varphi). \end{aligned} \tag{3.2}$$

Whenever $\ell(\varphi(\Gamma_{j,k})) \geq 2^{-j\beta}$, by (2.3) we have $j^\lambda \lesssim \log^\lambda\left(e + \ell(\varphi(\Gamma_{j,k}))\ell(\Gamma_{j,k})^{-1}\right)$. Therefore

$$\begin{aligned} \mathcal{E}_1'(p,\alpha,\lambda,\varphi) \lesssim & \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\right) \\ = & \mathcal{E}_2(p,\alpha,\lambda,\varphi). \end{aligned} \tag{3.3}$$

Moreover, by (2.3) we have that

$$\mathcal{E}_1''(p,\alpha,\lambda,\varphi) \leq \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{-\beta j p} 2^{j(p-2-\alpha)} j^\lambda = \sum_{j=1}^{+\infty} 2^{-j((\beta-1)p+1+\alpha)} j^\lambda < \infty. \tag{3.4}$$

We conclude from (3.2), (3.3) and (3.4) that there is a constant $C > 0$ such that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) \lesssim C + \mathcal{E}_2(p,\alpha,\lambda,\varphi). \tag{3.5}$$

From (3.1) and (3.5) it follows that

(3.6) $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ and $\mathcal{E}_2(p,\alpha,\lambda,\varphi)$ are comparable whenever $\lambda \geq 0$.

Analogously to (3.6), we have that $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ and $\mathcal{E}_2(p,\alpha,\lambda,\varphi)$ are comparable whenever $\lambda < 0$. □

Lemma 3.3. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the Poisson homeomorphic extension of φ. For any $p > 1$, we have that $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\alpha \in (-1,+\infty)$ and $\lambda \in \mathbb{R}$, while $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ is controlled by $I_1(p,\alpha,\lambda,h)$ for all $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. We first prove that $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ for all $\alpha > -1$ and all $\lambda \in \mathbb{R}$. Let $w_{\alpha,\lambda}$ be as in (2.7). For any $j \in \mathbb{N}$ and $1 \leq k \leq 2^j$, by (2.4) and (2.3) we have that

$$w_{\alpha,\lambda}(z) \approx 2^{-j\alpha} j^\lambda \approx \ell(\Gamma_{j,k})^\alpha j^\lambda \tag{3.7}$$

for all $z \in Q_{j,k}$. Hence

$$I_1(p,\alpha,\lambda,h) \approx \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{-j\alpha} j^\lambda \int_{Q_{j,k}} |Dh(z)|^p \, dz. \tag{3.8}$$

Let $\mathcal{P}(\Gamma_{j,k})$ be the technical decomposition of $\mathbb{S}^1$ based on $\Gamma_{j,k}$ in [9, Section 2.1]. As shown in [9, Proof (iv) ⇒ (i)], for any $j \in \mathbb{N}$ and $k = 1, ..., 2^j$ we have that

$$|Dh(z)| \lesssim \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}. \tag{3.9}$$

for all $z \in Q_{j,k}$. Here $\Gamma_{n,m} \in \mathcal{P}(\Gamma_{j,k})$ and $\sharp i_n \leq 3$ for all $n \leq j$, see [9, Section 2.1]. Let $\alpha > -1$. There is $q_0 > 1$ such that $p/q_0 - 1 - \alpha < 0$. Denote by p_0 the exponent conjugate to q_0. Via Hölder's inequality we derive from (3.9) that

$$|Dh(z)|^p \lesssim \left(\sum_{n\leq j} \sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n(\frac{1}{q_0}+\frac{1}{p_0})}} \right)^p \leq \left(\sum_{n\leq j} \sum_{m\in i_n} 2^{\frac{nq}{q_0}} \right)^{\frac{p}{q}} \sum_{n\leq j} \sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}}$$

$$\approx 2^{\frac{jp}{q_0}} \sum_{n\leq j} \sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \tag{3.10}$$

for all $z \in Q_{j,k}$. By (2.5), (3.8) and (3.10) we have that

$$I_1(p,\alpha,\lambda,h) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda \sum_{n\leq j} \sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{q_0}}}. \tag{3.11}$$

Moreover given a dyadic arc $\Gamma_{n,m}$, for any $j \geq n$ it is shown in [9, Section 2.1] that

$$\sharp\{\Gamma : \Gamma \text{ is a } j\text{-level dyadic arc and } \Gamma_{n,m} \in \mathcal{P}(\Gamma)\} \leq 3 \cdot 2^{j-n}. \tag{3.12}$$

From Fubini's theorem and (3.12) we obtain that

$$\begin{aligned} &\sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda \sum_{n\leq j} \sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \\ =& \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \sum_{n\leq j} \sum_{k} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda \\ \lesssim& \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \sum_{n\leq j} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda 2^{j-n} \\ =& \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \ell(\varphi(\Gamma_{n,m}))^p 2^{n(\frac{p}{p_0}-1)} \sum_{n\leq j} 2^{j(\frac{p}{q_0}-1-\alpha)} j^\lambda. \end{aligned} \tag{3.13}$$

Moreover when $p/q_0 - 1 - \alpha < 0$ we have that

$$\sum_{n\leq j} 2^{j(\frac{p}{q_0}-1-\alpha)} j^\lambda \approx 2^{n(\frac{p}{q_0}-1-\alpha)} n^\lambda. \tag{3.14}$$

By (3.11), (3.13), (3.14) and (2.3), we conclude that

$$I_1(p,\alpha,\lambda,h) \lesssim \sum_{n=1}^{\infty} \sum_{m=1}^{2^n} \ell(\varphi(\Gamma_{n,m}))^p 2^{n(p-2-\alpha)} n^\lambda \approx \mathcal{E}_1(p,\alpha,\lambda,\varphi).$$

We next prove that $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ is controlled by $I_1(p,\alpha,\lambda,h)$ for all $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$. By [9, (3.17)], there is $j_0 > 1$ such that

$$\ell(\varphi(\Gamma_{j,k})) \lesssim \frac{1}{\ell(\Gamma_{j,k})} \int_{CQ_{j,k}\cap\mathbb{D}} |Dh(z)|\, dz \tag{3.15}$$

for all $j \geq j_0$ and $k \in \{1,...,2^j\}$. Set $H(z) = |Dh(z)|\chi_{\mathbb{D}}(z)$. By (2.6) we have that

$$⨍_{CQ_{j,k}\cap\mathbb{D}} |Dh(z)|dz \leq ⨍_{CC'B_{j,k}} H(z)dz \leq ⨍_{Q_{j,k}} M_H(w)dw, \tag{3.16}$$

where the last inequality comes from the fact that $⨍_{CC'B_{j,k}} H(z)dz \leq M_H(w)$ for all $w \in Q_{j,k}$. Combining (3.15) with (3.16) implies that

$$\ell(\varphi(\Gamma_{j,k})) \lesssim \ell(\Gamma_{j,k}) ⨍_{Q_{j,k}} M_H(z)dz \tag{3.17}$$

for all $j \geq j_0$ and $k = 1,...,2^j$. By Jensen's inequality and (2.5), we deduce from (3.17) that

$$\ell(\varphi(\Gamma_{j,k}))^p \lesssim \ell(\Gamma_{j,k})^{p-2} \int_{Q_{j,k}} M_H^p(z)\, dz. \tag{3.18}$$

By (3.7) and (3.18), there is then a constant $C > 0$ such that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) \lesssim C + \sum_{j=j_0}^{+\infty}\sum_{k=1}^{2^j} \int_{Q_{j,k}} M_H^p(z) w_{\alpha,\lambda}(z)\, dz \leq C + \int_{\mathbb{R}^2} M_H^p(z) w_{\alpha,\lambda}(z)\, dz. \tag{3.19}$$

Moreover, for any $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$, from Proposition 2.3 and Remark 2.8 it follows that

$$\int_{\mathbb{R}^2} M_H^p(z) w_{\alpha,\lambda}(z)\, dz \lesssim \int_{\mathbb{R}^2} H^p(z) w_{\alpha,\lambda}(z)\, dz = I_1(p,\alpha,\lambda,h). \tag{3.20}$$

By (3.19) and (3.20) we conclude that $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ is controlled by $I_1(p,\alpha,\lambda,h)$ for all $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$. □

Lemma 3.4. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the Poisson homeomorphic extension of φ. For any $p > 1$, we have that $I_2(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\alpha \in (-1,+\infty)$ and $\lambda \in \mathbb{R}$, while $\mathcal{E}_2(p,\alpha,\lambda,\varphi)$ is controlled by $I_2(p,\alpha,\lambda,h)$ for all $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. We first consider that case $\lambda \geq 0$. Let $\Phi_{p,\lambda}$ be as in (2.21). Proposition 2.7 shows that $\Phi_{p,\lambda}(t)$ is increasing and satisfies Δ_2-property on $[0, +\infty)$. From (3.7) and (3.9) we have that

$$\begin{aligned} I_2(p,\alpha,\lambda,h) &= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi_{p,\lambda}(|Dh(z)|)w_{\alpha,0}(z)\,dz \\ &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Phi_{p,\lambda}\big(\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big). \end{aligned} \tag{3.21}$$

Moreover since $\ell(\varphi(\Gamma_{n,m})) \leq 2\pi$ for all $n \in \mathbb{N}$ and $m = 1, ..., 2^n$, it follows that

$$\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \lesssim \sum_{n\leq j}\frac{1}{2^{-n}} \lesssim 2^j.$$

for any $j \geq 1$. Therefore

$$\log^\lambda\big(e + \sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big) \lesssim \log^\lambda(e+2^j) \lesssim j^\lambda \tag{3.22}$$

for all $j \geq 1$. By (3.21) and (3.22) we obtain that

$$I_2(p,\alpha,\lambda,h) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\ell(\Gamma_{j,k})^{\alpha+2}j^\lambda\big(\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big)^p. \tag{3.23}$$

The analogous arguments as for $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ in Lemma 3.3 imply that

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\ell(\Gamma_{j,k})^{\alpha+2}j^\lambda\big(\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big)^p \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi). \tag{3.24}$$

We conclude from (3.23) and (3.24) that $I_2(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$.

Applying $\Phi_{p,\lambda}$ to the both sides of (3.17), via Proposition 2.7 and Jensen's inequality we have that

$$\Phi_{p,\lambda}\big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\big) \lesssim \Phi_{p,\lambda}\big(\fint_{Q_{j,k}} M_H(z)dz\big) \leq \fint_{Q_{j,k}} \Phi_{p,\lambda}(M_H(z))dz \tag{3.25}$$

for all $j \geq j_0$ and $k \in \{1, ..., 2^j\}$. By (2.5), (3.7) and (3.25), we then obtain that

$$\begin{aligned} \mathcal{E}_2(p,\alpha,\lambda,\varphi) &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}}\Phi_{p,\lambda}(M_H(z))w_{\alpha,0}(z)\,dz \\ &\leq \int_{\mathbb{R}^2}\Phi_{p,\lambda}(M_H(z))w_{\alpha,0}(z)\,dz. \end{aligned} \tag{3.26}$$

Moreover, for any $\alpha \in (-1, p-1)$ it follows from Remark 2.8 that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_H(z)) w_{\alpha,0}(z)\, dz \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(H(z)) w_{\alpha,0}(z)\, dz = I_2(p,\alpha,\lambda,h). \tag{3.27}$$

By (3.26) and (3.27) we conclude that $\mathcal{E}_2(p,\alpha,\lambda,\varphi) \lesssim I_2(p,\alpha,\lambda,h)$.

We next consider the case $\lambda < 0$. Let $\Psi_{p,\lambda}$ be as in (2.29). By the analogous arguments as for (3.21), we have that

$$\int_{\mathbb{D}} \Psi_{p,\lambda}(|Dh(z)|) w_{\alpha,0}(z)\, dz \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Psi_{p,\lambda}\big(\sum_{n \le j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \big). \tag{3.28}$$

Set $S_{j,k} = \sum_{n \le j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}$. It follows from (2.38) and (3.28) that

$$I_2(p,\alpha,\lambda,h) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}(S_{j,k}). \tag{3.29}$$

Since $\alpha > -1$, there is $\beta > 0$ such that $\beta p \le 1 + \alpha$. Define

$$\chi(j,k) = \begin{cases} 1 & \text{if } S_{j,k} < 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}(S_{j,k}) &= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}(\chi(j,k) S_{j,k}) \\ &\quad + \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Phi_{p,\lambda}((1-\chi(j,k)) S_{j,k}) =: \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned} \tag{3.30}$$

Since $\log^{\lambda}(e + S_{j,k}) \le \log^{\lambda}(e) = 1$, we have that

$$\sum\nolimits_1 \le \sum\nolimits_1 2^{-j(\alpha+2)} (S_{j,k})^p \le \sum_{j=1}^{\infty} 2^{j(p\beta-\alpha-1)} < \infty. \tag{3.31}$$

Whenever $S_{j,k} \ge 2^{j\beta}$, it follows that $\log^{\lambda}(e + S_{j,k}) \lesssim j^{\lambda}$. Via the analogous arguments as for $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ in Lemma 3.3, it then follows that

$$\sum\nolimits_2 \le \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^{\lambda} S_{j,k}^p \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi). \tag{3.32}$$

From (3.29), (3.30), (3.31) and (3.32), we conclude that there is a constant $C > 0$ such that $I_2(p, \alpha, \lambda, h) \lesssim C + \mathcal{E}_1(p, \alpha, \lambda, \varphi)$.

By the analogous arguments as for $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$ whenever $\lambda \geq 0$, we have that

$$\sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Psi_{p,\lambda}\Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big) \lesssim \int_{\mathbb{R}^2} \Psi_{p,\lambda}(|Dh|(z)) w_{\alpha,0}(z)\, dz. \tag{3.33}$$

It follows from (2.38) that $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$. □

Proof of Theorem 3.1 (1). From Lemma 3.3 and Lemma 3.4, we have that both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are dominated by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ for all $p > 1, \alpha \in (-1, +\infty)$ and each $\lambda \in \mathbb{R}$. Moreover since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ for all $j \geq 1$ and $1 \leq k \leq 2^j$, we have that

$$\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \leq (2\pi)^{p-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) = (2\pi)^p.$$

Therefore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are controlled by $\sum_{j=1}^{\infty} 2^{j(p-2-\alpha)} j^\lambda$ whenever $\alpha \in (-1, +\infty)$ and $\lambda \in \mathbb{R}$. Notice that $\sum_{j=1}^{\infty} 2^{j(p-2-\alpha)} j^\lambda < \infty$ whenever either $p - 2 < \alpha$ and $\lambda \in \mathbb{R}$, or $p - 2 = \alpha$ and $\lambda < -1$. We hence complete Theorem 3.1 (1). □

By Example 4.4, there are homeomorphisms $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that, for their harmonic extensions $P[\varphi]$, both $I_1(p, \alpha, \lambda, P[\varphi])$ and $I_2(p, \alpha, \lambda, P[\varphi])$ may be finite or infinite for either some $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or some $\alpha = p-2$ and $\lambda \in [-1, +\infty)$. How can we characterize both $I_1(p, \alpha, \lambda, P[\varphi]) < \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) < \infty$? As shown in [9], double integrals of the inverse mapping over the boundary are potential choices.

Lemma 3.5. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. For any $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, $\mathcal{V}(p, \alpha, \lambda, \varphi)$ is dominated by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $p \in (1, 2]$; while $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ is controlled by $\mathcal{V}(p, \alpha, \lambda, \varphi)$ if $p \in [2, +\infty)$.*

Proof. We first consider the case $p \in (1, 2]$. Given $\xi \in \mathbb{S}^1$ and $t \geq 0$, set

$$E_t(\xi) = \{\eta \in \mathbb{S}^1 : |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| < t\}.$$

By Fubini's theorem we have that

$$\int_{\mathbb{S}^1} \mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|)\,|d\eta| = \int_{\mathbb{S}^1}\int_{|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|}^{1} -\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt\,|d\eta|$$
$$= \int_0^1\int_{\mathbb{S}^1} -\mathcal{A}'_{p,\alpha,\lambda}(t)\chi_{E_t(\xi)}\,|d\eta|\,dt$$
$$= \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\,dt. \tag{3.34}$$

Moreover, from Jensen's inequality and Minkowski's inequality it follows that

$$\Big(\int_{\mathbb{S}^1}\big(\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\,dt\big)^{p-1}|d\xi|\Big)^{\frac{1}{p-1}}$$
$$\lesssim \Big(\int_{\mathbb{S}^1}\big(\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\,dt\big)^{\frac{1}{p-1}}|d\xi|\Big)^{p-1}$$
$$\le \int_0^1\Big(\int_{\mathbb{S}^1}\big(-\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\big)^{\frac{1}{p-1}}|d\xi|\Big)^{p-1}dt$$
$$= \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_t(\xi))^{\frac{1}{p-1}}|d\xi|\Big)^{p-1}dt. \tag{3.35}$$

Combining (3.34) with (3.35) implies that

$$\mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi) \lesssim \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_t(\xi))^{\frac{1}{p-1}}|d\xi|\Big)^{p-1}dt$$
$$\le \sum_{j=1}^{+\infty}\int_{2^{-j}}^{2^{1-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}|d\xi|\Big)^{p-1}. \tag{3.36}$$

Since $E_{2^{1-j}}(\xi)\subset\cup_{i=k-1}^{k+1}\varphi(\Gamma_{j,i})$ for all $j\in\mathbb{N},\ k=1,...,2^j$ and all $\xi\in\varphi(\Gamma_{j,k})$, we have that

$$\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}|d\xi|\Big)^{p-1} = \Big(\sum_{k=1}^{2^j}\int_{\varphi(\Gamma_{j,k})}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}|d\xi|\Big)^{p-1}$$
$$\le \Big(\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))\Big(\sum_{i=k-1}^{k+1}\ell(\varphi(\Gamma_{j,i}))\Big)^{\frac{1}{p-1}}\Big)^{p-1}$$
$$\le \sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^{p-1}\sum_{i=k-1}^{k+1}\ell(\varphi(\Gamma_{j,i})). \tag{3.37}$$

Moreover by Young's inequality we have that

$$\begin{aligned}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^{p-1}\ell(\varphi(\Gamma_{j,k-1})) &\le \sum_{k=1}^{2^j} \frac{1}{p}\ell(\varphi(\Gamma_{j,k}))^p + \frac{p}{p-1}\ell(\varphi(\Gamma_{j,k-1}))^p \\ &\lesssim \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^{p-1}\ell(\varphi(\Gamma_{j,k+1})) &\le \sum_{k=1}^{2^j} \frac{1}{p}\ell(\varphi(\Gamma_{j,k}))^p + \frac{p}{p-1}\ell(\varphi(\Gamma_{j,k+1}))^p \\ &\lesssim \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p. \end{aligned} \tag{3.39}$$

Combining (3.37), (3.38) with (3.39) implies that

$$\left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\right)^{p-1} \lesssim \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \tag{3.40}$$

for all $j \in \mathbb{N}$. Let

$$\Lambda_\lambda(t) = \begin{cases} t^{\lambda+1} & \lambda \neq -1, \\ \log t & \lambda = -1. \end{cases}$$

For any $j \in \mathbb{N}$, we have that

$$\int_{2^{-j}}^{2^{1-j}} t^{-1}\log_2^\lambda(t^{-1})\,dt \approx -\int_{2^{-j}}^{2^{1-j}} d\Lambda_\lambda(\log_2(t^{-1})) = \Lambda_\lambda(j) - \Lambda_\lambda(j-1) \approx j^\lambda. \tag{3.41}$$

It follows (3.41) and (2.3) that

$$\int_{2^{-j}}^{2^{1-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt \approx 2^{j(p-2-\alpha)} \int_{2^{-j}}^{2^{1-j}} \frac{1}{t}\log_2^\lambda(t^{-1})\,dt \approx \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda. \tag{3.42}$$

By combining (3.36), (3.40) with (3.42), we conclude that

$$\mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda = \mathcal{E}_1(p,\alpha,\lambda,\varphi).$$

We next consider the case $p \in [2,+\infty)$. By the analogous arguments as for (3.36) we have that

$$\mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi) \gtrsim \sum_{j=5}^{+\infty} \int_{\pi 2^{1-j}}^{\pi 2^{2-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t)\, dt \left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{\pi 2^{1-j}}(\xi))^{\frac{1}{p-1}} \, |d\xi| \right)^{p-1}. \tag{3.43}$$

Since $\varphi(\Gamma_{j,k}) \subset E_{\pi 2^{1-j}}(\xi)$ for all $j \geq 5$, $k \in \{1, ..., 2^j\}$ and all $\xi \in \varphi(\Gamma_{j,k})$, we have that

$$\begin{aligned}
\left(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{\pi 2^{1-j}}(\xi))^{\frac{1}{p-1}} \, |d\xi| \right)^{p-1} &= \Big(\sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \mathcal{L}^1(E_{\pi 2^{1-j}}(\xi))^{\frac{1}{p-1}} \, |d\xi| \Big)^{p-1} \\
&\geq \Big(\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) \ell(\varphi(\Gamma_{j,k}))^{\frac{1}{p-1}} \Big)^{p-1} \\
&\geq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p. \qquad (3.44)
\end{aligned}$$

By (3.42), (3.43) and (3.44), there is a constant $C > 0$ such that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) = \sum_{j=1}^{4} \sum_{k=1}^{2^j} + \sum_{j=5}^{+\infty} \sum_{k=1}^{2^j} \lesssim C + \mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi).$$

□

We next prove Theorem 3.1 (2).

Lemma 3.6. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. For any $p \in (1,+\infty)$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, we have that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ and $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ are comparable.*

Proof. We first prove that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ is controlled by $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$. Given $\xi \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$, let $\ell(\xi\eta)$ be the arc length of the shorter arc in $\mathbb{S}^1$ connecting ξ and η. Given $j \geq 1$ and $\xi \in \mathbb{S}^1$, set

$$A_j = \{(\xi,\eta) \in \mathbb{S}^1 \times \mathbb{S}^1 : \pi 2^{-j} < \ell(\xi\eta) \leq \pi 2^{1-j}\}$$

and $A_j(\xi) = \{\eta \in \mathbb{S}^1 : (\xi,\eta) \in A_j\}$. Notice that $\lambda_{\mathbb{D}}$ is the Euclidean distance. We have that

$$\begin{aligned}
\mathcal{U}(p,\alpha,\lambda,\varphi) &= \sum_{j=1}^{+\infty} \int_{A_j} \Phi_{p,\lambda}\Big(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \Big) |\xi - \eta|^{\alpha} \, |d\eta| \, |d\xi| \\
&= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{\Gamma_{j,k}} \int_{A_j(\xi)} \Phi_{p,\lambda}\Big(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|} \Big) |\xi - \eta|^{\alpha} \, |d\eta| \, |d\xi|. \qquad (3.45)
\end{aligned}$$

Notice that

$$|\xi-\eta| \approx \ell(\Gamma_{j,k}) \text{ and } |\varphi(\xi)-\varphi(\eta)| \le \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i})) \le 2\pi \tag{3.46}$$

for all $j \in \mathbb{N}$, $k \in \{1, ..., 2^j\}$, $\xi \in \Gamma_{j,k}$ and $\eta \in A_j(\xi)$. It then follows that

$$\begin{aligned}\Phi_{p,\lambda}\left(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\right)|\xi-\eta|^{\alpha} &\lesssim \left(\sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))\right)^{p} \ell(\Gamma_{j,k})^{\alpha-p} \log^{\lambda}(e+2\pi\cdot 2^j)\\ &\lesssim \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))^p \ell(\Gamma_{j,k})^{\alpha-p} j^{\lambda} \end{aligned}\tag{3.47}$$

for all $\lambda \ge 0$, $\xi \in \Gamma_{j,k}$ and $\eta \in A_j(\xi)$. Since

$$\mathcal{L}^1(A_j(\xi)) \approx \ell(\Gamma_{j,k}) \tag{3.48}$$

for all $j \in \mathbb{N}$, $k = 1, ..., 2^j$ and $\xi \in \Gamma_{j,k}$, we derive from (3.45) and (3.47) that

$$\begin{aligned}\mathcal{U}(p,\alpha,\lambda,\varphi) &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))^p \ell(\Gamma_{j,k})^{\alpha-p} j^{\lambda} \int_{\Gamma_{j,k}}\int_{A_j(\xi)} |d\eta|\,|d\xi|\\ &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^{\lambda} = \mathcal{E}_1(p,\alpha,\lambda,\varphi)\end{aligned}$$

whenever $\lambda \ge 0$.

Since φ is homeomorphic, for any $j \in \mathbb{N}$ and $k \in \{1, ..., 2^j\}$ there are $\xi'_{j,k} \in \Gamma_{j,k}$ and $\eta'_{j,k} \in A_j(\xi'_{j,k})$ such that

$$\begin{aligned}&\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)|\xi'_{j,k}-\eta'_{j,k}|^{\alpha}\\ &= \max\Big\{\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\Big)|\xi-\eta|^{\alpha} : \xi \in \Gamma_{j,k} \text{ and } \eta \in A_j(\xi)\Big\}.\end{aligned}\tag{3.49}$$

Since $0 < \alpha+1 < p$, there is $\beta \in (-1, 0)$ such that $0 < (1+\beta)p < \alpha+1$. Define

$$\chi(j,k) = \begin{cases} 1 & \text{if } |\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})| \le 2^{j\beta},\\ 0 & \text{otherwise.}\end{cases}$$

From (3.45), (3.49), (3.48), (2.3) and (3.46), we obtain that

$$
\begin{aligned}
\mathcal{U}(p,\alpha,\lambda,\varphi) &\leq \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big) \\
&= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)\chi(j,k) \\
&\quad + \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)(1-\chi(j,k)) =: \sum\nolimits_1 + \sum\nolimits_2 .
\end{aligned} \tag{3.50}
$$

Since $\log^\lambda(e+|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})||\xi'_{j,k}-\eta'_{j,k}|^{-1}) \leq 1$ for all $\lambda<0$, $j\in\mathbb{N}$ and $1\leq k\leq 2^j$, by (3.46) and (2.3) we have that

$$
\sum\nolimits_1 \lesssim \sum_{j=1}^{+\infty} 2^{-2j}\sum_{k=1}^{2^j} 2^{j((1+\beta)p-\alpha)} = \sum_{j=1}^{+\infty} 2^{j((1+\beta)p-\alpha-1)} < +\infty. \tag{3.51}
$$

Moreover we derive from (3.46) that

$$
\begin{aligned}
\sum\nolimits_2 &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha-p}\Big(\sum_{i=k-1}^{k+1}\ell(\varphi(\Gamma_{j,k}))\Big)^p \log^\lambda(2^{j(1+\beta)}) \\
&\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda = \mathcal{E}_1(p,\alpha,\lambda,\varphi).
\end{aligned} \tag{3.52}
$$

for all $\lambda<0$. Combining (3.50), (3.51) with (3.52) implies that there is a constant $C>0$ such that $\mathcal{U}(p,\alpha,\lambda,\varphi) \lesssim C+\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ for all $\lambda<0$.

We next prove that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ dominates $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$. Given $j\geq 3$ and $\xi\in\mathbb{S}^1$, set

$$
B_j = \{(\xi,\eta)\in\mathbb{S}^1\times\mathbb{S}^1 : \pi 2^{2-j} < \ell(\xi\eta) \leq \pi 2^{3-j} \text{ with } \arg\eta > \arg\xi\}
$$

and $B_j(\xi)=\{\eta\in\mathbb{S}^1 : (\xi,\eta)\in B_j\}$. We have that

$$
\sum_{j=3}^{+\infty}\sum_{k=1}^{2^j}\int_{\Gamma_{j,k-1}}\int_{B_j(\xi)} \Phi_{p,\lambda}\Big(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\Big)|\xi-\eta|^\alpha\,|d\eta|\,|d\xi| = \mathcal{U}(p,\alpha,\lambda,\varphi). \tag{3.53}
$$

Since φ is homeomorphic, for any $j \geq 3$ and $1 \leq k \leq 2^j$ there are $\xi''_{j,k} \in \Gamma_{j,k-1}$ and $\eta''_{j,k} \in B_j(\xi''_{j,k})$ such that

$$\begin{aligned} &\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|}\Big)|\xi''_{j,k} - \eta''_{j,k}|^{\alpha} \\ &= \min\Big\{\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|}\Big)|\xi - \eta|^{\alpha} : \xi \in \Gamma_{j,k-1} \text{ and } \eta \in B_j(\xi)\Big\}. \end{aligned} \tag{3.54}$$

Notice that

$$|\xi''_{j,k} - \eta''_{j,k}| \approx \ell(\Gamma_{j,k}) \text{ and } 2 \geq |\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})| \gtrsim \ell(\varphi(\Gamma_{j,k})) \tag{3.55}$$

whenever $j \geq 3$ and $k \in \{1, ..., 2^j\}$. Since $\mathcal{L}^1(B_j(\xi)) \approx \ell(\Gamma_{j,k})$ for all $j \geq 3,\ k = 1, ..., 2^j$ and $\xi \in \mathbb{S}^1$, it follows from (3.53), (3.54) and (3.55) that

$$\sum_{j=3}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|}\Big)| \lesssim \mathcal{U}(p, \alpha, \lambda, \varphi). \tag{3.56}$$

Moreover, for any $\lambda \leq 0$ we obtain from (3.55) that

$$j^{\lambda} \lesssim \log^{\lambda}(e + 2^{1+j}) \lesssim \log^{\lambda}\left(e + \frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|}\right) \tag{3.57}$$

for all $j \in \mathbb{N}$ and all $k = 1, ..., 2^j$. From (3.55), (3.56) and (3.57), there is a constant $C > 0$ such that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = C + \sum_{j=3}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^{\lambda} \lesssim C + \mathcal{U}(p, \alpha, \lambda, \varphi)$$

for all $\lambda \leq 0$. For any $\lambda > 0$, by (3.55) and (3.56) there is a constant $C > 0$ such that

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^{\lambda}\left(2^j \ell(\varphi(\Gamma_{j,k}))\right) \lesssim C + \mathcal{U}(p, \alpha, \lambda, \varphi). \tag{3.58}$$

Let β be same as in (3). Set

$$\chi_{j,k} = \begin{cases} 1 & \text{if } \ell(\varphi(\Gamma_{j,k})) \leq 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) = \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \chi_{j,k}$$

$$+ \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda (1-\chi_{j,k}) =: \sum{}^1 + \sum{}^2. \tag{3.59}$$

Moreover

$$\sum{}^1 \leq \sum_{j=1}^{+\infty} 2^{j((1+\beta)p-\alpha-1)} j^\lambda < \infty \tag{3.60}$$

and

$$\sum{}^2 \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda \left(2^j \ell(\varphi(\Gamma_{j,k}))\right). \tag{3.61}$$

From (3.59), (3.60), (3.61) and (3.58), we have that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ controls $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\lambda > 0$. □

Proof of Theorem 3.1 (2). By Lemma 3.2, Lemma 3.3 and Lemma 3.4, for any $p > 1$ we have that both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are comparable to $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$. By Lemma 3.6, we hence conclude comparability of both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ with $\mathcal{U}(p,\alpha,\lambda,\varphi)$ for all $p > 1, \alpha \in (-1,p-1)$ and every $\lambda \in \mathbb{R}$. By Lemma 3.5, we can dominate $\mathcal{V}(p,\alpha,\lambda,\varphi)$ by either $I_1(p,\alpha,\lambda,h)$ or $I_2(p,\alpha,\lambda,h)$ whenever $p \in (1,2]$, while both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are controlled by $\mathcal{V}(p,\alpha,\lambda,\varphi)$ for all $p \in [2,+\infty)$. Moreover from Example 4.2 and Example 4.3, we have that $\mathcal{V}(p,\alpha,\lambda,\varphi)$ is comparable to either $I_1(p,\alpha,\lambda,h)$ or $I_2(p,\alpha,\lambda,h)$ only when $p = 2$. □

Towards the proof of Theorem 3.1 (3), we have the following general result.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. For any $p > 1$, there is no diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_1(p,\alpha,\lambda,h) < +\infty$ for either $\alpha \in (-\infty,-1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1,+\infty)$; or for which $I_2(p,\alpha,\lambda,h) < +\infty$ for some $\alpha \in (-\infty,-1]$ and $\lambda \in \mathbb{R}$.*

Proof. Assume that there is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_1(p,\alpha,\lambda,h) < +\infty$ for either $\alpha \in (-\infty,-1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1,+\infty)$. Then $h \in W^{1,p}(\mathbb{D},\Omega)$. Let

$$\mathbb{S}_r = \{\xi \in \mathbb{R}^2 : |\xi| = r\} \text{ and } \operatorname{osc}_{\mathbb{S}_r} h = \sup\{|h(\xi_1) - h(\xi_2)| : \xi_1, \xi_2 \in \mathbb{S}_r\}.$$

By the ACL-property of Sobolev mappings, we have that

$$\operatorname{osc}_{\mathbb{S}_r} h \leq \int_{\mathbb{S}_r} |Dh(\xi)|\,|d\xi| \tag{3.62}$$

for $\mathcal{L}^1$-a.e. $r \in [0,1)$. By Jensen's inequality we derive from (3.62) that

$$\begin{aligned}(\operatorname{osc}_{\mathbb{S}_r} h)^p \leq &(\operatorname{osc}_{\mathbb{S}_r} h)^p r^{1-p} \lesssim \int_{\mathbb{S}_r} |Dh(\xi)|^p\,|d\xi| \\ =&w_{\alpha,\lambda}^{-1}(1-r) \int_{\mathbb{S}_r} |Dh(\xi)|^p w_{\alpha,\lambda}(1-r)\,|d\xi|. \end{aligned}\tag{3.63}$$

Let $\mathbb{D}_r = \{z \in \mathbb{R}^2 : |z| < r\}$. Since h is a homeomorphism, we have $\operatorname{osc}_{\mathbb{D}_r} h = \operatorname{osc}_{\mathbb{S}_r} h$. Hence

$$\operatorname{osc}_{\mathbb{S}_r} h \text{ is increasing with respect to } r \in [0,1). \tag{3.64}$$

Moreover $w_{\alpha,\lambda}(1-r) \approx 2^{-\alpha j} j^{\lambda}$ for all $j \geq 0$ and $r \in (1-2^{-j}, 1-2^{-j-1}]$. By (3.63), (3.64) and Fubini's theorem, we obtain that

$$\begin{aligned}\sum_{j=1}^{+\infty} (\operatorname{osc}_{\mathbb{S}_{1-2^{-j}}} h)^p 2^{-(\alpha+1)j} j^{\lambda} \leq &\sum_{j=1}^{+\infty} \int_{1-2^{-j}}^{1-2^{-j-1}} (\operatorname{osc}_{\mathbb{S}_r} h)^p w_{\alpha,\lambda}(1-r)\,dr \\ \lesssim &\sum_{j=1}^{+\infty} \int_{1-2^{-j}}^{1-2^{-j-1}} \int_{\mathbb{S}_r} |Dh(\xi)|^p w_{\alpha,\lambda}(1-r)\,|d\xi|\,dr \\ =&I_1(p,\alpha,\lambda,h). \end{aligned}\tag{3.65}$$

By the assumption at the beginning, we derive from (3.65) that

$$\sum_{j=1}^{+\infty} (\operatorname{osc}_{\mathbb{S}_{1-2^{-j}}} h)^p 2^{-(\alpha+1)j} j^{\lambda} < +\infty \tag{3.66}$$

for either $\alpha < -1$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \geq -1$. Hence by (3.64) we have that $\operatorname{osc}_{\mathbb{S}_{1-2^{-j}}} h = 0$ for all $j \geq 1$. Therefore there is a constant C such that $h(z) = C$ for all $z \in \mathbb{D}$. This contradicts the homeomorphicity of h. We conclude that the assumption at the beginning cannot hold.

We next assume that there is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_2(p,\alpha,\lambda,h) < +\infty$ for some $\alpha \in (-\infty,-1]$ and $\lambda \in \mathbb{R}$. It is not difficult to see that $h \in W^{1,1}(\mathbb{D},\Omega)$. We first let $\lambda \geq 0$. Proposition 2.7 shows that $\Phi_{p,\lambda}$ is convex. Analogously to (3.65), we have

$$\sum_{j=1}^{+\infty} \Phi_{p,\lambda}\left(\frac{\operatorname{osc}_{\mathbb{S}_{1-2^{-j}}} \operatorname{Re} h}{2\pi}\right) 2^{-(\alpha+1)j} \lesssim I_2(p,\alpha,\lambda,h). \tag{3.67}$$

Analogous arguments as below (3.66) imply that there is a contradiction under the above assumption. We next let $\lambda < 0$. Proposition 2.9 shows that $\Psi_{p,\lambda}$ is convex. Analogously to (3.67), we obtain from (2.38) that

$$\begin{aligned}\sum_{j=1}^{+\infty} \Psi_{p,\lambda}\left(\frac{\operatorname{osc}_{\mathbb{S}_{1-2^{-j}}} \operatorname{Re} h}{2\pi}\right) 2^{-(\alpha+1)j} \lesssim & \int_{\mathbb{D}} \Psi_{p,\lambda}(|Dh(z)|) w_{\alpha,0}(z)\, dz \\ \approx & I_2(p,\alpha,\lambda,h).\end{aligned}$$

Analogous arguments as below (3.66) imply that there is a contradiction under the above assumption. □

Proof of Theorem 1.1. Let λ_Ω be the internal distance and $|\cdot|$ be the Euclidean distance. As the proof of [9, Theorem 1] shows that there exist a bi-Lipschitz mapping $g : (\mathbb{S}^1, |\cdot|) \to (\partial\Omega, \lambda_\Omega)$ and a diffeomorphic bi-Lipschitz extension $\tilde{g} : (\mathbb{D}, |\cdot|) \to (\Omega, \lambda_\Omega)$ of g. Let $h = \tilde{g} \circ P[g^{-1} \circ \varphi]$. Then $h : \mathbb{D} \to \Omega$ is a diffeomorphic extension of φ. Moreover

$$I_1(p,\alpha,\lambda,h) \approx I_1(p,\alpha,\lambda,P[g^{-1}\circ\varphi]),\ \ I_2(p,\alpha,\lambda,h) \approx I_2(p,\alpha,\lambda,P[g^{-1}\circ\varphi]),$$

$$\mathcal{U}(p,\alpha,\lambda,\varphi) \approx \mathcal{U}(p,\alpha,\lambda,g^{-1}\circ\varphi),\ \ \mathcal{V}(p,\alpha,\lambda,\varphi) \approx \mathcal{V}(p,\alpha,\lambda,g^{-1}\circ\varphi).$$

Hence Theorem 1.1 (1) and (2) follow from Theorem 3.1. By Lemma 3.7, we complete the proof of Theorem 1.1. □

4 Examples

In this section, we give examples related to Theorem 3.1 (2). We first decompose $[0,1]$. For a given $s \in (0,+\infty)$, let

(4.1) $$j_n^s = [2^{\frac{n}{s}}]$$

be the largest integer less than $2^{n/s}$. There is $n_0^s \geq 1$ such that

(4.2) $$2^{-2-j_n^s} \geq 2^{-j_{n+1}^s} \text{ and } 2^{-j_n^s} \leq 4^{-n} \qquad \forall n \geq n_0^s - 1.$$

Step 1. Let

$$I_1 = I_{1,1} = (a_{1,1}, a_{1,2}) \qquad \text{where } a_{1,1} = 4^{-1} \text{ and } a_{1,2} = 1 - 4^{-1}.$$

Renumber the elements in $T_1 = \{0,1\} \cup \partial I_1$ as $\{b_{1,i_1} : i_1 = 1, ..., 4\}$ such that $b_{1,i_1'} < b_{1,i_1''}$ if $i_1' < i_1''$.

Step 2. Let

$$I_{2,1} = (b_{1,1} + 4^{-2}, b_{1,2} - 4^{-2}) \text{ and } I_{2,2} = (b_{1,3} + 4^{-2}, b_{1,4} - 4^{-2}).$$

Set $I_2 = \cup_{i=1}^{2} I_{2,i}$, and renumber the elements in $T_2 = T_1 \cup \partial I_2$ as $\{b_{2,i_2} : i_2 = 1, ..., 8\}$ such that $b_{2,i_2'} < b_{2,i_2''}$ if $i_2' < i_2''$.

After Step (n-1), we have $\{I_{n-1,k_{n-1}} : k_{n-1} = 1, ..., 2^{n-2}\}$, $I_{n-1} = \cup_{k_{n-1}=1}^{2^{n-2}} I_{n-1,k_{n-1}}$ and $T_{n-1} := T_{n-2} \cup \partial I_{n-1} = \{b_{n-1,i_{n-1}} : i_{n-1} = 1, ..., 2^n\}$ where $b_{n-1,i'_{n-1}} < b_{n-1,i''_{n-1}}$ if $i'_{n-1} < i''_{n-1}$. In the following Step n, set

$$I_{n,k_n} := (b_{n-1,2k_n-1} + 4^{-n}, b_{n-1,2k_n} - 4^{-n}) \qquad \text{for } k_n = 1, ..., 2^{n-1}. \tag{4.3}$$

and $I_n = \cup_{k_n=1}^{2^{n-1}} I_{n,k_n}$. After renumbering the elements in $T_n = T_{n-1} \cup \partial I_n$ as above, we can proceed to Step (n+1). Moreover we must replace I_{n,k_n} in (4.3) by

$$I_{n,k_n} = (b_{n-1,2k_n-1} + 2^{-j_n^s}, b_{n-1,2k_n} - 2^{-j_n^s}) \tag{4.4}$$

whenever $n \geq n_0^s$. Let $I = \cup_{n=1}^{\infty} I_n$ and $R = [0,1] \setminus I$. Then $R \neq \emptyset$. We finally decompose $[0,1]$ as

$$R \cup I. \tag{4.5}$$

We next give an estimate on the length of I_{n,k_n}. Since $\mathcal{L}^1(I_{n,k_n}) = 2^{-j_{n-1}} - 2^{1-j_n}$ for all $n \geq n_0 + 1$ and $k_n \in \{1, ..., 2^{n-1}\}$, by the first inequality in (4.2) we have that

$$\mathcal{L}^1(I_{n,k_n}) \geq 2^{-1-j_{n-1}}. \tag{4.6}$$

for all $n \geq n_0 + 1$ and $k_n \in \{1, ..., 2^{n-1}\}$. When $n = n_0$, from (4.4) and the second estimate in (4.2) we have that

$$\mathcal{L}^1(I_{n,k_n}) = 4^{1-n_0} - 2^{1-j_{n_0}} \geq 4^{-n_0+1/2} > 4^{-n_0} \tag{4.7}$$

for all $k_n = 1, ..., 2^{n-1}$. Whenever $1 \leq n \leq n_0 - 1$ and $k_n \in \{1, ..., 2^{n-1}\}$, we have $\mathcal{L}^1(I_{n,k_n}) = 4^{-n}$. Let $C_1(s) = \min\{2^{j_{n-1}-2n} : 1 \leq n \leq n_0\}$. Then

$$\mathcal{L}^1(I_{n,k_n}) \geq C_1(s) 2^{-j_{n-1}} \tag{4.8}$$

for all $1 \leq n \leq n_0$ and $k_n \in \{1, ..., 2^{n-1}\}$. By (4.6), (4.7) and (4.8), we obtain that there is a constant $C(s) > 0$ such that

$$\mathcal{L}^1(I_{n,k_n}) \geq C(s) 2^{-j_{n-1}} \tag{4.9}$$

for all $n \in \mathbb{N}$ and $k_n \in \{1, ..., 2^{n-1}\}$.

Define

$$f_{n,s}^1(x) = \sum_{k_n=1}^{2^{n-1}} \frac{2k_n - 1}{2^n} \chi_{\overline{I_{n,k_n}}}(x) \text{ and } f_s^1(x) = \sum_{n=1}^{+\infty} f_{n,s}^1(x). \tag{4.10}$$

For any $x \in R$ and any $n \geq n_0^s$, there is $b_n \in \partial I_n$ such that $|b_n - x| = \inf_{b \in \partial I_n} |b - x|$. By (4.4) and (4.10), we have that

$$|b_n - x| \leq 2^{-j_n} \text{ and } |f_s^1(b_{n+1}) - f_s^1(b_n)| < 2^{-n-1}.$$

It follows that $\lim_{n\to+\infty} b_n = x$ and $\{f^1(b_n)\}$ is a Cauchy sequence. Therefore

$$f_s(x) = \begin{cases} f_s^1(x) & \text{if } x \in I, \\ \lim\limits_{n\to+\infty} f_s^1(b_n) & \text{if } x \in R. \end{cases} \tag{4.11}$$

is a well-defined function on $[0, 1]$.

Proposition 4.1. *Let f_s be as in (4.11) with $s \in (0, +\infty)$. Then $f_s(0) = 0$, $f_s(1) = 1$ and f_s is increasing on $[0, 1]$. Moreover there is a constant $C(s) > 0$ such that*

$$|f(x) - f(y)| \log^s(|x - y|^{-1}) \leq C(s) \tag{4.12}$$

for all $x, y \in [0, 1]$ with $x \neq y$.

Proof. By (4.11), we have that $f_s(0) = \lim_{n\to\infty} f_s^1(2^{-j_n}) = \lim_{n\to\infty} 2^{-n} = 0$. Analogously $f_s(1) = 1$.

We next prove the monotonicity of f_s. Let $x_1 \in [0, 1]$, $x_2 \in [0, 1]$ with $x_1 \leq x_2$. If $x_1 \in I_{n,k_n'}$ and $x_2 \in I_{n,k_n''}$ with $k_n' \leq k_n''$, from (4.11) we have that

$$f_s(x_1) \leq f_s(x_2). \tag{4.13}$$

Assume $x_1 \in I_{n_1,k_{n_1}}$ and $x_2 \in I_{n_2,k_{n_2}}$ with $n_1 \neq n_2$. Let $q = |n_2 - n_1|$. If $n_1 < n_2$, from the construction of $\{I_{n,k_n}\}$ we have that $k_{n_2} \geq 2^q(k_{n_1} - 1) + 2^{q-1} + 1$. It then follows from (4.10) that

$$f_s(x_2) \geq \frac{2\big(2^q(k_{n_1} - 1) + 2^{q-1} + 1\big) - 1}{2^{n_1} 2^q} > f_s(x_1). \tag{4.14}$$

If $n_2 < n_1$, from the construction of $\{I_{n,k_n}\}$ we have that

$$k_{n_2} \geq \begin{cases} \left[\frac{k_{n_1}}{2^q}\right] + 1 & \text{if } 0 \leq \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] \leq 1/2, \\ \left[\frac{k_{n_1}}{2^q}\right] + 2 & \text{if } 1/2 < \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] < 1. \end{cases}$$

It follows that

$$(4.15) \qquad 2k_{n_2} - 1 \geq 2(\frac{k_{n_1}}{2^q} + 1/2) - 1 = 2\frac{k_{n_1}}{2^q} \qquad \text{if } 0 \leq \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] \leq 1/2$$

and

$$(4.16) \qquad 2k_{n_2} - 1 \geq 2(\frac{k_{n_1}}{2^q} + 1) - 1 = 2\frac{k_{n_1}}{2^q} + 1 \qquad \text{if } 1/2 < \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] < 1.$$

By combining (4.15) with (4.16), we deduce from (4.10) that

$$(4.17) \qquad f_s(x_2) > f_s(x_1).$$

Assume $x_1 \in R$ and $x_2 \in I$. By (4.11), there is $\{b_n\} \subset \partial I$ such that $\lim_{n\to\infty} b_n = x_1$. Together with $x_1 < x_2$, it follows that $b_n < x_2$ whenever $n \gg 1$. Via the arguments for (4.13), (4.14) and (4.17), we have that

$$(4.18) \qquad f_s^1(b_n) \leq f_s(x_2) \qquad \forall n \gg 1.$$

By taking limit for (4.18), we have that

$$(4.19) \qquad f_s(x_1) \leq f_s(x_2).$$

Assume either $x_1 \in I$ and $x_2 \in R$, or $x_1 \in R$ and $x_2 \in R$. Via analogous arguments as for (4.19), we can also prove $f_s(x_1) \leq f_s(x_2)$ at these two cases. By preceding arguments, we conclude that f_s is increasing on $[0, 1]$.

We next prove (4.12). Let $T_n = \{b_{n,i_n} : i_n = 1, ..., 2^{n+1}\}$ with $n \in \mathbb{N}$ and $f_{i,s}^1$ be as in (4.10). For a given $n \in \mathbb{N}$, define

$$f_{n,s}^2(x) = \sum_{i=1}^{2^n} \left(\frac{2^{j_n}}{2^n}(x - b_{n,2i-1}) + \frac{i-1}{2^n} \right) \chi_{[b_{n,2i-1}, b_{n,2i}]}(x),$$

$$(4.20) \qquad f_{n,s}(x) = f_{n,s}^2(x) + \sum_{i=1}^{n} f_{i,s}^1(x).$$

Then $f_{n,s}$ is piecewise affine, increasing and continuous on $[0, 1]$. Furthermore we claim:

(i) $\lim_{n\to\infty} f_{n,s}(x_0) = f_s(x_0)$ for all $x_0 \in [0, 1]$,

(ii) there are constant $C(s) > 0$ and $N(s) > 0$ such that

$$\sup\left\{|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) : x, y \in [0,1] \text{ and } x \neq y\right\} \leq C(s)$$

for all $n \geq N(s)$.

If both (i) and (ii) hold, we can prove (4.12).

We first prove (i). Let $x_0 \in [0,1]$. If $x_0 \in I$, without loss of generality we assume that $x_0 \in I_{n_0,k_{n_0}}$. From (4.11) and (4.20), we have that $f_n(x_0) = f(x_0)$ for all $n \geq n_0$. Therefore (i) holds. If $x_0 \in R$, from (4.11) there is $\{b_n\} \subset \partial I$ such that $\lim_{n\to\infty} b_n = x_0$ and $\lim_{n\to\infty} f_s^1(b_n) = f_s(x_0)$. Moreover by (4.20), we have that

$$|f_{n,s}(x_0) - f_s^1(b_n)| = |f_{n,s}(x_0) - f_{n,s}(b_n)| \leq 2^{-n}.$$

Together with $|f_{n,s}(x_0) - f_s(x_0)| \leq |f_{n,s}(x_0) - f_s^1(b_n)| + |f_s^1(b_n) - f_s(x_0)|$, we have that (i) also holds at this case.

We next prove (ii). Given $n \geq 1,\ x \in [0,1]$ and $y \in [0,1]$ with $x < y$, set

$$k_n(x,y) = \#\{I_{m,k_m} :\ I_{m,k_m} \subset [x,y] \text{ for } m = 1, ..., n \text{ and } k_m = 1, ..., 2^{m-1}\}.$$

Then $0 \leq k_n(x,y) \leq 2^n - 1$.

Assume $k_n(x,y) = 0$. If $x \in \cup_{m=1}^n I_m$, there are $m \in \{1, ..., n\}$ and $k_m \in \{1, ..., 2^{m-1}\}$ such that $x \in I_{m,k_m}$. For the location of y, possibly we have that

$$y \in I_{m,m_k},\ y \in I_{m,m_k+1},\ \text{or } y \in [0,1] \setminus (\cup_{m=1}^n I_m). \tag{4.21}$$

If $y \in I_{m,m_k}$, by (4.20) we have that

$$f_{n,s}(x) = f_{n,s}(y) \qquad \forall n \geq m.$$

If $y \in I_{m,m_k+1}$, then $|x-y| \geq 2^{-j_n}$. It follows from (4.20) that

$$|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \leq 2^{-n} \log^s(2^{j_n}) < 1. \tag{4.22}$$

If $y \in [0,1] \setminus (\cup_{m=1}^n I_m)$, there is $x_0 \in [x,y) \cap T_n$ such that

$$0 < y - x_0 < 2^{-j_n} \text{ and } f_{n,s}(x) = f_{n,s}(x_0). \tag{4.23}$$

Since there is $n_1^s > 0$ such that $\log(2^{j^s_{n_1^s}}) - s > 0$, we have that

$$t \log^s(t^{-1}) \leq 2^{-j_n^s} \log^s(2^{j_n^s}) < 2^{n-j_n} \tag{4.24}$$

for all $n \geq n_1^s$ and every $t \in (0, 2^{-j_n^s}]$. By (4.20), (4.23) and (4.24), we then have that

$$|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \leq \frac{|f_{n,s}(x_0) - f_{n,s}(y)|}{|x_0 - y|} |x_0 - y| \log^s(|x_0 - y|^{-1})$$

$$< \frac{2^{j_n}}{2^n} \frac{2^n}{2^{j_n}} = 1 \tag{4.25}$$

whenever $n \geq N(s) := \max\{n_0^s, n_1^s\}$. If $x \in [0,1] \setminus (\cup_{m=1}^n I_m)$, for the location of y we possibly have that

$$y \in [0,1] \setminus (\cup_{m=1}^n I_m), \ y \in \cup_{m=1}^n I_m.$$

If $y \in [0,1] \setminus (\cup_{m=1}^n I_m)$, then $0 < y - x < 2^{-j_n}$. By (4.20) and (4.24) we have that

$$|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) = \frac{2^{j_n}}{2^n} |x-y| \log^s(|x-y|^{-1}) < 1 \tag{4.26}$$

for all $n \geq N(s)$. If $y \in \cup_{m=1}^n I_m$, by analogous arguments as for (4.25) we have that

$$|f_{n,s}(y) - f_{n,s}(x)| \log^s(|x-y|^{-1}) < 1 \tag{4.27}$$

for all $n \geq N(s)$. By (4), (4.22), (4.25), (4.26) and (4.27), we conclude that

$$|f_{n,s}(y) - f_{n,s}(x)| \log^s(|x-y|^{-1}) < 1 \tag{4.28}$$

for all $n \geq N(s)$ and $k_n(x,y) = 0$.

Assume $k_n(x,y) \in \{1, ..., 2^n - 1\}$. Define

$$x' = \inf\{e \in I_{m,k_m} : I_{m,k_m} \subset [x,y] \text{ for } m = 1, ..., n \text{ and } k_m = 1, ..., 2^{m-1}\}$$

and

$$y' = \sup\{e \in I_{m,k_m} : I_{m,k_m} \subset [x,y] \text{ for } m = 1, ..., n \text{ and } k_m = 1, ..., 2^{m-1}\}.$$

If $k_n(x,y) = 1$, by (4.20) we have that

$$f_{n,s}(x') = f_{n,s}(y'). \tag{4.29}$$

If $2^m \leq k_n(x,y) \leq 2^{m+1} - 1$ for $m = 1, ..., n-1$, by (4.5), (4.9) and (4.20) we have that

$$|x - y| \geq \mathcal{L}^1(I_{n-m,k_{n-m}}) \geq C(s) 2^{-j_{n-m-1}^s}$$

and

$$|f_{n,s}(x') - f_{n,s}(y')| = \frac{2 + ... + 2^m}{2^n} < 2^{m+1-n}.$$

Whenever $n \geq n_0^s + 1$, it follows from (4.1) that

$$\begin{aligned} |f_{n,s}(x') - f_{n,s}(y')| \log^s(|x-y|^{-1}) \leq & 2^{m+1-n} \log^s \left(C^{-1} 2^{j^s_{n-m-1}}\right) \\ \leq & C(s) 2^{m+1-n} j^s_{n-m-1} < C(s). \end{aligned} \tag{4.30}$$

Notice that there are two cases for the location of x

$$x \in (x' - 2^{-j_n}, x'],\ x \in \cup_{m=1}^n I_m.$$

If $x \in (x' - 2^{-j_n}, x']$, by analogous arguments as for (4.26) we have that

$$|f_{n,s}(x) - f_{n,s}(x')| \log^s(|x-y|^{-1}) < 1 \qquad \text{whenever } n \geq N(s). \tag{4.31}$$

If $x \in \cup_{m=1}^n I_m$, same arguments as (4.25) imply (4.31). Analogously, we have that

$$|f_{n,s}(y') - f_{n,s}(y)| \log^s(|x-y|^{-1}) < 1 \qquad \text{whenever } n \geq N(s). \tag{4.32}$$

Since

$$\begin{aligned} & |f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \\ = & \big(|f_{n,s}(x) - f_{n,s}(x')| + |f_{n,s}(x') - f_{n,s}(y')| + |f_{n,s}(y') - f_{n,s}(y)|\big) \log^s(|x-y|^{-1}), \end{aligned}$$

by (4.29), (4.30), (4.31) and (4.32) there is a constant $C(s) > 0$ such that

$$|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \leq C(s) \tag{4.33}$$

whenever $n \geq N(s)$ and $k_n(x,y) \in \{1, ..., 2^n - 1\}$. By (4.28) and (4.33), we finish the proof of (ii). □

Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. In the following we denote by $P[\varphi] : \mathbb{D} \to \mathbb{D}$ the harmonic extension of φ.

Example 4.2. *For a given $p \in (1,2)$, there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\mathcal{V}(p, p-2, 0, \varphi) < \infty$, $I_1(p, p-2, 0, P[\varphi]) = \infty$ and $I_2(p, p-2, 0, P[\varphi]) = \infty$.*

Proof. We first introduce a class of self-homeomorphisms on $\mathbb{S}^1$ and their properties. Let f_s be as in (4.11) with $s \in (0, +\infty)$. Define

$$g_s(x) = \frac{f_s(x) + x}{2} \qquad x \in [0,1]. \tag{4.34}$$

Then $g_s : [0,1] \to [0,1]$ is strictly increasing and continuous, i.e. g_s is homeomorphic. Moreover by (4.1), there is a constant $C(s) > 0$ such that

$$|g_s(x) - g_s(y)| \leq C(s) \log^{-s}(|x-y|^{-1}) \tag{4.35}$$

for all $x, y \in [0,1]$ with $x \neq y$. Let $\arg z \in (-\pi, \pi]$ be the principal value of the argument z. Define

$$\varphi_s(z) = \exp\left(i2\pi \left[g_s\left(\frac{\arg z}{2\pi}\right) - g_s(\frac{1}{2}) + \frac{1}{2}\right]\right) \qquad z \in \mathbb{S}^1. \tag{4.36}$$

Then $\varphi_s : \mathbb{S}^1 \to \mathbb{S}^1$ is homeomorphic and $\varphi(e^{i\pi}) = e^{i\pi}$. Next we prove that

$$|\varphi_s(z_1) - \varphi_s(z_2)| \lesssim \log^{-s}(|z_1 - z_2|^{-1}) \tag{4.37}$$

for all $z_1, z_2 \in \mathbb{S}^1$ with $z_1 \neq z_2$. Let $\Gamma(z_1, z_2)$ be the arc in $\mathbb{S}^1$ joining z_1 to z_2 with smaller length. Denote by $\ell(\Gamma(z_1, z_2))$ the length of $\Gamma(z_1, z_2)$. In order to prove (4.37), it is enough to consider the case $\ell(\Gamma(z_1, z_2)) \ll 1$. If $e^{i\pi} \notin \Gamma(z_1, z_2)$, we have that

$$|\arg z_1 - \arg z_2| \approx |z_1 - z_2| \text{ and } \left|g_s\left(\frac{\arg z_1}{2\pi}\right) - g_s\left(\frac{\arg z_2}{2\pi}\right)\right| \approx |\varphi_s(z_1) - \varphi_s(z_2)|$$

whenever $\ell(\Gamma(z_1, z_2)) \ll 1$. Together with (4.35), we then have that

$$\begin{aligned} |\varphi_s(z_1) - \varphi_s(z_2)| \approx & \left|g_s\left(\frac{\arg z_1}{2\pi}\right) - g_s\left(\frac{\arg z_2}{2\pi}\right)\right| \\ \lesssim & \log^{-s}(|\arg z_1 - \arg z_2|^{-1}) \approx \log^{-s}(|z_1 - z_2|^{-1}). \end{aligned} \tag{4.38}$$

If $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi(z_1), \varphi(e^{i\pi}))) > \ell(\Gamma(\varphi(e^{i\pi}), \varphi(z_2)))$, there is $z_0 \in \Gamma(z_1, e^{i\pi})$ such that

$$|\varphi_s(z_1) - \varphi_s(z_2)| \lesssim |\varphi_s(z_1) - \varphi_s(z_0)|. \tag{4.39}$$

Same arguments as for (4.38) imply that

$$|\varphi_s(z_1) - \varphi_s(z_0)| \lesssim \log^{-s}(|z_1 - z_0|^{-1}) \lesssim \log^{-s}(|z_1 - z_2|^{-1}). \tag{4.40}$$

Combining (4.39) with (4.40) therefore implies that (4.37) holds when $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi_s(z_1), \varphi_s(e^{i\pi}))) > \ell(\Gamma(\varphi(e^{i\pi})$. Analogously, we can prove that (4.37) holds when $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi_s(z_1), \varphi_s(e^{i\pi}))) \leq \ell(\Gamma(\varphi_s(e^{i\pi}), \varphi_s(z_2)))$.

Let $p \in (1,2)$. There is $s \in (1, +\infty)$ such that $p - 1 < 1/s < 1$. Based on this s, we obtain a homeomorphism $\varphi = \varphi_s : \mathbb{S}^1 \to \mathbb{S}^1$, where φ_s is from (4.36). By

Jensen's inequality and (4.37), we have that

$$\begin{aligned}\mathcal{V}(p,p-2,0,\varphi) =&\int_{\mathbb{S}^1}\Big(\int_{\mathbb{S}^1}\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\Big)^{p-1}|d\xi|\\ \leq&\Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi|\Big)^{p-1}\\ \lesssim&\Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}|\xi-\eta|^{-\frac{1}{s}}\,|d\eta|\,|d\xi|\Big)^{p-1}<+\infty.\end{aligned}$$

Let n_0^s be as in (4.2) with s chosen above. For any $n\geq n_0^s$ and any $j_n<j\leq j_{n+1}$, by (4.34) and (4.11) we have that

$$\begin{aligned}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^p =&2\pi\sum_{k=1}^{2^j}\mathcal{L}^1(g_s([(k-1)2^{-j},k2^{-j}]))^p\\ \gtrsim&\sum_{k=1}^{2^j}(f_s(k2^{-j})-f_s((k-1)2^{-j}))^p=2^{(n+1)(1-p)}.\end{aligned}\tag{4.41}$$

Notice that $j_{n+1}-j_n\approx 2^{n/s}$ whenever $n\geq n_0$. We then derive from (4.41) that

$$\mathcal{E}_1(p,p-2,0,\varphi)\geq\sum_{n=n_0}^{+\infty}\sum_{j_n<j\leq j_{n+1}}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^p\gtrsim\sum_{n=n_0}^{+\infty}2^{n(1-p+\frac{1}{s})}=+\infty.$$

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, it follows that $I_1(p,p-2,0,P[\varphi])=\infty$ and $I_2(p,p-2,0,P[\varphi])=\infty$. □

Example 4.3. *For a given $p\in(2,+\infty)$, there is a homeomorphism $\varphi:\mathbb{S}^1\to\mathbb{S}^1$ such that $\mathcal{V}(p,p-2,0,\varphi)=\infty$, $I_1(p,p-2,0,P[\varphi])<\infty$ and $I_2(p,p-2,0,P[\varphi])<\infty$.*

Proof. Since $p\in(2,+\infty)$, there is $s\in(0,1)$ such that $p-1>1/s>1$. Based on this chosen s, we obtain a homeomorphism $\varphi=\varphi_s:\mathbb{S}^1\to\mathbb{S}^1$, where φ_s is from (4.36). In order to prove $\mathcal{V}(p,p-2,0,\varphi)=\infty$, by Jensen's inequality it suffices to prove that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi|=+\infty.\tag{4.42}$$

For any $\sigma\in\mathbb{S}^1$ and $\tau\in\mathbb{S}^1$, let $\ell(\sigma,\tau)$ be the arc length of the shorter arc in $\mathbb{S}^1$ joining σ and τ. Let n_0^s be from (4.2) with s chosen above. For any $n\geq n_0^s$, set

$$\Gamma_n=\{(\xi,\eta)\in\mathbb{S}^1\times\mathbb{S}^1:\pi 2^{1-j_{n+1}}<\ell(\varphi^{-1}(\xi),\varphi^{-1}(\eta))\leq\pi 2^{1-j_n}\}.$$

We have that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi| \geq \sum_{n=n_0}^{+\infty}\int_{\Gamma_n} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi|$$

$$\gtrsim \sum_{n=n_0}^{+\infty} j_n \int_{\Gamma_n} |d\eta|\,|d\xi|. \tag{4.43}$$

Given $n \geq n_0^s$ and $k = 1, ..., 2^n$, let

$$\Gamma'_{n,k} = \exp(i2\pi[b_{n,2k-1}, 2^{-j_{n+1}} + b_{n,2k-1}]),$$

$$\Gamma''_{n,k} = \exp(i2\pi[2^{-j_n} - 2^{-j_{n+1}} + b_{n,2k-1}, 2^{-j_n} + b_{n,2k-1}]).$$

For any $\xi \in \varphi(\Gamma'_{n,k})$ and $\eta \in \varphi(\Gamma''_{n,k})$, we have that

$$2\pi(2^{-j_n} - 2^{1-j_{n+1}}) \leq \ell(\varphi^{-1}(\xi), \varphi^{-1}(\eta)) \leq \pi \cdot 2^{1-j_n}. \tag{4.44}$$

Notice that by (4.2) we have that $2^{-j_{n+1}} < 2^{-j_n} - 2^{1-j_{n+1}}$ whenever $n \geq n_0^s$. It then follows from (4.44) that

$$\varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k}) \subset \Gamma_n \tag{4.45}$$

for all $n \geq n_0^s$ and all $k = 1, ..., 2^n$. Moreover from (4.36), (4.34) and (4.11), it follows that

$$\ell(\varphi(\Gamma'_{n,k})) = 2\pi\mathcal{L}^1(g([b_{n,2k-1}, 2^{-j_{n+1}} + b_{n,2k-1}]))$$

$$\geq \pi(f_s(2^{-j_{n+1}}) - f_s(0)) = \pi 2^{-n-1}. \tag{4.46}$$

for all $n \geq n_0^s$ and all $k = 1, ..., 2^n$. Similarly

$$\ell(\varphi(\Gamma''_{n,k})) \geq \pi 2^{-n-1}. \tag{4.47}$$

Since $(\varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k})) \cap (\varphi(\Gamma'_{n,j}) \times \varphi(\Gamma''_{n,j})) = \emptyset$ for all $n \geq n_0^s$ and $k, j \in \{1, ..., 2^n\}$ with $k \neq j$, it follows (4.45), (4.46) and (4.47) that

$$\int_{\Gamma_n} |d\eta|\,|d\xi| \geq \sum_{k=1}^{2^n} \int_{\varphi(\Gamma'_{n,k})\times\varphi(\Gamma''_{n,k})} |d\xi|\,|d\eta| \geq \pi^2 2^{-n-2} \tag{4.48}$$

for all $n \geq n_0^s$. Combining (4.43) with (4.48) hence implies that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi| \gtrsim \sum_{n=n_0}^{+\infty} \frac{j_n}{2^n} \approx \sum_{n=n_0}^{+\infty} \frac{2^{\frac{n}{s}}}{2^n} = +\infty.$$

Therefore (4.42) is complete.

For any $n \geq n_0$ and $j_n < j \leq j_{n+1}$, by (4.36), (4.34), (4.11) and Jensen's inequality we have that

$$\begin{aligned}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p =& 2\pi \sum_{k=1}^{2^j} \mathcal{L}^1(g_s([(k-1)2^{-j}, k2^{-j}]))^p \\ \lesssim& \sum_{k=1}^{2^j} (f_s(k2^{-j}) - f_s((k-1)2^{-j}))^p + \sum_{k=1}^{2^j} 2^{-pj} \\ =& 2^{(1-p)(n+1)} + 2^{(1-p)j}\end{aligned} \tag{4.49}$$

Notice $j_{n+1} - j_n \approx 2^{n/s}$ whenever $n \geq n_0^s$. We then derive from (4.49) that

$$\begin{aligned}\sum_{j=j_{n_0}+1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \lesssim& \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} 2^{(1-p)(n+1)} + \sum_{j=j_{n_0}+1}^{+\infty} 2^{(1-p)j} \\ \approx& \sum_{n=n_0}^{+\infty} 2^{n(1-p+\frac{1}{s})} + \sum_{j=j_{n_0}+1}^{+\infty} 2^{(1-p)j} < +\infty.\end{aligned} \tag{4.50}$$

Since $\sum_{j=1}^{j_{n_0}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p$ is finite, it follows from (4.50) that $\mathcal{E}_1(p, p-2, 0, \varphi) < +\infty$. Moreover by Lemma 3.3 and Lemma 3.4 we have that $I_1(p, p-2, 0, P[\varphi]) < +\infty$ and $I_2(p, p-2, 0, P[\varphi]) < +\infty$. □

Example 4.4. *There is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that both $I_1(p, \alpha, \lambda, P[\varphi]) < +\infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) < +\infty$ hold for all $p > 1$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. Moreover for any $p > 1$, there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $I_1(p, \alpha, \lambda, p[\varphi]) = \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) = \infty$ whenever either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$.*

Proof. Take $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ as the identity mapping. We have that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) \approx \sum_{j=1}^{+\infty} 2^{j(p-2-\alpha)} j^\lambda 2^j (2^{1-j}\pi)^p \approx \sum_{j=1}^{+\infty} 2^{-j(1+\alpha)} j^\lambda < +\infty$$

whenever $p > 1$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. Therefore by Lemma 3.3 and Lemma 3.4 both $I_1(p, \alpha, \lambda, P[\varphi])$ and $I_2(p, \alpha, \lambda, P[\varphi])$ are finite now.

For a given $p > 1$, set j_n in (4.1) as $[e^{2^{n(p-1)}}]$. There is $n_0 \geq 1$ such that (4.2) holds for all $n \geq n_0 - 1$. By following the arguments for (4.5), we have f as in (4.11). Moreover by same arguments as in the proof of Proposition 4.1, there is a constant $C > 0$ depending only on p such that

$$|f(x) - f(y)| \log^{\frac{1}{p-1}} \log(|x-y|^{-1}) \leq C$$

for all $x, y \in [0,1]$ with $x \neq y$. As in (4.36), we finally obtain a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. For any $n \geq n_0$ and $j_n < j \leq j_{n+1}$, by analogous arguments for (4.41) we have that

$$\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \gtrsim 2^{n(1-p)}. \tag{4.51}$$

Notice that $\sum_{j_n<j\leq j_{n+1}} j^{-1} \approx \log j_{n+1} - \log j_n \gtrsim 2^{n(p-1)}$ for all $n \geq n_0$. For any $\lambda \in [-1, +\infty)$ it then follows from (4.51) that

$$\begin{aligned}\mathcal{E}_1(p, p-2, \lambda, \varphi) &\geq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p j^{-1} \geq \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} j^{-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \\ &\gtrsim \sum_{n=n_0}^{+\infty} 2^{n(p-1)} \cdot 2^{n(1-p)} = +\infty. \end{aligned} \tag{4.52}$$

For any $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$, we have that $2^{j(p-2-\alpha)} j^\lambda \gtrsim j^{-1}$ whenever $j \gg 1$. Without loss of generality, we assume that $2^{j(p-2-\alpha)} j^\lambda \gtrsim j^{-1}$ for all $n \geq n_0$ and $j_n < j \leq j_{n+1}$. Hence from (4.52) we have that

$$\begin{aligned}\mathcal{E}_1(p, \alpha, \lambda, \varphi) &\geq \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p 2^{j(p-2-\alpha)} j^\lambda \\ &\gtrsim \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} \frac{1}{j} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p = +\infty \end{aligned} \tag{4.53}$$

for all $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we conclude from (4.52) and (4.53) that for any $p > 1$ there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $I_1(p, \alpha, \lambda, P[\varphi]) = \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) = \infty$ whenever either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$. □

Acknowledgment

The author has been supported by China Scholarship Council (project No. 201706340060) and by National Natural Science Foundation of China (project No. 11571333). This paper is a part of the author's doctoral thesis. The author thanks his advisor Professor Pekka Koskela for posing this question and for valuable discussions.

References

[1] K. Astala, T. Iwaniec, G. J. Martin, and J. Onninen: Extremal mappings of finite distortion, Proc. London Math. Soc. 91 (2005), no. 3, 655-702.

[2] O. J. Broch: Extension of internally bilipschitz maps in John disks. Ann. Acad. Sci. Fenn. Math. Volumen 31, (2006), 13-31.

[3] G. Choquet: Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, Bull. Sci. Math. (2) 69 (1945) 156-165

[4] R. R. Coifman, and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. Studia Math. 51 (1974), 241-250.

[5] T. Iwaniec, G. J. Martin, and C. Sbordone: L^p-integrability & weak type L^2-estimates for the gradient of harmonic mappings of $\mathbb{D}$. Discrete Contin. Dyn. Syst. Ser. B 11 (2009), no. 1, 145-152.

[6] P. W. Jones: Factorization of A_p weights. Ann. of Math. (2) 111 (1980), no. 3, 511-530.

[7] H. Kneser: Lösung der Aufgabe 41, Jahresber. Deutsch. Math.-Verein. 35 (1926), 123-124.

[8] V. Kokilashvili, and M. Krbec: Weighted inequalities in Lorentz and Orlicz spaces. World Scientific Publishing Co., Inc., River Edge, NJ, 1991. xii+233 pp.

[9] P. Koskela, Z. Wang, and H. Xu: Controlled diffeomorphic extension of homeomorphisms. Nonlinear Anal. 177 (2018), part B, 586-600.

[10] H. Lewy: On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc. 42 (1936) 689-692.

[11] B. Muckenhoupt: Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc. 165 (1972), 207-226.

[12] R. Näkki, and J. Väisälä: John disks. Expo. Math. 9, (1991), 9-43.

[13] T. Radó: Aufgabe 41, Jahresber. Deutsch. Math.-Verein. 35 (1926) 49.

[14] G. C. Verchota: Harmonic homeomorphisms of the closed disc to itself need be in $W^{1,p}$, $p < 2$, but not $W^{1,2}$. Proc. Amer. Math. Soc. 135 (2007), no. 3, 891-894.