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ABSTRACT

In this article, we construct a kind of three-dimensional piecewise linear (PWL) system with three switching manifolds and obtain four theorems with regard to the existence of a homoclinic orbit and a heteroclinic cycle in this class of PWL system. The first theorem studies the existence of a heteroclinic cycle connecting two saddle-foci. The existence of a homoclinic orbit connecting one saddle-focus is investigated in the second theorem, and the third theorem examines the existence of a homoclinic orbit connecting another saddle-focus. The last one proves the coexistence of the heteroclinic cycle and two homoclinic orbits for the same parameters. Numerical simulations are given as examples and the results are consistent with the predictions of theorems.

From the Shil'nikov theorems, it is known that the existence of a homoclinic orbit and a heteroclinic cycle play a key and important role in the chaos research of dynamic systems. Moreover, chaos and piecewise linear (PWL) systems have important applications in many fields such as electronic circuits, biology, machinery, and so on. Naturally, the research on the existence of homoclinic orbits and heteroclinic cycles of piecewise linear systems is very meaningful. However, it is not easy to find the homoclinic orbit and the heteroclinic cycle of smooth dynamic systems, especially for higher-dimensional piecewise linear systems with multiple switching manifolds, which will inevitably make the exploration of this problem more complicated. Therefore, this article studies the existence of homoclinic orbits and heteroclinic cycles in a style of the piecewise linear system with three switching manifolds.

I. INTRODUCTION

In the past 50 years, chaos has been a hot research field in nonlinear science. It is well known that Lorenz discovered the first chaotic attractor in the literature,[1] which indicates that it is unrealistic to predict weather conditions for a long time afterward and leads the trend of researching chaos by numerical simulations. Since then, more and more scholars have begun to devote themselves to the study of chaos. Therefore, the research results of chaos have been applied not only in meteorology but also in many other fields, including but not limited to the fields of communication, biology, machinery, and circuit.[2–6] However, most of the literature have no strict mathematical proof for the existence of chaos, which can only be verified by computers. One of the main reasons for this is the difficulty in proving the existence of chaos.

With the efforts of many scholars, some achievements have been made in the mathematical proof of chaos in smooth systems, such as the famous Smale Horseshoe[7] and Shilnikov's theorem.[8] One of the key issues is the existence of the homoclinic orbit or the heteroclinic cycle, which has been given in some studies. For example, the perturbation is used to study the homoclinic orbit and the heteroclinic cycle of systems.[9,10] Leonov showed the fishing method is a good way to prove the existence of the homoclinic orbit and the heteroclinic cycle.[11]

In the past few years, with the development of the study of smooth systems, some limitations of smooth systems in the mathematical modeling of more complex phenomena in the real world have gradually been realized. Therefore, researchers are paying more and more attention to non-smooth dynamic systems, especially the simpler PWL systems. Compared with smooth systems, PWL systems can more accurately characterize complex phenomena in the real world, such as collisions in mechanical systems[12,13] and switching in the circuit.[14]

Although smooth systems are similar to PWL systems in some aspects, basic concepts and definitions cannot be copied directly. Therefore, the study of the basic definitions has become one of the hotspots for PWL systems. Some corresponding results were reported in Refs. 15 and 16. On the other hand, some unique complex bifurcations caused by discontinuities in PWL systems, such as boundary collision bifurcations and sliding bifurcations, have also attracted the attention of many scholars. In recent years, relevant research results of this research direction have been reflected in Refs. 17 and 18. Inspired by Shil'nikov's theorem in smooth systems, the existence of the homoclinic orbit and the heteroclinic cycle is crucial to PWL systems. For PWL systems with low-dimensional or less switching manifolds, great progress has been made in the study of the existence of homoclinic orbits or heteroclinic cycles.[19–26] Some researchers have also studied the existence of homoclinic orbits, heteroclinic cycles, and chaos in higher-dimensional PWL systems, and obtained some results[27–31] by constructing a Poincaré map and proving the existence of topological horseshoes.

However, there are few studies on the existence of homoclinic orbits or heteroclinic cycles in PWL systems with multiple manifolds. In Ref. 32, Chen *et al.* studied the existence of heteroclinic cycles in several kinds of 3D three-region PWL systems with two switching planes. In Ref. 33, Lu *et al.* proposed a new 3D three-region PWL system with two discontinuous boundaries. For three different situations, (i) one saddle and two foci, (ii) two saddles and one focus, and (iii) three saddles, some criteria for the existence of heteroclinic cycles are provided. In addition, sufficient conditions for the existence of chaos are obtained. In Ref. 34, Lu *et al.* studied the coexistence problems of homoclinic orbit connected with one saddle point and heteroclinic cycle connected with two saddle points for a new class of 3D three-region piecewise affine systems (PASs). Recently, Lu *et al.* further proposed some criteria to locate the coexistence of homoclinic cycles and heteroclinic cycles in a class of 3D PASs and gave a mathematical proof of chaos by analyzing the constructed Poincaré map.[35]

As far as we know, few people have studied the existence of homoclinic orbits and heteroclinic cycles in PWL systems with three or more switching manifolds. The purpose of this paper is to explore the existence of homoclinic orbits and heteroclinic cycles in a class of 3D PWL system with three switching manifolds. The main idea is to obtain the stable and unstable manifolds, as well as the intersections of the stable manifolds and unstable manifolds with switching manifolds, respectively, and then obtain the corresponding theorems by basic mathematical analyses.

This article is organized as follows. In Sec. II, a novel PWL system with four regions is introduced. Next, the existence theorems of homoclinic orbits and heteroclinic cycles are given in Sec. III. In order to verify the correctness of these theorems, a concrete example and its numerical simulation are given in Sec. IV. Finally, Sec. V discusses the research content of this paper and these problems are worthy of further research in the future.

II. THE PIECEWISE LINEAR SYSTEM WITH THREE SWITCHING MANIFOLDS

This section provides a new class of 3D PWL system with four regions and gets some basic dynamic properties.

Consider the following 3D PWL system with four regions,

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \begin{cases} A_1X + a_1, & X \in S_1 = \{X \in \mathbf{R}^3 | c'X < d_1\}, \\ A_2X + a_2, & X \in S_2 = \{X \in \mathbf{R}^3 | d_1 < c'X < d_2\}, \\ A_3X + a_3, & X \in S_3 = \{X \in \mathbf{R}^3 | d_2 < c'X < d_3\}, \\ A_4X + a_4, & X \in S_4 = \{X \in \mathbf{R}^3 | d_3 < c'X\}, \end{cases} \tag{1}$$

where $X = (x, y, z)' \in \mathbf{R}^3$, A_i are the 3×3 matrices and $a_i \in \mathbf{R}^3$ are the constant vectors, with $i = 1, 2, 3, 4$. The four areas are S_1, S_2, S_3, and S_4, which are separated by three switching manifolds $\Sigma_{j,j+1} = \bar{S}_j \cap \bar{S}_{j+1}$, with $j = 1, 2, 3$. $c' = (c_0, c_1, c_2) \in \mathbf{R}^3$ is a constant vector. d_1, d_2 and d_3 are constants, and $d_1 < d_2 < d_3$.

Let $E_i = -A_i^{-1}a_i \in S_i$ be the equilibria of the following subsystems of system (1):

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A_iX + a_i, \quad i = 1, 2, 3, 4, \tag{2}$$

and the eigenvalues of A_2 are $\alpha_{A_2} \pm i\beta_{A_2}$, λ_{A_2}; the eigenvalues of A_3 are $\alpha_{A_3} \pm i\beta_{A_3}$, λ_{A_3}; the eigenvalues of A_1 are α_{A_1}, β_{A_1}, λ_{A_1}; and the eigenvalues of A_4 are α_{A_4}, β_{A_4}, λ_{A_4}, where $\alpha_{A_2}, \alpha_{A_3}, \beta_{A_2}, \beta_{A_3} > 0$, $\lambda_{A_2}, \lambda_{A_3} < 0$, and $\alpha_{A_1}, \beta_{A_1}, \lambda_{A_1}, \alpha_{A_4}, \beta_{A_4}, \lambda_{A_4} \neq 0$. Then, $c'E_1 < d_1$, $d_1 < c'E_2 < d_2$, $d_2 < c'E_3 < d_3$, $d_4 < c'E_4$. There exist invertible matrices $P_1 = (\zeta_1, \zeta_2, \zeta_3)$, $P_2 = (\xi_1, \xi_2, \xi_3)$, $P_3 = (\eta_1, \eta_2, \eta_3)$, and $P_4 = (\rho_1, \rho_2, \rho_3)$ such that

$$J_{A_i} = P_i^{-1}A_iP_i, i = 1, 2, 3, 4,$$

where

$$J_{A_2} = \begin{pmatrix} \alpha_{A_2} & -\beta_{A_2} & 0 \\ \beta_{A_2} & \alpha_{A_2} & 0 \\ 0 & 0 & \lambda_{A_2} \end{pmatrix}, \quad J_{A_3} = \begin{pmatrix} \alpha_{A_3} & -\beta_{A_3} & 0 \\ \beta_{A_3} & \alpha_{A_3} & 0 \\ 0 & 0 & \lambda_{A_3} \end{pmatrix},$$

while J_{A_1}, J_{A_4} have one of the following three forms:

$$J_1 = \begin{pmatrix} \alpha_{A_{1,4}} & 0 & 0 \\ 0 & \beta_{A_{1,4}} & 0 \\ 0 & 0 & \lambda_{A_{1,4}} \end{pmatrix}, \quad J_2 = \begin{pmatrix} \alpha_{A_{1,4}} & 1 & 0 \\ 0 & \alpha_{A_{1,4}} & 0 \\ 0 & 0 & \lambda_{A_{1,4}} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} \alpha_{A_{1,4}} & 1 & 0 \\ 0 & \alpha_{A_{1,4}} & 1 \\ 0 & 0 & \alpha_{A_{1,4}} \end{pmatrix}.$$

For convenience, this article only considers J_{A_1}, J_{A_4} as the first form J_1.

Choose initial points $x_0 \in S_1$, $y_0 \in S_2$, $z_0 \in S_3$, and $w_0 \in S_4$, which have the following forms:

$$x_0 = E_1 + \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} \begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix}', \tag{3}$$

$$y_0 = E_2 + \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix} \begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix}', \tag{4}$$

$$z_0 = E_3 + \begin{pmatrix}\eta_1 & \eta_2 & \eta_3\end{pmatrix}\begin{pmatrix}x_3 & y_3 & z_3\end{pmatrix}', \tag{5}$$

and

$$w_0 = E_4 + \begin{pmatrix}\rho_1 & \rho_2 & \rho_3\end{pmatrix}\begin{pmatrix}x_4 & y_4 & z_4\end{pmatrix}'. \tag{6}$$

Denote $\psi_{A_1}(t, x_0)$, $\psi_{A_2}(t, y_0)$, $\psi_{A_3}(t, z_0)$ and $\psi_{A_4}(t, w_0)$ as the solutions of subsystem (2) with the initial values x_0, y_0, z_0, and w_0, respectively. Therefore, we can get

$$\begin{aligned}\psi_{A_1}(t, x_0) &= e^{A_1 t}(x_0 - E_1) + E_1\\ &= \begin{pmatrix}\zeta_1 & \zeta_2 & \zeta_3\end{pmatrix}\begin{pmatrix}e^{\alpha_{A_1} t}x_1\\ e^{\beta_{A_1} t}y_1\\ e^{\lambda_{A_1} t}z_1\end{pmatrix} + E_1,\end{aligned} \tag{7}$$

$$\begin{aligned}\psi_{A_2}(t, y_0) &= e^{A_2 t}(y_0 - E_2) + E_2\\ &= \begin{pmatrix}\xi_1 & \xi_2 & \xi_3\end{pmatrix}\begin{pmatrix}e^{\alpha_{A_2} t}[x_2\cos(\beta_{A_2}t) - y_2\sin(\beta_{A_2}t)]\\ e^{\alpha_{A_2} t}[x_2\sin(\beta_{A_2}t) + y_2\cos(\beta_{A_2}t)]\\ e^{\lambda_{A_2} t}z_2\end{pmatrix} + E_2,\end{aligned} \tag{8}$$

$$\begin{aligned}\psi_{A_3}(t, z_0) &= e^{A_3 t}(z_0 - E_3) + E_3\\ &= \begin{pmatrix}\eta_1 & \eta_2 & \eta_3\end{pmatrix}\begin{pmatrix}e^{\alpha_{A_3} t}[x_3\cos(\beta_{A_3}t) - y_3\sin(\beta_{A_3}t)]\\ e^{\alpha_{A_3} t}[x_3\sin(\beta_{A_3}t) + y_3\cos(\beta_{A_3}t)]\\ e^{\lambda_{A_3} t}z_3\end{pmatrix} + E_3,\end{aligned} \tag{9}$$

and

$$\begin{aligned}\psi_{A_4}(t, w_0) &= e^{A_4 t}(w_0 - E_4) + E_4\\ &= \begin{pmatrix}\rho_1 & \rho_2 & \rho_3\end{pmatrix}\begin{pmatrix}e^{\alpha_{A_4} t}x_4\\ e^{\beta_{A_4} t}y_4\\ e^{\lambda_{A_4} t}z_4\end{pmatrix} + E_4.\end{aligned} \tag{10}$$

Denote $W^u(E_s)$ and $W^s(E_s)$ are the unstable and stable manifolds of $E_s (s = 2, 3)$, respectively. From formulas (8) and (9), we can see $W^u(E_s)$ is two-dimensional and $W^s(E_s)$ is one-dimensional. Therefore, we obtain the following formulas:

$$\begin{aligned}W^u(E_2) &= \{E_2 + k_1\xi_1 + k_2\xi_2 | k_1, k_2 \in \mathbf{R}\},\\ W^u(E_3) &= \{E_3 + k_1'\eta_1 + k_2'\eta_2 | k_1', k_2' \in \mathbf{R}\},\end{aligned}$$

and

$$\begin{aligned}W^s(E_2) &= \{E_2 + k_3\xi_3 | k_3 \in \mathbf{R}\},\\ W^s(E_3) &= \{E_3 + k_3'\eta_3 | k_3' \in \mathbf{R}\}.\end{aligned} \tag{11}$$

Suppose $p_2 = W^s(E_2) \cap W^u(E_3) \cap \Sigma_{2,3}$, $q_2 = W^u(E_2) \cap W^s(E_3) \cap \Sigma_{2,3}$, $L_1 = W^u(E_2) \cap \Sigma_{1,2}$, $q_1 = W^s(E_2) \cap \Sigma_{1,2}$, $L_2 = W^u(E_3) \cap \Sigma_{3,4}$, and $p_3 = W^s(E_3) \cap \Sigma_{3,4}$, then we have

$$q_1 = \begin{pmatrix} x_{E_2} + \dfrac{(d_1 - c'E_2)\xi_{31}}{c_0\xi_{31} + c_1\xi_{32} + c_2\xi_{33}}\\ y_{E_2} + \dfrac{(d_1 - c'E_2)\xi_{32}}{c_0\xi_{31} + c_1\xi_{32} + c_2\xi_{33}}\\ z_{E_2} + \dfrac{(d_1 - c'E_2)\xi_{33}}{c_0\xi_{31} + c_1\xi_{32} + c_2\xi_{33}}\end{pmatrix},$$

$$p_3 = \begin{pmatrix} x_{E_3} + \dfrac{(d_3 - c'E_3)\eta_{31}}{c_0\eta_{31} + c_1\eta_{32} + c_2\eta_{33}}\\ y_{E_3} + \dfrac{(d_3 - c'E_3)\eta_{32}}{c_0\eta_{31} + c_1\eta_{32} + c_2\eta_{33}}\\ z_{E_3} + \dfrac{(d_3 - c'E_3)\eta_{33}}{c_0\eta_{31} + c_1\eta_{32} + c_2\eta_{33}}\end{pmatrix},$$

and

$$\begin{aligned}L_1 = \{&E_2 + k_1\xi_1 + k_2\xi_2 | c_0(k_1\xi_{11} + k_2\xi_{21})\\ &+ c_1(k_1\xi_{12} + k_2\xi_{22}) + c_2(k_1\xi_{13} + k_2\xi_{23})\\ &= d_1 - c'E_2, k_1, k_2 \in \mathbf{R}\},\end{aligned} \tag{12}$$

$$\begin{aligned}L_2 = \{&E_3 + k_1'\eta_1 + k_2'\eta_2 | c_0(k_1'\eta_{11} + k_2'\eta_{21})\\ &+ c_1(k_1'\eta_{12} + k_2'\eta_{22}) + c_2(k_1'\eta_{13} + k_2'\eta_{23})\\ &= d_3 - c'E_3, k_1', k_2' \in \mathbf{R}\},\end{aligned} \tag{13}$$

where ζ_{ij}, ξ_{ij}, η_{ij}, and ρ_{ij} are the jth coordinate of ζ_i, ξ_i, η_i, ρ_i, $i = 1, 2, 3$, respectively. For points q_1 and p_3, according to Eqs. (3) and (6), we know that there exist σ_i and τ_i $(i = 1, 2, 3)$ such that

$$q_1 - E_1 = P_1\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}, \quad p_3 - E_4 = P_4\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\end{pmatrix}. \tag{14}$$

Choosing points $p_1 \in L_1$ and $q_3 \in L_2$, there exist σ_i' and τ_i' $(i = 1, 2, 3)$ similarly such that

$$p_1 - E_1 = P_1\begin{pmatrix}\sigma_1'\\ \sigma_2'\\ \sigma_3'\end{pmatrix}, \quad q_3 - E_4 = P_4\begin{pmatrix}\tau_1'\\ \tau_2'\\ \tau_3'\end{pmatrix}. \tag{15}$$

III. MAIN RESULTS

In this section, the main results of this paper are given. That is, the conditions for the existence of heteroclinic cycle, homoclinic orbits in the system (1) and related proofs.

First, we will ensure that system (1) has a heteroclinic cycle Γ_1. The general idea is given here. As shown in Fig. 1, to prove that Theorem 3.1 holds, only the following conditions are satisfied:

$(c1.1)$ $\{\psi_{A_2}(t, p_2) | t > 0\} \subset S_2, \{\psi_{A_3}(t, q_2) | t > 0\} \subset S_3$,
$(c1.2)$ $\{\psi_{A_2}(t, q_2) | t < 0\} \subset S_2, \{\psi_{A_3}(t, p_2) | t < 0\} \subset S_3$,
$(c1.3)$ $c'(A_3q_2 + a_3) > 0, c'(A_3p_2 + a_3) < 0,$
$c'(A_2p_2 + a_2) < 0, c'(A_2q_2 + a_2) > 0.$

Theorem 3.1. *Suppose that there exist constants k_i, k_i', $i = 1, 2, 3$ and points p_2, q_2 such that the following conditions* (i)–(ii) *hold*:

(i)

$$\begin{aligned}p_2 &= E_3 + k_1'\eta_1 + k_2'\eta_2 = E_2 + k_3\xi_3,\\ q_2 &= E_2 + k_1\xi_1 + k_2\xi_2 = E_3 + k_3'\eta_3,\end{aligned}$$

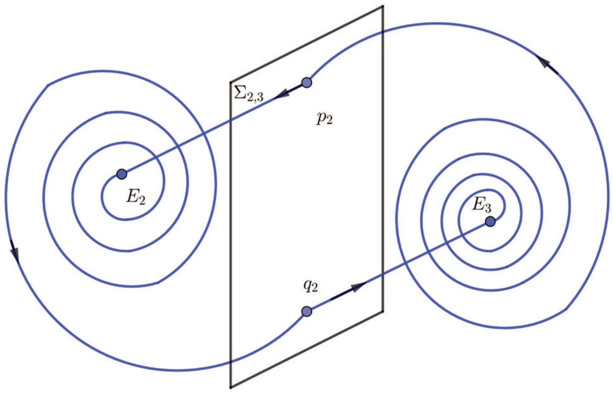

FIG. 1. Schematic diagram of the heteroclinic cycle Γ_1 satisfying Theorem 3.1.

(ii)

$$\alpha_{A_2}(d_2 - c_0x_{E_2} - c_1y_{E_2} - c_2z_{E_2}) + \beta_{A_2}(m_2k_1 - m_1k_2) > 0,$$

$$M_1e^{-\alpha_{A_2}T_1}\frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} < d_2 - c'E_2,$$

$$M_1e^{-\alpha_{A_2}T_1'}\frac{-\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} > d_1 - c'E_2,$$

$$\alpha_{A_3}(d_2 - c_0x_{E_3} - c_1y_{E_3} - c_2z_{E_3}) + \beta_{A_3}(m_2'k_1' - m_1'k_2') < 0,$$

$$M_2e^{-\alpha_{A_3}T_2}\frac{-\beta_{A_3}}{\sqrt{\alpha_{A_3}^2 + \beta_{A_3}^2}} > d_2 - c'E_3,$$

$$M_2e^{-\alpha_{A_3}T_2'}\frac{\beta_{A_3}}{\sqrt{\alpha_{A_3}^2 + \beta_{A_3}^2}} < d_3 - c'E_3,$$

where

$$m_1 = c_0\xi_{11} + c_1\xi_{12} + c_2\xi_{13}, m_2 = c_0\xi_{21} + c_1\xi_{22} + c_2\xi_{23},$$

$$m_1' = c_0\eta_{11} + c_1\eta_{12} + c_2\eta_{13},$$

$$m_2' = c_0\eta_{21} + c_1\eta_{22} + c_2\eta_{23},$$

$$M_1 = \sqrt{(d_2 - c'E_2)^2 + (m_2k_1 - m_1k_2)^2},$$

$$M_2 = \sqrt{(d_2 - c'E_3)^2 + (m_2'k_1' - m_1'k_2')^2},$$

$$T_1 = \frac{\pi}{\beta_{A_2}} + \frac{1}{\beta_{A_2}}\arcsin\frac{(d_2 - c'E_2)}{M_1} - \frac{1}{\beta_{A_2}}\arctan\frac{-\beta_{A_2}}{\alpha_{A_2}},$$

$$T_1' = \frac{1}{\beta_{A_2}}\arcsin\frac{(d_2 - c'E_2)}{M_1} - \frac{1}{\beta_{A_2}}\arctan\frac{-\beta_{A_2}}{\alpha_{A_2}},$$

$$T_2 = \frac{\pi}{\beta_{A_3}} + \frac{1}{\beta_{A_3}}\arcsin\frac{(d_2 - c'E_3)}{M_2} - \frac{1}{\beta_{A_3}}\arctan\frac{-\beta_{A_3}}{\alpha_{A_3}},$$

$$T_2' = \frac{1}{\beta_{A_3}}\arcsin\frac{(d_2 - c'E_3)}{M_2} - \frac{1}{\beta_{A_3}}\arctan\frac{-\beta_{A_3}}{\alpha_{A_3}},$$

then, system (1) has a heteroclinic cycle Γ_1 connecting equibria E_2 and E_3. In addition, Γ_1 transversally intersects $\Sigma_{2,3}$ at points p_2 and q_2, as shown in Fig. 1.

Proof. According to the first equation of (11), $\{\psi_{A_2}(t, p_2)| t > 0\}$ is a straight line and $\psi_{A_2}(t, p_2) \to E_2$, when $t \to +\infty$. From the previous assumption $E_2 \in S_2$, we can see that

$$\{\psi_{A_2}(t, p_2)|t > 0\} \subset S_2. \tag{16}$$

Similarly, to prove formula (16), from the second equation of (11), $\{\psi_{A_3}(t, q_2)|t > 0\}$ is a straight line and $\psi_{A_3}(t, q_2) \to E_3$, when $t \to +\infty$. Due to $E_3 \in S_3$, then

$$\{\psi_{A_3}(t, q_2)|t > 0\} \subset S_3. \tag{17}$$

Next, we have to prove that $\{\psi_{A_2}(t, q_2)|t < 0\} \subset S_2$, which is equivalent to

$$\{\psi_{A_2}(-t, q_2)|t > 0\} \subset S_2. \tag{18}$$

Denote a function $f_1(t) = c'(\psi_{A_2}(-t, q_2) - E_2)$. From representation (8) and the second equation of condition (i) of Theorem 3.1, then $f_1(t)$ has the following form:

$$f_1(t) = M_1e^{-\alpha_{A_2}t}\sin(-\beta_{A_2}t + \theta_1), \tag{19}$$

where

$$\sin\theta_1 = \frac{d_2 - c_0x_{E_2} - c_1y_{E_2} - c_2z_{E_2}}{M_1},$$

$$\cos\theta_1 = \frac{m_2k_1 - m_1k_2}{M_1},$$

$$M_1 = \sqrt{(d_2 - c'E_2)^2 + (m_2k_1 - m_1k_2)^2}.$$

In order to prove $\{\psi_{A_2}(-t, q_2)|t > 0\} \subset S_2$, it needs to be verified that $d_1 - c'E_2 < f_1(t) < d_2 - c'E_2$ for $t > 0$. From Eqs. (8) and (19), we can get the following formulas:

$$f_1(0) = d_2 - c'E_2 > 0,$$

$$f_1'(t) = M_1e^{-\alpha_{A_2}t}[-\alpha_{A_2}\sin(-\beta_{A_2}t + \theta_1) - \beta_{A_2}\cos(-\beta_{A_2}t + \theta_1)],$$

$$f_1''(t) = M_1e^{-\alpha_{A_2}t}[(\alpha_{A_2}^2 - \beta_{A_2}^2)\sin(-\beta_{A_2}t + \theta_1) + 2\alpha_{A_2}\beta_{A_2}\cos(-\beta_{A_2}t + \theta_1)].$$

According to the first inequality in condition (ii) of Theorem 3.1, we have

$$f_1'(0) = -\alpha_{A_2}(d_1 - c'E_2) - \beta_{A_2}(m_2k_1 - m_1k_2) < 0,$$

which shows that $f_1(t) < d_2 - c'E_2$ for $t \in (0, \varepsilon)$, where ε is a small enough positive constant.

Denote $T' = -\frac{1}{\alpha_{A_2}}\ln\frac{\varepsilon}{M_1}$, so that $f_1(t) < d_2 - c'E_2$ for $t \geq T'$. Now, we just need to prove that $f_1(t) < d_2 - c'E_2$ for $t \in [\frac{\varepsilon}{2}, T' + \frac{2\pi}{\beta_{A_2}}]$.

Define the solution of equation $f_1(t) = 0$ for $t \in [\frac{\varepsilon}{2}, T' + \frac{2\pi}{\beta_{A_2}}]$ as t_0. Then, t_0 satisfies $\tan(-\beta_{A_2} t_0 + \theta_1) = -\frac{\beta_{A_2}}{\alpha_{A_2}}$, i.e., either

$$\begin{aligned} \sin(-\beta_{A_2} t_0 + \theta_1) &= -\frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}}, \\ \cos(-\beta_{A_2} t_0 + \theta_1) &= \frac{\alpha_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}}, \end{aligned} \tag{20}$$

or

$$\begin{aligned} \sin(-\beta_{A_2} t_0 + \theta_1) &= \frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}}, \\ \cos(-\beta_{A_2} t_0 + \theta_1) &= -\frac{\alpha_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}}. \end{aligned} \tag{21}$$

If t_0 satisfies (20), then $f_1''(t_0) > 0$, t_0 is a local minimum point of function $f_1(t)$. If t_0 satisfies (21), then $f_1''(t_0) < 0$, t_0 is a local maximum point of function $f_1(t)$.

So, we can get local maximum points

$$t_{0i} = -\frac{1}{\beta_{A_2}}\arctan\frac{-\beta_{A_2}}{\alpha_{A_2}} + \frac{1}{\beta_{A_2}}\arcsin\frac{d_2 - c'E_2}{M_1} + \frac{\pi + 2i\pi}{\beta_{A_2}},$$

$i = 0, 1, 2, \ldots, r = [\frac{\beta_{A_2} T'}{\pi}]$ by calculation. Thus, the corresponding maximum values are

$$f_1(t_{0i}) = M_1 e^{-\alpha_{A_2} t_{0i}} \frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}}.$$

According to $\alpha_{A_2} > 0$ and the second inequality in condition (ii) of Theorem 3.1, then

$$\begin{aligned} f_1(t_{0r}) < \cdots < f_1(t_{00}) &= M_1 e^{-\alpha_{A_2} T_1} \frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} \\ &< d_2 - c'E_2, \end{aligned}$$

where $T_1 = t_{00}$.

Similarly, to prove the maximum values of $f_1(t)$, the minimum point of $f_1(t)$ is

$$T_1' = \frac{1}{\beta_{A_2}}\arcsin\frac{(d_2 - c'E_2)}{M_1} - \frac{1}{\beta_{A_2}}\arctan\frac{-\beta_{A_2}}{\alpha_{A_2}},$$

and the minimum value of $f_1(t)$ is

$$f_1(T_1') = M_1 e^{-\alpha_{A_2} T_1'} \frac{-\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} > d_1 - c'E_2.$$

Due to the third inequality of condition (ii) of Theorem 3.1. Then,

$$\{\psi_{A_2}(t, q_2) | t < 0\} \subset S_2 \tag{22}$$

has been proven.

Using the analysis method similar to the proof of (22), we can also obtain

$$\{\psi_{A_3}(t, p_2) | t < 0\} \subset S_3. \tag{23}$$

By combining formulas (16), (17), (22), and (23), system (1) has a heteroclinic cycle Γ_1 connecting equilibria E_2 and E_3, and Γ_1 intersects $\Sigma_{2,3}$ at points p_2 and q_2.

According to the inequalities of Theorem 3.1, one gets

$$\begin{aligned} &c'(A_3 q_2 + a_3) = \lambda_{A_3}(d_2 - c'E_3) > 0, \\ &c'(A_3 p_2 + a_3) = \alpha_{A_3}(d_2 - c'E_3) + \beta_{A_3}(m_2' k_1' - m_1' k_2') < 0, \\ &c'(A_2 p_2 + a_2) = \lambda_{A_2}(d_2 - c'E_2) < 0, \\ &c'(A_2 q_2 + a_2) = \alpha_{A_2}(d_2 - c'E_2) + \beta_{A_2}(m_2 k_1 - m_1 k_2) > 0, \end{aligned}$$

which mean that Γ_1 transversally intersects $\Sigma_{2,3}$.

In summary, the proof of Theorem 3.1 is completed. □

Next, the theorem that system (1) has the homoclinic orbit Γ_2 is given.

Before formally proving Theorem 3.2, the general idea of the proof is given here. As shown in Fig. 2, to prove Theorem 3.2 is holds, only the following conditions are satisfied:

(c2.1) $\{\psi_{A_2}(t, q_1) | t > 0\} \subset S_2, \{\psi_{A_2}(t, p_1) | t < 0\} \subset S_2$,
(c2.2) $\psi_{A_1}(T_0, p_1) = q_1, \{\psi_{A_1}(t, p_1) | 0 < t < T_0\} \subset S_1$,
(c2.3) $c'(A_1 p_1 + a_1) < 0, c'(A_1 q_1 + a_1) > 0,$
$c'(A_2 q_1 + a_2) > 0, c'(A_2 p_1 + a_2) < 0.$

Theorem 3.2. *Suppose that there exist constants $l_i, i = 1, 2, 3$ and points p_1, q_1 such that the following conditions* (i)–(iii) *hold*:

(i) $p_1 = E_2 + l_1\xi_1 + l_2\xi_2 \in L_1$,

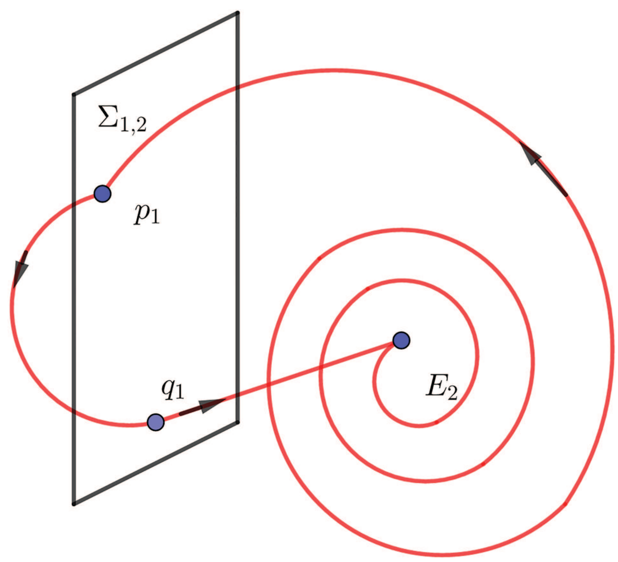

FIG. 2. Schematic diagram of the homoclinic orbit Γ_2 satisfying Theorem 3.2.

(ii) *there exists a constant $T_0 > 0$ such that*

$$e^{J_{A_1}T_0}\begin{pmatrix}\sigma_1'\\ \sigma_2'\\ \sigma_3'\end{pmatrix} = \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix},\ c'P_1J_{A_1}\begin{pmatrix}\sigma_1'\\ \sigma_2'\\ \sigma_3'\end{pmatrix} < 0,\ c'P_1J_{A_1}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} > 0,$$

(iii)

$$\alpha_{A_2}(d_1 - c_0x_{E_2} - c_1y_{E_2} - c_2z_{E_2}) + \beta_{A_2}(m_2''l_1 - m_1''l_2) < 0,$$
$$M_3e^{-\alpha_{A_2}T_3}\frac{-\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} > d_1 - c'E_2,$$
$$M_3e^{-\alpha_{A_2}T_3'}\frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} < d_2 - c'E_2,$$

where

$$m_1'' = c_0\xi_{11} + c_1\xi_{12} + c_2\xi_{13},$$
$$m_2'' = c_0\xi_{21} + c_1\xi_{22} + c_2\xi_{23},$$
$$M_3 = \sqrt{(d_1 - c'E_2)^2 + (m_2''l_1 - m_1''l_2)^2},$$
$$T_3 = \frac{\pi}{\beta_{A_2}} - \frac{1}{\beta_{A_2}}arcsin\frac{(d_1 - c'E_2)}{M_3} + \frac{1}{\beta_{A_2}}arctan\frac{\beta_{A_2}}{\alpha_{A_2}},$$
$$T_3' = -\frac{1}{\beta_{A_2}}arcsin\frac{(d_1 - c'E_2)}{M_3} + \frac{1}{\beta_{A_2}}arctan\frac{\beta_{A_2}}{\alpha_{A_2}},$$

then, system (1) has a homoclinic orbit Γ_2 connecting equilibrium E_2. Moreover, the homoclinic orbit Γ_2 transversally intersects switching manifold $\Sigma_{1,2}$ at points p_1 and q_1, as shown in Fig. 2.

Proof. Similar to the proof of formulas (16) and (22), we can prove

$$\{\psi_{A_2}(t, q_1)|t > 0\} \subset S_2, \tag{24}$$

and

$$\{\psi_{A_2}(t, p_1)|t < 0\} \subset S_2. \tag{25}$$

Next, it is shown that for a positive constant T_0

$$\psi_{A_1}(T_0, p_1) = q_1, \quad \{\psi_{A_1}(t, p_1)|0 < t < T_0\} \subset S_1.$$

According to the condition (ii) of Theorem 3.2, there exists $T_0 > 0$ such that

$$\psi_{A_1}(T_0, p_1) = q_1. \tag{26}$$

Denote a function $f_4(t) = c'(\psi_{A_1}(t, p_1) - E_1)$ for $J_{A_1} = J_1$. Therefore, in order to prove that $\{\psi_{A_1}(t, p_1)|0 < t < T_0\} \subset S_1$, we only need to prove that $f_4(t) < d_1 - c'E_1$ for $0 < t < T_0$.

Due to formulas (7), (14), and (15), we can deduce

$$f_4(t) = \sigma_1'c'\zeta_1e^{\alpha_{A_1}t} + \sigma_2'c'\zeta_2e^{\beta_{A_1}t} + \sigma_3'c'\zeta_3e^{\lambda_{A_1}t},$$
$$f_4(t) = e^{\alpha_{A_1}t}F_1(t),$$

where

$$F_1(t) = \alpha_{A_1}\sigma_1'c'\zeta_1 + \beta_{A_1}\sigma_2'c'\zeta_2e^{(\beta_{A_1}-\alpha_{A_1})t} + \lambda_{A_1}\sigma_3'c'\zeta_3e^{(\lambda_{A_1}-\alpha_{A_1})t}.$$

Taking the derivation of function $F_1(t)$ with repect to t, we can obtain

$$F_1'(t) = e^{(\beta_{A_1}-\alpha_{A_1})t}[\beta_{A_1}\sigma_2'c'\zeta_2(\beta_{A_1} - \alpha_{A_1}) + (\lambda_{A_1} - \alpha_{A_1}) \times \lambda_{A_1}\sigma_3'c'\zeta_3e^{(\lambda_{A_1}-\beta_{A_1})t}]. \tag{27}$$

According to condition (ii) of Theorem 3.2, we can get

$$f_4(0) = f_4(T_0) = d_1 - c'E_1,$$
$$f_4'(0) = c'P_1J_{11}\begin{pmatrix}\sigma_1' & \sigma_2' & \sigma_3'\end{pmatrix}' < 0,$$
$$f_4'(T_0) = c'P_1J_{11}\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}' > 0,$$

then, $F_1(0) < 0, F_1(T_0) > 0$; that is to say, $F_1(t) = 0$ has a solution in the interval $(0, T_0)$. From formula (27), $F_1'(t) = 0$ in interval $(0, T_0)$ has one root at most. If $F'(t) \neq 0$ for $t \in (0, T_0)$, due to the monotonicity and $F_1(0) < 0, F_1(T_0) > 0$, $F_1(t) = 0$ has only one root in $(0, T_0)$. Otherwise, there exists a unique root $\bar{T}_0$ of equation $F_1'(t) = 0$, so that $F_1'(t) > 0, t \in (0, \bar{T}_0), F_1'(t) < 0, t \in (\bar{T}_0, T_0)$ or $F_1'(t) < 0, t \in (0, \bar{T}_0), F_1'(t) > 0, t \in (\bar{T}_0, T_0)$, which indicates that $F_1(t) = 0$ has a unique root in $(0, T_0)$. In other words, $f_4'(t) = 0$ in $(0, T_0)$ has a unique root. According to $f_4(0) = f_4(T_0) = d_1 - c'E_1, f_4'(0) < 0$, and $f_4'(T_0) > 0$; then, $f_4(t) < d_1 - c'E_1$ for $t \in (0, T_0)$.

Therefore,

$$\{\psi_{A_1}(t, p_1)|0 < t < T_0\} \subset S_1. \tag{28}$$

According to formulas (24), (25), (26), and (28), system (1) has a homoclinic orbit Γ_2 connecting equilibrium E_2, and homoclinic orbit Γ_2 intersects $\Sigma_{1,2}$ at points p_1 and q_1.

Due to the inequalities of Theorem 3.2, there exist

$$c'(A_1p_1 + a_1) = c'P_1J_{A_1}\begin{pmatrix}\sigma_1' & \sigma_2' & \sigma_3'\end{pmatrix}' < 0,$$
$$c'(A_1q_1 + a_1) = c'P_1J_{A_1}\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}' > 0,$$
$$c'(A_2q_1 + a_2) = \lambda_{A_2}(d_1 - c'E_2) > 0,$$
$$c'(A_2p_1 + a_2) = \alpha_{A_2}(d_1 - c'E_2) + \beta_{A_2}(m_2k_1 - m_1k_2) < 0,$$

which means that the homoclinic orbit Γ_2 transversally intersects switching manifold $\Sigma_{1,2}$.

Hence, the proof of Theorem 3.2 is completed. □

Remark 3.1. *In Sec. II, regarding the position of equilibrium E_1, it is assumed that E_1 is located in S_1 for convenience. In fact, even if E_1 is the virtual equilibrium point, where E_1 lies in S_2, S_3, or S_4, we can get something similar to Theorem 3.2.*

Then, the theorem that system (1) has the homoclinic Γ_3 is also given below.

Before formally proving Theorem 3.3, the general idea of the proof is given here. As shown in Fig. 3, to prove that Theorem 3.3 holds, only the following conditions are satisfied:

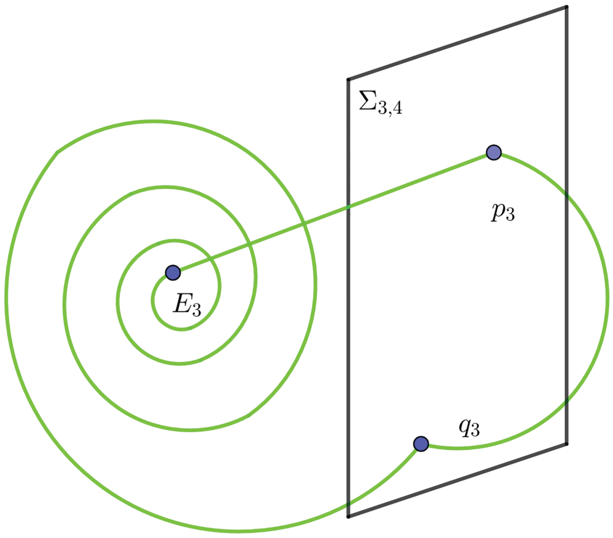

FIG. 3. Schematic diagram of the homoclinic orbit Γ_3 satisfying Theorem 3.3.

(c3.1) $\{\psi_{A_3}(t,p_3)|t>0\}\subset S_3, \{\psi_{A_3}(t,q_3)|t<0\}\subset S_3$,
(c3.2) $\psi_{A_4}(T_4,q_3)=p_3, \{\psi_{A_4}(t,q_3)|0<t<T_4\}\subset S_4$,
(c3.3) $c'(A_4q_3+a_4)>0, c'(A_4p_3+a_4)<0$,
$c'(A_3p_3+a_3)<0, c'(A_3q_3+a_3)>0$.

Theorem 3.3. *Suppose that there exist constants $l_i', i=1,2,3$ and points p_3, q_3 such that the following conditions* (i)–(iii) *hold*:

(i)

$$q_3=E_3+l_1'\eta_1+l_2'\eta_2\in L_2,$$

(ii) *there exist a constant $T_4>0$ such that*

$$e^{J_{A_4}T_4}\begin{pmatrix}\tau_1'\\ \tau_2'\\ \tau_3'\end{pmatrix}=\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\end{pmatrix},\quad c'P_4J_{A_4}\begin{pmatrix}\tau_1'\\ \tau_2'\\ \tau_3'\end{pmatrix}>0,$$
$$c'P_4J_{A_4}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\end{pmatrix}<0,$$

(iii)

$$\alpha_{A_3}(d_3-c_0x_{E_3}-c_1y_{E_3}-c_2z_{E_3})+\beta_{A_3}(m_2'''l_1'-m_1'''l_2')>0,$$
$$M_4e^{-\alpha_{A_3}T_5}\frac{\beta_{A_3}}{\sqrt{\alpha_{A_3}^2+\beta_{A_3}^2}}<d_3-c'E_3,$$
$$M_4e^{-\alpha_{A_3}T_5'}\frac{-\beta_{A_3}}{\sqrt{\alpha_{A_3}^2+\beta_{A_3}^2}}>d_2-c'E_3,$$

where

$$m_1'''=c_0\eta_{11}+c_1\eta_{12}+c_2\eta_{13},$$
$$m_2'''=c_0\eta_{21}+c_1\eta_{22}+c_2\eta_{23},$$
$$M_4=\sqrt{(d_3-c'E_3)^2+(m_2'''l_1'-m_1'''l_2')^2},$$
$$T_5=\frac{\pi}{\beta_{A_3}}+\frac{1}{\beta_{A_3}}arcsin\frac{(d_3-c'E_3)}{M_4}+\frac{1}{\beta_{A_3}}arctan\frac{\beta_{A_3}}{\alpha_{A_3}},$$
$$T_5'=\frac{1}{\beta_{A_3}}arcsin\frac{(d_3-c'E_3)}{M_4}+\frac{1}{\beta_{A_3}}arctan\frac{\beta_{A_3}}{\alpha_{A_3}},$$

then, system (1) has a homoclinic orbit Γ_3 connecting equilibrium E_3. Moreover, homoclinic orbit Γ_3 intersects switching manifold $\Sigma_{3,4}$ at points p_3 and q_3 transversally, as shown in Fig. 3.

Proof. Similar to the proof of Theorem 3.2, if conditions of Theorem 3.3 are satisfied, we can get

$$\{\psi_{A_3}(t,p_3)|t>0\}\subset S_3, \tag{29}$$

$$\{\psi_{A_3}(t,q_3)|t<0\}\subset S_3, \tag{30}$$

and there exists a constant $T_4>0$ such that

$$\psi_{A_4}(T_4,q_3)=p_3 \tag{31}$$

and

$$\{\psi_{A_4}(t,q_3)|0<t<T_4\}\subset S_4. \tag{32}$$

Due to formulas (29), (30), (31), and (32), system (1) has a homoclinic orbit Γ_3 connecting equilibrium E_3, and the homoclinic orbit Γ_3 intersects switching manifold $\Sigma_{3,4}$ at points p_3 and q_3.

According to the inequalities of Theorem 3.3, we can obtain

$$c'(A_4q_3+a_4)=c'P_4J_{A_4}\begin{pmatrix}\tau_1' & \tau_2' & \tau_3'\end{pmatrix}'>0,$$
$$c'(A_4p_3+a_4)=c'P_4J_{A_4}\begin{pmatrix}\tau_1 & \tau_2 & \tau_3\end{pmatrix}'<0,$$
$$c'(A_3p_3+a_3)=\lambda_{A_3}(d_3-c'E_3)<0,$$
$$c'(A_3q_3+a_3)=\alpha_{A_3}(d_3-c'E_3)+\beta_{A_3}(m_2'''l_1'-m_1'''l_2')>0,$$

which indicate that the homoclinic orbit Γ_3 transversally intersects switching manifold $\Sigma_{3,4}$ at points p_3 and q_3.

So, the proof of Theorem 3.3 is completed. □

Remark 3.2. *Same as Remark 3.1, in Sec. II, regarding the position of equilibrium E_4, it is assumed that E_4 is located in S_4 for convenience. In fact, even if E_4 is the virtual equilibrium point, where E_4 lies in S_1, S_2, or S_3, we can get something similar to Theorem 3.3.*

At last, the theorem of coexistence of heteroclinic cycle Γ_1, homoclinic orbits Γ_2 and Γ_3 of system (1) is given.

As shown in Fig. 4, in order to prove Theorem 3.4, it need to combine the proof ideas of the previous Theorem 3.1–3.3.

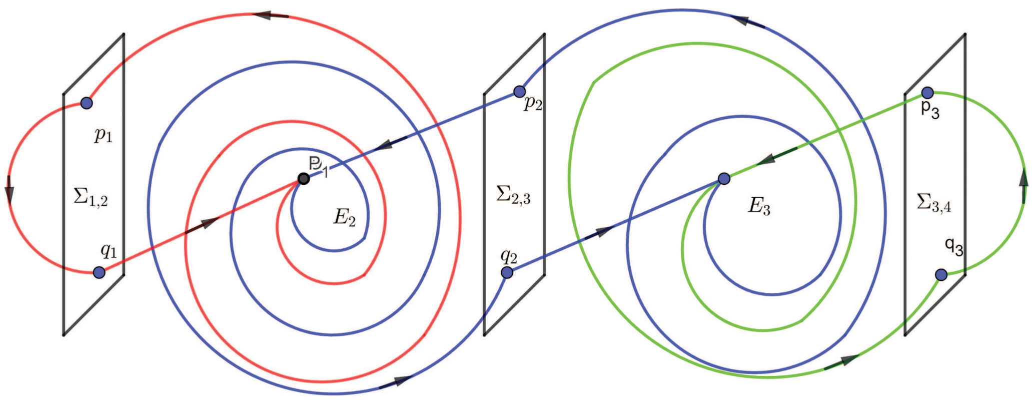

FIG. 4. Schematic diagram of the co-existence of the heteroclinic cycle Γ_1, the homoclinic orbit Γ_2, and the homoclinic orbit Γ_3 satisfying Theorem 3.4. The blue line represents the heteroclinic cycle Γ_1, and the red and green lines represent the homoclinic orbit Γ_2 and the homoclinic orbit Γ_3, respectively.

Theorem 3.4. *Suppose that conditions of Theorems* 3.1–3.3 *all hold, then the heteroclinic cycle* Γ_1 *in Theorem* 3.1, *the homoclinic orbit* Γ_2 *in Theorem* 3.2 *and the homoclinic orbit* Γ_3 *in Theorem* 3.3 *of system* (1) *coexist. In addition, they transversally intersect the switching manifolds* $\Sigma_{2,3}$, $\Sigma_{1,2}$, *and* $\Sigma_{3,4}$ *at points* p_2, q_2, p_1, q_1, p_3, *and* q_3 *respectively, as shown in Fig.* 4.

Proof. Combining with Theorems 3.1–3.3, we can easily get the proof of Theorem 3.4. For the sake of simplicity, we will not repeat them in detail.

Thus, the proof of Theorem 3.4 is completed. □

Remark 3.3. *Same as Remarks 3.1 and 3.2, in Sec. II, regarding the position of equilibrium* E_1 *and* E_4, *it is assumed that* E_1 *is located in* S_1 *and* E_4 *is located in* S_1 *for convenience. In fact, even if* E_1 *and* E_4 *are the virtual equilibrium, where* E_1 *lies in* S_2, S_3, *or* S_4 *and* E_4 *lies in* S_1, S_2, *or* S_3, *we can get something similar to Theorem 3.4.*

IV. NUMERICAL SIMULATIONS FOR THEORETICAL RESULTS AND CHAOS

In this section, the correctness of theorems is verified by some numerical simulations of a specific case that conforms to theorems.

Considering the 3D PWL system with four regions,

$$A_1 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 2 & -2 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -5 & -4 & -1 \\ 4/3 & -11/3 & -12 \\ -248/9 & 28/9 & -13/3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -39/2 & -10 & -1/4 \\ 29/2 & 21 & -131/4 \\ -21 & 20 & -59/2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$a_1 = \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; a_3 = \begin{pmatrix} 39 \\ -29 \\ 42 \end{pmatrix}, \; a_4 = \begin{pmatrix} -7/2 \\ 127/20 \\ 183/20 \end{pmatrix},$$

$$c' = (c_0, \quad c_1, c_2) = (1, 0, 0),\ d_1 = -1, d_2 = 1, d_3 = 3. \qquad (33)$$

The equilibria of the four subsystems are

$$E_1 = \begin{pmatrix} -10 \\ -5 \\ -15 \end{pmatrix} \in S_1, \quad E_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in S_2,$$

$$E_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \in S_3, \quad E_4 = \begin{pmatrix} 563/20 \\ 127/20 \\ 183/20 \end{pmatrix} \in S_4.$$

There are invertible matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ -2 & 1 & -1 \\ 0 & -2 & -1 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 1/2 & 1/2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -2 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that

$$J_1 = P_1^{-1} A_1 P_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_2 = P_2^{-1} A_2 P_2 = \begin{pmatrix} 1 & -10 & 0 \\ 10 & 1 & 0 \\ 0 & 0 & -15 \end{pmatrix},$$

$$J_3 = P_3^{-1} A_3 P_3 = \begin{pmatrix} 1 & -20 & 0 \\ 20 & 1 & 0 \\ 0 & 0 & -30 \end{pmatrix},$$

$$J_4 = P_4^{-1} A_4 P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

There exist

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} k_1' \\ k_2' \\ k_3' \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} l_1 \\ l_2 \\ T_0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \ln 2 \end{pmatrix}, \quad \begin{pmatrix} l_1' \\ l_2' \\ T_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \ln \dfrac{1293}{829} \end{pmatrix},$$

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 3/2 \end{pmatrix}, \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} \tau_1' \\ \tau_2' \\ \tau_3' \end{pmatrix} = \begin{pmatrix} \dfrac{-1577}{321} \\ \dfrac{-751}{180} \\ \dfrac{-4918}{441} \end{pmatrix}, \quad \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} \dfrac{-613}{80} \\ \dfrac{-107}{40} \\ \dfrac{-143}{20} \end{pmatrix},$$

then

$$q_1 = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}, \quad p_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \; q_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix},$$

$$p_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}, \; p_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix},$$

$$e^{J_{A_1} T_0} \begin{pmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \quad e^{J_{A_4} T_4} \begin{pmatrix} \tau_1' \\ \tau_2' \\ \tau_3' \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix},$$

$$c' P_1 J_{A_1} \begin{pmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \end{pmatrix} = -3 < 0, \quad c' P_1 J_{A_1} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 3 > 0,$$

$$c' P_4 J_{A_4} \begin{pmatrix} \tau_1' \\ \tau_2' \\ \tau_3' \end{pmatrix} = \frac{1974}{359} > 0, \quad c' P_4 J_{A_4} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = -\frac{11}{2} < 0,$$

$$M_1 e^{-\alpha_{A_2} T_1} \frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} < d_2 - c' E_2,$$

$$M_1 e^{-\alpha_{A_2} T_1'} \frac{-\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} > d_1 - c' E_2,$$

$$M_2 e^{-\alpha_{A_3} T_2} \frac{-\beta_{A_3}}{\sqrt{\alpha_{A_3}^2 + \beta_{A_3}^2}} > d_2 - c' E_3,$$

$$M_2 e^{-\alpha_{A_3} T_2'} \frac{\beta_{A_3}}{\sqrt{\alpha_{A_3}^2 + \beta_{A_3}^2}} < d_3 - c' E_3,$$

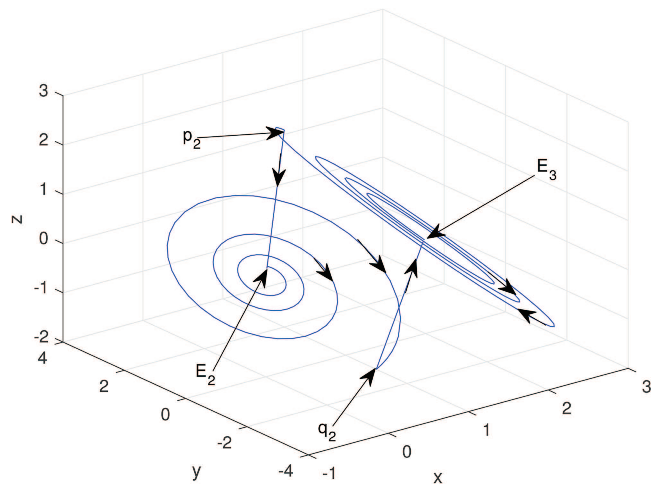

FIG. 5. Phase diagram of the heteroclinic cycle Γ_1 satisfying Theorem 3.1.

$$M_3 e^{-\alpha_{A_2} T_3} \frac{-\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} > d_1 - c' E_2,$$

$$M_3 e^{-\alpha_{A_2} T_3'} \frac{\beta_{A_2}}{\sqrt{\alpha_{A_2}^2 + \beta_{A_2}^2}} < d_2 - c' E_2,$$

$$M_4 e^{-\alpha_{A_3} T_5} \frac{-\beta_{A_3}}{\sqrt{\alpha_{A_3}^2 + \beta_{A_3}^2}} < d_3 - c' E_3,$$

$$M_4 e^{-\alpha_{A_3} T_5'} \frac{\beta_{A_3}}{\sqrt{\alpha_{A_3}^2 + \beta_{A_3}^2}} > d_2 - c' E_3,$$

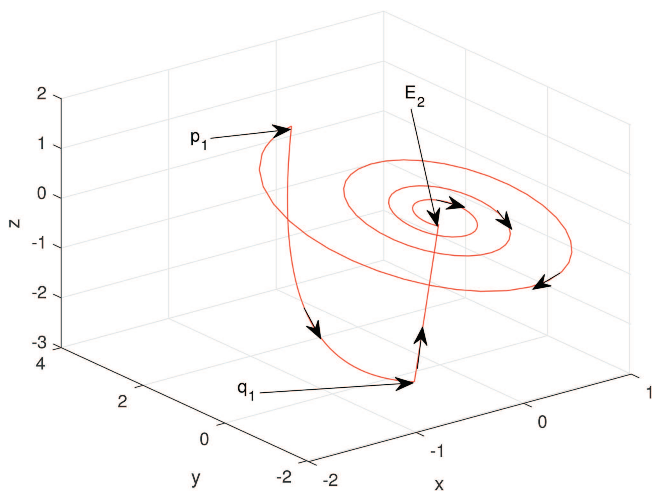

FIG. 6. Phase diagram of the homoclinic orbit Γ_2 satisfying Theorem 3.2.

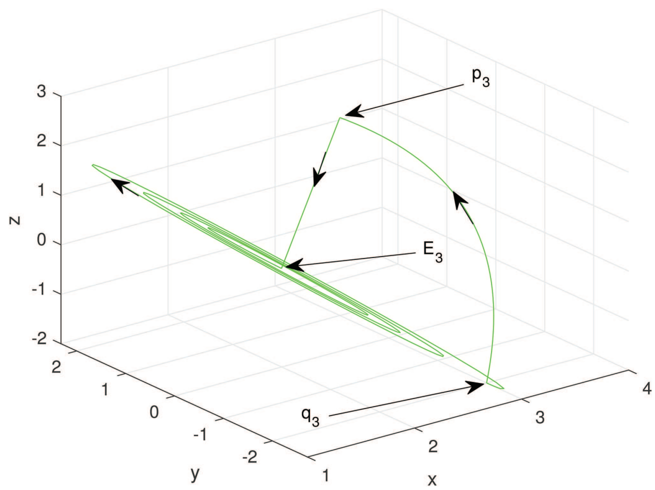

FIG. 7. Phase diagram of the homoclinic orbit Γ_3 satisfying Theorem 3.3.

and

$$\alpha_{A_2}(d_2 - c_0x_{E_2} - c_1y_{E_2} - c_2z_{E_2}) + \beta_{A_2}(m_2k_1 - m_1k_2) = 1 > 0,$$
$$\alpha_{A_3}(d_2 - c_0x_{E_3} - c_1y_{E_3} - c_2z_{E_3}) + \beta_{A_3}(m_2'k_1' - m_1'k_2') = -1 < 0,$$
$$\alpha_{A_2}(d_1 - c_0x_{E_2} - c_1y_{E_2} - c_2z_{E_2}) + \beta_{A_2}(m_2''l_1 - m_1''l_2) = -1 < 0,$$
$$\alpha_{A_3}(d_3 - c_0x_{E_3} - c_1y_{E_3} - c_2z_{E_3}) + \beta_{A_3}(m_2'''l_1' - m_1'''l_2') = 1 > 0.$$

Therefore, this instance satisfies Theorem 3.1, the heteroclinic cycle Γ_1 of this PWL system exists and Γ_1 transversally intersects the switching manifold $\Sigma_{2,3}$ at points p_2, q_2, as shown in Fig. 5. The case satisfies Theorem 3.2 too. So, the homoclinic orbit Γ_2 of the

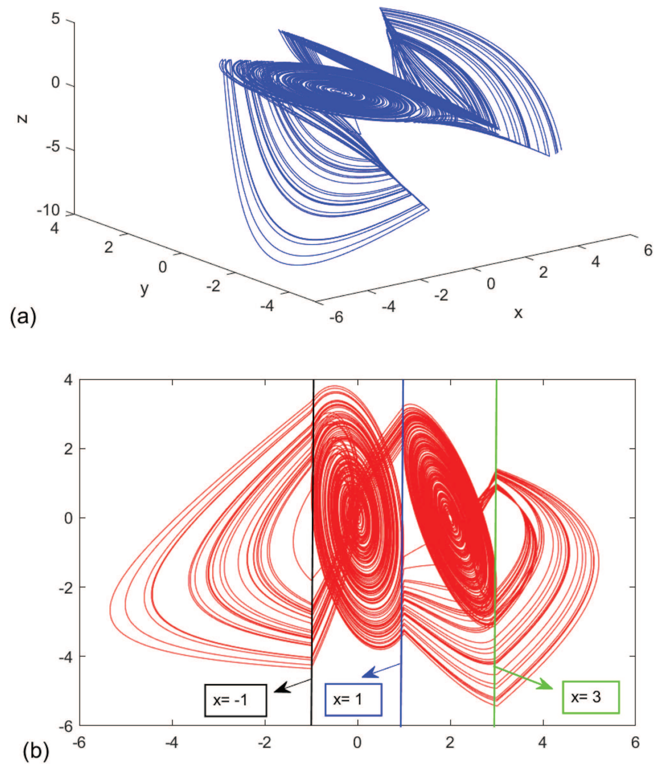

FIG. 9. Chaos in system (33) satisfying Theorem 3.4 when the initial point is $(3, -2.1, -2)$ near q_3: (a) phase diagram in the x–y–z space and (b) projection of (a) on the x–y plane.

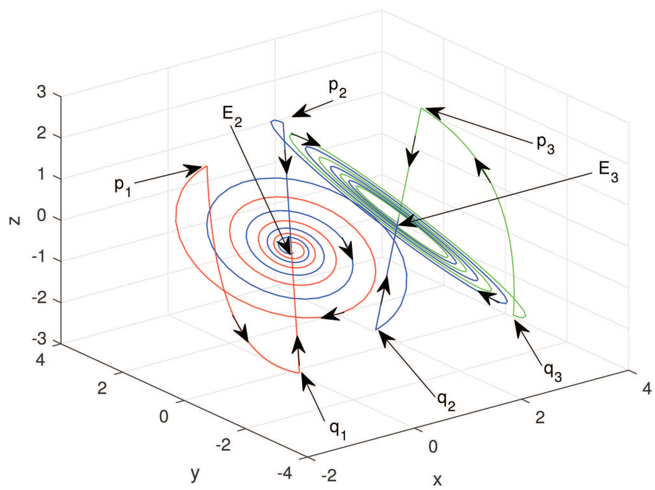

FIG. 8. Phase diagram of the coexistence of the heteroclinic cycle Γ_1, the homoclinic orbit Γ_2, and the homoclinic orbit Γ_3 in system (33) satisfying Theorem 3.4. Among them, the blue line is heteroclinic cycle, and the red and green lines are homoclinic orbits.

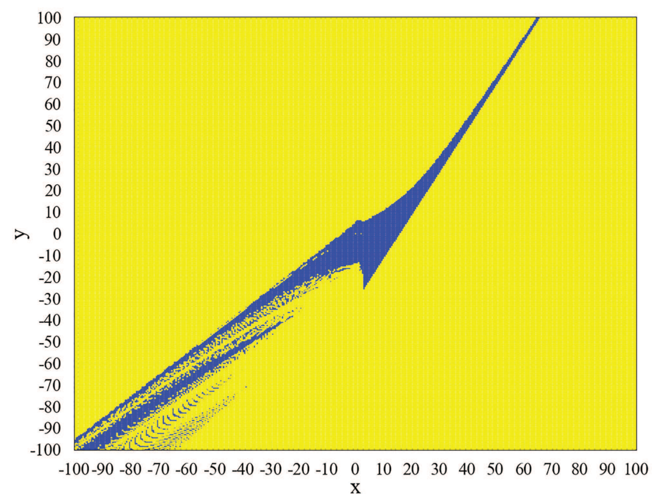

FIG. 10. The blue dots on the plane $\{X \in \mathbb{R}^3 : z = 0\}$ have been evaluated and belong to the basin of attraction.

system exists and Γ_2 transversally intersects the switching manifold $\Sigma_{1,2}$ at points p_1 and q_1, as shown in Fig. 6. The example also satisfies Theorem 3.3, the homoclinic orbit Γ_3 of the PWL systems exists, and Γ_3 transversally intersects the switching manifold $\Sigma_{3,4}$ at points p_3 and q_3, as shown in Fig. 7. It is clear that Theorem 3.4 holds for the system, then the heteroclinic cycle Γ_1 and the homoclinic orbits Γ_2 and Γ_3 coexist, as shown in Fig. 8. Furthermore, the chaotic invariant set is shown in Fig. 9. The largest Lyapunov exponent with the Wolf's algorithm is 1.039 for $t \in [0, 967\,000]$ with a RK4 with a step of 0.001.

A subset of the basin of attraction on the plane $\{X \in \mathbb{R}^3 | z = 0\}$ has been numerically found and it is shown in Fig. 10. The blue dots are initial conditions that belong to the basin of attraction while the yellow dots do not.

V. CONCLUSION

This paper introduces a new class of 3D PWL systems with four regions. The analysis on the existence of homoclinic orbits or heteroclinic cycles is presented with four subsystems. We establish sufficient conditions for the coexistence of homoclinic orbits and heteroclinic cycle of 3D PWL systems by rigorous proof. A numerical example with homoclinic orbits, heteroclinic cycle, and chaos is given to illustrate the validity of the presented method and the obtained theoretical results. In addition, the basin of attraction for the chaotic attractor is given. However, for higher-dimensional PWL systems with more switching manifolds, the existence of homoclinic orbit, heteroclinic cycle, and chaos needs further study.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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