JYU DISSERTATIONS 335

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Quasisymmetric Uniformization via Metric Doubling Measures

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Esitetään Jyväskylän yliopiston matemaattis-luonnontieteellisen tiedekunnan suostumuksella julkisesti tarkastettavaksi joulukuun 12. päivänä 2020 kello 12.

Academic dissertation to be publicly discussed, by permission of the Faculty of Mathematics and Science of the University of Jyväskylä,, on December 12, 2020 at 12 o'clock noon.

JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

JYVÄSKYLÄ 2020

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Permanent link to this publication: http://urn.fi/URN:ISBN:978-951-39-8357-4

ISBN 978-951-39-8357-4 (PDF)
URN:ISBN:978-951-39-8357-4
ISSN 2489-9003

Acknowledgements

I wish to express my deepest gratitude for my advisor Kai Rajala for his guidance, hospitality and endless patience over the past years. I also want to thank everyone at the department for the great environment and for being a part of my journey towards this dissertation.

I thank my collaborators Atte Lohvansuu and Matthew Romney and professors Sergiy Merenkov and Hrant Hakobyan for their careful pre-examination of this thesis. I have received financial support for this work from the Department of Mathematics and Statistics and the Academy of Finland.

Finally, I want to thank my family and my wife Minna for all their love, support and encouragement.

Espoo, November 23, 2020
Martti Rasimus

List of included articles

This dissertation consists of an introductory part and the following three articles:

[A] Atte Lohvansuu, Kai Rajala and Martti Rasimus, *Quasispheres and metric doubling measures*, Proc. Amer. Math. Soc. 146 (2018), no. 7, 2973–2984.

[B] Kai Rajala, Martti Rasimus and Matthew Romney, *Uniformization with infinitesimally metric measures*, arXiv e-prints (2019), arXiv:1907.07124.

[C] Kai Rajala and Martti Rasimus, *Quasisymmetric Koebe uniformization with weak metric doubling measures*, arXiv e-prints (2020), arXiv:2005.01700.

The author of this dissertation has actively taken part in the research of the joint articles [A], [B] and [C].

INTRODUCTION

In this thesis we study uniformization problems, which are central in analysis in metric spaces. Our focus lies in the two-dimensional case, where we give characterizations for the existence of a quasisymmetric (or quasiconformal) mapping from a given metric space $X = (X, d)$ to the Euclidean plane $\mathbb{R}^2$ or sphere $\mathbb{S}^2$.

Given necessary topological and geometric conditions, our main results show that such a map exists exactly when X carries a measure μ that deforms the metric in a suitably controlled manner (see Sections 3 and 5). If we know that a quasisymmetric map $f : X \to \mathbb{R}^2$ exists, then $\mu = f^* m_2$, i.e. the pullback of the Lebesgue measure under f, has this property.

In the other direction, we show that a given μ induces a new metric q on X which is quasisymmetrically equivalent to d and also has strong geometric properties which allow the application of existing uniformization tools to (X, q). These combined with geometric estimates involving μ are powerful enough to guarantee the existence of the desired quasisymmetric map f.

Our results generalize earlier uniformization theorems concerning Ahlfors 2-regular spaces X. The novelty of our method is that it applies to the fractal case where (X, d) has Hausdorff dimension strictly greater than two. Finding quasisymmetric parametrizations in the fractal setting is among the most important open problems in analysis in metric spaces. We next give some more background.

1. Uniformization of metric spaces

In the classical setting the Riemann mapping theorem tells us that every simply connected domain in the complex plane other than the whole plane can be mapped conformally onto the unit disk. More generally, the uniformization theorem states that each simply connected Riemann surface is conformally equivalent to either the Riemann sphere, complex plane or the unit disk.

Motivated largely by the need for similar results in more general contexts, a rich theory of geometric analysis has been developed. Much of the theory is concerned with finding suitable parametrizations, or uniformizations, for metric spaces satisfying varying assumptions. Typically one seeks to classify spaces with similar geometries, and furthermore to quantify the differences between these geometries.

For example, conformal mappings preserve infinitesimal shapes, whereas *quasiconformal mappings* are allowed to distort them by a bounded amount (see Definition 5.5). Our main focus is in estimating deformations in the global scale, and for this purpose the correct mapping class is that of *quasisymmetric mappings.*

Definition 1.1. A homeomorphism $f \colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is quasisymmetric or η-quasisymmetric if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such that for all $x, y, z \in X$ and $t \geq 0$

with

$$d_X(x, y) \leq t d_X(x, z)$$

we have

$$d_Y(f(x), f(z)) \leq \eta(t) d_Y(f(x), f(z)).$$

We call η the distortion function of a η-quasisymmetric mapping. Metric spaces X and Y are called quasisymmetrically equivalent if there exists a quasiymmetric mapping f from X onto Y. It is easy to see that inverses and compositions of quasisymmetric mappings are also quasisymmetric. Thus it is a natural question to classify metric spaces up to quasisymmetric equivalence. See Chapters 10 and 11 in [16] for more properties of quasisymmetric mappings.

In the general metric setting, where there is no given differential structure, quasisymmetric mappings offer a natural generalization to conformal mappings. They are global versions of quasiconformal mappings in the sense that they distort relative distances in a bounded way at all scales. Quasisymmetric mappings have several useful geometric properties and are also more flexible than bi-Lipschitz mappings, which must preserve also absolute distances up to a multiplicative factor.

Quasisymmetric mappings were first studied in the general metric setting by Tukia and Väisälä [33]. The definition originates from the work of Beurling and Ahlfors [1] on the boundary behavior of two-dimensional quasiconformal maps. In their article, Tukia and Väisälä established fundamental properties of quasisymmetric mappings. In particular, they gave the first uniformization result involving them: a full characterization of quasisymmetric circles in terms of the following intrinsic properties of the given metric space.

Definition 1.2. A metric space (X, d) is *doubling* if there exists a constant $N \in \mathbb{N}$ such that every ball $B(x, r) \subset X$ can be covered with at most N balls $B(x_i, r/2)$, $i = 1, \ldots, N$.

Recall that a *continuum* is a connected and compact set containing more than one point.

Definition 1.3. A metric space (X, d) is *linearly locally connected* or *LLC* if there exists a constant $\lambda \geq 1$ such that the following properties hold

- For any $x \in X$, $r > 0$ and $y, z \in B(x, r)$ there exists a continuum $K \subset B(x, \lambda r)$ with $y, z \in K$.
- For any $x \in X$, $r > 0$ and $y, z \in X \setminus B(x, r)$ there exists a continuum $K \subset X \setminus B(x, r/\lambda)$ with $y, z \in K$.

THEOREM 1.4 ([33], Theorem 4.9)**.** *A metric space X homeomorphic to $\mathbb{S}^1 = \{z \in \mathbb{R}^2 : |z| = 1\}$ is quasisymmetrically equivalent to $\mathbb{S}^1$ equipped with the Euclidean metric if and only if it is doubling and linearly locally connected.*

Tukia and Väisälä formulated this result with slightly different but equivalent assumptions on the space. See [29] and [19] for more theory and classification of bi-Lipschitz equivalent quasisymmetric circles.

The doubling and LLC properties are both preserved by quasisymmetric mappings. Thus they are a natural starting point for quasisymmetric uniformization problems, as many of the standard model spaces such as the Euclidean spaces, balls and spheres satisfy these properties.

The theory of quasisymmetric mappings between metric spaces has grown rapidly after the work of Tukia and Väisälä. A central problem is to find extensions of Theorem 1.4 to more general metric spaces. In particular, one seeks to characterize spaces quasisymmetrically equivalent to $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ equipped with the Euclidean metric for $n \geq 2$. This problem has proved to be extremely challenging, and a full characterization is still missing in spite of extensive efforts during the last 25 years.

As stated above, it is necessary for a space quasisymmetrically equivalent to $\mathbb{S}^n$ or $\mathbb{R}^n$ to be doubling and LLC, but these properties are not sufficient. A fundamental counterexample is the Rickman rug: Let $X = \mathbb{R}^2$, $\varepsilon \in (0,1)$, and consider the product metric

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|^{\varepsilon}$$

for $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. This space can be realized as a metric product of the Euclidean real line and the (unbounded) *von Koch snowflake curve*, which is quasisymmetrically equivalent to $\mathbb{R}$. The Rickman rug is doubling and LLC but not quasisymmetrically equivalent to the Euclidean plane. Roughly speaking, the reason for this is that the two curves in the product are intrinsically different although quasisymmetrically equivalent.

In dimensions three and higher there are quite well-behaved spaces without quasisymmetric parametrizations. For each $n \geq 3$ there exist spaces homeomorphic to $\mathbb{S}^n$ that are doubling, LLC, smooth outside small singular sets and with Euclidean-type mass bounds but for which there exist no quasisymmetric mapping onto $\mathbb{S}^n$, see [32], [18], [27] and [26]. In contrast, a fundamental theorem of Bonk and Kleiner [5] shows that such examples cannot exist in dimension two. More precisely, the following holds.

Definition 1.5. Let (X,d) be a metric space. A Borel measure μ on X is Ahlfors regular of dimension Q or Q-regular if there exists a constant $C > 0$ such that

$$r^Q/C \leq \mu(B(x,r)) \leq C r^Q$$

for every $x \in X$ and $0 < r < \operatorname{diam} X$. The space is called Q-regular if it supports a Q-regular measure.

THEOREM 1.6 ([5] Theorem 1.1)**.** *Suppose (X,d) is homeomorphic to $\mathbb{S}^2$ and Ahlfors 2-regular. Then (X,d) is quasisymmetrically equivalent to $\mathbb{S}^2$ if and only if it is linearly locally connected.*

In [5] Bonk and Kleiner also gave a necessary and sufficient condition for spaces quasisymmetrically equivalent to $\mathbb{S}^2$ in terms of a combinatorially defined modulus. The precise statement of this condition is however quite technical and not easily applicable. Similar result for Ahlfors 2-regular spaces homeomorphic to $\mathbb{R}^2$ was given by Wildrick [34].

Ahlfors Q-regular spaces are in particular doubling and have Hausdorff dimension Q. We call a metric space (X, d) homeomorphic to $\mathbb{S}^n$ or $\mathbb{R}^n$ *fractal* if it has Hausdorff dimension strictly greater than n. Fractal spaces are far from being Ahlfors n-regular, and already in dimension two this poses great difficulty for finding a characterization for quasisymmetric parametrization that is verifiable in concrete settings. Examples of such fractals are the snowflake curve and the Rickman rug mentioned above.

The quasisymmetric uniformization problem in the two-dimensional fractal case is both difficult and highly important due to connections to different branches of analysis. In addition to spaces homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$, also different carpets, such as spaces homeomorphic to the standard Sierpiński carpet, are extensively studied in terms of quasisymmetric equivalence (see for example [3] and [14]). A better understanding of the problem also provides information on questions in geometric group theory (see for example [6], [2] and [7]) and complex dynamics (see for example [13] and [8]).

In geometric group theory a major open problem directly tied to quasisymmetric uniformization is Cannon's conjecture. In the original form the conjecture states that if the boundary at infinity of a Gromov hyperbolic group G is homeomorphic to $\mathbb{S}^2$, then G acts properly discontinuously, cocompactly and isometrically on the three dimensional hyperbolic space $\mathbb{H}^3$. It follows from results by Sullivan and Tukia that the conjecture is equivalent with the statement that if the boundary at infinity of a Gromov hyperbolic group is homeomorphic to $\mathbb{S}^2$, then it is quasisymmetrically equivalent to $\mathbb{S}^2$. This boundary has a natural family of so-called visual metrics which are all quasisymmetrically equivalent. The resulting metric spaces are always LLC but typically fractal. Therefore the conjecture is a natural motivation and also one of the greatest goals in the search of tools for quasisymmetric parametrizations of fractal spheres. See for example [2] for more details on Cannon's conjecture and Gromov hyperbolic groups.

2. Strong A_∞ weights and metric doubling measures

Our approach to the uniformization problem is to consider deformations of the space with a suitable *doubling measure*. Recall that a Radon measure μ on a metric space (X, d) is doubling if there exists a constant $C \geq 1$ such that for every ball $B(x, r) \subset X$ we have

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty. \tag{1}$$

The idea of inducing a new geometry on a metric space using a doubling measure was first introduced by David and Semmes [10]. Originally they studied this in the form of *strong A_∞ weights*. Recall that a non-negative, locally integrable function ω on $\mathbb{R}^n$ is an A_p weight for $p \in [1, \infty]$ if

- $p = \infty$ and there exist constants $\gamma, C > 0$ such that for every ball $B \subset \mathbb{R}^n$ and measurable subset $E \subset B$
$$\frac{\int_E \omega}{\int_B \omega} \leq C \left(\frac{m_n(E)}{m_n(B)}\right)^{\gamma},$$
- $1 < p < \infty$ and there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{R}^n$
$$\left(\fint_B \omega\right)\left(\fint_B \omega^{\frac{1}{1-p}}\right)^{p-1} \leq C$$
or
- $p = 1$ and there exists a constant such that for every ball $B \subset \mathbb{R}^n$
$$\fint_B \omega \leq C \operatorname{essinf}_B \omega.$$

Here $\fint_A \omega$ is the integral average $\frac{1}{m_n(A)} \int_A \omega$ of ω over a measurable set $A \subset \mathbb{R}^n$ with finite and positive measure. Here we denote m_n for the Lebesgue measure in $\mathbb{R}^n$.

A_p weights satisfy $A_q \subset A_r$ for $1 \leq q \leq r \leq \infty$ and $A_\infty = \cup_{p \geq 1} A_p$. The family of A_∞ weights is also characterized by the following *reverse Hölder inequality*: $\omega \in A_\infty$ if and only if there exists $C > 0$ and $p > 1$ such that for every ball $B \subset \mathbb{R}^n$
$$\left(\fint_B \omega^p\right)^{\frac{1}{p}} \leq C \fint_B \omega.$$

If ω is an A_∞ weight, then $\mu = \omega\, dm_n$ is a doubling measure on $\mathbb{R}^n$. David and Semmes study the geometry given by this type of doubling measure via the *quasimetric* function
$$D_\mu(x,y) = \mu(B_{xy})^{1/n}.$$
Here and later we denote $B_{xy} = B(x, d(x,y)) \cup B(y, d(x,y))$ in any metric space (X, d).

An A_∞ weight ω is called a strong A_∞ weight if there exists a metric d_ω on $\mathbb{R}^n$ comparable to $D_\mu = D_{\omega dm_n}$, that is
$$\frac{1}{C} d_\omega(x,y) \leq D_\mu(x,y) \leq C d_\omega(x,y) \tag{2}$$
for some $C \geq 1$ and all $x, y \in \mathbb{R}^n$. The main examples of strong A_∞ weights are A_1 weights and Jacobian determinants of quasiconformal mappings on $\mathbb{R}^n$. The class of A_p weights is intimately connected to the quasiconformal Jacobian problem, see for example [4].

If μ is a doubling measure such that D_μ is comparable to a metric on $\mathbb{R}^n$, then μ is necessarily absolutely continuous and has an A_∞ weight ω as a density, see [12] and [31]. Using the reverse Hölder inequality one can also show that the first inequality in (2) always holds for any A_∞ weight ω

and for the geodesic distance q_ω associated with ω. In the case when ω is continuous, this geodesic distance can be realized by

$$q_\omega(x,y) = \inf \int_\gamma \omega^{1/n}\, ds,$$

where the infimum is taken over all rectifiable paths γ connecting x and y. We give a definition in Section 3 for the general case. See [31] for more properties and results related to strong A_∞ weights, and [9] and [23] for more recent results on these weights in the metric setting.

Strong A_∞ weights correspond to *metric doubling measures*. We give a general definition in the metric setting.

Definition 2.1. A doubling measure μ on an Ahlfors n-regular metric space (X,d) is a metric doubling measure if there exists a metric d_μ and a constant $C \geq 1$ such that

$$\frac{1}{C} d_\mu(x,y) \leq \mu(B_{xy})^{1/n} \leq C d_\mu(x,y)$$

for every $x, y \in X$.

Metric doubing measures are naturally related to quasisymmetric mappings. In particular, if $f\colon X \to Y$ is a quasisymmetric map between n-regular spaces, then the pullback $f^*\mathcal{H}^n$ of the Hausdorff n-measure $\mathcal{H}^n$ is a metric doubling measure.

More generally we can consider the map $D_{\mu,s}(x,y) = \mu(B_{xy})^{1/s}$ for a given $s > 0$ and a doubling measure μ similarly as in $\mathbb{R}^n$, with no regularity assumption on the metric space X. This map $D_{\mu,s}$ is a *quasimetric* on X for any doubling measure μ. This means that there exists a constant $K \geq 1$ such that

- $D_{\mu,s}(x,y) = D_{\mu,s}(y,x) \geq 0$ for all $x, y \in X$,
- $D_{\mu,s}(x,y) = 0$ if only if $x = y$, and
- $D_{\mu,s}(x,y) \leq K(D_{\mu,s}(x,z) + D_{\mu,s}(z,y))$ for all $x, y, z \in X$.

The last condition means that $D_{\mu,s}$ satisfies the usual triangle inequality up to a multiplicative constant, and this follows easily from the doubling condition (1). Note that if $K = 1$, then this condition is the usual triangle inequality.

The reason for considering $D_{\mu,s}$ is that this is a way of introducing a new geometry for any metric space starting from a doubling measure μ. Indeed, when s is large enough depending on the doubling constant of μ, then D_μ is comparable to a genuine metric. That is, there exists a metric d_μ on X and a constant $C \geq 1$ such that

$$\frac{1}{C} d_\mu(x,y) \leq D_{\mu,s}(x,y) \leq C d_\mu(x,y) \tag{3}$$

for all $x, y \in X$. If X is moreover connected, then the metric space (X, d_μ) is Ahlfors s-regular and the identity mapping $\mathrm{id}\colon (X,d) \to (X,d_\mu)$ is quasisymmetric. When X is LLC and homeomorphic to $\mathbb{S}^2$, the best deformation one can hope for is a μ on X for which (3) holds with $s = 2$. In this case (X, d_μ) is

2-regular and thus quasisymmetrically equivalent to $\mathbb{S}^2$ by the Bonk-Kleiner Theorem 1.6. See Chapter 16 in [11] and Chapter 14 in [16] for proofs of the above claims and more basic properties of quasimetrics.

3. Weak metric doubling measures

We would like to generalize the idea of metric doubling measures to fractal spaces as a tool for studying their geometry. In [A] and [C] we show that a necessary and sufficient condition for a quasisymmetric uniformization of certain fractal spaces is the existence of a *weak metric doubling measure.*

We now give a version of the geodesic distance associated with a doubling measure discussed in Section 2. Let (X, d) be a metric space and $x, y \in X$. We call a finite sequence of points $x_0, \dots, x_m$ a *δ-chain from x to y*, if $\delta > 0$, $x_0 = x$, $x_m = y$ and $d(x_j, x_{j+1}) \leq \delta$ for every $j = 0, \dots, m-1$. It is easy to see that if X is connected, then any pair $x, y \in X$ can be connected by a δ-chain for any $\delta > 0$.

Definition 3.1. Let (X, d) be a connected metric space, μ a doubling measure on X and $s > 0$. The *μ-length* $q = q_{\mu,s}$ between two points $x, y \in X$ is

$$q_{\mu,s}(x, y) = \limsup_{\delta \to 0} q^{\delta}_{\mu,s}(x, y),$$

where

$$q^{\delta}_{\mu,s}(x, y) = \inf \sum_j \mu(B_{x_j x_{j+1}})^{1/s}$$

and the infimum is taken over all δ-chains $(x_j)_j$ from x to y.

It follows from the definition that q is symmetric and satisfies the triangle inequality, but $q(x, y) \in (0, \infty)$ for $x \neq y$ may fail. If ω is a strong A_∞ weight on $\mathbb{R}^n$, then any metric d_ω satisfying (2) is comparable to $q_{\mu,n}$ where $\mu = \omega \, dm_n$. This fact is the motivation behind the following definition.

Definition 3.2. Let X, d be a connected metric space and $s > 0$. A doubling measure μ on X is a C_W-weak metric doubling measure of dimension s if

$$\mu(B_{xy})^{1/s} \leq C_W q_{\mu,s}(x, y) \tag{4}$$

for all $x, y \in X$.

As mentioned in Section 2, a metric space homeomorphic to $\mathbb{S}^2$ is quasisymmetrically equivalent to $\mathbb{S}^2$ if and only if it is LLC and there exists a metric doubling measure on X. As the main result in [A] we show that this characterization also holds with only assuming the existence of a weak metric doubling measure.

THEOREM 3.3 ([A] Theorem 1.2)**.** *Let (X, d) be a metric space homeomorphic to $\mathbb{S}^2$. Then (X, d) is quasisymmetrically equivalent to $\mathbb{S}^2$ if and only if it is linearly locally connected and there exists a weak metric doubling measure μ of dimension 2 on X.*

We prove Theorem 3.3 by showing that also the reverse inequality of (4) holds under these assumptions. This implies that $q = q_{\mu,2}$ is a metric and μ is a metric doubling measure on the deformed space (X, q). The uniformization of (X, d) then follows from the Bonk-Kleiner Theorem 1.6 since the metrics q and d are quasisymmetrically equivalent as discussed above.

The proof relies on the LLC condition and the topology and separation properties of $\mathbb{S}^2$. In [C] we generalize the result to surfaces homeomorphic to finitely connected planar domains, and in this case we need to utilize a quasiconformal uniformization rather than showing a global reverse of (4).

Our proof applies only in the two-dimensional case. One step in the proof is separating the center of a ball from its complement by continuum that contains a chain of points with controlled μ-length. This estimate is a form of coarea formula in metric spaces with dimension and codimension both equal to one. Extending our proof to higher dimensions would require a suitable analogue of this estimate. Recall from Section 1 that the Bonk-Kleiner Theorem 1.6 does not hold in higher dimensions. Thus a natural generalization of our result to higher dimension would be the following problem of minimizing the *conformal dimension* (see [24]) of a space with a weak metric doubling measure. Recall that a metric space Z is *linearly locally contractible* if there exists $\lambda' \geq 1$ such that every ball $B(z, r) \subset Z$ with $r < \operatorname{diam} Z/\lambda'$ is contractible in $B(z, \lambda' r)$.

Question 3.4. Let (X, d) be a linearly locally contractible metric space homeomorphic to $\mathbb{R}^n$ or $\mathbb{S}^n$ with $n \geq 3$ and suppose there exists a weak metric doubling measure of dimension n on X. Is there a n-regular metric on X quasisymmetrically equivalent d?

4. Quasisymmetric Koebe uniformization

The classical Koebe conjecture [21] states that every domain in the complex plane is conformally equivalent to a *circle domain*. A circle domain is a domain in the Riemann sphere $\mathbb{S}^2$ such that every component of its boundary is either a circle or a point. The conjecture was confirmed by Koebe [22] in the case of finite number of complementary components, and by He and Schramm [15] in the case of countably many complementary components.

Merenkov and Wildrick [25] gave a characterization for metric spaces quasisymmetrically equivalent to a circle domain assuming Ahlfors 2-regularity and a bound on the relative accumulation of boundary components of the space. In particular, their result implies that if a 2-regular space is homeomorphic to a domain in $\mathbb{S}^2$ and has finitely many boundary components, it is quasisymmetrically equivalent to a circle domain if and only if it is LLC and has compact completion. In [C] we generalize this result using weak metric doubling measures, with no assumption on the regularity of the given metric.

We denote by $\overline{X}$ the metric completion of a metric space X and call $\partial X = \overline{X} \setminus X$ the metric boundary of X.

THEOREM 4.1 ([C] Theorem 1.1). *Let X be a metric space homeomorphic to a domain in $\mathbb{S}^2$ such that $\overline{X} \setminus X$ contains finitely many components. Then*

X is quasisymmetrically equivalent to a circle domain if and only if it is LLC, carries a weak metric doubling measure of dimension 2 and has compact completion. The distortion function η related to the quasisymmetry condition only depends on the data of μ and X.

By the data of μ and X we mean the constants in the LLC, doubling and weak metric doubling measure conditions and the number and minimal relative distance of the boundary components of X.

Our strategy is to first deform the space with the μ-length q as in [A], and then apply a quasiconformal uniformization on the deformed space (X, q). This is guaranteed by showing that (X, q) is *reciprocal* and applying the recent works by Rajala [28], Romney [30] and Ikonen [20]. We then show that the quasiconformal mapping from (X, q) onto a circle domain thus obtained is in fact quasisymmetric in terms of the original metric d.

In [25] Merenkov and Wildrick give a counterexample for the quasisymmetric uniformization of a general metric surface with countably many boundary components. In this space a necessary control on the accumulation of the boundary components fails. A natural follow-up problem to our result is whether a weak metric doubling measure is sufficient for the uniformization of surfaces with countably many boundary components, for which some steps of our proof fail.

Question 4.2. Does a version of 4.1 hold for spaces with countably many boundary components?

5. μ-quasiconformal mappings and infinitesimally metric measures

In [B] we consider infinitesimal versions of the previous methods and introduce the concept of *μ-quasiconformal mappings*, where μ is a Radon measure on the given metric space. Similarly as in Section 4 we would like to use the theory of quasiconformal mappings in metric spaces in order to find quasisymmetric parametrizations. A fundamental difficulty for this method in fractal spaces is the lack of rectifiable paths. These are needed for the use of the *conformal modulus*, a powerful geometric tool and a starting point in the geometric definition of quasiconformal mappings.

We propose a different approach to defining the modulus of a path family and quasiconformal mappings by using a measure as follows. For simplicity we discuss only the two-dimensional case as in [B], and assume throughout this section that (X, d) is a metric space homeomorphic to $\mathbb{R}^2$.

Definition 5.1. Let μ be a Radon measure on X and $\mathcal{B}$ a collection of balls with the following property: For every $x \in X$ there exists $r_x > 0$ such that $B(x, r) \in \mathcal{B}$ for every $r \in (0, r_x)$. The *μ-length measure* of a Borel set $A \subset X$ is

$$\ell_\mu(A) = \limsup_{\delta \to 0} \inf\{\sum_j \mu(B_j)^{1/2}\},$$

where the infimum is taken over all δ-covers $(B_j)_j \subset \mathcal{B}$ of A, i.e. sequences of balls from $\mathcal{B}$ with diameter less than δ and whose union covers A.

If μ is 2-regular, then ℓ_μ is clearly comparable to the Hausdorff 1-measure $\mathcal{H}^1$. Recall that using $\mathcal{H}^1$ or arc length in place of ℓ_μ and the Lebesgue measure m_2 in place of μ in the following definition gives the standard conformal 2-modulus mod_2 in $\mathbb{R}^2$.

Definition 5.2. Let Γ be a family of curves on X. The *μ-modulus* of Γ is

$$\mathrm{mod}_\mu(\Gamma) = \inf \int_X \rho^2 \, d\mu,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0, \infty]$ satisfying $\int_\gamma \rho \, d\ell_\mu \geq 1$ for all $\gamma \in \Gamma$ with locally finite ℓ_μ-measure.

Our next definition generalizes the notion of quasiconformal mappings using the μ-modulus. For a mapping f and a curve family Γ we denote $f\Gamma = \{f(\gamma) : \gamma \in \Gamma\}$.

Definition 5.3. Let $f\colon X \to \Omega \subset \mathbb{R}^2$ be a homeomorphism. The mappings f and f^{-1} are μ-quasiconformal, if there exists $K \geq 1$ such that

$$\frac{1}{K} \mathrm{mod}_\mu(\Gamma) \leq \mathrm{mod}_2(f\Gamma) \leq K \, \mathrm{mod}_\mu(\Gamma)$$

for every curve family Γ in X.

Recall that using the standard conformal modulus mod_2 in place of mod_μ in Definition 5.3 gives the standard geometric definition of quasiconformal mappings.

Using these definitions we are able to extend tools from quasiconformal analysis to fractal spaces. A first step in this direction is to determine when a μ-quasiconformal mapping exists. As our first main result in [B] we show that a sufficient condition is given by *infinitesimally metric measures*. In short these are measures for which the geodesic distance q defined in terms of ℓ_μ is comparable to $\mu^{1/2}$ at the infinitesimal scale. Hence they can be seen as infinitesimal versions of the metric doubling measures of David and Semmes. For the formal definition see 3.1 in [B].

THEOREM 5.4 ([B] Theorem 1.1)**.** *If μ is an infinitesimally metric measure on X, then there exists a μ-quasiconformal map $f\colon X \to \Omega \subset \mathbb{R}^2$.*

Deforming the space similarly as in the previous sections, we get a space that is *infinitesimally Ahlfors 2-regular*. This means that the Hausdorff 2-measure of a ball $B(x, r)$ in (X, q) is comparable to r^2 for small r, depending on the point x. We show that this is enough for reciprocality (see again [28]) and thus there exists a quasiconformal mapping f from (X, q) that is furthermore μ-quasiconformal as a map from (X, d, μ).

Next we study the metric properties of μ-quasiconformal mappings. The global quasisymmetry condition turns out to be too strong a conclusion under only our infinitesimal assumptions. A natural alternative would be to consider the metric definition of quasiconformality, which is an infinitesimal property.

Definition 5.5. A homeomorphism $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is metrically H-quasiconformal if

$$H_f(x,r) = \limsup_{r\to 0} \frac{\sup_{d_X(x,y)\le r} d_Y(f(y), f(x))}{\inf_{d_X(x,y)\ge r} d_Y(f(y), f(x))} \le H \quad \text{for all } x \in X.$$

Clearly quasisymmetric mappings are metrically quasiconformal. Recall also that in $\mathbb{R}^n$ the geometric and metric definitions of quasiconformal mappings agree. Though easier to state, the metric definition is hard to use in practice and too weak as a basis for doing analysis on general metric spaces. We introduce *infinitesimally quasisymmetric mappings* as an intermediate class between metrically quasiconformal and quasisymmetric mappings.

Definition 5.6. A homeomorphism $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is infinitesimally quasisymmetric if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ and for every $x \in X$ there exists a radius $r_x > 0$ such that if $y, z \in B(x, r_x)$ and $t \ge 0$ with

$$d_X(x,y) \le t d_X(x,z)$$

then

$$d_Y(f(x), f(z)) \le \eta(t) d_Y(f(x), f(z)).$$

In our second main result of [B] we characterize spaces for which there exists an infinitesimally quasisymmetric mapping into $\mathbb{R}^2$. As necessary and sufficient conditions we introduce infinitesimal versions of the LLC condition and the *Loewner property* coined by Heinonen and Koskela [17].

THEOREM 5.7 ([B] Theorem 1.2)**.** *There exists an infinitesimally quasisymmetric mapping $f\colon X \to \Omega \subset \mathbb{R}^2$ if and only if X is infinitesimally LLC and supports an infinitesimally metric measure μ such that (X, μ) is infinitesimally Loewner.*

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Included articles

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Quasispheres and metric doubling measures

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First published in *Proc. Amer. Math. Soc.* 146 (2018), no. 7,
published by the American Mathematical Society.

QUASISPHERES AND METRIC DOUBLING MEASURES

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Abstract. Applying the Bonk-Kleiner characterization of Ahlfors 2-regular quasispheres, we show that a metric two-sphere X is a quasisphere if and only if X is linearly locally connected and carries a *weak metric doubling measure*, i.e., a measure that deforms the metric on X without much shrinking.

1. Introduction

A homeomorphism $f\colon (X,d) \to (Y,d')$ between metric spaces is *quasisymmetric*, if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d(x_1,x_2)}{d(x_1,x_3)} \leqslant t \text{ implies } \frac{d'(f(x_1),f(x_2))}{d'(f(x_1),f(x_3))} \leqslant \eta(t)$$

for all distinct $x_1,x_2,x_3 \in X$. Applying the definition with $t=1$ shows that quasisymmetric homeomorphisms map all balls to sets that are uniformly round. Therefore, the condition of quasisymmetry can be seen as a global version of conformality or quasiconformality.

Starting from the work of Tukia and Väisälä [26], a rich theory of quasisymmetric maps between metric spaces has been developed. An overarching problem is to characterize the metric spaces that can be mapped to a given space S by a quasisymmetric map.

This problem is particularly appealing when S is the two-sphere $\mathbb{S}^2$. There are connections to geometric group theory, (cf. [3], [5], [6]), complex dynamics ([7], [8], [13]), as well as minimal surfaces ([17]).

Bonk and Kleiner [4] solved the problem in the setting of two-spheres with "controlled geometry", see also [17], [18], [22], [23], [29]. We say that (X,d) is a *quasisphere*, if there is a quasisymmetric map from (X,d) to $\mathbb{S}^2$. See Section 2 for further definitions.

THEOREM 1.1 ([4], Theorem 1.1)**.** *Suppose (X,d) is homeomorphic to $\mathbb{S}^2$ and Ahlfors 2-regular. Then (X,d) is a quasisphere if and only if it is linearly locally connected.*

Research supported by the Academy of Finland, project number 308659.
2010 Mathematics Subject Classification. Primary 30L10, Secondary 30C65, 28A75.

Finding generalizations of the Bonk-Kleiner theorem beyond the Ahlfors 2-regular case and to fractal surfaces is important; applications include Cannon's conjecture on hyperbolic groups, cf. [2], [16] (by [9] the boundary of a hyperbolic group is Ahlfors Q-regular with Q greater than or equal to the topological dimension of the boundary). A characterization of general quasispheres in terms of combinatorial modulus is given in [4, Theorem 11.1]. However, this result is difficult to apply in practice and in fact an easily applicable characterization is not likely to exist. Several types of fractal quasispheres have been found (cf. [1], [12], [19], [27], [28], [30]), showing the difficulty of the problem.

In this paper we characterize quasispheres in terms of a condition related to *metric doubling measures* of David and Semmes [10], [11]. These are measures that deform a given metric in a controlled manner. More precisely, a (doubling) Borel measure μ is a metric doubling measure of dimension 2 on (X, d) if there is a metric q on X and $C \geqslant 1$ such that for all $x, y \in X$,

$$C^{-1}\mu(B(x, d(x, y)))^{1/2} \leqslant q(x, y) \leqslant C\mu(B(x, d(x, y)))^{1/2}. \tag{1}$$

It is well-known that metric doubling measures induce quasisymmetric maps $(X, d) \to (X, q)$. Our main result shows that quasispheres can be characterized using a weaker condition where we basically only assume the first inequality of (1). We call measures satisfying such a condition *weak metric doubling measures*, see Section 2.

THEOREM 1.2. *Let (X, d) be a metric space homeomorphic to $\mathbb{S}^2$. Then (X, d) is a quasisphere if and only if it is linearly locally connected and carries a weak metric doubling measure of dimension* 2*.*

To prove Theorem 1.2 we show, roughly speaking, that the first inequality in (1) actually implies the second inequality. It follows that μ induces a quasisymmetric map $(X, d) \to (X, q)$, and (X, q) is 2-regular and linearly locally connected. Applying Theorem 1.1 to (X, q) and composing then gives the desired quasisymmetric map. It would be interesting to find higher-dimensional as well as quasiconformal versions of Theorem 1.2. See Section 6 for further discussion.

2. Preliminaries

We first give precise definitions. Let $X = (X, d)$ be a metric space. As usual, $B(x, r)$ is the open ball in X with center x and radius r, and $S(x, r)$ is the set of points whose distance to x equals r.

We say that X is λ-*linearly locally connected* (LLC), if for any $x \in X$ and $r > 0$ it is possible to join any two points in $B(x, r)$ with a continuum in $B(x, \lambda r)$, and any two points in $X \setminus B(x, r)$ with a continuum in $X \setminus B(x, r/\lambda)$.

A Radon measure μ on X is *doubling*, if there exists a constant $C_D \geqslant 1$ such that for all $x \in X$ and $R > 0$,

$$\mu(B(x,2R)) \leqslant C_D \mu(B(x,R)), \tag{2}$$

and *Ahlfors s-regular*, $s > 0$, if there exists a constant $A \geqslant 1$ such that for all $x \in X$ and $0 < R < \operatorname{diam} X$,

$$A^{-1}R^s \leqslant \mu(B(x,R)) \leqslant AR^s.$$

Moreover, X is Ahlfors s-regular if it carries an s-regular measure μ.

We now define weak metric doubling measures. In what follows, we use notation $B_{xy} = B(x, d(x,y)) \cup B(y, d(x,y))$.

Let μ be a doubling measure on (X,d). For $x, y \in X$ and $\delta > 0$, a finite sequence of points $x_0, x_1, \ldots, x_m$ in X is a *δ-chain from x to y*, if $x_0 = x$, $x_m = y$ and $d(x_j, x_{j-1}) \leqslant \delta$ for every $j = 1, \ldots, m$.

Now fix $s > 0$ and define a "μ-length" $q_{\mu,s}$ as follows: set

$$q^{\delta}_{\mu,s}(x,y) := \inf \left\{ \sum_{j=1}^{m} \mu(B_{x_j x_{j-1}})^{1/s} \colon (x_j)_{j=0}^m \text{ is a } \delta\text{-chain from } x \text{ to } y \right\}$$

and

$$q_{\mu,s}(x,y) := \limsup_{\delta \to 0} q^{\delta}_{\mu,s}(x,y) \in [0,\infty].$$

Definition 2.1. A doubling measure μ on (X,d) is a *weak metric doubling measure of dimension s*, if there exists $C_W \geqslant 1$ such that for all $x, y \in X$,

$$\frac{1}{C_W}\mu(B_{xy})^{1/s} \leqslant q_{\mu,s}(x,y). \tag{3}$$

In what follows, if the dimension s is not specified then it is understood that $s = 2$, and $q_{\mu,2}$ is shortened to q_μ.

3. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2, assuming Proposition 3.1 to be proved in the following sections. First, it is not difficult to see that if there exists a quasisymmetric map $\varphi : X \to \mathbb{S}^2$, then X is LLC, and

$$\mu(E) := \mathcal{H}^2(\varphi(E))$$

defines a weak metric doubling measure on X. Therefore, the actual content in the proof of Theorem 1.2 is the existence of a quasisymmetric parametrization, assuming LLC and the existence of a weak metric doubling measure (of dimension 2). The proof is based on the following result.

Proposition 3.1. *Let (X,d) be LLC and homeomorphic to $\mathbb{S}^2$. Moreover, assume that (X,d) carries a weak metric doubling measure μ of dimension 2. Then q_μ is a metric on X and μ is a metric doubling measure in (X,q_μ), that is there exists a constant $C_S \geqslant 1$ such that also the bound*

$$q_\mu(x,y) \leqslant C_S \mu(B_{xy})^{1/2}$$

holds for all $x,y \in X$.

We will apply the well-known growth estimates for doubling measures. The proof is left as an exercise, see [14, ex. 13.1].

Lemma 3.2. *Let X be as in Proposition 3.1 and let μ be a doubling measure on X. Then there exist constants $C,\alpha > 1$ depending only on the doubling constant C_D of μ such that*

$$\frac{\mu(B(x,r_2))}{\mu(B(x,r_1))} \leqslant C \max\left\{ \left(\frac{r_2}{r_1}\right)^{\alpha}, \left(\frac{r_2}{r_1}\right)^{1/\alpha} \right\}$$

for all $0 < r_1, r_2 < \operatorname{diam}(X)$.

Combining Proposition 3.1 and Lemma 3.2 shows that q_μ induces a quasisymmetric map. This is essentially Proposition 14.14 of [14]. We include a proof for completeness.

Corollary 3.3. *Let X and μ be as in Proposition 3.1. Then the identity mapping $i\colon (X,d) \to (X,q_\mu)$ is quasisymmetric, and (X,q_μ) is Ahlfors 2-regular.*

Proof. We denote $q = q_\mu$. We first show that i is a homeomorphism. Since (X,d) is a compact metric space, it suffices to show that i is continuous, i.e., that any q-ball $B^q(x,r)$ contains a d-ball $B^d(x,\delta)$ for some $\delta = \delta(x,r)$. Suppose that this does not hold for some $x \in X$ and $r > 0$. Then there exists a sequence $(x_n)_{n=1}^\infty$ such that $d(x_n,x) \to 0$ but $q(x_n,x) \geqslant r$ for all $n \in \mathbb{N}$. Now Proposition 3.1 implies

$$r \leqslant q(x_n,x) \leqslant C\mu(B^d(x,2d(x,x_n)))^{1/2} \overset{n\to\infty}{\longrightarrow} 0,$$

which is a contradiction. Thus i is a homeomorphism. Let $x,y,z \in X$ be distinct. By Proposition 3.1 and Lemma 3.2 we have

$$\frac{q(x,y)}{q(x,z)} \leqslant C\frac{\mu(B_{xy})^{1/2}}{\mu(B_{xz})^{1/2}} \leqslant C\frac{\mu(B(x,2d(x,y)))^{1/2}}{\mu(B(x,2d(x,z)))^{1/2}} \leqslant \eta\left(\frac{d(x,y)}{d(x,z)}\right),$$

where $\eta : [0,\infty) \to [0,\infty)$ is the homeomorphism

$$\eta(t) = C\max\{t^{\alpha/2}, t^{1/2\alpha}\}.$$

Thus i is η-quasisymmetric.

We next claim that μ is Ahlfors 2-regular on (X, q). Fix $x \in X$ and $0 < r < \operatorname{diam}(X, q)/10$. Since (X, q) is connected, there exists $y \in S^q(x, r)$. Now by Proposition 3.1,

$$C_S^{-2} r^2 \leqslant \mu(B_{xy}) \leqslant C_W^2 r^2.$$

On the other hand, the quasisymmetry of the identity map i and the doubling property of μ give

$$C^{-1}\mu(B^q(x, r)) \leqslant \mu(B_{xy}) \leqslant C\mu(B^q(x, r)),$$

where C depends only on C_D and η. Combining the estimates gives the 2-regularity. $\square$

We are now ready to finish the proof of Theorem 1.2, modulo Proposition 3.1. Indeed, Corollary 3.3 shows that there is a quasisymmetric map from (X, d) onto the 2-regular (X, q_μ). It is not difficult to see that the quasisymmetric image of a LLC space is also LLC. Hence, by Theorem 1.1, there exists a quasisymmetric map from (X, q_μ) onto $\mathbb{S}^2$. Since the composition of two quasisymmetric maps is quasisymmetric, Theorem 1.2 follows.

4. Separating chains in annuli

We prove Proposition 3.1 in two parts. In this section we find short chains in annuli (Lemma 4.3). In the next section we take suitable unions of these chains to connect given points.

We first show that it suffices to consider δ-chains with sufficiently small δ. In what follows, we use notation

$$cB_{xy} = B(x, cd(x, y)) \cup B(y, cd(x, y)).$$

Lemma 4.1. *Let (X, d) be a compact, connected metric space admitting a weak metric doubling measure μ of some dimension $s > 0$. Then for any $r > 0$ there exists $\delta_r > 0$ such that if $x, y \in X$ with $d(x, y) \geqslant r$ then we have*

$$2C_W C_D^{2/s} q_{\mu,s}^{\delta_r}(x, y) \geqslant \mu(B_{xy})^{1/s}, \tag{4}$$

where C_W and C_D are the constants in (3) *and* (2)*, respectively.*

Proof. Suppose to the contrary that (4) does not hold for some $r > 0$. Then there exists a sequence of pairs of points $(x_j, y_j)_j$ for which $d(x_j, y_j) \geqslant r$ and

$$q_{\mu,s}^{1/j}(x_j, y_j) < \frac{1}{2C_W C_D^{2/s}} \mu(B_{x_j y_j})^{1/s}$$

for all $j = 1, 2, 3, \ldots$. Then by compactness we can, after passing to a subsequence, assume that $x_j \to x$ and $y_j \to y$ where also $d(x, y) \geqslant r$. Let then $k \in \mathbb{N}$ be arbitrary and $j \geqslant k$ so large that $B_{x_j y_j} \subset 4B_{xy}$,

$$d(x, x_j), d(y, y_j) \leqslant \frac{1}{k}$$

and

$$\mu(B_{xx_j})^{1/s} + \mu(B_{yy_j})^{1/s} < \frac{1}{3C_W}\mu(B_{xy})^{1/s}. \tag{5}$$

The last estimate is made possible by the fact that $\mu(\{z\}) = 0$ for every point z in the case of a doubling measure and a connected space, or more generally when the space is *uniformly perfect* (see [11, 5.3 and 16.2]). Now choose a $\frac{1}{j}$-chain $z_0, \ldots, z_m$ from x_j to y_j satisfying

$$\sum_{i=1}^{m} \mu(B_{z_i z_{i-1}})^{1/s} < \frac{1}{2C_W C_D^{2/s}} \mu(B_{x_j y_j})^{1/s} \leqslant \frac{1}{2C_W} \mu(B_{xy})^{1/s} \tag{6}$$

so that $x, z_0, \ldots, z_m, y$ is in particular a $\frac{1}{k}$-chain from x to y. Combining (5) and (6), we have

$$q_{\mu,s}^{1/k}(x, y) < \frac{5}{6C_W}\mu(B_{xy})^{1/s}.$$

This contradicts (3) when $k \to \infty$. □

In what follows, we will abuse terminology by using a non-standard definition for separating sets.

Definition 4.2. Given $A, B, K \subset X$, we say that K *separates* A and B if there are distinct connected components U and V of $X \setminus K$ such that $A \subset U$ and $B \subset V$.

Lemma 4.3. *Suppose (X, d) is λ-LLC and homeomorphic to $\mathbb{S}^2$, and μ a weak metric doubling measure on X. Let k be the smallest integer such that $2^k > \lambda$. Then there exists $C > 1$ depending only on λ, C_D and C_W such that for any $x \in X, 0 < r < 2^{-8k}\operatorname{diam} X$ and $\delta > 0$ there exists a δ-chain $x_0, \ldots, x_p$ in the annulus $\overline{B(x, 2^{5k}r)} \setminus B(x, 2^{2k}r)$ such that*

$$\sum_{j=1}^{p} \mu(B_{x_j x_{j-1}})^{1/2} \leqslant C\mu(B(x, r))^{1/2}$$

and the union $\cup_j \overline{5B_{x_j x_{j-1}}}$ contains a continuum separating $B(x, r)$ and $X \setminus \overline{B(x, 2^{7k}r)}$.

Proof. Let $x \in X, 0 < r < 2^{-8k}\operatorname{diam} X$ and $\delta > 0$ be arbitrary. By Lemma 4.1 we may assume without loss of generality that

$$q_\mu^\delta(y,z) \geqslant \frac{1}{C'}\mu(B_{yz})^{1/2} \tag{7}$$

for any $y \in S(x, 2^{3k}r), z \in S(x, 2^{4k}r)$ and also $\delta < r$ by finding a finer chain than possibly asked.

Next we cover the annulus $A = \overline{B(x, 2^{5k}r)} \setminus B(x, 2^{2k}r)$ as follows: Let $\varepsilon > 0$ be small enough so that $\mu(B(w, \delta/10)) > \varepsilon^2$ for every $w \in X$ (see again [11, 16.2]). Then for every $w \in A$ we can choose a radius $0 < r_w < \delta/10$ with

$$\frac{\varepsilon^2}{2C_D} \leqslant \mu(B(w, r_w)) \leqslant \varepsilon^2.$$

Using the $5r$-covering theorem, we find a finite number m of pairwise disjoint balls $B_j = B(w_j, r_j)$, $r_j = r_{w_j}$ from the cover $\{B(w, r_w)\}_{w\in A}$, such that

$$A \subset \bigcup_{j=1}^{m} 5B_j \subset B(x, 2^{6k}r) \setminus \overline{B(x, 2^k r)}.$$

Observe that for any point z in the thinner annulus $A' = \overline{B(x, 2^{4k}r)} \setminus B(x, 2^{3k}r)$ there exists a continuum in A joining z to some point $y \in S(x, 2^{3k}r)$ by the LLC-property. Hence there exists a subcollection $B'_1, \ldots, B'_n$ of the cover $(5B_j)$ forming a ball chain from this y to z, meaning that $y \in B'_1, z \in B'_n$ and $B'_j \cap B'_{j+1} \neq \emptyset$. Thus we can define a “counting” function u for this cover on A' by setting $u(z)$ to be the smallest $n \in \{1, \ldots, m\}$ so that there exists a ball chain $(B'_i)_{i=1}^n$ from some $y \in S(x, 2^{3k}r)$ to z.

Using (7), we find a lower bound for u on $S(x, 2^{4k}r)$: Let $y \in S(x, 2^{3k}r), z \in S(x, 2^{4k}r)$ be arbitrary and $(B'_i)_{i=1}^n = (B(w'_i, 5r'_i))_{i=1}^n$ the corresponding chain. Then $y = w'_0, w'_1, \ldots, w'_n, z = w'_{n+1}$ is also a δ-chain. Hence

$$\mu(B_{yz})^{1/2} \leqslant C' \sum_{i=1}^{n+1} \mu(B_{w'_i w'_{i-1}})^{1/2} \leqslant C' C_D^3 n\varepsilon$$

as every $B_{w'_i w'_{i-1}}$ is contained in $B(w'_l, 20 r_{w'_l})$, $l = i$ or $i-1$. On the other hand $B(x, 2^{7k}r) \subset B(y, 2^{7k+1}r)$, and since the balls B_j are disjoint,

$$m\varepsilon^2 \leqslant \mu(B(y, 2^{7k+1}r)) \leqslant C_D^{7k+1}\mu(B_{yz}),$$

implying $n^2 \geqslant m/C''$ or $u(z) \geqslant \sqrt{m}/C''$.

Let then n be the minimal value of u on $S(x, 2^{4k}r)$ and for $j = 1, 2, \ldots, n$ define

$$A_j = \bigcup_{5B_i \cap u^{-1}(j) \neq \emptyset} 5B_i.$$

By the definition of u each ball $5B_i$ can be contained in at most two "level sets" A_j and so we obtain a constant $C \geqslant 1$ such that

$$\begin{aligned}
\min_{1 \leqslant j \leqslant n} \sum_{5B_i \subset A_j} \mu(5B_i)^{1/2} &\leqslant \frac{1}{n} \sum_{j=1}^{n} \sum_{5B_i \subset A_j} \mu(5B_i)^{1/2} \\
&\leqslant \frac{1}{n} C_D^3 \varepsilon \cdot 2m \\
&\leqslant 2C_D^3 \frac{\sqrt{m}}{n} \sqrt{\varepsilon^2 m} \\
&\leqslant C\mu(B(x,r))^{1/2}.
\end{aligned}$$

Let $j \in \{1, \ldots, n\}$ be the index for which the above left hand sum is smallest. Since by construction A_j necessarily intersects any curve joining $B(x, 2^k r)$ and $X \setminus \overline{B(x, 2^{6k} r)}$, it separates $B(x,r)$ and $X \setminus \overline{B(x, 2^{7k} r)}$ by the LLC-property as $2^k > \lambda$. Hence the closed set $\overline{A_j}$ contains a continuum K separating these sets by topology of $\mathbb{S}^2$, see for example [20] V 14.3.. Now K is covered by a ball chain $\overline{B(w_0', 5r_0')}, \ldots, \overline{B(w_p', 5r_p')}$ of closures of balls $5B_i$ contained in A_j. Hence these points $w_0', \ldots, w_p'$ are the desired δ-chain, since clearly $d(w_i', w_{i+1}') \leqslant 5r_i' + 5r_{i+1}' < \delta$ and

$$\sum_{i=1}^{p} \mu(B_{w_i' w_{i-1}'})^{1/2} \leqslant C\mu(B(x,r))^{1/2}$$

by our choice of j. □

Remark 4.4. Note that in the claim of the above lemma the constant C is uniform with respect to the required step δ of the chain; we can in fact find arbitrarily fine chains and have the same estimate from above for $\sum \mu(B_j)^{1/2}$. This is essentially obtained by the doubling property and the $5r$-covering theorem. We also work with dimension $s = 2$, since passing from the lower estimate of 4.1 to the upper in the claim we actually switch the power $1/s$ of the measure to $(s-1)/s$, both $1/2$ in the proof. Thus this argument seems not to apply for higher dimension (see Question 6.3). Moreover the topology of $\mathbb{S}^2$ is used for finding a single separating component, which is not always possible for example on a torus.

5. Proof of Proposition 3.1

In this section (X, d, μ) satisfies the assumptions of Proposition 3.1. Lemma 4.3 and the $5r$-covering lemma then give the following: For any given $B = B(x, R) \subset X$ and $\delta > 0$ there is a cover of the x-component U of B by at most $M = M(\lambda, C_D, L)$ balls $\{B_i\}_{i=1}^m$ with centers in U such that for every i

(1) $L^{-2}\mu(B) \leqslant \mu(B_i) \leqslant L^{-1}\mu(B)$
(2) A continuum $K_i \subset \overline{2^{7k}B_i} \setminus B_i$ separates B_i and $X \setminus \overline{2^{7k}B_i}$
(3) $K_i \subset \bigcup_p \overline{5B_{x_p^i x_{p-1}^i}}$, where $(x_p^i)_p$ is a δ-chain
(4) $\sum_p \mu(B_{x_p^i x_{p-1}^i})^{1/2} \leqslant C\mu(B_i)^{1/2}$.

Here k is as in Lemma 4.3, $L > C_D^{8k}$ and $C = C(\lambda, C_D, C_W)$.

We would like to take unions of the continua K_i to join points. However, the union $\cup_i K_i$ need not be a connected set. The following lemma takes care of this problem. We denote by $\hat{K}_i$ the *interior* of K_i, i.e., the component of $X \setminus K_i$ that contains B_i.

Lemma 5.1. *Let $i \in \{1, 2\}$. Let $B_i = B(x_i, r_i) \subset X$ be a (small) ball and let $K_i \subset \overline{2^{7k}B_i} \setminus B_i$ be a continuum that separates B_i and $X \setminus \overline{2^{7k}B_i}$. Suppose $\hat{K}_1 \cap \hat{K}_2 \neq \emptyset$. If $K_1 \cap K_2 = \emptyset$, then either $K_1 \subset \hat{K}_2$ and $\hat{K}_1 \subset \hat{K}_2$ or $K_2 \subset \hat{K}_1$ and $\hat{K}_2 \subset \hat{K}_1$.*

Proof. Since X is homeomorphic to $\mathbb{S}^2$, path components of an open set in X are exactly its components. In addition such components are open. Since K_1 and K_2 are nonempty disjoint compact sets, there exist path connected open sets $U_1, U_2 \subset X$ such that $K_1 \subset U_1 \subset X \setminus K_2$ and $K_2 \subset U_2 \subset X \setminus K_1$. Let $w \in \hat{K}_1 \cap \hat{K}_2$. Let $\gamma : [0, 1] \to X$ be a path from w to $z \in X \setminus (\overline{2^{7k}B_1} \cup \overline{2^{7k}B_2})$. By the separation properties $\gamma([0, 1])$ intersects K_1 and K_2. Let

$$s = \inf\{t \in [0, 1] \mid \gamma(t) \in K_1 \cup K_2\}.$$

Now $s > 0$ and $\tilde{\gamma} := \gamma|_{[0,s]}$ is a path that intersects $K_1 \cup K_2$ exactly once. Without loss of generality we may assume $\gamma(s) \in K_1$. By construction of U_1 the point w can be connected to any point in K_1 inside $X \setminus K_2$. Thus $K_1 \subset \hat{K}_2$. Now let $y \in \hat{K}_1$. It suffices to show that there exists a path in $\hat{K}_2$ from y to w. Suppose there is no such path. Now the argument of the first part of this proof implies that $K_2 \subset \hat{K}_1$. Let S be the number obtained by changing the infimum in the definition of s to the respective supremum. Necessarily $\gamma(S) \in K_2$, since otherwise we could construct a path in $\hat{K}_2$ from w to z. Since $K_2 \subset U_2 \subset \hat{K}_1$, there exists a path connecting w to $\gamma(S)$ in $\hat{K}_1$, i.e., there exists a path from w to z in $\hat{K}_1$, which is impossible. Thus $\hat{K}_1 \subset \hat{K}_2$. □

Motivated by Lemma 5.1 we say that a continuum K_i is *maximal* (in $\{K_i\}_{i=1}^m$) if it is not contained in the interior of some other K_j. Define K to be the union of all maximal continua in $\{K_i\}_{i=1}^m$. Clearly K is compact. Let us show that it is also connected. Suppose K_i and K_j are distinct maximal continua. Let $B_{(i)}$ and $B_{(j)}$ be the balls in $\{B_i\}$ that are contained in the interiors $\hat{K}_i$ and $\hat{K}_j$, respectively. Since $\{B_i\}$ is a cover of the x-component of B, we can find a chain of balls in $\{B_i\}$ connecting any pair of points in the component. On the other hand, every ball B_i intersects the x-component, so it suffices to consider the case where $B_{(i)} \cap B_{(j)} \neq \emptyset$. By Lemma 5.1 either $K_i \cap K_j \neq \emptyset$ or we may assume that $K_i \subset \hat{K}_j$, but the latter contradicts maximality. Thus K is a continuum. We have now proved the following proposition.

Proposition 5.2. *Fix $L > C_D^{8k}$, $\delta > 0$, and $B = B(x, R) \subset X$. Then there are at most $M = M(\lambda, C_D, L) < \infty$ balls B_i centered at the x-component U of B such that*

(1) $U \subset \cup_i B_i$
(2) $\mu(B_i) \leqslant \frac{1}{L}\mu(B)$ *for all* i
(3) *For every* i *there is a continuum* $K_i \subset \overline{2^{7k}B_i} \setminus B_i$ *which separates* B_i *and* $X \setminus \overline{2^{7k}B_i}$
(4) $K_i \subset \bigcup_p \overline{5B_{x_p^i x_{p-1}^i}}$, *where* $(x_p^i)_p$ *is a finite* δ*-chain*
(5) $\sum_p \mu(B_{x_p^i x_{p-1}^i})^{1/2} \leqslant C\mu(B)^{1/2}$, $C = C(\lambda, C_D, C_W)$
(6) *the union* K *of all maximal continua in* $\{K_i\}$ *is a continuum.*

Now we can finish the proof of Proposition 3.1 with the following:

Lemma 5.3. *There exists a constant $C = C(\lambda, C_D, C_W)$ such that for any $\delta > 0$ and $x, y \in X$,*

$$q_\mu^\delta(x, y) \leqslant C\mu(B_{xy})^{1/2}.$$

Proof. Fix $x, y \in X$ and apply Proposition 5.2 to $B^1 = B(x, 2^{2k}d(x, y))$ with $L = C_D^{15k}$. Note that x and y belong to the same component of B^1. Let $z = x$ or $z = y$. Let us define balls $B^{l,z}$ recursively for $l \geqslant 2$. Define $B^{1,z} = B^1$. Suppose we have defined the set $B^{n,z}$ for all $n \leqslant l$. Apply Proposition 5.2 with the same L to $B^{l,z}$ to find a ball $B_j^{l,z}$ which contains z. By Lemma 5.1 $B_j^{l,z}$ is contained in the interior of some maximal continuum $K_{j'}^{l,z}$. Define $B^{l+1,z} = 2^{7k}B_{j'}^{l,z}$. Note that Proposition 5.2 also yields the balls $B^{n,z}$ and $B_i^{n,z}$ and continua $K_i^{n,z}$

and $K^{n,z}$. Also, by the separation properties and Lemma 5.1

$$z \in B_j^{l,z} \subset \hat{K}_j^{l,z} \subset \hat{K}_{j'}^{l,z} \subset 2^{7k} B_{j'}^{l,z} = B^{l+1,z}.$$

Let $\varepsilon > 0$ and let $B_z = B(z, r_z)$ be a ball with $r_z \leqslant 6\delta$ and $\mu(B_z) \leqslant C_D^{-1}\varepsilon^2$. Define

$$K_z := \bigcup_{n=1}^{l_z^\varepsilon} K^{n,z}$$

where l_z^ε is the smallest integer l that satisfies $K^{l,z} \subset B(z, 100^{-1} r_z)$. Such a number exists, since $z \in B^{l,z}$ for all l. Moreover, our choice of L gives $C_D^{7k} L^{-1} = \tau < 1$ and

$$\mu(B^{l,z}) \leqslant C_D^{7k} L^{-1} \mu(B^{l-1,z}) \leqslant \tau \mu(B^{l-1,z}) \leqslant \ldots \leqslant \tau^{(l-1)} \mu(B^1). \tag{8}$$

In particular, $\operatorname{diam}(B^{l,z}) \overset{l\to\infty}{\longrightarrow} 0$. We next show that K_z is a continuum. It is clearly compact, and connectedness follows if

$$K^{n,z} \cap K^{n+1,z} \neq \emptyset. \tag{9}$$

Let j be the index for which $2^{7k} B_j^{n,z} = B^{n+1,z}$. To show (9) it suffices to show that $K_j^{n,z} \cap K_i^{n+1,z} \neq \emptyset$ for some maximal $K_i^{n+1,z}$. By Lemma 5.1 there exists a maximal continuum $K_i^{n+1,z}$ such that the interiors of $K_i^{n+1,z}$ and $K_j^{n,z}$ intersect. Moreover either (9) holds or one of $K_i^{n+1,z} \subset \hat{K}_j^{n,z}$, $K_j^{n,z} \subset \hat{K}_i^{n+1,z}$ is true for any such i. Suppose $K_j^{n,z} \subset \hat{K}_i^{n+1,z}$. By separation properties $B_j^{n,z} \subset 2^{7k} B_i^{n+1,z}$, which together with our choice of L leads to a contradiction:

$$\begin{aligned} \mu(B_j^{n,z}) &\leqslant \mu(2^{7k} B_i^{n+1,z}) \leqslant C_D^{7k} \mu(B_i^{n+1,z}) \leqslant C_D^{7k} L^{-1} \mu(B^{n+1,z}) \\ &= C_D^{7k} L^{-1} \mu(2^{7k} B_j^{n,z}) \leqslant C_D^{14k} L^{-1} \mu(B_j^{n,z}) < \mu(B_j^{n,z}). \end{aligned}$$

Now if (9) were not true, $K_i^{n+1,z} \subset \hat{K}_j^{n,z}$ for every i for which the interiors of $K_i^{n+1,z}$ and $K_j^{n,z}$ intersect. This is impossible, since *every* ball $B_i^{n+1,z}$ lies in the interior of some maximal continuum and at least one of them intersects $K_j^{n,z}$. Hence (9) holds and K_z is a continuum.

Finally, define

$$K = K_x \cup K_y.$$

Note that K is a continuum, since by construction $K^{1,x} = K^{1,y}$. Recall that for all i, j, z there exists a finite δ-chain $(x_p^{i,j,z})_p$ in $2^{7k} B_j^{i,z} \setminus B_j^{i,z}$ such that

$$K_j^{i,z} \subset \bigcup_p \overline{5B_{x_p^{i,j,z} x_{p-1}^{i,j,z}}} \subset \bigcup_p 6B_{x_p^{i,j,z} x_{p-1}^{i,j,z}},$$

and

$$\sum_p \mu(B_{x_p^{i,j,z} x_{p-1}^{i,j,z}})^{1/2} \leqslant C\mu(B_j^{i,z})^{1/2}.$$

Since the set of balls

$$\mathcal{B} := \left\{ B(x_p^{i,j,z}, 6d(x_p^{i,j,z}, x_{p-1}^{i,j,z})), B(x_{p-1}^{i,j,z}, 6d(x_p^{i,j,z}, x_{p-1}^{i,j,z})) \right\}_{i,j,p,z}$$

forms an open cover for the continuum K, we may extract a finite chain of balls $(A_i)_{i=1}^{N-1}$ of the set $\mathcal{B}$ so that, denoting $A_0 = B_x$, $A_N = B_y$ we have $A_i \cap A_{i-1} \neq \emptyset$ for $i = 1, \dots N$. Let $x_0 = x$, $x_{2N} = y$ and for other indices choose $x_{2i} \in A_i$ so that $A_i = B(x_{2i}, r_i)$ for some $r_i \leqslant 6\delta$. Let $x_{2i-1} \in A_i \cap A_{i-1}$ for $i = 1, \dots, N$. Now $(x_i)_{i=0}^{2N}$ is a 6δ-chain between the points x and y. Moreover, by (8)

$$\begin{aligned}
&\sum_{i=1}^{2N} \mu(B_{x_i x_{i-1}})^{1/2} \leqslant 2\sum_{i=0}^{N} \mu(2A_i)^{1/2} \leqslant C\sum_{i=1}^{N-1} \mu(A_i)^{1/2} + 4\varepsilon \\
&\leqslant C\sum_{B\in\mathcal{B}} \mu(B)^{1/2} + 4\varepsilon \leqslant C\sum_{z,i,j,p} \mu(B(x_p^{i,j,z}, d(x_p^{i,j,z}, x_{p-1}^{i,j,z})))^{1/2} + 4\varepsilon \\
&\leqslant C\sum_{z,i,j}\sum_p \mu(B_{x_p^{i,j,z} x_{p-1}^{i,j,z}})^{1/2} + 4\varepsilon \leqslant C\sum_{z,i}\sum_j \mu(B_j^{i,z})^{1/2} + 4\varepsilon \\
&\leqslant C\sum_z\sum_i M\mu(B^{i,z})^{1/2} + 4\varepsilon \leqslant CM\sum_z\sum_i \tau^{(i-1)/2}\mu(B^1)^{1/2} + 4\varepsilon \\
&\leqslant CM\mu(B^1)^{1/2} + 4\varepsilon = CM\mu(B(x, 2^{2k}d(x,y)))^{1/2} + 4\varepsilon \\
&\leqslant CM\mu(B_{xy})^{1/2} + 4\varepsilon.
\end{aligned}$$

Since ε is arbitrary, the claim follows.

□

6. Concluding remarks

It is natural to ask if Theorem 1.2 remains valid with weak metric doubling measures of dimension $s \neq 2$. The two lemmas below show that it does not.

Lemma 6.1. *Let (X, d) be a linearly locally connected metric space homeomorphic to $\mathbb{S}^2$, and $0 < s < 2$. Then X does not carry weak metric doubling measures of dimension s.*

Proof. Assume towards a contradiction that X carries such as measure μ. Then there exists $C > 0$ such that for every $x, y \in X$ the following holds: if $(x_i)_{i=0}^m$ is a δ-chain from x to y and if δ is small enough, then

$$\begin{aligned}\mu(B_{xy})^{1/2} &= \mu(B_{xy})^{1/2-1/s}\mu(B_{xy})^{1/s} \leqslant C\mu(B_{xy})^{1/2-1/s}\sum_{i=1}^{m}\mu(B_{x_ix_{i-1}})^{1/s} \\ &\leqslant C\mu(B_{xy})^{1/2-1/s}\max_i \mu(B_{x_ix_{i-1}})^{1/s-1/2}\sum_{i=1}^{m}\mu(B_{x_ix_{i-1}})^{1/2}.\end{aligned}$$

Notice that

$$\max_i \mu(B_{x_ix_{i-1}})^{1/s-1/2} \to 0 \quad \text{as } \delta \to 0.$$

Applying the estimates to all δ-chains and letting $\delta \to 0$, we conclude that μ is a weak metric doubling measure of dimension 2 and

$$\mu(B_{xy})^{1/2} \leqslant \epsilon q_{\mu,2}(x,y) \quad \text{for all } \epsilon > 0.$$

Since $\mu(B_{xy}) > 0$ for all distinct x and y, if follows that $q_{\mu,2}(x,y) = \infty$. This contradicts Theorem 1.2. □

Lemma 6.2. *Fix $s > 2$. Then there exists a metric space (X,d), homeomorphic to $\mathbb{S}^2$ and LLC, such that X carries a weak metric doubling measure of dimension s but there is no quasisymmetric $f : X \to \mathbb{S}^2$.*

Proof. Let $(\mathbb{R}^2, d)$ be a Rickman rug; d is the product metric

$$d((x_1,y_1),(x_2,y_2)) = \Big(|x_1-x_2|^2 + |y_1-y_2|^{2/(s-1)}\Big)^{1/2}.$$

It is well-known that there are no quasisymmetric maps from $(\mathbb{R}^2, d)$ onto the standard plane. Moreover, it is not difficult to show that $\mu = \mathcal{H}^1 \times \mathcal{H}^{s-1}$ is a weak metric doubling measure of dimension s on $(\mathbb{R}^2, d)$. To construct a similar example homeomorphic to $\mathbb{S}^2$, one can apply a suitable stereographic projection. □

It would be interesting to extend Theorem 1.2 to higher dimensions. Recall that the Bonk-Kleiner theorem (Theorem 1.1) does not extend to dimensions higher than 2, see [24], [15], [21].

Question 6.3. Let (X,d) be a metric space homeomorphic to $\mathbb{S}^n$, $n \geqslant 3$. Assume that X is linearly locally contractible and carries a weak metric doubling measure of dimension n. Is there a quasisymmetric $f : (X,d) \to (X,d')$, where (X,d') is Ahlfors n-regular?

Recall that (X,d) is linearly locally contractible if there exists $\lambda' \geqslant 1$ such that $B(x,R) \subset X$ is contractible in $B(x,\lambda' R)$ for every $x \in X$, $0 < R < \operatorname{diam} X/\lambda'$. Linear local contractibility is equivalent to the LLC condition when X is homeomorphic to $\mathbb{S}^2$, see [4].

The basic tool in the proof of Theorem 1.2 was a coarea-type estimate for real-valued functions. Extending our method to higher dimensions would require similar estimates for suitable maps with values in $\mathbb{R}^{n-1}$, which are difficult to construct when $n \geqslant 3$. This problem is related to the deep results of Semmes [25] on Poincaré inequalities in Ahlfors n-regular and linearly locally contractible n-manifolds.

Finally, it is also desirable to characterize the metric spheres that can be uniformized by *quasiconformal* homeomorphisms which are more flexible than quasisymmetric maps, see [22]. However, it is not clear which definition of quasiconformality should be used in the generality of possibly fractal surfaces. Our methods suggest a measure-dependent modification to the familiar geometric definition. More precisely, given a measure μ, conformal modulus should be defined applying not the usual path length but a *μ-length* as in Section 2.

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[B]

Uniformization with infinitesimally metric measures

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Preprint

UNIFORMIZATION WITH INFINITESIMALLY METRIC MEASURES

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Abstract. We consider extensions of quasiconformal maps and the uniformization theorem to the setting of metric spaces X homeomorphic to $\mathbb{R}^2$. Given a measure μ on such a space, we introduce *μ-quasiconformal maps* $f : X \to \mathbb{R}^2$, whose definition involves deforming lengths of curves by μ. We show that if μ is an *infinitesimally metric measure*, i.e., it satisfies an infinitesimal version of the metric doubling measure condition of David and Semmes, then such a μ-quasiconformal map exists. We apply this result to give a characterization of the metric spaces admitting an *infinitesimally quasisymmetric* parametrization.

1. Introduction

The *quasisymmetric uniformization problem* asks one to characterize, as meaningfully as possible, those metric spaces which may be mapped onto a domain in the Euclidean plane, or the 2-sphere, by a quasisymmetric homeomorphism. Informally, a mapping is *quasisymmetric* if it roughly preserves the relative distance between triples of points. See Section 4 for the precise definition.

Significant results on the uniformization problem, such as the Bonk–Kleiner theorem [4] and its extensions in [21] and [22], have been obtained for surfaces that are non-fractal, i.e., their 2-dimensional Hausdorff measure is locally finite. These spaces carry enough rectifiable paths for classical methods such as conformal modulus to be applicable. By *surface*, we mean a 2-manifold equipped with a continuous metric.

In contrast, the class of fractal surfaces is too general for the standard methods. Consequently, understanding the quasisymmetric uniformization of fractal surfaces has proved extremely difficult. Any progress is desirable, especially due to applications to geometric group theory (cf. [3], [12]) and complex dynamics (cf. [5]).

The usual method for constructing quasisymmetric maps is to first show the existence of some *conformal* or *quasiconformal* map in the spirit of the classical uniformization theorem. Then, if the underlying surface has good geometric properties, one can use quasiconformal invariants to show that such a map is actually quasisymmetric.

A fundamental difficulty in extending this method to fractal surfaces is the lack of a suitable definition of quasiconformality. The classical metric definition (see Section 4) is too weak to lead to a satisfactory theory in this

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Mathematics Subject Classification 2010: Primary 30L10, Secondary 30C65, 28A75, 51F99.

All authors were supported by the Academy of Finland, project number 308659.

generality. The geometric definition (see Section 2) requires the existence of many rectifiable paths, which need not be the case for fractal surfaces.

In Section 2 we propose the definition of *μ-quasiconformality* for homeomorphisms $f\colon X \to \mathbb{R}^2$, depending on a measure μ on X. This is a modification of the geometric definition: we deform the metric on X using μ to obtain the *μ-length* of a curve, and we define the corresponding *μ-modulus* of a family of curves in X. A homeomorphism f is μ-quasiconformal if the μ-modulus of every family of curves in X is comparable to the conformal modulus of its image under f in $\mathbb{R}^2$.

A quasisymmetric map $f\colon X \to \mathbb{R}^2$ is μ-quasiconformal when μ is the pullback of the Lebesgue measure on $\mathbb{R}^2$. Our goal is to find measures μ on a given space X for which the existence of μ-quasiconformal maps can be shown.

In Section 3 we introduce the notion of *infinitesimally metric measure* on X. These correspond to the metric doubling measures of David and Semmes [6], [13], the correspondence being similar to the one between metrically quasiconformal (MQC) maps and quasisymmetric (QS) maps, where the former is an infinitesimal condition and the latter is a global condition. Metric doubling measures can be used to produce quasisymmetric maps via deformation of the metric on X. Our first main result shows that a μ-quasiconformal map exists if μ is an infinitesimally metric measure.

THEOREM 1.1. *Let X be a metric space homeomorphic to $\mathbb{R}^2$ which supports an infinitesimally metric measure μ. Then there exists a μ-quasiconformal map $f\colon X \to \Omega$, where $\Omega = \mathbb{D} \subset \mathbb{R}^2$ or $\Omega = \mathbb{R}^2$.*

To prove Theorem 1.1, we first show that the metric d on X can be deformed using μ to yield a "quasiconformally equivalent" metric q that has locally finite Hausdorff 2-measure. Then, we apply the uniformization theorem in [14] to obtain a quasiconformal map $(X, q) \to \mathbb{R}^2$. Composing, we then get the desired μ-quasiconformal map.

In view of the correspondence between infinitesimally metric measures and metric doubling measures, it is natural to attempt to characterize the class of metric spaces X that admit metrically quasiconformal maps $f\colon X \to \mathbb{R}^2$ in terms of infinitesimally metric measures. However, it turns out that the existence of such maps can be rather arbitrary unless strong conditions are imposed on X.

Instead, we consider the notion of *infinitesimally quasisymmetric* (I-QS) mapping (Definition 4.1). Such maps form an intermediate class between those of MQC and QS maps. In our second main result, we characterize the metric spaces which admit such maps into $\mathbb{R}^2$ as the spaces that carry infinitesimally metric measures with suitable properties.

THEOREM 1.2. *Let X be a metric space homeomorphic to $\mathbb{R}^2$. There exists an infinitesimally quasisymmetric map $f\colon X \to \Omega$, where $\Omega = \mathbb{D}$ or $\Omega = \mathbb{R}^2$, if and only if X is infinitesimally linearly locally connected and supports an infinitesimally metric measure μ such that (X, μ) is infinitesimally Loewner.*

See Section 4 for definitions. The proof combines Theorem 1.1 with estimates for the μ-modulus that generalize the modulus estimates in [11].

One motivation for our work is to understand the conformal geometry of metric surfaces in the absence of strong geometric assumptions such as Ahlfors regularity, linear local connectedness and the Loewner condition (see Section 4). In Section 5, we present four examples to illustrate possible behaviors of metric surfaces under weaker geometric assumptions. We remark that, while the main theorems of this paper are applicable to any metric space homeomorphic to $\mathbb{R}^2$, including fractal spaces, all of these examples have locally finite Hausdorff 2-measure. The four examples are summarized here, listed by section in which they appear.

5.1. A surface that admits an MQC parametrization by $\mathbb{R}^2$ but not an I-QS parametrization. This surface is linearly locally connected (LLC) but not Loewner. This example also illustrates how metric quasiconformality is not preserved under taking inverses or precomposing with a QS map.

5.2. A surface that admits a geometrically quasiconformal (QC) parametrization by $\mathbb{R}^2$ but not a MQC parametrization. This surface is upper Ahlfors 2-regular but not infinitesimally LLC.

5.3. A surface that admits an I-QS parametrization by $\mathbb{R}^2$ but not a quasisymmetric parametrization. This surface is upper Ahlfors 2-regular but not LLC.

5.4. A surface that, despite being a geodesic space of locally finite Hausdorff 2-measure, violates infinitesimal upper Ahlfors 2-regularity at every point along a nondegenerate continuum. This surface is LLC, and it admits a QC parametrization by $\mathbb{R}^2$ but not a MQC parametrization.

In particular, these examples show that the class of I-QS maps from $\mathbb{R}^2$ onto a metric space differs from both the class of QS maps and the class of MQC maps.

2. μ-quasiconformal maps

We assume throughout the paper that (X, d) is a metric space homeomorphic to the Euclidean plane $\mathbb{R}^2$. We denote $B(x,r) = \{y \in X : d(x,y) < r\}$, $\overline{B}(x,r) = \{y \in X : d(x,y) \leqslant r\}$, and $S(x,r) = \{y \in X : d(x,y) = r\}$. If B is a ball of radius r, we denote by λB the ball with the same center and radius λr. A *path* in X is a continuous map $\gamma : I \to X$, where I is an interval. The image of such a path is called a *curve* in X.

We recall the Carathéodory construction of measures, cf. [7, 2.10]. Let $\mathcal{F}$ be a family of subsets of X, and $\varphi : \mathcal{F} \to [0, \infty]$. For $A \subset X$ and $\delta > 0$, the *δ-content* $\phi_\delta(A)$ is

$$\phi_\delta(A) = \inf \sum_{S \in \mathcal{G}} \varphi(S),$$

where the infimum is taken over all countable

$$\mathcal{G} \subset \{S \in \mathcal{F} : \operatorname{diam}(S) \leqslant \delta\} \quad \text{such that} \quad A \subset \bigcup_{S \in \mathcal{G}} S.$$

Then, since $\phi_\delta(A)$ is decreasing with respect to δ, the limit

$$\psi(A) = \lim_{\delta \to 0^+} \phi_\delta(A) \in [0, \infty]$$

exists. Moreover, if every $S \in \mathcal{F}$ is a Borel set, then ψ is a Borel regular measure in X.

Applying the Carathéodory construction with $\mathcal{F}$ all the non-empty subsets of X and $\phi(S) = \alpha(m)2^{-m} \operatorname{diam}(S)^m$ gives the *m-dimensional Hausdorff measure* $\mathcal{H}^m$ in X, where $\alpha(1) = 2$ and $\alpha(2) = \pi$.

Before defining μ-quasiconformal maps, we review the classical geometric definition of quasiconformality. However, we replace the standard modulus of path families with the modulus of curve families, which lead to equivalent definitions but are easier to work with in our setting.

Let Γ be a family of curves (i.e., images of paths) in X. A Borel function $\rho\colon X \to [0, \infty]$ is *admissible* for Γ if $\int_{\mathcal{C}} \rho \, d\mathcal{H}^1 \geqslant 1$ for all $\mathcal{C} \in \Gamma$ with locally finite $\mathcal{H}^1$-measure. The *(conformal) modulus* of Γ is defined as

$$\operatorname{mod} \Gamma = \inf \int_X \rho^2 \, d\mathcal{H}^2, \tag{1}$$

where the infimum is taken over all admissible functions ρ.

Let X, Y be metric spaces homeomorphic to $\mathbb{R}^2$ and $f\colon X \to Y$ a homeomorphism. Then f is *geometrically quasiconformal* (QC), if there exists $K \geqslant 1$ such that

$$K^{-1} \operatorname{mod} \Gamma \leqslant \operatorname{mod} f\Gamma \leqslant K \operatorname{mod} \Gamma$$

for all curve families Γ in X. In this case, we also say that f is *geometrically K-quasiconformal* (K-QC).

We now define μ-quasiconformal maps. Let μ be a Radon measure in X with no atoms such that $\mu(B) > 0$ for every open ball $B \subset X$. Recall that a Borel regular measure μ is *Radon* if it is finite on compact sets.

We associate with μ a collection $\mathcal{B}$ of open balls in X such that for every point $x \in X$ there is $r_x > 0$ such that $B(x, r) \in \mathcal{B}$ for every $r < r_x$. We also make the requirement that $\overline{B}(x, r_x)$ is compact for all x. We refer to such a collection $\mathcal{B}$ as an *admissible cover*. From now on we use the convention that every measure μ comes equipped with an admissible cover $\mathcal{B}$.

Definition 2.1. The *μ-length measure* ℓ_μ in X is defined by the Carathéodory construction with $\mathcal{F} = \mathcal{B}$ and $\varphi\colon \mathcal{B} \to [0, \infty]$, $\varphi(B) = 2\pi^{-1/2}\mu(B)^{1/2}$.

The ℓ_μ is normalized so that if $X = \mathbb{R}^2$ and μ the Lebesgue measure, then $\ell_\mu = \mathcal{H}^1$ (for any choice of $\mathcal{B}$).

Definition 2.2. Let Γ be a family of curves in X. We say that a Borel function $\rho\colon X \to [0, \infty]$ is *μ-admissible* for Γ if $\int_{\mathcal{C}} \rho \, d\ell_\mu \geqslant 1$ for all $\mathcal{C} \in \Gamma$ with locally finite ℓ_μ-measure. We denote the set of such functions by $\Phi_\mu(\Gamma)$. The *μ-modulus* of Γ is

$$\operatorname{mod}_\mu \Gamma = \inf_{\rho \in \Phi_\mu(\Gamma)} \int_X \rho^2 \, d\mu.$$

Notice that if $\ell_\mu(\mathcal{C}) = 0$ for some $\mathcal{C} \in \Gamma$, then there are no μ-admissible functions for Γ and thus $\operatorname{mod}_\mu \Gamma = \infty$. On the other hand, if ℓ_μ is not locally finite on any $\mathcal{C} \in \Gamma$, then $\operatorname{mod}_\mu \Gamma = 0$. Definition 2.2 coincides with (1) when $X = \mathbb{R}^2$ and μ the Lebesgue measure.

Definition 2.3. Let $f\colon X \to \Omega$ be a homeomorphism, where Ω is a domain in $\mathbb{R}^2$. We say that f and f^{-1} are *μ-quasiconformal*, if there exists $K \geqslant 1$ such that

$$K^{-1} \operatorname{mod}_\mu \Gamma \leqslant \operatorname{mod} f\Gamma \leqslant K \operatorname{mod}_\mu \Gamma$$

for every curve family Γ in X.

Definition 2.3 naturally leads to the following questions:

(1) How to decide if a given metric space X carries a measure μ for which there exists a μ-quasiconformal map into $\mathbb{R}^2$?
(2) How to decide if there exists a μ-quasiconformal map for a given (X, μ)?

Concerning Question (2), it is reasonable to ask if the *reciprocality* condition (Definition 3.7 below) can be modified to yield a characterization similar to the one obtained in [14] for the 2-dimensional Hausdorff measure. In the next section we introduce infinitesimally metric measures and show that they lead to the existence of μ-quasiconformal maps.

3. Infinitesimally metric measures

We now define the infinitesimally metric measures. Let X, μ, $\mathcal{B}$ and ℓ_μ be as above. Moreover, for $x, y \in X$ let

$$q(x, y) = \inf \ell_\mu(\mathcal{C}(x, y)),$$

where the infimum is taken over all curves $\mathcal{C}(x, y)$ that join x and y in X. Thus q defines a pseudometric on X. In the following, we use the subscripts d and q to indicate which (pseudo)metric is being used in our notation for balls, spheres, and diameter.

Definition 3.1. The measure μ is *infinitesimally metric* (I-MM) if there exist $\Lambda > 1$, $C_i \geqslant 1$ such that

$$C_i^{-1} q(y, z) \leqslant \mu(B_d(x, r))^{1/2} \leqslant C_i q(y, z) \tag{2}$$

for every $B_d(x, r) \in \mathcal{B}$, $y \in B_d(x, r/\Lambda)$ and $z \in S_d(x, r)$.

It follows immediately from the definition that if μ is I-MM, then q is a metric on X.

Recall that a metric space X is *(Ahlfors) 2-regular* if there exists $C \geqslant 1$ such that $C^{-1} r^2 \leqslant \mathcal{H}^2(B(x, r)) \leqslant C r^2$ for all $x \in X$, $r \in (0, \operatorname{diam} X)$. We say that X is *lower* or *upper 2-regular* if, respectively, the first or second of these inequalities holds. Definition 3.1 imposes a similar infinitesimal condition on the measure μ. In fact, we show in Lemma 3.4 and Lemma 3.5 that (X, q) is infinitesimally Ahlfors 2-regular.

The remainder of this section is dedicated to the proof of Theorem 1.1. We first restate the theorem.

THEOREM 3.2. *Let X be a metric space homeomorphic to $\mathbb{R}^2$ which supports an I-MM μ. Then there exists a μ-quasiconformal map $f\colon X \to \Omega$, where $\Omega = \mathbb{D} \subset \mathbb{R}^2$ or $\Omega = \mathbb{R}^2$.*

As groundwork, we require several lemmas to estimate the 1- and 2-dimensional Hausdorff measures corresponding to the metric q.

We fix an I-MM μ. Let $\mathcal{B} = \{B_d(x,r) : x \in X, r < r_x\}$ be the admissible cover associated with μ. The assumption that μ has no atoms implies that $\lim_{r\to 0} \mu(B_d(x,r)) = 0$ for all $x \in X$. Definition 3.1 then implies that the metrics d and q are topologically equivalent.

Lemma 3.3. *We have*

$$\mu(\overline{B}_d(x,r)) \leqslant C_i^2 \mu(B_d(x,r))$$

for every $B_d(x,r) \in \mathcal{B}$, where C_i is the constant in Definition 3.1.

Proof. Since $\overline{B}_d(x,r)$ is compact and X homeomorphic to $\mathbb{R}^2$, there exists a point $z \in \partial(X \setminus \overline{B}_d(x,r))$. Observe that $z \in S_d(x,r)$. Let $\varepsilon > 0$, and let $w \in B_q(z,\varepsilon)$ such that $r < d(x,w) < r_x$. Now,

$$\begin{aligned}\mu(\overline{B}_d(x,r))^{1/2} &\leqslant \mu(B_d(x,d(x,w))^{1/2} \\ &\leqslant C_i q(x,w) \leqslant C_i q(x,z) + C_i\varepsilon \\ &\leqslant C_i^2 \mu(B_d(x,r))^{1/2} + C_i\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ proves the claim. □

Lemma 3.4. *We have*

$$C_i^{-2} r^2 \leqslant \mu(B_q(x,r)) \leqslant C_i^3 r^2$$

for every ball $B_q(x,r)$ contained in $B_d(x,r_x/2)$, where C_i is the constant in Definition 3.1.

Proof. Let

$$s = \inf_{y \in X \setminus B_q(x,r)} d(x,y) \quad \text{and} \quad t = \sup_{z \in B_q(x,r)} d(x,z).$$

Clearly $B_d(x,s) \subset B_q(x,r)$. We claim that there exists $y \in S_d(x,s)$ such that $q(x,y) \geqslant r$. If not, then $X \setminus B_q(x,r)$ and $\overline{B}_d(x,s)$ are disjoint closed sets, with $\overline{B}_d(x,s)$ compact. This implies that $\operatorname{dist}(X \setminus B_q(x,r), \overline{B}_d(x,s)) > 0$, contradicting the definition of s. Since μ is assumed to be I-MM, we have $\mu(B_q(x,r)) \geqslant \mu(B_d(x,s)) \geqslant C_i^{-2} r^2$.

Likewise, $B_q(x,r) \subset \overline{B}_d(x,t)$. Similarly to the first part of the proof, we note that $(X \setminus B_d(x,t)) \cap \overline{B}_q(x,r) \neq \emptyset$. Thus, there exists $z \in S_d(x,t)$ such that $q(x,z) \leqslant r$. Since μ is I-MM, Lemma 3.3 gives

$$\mu(B_q(x,r)) \leqslant \mu(\overline{B}_d(x,t)) \leqslant C_i^2 \mu(B_d(x,t)) \leqslant C_i^3 r^2.$$ □

For $s, \delta > 0$, let $\mathcal{H}_q^s$ and $\mathcal{H}_{q,\delta}^s$ denote the s-dimensional Hausdorff measure and Hausdorff δ-content on (X,q), respectively.

Lemma 3.5. *We have*

$$\frac{\pi}{4C_i^2}\mu(A) \leqslant \mathcal{H}_q^2(A) \leqslant 100\pi C_i^2 \mu(A)$$

for any Borel set $A \subset X$, where C_i is the constant in Definition 3.1.

Proof. Let $\delta > 0$, and let $U \subset X$ be an open set with $A \subset U$ and $\mu(U) \leqslant \mu(A) + \delta$. Using the basic covering theorem (see [9, Thm. 1.2]), choose a sequence of pairwise disjoint balls $B_j = B_q(x_j, r_j)$ with $B_j \subset U$, $B_j \subset B_d(x_j, r_{x_j}/2)$ and $10r_j < \delta$ for all j, such that $U \subset \cup_{j=1}^{\infty} 5B_j$. Then

$$\mathcal{H}_{q,\delta}^2(A) \leqslant \pi \sum_{j=1}^{\infty} (10r_j)^2 \leqslant C\pi \sum_{j=1}^{\infty} \mu(B_j) \leqslant C\pi\mu(U) \leqslant C\pi(\mu(A) + \delta),$$

where $C = 100C_i^2$ (the π comes from the normalization of $\mathcal{H}^2$). The upper bound for $\mathcal{H}_q^2(A)$ follows.

For the lower bound, fix n and define the Borel set

$$A_n = \overline{\{x \in A : B_q(x, 1/n) \subset B_d(x, r_x/2)\}} \cap A.$$

Let $\{E_j\}$ be a cover for A_n with $\operatorname{diam}_q(E_j) < \frac{1}{2n}$ for all j. Removing sets from the cover if necessary, we may assume that for every j there exists $x_j \in A_n$ such that $E_j \subset B_q(x_j, 2\operatorname{diam}_q E_j)$ and $B_q(x_j, 1/n) \subset B_d(x_j, r_{x_j}/2)$. Since

$$\mu(A_n) \leqslant \sum_{j=1}^{\infty} \mu(B_q(x_j, 2\operatorname{diam}_q E_j)) \leqslant 4C_i^2 \sum_{j=1}^{\infty} \operatorname{diam}_q(E_j)^2,$$

we get

$$\frac{\pi}{4C_i^2}\mu(A_n) \leqslant \mathcal{H}_{q,1/2n}^2(A_n) \leqslant \mathcal{H}_q^2(A).$$

Since $\mu(A) = \lim_{n\to\infty} \mu(A_n)$, the claim follows. □

Lemma 3.6. *We have*

$$\frac{2}{C_i\sqrt{\pi}}\mathcal{H}_q^1(A) \leqslant \ell_\mu(A) \leqslant \frac{4C_i^3}{\sqrt{\pi}}\mathcal{H}_q^1(A)$$

for any Borel set $A \subset X$, where C_i is the constant in Definition 3.1.

Proof. Since X is homeomorphic to $\mathbb{R}^2$, it is locally compact and can be exhausted by compact sets X^j. We can also approximate both $\ell_\mu(A)$ and $\mathcal{H}_q^1(A)$ from below with the measures of the sets $A^j = A \cap X^j$, and by considering some compact neighbourhood X^{j+k} of A^j we can assume that

$$\sup_{x\in X} \operatorname{diam}_q(B_d(x,r)), \sup_{x\in X} \operatorname{diam}_d(B_q(x,r)) \to 0 \text{ as } r \to 0.$$

We first consider Borel sets

$$A_n = \overline{\{x \in A : 1/n < r_x\}} \cap A,\ n \in \mathbb{N}.$$

Let $\sigma > 0$ be arbitrary and $\delta > 0$ small enough so that $\operatorname{diam}_d(B_q(x, 2\delta)) < \min\{\sigma, 1/n\}$ for every x. Fix any cover $\{E_j\}$ of A_n with $\operatorname{diam}_q(E_j) < \delta$

for all j. Removing sets from the cover if necessary, we may assume that for every j there exists $x_j \in A_n$ such that $\operatorname{dist}(\{x_j\}, E_j) < \operatorname{diam}_q(E_j)$ and $r_{x_j} > 1/n$. Let

$$t_j = \inf\{t > 0 : E_j \subset B_d(x_j, t)\}.$$

Then for every j we have $E_j \subset \overline{B}_d(x_j, t_j)$. Moreover, since

$$E_j \subset B_q(x_j, 2 \operatorname{diam}_q E_j) \subset B_q(x_j, 2\delta),$$

we have $t_j < \min\{\sigma, 1/n\}$.

For every $j, m \in \mathbb{N}$ there exists $y_m^j \in E_j \setminus B_d(x_j, t_j - 1/m)$, so that

$$\mu(B_d(x_j, t_j - 1/m))^{1/2} \leqslant C_i q(x_j, y_m^j).$$

Since $y_m^j \in B_q(x_j, 2 \operatorname{diam}_q E_j)$, we have $\mu(B_d(x_j, t_j))^{1/2} \leqslant 2C_i \operatorname{diam}_q(E_j)$. Recall that ℓ_μ is defined by the Carathéodory construction: $\ell_\mu(A_n) = \lim_{\sigma \to 0} \ell_{\mu,\sigma}(A_n)$, where $\ell_{\mu,\sigma}$ is the corresponding σ-content. By Lemma 3.3 we get

$$\begin{aligned} \ell_{\mu,\sigma}(A_n) &\leqslant 2\pi^{-1/2} \sum_j \mu(\overline{B}_d(x_j, t_j))^{1/2} \leqslant 2\pi^{-1/2} C_i^2 \sum_j \mu(B_d(x_j, t_j))^{1/2} \\ &\leqslant 4\pi^{-1/2} C_i^3 \sum_j \operatorname{diam}_q(E_j) \end{aligned}$$

(the $2\pi^{-1/2}$ comes from the normalization of ℓ_μ) and hence $\ell_{\mu,\sigma}(A_n) \leqslant 2\pi^{-1/2} C_i^3 \mathcal{H}_q^1(A_n)$. This holds for all $\sigma > 0$ and $n \in \mathbb{N}$, so we have $\ell_\mu(A) \leqslant 4\pi^{-1/2} C_i^3 \mathcal{H}_q^1(A)$.

The other inequality can be proved more directly, with similar arguments but without the need to consider the sets A_n. □

We will apply the main result in [14]. It depends on the following definition. A *quadrilateral* $Q = Q(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is a set homeomorphic to a closed square in $\mathbb{R}^2$, with boundary edges $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ (in cyclic order). For sets $E, F \subset G$, $\Gamma(E, F; G)$ denotes the family of curves in G that join E and F. While path families were considered in [14], the results applied below remain valid when they are replaced with curve families.

Definition 3.7. Let Y be a metric space homeomorphic to $\mathbb{R}^2$ with locally finite Hausdorff 2-measure. The space Y is *reciprocal* if there exists $\kappa \geqslant 1$ such that for all quadrilaterals $Q = Q(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ in X,

$$\operatorname{mod} \Gamma(\zeta_1, \zeta_3; Q) \operatorname{mod} \Gamma(\zeta_2, \zeta_4; Q) \leqslant \kappa \tag{3}$$

and for all $x \in X$ and $R > 0$ such that $X \setminus B(x, R) \neq \emptyset$,

$$\lim_{r \to 0} \operatorname{mod} \Gamma(B(x, r), X \setminus B(x, R); B(x, R)) = 0. \tag{4}$$

It was shown in [15] that the inequality opposite to (3) holds in every Y. That is, there exists a universal constant $\kappa' > 0$ such that

$$\operatorname{mod} \Gamma(\zeta_1, \zeta_3; Q) \operatorname{mod} \Gamma(\zeta_2, \zeta_4; Q) \geqslant \kappa'$$

for all quadrilaterals $Q \subset Y$.

THEOREM 3.8 (Theorem 1.4 [14]). *Let Y be a metric space homeomorphic to $\mathbb{R}^2$, with locally finite Hausdorff 2-measure. There exists a QC map $h\colon Y \to \Omega \subset \mathbb{R}^2$ if and only if Y is reciprocal.*

The next proposition is a generalization of Theorem 1.6 from [14], where the mass upper bound is assumed for every radius.

Proposition 3.9. *Let Y be a metric space homeomorphic to $\mathbb{R}^2$. Suppose there exist $C_U > 0$ and for every $y \in Y$ a radius $r_y > 0$ such that*

$$\mathcal{H}^2(B(y,r)) \leqslant C_U r^2 \tag{5}$$

for every $r < r_y$. Then Y is reciprocal.

Proof. Condition (4) follows by considering the admissible function

$$\rho(z) = \frac{1}{\log(R/r)d(y,z)}.$$

To prove (3), we modify the proof of [14, Proposition 15.5]. We give the main steps and refer to [14] for the missing details. Let $Q = Q(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ be a quadrilateral. Then there exists a ρ that is *weakly admissible* (admissible outside an exceptional curve family of zero modulus) for $\Gamma(\zeta_1, \zeta_3; Q)$, such that

$$\int_Y \rho^2 \, d\mathcal{H}^2 = \operatorname{mod} \Gamma(\zeta_1, \zeta_3; Q).$$

Fix a curve $\mathcal{C} \in \Gamma(\zeta_2, \zeta_4; Q)$. We may assume that $\mathcal{C}$ is homeomorphic to $[0,1]$ and has finite length. Using the basic covering theorem, we find a finite cover $\{5B_j\} = \{B(y_j, 5r_j)\}$ of $\mathcal{C}$ such that $y_j \in \mathcal{C}$ and $36 r_j < r_y$ for all j, and such that the balls B_j are pairwise disjoint. Moreover, let $g\colon Q \to [0,\infty]$,

$$g(y) = \sum_j r_j^{-1} \chi_{6B_j \cap Q}(y). \tag{6}$$

Since every $\mathcal{C}'$ in $\Gamma(\zeta_1, \zeta_3; Q)$ intersects at least one of the balls $5B_j$, it follows that g is admissible for $\Gamma(\zeta_1, \zeta_3; Q)$. Moreover, since ρ is a minimizer for $\operatorname{mod} \Gamma(\zeta_1, \zeta_3; Q)$, applying the weak admissibility of $(1-t)\rho + tg$ and letting $t \to 0$ leads to

$$\operatorname{mod} \Gamma(\zeta_1, \zeta_3; Q) \leqslant \int_Q \rho g \, d\mathcal{H}^2 = \sum_j r_j^{-1} \int_{6B_j \cap Q} \rho \, d\mathcal{H}^2. \tag{7}$$

For the maximal function $\mathcal{M}\rho : Q \to [0,\infty]$,

$$\mathcal{M}\rho(z) = \sup_{r>0} \frac{1}{\mathcal{H}^2(B(z,5r))} \int_{B(z,r)\cap Q} \rho \, d\mathcal{H}^2,$$

standard arguments show that

$$\int_Q (\mathcal{M}\rho)^2 \, d\mathcal{H}^2 \leqslant 8 \int_Q \rho^2 \, d\mathcal{H}^2. \tag{8}$$

Now we apply (5) to estimate the right hand term of (7) from above by

$$1296C_U \sum_j \frac{r_j}{\mathcal{H}^2(B(y_j, 36B_j))} \int_{B(y_j,6_j)\cap Q} \rho \, d\mathcal{H}^2$$
$$\leqslant 1296C_U \sum_j r_j \inf_{y\in\mathcal{C}\cap B_j} \mathcal{M}\rho(y).$$

Since the right hand term is bounded from above by $1296C_U \int_{\mathcal{C}} \mathcal{M}\rho \, d\mathcal{H}^1$, we conclude that

$$y \mapsto \frac{1296C_U\mathcal{M}\rho(y)}{\operatorname{mod}\Gamma(\zeta_1,\zeta_3;Q)}$$

is admissible for $\Gamma(\zeta_2,\zeta_4;Q)$. Combining the admissibility with (6) and (8), we have

$$\operatorname{mod}\Gamma(\zeta_2,\zeta_4;Q) \leqslant \frac{8\cdot 1296^2 C_U^2}{\operatorname{mod}\Gamma(\zeta_1,\zeta_3;Q)},$$

from which (4) follows. □

Proof of Theorem 1.1. By Lemmas 3.4 and 3.5, the space (X,q) satisfies the assumption of Proposition 3.9. Thus by Theorem 3.8 there exists a QC map $h\colon (X,q)\to\Omega\subset\mathbb{R}^2$. By the Riemann mapping theorem, we can choose h such that $\Omega=\mathbb{D}$ or $\Omega=\mathbb{R}^2$. Moreover, by Lemmas 3.5 and 3.6 the μ-modulus $\operatorname{mod}_\mu(\Gamma)$ and the conformal 2-modulus $\operatorname{mod}_2(\Gamma)$ in X are comparable for any curve family Γ, so h precomposed with the identity map from (X,d) to (X,q) is μ-quasiconformal. □

4. Infinitesimally quasisymmetric maps

In this section we introduce the notion of infinitesimally quasisymmetric map and apply our results on infinitesimally metric measures to give a characterization for the spaces that admit such a parametrization by a Euclidean planar domain.

Recall that a homeomorphism $f\colon (X,d)\to(Y,d')$ between metric spaces is *quasisymmetric* (QS) if there exists a homeomorphism $\eta\colon[0,\infty)\to[0,\infty)$ such that

$$\frac{d(x,y)}{d(x,z)} \leqslant t \text{ implies } \frac{d'(f(x),f(y))}{d'(f(x),f(z))} \leqslant \eta(t) \tag{9}$$

for all distinct points $x,y,z\in X$. Closely related is the following definition. A homeomorphism $f\colon(X,d)\to(Y,d')$ between metric spaces is *metrically quasiconformal* (MQC) if there exists $H\geqslant 1$ such that

$$\limsup_{r\to 0} \frac{\sup\{d'(f(x),f(y)) : d(x,y)\leqslant r\}}{\inf\{d'(f(x),f(y)) : d(x,y)\geqslant r\}} \leqslant H$$

for all $x\in X$.

Definition 4.1. A homeomorphism $f\colon(X,d)\to(Y,d')$ is *infinitesimally quasisymmetric* (I-QS) if there exists a homeomorphism $\eta\colon[0,\infty)\to[0,\infty)$ such that for every $x\in X$ there exists a radius $r_x>0$ such that (9) holds for all $y,z\in B(x,r_x)$.

It is a standard exercise to show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are QS, then $g \circ f$ and f^{-1} are also QS. These properties also hold for the class of I-QS maps. Note that both properties may fail for MQC maps, even for metric spaces homeomorphic to $\mathbb{R}^2$. In Section 5.1, we give an example of this.

It is immediate from the definitions that any QS map is I-QS, and any I-QS is MQC. Thus infinitesimal quasisymmetry is an intermediate condition between quasisymmetry and metric quasiconformality. In Section 5.3, we give an example of a map which is I-QS but not QS.

Recall that a metric space (X, d) is *linearly locally connected* (LLC) if there exists $\lambda \geqslant 1$ such that the following properties hold:

(1) For any $x \in X$, $r > 0$ and $y, z \in B(x, r)$ there exists a continuum $K \subset B(x, \lambda r)$ with $y, z \in K$.
(2) For any $x \in X$, $r > 0$ and $y, z \in X \setminus B(x, r)$ there exists a continuum $K \subset X \setminus B(x, \lambda^{-1} r)$ with $y, z \in K$.

Definition 4.2. A metric space (X, d) is *infinitesimally linearly locally connected* (I-LLC) if there exists $\Lambda \geqslant 1$ such that for every $x \in X$ there exists a radius $r_x > 0$ such that the above properties hold for all $r < r_x$.

It is easy to see that the LLC property is preserved under QS maps. Similarly, I-QS maps preserve the I-LLC property. Since every planar domain is I-LLC, any metric space that admits an I-QS map to such a domain must also be I-LLC.

Finally, we introduce a modification of the Loewner condition of Heinonen and Koskela [11]. We denote by $\Gamma(A, B)$ the family of curves which join sets A and B in X. Recall that X (equipped with $\mathcal{H}^2$) is *Loewner* if there exists a decreasing function $\phi\colon (0, \infty) \to (0, \infty)$ such that $\operatorname{mod} \Gamma(E, F) \geqslant \phi(t)$ for all disjoint nondegenerate continua E, F satisfying

$$\frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}} \leqslant t. \tag{10}$$

Also, recall our convention that any measure μ comes equipped with an admissible cover $\mathcal{B} = \{B(x, r) : 0 < r < r_x\}$.

Definition 4.3. A metric space X equipped with a measure μ is *infinitesimally Loewner* (I-Loewner) if there exists a decreasing function $\phi\colon (0, \infty) \to (0, \infty)$ such that $\operatorname{mod}_\mu \Gamma(E, F) \geqslant \phi(T)$ for all disjoint continua E, F such that E joins x and $S(x, t)$, $F \supset S(x, r_x)$ joins $S(x, s)$ and $S(x, r_x)$, and $0 < s, t < r_x/2$, $s/t \leqslant T$.

It follows from the Loewner property of $\mathbb{R}^2$ that every planar domain, equipped with Lebesgue measure and any admissible cover, is I-Loewner. The remainder of this section is dedicated to the proof of Theorem 1.2. We first restate the theorem.

THEOREM 4.4. *Let X be a metric space homeomorphic to $\mathbb{R}^2$. There exists an I-QS map $f\colon X \to \Omega$, where $\Omega = \mathbb{D}$ or $\Omega = \mathbb{R}^2$, if and only if X is I-LLC and supports an I-MM μ such that (X, μ) is I-Loewner.*

To prove the theorem, we first show in Lemma 4.5 and Proposition 4.7 that if $f\colon X \to \Omega$ is I-QS, then the pullback of Lebesgue measure satisfies the conditions of the theorem (we already noticed that the existence of f forces X to be I-LLC). For the other direction, we show in Proposition 4.8 that μ-quasiconformal maps $X \to \Omega \subset \mathbb{R}^2$, such as the map in Theorem 1.1, are I-QS under these conditions. Proposition 4.8 can be seen as an infinitesimal analog of [11, Theorem 4.7], and it is proved using similar arguments.

Lemma 4.5. *Let $f\colon X \to \Omega \subset \mathbb{R}^2$ be an I-QS map, and $\mu = f^*\mathcal{L}_2$ the pullback measure of the Lebesgue measure $\mathcal{L}_2$. Equip μ with admissible cover $\mathcal{B} = \{B(x,r) : 0 < r < r_x\}$, where the r_x are the radii in Definition 4.1 of I-QS maps. Then*

$$\eta(1)^{-1}\mathcal{H}^1(f(\mathcal{C})) \leqslant \ell_\mu(\mathcal{C}) \leqslant 4\eta(5)\mathcal{H}^1(f(\mathcal{C}))$$

for any curve $\mathcal{C} \subset X$.

Proof. We may assume that the curve $\mathcal{C}$ is simple and compact. As in Lemma 3.5, it suffices to prove the claim for sets $\mathcal{C}$ for which there exists $\delta > 0$ such that the set of points x satisfying $B(x,\delta) \in \mathcal{B}$ is dense in $\mathcal{C}$.

Fix such a δ and a sequence $(B_j) = (B(x_j, r_j))$ of disjoint balls such that $x_j \in \mathcal{C}$, $5B_j \in \mathcal{B}$, $5r_j < \delta$ and $\mathcal{C} \subset \cup_j 5B_j$, ordered so that if γ is any injective parametrization of $\mathcal{C}$ then $s_j = \gamma^{-1}(x_j)$ is a monotone sequence.

Let

$$T_j = \sup\{t > 0 : B(f(x_j), t) \subset f(B_j)\}$$

for every j. Then there exists $z_j \in X$ with $d(x_j, z_j) \geqslant r_j$ and $|f(x_j) - f(z_j)| \leqslant 2T_j$. Using the infinitesimal quasisymmetry of f we find that for any $y_j \in 5B_j$

$$|f(x_j) - f(y_j)| \leqslant \eta(5)|f(x_j) - f(z_j)|$$

so that $f(5B_j) \subset B(f(x), 2\eta(5)T_j)$. By the choice of T_j also $B(f(x), T_j) \subset f(B_j)$ and thus

$$\mu(5B_j) = \mathcal{L}_2(f(5B_j)) \leqslant 4\pi\eta(5)^2 T_j^2 \leqslant 4\pi\eta(5)^2|f(x_j) - f(x_k)|^2$$

for all $k \neq j$ as the balls B_j are disjoint. Now the δ-content $\ell_{\mu,\delta}$ satisfies

$$\ell_{\mu,\delta}(\mathcal{C}) \leqslant 2\pi^{-1/2}\sum_j \mu(5B_j)^{1/2} \leqslant 4\eta(5)\sum_j |f(x_j) - f(x_{j+1})|.$$

Since $f(\mathcal{C})$ is the nonoverlapping union of the subcurves connecting $f(x_j)$ and $f(x_{j+1})$, we have $\ell_{\mu,\delta}(\mathcal{C}) \leqslant 4\eta(5)\mathcal{H}^1(f(\mathcal{C}))$ for any $\delta > 0$ and thus $\ell_\mu(\mathcal{C}) \leqslant 4\eta(5)\mathcal{H}^1(f(\mathcal{C}))$.

To prove the other inequality, fix $\varepsilon > 0$ and let $B_j = B(x_j, r_j)$ be a sequence of balls in $\mathcal{B}$ covering $\mathcal{C}$ with $\operatorname{diam} B_j < \sigma$ and $B_j \cap \mathcal{C} \neq \emptyset$ for all j and some $\sigma > 0$. Since X is locally compact and $\mathcal{C}$ is compact, $\operatorname{diam} f(B_j) < \varepsilon$ for all j when σ is sufficiently small.

By the infinitesimal quasisymmetry of f we have

$$\operatorname{diam} f(B_j)^2 \leqslant 4\pi^{-1}\eta(1)^2\mathcal{L}_2(f(B_j)) = 4\pi^{-1}\eta(1)^2\mu(B_j)$$

for every j, and hence

$$\mathcal{H}^1_\varepsilon(f(\mathcal{C})) \leqslant 2\pi^{-1/2}\eta(1)\sum_j \mu(B_j)^{1/2}.$$

Thus $\mathcal{H}^1_\varepsilon(f(\mathcal{C})) \leqslant \eta(1)\ell_{\mu,\sigma}(\mathcal{C}) \leqslant \eta(1)\ell_\mu(\mathcal{C})$, and the same upper bound holds for $\mathcal{H}^1$ since ε was arbitrary. □

Corollary 4.6. *Let f and μ be as in Lemma 4.5. Then f is μ-quasiconformal.*

Proof. Let Γ be a curve family in X and $\varepsilon > 0$. We choose a μ-admissible function ρ with $\int_X \rho^2\, d\mu \leqslant \operatorname{mod}_\mu(\Gamma) + \varepsilon$ and define $\tilde{\rho} = \rho \circ f^{-1}$ in Ω. If a curve $\mathcal{C} \in \Gamma$ has locally finite ℓ_μ-measure, then by Lemma 4.5 and a change of variables

$$\int_{f(\mathcal{C})} \tilde{\rho}\, d\mathcal{H}^1 \geqslant \frac{1}{4\eta(5)}\int_{\mathcal{C}} \rho\, d\ell_\mu,$$

so that $4\eta(5)\tilde{\rho}$ is admissible for $f(\Gamma)$. Thus using the definition of μ and a change of variables we have

$$\operatorname{mod}(f(\Gamma)) \leqslant 16\eta(5)^2\int_\Omega \tilde{\rho}^2\, d\mathcal{L}_2 = 16\eta(5)^2\int_X \rho^2\, d\mu \leqslant 16\eta(5)^2\left(\operatorname{mod}_\mu(\Gamma) + \varepsilon\right).$$

The other direction can be proved similarly using the other inequality of Lemma 4.5. □

Proposition 4.7. *Let f, μ and $\mathcal{B}$ be as in Lemma 4.5. Then μ is I-MM and satisfies the I-Loewner condition.*

Proof. Let $\Lambda > 1$ be large enough so that $\eta(1/\Lambda) \leqslant \frac{1}{2}$. Fix $x \in X$ and $0 < r < r_x/2$ so that $\overline{B}(f(x), \operatorname{diam} f(B(x,r))) \subset \Omega$. In order to prove the I-MM condition (2), fix $y \in B(x, r/\Lambda)$ and $z \in S(x,r)$. Then the segment $[f(y), f(z)]$ is contained in Ω. Let $\mathcal{C} = f^{-1}([f(y), f(z)])$, which is a curve connecting y and z.

Now let

$$T = \sup\{t > 0 : B(f(x), t) \subset f(B(x,r))\}.$$

Using Lemma 4.5 and infinitesimal quasisymmetry, we have

$$\begin{aligned}
\ell_\mu(\mathcal{C}) &\leqslant 4\eta(5)\mathcal{H}^1(f(\mathcal{C})) = 4\eta(5)|f(y) - f(z)| \leqslant 4\eta(5)\operatorname{diam} fB(x,r)\\
&\leqslant 8\eta(1)\eta(5)T \leqslant \frac{8\eta(1)\eta(5)}{\sqrt{\pi}}\mathcal{L}_2(f(B(x,r)))^{1/2}\\
&= \frac{8\eta(1)\eta(5)}{\sqrt{\pi}}\mu(B(x,r))^{1/2},
\end{aligned}$$

so the first inequality in (2) holds.

For the reverse inequality, notice first that our choice of Λ implies that $|f(x)-f(y)| \leqslant \frac{1}{2}|f(x)-f(z)|$ and thus $|f(y)-f(z)| \geqslant \frac{1}{2}|f(x)-f(z)|$. Let $\mathcal{C}$ be any curve connecting y and z. Now by Lemma 4.5

$$\begin{aligned}\ell_\mu(\mathcal{C}) &\geqslant \eta(1)^{-1}\mathcal{H}^1(f(\mathcal{C}))\\ &\geqslant \eta(1)^{-1}|f(y)-f(z)| \geqslant \frac{1}{2\eta(1)}|f(x)-f(z)|\\ &\geqslant \frac{1}{2\sqrt{\pi}\eta(1)^2}\mathcal{L}_2(f(B(x,r)))^{1/2} = \frac{1}{2\sqrt{\pi}\eta(1)^2}\mu(B(x,r))^{1/2},\end{aligned}$$

since $f(B(x,r)) \subset B(f(x),\eta(1)|f(x)-f(z)|)$. Hence also the second inequality in (2) holds. We conclude that μ is I-MM.

Finally, we show the I-Loewner condition. Fix $x \in X$ and disjoint continua E and F as in Definition 4.3, so that there are $y \in F \cap S(x,s)$ and $z \in E \cap S(x,t)$. By infinitesimal quasisymmetry,

$$\frac{\operatorname{dist}(fE,fF)}{\operatorname{diam} E} \leqslant \frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leqslant \eta(s/t).$$

By definition, F contains $S(x,r_x)$. In particular, $fS(x,r_x)$ surrounds $f(x)$, and we have $\operatorname{dist}(fE,fF) \leqslant \operatorname{diam} fF$. Combining the estimates yields

$$\frac{\operatorname{dist}(fE,fF)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}} \leqslant \max\{\eta(s/t),1\}.$$

Since $\mathbb{R}^2$ is Loewner, there is ϕ' such that

$$\operatorname{mod}\Gamma(fE,fF) \geqslant \phi'(\max\{\eta(s/t),1\}).$$

On the other hand f is μ-quasiconformal by Theorem 1.1, so

$$\operatorname{mod}_\mu \Gamma(E,F) \geqslant K^{-1}\operatorname{mod}\Gamma(fE,fF)$$

for some $K \geqslant 1$. We conclude that the I-Loewner condition holds with $\phi(T) = K^{-1}\phi'(\max\{\eta(T),1\})$. □

Proposition 4.8. *Let μ be an I-MM on X, and $f\colon X \to \Omega$ a μ-quasiconformal homeomorphism. Suppose that X is I-LLC and μ satisfies the I-Loewner condition. Then f is I-QS.*

Proof. Let Λ and λ be the constants in Definitions 3.1 and 4.2 of I-MM and I-LLC, respectively. We will prove the equivalent statement that $g = f^{-1}$ is I-QS. In this proof, for a point $a \in \Omega$ and set $A \subset \Omega$, let $a' = g(a)$ and $A' = g(A)$.

Fix $x \in \Omega$ and $r > 0$ so that

$$B(x,3r) \subset \Omega \cap g^{-1}\big(B(x',r_{x'}/(10\lambda^4\Lambda^4))\big),$$

and $y,z \in B(x,r)$. By our choice of r, we can choose $w \in g^{-1}S(x',r_{x'})$ so that the segment $[x,w]$ contains z. Moreover, taking r to be sufficiently small, we can ensure that the segment $[x,w]$ lies in Ω. Notice that $w \notin$

$B(x, 3r)$. Let $m = d(x', y')$ and $\ell = d(x', z')$. Let $t > 0$. We must find an upper bound $\eta(t)$ on m/ℓ that holds whenever $|x-y|/|x-z| \leqslant t$, such that $\eta(t) \to 0$ as $t \to 0$. Assume then that y, z satisfy $|x-y|/|x-z| \leqslant t$.

Suppose first that $m/\ell \geqslant \Lambda\lambda^2$. Then, by the I-LLC property, we can connect x' to z' by a continuum E' contained in $B(x', \lambda\ell)$, and y' to w' by a continuum F' contained in $X \setminus B(x', m/\lambda)$. Let $k = \lceil \log_\Lambda(m/(\ell\lambda^2)) \rceil$,

$$B_j = B(x', \Lambda^j \ell/\lambda), \text{ and } A_j = B(x', \Lambda^j \ell/\lambda) \setminus B(x', \Lambda^{j-1}\ell/\lambda).$$

Then, by the definition of I-MM,

$$\rho = \frac{1}{k} \sum_{j=1}^{k} \frac{C_i \chi_{A_j}}{\mu(B_j)^{1/2}}$$

is μ-admissible for $\Gamma(E', F')$. Thus

$$\operatorname{mod}_\mu \Gamma(E', F') \leqslant \int_X \rho^2 \, d\mu \leqslant \frac{1}{k^2} \sum_{j=1}^{k} \frac{C_i^2 \mu(A_j)}{\mu(B_j)} \leqslant \frac{C_i^2}{k} \leqslant \frac{C_i^2}{\log_\Lambda(m/(\ell\lambda^2))}.$$

Hence $\operatorname{mod}_\mu \Gamma(E', F')$ becomes arbitrarily small as m/ℓ increases to infinity.

Since g is μ-quasiconformal, $\operatorname{mod} \Gamma(E, F)$ is also small, where $E = g^{-1}(E')$ and $F = g^{-1}(F')$. But these sets connect x to z and y to w, respectively, and have relative distance

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}} \leqslant \frac{|x-y|}{|x-z|}.$$

Thus, by the Loewner property of $\mathbb{R}^2$, we have $|x-y|/|x-z| \to \infty$ as $m/\ell \to \infty$, establishing the distortion inequality in this case.

Suppose then that $0 < m/\ell < \Lambda\lambda^2$. In this case we choose $E = [x, y]$ and $F = [z, w] \cup g^{-1}S(x', r_{x'})$. We may assume that $2|x-y| < |x-z|$, since otherwise there is nothing to prove. Applying the I-Loewner condition to E' and F', we have

$$\operatorname{mod}_\mu \Gamma(E', F') \geqslant \phi(\ell/m).$$

Combining with the μ-quasiconformality of g, we get $\operatorname{mod} \Gamma(E, F) \geqslant K^{-1}\phi(\ell/m)$. On the other hand, by our choice of w we can estimate $\operatorname{mod} \Gamma(E, F)$ from above as follows:

$$\operatorname{mod} \Gamma(E, F) \leqslant \operatorname{mod} \Gamma(S(x, |x-z|), S(x, |x-y|)) = 2\pi \Big(\log \frac{|x-z|}{|x-y|} \Big)^{-1}.$$

Combining the estimates, we see that $\phi(\ell/m) \leqslant 2\pi K (\log(1/t))^{-1}$. Observe that this bound becomes arbitrarily small as $t \to 0$. Since ϕ is decreasing, this yields an upper bound $\eta(t)$ on m/ℓ that goes to zero as $t \to 0$. □

5. Examples

In this section, we work out in detail a number of specific examples of metric spaces homeomorphic to the plane. All of our examples have locally finite Hausdorff 2-measure, and we assume throughout this section that a given metric space is equipped with the Hausdorff 2-measure. We write a point x in coordinates as $x = (x_1, x_2)$ if $x \in \mathbb{R}^2$ or $x = (x_1, x_2, x_3)$ if $x \in \mathbb{R}^3$.

In addition to the examples of this section, we refer the reader to Example 4.7 of [10] for a family of uniformly LLC surfaces in $\mathbb{R}^3$, equipped with the ambient Euclidean metric, that are conformally equivalent but not uniformly QS equivalent to the Euclidean plane. We also refer to Example 2.1 of [14] for an example of a non-reciprocal metric on the plane, and to Example 17.1 of [14] for a non-rectifiable surface in $\mathbb{R}^3$ that is QC equivalent to the Euclidean plane. Finally, see [17] for the construction of a surface of locally finite Hausdorff 2-measure that is QS equivalent to the plane but not QC equivalent.

5.1. **Conformal weight that decreases rapidly near the origin.** Define a metric d on the Riemann sphere $\widehat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$ via the conformal weight

$$\omega(x) = \begin{cases} e^{-1/|x|}/|x|^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \infty \end{cases}.$$

That is, for all $x, y \in \widehat{\mathbb{R}}^2$, the metric d is given by $d(x,y) = \inf_\gamma \int_\gamma \omega \, ds$, where the infimum is taken over all absolutely continuous paths $\gamma \colon [0,1] \to \widehat{\mathbb{R}}^2$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

It is easy to check that $d(0,x) = e^{-1/|x|}$ for all $x \in \widehat{\mathbb{R}}^2 \setminus \{0\}$. In particular, $d(0,\infty) = 1$ and we see that d is finite. Next, let $x, y \in \widehat{\mathbb{R}}^2 \setminus \{0\}$ and assume that $|x| \leqslant |y|$. By considering the concatenation of the straight-line path from x to $(|y|/|x|)x$ and a circular arc from $(|y|/|x|)x$ to y, we obtain the estimate

$$d(x,y) \leqslant e^{-1/|y|} - e^{-1/|x|} + \frac{2\pi e^{-1/|y|}}{|y|}.$$

As a consequence, if (x_j) and (y_j) are sequences in $\widehat{\mathbb{R}}^2$ such that $x_j \to \infty$ and $y_j \to \infty$, then $d(x_j, y_j) \to 0$. This is sufficient to conclude that $(\widehat{\mathbb{R}}^2, d)$ is homeomorphic to the Riemann sphere.

In fact, by considering the pushforward of ω under the inversion map $x \mapsto x/|x|^2$, we see that $(\widehat{\mathbb{R}}^2, d)$ is isometric to the metric space $(\widehat{\mathbb{R}}^2, \widetilde{d})$, where $\widetilde{d}$ is the metric generated by the conformal weight $\widetilde{\omega}(x) = e^{-|x|}$. In particular, any ball in $(\widehat{\mathbb{R}}^2, d)$ centered at ∞ not containing the origin is bi-Lipschitz equivalent to a Euclidean disk.

In Figure 1, a number of geodesics emanating from the point $p = (.3, 0)$ are plotted. Observe that the length-minimizing path from p to a point q in the upper left region of the plot is the concatenation of the straight-line path from p to the origin and the straight-line path from the origin to q.

This example illustrates how metric quasiconformality is not preserved in general under taking inverses or under precomposition with a quasisymmetry, as the following proposition shows.

Proposition 5.1. *Let $\iota\colon (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, d)$ be the identity map, and let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by $h(x_1, x_2) = (x_1/2, x_2)$.*

(a) ι is MQC with $H = 1$, as is its inverse.
(b) ι is 1-QC.
(c) ι is not I-QS.
(d) $(\iota \circ h)^{-1}$ is MQC.
(e) $\iota \circ h$ is not MQC.

Proof. Claim (a) is immediate for all $x \neq 0$ by virtue of ω being a conformal weight, and it also holds for $x = 0$ by the radial symmetry of ω.

Claim (b) is also immediate if we exclude $x = 0$. However, observe that reciprocality condition (4) holds for the metric d at the origin. Thus the geometric definition is unaffected by adding the origin back in, so the claim holds on all of $\mathbb{R}^2$.

For claim (c), let (t_j) be a sequence of positive numbers converging to zero, and let $y_j = (2t_j, 0)$, $z_j = (t_j, 0)$. Then $|y_j - 0| = 2t_j$, $|z_j - 0| = t_j$, $d(y_j, 0) = \sqrt{e^{-1/t_j}}$, and $d(z_j, 0) = e^{-1/t_j}$. But then $|y_j - 0|/|z_j - 0| = 2$ while $d(y_j, 0)/d(z_j, 0) \to \infty$, violating the I-QS condition.

For claim (d), note that $(\iota \circ h)^{-1} = h^{-1} \circ \iota^{-1}\colon (\mathbb{R}^2, d) \to (\mathbb{R}^2, |\cdot|)$ is the postcomposition of a MQC map by a QS map, which is always MQC.

Claim (e) follows from a variation of the argument for (c). Let (t_j) again be a sequence of positive numbers converging to zero, and let $y_j = (t_j, 0)$ and $z_j = (0, t_j)$. Then $h(y_j) = (t_j/2, 0)$ and $h(z_j) = z_j$. This gives $d(h(y_j), 0) = \sqrt{e^{-1/t_j}}$ and $d(z_j, 0) = e^{-1/t_j}$, showing that $\iota \circ h$ is not MQC. □

Claim (c) of Proposition 5.1 can be strengthened to the following.

Proposition 5.2. *There is no I-QS map $f\colon (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, d)$.*

Proof. Suppose that such an I-QS map f exists. Then f^{-1} is also I-QS. Since metric quasiconformality is preserved under postcomposition by an I-QS map, it follows that $f^{-1} \circ \iota$ is an MQC map of the Euclidean plane. By the equivalence of definitions of quasiconformality in the Euclidean setting (for example, see [20, Thm. 34.1]), we conclude that $f^{-1} \circ \iota$ is QS and thus that ι itself is I-QS. This contradicts claim (c) of Proposition 5.1. □

Note that the claims in Proposition 5.1 all hold if we replace $\mathbb{R}^2$ with $\widehat{\mathbb{R}}^2$ equipped with the spherical metric. We also observe that $(\mathbb{R}^2, d)$ is not upper 2-regular: The Hausdorff 2-measure of the ball $B_r = B(0, r)$, where $r \in [0, 1]$, is given by

$$\mathcal{H}^2(B_r) = \int_{B_r} \omega^2 \, d\mathcal{L}^2 = 2\pi \int_0^R e^{-2/t}/t^3 \, dt,$$

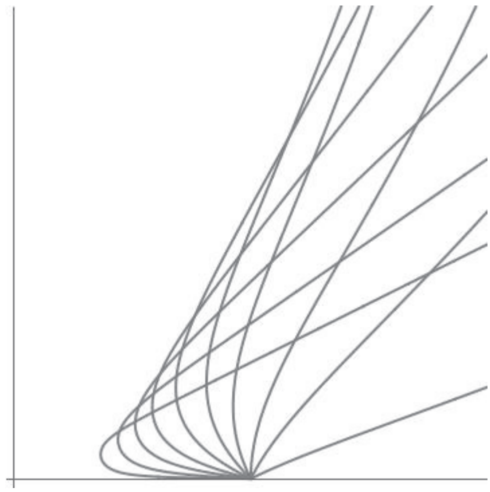

FIGURE 1. Geodesics emanating from the point $(.3,0)$

where $R = -(\log r)^{-1}$. This evaluates to

$$\mathcal{H}^2(B_r) = 2\pi e^{-2/R}\left(\frac{1}{4}+\frac{1}{2R}\right) = 2\pi r^2\left(\frac{1}{4}-\frac{\log r}{2}\right).$$

Since $-\log r \to \infty$ as $r \to 0$, we see that upper 2-regularity fails.

Proposition 5.3. *The space $(\widehat{\mathbb{R}}^2, d)$ is linearly locally connected. However, it is not a Loewner space.*

The proof of linear local connectedness uses the following lemma.

Lemma 5.4. *Let $x \in \mathbb{R}^2$ and $r > 0$ be such that $B(x,r) \subset B(0,e^{-2})$. Then $B(x,r)$ is simply connected.*

Proof. The claim is obvious when $x = 0$, so we assume that $x \neq 0$. We argue by contradiction. Suppose that $B = B(x,r)$ is not simply connected. Since $(\widehat{\mathbb{R}}^2, d)$ is a geodesic space, all metric balls are connected. Hence the failure of simple connectivity implies that there exists a component V of $\widehat{\mathbb{R}}^2 \setminus B$ not containing ∞.

Observe that $B(0,e^{-2})$ coincides with the Euclidean ball $B(0,1/2)$. In this region, ω is increasing as a function of the radius. Let L be the Euclidean straight line which contains x and the origin. The increasing property of ω implies that $L \cap B(0,e^{-2})$ is a geodesic segment. Thus $L \cap B(x,r)$ is connected, and in particular $V \cap L = \emptyset$.

It follows that V is contained in one of the two open half-planes defined by the line L, denoted by W. Let $z \in V$ and let S denote the Euclidean circle of radius $|z|$ centered at the origin. Let L' denote the Euclidean straight line containing 0 and z. Then $W \setminus L'$ consists of two disjoint open sets W_1, W_2, where $x \in \partial W_1$. We observe that there exists a point $y \in S \cap B \cap \overline{W}_2$. A length-minimizing curve from x to y must cross L' at some point v. However, the radial symmetry of ω implies that $d(v,z) \leqslant d(v,y)$, and thus that $d(x,z) \leqslant d(x,y)$. This gives a contradiction, and we conclude that B is simply connected. □

Proof of Proposition 5.3. That $(\widehat{\mathbb{R}}^2, d)$ is linearly locally connected can be shown from Lemma 5.4 as follows. By Lemma 2.5 in [4], it suffices to show that there exists $r_0 > 0$ and $\lambda \geqslant 1$ such that every ball $B(x,r)$ of radius $r \in (0, r_0)$ is contractible inside the ball $B(x, \lambda r)$.

Let $s = e^{-2}/4$ and let $L \geqslant 1$ be such that $(\widehat{\mathbb{R}}^2 \setminus B(0,s), d)$ is L-bi-Lipschitz equivalent to a Euclidean disk. Let $r_0 = e^{-2}/(4L^2)$ and $\lambda = L^2$. For any $r \in (0, r_0)$ and $x \in \widehat{\mathbb{R}}^2$, the ball $B(x, \lambda r)$ is contained in $B(0, e^{-2})$ or it is contained in $(\widehat{\mathbb{R}}^2 \setminus B(0,s), d)$. In the first case, $B(x,r)$ is simply connected by Lemma 5.4 and hence contractible. In the second case, the L-bi-Lispchitz equivalence of $(\widehat{\mathbb{R}}^2 \setminus B(0,s), d)$ with a Euclidean disk implies that $B(x,r)$ is contractible inside $B(x, \lambda r)$. We conclude that $(\widehat{\mathbb{R}}^2, d)$ is linearly locally connected.

We now show that $(\widehat{\mathbb{R}}^2, d)$ is not Loewner. Let $E = (-\infty, 0) \times \{0\}$ and let $F_t = [r_t, R_t] \times \{0\}$ for $t \in (0,1)$, where $r_t = -1/\log(t/2)$ and $R_t = -1/\log t$. Then $\operatorname{dist}(E, F_t) = \operatorname{diam}(F_t) = t$, so that $\Delta(E, F_t) = 1$ for all t. Observe that $\lim_{t \to 0} R_t / r_t = 1$.

Since the identity map $\iota \colon (\mathbb{R}^2, |\cdot|) \to (\mathbb{R}^2, d)$ is 1-QC, the modulus of $\Gamma(E, F_t)$ relative to the metric d is the same as the modulus of the same curve family relative to the Euclidean metric. These curve families arise classically in the Teichmüller ring problem [1, Chapter III]. One can give an upper bound on their modulus as follows. Let Γ_t denote the family of curves which span the open Euclidean annulus $A_t = A((r_t, 0); R_t - r_t, r_t)$, where t is sufficiently small so that $R_t < 2r_t$. For sufficiently small t, the annulus A_t does not intersect E. The family Γ_t majorizes $\Gamma(E, F_t)$ and has modulus $2\pi / \log(r_t/(R_t - r_t))$.

As $t \to 0$, we have that $\operatorname{mod} \Gamma(E, F_t)$ goes to zero. Hence $(\mathbb{R}^2, d)$ is not Loewner. □

The Loewner condition and linear local connectedness are conceptually similar in that they both rule out the existence of cusps and sequences of bottlenecks that become arbitrarily thin. In fact, the two properties are equivalent for the class of Ahlfors 2-regular metric spheres. This follows from Theorem 1.1 and Theorem 1.2 in [4] together with the quasisymmetric invariance of the Loewner condition [19, Cor. 1.6]. This example illustrates how, for metric spheres of finite Hausdorff 2-measure, linear local connectedness does not imply the Loewner condition without the assumption of Ahlfors regularity.

5.2. **An accumulation of spikes, I.** The purpose of this example is to give a metric surface X so that the Hausdorff 2-measure on X is upper 2-regular but X fails to be I-LLC. Upper regularity implies, by Proposition 3.9, that there is a QC parametrization of X by the Euclidean plane. However, X does not admit an MQC parametrization by the Euclidean plane, as shown by the following simple lemma.

Lemma 5.5. *Suppose there is an MQC map $g\colon \Omega \to X$, where Ω is a domain in $\mathbb{R}^2$. Then X is I-LLC.*

Proof. Let $x \in X$ and $x' = g^{-1}(x)$. Let $R_x > 0$ be sufficiently small so that $H_g(x', R) \leqslant 2H$ for all $R < R_x$.

For small $r > 0$, $g^{-1}(B(x,r)) \subset B(x', R_x)$. Let $y, z \in B(x,r)$, $y' = g^{-1}(y)$, $z' = g^{-1}(z)$, and $R' = \sup\{|x' - w'| : w' \in g^{-1}(B(x,r))\}$. Then there is a curve $\mathcal{C}$ from y' to z' which is contained in $B(x', R')$. The metric quasiconformality implies that $g(\mathcal{C})$ is a curve from y to z contained in $B(x, 2Hr)$.

Similarly, let $y, z \in X \setminus B(x,r)$, with $y' = g^{-1}(y)$ and $z' = g^{-1}(z)$. Now, let $R' = \inf\{|x' - w'| : w' \in \Omega \setminus g^{-1}(B(x,r))\}$. Connect y' to z' by a curve $\mathcal{C}$ in $\Omega \setminus B(x', R')$. Then metric quasiconformality implies that $g(\mathcal{C})$ is a curve from y to z contained in $X \setminus B(x, r/(2H))$. This establishes that X is I-LLC. □

We construct this example as a surface in $\mathbb{R}^3$ containing a sequence of spikes that become progressively smaller and converge to a point. For all $n \in \mathbb{N}$, let $t_n = 2^{-n}$, $h_n = 2^{-n/2}$, and $r_n = 2^{-2} \cdot 2^{-3n/2}$. The surface X is constructed by removing each Euclidean disk $B((t_n, 0), r_n)$ from $\mathbb{R}^2$, identified here with $\mathbb{R}^2 \times \{0\}$, and replacing it with a cone S_n of height h_n. That is, S_n has vertex $(t_n, 0, h_n)$ and joins to $\mathbb{R}^2$ along the circle $S((t_n, 0), r_n)$. We equip X with the ambient Euclidean metric from $\mathbb{R}^3$, though the example works just as well if we were to take the induced length metric.

We check that X is upper 2-regular. Let $x \in X$ and $r > 0$. In the first case, assume that $r \leqslant |x|/20$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$. A computation shows that $B(x,r)$ intersects at most one of the cones S_n. It is clear that $\mathcal{H}^2(B(x,r) \cap (\mathbb{R}^2 \times \{0\})) \leqslant \pi r^2$. By the elementary geometry of cones in $\mathbb{R}^3$, it also holds that $\mathcal{H}^2(B(x,r) \cap S_n) \leqslant \pi r^2$. We conclude that $\mathcal{H}^2(B(x,r) \leqslant 2\pi r^2$.

In the second case, assume that $r > |x|/20$. Then $B(x,r) \subset B(0, 21r)$, writing 0 to denote the origin in $\mathbb{R}^3$. For this, we compute

$$\mathcal{H}^2(B(0, 2^{-n})) \leqslant \pi 2^{-2n} + \sum_{k=n}^{\infty} \mathcal{H}^2(S_n)$$

$$\leqslant \pi 2^{-2n} + \pi \sum_{k=n}^{\infty} 2^{-3n/2}\sqrt{2^{-n}+2^{-3n}}$$
$$\leqslant \pi 2^{-2n} + \pi \sum_{k=n}^{\infty} 2^{-3n/2}(2^{-n/2}+2^{-3n/2}) \lesssim 2^{-2n}.$$

We deduce that $\mathcal{H}^2(B(0,21r)) \lesssim r^2$, and therefore that X is upper 2-regular.

Finally, the point $y_n = (t_n, 0, h_n)$ lies outside the ball $B_n = B(0, |y_n|/2)$. Any continuum connecting y_n to the unbounded component of $\mathbb{R}^2 \setminus B_n$ must pass through the smaller ball $X \setminus B(0, 2t_n)$. However, $\lim_{n\to\infty} t_n/|y_n| = 0$, violating the I-LLC property.

5.3. **An accumulation of spikes, II.** By modifying the previous example, we construct a space which is I-QS equivalent to the plane but not QS equivalent.

We carry out the same construction as above, now taking $t_n = 2^{-n}$, $h_n = 2^{-n}$, and $r_n = 2^{-2} \cdot 2^{-2n}$. Instead of cones, we replace the Euclidean disks $B((t_n,0),r_n)$ with cylinders C_n of height h_n. More precisely, $C_n = E_n \cup F_n$, where $E_n = \{(x_1,x_2,x_3) : (x_1,x_2) \in S((t_n,0),r_n),\ 0 \leqslant x_3 \leqslant h_n\}$ and $F_n = B((t_n,0),r_n) + (0,0,h_n)$. Again, we equip the resulting space X with the restriction of the ambient Euclidean metric to X to get (X,d).

The space X is not LLC because the cylinders get progressively narrower; thus X is not QS equivalent to the Euclidean plane. However, we claim that X equipped with $\mu = \mathcal{H}^2$ satisfies the conditions of Theorem 1.2 and therefore admits an I-QS map from $\mathbb{R}^2$.

First, notice that for every $x \in X \setminus \{(0,0,0)\}$ there is $r_x > 0$ so that $B(x, r_x) \subset X$ is 10-bi-Lipschitz equivalent to a planar disk. In particular, the conditions of Theorem 1.2 hold for all such points x.

We still need to verify the conditions of Theorem 1.2 for $x = \mathbf{0} = (0,0,0)$. Take $r_{\mathbf{0}} = 1/2$. The I-LLC condition follows from our choices of t_n, h_n and r_n. Also, calculating as in Section 5.2, we conclude that $r^2 \lesssim \mathcal{H}^2(B(\mathbf{0},r)) \lesssim r^2$ for all $r > 0$. Therefore, the q-metric on X is comparable to the metric d, and μ is I-MM.

Finally, we show that the I-Loewner condition is satisfied at $\mathbf{0}$. For a fixed $T > 0$, let $s,t > 0$ satisfy $s/t \leqslant T$. Let $n \in \mathbb{N}$ be such that $2^{-n-1} \leqslant s < 2^{-n}$. Consider two disjoint continua $E, F \subset X$ as in Definition 4.3. We make the observation that the cylinders C_n and C_{n-1} are separated by a distance of at least 2^{-n-1}. Thus $F \cap (\mathbb{R}^2 \times \{0\}) \cap B(\mathbf{0}, 2^{-n+1})$ contains a continuum F' of diameter at least 2^{-n-1}. Next, we split into two cases. If $t \geqslant s$, then by similar reasoning $E \cap (\mathbb{R}^2 \times \{0\}) \cap B(\mathbf{0}, 2^{-n})$ contains a continuum E' of diameter at least 2^{-n-2}. If $t \leqslant s$, then take E' to be a continuum in $E \cap (\mathbb{R}^2 \times \{0\}) \cap B(\mathbf{0}, t)$ of diameter at least $t/16$.

Then $d(E',F') \leqslant 2^{-n+2}$, and E' and F' have relative distance

$$\Delta(E',F') \leqslant \frac{2^{-n+2}}{\min\{2^{-n-2}, t/16\}} \leqslant \max\{128T, 16\}.$$

Let $T' = \max\{128T, 16\}$, so that $\Delta(E', F') \leqslant T'$. Consider the domain

$$G = \left(\mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \overline{B}((t_n, 0), r_n) \right) \times \{0\} \subset X.$$

The domain G is Loewner; let $\widetilde{\varphi}$ be the associated Loewner function. We have then the inequality

$$\operatorname{mod} \Gamma(E, F) \geqslant \operatorname{mod} \Gamma(E', F'; G) \geqslant \widetilde{\varphi}(T').$$

We conclude that the I-Loewner condition is satisfied at $\mathbf{0}$.

5.4. **Gluing a Grushin half-plane to a Euclidean half-plane.** The *Grushin plane* is a basic example of a sub-Riemannian manifold. See [2, Sec. 3.1] for an overview. One approach to the Grushin plane, studied in [16], is given by the following definition. For each $\beta \in (0, 1)$, the *β-Grushin plane* is $\mathbb{R}^2$ equipped with the metric $\widetilde{d}$ obtained from the singular conformal weight $\widetilde{\omega}\colon \mathbb{R}^2 \to [0, \infty]$ defined by $\widetilde{\omega}(x) = |x_1|^{-\beta}$. The standard Grushin plane is obtained by taking $\beta = 1/2$. Note that the standard Grushin plane does not have locally finite Hausdorff 2-measure. However, in the case when $\beta \in (0, 1/2)$, it was shown in [18] and [23] that the β-Grushin plane is bi-Lipschitz equivalent to the Euclidean plane. In particular, the β-Grushin plane is Ahlfors 2-regular. Moreover, the identity map $\mathbb{R}^2 \to (\mathbb{R}^2, \widetilde{d})$ is QS. A proof of this can be found in [16, Thm. 4.3].

Here, we present a modified version of the Grushin plane. Let $\beta \in (0, 1/2)$. Define the conformal weight $\omega\colon \mathbb{R}^2 \to [0, \infty]$ by

$$\omega(x) = \begin{cases} |x_1|^{-\beta} & \text{if } x_1 > 0 \\ 1 & \text{if } x_1 \leqslant 0 \end{cases}.$$

Let d denote the resulting metric.

First, we establish a ball-box relationship. For all $r \leqslant 1$, let

$$D_r = [-r, (1-\beta) r^{1/(1-\beta)}] \times [-r, r].$$

Note that, for all $x_2 \in \mathbb{R}$, the straight-line curve from $(0, x_2)$ to $((1-\beta)r^{1/(1-\beta)}, x_2)$ has length r. Observe further that $\omega \geqslant 1$ on D_r. From this, it follows that $d(x, 0) \geqslant r$ for all $x \in \partial D_r$. Next, by considering the concatenation of the vertical line segment from 0 to $(0, x_2)$ with the horizontal line segment from $(0, x_2)$ to x, we see that $d(x, 0) \leqslant 2r$ for all $x \in \partial D_r$. We conclude that

$$B_d((0,0), r) \subset D_r \subset B_d((0,0), 2r) \tag{11}$$

for all $r \leqslant 1$.

Next, observe that $\mathcal{H}^2(B_d(0, 2r))$ is bounded from below by

$$\int_{D_r} \omega^2 \, d\mathcal{L}^2 = 2r^2 + 2r \int_0^R t^{-2\beta} \, dt = 2r^2 + \frac{r^{(2-3\beta)/(1-\beta)}}{1 - 2\beta}. \tag{12}$$

For $\beta \in (0, 1/2)$, the inequality $(2 - 3\beta)/(1 - \beta) < 2$ holds, from which we conclude that

$$\liminf_{r \to 0} \frac{\mathcal{H}^2(B_d(x, r))}{r^2} = \infty$$

for all x lying on the vertical axis. On the other hand, (12) is an upper bound on $\mathcal{H}^2(B_d(x,r))$, showing that $(\mathbb{R}^2, d)$ has locally finite Hausdorff 2-measure.

Since ω is constant on each vertical line, we see that metric balls are simply connected. In particular, $(\mathbb{R}^2, d)$ is LLC.

This example illustrates how a metric surface with locally finite 2-measure can violate infinitesimal upper 2-regularity at every point in a fairly large set, namely a nondegenerate continuum. Since any metric surface that is infinitesimally upper 2-regular is reciprocal, this suggests the following question.

Question 5.6. Is there a metric surface for which reciprocality condition (4) fails at every point on a nondegenerate continuum?

The space $(\mathbb{R}^2, d)$ in this example is reciprocal and hence does not answer this question. In fact, the identity map onto the Euclidean plane is 1-QC. This can be shown by a change of variables argument; see also Proposition 3.5 in [8], where the corresponding fact is proved for the β-Grushin plane. In contrast, we have the following.

Proposition 5.7. *There is no MQC map from the Euclidean plane to $(\mathbb{R}^2, d)$.*

Proof. Assume there is a MQC map $f\colon \mathbb{R}^2 \to (\mathbb{R}^2, d)$. Observe that the identity map $\iota\colon \mathbb{R}^2 \to (\mathbb{R}^2, d)$ is locally quasisymmetric outside of the vertical axis Z. This implies that $F = \iota^{-1} \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is MQC outside of the set $f^{-1}(Z)$. By a classical removability theorem for planar quasiconformal mappings [20, Thm. 35.1], it follows that F is globally QS; see also Proposition 2.5 of [8].

Let $x \in f^{-1}(Z)$. By quasisymmetry, there exists $H \geqslant 1$ such that

$$B_{\mathrm{Euc}}(F(x), s(r)) \subset F(B_{\mathrm{Euc}}(x,r)) \subset B_{\mathrm{Euc}}(F(x), Hs(r))$$

for all $r > 0$, where $s(r) = \inf\{|F(x) - F(y)| : y \in \mathbb{R}^2 \setminus B_{\mathrm{Euc}}(x,r)\}$. Comparing this with the ball-box relationship (11), we conclude that f is not MQC. This is a contradiction. □

A similar argument shows that there is no MQC map from $(\mathbb{R}^2, d)$ to $\mathbb{R}^2$.

Acknowledgment. We thank the referee for a careful reading of the paper and detailed feedback, which helped us improve the text significantly.

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[C]

Quasisymmetric Koebe uniformization with weak metric doubling measures

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Preprint

QUASISYMMETRIC KOEBE UNIFORMIZATION WITH WEAK METRIC DOUBLING MEASURES

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Abstract. We give a characterization of metric spaces quasisymmetrically equivalent to a finitely connected circle domain. This result generalizes the uniformization of Ahlfors 2-regular spaces by Merenkov and Wildrick [7].

1. Introduction

A homeomorphism f between metric spaces (X,d) and (Y,d') is *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d'(f(x),f(y))}{d'(f(x),f(z))} \leqslant \eta\left(\frac{d(x,y)}{d(x,z)}\right)$$

for all distinct points $x, y, z \in X$. Quasisymmetric maps form a natural generalization of conformal maps to the setting of abstract metric spaces. In particular, the *uniformization problem* for quasisymmetric maps is important due to applications in areas such as geometric group theory, complex dynamics, and geometric topology.

The uniformization problem asks which spaces admit quasisymmetric maps onto some *standard* space such as $\mathbb{S}^2$. Bonk and Kleiner [1] were able to solve the problem for *Ahlfors* 2*-regular spheres* (X,d), i.e., topological spheres for which the two-dimensional Hausdorff measure $\mathcal{H}^2_d$ satisfies

$$C^{-1}r^2 \leqslant \mathcal{H}^2_d(B_d(x,r)) \leqslant Cr^2 \quad \text{for all } x \in X,\ 0 < r < \operatorname{diam} X.$$

Bonk and Kleiner showed that *linear local connectedness* (see Section 2) is a necessary and sufficient condition for 2-regular spheres to be quasisymmetrically equivalent to $\mathbb{S}^2$.

Later, Merenkov and Wildrick [7] considered the *multiply connected* setting, generalizing the classical Koebe uniformization. They gave characterizations for the Ahlfors 2-regular surfaces that are quasisymmetrically equivalent to some finitely or countably connected circle domains in $\mathbb{S}^2$. Here a *circle domain* is an open and connected subset whose complementary components are geometric disks/points. We refer to [7] for further motivation and background.

Mathematics Subject Classification 2010: Primary 30L10, Secondary 30C65, 28A75.

The authors were supported by the Academy of Finland, project number 308659.

Our aim is to find similar characterizations for surfaces that need not be 2-regular, such as *fractal surfaces*. Uniformization results for fractal surfaces are of great importance in view of applications, cf. [2], [7, Section 2], but one cannot expect results as strong as above to hold.

In [6], Lohvansuu and the authors introduced the *weak metric doubling measures*, generalizing the metric doubling measures, or Strong A_∞-weights, of David and Semmes [3]. These are, roughly speaking, measures that can be used to construct quasisymmetric deformations for a given metric, see Section 2 for the precise definition. We then gave a version of the Bonk-Kleiner theorem in terms of the existence of such measures.

In this paper we apply the weak metric doubling measures to finitely connected surfaces. Namely, we have the following generalization of the characterization given by Merenkov and Wildrick.

THEOREM 1.1. *Let X be a metric space homeomorphic to a domain in $\mathbb{S}^2$ such that $\overline{X} \setminus X$ contains finitely many components. Then X is quasisymmetrically equivalent to a circle domain if and only if it is linearly locally connected, carries a weak metric doubling measure and $\overline{X}$ is compact.*

Here $\overline{X}$ is the completion of X. The "only if" part of Theorem 1.1 follows from the definitions in a straightforward manner. Theorem 2.2 below is a quantitative version of the "if" part.

To prove this, we first apply the weak metric doubling measure to suitably deform the metric on X. We show that the deformed space is *reciprocal* in the sense of [9], and therefore admits a *quasiconformal* map into $\mathbb{S}^2$ by a recent result of Ikonen [5]. We then apply geometric estimates to show that this map, when suitably normalized, is quasisymmetric. Our approach is different from those in [7] and [6], both of which apply the Bonk-Kleiner theorem.

2. Weak metric doubling measures

For $x, y \in X$ and $\delta > 0$, a finite sequence of points $x_0, x_1, \ldots, x_m$ in X is a *δ-chain from x to y*, if $x_0 = x$, $x_m = y$ and $d(x_j, x_{j-1}) \leqslant \delta$ for every $j = 1, \ldots, m$. Notice that in every connected metric space each pair of points can be connected by a δ-chain for any $\delta > 0$.

Recall that a measure μ in a metric space (X, d) is *doubling* if there is $C_D \geqslant 1$ such that

$$\mu(B_d(x, 2r)) \leqslant C_D \mu(B_d(x, r)) \quad \text{for all } x \in X,\ r > 0.$$

From now on we assume that μ is a Radon measure in X that is doubling with constant C_D.

In what follows, we use notation

$$B_{xy} = B_d(x, d(x, y)) \cup B_d(y, d(x, y)).$$

Given μ and a "dimension" $s > 0$, we define the *μ-length* $q_{\mu,s}$ of points $x, y \in X$ as follows: set

$$q^{\delta}_{\mu,s}(x,y) := \inf \left\{ \sum_{j=1}^{m} \mu(B_{x_j x_{j-1}})^{1/s} \colon (x_j)_{j=0}^{m} \text{ is a } \delta\text{-chain from } x \text{ to } y \right\}$$

and

$$q_{\mu,s}(x,y) := \limsup_{\delta \to 0} q^{\delta}_{\mu,s}(x,y).$$

Definition 2.1. We say that μ is a *C_W-weak metric doubling measure*, or *WMDM*, of dimension $s > 0$ in (X,d), if for all $x, y \in X$,

$$\frac{1}{C_W} \mu(B_{xy})^{1/s} \leqslant q_{\mu,s}(x,y). \tag{1}$$

From now on we assume that μ is a C_W-WMDM of dimension 2, and we abbreviate $q = q_{\mu,2}$. See [6] for examples and further discussion.

Weak metric doubling measures should be compared to the *metric doubling measures* of David and Semmes. They are essentially defined by requiring that in addition to (1) also the reverse inequality holds.

Recall that a metric space (X,d) is *λ-linearly locally connected*, or *LLC*, if for any $x \in X$ and $r > 0$,

(i) if $y, z \in B_d(x,r)$ then there exists a continuum $K \subset B_d(x, \lambda r)$ with $y, z \in K$, and
(ii) if $y, z \in X \setminus B_d(x,r)$ then there exists a continuum $K \subset X \setminus B_d(x, r/\lambda)$ with $y, z \in K$.

From now on we assume that (X,d) is λ-LLC and homeomorphic to a circle domain such that $\overline{X}$ is compact and $\overline{X} \setminus X$ contains $M < \infty$ components. We denote by C_X the ratio of the diameter of (X,d) to the minimum distance between the components of $\overline{X} \setminus X$. We are now ready to state the main result of this paper.

THEOREM 2.2. *There is an η-quasisymmetric homeomorphism from (X,d) onto a circle domain $\Omega \subset \mathbb{S}^2$, where η depends only on λ, C_X, C_D, C_W, and M.*

Here and in what follows, $\mathbb{S}^2$ is equipped with the usual chordal metric and $\mathbb{R}^2$ with the euclidean metric.

As pointed out in the introduction, Theorem 1.1 is a straightforward consequence of Theorem 2.2. We do not know if the dependence on the number of components M is necessary in Theorem 2.2, and if it admits extensions to countably connected domains corresponding to [7, Theorem 1.4]. The rest of the paper is dedicated to the proof of Theorem 2.2.

3. Deformation of the metric

Theorem 2.2 is proved by showing that the μ-length q is a metric on X with strong geometric properties. Our approach is based on the following reverse inequality for WMDMs.

Proposition 3.1. *For every $x \in X$ there is $r_x > 0$ such that*

$$q(x,y) \leqslant C_S \mu(B_{xy})^{1/2}$$

for all $y \in B_d(x, r_x)$, where $C_S = 16 C_W C_D^{28+16\lceil \log_2 \lambda \rceil}$.

Before proving Proposition 3.1, we state some consequences. We will apply the following elementary property of doubling measures, see [4, 13.1]: For all $x \in X$ and $0 < r \leqslant R < \operatorname{diam}_d(X)$,

$$\frac{1}{C}\left(\frac{R}{r}\right)^{1/\alpha} \leqslant \frac{\mu(B_d(x,R))}{\mu(B_d(x,r))} \leqslant C\left(\frac{R}{r}\right)^{\alpha}. \tag{2}$$

Here C and α depend only on C_D.

Corollary 3.2. *(X,q) is a metric space homeomorphic to (X,d).*

Proof. Combine the definitions with Proposition 3.1 and (2). □

We use notations B_d and B_q for the open balls in (X,d) and (X,q), respectively. We next give estimates for measures of balls in (X,q).

Lemma 3.3. *Let $x \in X$ and $s > 0$. Then*

$$\mu(B_q(x,s)) \leqslant C_W^2 s^2. \tag{3}$$

Moreover, if $r_x > 0$ is as in Proposition 3.1 and $B_q(x,s) \subset B_d(x,r_x)$, then

$$\frac{s^2}{2C_S^2 C_D} \leqslant \mu(B_q(x,s)). \tag{4}$$

Proof. First, we apply the WMDM-definition 2.1 to establish the inclusions

$$\begin{aligned} B_q(x,s) &\subset \{y : \mu(B_{xy})^{1/2} < C_W s\} \\ &\subset \{y : \mu(B_d(x,d(x,y)))^{1/2} < C_W s\} = B_d(x,r_s) \end{aligned}$$

for some $r_s > 0$. Since $\mu(B_d(x,r_s)) \leqslant C_W^2 s^2$, (3) follows. Similarly, Proposition 3.1 and doubling yield

$$\begin{aligned} B_q(x,s) &\supset \{y : C_S \mu(B_{xy})^{1/2} < s\} \\ &\supset \{y : C_D^{1/2} C_S \mu(B_d(x,d(x,y)))^{1/2} < s\}, \end{aligned}$$

from which (4) follows. □

It follows from the above estimates that μ is in fact comparable to the 2-dimensional Hausdorff measure $\mathcal{H}_q^2$ in (X,q). We normalize $\mathcal{H}_q^2$ so that it coincides with the Lebesgue measure if q is the euclidean metric in $\mathbb{R}^2$.

Corollary 3.4. *We have*

$$\frac{1}{2\pi C_S^2 C_D^4}\mathcal{H}_q^2(E) \leqslant \mu(E) \leqslant \frac{C_W^2}{\pi}\mathcal{H}_q^2(E) \tag{5}$$

for all Borel sets $E \subset X$. In particular,

$$\mathcal{H}_q^2(B_q(x,s)) \leqslant 2\pi C_S^2 C_D^4 C_W^2 s^2 \tag{6}$$

for all $x \in X$ and $s > 0$.

Proof. The second inequality in (5) follows directly from (3) and the definition of $\mathcal{H}_q^2$. Also, (6) follows directly from (3) and the first inequality in (5).

For the first inequality in (5), we may assume that E is open since μ is Radon. Given $\delta > 0$, we can apply the $5r$-covering lemma to cover E with balls $B_q^j(x_j, s_j) \subset E$ satisfying (4) such that the balls $B_q^j(x_j, s_j/5)$ are pairwise disjoint and each $s_j < \delta$. We denote the corresponding δ-content by $\mathcal{H}_{q,\delta}^2$. Then by (4), the doubling property of μ, and the disjointness give

$$\begin{aligned} \mathcal{H}_{q,\delta}^2(E) &\leqslant \pi \sum_j s_j^2 \leqslant 2\pi C_S^2 C_D \sum_j \mu(B_q(x_j, s_j)) \\ &\leqslant 2\pi C_S^2 C_D^4 \sum_j \mu(B_q(x_j, s_j/5)) \leqslant 2\pi C_S^2 C_D^4 \mu(E). \end{aligned}$$

The claim follows by taking $\delta \to 0$. □

4. Proof of Proposition 3.1

We prove Proposition 3.1 by constructing a continuum connecting the given points with controlled q-diameter. We define the q-diameter with

$$\operatorname{diam}_q(A) = \sup_{a,b\in A} q(a,b)$$

for $A \subset X$, which makes sense even though we have not yet proved that q is a finite distance. Note also that the definition of q implies that it satisfies the standard triangle inequality.

As a first step of the construction we find separating continua in small annuli. We denote $\ell = \lceil \log_2 \lambda \rceil$ for the rest of this section.

Lemma 4.1. *Let $x \in X$ and $r > 0$ such that $\overline{B}_d(x,(2\lambda)^7 r)$ is compact and contained in a topological disk $U \subset X$. Then there exists a continuum $K \subset \overline{B}_d(x,(2\lambda)^6 r) \setminus B_d(x, 2\lambda r)$ separating $B_d(x,r)$ and $X \setminus \overline{B}_d(x,(2\lambda)^7 r)$ with*

$$\operatorname{diam}_q(K) \leqslant 8 C_W C_D^{12+4\ell} \mu(B_d(x,r))^{1/2}. \tag{7}$$

Proof. We use notation $S_d(x,r) = \{y \in X : d(x,y) = r\}$. Let $E = S_d(x,(2\lambda)^3 r)$ and $F = S_d(x,(2\lambda)^4 r)$. By (1) and a standard compactness argument (see [6, 4.1]) there exists $\delta_{x,r} > 0$ such that for all $y \in E$, $z \in F$ and $0 < \delta < \delta_{x,r}$

$$q^\delta(y,z) \geqslant \frac{1}{2C_W C_D}\mu(B_{yz})^{1/2}. \tag{8}$$

Fix $0 < \delta < \min(\delta_{x,r}, r)$. Using the doubling condition, the $5r$-covering lemma and (2) we can find a cover

$$\mathcal{B} = \{B_1^i\}_{i=1}^m = \{B_d(x_i, r_i)\}_{i=1}^m \tag{9}$$

for the annulus

$$A = B_d(x, (2\lambda)^5 r) \setminus \overline{B}_d(x, (2\lambda)^2 r)$$

such that the balls $B_d(x_i, r_i/5)$ are pairwise disjoint, $r_i < \delta/2$ and

$$\varepsilon^2 \leqslant \mu(B_d(x_i, r_i/5)) \leqslant C_D \varepsilon^2$$

for every i and some fixed $\varepsilon > 0$. The balls in the cover are contained in $B_d(x, (2\lambda)^6 r) \setminus \overline{B}_d(x, (2\lambda) r)$ by the choice of r_i and δ.

If $z \in F$, there exists by the LLC-condition a continuum contained in A that connects z to E. Thus there is a chain of balls $B_1, \ldots, B_n \in \mathcal{B}$ such that for some $y \in E$ we have $y \in B_1$, $z \in B_n$ and $B_j \cap B_{j+1} \neq \emptyset$ for every $j = 1, \ldots, n-1$. With this in mind, we define

$$\mathcal{B}_1 = \{B \in \mathcal{B} : B \cap E \neq \emptyset\}$$

and

$$\mathcal{B}_j = \{B \in \mathcal{B} \setminus \left(\bigcup_{k=1}^{j-1} \mathcal{B}_k\right) : B \cap \left(\bigcup_{B' \in \mathcal{B}_{j-1}} B'\right) \neq \emptyset\}.$$

The collections $\mathcal{B}_j$ form layers selected from the cover $\mathcal{B}$, the first containing those balls that intersect E and the subsequent ones those not previously selected which intersect with the previous layer.

Recall that each $z \in F$ is contained in some $B_z \in \mathcal{B}_n$, where n depends on z. We claim that

$$n \geqslant \frac{\sqrt{m}}{4C_W C_D^{6+\ell}} \tag{10}$$

for all such z, where m is the number of balls in the cover (9). Indeed, if $B_1, \ldots, B_n$ is a chain of balls as above, then their centers and the points y and z form a δ-chain, and by (8)

$$\mu(B_{yz})^{1/2} \leqslant 4C_W C_D^2 \sum_{j=1}^n \mu(B_j)^{1/2} \leqslant 4C_W C_D^4 n\varepsilon.$$

But

$$B_d(x, (2\lambda)^6 r) \subset B_d(y, 4(2\lambda)^2 d(y,z))$$

and as the m balls $B_d(x_i, r_i/5)$ in $\mathcal{B}$ are pairwise disjoint, we have

$$\mu(B_{yz}) \geqslant \mu(B_d(x, (2\lambda)^6 r))/C_D^{4+2\ell} \geqslant m\varepsilon^2/C_D^{4+2\ell},$$

so (10) follows.

Let $n_0 = \lceil \sqrt{m}/4C_W C_D^{6+\ell} \rceil$. As the layers $\mathcal{B}_j$ are pairwise disjoint, we have

$$\begin{aligned}\sum_{j=1}^{n_0} \sum_{B_i \in \mathcal{B}_j} \mu(B_i)^{1/2} &\leqslant C_D^2 m\varepsilon \\ &\leqslant 4C_W C_D^{8+\ell} n_0 \sqrt{m}\varepsilon \\ &\leqslant 4C_W C_D^{11+4\ell} n_0 \mu(B_d(x,r))^{1/2}.\end{aligned}$$

Hence for some $1 \leqslant j_0 \leqslant n_0$ we have

$$\sum_{B_i \in \mathcal{B}_{j_0}} \mu(B_i)^{1/2} \leqslant C' \mu(B_d(x,r))^{1/2},$$

where $C' = 4C_W C_D^{11+4\ell}$. We denote by K_1' the compact set $\cup_{B_i \in \mathcal{B}_{j_0}} \overline{B}_i$. By the choice of n_0 and the LLC-condition K_1' separates $B_d(x,r)$ and $X \setminus \overline{B}_d(x,(2\lambda)^7 r)$. Moreover, since $\overline{B}_d(x,(2\lambda)^7 r) \subset U$ for some $U \subset X$ homeomorphic to a disk, a component K_1 of K_1' also separates the same sets (see for example [8, V 14.3]).

By repeating the above construction for δ/j, $j = 2, 3, \ldots$ we obtain continua K_j, each separating $B_d(x,r)$ and $X \setminus \overline{B}_d(x,(2\lambda)^7 r)$. By connectedness, between any two points of K_j there exists a δ/j-chain among the centers of the balls B_j^i covering K_j. For each j we have the same estimate

$$\sum_i \mu(B_j^i)^{1/2} \leqslant C' \mu(B_d(x,r))^{1/2}.$$

Then using compactness in the Hausdorff metric for compact sets we find a subsequence of (K_j) converging to a compact set K'. Now also K' and hence one of its components K again separates $B_d(x,r)$ and $X \setminus \overline{B}_d(x,(2\lambda)^7 r)$.

If $a, b \in K$ and $\delta' > 0$, we pick a large j such that $\delta/j < \delta'$ and the Hausdorff distance between K and K_j is less than δ'. Then from K_j we find points $p_1, \ldots, p_{l-1}$ so that a and b are connected by the δ'-chain $a = p_0, p_1, \ldots, p_{l-1}, p_l = b$ with

$$q^{\delta'}(a,b) \leqslant \sum_{i=1}^{l} \mu(B_{p_i p_{i-1}})^{1/2} \leqslant 2C_D C' \mu(B_d(x,r))^{1/2}.$$

Since the upper bound holds for all $\delta' > 0$ the estimate is true also for $q(a,b)$. □

For $x \in X$, $r > 0$ and K as in Lemma 4.1, we let

$$\begin{aligned} K(x,r) &= K \quad \text{and} \\ \hat{K}(x,r) &= \text{the component of } X \setminus K(x,r) \text{ containing } x. \end{aligned}$$

The following lemma on basic planar topology allows us to connect $K(x_1,r_1)$ and $K(x_2,r_2)$ for correctly chosen adjacent balls $B_d(x_1,r_1)$ and $B_d(x_2,r_2)$. We refer to [6, 5.1] for a proof.

Lemma 4.2. *Let $x_1, x_2 \in X$ and $r_1, r_2 > 0$ be as in Lemma 4.1 such that*

(1) $\hat{K}(x_1, r_1)$ and $\hat{K}(x_2, r_2)$ intersect,
(2) $\hat{K}(x_1, r_1) \not\subset \hat{K}(x_2, r_2)$,
(3) $\hat{K}(x_2, r_2) \not\subset \hat{K}(x_1, r_1)$.

Then the continua $K(x_1, r_1)$ and $K(x_2, r_2)$ intersect.

With these lemmas we are ready to construct the desired continuum between the given points.

Proof of Proposition 3.1. Let $y \in X$ be such that $B_d(x, 2\lambda d(x, y))$ is contained in a topological disk. Then, as in the proof of Lemma 4.1, we can cover the ball $B_1 = B_d(x, \lambda d(x, y))$ with M_1 balls

$$B_1^i = B_d(z_i^1, r_i^1)$$

such that $z_i^1 \in B_1$, the balls $\frac{1}{5}B_1^i$ are pairwise disjoint, and

$$\mu(B_1)/4C_D^{8+7\ell} \leqslant \mu((2\lambda)^7 B_1^i) \leqslant \mu(B_1)/4C_D^{7+7\ell} \tag{11}$$

for each i. The doubling condition and (11) now imply that

$$M_1 \leqslant C_D^{19+14\ell} \tag{12}$$

and that

$$(2\lambda)^7 r_i^1 \leqslant \frac{\lambda \operatorname{radius}(B_1)}{4}.$$

Furthermore, Lemma 4.1 can be applied with z_i^1 and r_i^1 for each i.

Let $\mathcal{I}_1$ be the set of indices i such that B_1^i intersects the component D_1 of B_1 containing x and $\hat{K}(z_i^1, r_i^1) \not\subset \hat{K}(z_j^1, r_j^1)$ for all $j \neq i$. For future reference, notice that $y \in D_1$ by the LLC-condition. Then the compact set

$$K_1 = \bigcup_{i \in \mathcal{I}_1} K(z_i^1, r_i^1) \subset 2B_1$$

is connected. Indeed, if $k, l \in \mathcal{I}_1$, there exists a path from $\hat{K}(z_k^1, r_k^1)$ to $\hat{K}(z_l^1, r_l^1)$ in B_1. This path is now covered by a chain of sets $\hat{K}(z_i^1, r_i^1)$, $i \in \mathcal{I}_1$, so that for consecutive members in the chain the corresponding continua $K(z_i^1, r_i^1)$ intersect by Lemma 4.2 and the choice of $\mathcal{I}_1$.

Next we choose $h \in \mathcal{I}_1$ such that $x \in \hat{K}(z_h^1, r_h^1)$ and denote

$$B_2 = B_d(z_h^1, (2\lambda)^7 r_h^1).$$

Then $x \in B_2$ and $2B_2 \subset \frac{1}{2}B_1$. We cover B_2 with M_2 balls

$$B_2^i = B_d(z_i^2, r_i^2)$$

so that all the properties above remain valid with the balls B_1^i replaced by the balls B_2^i. In particular, (11) takes the form

$$\mu(B_2)/4C_D^{8+7\ell} \leqslant \mu((2\lambda)^7 B_2^i) \leqslant \mu(B_2)/4C_D^{7+7\ell}.$$

Repeating the previous construction then yields continuum

$$K_2 = \bigcup_{i \in \mathcal{I}_2} K(z_i^2, r_i^2) \subset 2B_2. \tag{13}$$

We next show that

$$K_1 \cap K_2 \neq \emptyset. \tag{14}$$

First, if $K(z_i^2, r_i^2)$ is one of the continua in (13), then $\hat{K}(z_h^1, r_h^1) \not\subset \hat{K}(z_i^2, r_i^2)$, since otherwise we would have

$$B_d(z_h^1, r_h^1) \subset B_d(z_i^2, (2\lambda)^7 r_i^2)$$

and by (11)

$$\mu(B_d(z_h^1, r_h^1)) \leqslant \mu(B_2)/4C_D^{7+7\ell} \leqslant \mu(B_d(z_h^1, r_h^1))/4,$$

a contradiction.

Secondly, if

$$w \in \overline{\hat{K}(z_h^1, r_h^1)} \cap K(z_h^1, r_h^1)$$

then at least one of the sets $K(z_i^2, r_i^2)$ in (13) satisfies $w \in \hat{K}(z_i^2, r_i^2)$. Then also

$$\hat{K}(z_i^2, r_i^2) \not\subset \hat{K}(z_h^1, r_h^1) \quad \text{and} \quad \hat{K}(z_h^1, r_h^1) \cap \hat{K}(z_i^2, r_i^2) \neq \emptyset.$$

Thus

$$K(z_h^1, r_h^1) \cap K(z_i^2, r_i^2) \neq \emptyset$$

by Lemma 4.2, and (14) follows.

We continue the above process to obtain continua

$$K_j = \bigcup_{i \in \mathcal{I}_j} K(z_i^j, r_i^j) \subset 2B_j$$

for each $j \in \mathbb{N}$, such that $K_j \subset 2B_j \ni x$ for all j, and

$$\operatorname{diam}_d(B_j) \to 0 \quad \text{as } j \to \infty.$$

Moreover, applying the constructions of the balls B_j together with estimates (11) applied to these balls, we get

$$\mu(B_{j+1})^{1/2} \leqslant \frac{1}{2}\mu(B_j)^{1/2}. \tag{15}$$

Repeating the argument in the previous paragraphs, we see that $K_j \cap K_{j+1} \neq \emptyset$ for all j. Therefore,

$$K = \cup_{j=1}^{\infty} K_j \cup \{x\}$$

is a continuum.

We now apply (7) and the construction of the set K to estimate its q-diameter. First, if $a, b \in K_1$, then for some $i_1, \ldots, i_{m+1} \in \mathcal{I}_1$ and points $x_1, \ldots, x_m \in K_1$ we have $a \in K(z_{i_1}^1, r_{i_1}^1)$,

$$x_1 \in K(z_{i_1}^1, r_{i_1}^1) \cap K(z_{i_2}^1, r_{i_2}^1), \ldots, x_m \in K(z_{i_m}^1, r_{i_m}^1) \cap K(z_{i_{m+1}}^1, r_{i_{m+1}}^1),$$

and $b \in K(z^1_{i_{m+1}}, r^1_{i_{m+1}})$. By (7),

$$q(a,b) \leqslant q(a,x_1) + q(b,x_m) + \sum_{j=1}^{m-1} q(x_j,x_{j+1})$$
$$\leqslant 8C_W C_D^{12+4\ell} \sum_{j=1}^{m+1} \mu(B(z^1_{i_j}, r^1_{i_j}))^{1/2}.$$

Since $m+1 \leqslant M_1$, combining with (11) and (12) gives

$$q(a,b) \leqslant C_2 \mu(B_1)^{1/2},$$

where $C_2 = 4C_W C_D^{28+15\ell}$. In particular, we get an upper bound for the q-diameter of K_1. Repeating the argument, we get

$$\mathrm{diam}_q(K_j) \leqslant C_2 \mu(B_j)^{1/2}$$

for all j. Combining with (15), we moreover have

$$\mathrm{diam}_q(K_j) \leqslant 2^{1-j} C_2 \mu(B_1)^{1/2} \tag{16}$$

for each j.

Now let $w_0 \in K_1$. Fix $\delta > 0$, $\varepsilon > 0$, and

$$w_j \in K_j \cap K_{j+1}$$

for each $j \geqslant 1$. Since $d(w_j, x) \to 0$, we find $k \in \mathbb{N}$ such that $d(w_k, x) < \delta$ and $\mu(B_{w_k x})^{1/2} < \varepsilon$. Then, by (16),

$$q^\delta(w_0,x) \leqslant \sum_{j=1}^{k} \mathrm{diam}_q(K_j) + q^\delta(w_k,x) \leqslant 2C_2\mu(B_1)^{1/2} + \varepsilon$$

and hence

$$q(w_0,x) \leqslant 2C_2\mu(B_1)^{1/2}. \tag{17}$$

Finally, recall that our goal is to bound $q(x,y)$. Since $y \in D_1$, we can repeat the argument above with the same cover for B_1 to find that (17) holds with x replaced by y. By triangle inequality, we conclude that

$$q(x,y) \leqslant 4C_2\mu(B_1)^{1/2} \leqslant 4C_D^\ell C_2 \mu(B_{xy})^{1/2}.$$

The proof is complete. □

5. Quasiconformal uniformization

Our strategy for proving Theorem 2.2 is to apply the existence of a quasiconformal homeomorphism f from (X,q) to a circle domain Ω. This is guaranteed by a recent result of Ikonen [5] and the classical Koebe uniformization of finitely connected Riemann surfaces. We will show in Sections 6 and 7 that f is in fact quasisymmetric, with respect to the original metric d, under a suitable normalization.

We recall the geometric definition of quasiconformal maps. Let $Y = (Y,d)$ be a metric space such that $\mathcal{H}^2_d$ is finite on compact subsets. We moreover assume that Y is a topological 2-manifold. It then follows that $\mathcal{H}_d^2$ is positive on open sets, cf. [9].

Let Γ be a family of paths in Y. We say that a Borel function $\rho \geqslant 0$ in Y is *admissible* for Γ, if

$$\int_\gamma \rho \, ds \geqslant 1 \quad \text{for all locally rectifiable } \gamma \in \Gamma.$$

The *(conformal) modulus* of Γ is

$$\operatorname{mod}(\Gamma) = \inf \int_Y \rho^2 \, d\mathcal{H}_d^2,$$

where the infimum is taken over all admissible functions.

A homeomorphism $f : Y \to Z$ between spaces as above is (geometric) K-*quasiconformal*, if

$$K^{-1} \operatorname{mod}(\Gamma) \leqslant \operatorname{mod}(f\Gamma) \leqslant K \operatorname{mod}(\Gamma)$$

for all path families Γ in Y, where $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$.

It is shown in [9] and [10] that if Y is a topological disk for which there exists $C > 0$ such that

$$\mathcal{H}_d^2(B_d(y,r)) \leqslant Cr^2 \quad \text{for all } y \in Y,\, r > 0,$$

then there exists a $\pi/2$-quasiconformal homeomorphism from Y into the euclidean plane. Recently Ikonen [5] generalized this result to the case of non-simply connected surfaces. In particular, he showed that the upper bound (6) guarantees that there is a $\pi/2$-quasiconformal homeomorphism from our space (X,q) onto a Riemann surface Z. Moreover, by the classical uniformization theorem for finitely connected Riemann surfaces, there is a conformal map from Z onto a circle domain Ω. Recall that conformal maps are 1-quasiconformal in the sense of the geometric definition above, and that the composition of a K_1- and a K_2-quasiconformal map is K_1K_2-quasiconformal. Thus we have the following.

Proposition 5.1. *There is a $\pi/2$-quasiconformal homeomorphism $f : (X,q) \to \Omega$, where $\Omega \subset \mathbb{S}^2$ is a circle domain. If moreover (X,q) is not homeomorphic to $\mathbb{S}^2$, then the statement remains valid with circle domain $\Omega \subset \mathbb{R}^2$.*

The second statement follows from the first simply by postcomposing f with a suitable Möbius transformation followed with the stereographic projection.

6. Modulus estimate in circle domains

In this section we assume that $\overline{X} \setminus X$ has at least two components. Let $\Omega \subset \mathbb{R}^2$ be the circle domain in Proposition 5.1. We now give a modulus estimate which, along with Proposition 3.1, is the main technical result towards Theorem 2.2.

In what follows, we denote by $\Gamma(A, B; G)$ the family of paths joining sets $A, B \subset \overline{G}$ in G, i.e., all the paths $\gamma : [a, b] \to \overline{G}$ such that $\gamma(a) \in A$, $\gamma(b) \in B$, and $\gamma(t) \in G$ for all $a < t < b$. We abbreviate $\operatorname{mod}(A, B; G) = \operatorname{mod}(\Gamma(A, B; G))$.

Proposition 6.1. *Let $E_1, E_2 \subset \Omega$ be disjoint continua such that*

$$\frac{\min\{\operatorname{diam}(E_1), \operatorname{diam}(E_2)\}}{\operatorname{dist}(E_1, E_2)} \geqslant 1. \tag{18}$$

Then

$$\operatorname{mod}(E_1, E_2; \Omega) \geqslant \frac{\alpha^M}{2\pi(10M)^M(M+2)^2}, \tag{19}$$

where

$$\alpha = 2^{-2-2M-\pi^2 C_W^2 C_D^{1+\log_2 C_X}/8\log 2}.$$

The rest of this section is devoted to the proof of Proposition 6.1. We denote the complementary components of Ω by

$$D_1, \ldots, D_M, \quad D_i = \overline{D}(z_i, r_i).$$

Complementary point-components do not have effect on the modulus. Therefore, we can assume that $r_i > 0$ for all i. We use notation

$$\Delta(i, j) = \frac{\operatorname{dist}(D_i, D_j)}{\min\{r_i, r_j\}}$$

for the relative distances. The homeomorphism f in Proposition 5.1 uniquely extends to a bijection from the set of components of $\overline{X} \setminus X$ to the set $\{D_i\}$. We denote by A_i the component corresponding to D_i under this bijection.

Lemma 6.2. *We have $\Delta(i, j) \geqslant \alpha$ for every $i \neq j$, where α is the constant in Proposition 6.1.*

Proof. Fix $i \neq j$ such that $r_i \leqslant r_j$. We consider $\operatorname{mod}(D_i, D_j; \Omega)$. We first claim that

$$\operatorname{mod}(D_i, D_j; \Omega) \leqslant \frac{\pi}{2} \operatorname{mod}(A_i, A_j; X) \leqslant \frac{\pi C_W^2 C_D^{1+\log_2 C_X}}{2}. \tag{20}$$

The first inequality follows from the quasiconformality of f. Towards the second inequality, recall that C_X is the ratio of the diameter of (X, d) to the minimum d-distance D between the components A_i. Let $m \geqslant 1$ be the smallest integer such that $C_X \leqslant 2^m$. Then, by the WMDM-condition and the doubling property of μ, the length of every path in $\Gamma(A_i, A_j; X)$ is at least

$$C_W^{-1} \inf_{x \in X} \mu(B(x, D))^{1/2} \geqslant C_W^{-1} C_D^{-m/2} \mu(X)^{1/2}.$$

Therefore,

$$\operatorname{mod}(A_i, A_j; X) \leqslant \int_X C_W^2 C_D^m \mu(X)^{-1} \, d\mu = C_W^2 C_D^m,$$

and (20) follows. We prove the lower bound for $\Delta(i,j)$ by showing that the opposite of (20) holds if $\Delta(i,j) < \alpha$.

Let $s = \mathrm{dist}(D_i, D_j)$ and

$$w = z_i + \frac{(r_i + \frac{s}{2})(z_j - z_i)}{|z_j - z_i|}$$

be the point in the middle of D_i and D_j. If $\Delta(i,j) < \alpha$, we have

$$2^{N+2}s \leqslant r_i, \tag{21}$$

where $N = \lfloor 2M + \pi^2 C_W^2 C_D^{1+\log_2 C_X}/8\log 2 \rfloor$. We consider the path families

$$\Psi_n = \{\text{components of } S(w, ts) \cap \overline{\Omega} : 2^{n-1} < t < 2^n\}$$

for $n = 1, \ldots, N$. Every path in $\Psi_0 \cup \cdots \cup \Psi_N$ either connects D_i and D_j or intersects some D_k, $k \neq i, j$. We claim that any such $D_k = \overline{D}(z_k, r_k)$ intersects paths from at most two families Ψ_n.

Suppose to the contrary that D_k intersects paths from Ψ_n and Ψ_{n+2} for some n. Then there exist $w_1, w_2 \in D_k$ with

$$|w_1 - w| < 2^n s \quad \text{and} \quad |w_2 - w| > 2^{n+1}s$$

so that

$$2^{n-1}s \leqslant r_k. \tag{22}$$

Since $r_i \leqslant r_j$ we can assume $|z_k - z_i| \leqslant |z_k - z_j|$, and now using basic planar geometry, (21) and (22) we have

$$|z_k - z_i|^2 \leqslant (r_i + \frac{s}{2})^2 + (r_k + 2^{n-1}s)^2 < (r_i + r_k)^2.$$

But this is impossible since D_i and D_k are disjoint.

Since the number of disks D_k, $k \neq i, j$ is at most M, we have shown that for at least $N - 2M + 2$ different indices n all the paths in Ψ_n connect D_i and D_j. Using standard properties of the modulus we have then the lower bounds

$$\mathrm{mod}(\Psi_n) \geqslant 4\,\mathrm{mod}(\{S(0,t) : 1 < t < 2\}) \geqslant \frac{2\log 2}{\pi}$$

for every n, see [11, Theorem 10.12], and

$$\mathrm{mod}(D_i, D_j; \Omega) \geqslant (N - 2M + 2)\frac{2\log 2}{\pi} > \frac{\pi C_W^2 C_D^{1+\log_2 C_X}}{4}. \tag{23}$$

We have thus proved that $\Delta(i,j) < \alpha$ leads to a contradiction with (20), and the lemma follows. □

Recall that $D_i = \overline{D}(z_i, r_i)$. We next consider the sets

$$\Phi_i = \{1 < t < 1 + \alpha :\ S(z_i, tr_i) \subset \Omega\},$$

where α is the constant in Proposition 6.1, and the family Γ_i of all the (parameterized) circles $S(z_i, tr_i)$, $t \in \Phi_i$.

Lemma 6.3. *We have*

$$m_1(\Phi_i) \geqslant \frac{\alpha^M}{(10M)^M} \tag{24}$$

for all $i = 1, \dots, M$. *In particular,*

$$\mathrm{mod}(\Gamma_i) \geqslant \beta = \frac{\alpha^M}{2\pi(10M)^M}. \tag{25}$$

Proof. Enumerate the disks according to decreasing radius, and fix D_i. By Lemma 6.2, $\mathrm{dist}(D_i, D_j) \geqslant \alpha r_i$ for every $j < i$. Now, if

$$\sum_{j \geqslant i+1} r_j \leqslant \alpha r_i/10,$$

then (24) holds. Otherwise $r_{i+1} \geqslant \alpha r_i/(10M)$. Continuing inductively, either

$$\sum_{j \geqslant i+k+1} r_j \leqslant \alpha r_{i+k}/10, \tag{26}$$

for some k, or

$$r_{i+k+1} \geqslant \alpha r_{i+k}/(10M) \geqslant \ldots \geqslant r_i \alpha^{k+1}/(10M)^{k+1}$$

for all k. In the latter case,

$$r_j \geqslant r_i \alpha^{M-1}/(10M)^{M-1} \quad \text{for all } j = 1, \dots, M, \tag{27}$$

and (24) follows from Lemma 6.2. On the other hand, if (26) occurs then Lemma 6.2 shows that

$$r_i m_1(\Phi_i) \geqslant \min_{j \leqslant i+k} \mathrm{dist}(D_i, D_j) - \sum_{j \geqslant i+k+1} 2r_j \geqslant \frac{\alpha r_{i+k}}{10}.$$

If moreover k is the smallest index for which (26) occurs, then (27) holds for j replaced with $i+k$ and we conclude (24) also in this case. Finally, (25) follows from (24) by a standard application of Hölder's inequality and polar coordinates. □

Now fix continua E_1, E_2 as in Proposition 6.1. First, an elementary geometric argument (cf. [11, Theorem 11.7]) applying (18) shows that there exist $z_0 \in \mathbb{R}^2$ and $r_0 > 0$ such that both E_1 and E_2 intersect $S(z_0, t_0 r_0)$ for all $1 \leqslant t_0 \leqslant \sqrt{3}$.

Let Φ_i, Γ_i, $i = 1, \dots, M$, be as in Lemma 6.3. Moreover, we denote $\Phi_0 = (1, \sqrt{3})$ and

$$\Gamma_0 = \{S(z_0, t_0 r_0) : t_0 \in \Phi_0\}, \tag{28}$$

so that (25) holds for all $i = 0, \dots, M$. By construction, Γ_i is a family of paths in Ω when $i = 1, \dots, M$, while the paths in Γ_0 are not required to lie in Ω. The proposition is proved by modifying Γ_0 to obtain a family of paths in Ω such that the lower bound for modulus is still valid. This is based on the following property.

Lemma 6.4. *Given*

$$T = (t_0, t_1, \ldots, t_M) \in \Phi := \Phi_0 \times \Phi_1 \times \cdots \times \Phi_M,$$

there is an injective path γ_T *connecting* E_1 *and* E_2 *in*

$$G_T := \bigcup_{j=0}^{M} S(z_j, t_j r_j) \cap \Omega.$$

Proof. By construction, there are $p_1, p_2 \in S(z_0, t_0 r_0)$ such that $p_1 \in E_1$ and $p_2 \in E_2$. On the other hand, the components of $S(z_0, t_0 r_0) \setminus \Omega$ are of the form

$$S(z_0, t_0 r_0) \cap D_j = S(z_0, t_0 r_0) \cap \overline{D}(z_j, r_j).$$

Therefore, the p_1-component F_T of G_T contains all of $S(z_0, t_0 r_0) \cap \Omega$. In particular, it contains p_2. We can choose γ_T to be a shortest path joining p_1 and p_2 in F_T. □

Let

$$\Gamma = \{\gamma_T : T \in \Phi\},$$

where γ_T is any path satisfying the conditions of Lemma 6.4. The proposition follows if we can bound $\mathrm{mod}(\Gamma)$ from below.

Let $\rho \geqslant 0$ be a Borel function in Ω such that

$$\int_\Omega \rho^2 \, dA = \frac{\beta}{(M+2)^2}, \tag{29}$$

where β is the constant in (25). The desired lower bound follows if we can show that such a ρ cannot be admissible for Γ. By (25), (28), and (29), we find that $(M + 3/2)\rho$ cannot be admissible for any of the path families Γ_i, $i = 0, \ldots, M$. Hence there is at least one $T = (t_0, t_1, \ldots, t_M) \in \Phi$ such that

$$\int_{S(z_i, t_i r_i)} \rho \, ds < \frac{1}{M + 3/2}$$

for each $i = 0, \ldots, M$. Applying the injectivity of γ_T, we moreover get

$$\int_{\gamma_T} \rho \, ds \leqslant \sum_{i=0}^{M} \int_{S(z_i, t_i r_i)} \rho \, ds \leqslant \frac{M+1}{M+3/2} < 1.$$

We conclude that ρ is not admissible for Γ. The proof of Proposition 6.1 is complete.

7. Proof of Theorem 2.2

Suppose that the assumptions of Theorem 2.2 are valid. By Proposition 5.1, there is a $\pi/2$-quasiconformal map $f : (X, q) \to \Omega$, where $\Omega \subset \mathbb{S}^2$ is a circle domain. We prove Theorem 2.2 by showing that, after a normalization, f is quasisymmetric with respect to the original metric d.

If $\Omega = \mathbb{S}^2$, then the theorem is proved in [6]. The proof in the case of one complementary component is easier than the one below and is omitted. We consider the remaining case where there are at least two complementary components.

Recall that the assumptions of Väisälä's theorem ([4, Theorem 10.17]) hold in our setting, so the quasisymmetry of f follows if we can prove the *weak quasisymmetry* of $h = f^{-1}$: there is $t \geqslant 1$ such that for every disjoint $y_0, y_1, y_2 \in \Omega$ with

$$|y_0 - y_1| \leqslant |y_0 - y_2| \leqslant \frac{1}{10},$$

we have

$$d(h(y_0), h(y_1)) \leqslant t d(h(y_0), h(y_2)). \tag{30}$$

To prove (30), we first normalize h. Namely, we precompose h with a suitable Möbius transformation, if necessary, so that

$$\min_{i \neq j} d(h(a_i), h(a_j)) \geqslant \mathrm{diam}_d(X)/10, \tag{31}$$

where $\{a_0, a_1, a_\infty\} \in \Omega$ correspond to the points 0, e_1, and ∞ under the stereographic projection.

Fix $y_0, y_1, y_2 \in \Omega$ as in (30), and denote

$$A = d(h(y_0), h(y_2)), \quad B = d(h(y_0), h(y_1)).$$

We need to show that $B \leqslant tA$. We may assume that

$$A \leqslant B/100\lambda^3 \leqslant \mathrm{diam}_d(X)/100\lambda^3,$$

otherwise there is nothing to prove. By (31) and triangle inequality, we find that for some $j \in \{0, 1, \infty\}$,

$$d(h(y_0), h(a_j)) \geqslant \mathrm{diam}_d(X)/20 \quad \text{and} \quad |y_1 - a_j| \geqslant 1/10.$$

Moreover, by the LLC-condition, we find a continuum

$$F_1 \subset B_d(h(y_0), \lambda A) \subset X$$

joining $h(y_0)$ and $h(y_2)$. Similarly, we find a continuum

$$F_2 \subset X \setminus B_d(h(y_0), B/\lambda)$$

joining $h(y_1)$ and $h(a_j)$.

Denote $E_\ell = f(F_\ell) = h^{-1}(F_\ell)$ for $\ell = 1, 2$. Then, if τ is a rotation of $\mathbb{S}^2$ sending y_0 to 0 and ϕ the stereographic projection, we see that $(\phi \circ \tau)(E_1)$ and $(\phi \circ \tau)(E_2)$ satisfy the conditions of Proposition 6.1. Since $\phi \circ \tau$ is conformal, we conclude that the lower bound in Proposition 6.1 holds also for the continua E_1 and E_2.

Next, we estimate $\mathrm{mod}(F_1, F_2; X)$ from above (recall the definition from Section 5). Denote

$$U_1 = \overline{B}_d(h(y_0), \lambda A), \quad U_2 = X \setminus B_d(h(y_0), B/\lambda).$$

Then, since $F_1 \subset U_1$ and $F_2 \subset U_2$, we have

$$\mathrm{mod}(F_1, F_2; X) \leqslant \mathrm{mod}(U_1, U_2; X).$$

Let $k \geqslant 2$ be the largest integer such that

$$B \geqslant 2^k \lambda^2 A,$$

and denote

$$A_j = \overline{B}_d(h(y_0), 2^j \lambda A) \setminus B_d(h(y_0), 2^{j-1}\lambda A), \quad j = 1, \dots, k.$$

The WMDM-condition and the doubling property of μ then guarantee that for every $\gamma \in \Gamma(U_1, U_2; X)$ the length in (X, q) of the restriction of γ to A_j is at least

$$\frac{\mu(B_d(h(y_0), 2^j\lambda A))^{1/2}}{C_W C_D}.$$

It follows that $\rho : U_2 \setminus U_1 \to [0, \infty]$,

$$\rho(w) = \frac{1}{k}\sum_{j=1}^{k} \frac{C_W C_D \chi_{A_j}(w)}{\mu(B_d(h(y_0), 2^j\lambda A))^{1/2}}$$

is admissible for $\Gamma(U_1, U_2; X)$. Integrating and applying (5), this yields

$$\begin{aligned} \mathrm{mod}(U_1, U_2; X) &\leqslant \frac{C_W^2 C_D^2}{k^2} \sum_{j=1}^{k} \frac{\mathcal{H}_q^2(A_j)}{\mu(B_d(h(y_0), 2^j\lambda A))} \\ &\leqslant \frac{2\pi C_W^2 C_S^2 C_D^6}{k}. \end{aligned} \tag{32}$$

Finally, combining Proposition 5.1, Proposition 6.1, and (32), we get

$$k \leqslant \frac{2\pi^3 C_W^2 C_S^2 C_D^6 (10M)^M (M+2)^2}{\alpha^M},$$

where α is the constant in Proposition 6.1. In particular, we have the desired bound for the ratio B/A. The proof is complete.

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