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The Egan problem
on the pull-in range of type 2 PLLs

N.V. Kuznetsov, M.Y. Lobachev, M.V. Yuldashev, R.V. Yuldashev

Abstract—In 1981, famous engineer William F. Egan conjectured that a higher-order type 2 PLL with an infinite hold-in range also has an infinite pull-in range, and supported his conjecture with some third-order PLL implementations. Although it is known that for the second-order type 2 PLLs the hold-in range and the pull-in range are both infinite, the present paper shows that the Egan conjecture may be not valid in general. We provide an implementation of the third-order type 2 PLL, which has an infinite hold-in range and experiences stable oscillations. This implementation and the Egan conjecture naturally pose a problem, which we will call the Egan problem: *to determine a class of type 2 PLLs for which an infinite hold-in range implies an infinite pull-in range*. Using the direct Lyapunov method for the cylindrical phase space we suggest a sufficient condition of the pull-in range infiniteness, which provides a solution to the Egan problem.

Index Terms—Phase-locked loop, PLL, type II, type 2, hold-in range, Egan conjecture, Egan problem on the pull-in range, Gardner problem on the lock-in range, Lyapunov functions, nonlinear analysis, global stability, describing function, harmonic balance method.

I. INTRODUCTION

PHASE-LOCKED LOOPS (PLLs) are classical nonlinear control systems for phase and frequency synchronization in electrical circuits [1]–[3]. A PLL includes the following three key elements: a phase detector (PD), a loop filter, and a voltage-controlled oscillator (VCO). The phase detector extracts a phase error, which is the phase difference between the reference signal and the output signal of the VCO, and then the resulting signal is filtered by the loop filter. The output of the loop filter controls the VCO frequency in such a way as to reduce the phase error, i.e., to synchronize the signals. In order to characterize the synchronization properties, the following frequency difference concepts are widely used: a hold-in range, a pull-in range (or an acquisition range), and a lock-in range [4]–[7].

In engineering practice, the so-called *type 2 PLLs* having loop filters with exactly one pole at the origin are most often used [3]. In 1959, Andrew J. Viterbi applied the phase-plane analysis and stated that the second-order type 2 PLL models have infinite (theoretically) hold-in and pull-in ranges for any loop parameters [8, p.12], [1]. However, his proof was incomplete (see, e.g. discussion in [9]). Later, Viterbi's

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statement was rigorously proved using the direct Lyapunov method ideas [10], [11].

Since the second-order PLLs may not provide the required noise reductions [3], [5], [12], [13], some extra poles with negative real parts are often added in the loop filter [3], [6], [14]. In practice, the higher-order loops are widely used. For a type 2 PLL of any order, the each of the hold-in and pull-in ranges is either infinite or empty (see Section III). In 1981, William F. Egan conjectured [15, p.176] that a higher-order *type 2 PLL with an infinite hold-in range also has an infinite pull-in range*, and supported it with some third-order PLL implementations (see also [16, p.192], [6, p.161], [17, p.245]). Nowadays, similar claims can be found in various publications (see, e.g. [18, p.96], [19, p.171], [20, p.198], [21, p.6], and others).

The present paper introduces a counterexample to the Egan conjecture: a type 2 PLL with an infinite hold-in range and a persistent oscillation, which indicates the emptiness of the pull-in range. The observed stable oscillation and the Egan conjecture naturally pose the following problem, which we will call *the Egan problem*: *to determine a class of type 2 PLLs for which an infinite hold-in range implies an infinite pull-in range*. For such a class of PLLs the global stability conditions can be obtained by straightforward linear methods. Notice that similar problems are well-known in the mathematical control theory (see, e.g. the Aizerman and Kalman conjectures [22]–[24]). Using the direct Lyapunov method for the cylindrical phase space we obtain a sufficient condition of the pull-in range infiniteness, thereby providing a solution to the Egan problem.
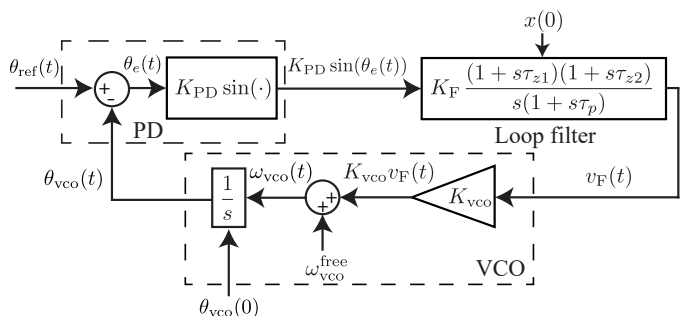
II. COUNTEREXAMPLE TO THE EGAN CONJECTURE

Fig. 1. Baseband model of the third-order analog PLL.

Consider the PLL baseband model [1], [3], [5], [25] in Fig.1 and its realization in MATLAB Simulink in Fig. 2. Here $\theta_{\text{ref}}(t) = \omega_{\text{ref}}t + \theta_{\text{ref}}(0)$ is a phase of the reference signal, a
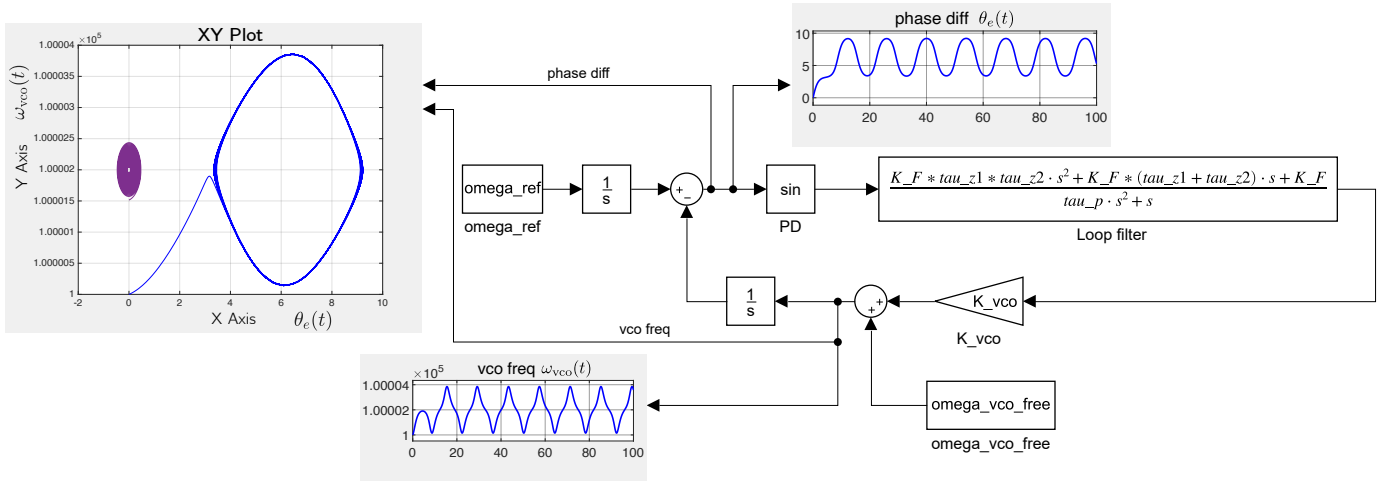
Fig. 2. Simulation of a type 2 PLL with infinite hold-in range in MATLAB Simulink (baseband model). The curve from the zero initial state $x_1(0) = x_2(0) = \theta_{ref}(0) = \theta_{vco}(0) = 0$ tends to a persistent oscillation (the blue color); the curve with the initial state $x_1(0) = 1.5$, $x_2(0) = 0$, $\theta_{ref}(0) = \theta_{vco}(0) = 0$ tends to the zero locked state (the purple color). Parameters: $\omega_{ref} = 2 + 10^5$, $\omega_{vco}^{free} = 10^5$, $F(s) = 1.01 \frac{(1+0.4s)(1+0.4s)}{s(1+0.9s)}$, $K_{vco} = K_{PD} = 1$.

phase of the VCO is $\theta_{vco}(t)$, $\theta_e(t) = \theta_{ref}(t) - \theta_{vco}(t)$ is a phase error. The phase detector generates a signal $K_{PD} \sin(\theta_e(t))$ where K_{PD} is a gain and $\sin(\cdot)$ is a characteristic of the phase detector. The state of the loop filter is represented by $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ and the transfer function [26] is

$$F(s) = K_F \frac{(1 + s\tau_{z1})(1 + s\tau_{z2})}{s(1 + s\tau_p)}, \tag{1}$$
$$K_F > 0, \ \tau_{z1} > 0, \ \tau_{z2} > 0, \ \tau_p > 0, \ \tau_{z1} \neq \tau_p, \ \tau_{z2} \neq \tau_p.$$

The output of the loop filter $v_F(t) = K_F(x_1(t) + x_2(t) + \frac{\tau_{z1}\tau_{z2}}{\tau_p} K_{PD} \sin(\theta_e(t)))$ is used to control the VCO frequency $\omega_{vco}(t)$, which is proportional to the control voltage:

$$\omega_{vco}(t) = \dot\theta_{vco}(t) = \omega_{vco}^{free} + K_{vco}v_F(t)$$

where K_{vco} is a gain and ω_{vco}^{free} is a free-running frequency of the VCO. Observe that the phase error $\theta_e(t)$ depends on the frequency error $\omega_e^{free} = \omega_{ref} - \omega_{vco}^{free}$ and not on the frequencies themselves. Therefore, transient processes in the model can be studied with respect to ω_e^{free}. For the loop filter transfer function (1), one of the possible implementations[1] is shown in Fig. 3. Since the loop filter transfer function has exactly one pole at the origin, the considered PLL model is a *type 2* one [3, p.12], [6, p.59].
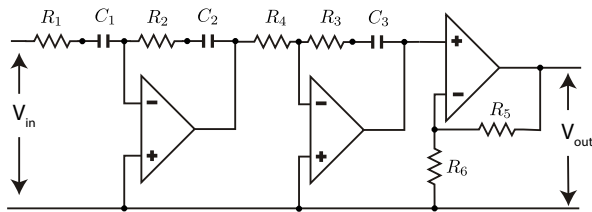

Fig. 3. An active loop filter with the transfer function $F(s) = (1 + \frac{R_5}{R_6}) \frac{C_1}{C_2 C_3 R_4} \frac{(1+sR_2C_2)(1+sR_3C_3)}{s(1+sR_1C_1)}$.

[1]There are other implementations of transfer function (1) (e.g. the passive ones), some of which pose additional physical restrictions on parameters and lead to an infinite pull-in range.

The PLL baseband model in Fig. 2 is *locked* if the phase error $\theta_e(t)$ is constant (phase-locked condition). For the locked states of practically used PLLs, the loop filter state is constant too (see Section III for type 2 PLLs). Thus, for arbitrary ω_e^{free}, the locked states correspond to the stationary states of the model. Such locked states, which the model returns to after a small perturbation of the loop filter state and the phase error, are called (locally) *asymptotically stable* and observed in practice.

Definition 1 (Hold-in range [7], [27], [28]): A *hold-in range* is the largest symmetric interval of frequency errors $|\omega_e^{free}| \in [0, \omega_h)$ such that an asymptotically stable locked state exists and varies continuously while ω_e^{free} varies continuously within the interval; ω_h is called a *hold-in frequency*.

In the above definition, the locked states are considered as a function of ω_e^{free} having a continuous single-valued branch (to exclude a case of multi-valued branch when some of the locked states may appear or disappear while the frequency error ω_e^{free} varies).

Definition 2 (Pull-in range [7], [27], [28]): A *pull-in range* is the largest symmetric interval of frequency errors from the hold-in range $|\omega_e^{free}| \in [0, \omega_p)$ such that a locked state is acquired for an arbitrary initial state; ω_p is called a *pull-in frequency*.

Remark that from some initial states the model state can tend to an unstable stationary state. To avoid such situation for all initial states (see, e.g. the discussion of corresponding artificial examples in [29]) the condition $[0, \omega_p) \subset [0, \omega_h)$ is required in the pull-in range definition.

For the counterexample to the Egan conjecture in Fig. 2, we consider the following parameters:

$$K_{vco} = K_{PD} = 1, \ K_F = 1.01, \ \tau_{z1} = \tau_{z2} = 0.4, \ \tau_p = 0.9,$$
$$\omega_{ref} = 2 + 10^5, \ \omega_{vco}^{free} = 10^5. \tag{2}$$

The model state with initial state $x_1(0) = 2$, $x_2(0) = 0$, $\theta_{ref}(0) = 0$, $\theta_{vco}(0) = -0.7$ in Fig. 2 tends to the zero locked

state. Since for type 2 PLLs the hold-in and the pull-in ranges are either infinite or empty, the simulation indicates the hold-in range infiniteness (see Section III for a rigorous analysis). Following the Egan conjecture, the pull-in range should be infinite too. However, for the zero initial state $x(0) = 0$, $\theta_{\text{ref}}(0) = \theta_{\text{vco}}(0) = 0$ the simulation shows persistent oscillation of the control signal (see the plots of the VCO frequency $\omega_{\text{vco}}(t)$, the phase error $\theta_e(t)$, and their combined XY plot in Fig. 2).

III. NONLINEAR ANALYSIS AND SOLUTION TO THE EGAN PROBLEM

Consider a state-space representation of the third-order type 2 PLL in Fig. 1:

$$
\begin{cases}
\dot{x}_1 = K_{\text{PD}} \sin \theta_e, \\
\dot{x}_2 = -\dfrac{1}{\tau_p} x_2 + \dfrac{(\tau_{z1} - \tau_p)(\tau_p - \tau_{z2})}{\tau_p^2} K_{\text{PD}} \sin \theta_e, \\
\dot{\theta}_e = \omega_e^{\text{free}} - K_{\text{F}} K_{\text{vco}} \left(x_1 + x_2 + \dfrac{\tau_{z1} \tau_{z2}}{\tau_p} K_{\text{PD}} \sin \theta_e \right).
\end{cases}
\tag{3}
$$

System (3) is not changed under the transformation

$$
\left(\omega_e^{\text{free}}, x_1(t), x_2(t), \theta_e(t) \right) \to \left(-\omega_e^{\text{free}}, -x_1(t), -x_2(t), -\theta_e(t) \right),
$$

and, thus, it can be studied for $\omega_e^{\text{free}} \ge 0$ only. Besides using the linear change of variables

$$
z(t) = \omega_e^{\text{free}} - K_{\text{F}} K_{\text{vco}} x_1(t),
$$

we can exclude ω_e^{free} from the system. Thus, its stability does not depend on ω_e^{free} and the hold-in and pull-in ranges are either infinite or empty.

The stationary states of system (3) are $(\omega_e^{\text{free}}/(K_{\text{F}} K_{\text{vco}}), 0, \pi k)$, $k \in \mathbb{Z}$. The characteristic equation of system (3) linearized at stationary states is

$$
\chi(s) = s^2 (1 + s \tau_p) + K_{\text{PD}} K_{\text{F}} K_{\text{vco}} (1 + s \tau_{z1})(1 + s \tau_{z2}) \cos(\pi k).
$$

From the Routh-Hurwitz criterion, it follows that the stationary states $(\omega_e^{\text{free}}/(K_{\text{F}} K_{\text{vco}}), 0, 2\pi k)$, $k \in \mathbb{Z}$ are asymptotically stable for any $k \in \mathbb{Z}$ if and only if the following condition is satisfied:

$$
K_{\text{PD}} K_{\text{F}} K_{\text{vco}} \tau_{z1} \tau_{z2} (\tau_{z1} + \tau_{z2}) > \tau_p - \tau_{z1} - \tau_{z2},
\tag{4}
$$

and the stationary states $(\omega_e^{\text{free}}/(K_{\text{F}} K_{\text{vco}}), 0, \pi + 2\pi k)$ are unstable. As a result, condition (4) is necessary and sufficient for the hold-in range infiniteness.

Since the pull-in range definition requires to analyse all the initial states of system (3), the conditions for the pull-in range infiniteness cannot be determined by linear analysis in general. In the following subsections, the Lyapunov direct method and the harmonic balance method are used to study the pull-in range and to solve the Egan problem.

A. Solution to the Egan problem: direct Lyapunov method

To analyse the pull-in range of system (3), we apply the direct Lyapunov method and the corresponding theorem on global stability for the cylindrical phase space (see, e.g. [29],

[30]). If there is a continuous function $V(x, \theta_e) : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $V(x, \theta_e + 2\pi) = V(x, \theta_e) \quad \forall x \in \mathbb{R}^{n-1}, \forall \theta_e \in \mathbb{R}$;

(ii) for any solution $(x(t), \theta_e(t))$ of system (3) the function $V(x(t), \theta_e(t))$ is nonincreasing;

(iii) if $V(x(t), \theta_e(t)) \equiv V(x(0), \theta_e(0))$, then $(x(t), \theta_e(t)) \equiv (x(0), \theta_e(0))$;

(iv) $V(x, \theta_e) + \theta_e^2 \to +\infty$ as $||x|| + |\theta_e| \to +\infty$

then any trajectory of system (3) tends to the stationary set (i.e., the system is globally stable).

Consider the following Lyapunov function

$$
V(x_1, x_2, \theta_e) = \frac{K_{\text{F}} K_{\text{vco}}}{2} \left(x_1 - \frac{\omega_e^{\text{free}}}{K_{\text{F}} K_{\text{vco}}} \right)^2 +
$$
$$
+ \frac{K_{\text{F}} K_{\text{vco}}}{2} \frac{\tau_p^2 (\tau_{z1} \tau_{z2} + \tau_p (\tau_{z1} + \tau_{z2} - \tau_p))}{(\tau_{z1} - \tau_p)^2 (\tau_p - \tau_{z2})^2} x_2^2 + \int\limits_0^{\theta_e} K_{\text{PD}} \sin \sigma d\sigma.
\tag{5}
$$

If the loop filter parameters satisfy the inequality

$$
\tau_{z1} + \tau_{z2} - \tau_p > 0
\tag{6}
$$

then the Lyapunov function derivative along the trajectories of system (3) is as follows:

$$
\dot{V}(x_1, x_2, \theta_e) =
$$
$$
= -K_{\text{F}} K_{\text{vco}} \left(\frac{\tau_p \tau_{z1} \tau_{z2} + \tau_p^2 (\tau_{z1} + \tau_{z2} - \tau_p)}{(\tau_{z1} - \tau_p)^2 (\tau_p - \tau_{z2})^2} x_2^2 - \right.
$$
$$
\left. - \frac{2 \tau_{z1} \tau_{z2}}{(\tau_{z1} - \tau_p)(\tau_p - \tau_{z2})} x_2 K_{\text{PD}} \sin \theta_e + \frac{\tau_{z1} \tau_{z2}}{\tau_p} K_{\text{PD}}^2 \sin^2 \theta_e \right) < 0,
$$
$$
x_2 \ne 0, \ \theta_e \ne \pi k, \ k \in \mathbb{Z}.
$$

Since the derivative along any solution other than stationary states is not identically zero, condition (6) provides the global stability of the system for any ω_e^{free}. Observe that condition (4) of local stability is also valid in this case. Therefore, under condition (6) the pull-in and the hold-in ranges of system (3) are infinite:

$$
[0, \omega_p) = [0, \omega_h) = [0, +\infty).
$$

Notice that in this case the classical Barbashin-Krasovsky and LaSalle theorems cannot be directly used with the Lyapunov functions of type (5) to study global stability, because for that the Lyapunov function $V(x, \theta_e)$ must be radially unbounded while $V(0, \theta_e) \not\to +\infty$ as $|\theta_e| \to +\infty$. The discussion of radial unboundedness is sometimes omitted in the existing literature (see, e.g. [26], [31], [32]).

Condition (6) determines a class of type 2 PLLs for which an infinite hold-in range implies an infinite pull-in range, thus providing a *solution to the Egan problem*. The above reasoning provides also an infinite pull-in range of the second-order type 2 PLL: Lyapunov function (5) without its second term and the corresponding model in Fig. 1 with $\tau_{z2} = \tau_p = 0$, $\tau_{z1} > 0$ satisfy the conditions of the above theorem.

Notice that the same Lyapunov function can be used to analyse transient processes and to provide *a solution to the Gardner problem on the lock-in range* [3, p. 187–188], [7], [27], [28], [33]: for the considered model a subinterval $[0, \omega_l)$ of the pull-in range over which the PLL re-establishes an

asymptotically stable locked state without cycle slipping may be estimated as $[0, \omega_l) \supset [0, \sqrt{K_{PD}K_F K_{vco}})$ (see, e.g. [33]).

Relation (6) is not satisfied for parameters (2). The gap between conditions (4) and conditions (6) poses the problem of studying this domain of parameters and the boundary of global stability (which parts can be either *trivial*, i.e., determined by local bifurcations, or *hidden*, i.e., determined by non-local bifurcations and the birth of hidden oscillations) [24], [29].

B. Analysis of periodic solutions: the harmonic balance method and phase portrait analysis

To analyse the PLL model behavior in the gap between conditions (4) and (6) (particularly to analyse oscillations in Fig. 2), the harmonic balance method and phase portrait analysis are used in this section. The harmonic balance method is widely used by engineers to predict oscillations in nonlinear systems and to check the global stability of phase-locked loops (see, e.g. [2], [34]–[38]), while the method is known to be an approximate one and to work well only when the *filter conjecture* is valid [39, p. 164], [40, p.329].

Following the harmonic balance method, we suppose that system (3) has a periodic solution (cycle)

$$\theta_e(t) \approx \delta\omega_0 t + a_0 \sin(\omega_0 t)$$

where a_0 is an amplitude and $\omega_0 > 0$ is a frequency, $\delta = 0$ for the cycles of the first kind, and $\delta = 1$ for the cycles of the second kind. Solving the describing function equation [38], one gets the following solution:

$$
\begin{aligned}
\omega_0 &= \sqrt{\frac{\tau_p - \tau_{z1} - \tau_{z2}}{\tau_{z1}\tau_p\tau_{z2}}}, \\
\frac{J_{1-\delta}(a_0) - J_{1+\delta}(-a_0)}{a_0} &= \frac{\tau_p - \tau_{z1} - \tau_{z2}}{K_F K_{vco}\tau_{z1}\tau_{z2}(\tau_{z1}+\tau_{z2})}
\end{aligned}
\tag{7}
$$

where $J_{0,1,2}(a)$ are the Bessel functions of the first kind:

$$J_k(a) = \frac{1}{\pi}\int_0^\pi \cos(kt - a\sin t)dt, \quad k \in \mathbb{Z}.$$

In this case, according to the harmonic balance method, a periodic solution does not develop in system (3) if condition (6) is met (there is no positive frequency $\omega_0 > 0$). Otherwise, if condition (6) is not fulfilled, then the frequency $\omega_0 \geq 0$ and amplitude a_0 satisfying equation (7) exist. Thus, the harmonic balance method does not improve the condition of global stability obtained by the direct Lyapunov method for the considered model.

The phase portrait of the type 2 PLL with parameters (2), considered for the numerical counterexample, is shown in Fig. 4. The trajectory with initial state $x_1(0) = 2$, $x_2(0) = 0$, $\theta_e(0) = 0.7 + 2\pi$ tends to an asymptotically stable equilibrium.

The dashed curve depicts possible unstable cycle, predicted by harmonic balance equations (7) with $\delta = 0$, $\omega_0 \approx 0.83$, $a_0 \approx 1.4$. Notice that equations (7) give only possible frequency and amplitude of a periodic solution but do not give any initial data of the solution. To visualize this trajectory in Fig. 4 we use the initial data provided by theorem from [22].
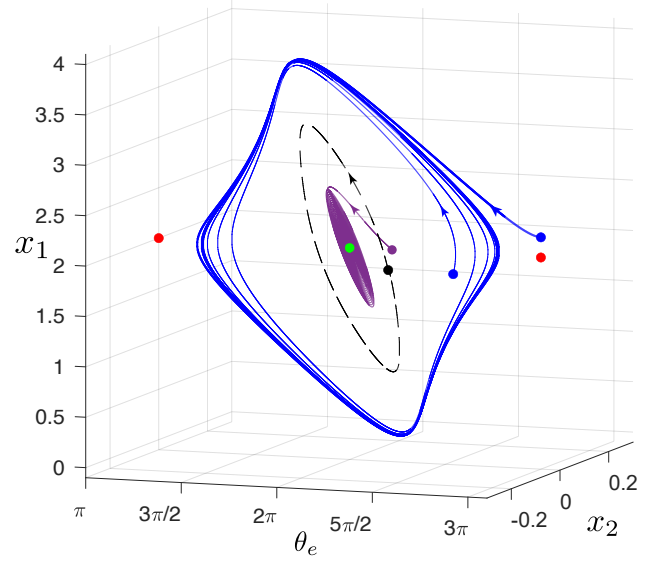

Fig. 4. Phase portrait of system (3). The green and red dots are asymptotically stable and unstable equilibria, respectively. The purple trajectory tends to an asymptotically stable equilibrium (green dot). The black dashed line depicts possible unstable periodic orbit, predicted by harmonic balance equations (7). The blue trajectories tend to stable limit cycle. Parameters: $\omega_e^{free} = 2$, $K_F = 1.01$, $\tau_{z1} = \tau_{z2} = 0.4$, $\tau_p = 0.9$, $K_{vco} = K_{PD} = 1$.

The trajectories with initial states $x_1(0) = 2.2$, $x_2(0) = 0$, $\theta_e(0) = 3\pi$ and $x_1(0) = 2$, $x_2(0) = -0.2$, $\theta_e(0) = 2.5 + 2\pi$ in Fig. 4 tend to a stable limit cycle, which can be classified as a self-excited oscillation [22]. This stable limit cycle corresponds to the periodic oscillation shown in Fig. 2. As it is stated above, this oscillation is not predicted by the harmonic balance and equations (7). A probable reason is that the stable limit cycle has a complex structure, which is far from the shape of a simple harmonic oscillation. As a result, the harmonic balance method is unable to predict this stable limit cycle, whereas it is necessary to avoid such oscillations in practice.

IV. CONCLUSION

Since PLLs are essentially nonlinear control systems, their non-local analysis and the pull-in range estimation are challenging tasks. Therefore, classes of PLL systems for which conditions of local (linear) and global (nonlinear) stability coincide (thus, providing a coincidence of the hold-in and the pull-in ranges), are of interest. For type 2 PLLs the Egan conjecture states that an infinite hold-in range implies an infinite pull-in range. While this conjecture is valid for the second-order loops, the third-order loop considered in this paper provides a counterexample to the Egan conjecture. It is shown that stable periodic oscillations in PLLs may not be revealed by the classical harmonic balance method. A sufficient condition, guaranteeing an infinite pull-in range, is obtained by the direct Lyapunov method for the cylindrical phase space. This condition determines a class of type 2 PLLs for which both the hold-in and the pull-in ranges are infinite, thus, providing a solution to the Egan problem.

Notice that similar conjectures on the pull-in range for some other types of PLLs are also known: for example, *the*

Kapranov conjecture on the pull-in range of the second-order type 1 PLLs [41] and *the Gardner conjecture* on the similarity of the transient response of charge-pump PLL (CP-PLL) and equivalent classical PLL [42, p. 1856] (see, e.g. discussions of corresponding counterexamples in [43]–[45]).

ACKNOWLEDGMENT

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