JYU DISSERTATIONS 318

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Uniqueness Results for Fractional Calderón Problems

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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List of included articles

This dissertation consists of an introduction and the following five articles:

(A) Giovanni Covi. *Inverse problems for a fractional conductivity equation*. *Nonlinear Analysis* 193 (2020), special issue: Nonlocal and Fractional Phenomena, 111418.

(B) Giovanni Covi. *An inverse problem for the fractional Schrödinger equation in a magnetic field*. *Inverse Problems* 36, no. 4 (2020).

(C) Giovanni Covi, Keijo Mönkkönen and Jesse Railo. *Unique continuation property and Poincaré inequality for higher order fractional Laplacians with applications in inverse problems*. Preprint (January 2020), arXiv: 2001.06210v2.

(D) Giovanni Covi and Angkana Rüland. *On some partial data Calderón type problems with mixed boundary conditions*. Preprint (June 2020), arXiv: 2006.03252v2.

(E) Giovanni Covi, Keijo Mönkkönen, Jesse Railo and Gunther Uhlmann. *The higher order fractional Calderón problem for linear local operators: uniqueness*. Preprint (August 2020), arXiv: 2008.10227.

The authour of this dissertation has actively taken part in the research of the joint articles (C), (D) and (E).

Abstract

This dissertation studies the inverse problem for a specific partial differential equation, the so called fractional Calderón problem or inverse problem for the fractional Schrödinger equation. The dissertation focuses mainly on uniqueness results for inverse problems involving the Dirichlet to Neumann map, the object encoding exterior measurements in the model. The included articles show how this information suffices to determine the parameters involved in the problems considered.

The first article considers a fractional version of the inverse problem for the conductivity equation, showing that the unknown conductivity can be recovered from the DN map even in the case of a single measurement. The technique employed is the fractional Liouville reduction, which allows one to state the problem in terms of the fractional Schrödinger equation. The second article extends the known result for the fractional Schrödinger equation to the magnetic case, showing how a nonlocal perturbation and a potential can be both recovered up to a natural gauge. This resembles the results known for the local case. The third article explores the fractional Schrödinger equation in a high order regime, proving the injectivity of the relative DN map in both the perturbed and unperturbed cases. This requires a high order Poincaré inequality, which has been studied in the same paper. The fifth article follows the third one, extending the study to general local high-order perturbations: the coefficients of any local lower order operator are shown to be recoverable from the DN map. The fourth article studies the perturbed fractional Calderón problem by means of the Caffarelli-Silvestre extension, transforming it into a local problem with mixed Robin boundary conditions, eventually showing that the bulk and boundary potentials can be recovered simultaneously. This requires some technical Carleman estimates and the construction of a new class of CGO solutions.

The introduction of the dissertation contains a survey of the literature related to both the classical and fractional Calderón problems, as well as a collection of the definitions of the function spaces appearing in the articles. The appendix is an informal introduction to key concepts in inverse problems and EIT, thought for the use of the general public.

Tiivistelmä

Tämän väitöskirjan tarkoitus on syventää ymmärrystä tietystä osittaisdifferentiaaliyhtälöiden inversio-ongelmasta, niin sanotusta fraktionaalisesta Calderónin ongelmasta tai fraktionaalisen Schrödingerin yhtälön inversio-ongelmasta. Väitöskirja keskittyy pääasiassa mallin ulkomittauksia karakterisoivan objektin eli Dirichlet-to-Neumann -kuvauksen (DN-kuvauksen) injektiivisyyteen. Väitöskirjaan sisältyvät artikkelit osoittavat, kuinka DN-kuvaus riittää määräämään ongelman tuntemattomat aineparametrit.

Ensimmäisessä artikkelissa tarkastellaan johtavuusyhtälöä koskevan inversio-ongelman fraktionaalista versiota ja osoitetaan, että tuntematon johtavuus voidaan määrittää DN-kuvauksesta jopa yhden mittauksen tapauksessa. Käytetty tekniikka on fraktionaalinen Liouvillen reduktio, jonka avulla ongelman voi lausua fraktionaalisen Schrödingerin yhtälön muodossa. Toinen artikkeli laajentaa fraktionaalisen Schrödingerin yhtälön tunnetun tuloksen magneettiseen tapaukseen osoittaen, kuinka epälokaali perturbaatio ja potentiaali voidaan molemmat määrittää luonnollista epäyksikäsitteisyyttä lukuunottamatta. Tämä muistuttaa lokaalin tapauksen tunnettuja tuloksia. Kolmannessa artikkelissa tutkitaan korkean kertaluvun fraktionaalista Schrödingerin yhtälöä ja osoitetaan DN-kuvauksen injektiivisyys sekä perturboidussa että ei-perturboidussa tilanteessa. Tähän tarvitaan korkean kertaluvun Poincarén epäyhtälöä, jota on tutkittu samassa artikkelissa. Viides artikkeli on jatkoa kolmannelle laajentaen tarkastelun yleisille lokaaleille korkean asteen perturbaatioille: minkä tahansa lokaalin alemman asteen operaattorin kertoimien osoitetaan olevan määritettävissä DN-kuvauksesta. Neljännessä artikkelissa tutkitaan perturboitua fraktionaalista Calderónin ongelmaa Caffarellin-Silvestren laajennuksen avulla muuttamalla se lokaaliksi ongelmaksi, jolla on sekoitetut Robin-reunaehdot. Lopulta osoitetaan, että sisä- ja reunapotentiaalit voidaan määrittää samanaikaisesti. Tämä vaatii joitain teknisiä Carlemanin estimaatteja ja CGO-ratkaisujen uuden luokan rakentamista.

Väitöskirjan johdanto sisältää kirjallisuuskatsauksen sekä klassisesta että fraktionaalisesta Calderónin ongelmasta ja kokoelman artikkeleissa esiintyvien funktioavaruuksien määritelmistä. Liite on epävirallinen suurelle yleisölle tarkoitettu johdanto inversio-ongelmien ja sähköimpedanssitomografian (EIT) avainkäsitteisiin.

Contents

1. Introduction

1.1. **The classical Calderón problem.** The problem of whether it is possible to determine the electrical conductivity inside of a domain by measurements performed on its boundary is one of the oldest and most classical inverse problems for partial differential equations. It first entered the mathematical literature in the year 1980, when the prominent Argentinian mathematician Alberto Calderón published his results about the method nowadays called Electric Impedance Tomograhy (EIT) as a way of prospecting for minerals. The famous analyst considered this problem in the 1940's while working as an engineer at YPF (*Yacimientos Petrolíferos Fiscales*, or Fiscal Oilfields), but did not publish the obtained results until many years later. The idea consists in first delivering current to the ground by means of some aptly placed electrodes, and then measuring the resulting voltage. The measurements should contain information about the composition of the materials hidden underground, since each substance is characterized by a specific electric conductivity and thus can influence the flow of current ([124, 138]).

The main characteristic that makes this method interesting is that it is *non-invasive*, meaning that it allows the recovery of information about the inside of an object from measurements on its surface. It is easy to see how this may be applied in the field of medical imaging: taking advantage of the fact that the tissues composing the human body have different electric conductivities ([66]), it is possible to obtain a representation of the internal structure of the body of a patient using electric measurements performed on his skin. This has lead to great advances in various occasions (see for instance [49] for cancer detection, [25] for the monitoring of vital functions, and many more [56]). Other applications of the EIT method were invented in seismic and industrial imaging (see e.g. [52]).

Let us now introduce the Calderón problem in mathematical language. We represent the object whose electric properties we want to study (may it be an industrial product, the body of a patient or the whole Earth) with a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. The unknown will be the electric conductivity $\gamma : \Omega \to (0, \infty)$ of the object. Next, we consider the Dirichlet problem for the conductivity equation

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases},$$

and enconde the boundary measurements in the so called Dirichlet-to-Neumann (DN) map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Λ_γ associates prescribed voltages to measured currents. The inverse problem thus consists in deducing γ from the knowledge of Λ_γ. Using the substitution $q = (\Delta\sqrt{\gamma})/\sqrt{\gamma}$, one can reformulate the Calderón problem as an inverse problem for the Schrödinger equation:

$$\begin{cases} (-\Delta + q)u &= 0 \ \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases}$$

This method is known as *Liouville reduction* [124]. In order to have unique solutions to the above equation, it is typical to assume that 0 is not a Dirichlet eigenvalue of the operator $(-\Delta + q)$. Given that one can express the DN map Λ_q for the Schrödinger equation in terms of the DN map Λ_γ for the conductivity equation, the inverse problem now requires to determine the potential q uniquely from Λ_q.

The Calderón problem can be generalized to contain first order perturbations. The result is the inverse problem for the magnetic Schrödinger equation (see [104]), which requires to find the electric and magnetic potentials existing in a medium using information derived solely from voltage and current measurements on its boundary. The components of such field are A for the magnetic potential and q for the electric one. The Dirichlet problem for the magnetic Schrödinger equation looks like

$$\begin{cases} (-\Delta)_A u + qu := -\Delta u - i\nabla\cdot(Au) - iA\cdot\nabla u + (|A|^2 + q)u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases},$$

where f is the prescribed boundary value for the voltage u. As in the conductivity case, the DN map $\Lambda_{A,q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ encodes the boundary measurements. The inverse problem thus consists in finding A, q in Ω knowing just $\Lambda_{A,q}$. This is however impossible to do in general, because the problem contains a natural gauge: while the electric potential q can be recovered completely, the magnetic potential A can only be recovered up to a gradient if Ω is known to be simply connected (see [104] for the case $n \geq 3$ and [82] for $n = 2$). It is however interesting to note that in the recent result [86] the authors proved the possibility to recover both the electric and the magnetic potential in a *nonlinear* magnetic Schrödinger equation from partial data.

This perturbed version of the Calderón problem has a connexion with the inverse scattering problem with a fixed energy ([104]). Moreover it arises by reduction in the study of the Maxwell ([98]), Schrödinger ([40]), Dirac ([105]) and Stokes equations ([55]), as well as in the study of isotropic elasticity ([106]).

1.2. **The fractional Calderón problem.** Another generalization of the Calderón problem consists in replacing the Laplace operator $(-\Delta)$ with the fractional Laplacian $(-\Delta)^s$, where $s \in (0, 1)$. This new operator, which will be defined in section 3.6, is in many ways different from the classical Laplacian: the main difference is that $(-\Delta)^s$ is a *nonlocal* operator, in the sense that it does not preserve supports. Because of this, the fractional Laplacian enjoys properties of unique continuation and approximation which are impossible for the classical Laplacian. Eventually, this means that stronger results are possible for the associated inverse problem (see 2).

As we have seen in section 1.1, the Calderón problem is eventually reduced to the study of the inverse problem for the Schrödinger equation. For this reason it is considered appropriate to use the name *fractional Calderón problem* when referring to the inverse problem for the fractional Schrödinger equation (see however our article (A) for a deeper understanding of this connection). This problem was introduced in the seminal paper

[46] in the following form. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $s \in (0,1)$, and define $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$ as the exterior of Ω. Consider the direct problem

$$\begin{cases} ((-\Delta)^s + q)u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{cases}$$

and its associated DN map $\Lambda_q \colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$. We see that Λ_q is well-defined as a bounded linear operator as soon as the potential q is such that 0 is not a Dirichlet eigenvalue of $((-\Delta)^s + q)$, that is

$$\text{If } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + q)u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u = 0.$$

Moreover, one proves that under stronger assumptions the DN map has the form $\Lambda_q f = (-\Delta)^s u|_{\Omega_e}$ (see lemma 3.1 in [46]). The inverse problem now consists in recovering the potential q from the knowledge of the DN map Λ_q. Many results were reached in uniqueness, stability and recoverability for the fractional Calderón problem (see section 2.2).

Fractional inverse problems have recently attracted the interest of numerous fields of science. This is mainly due to the fact that the fractional Laplace operator can be related to the process of anomalous diffusion ([139]), and thus the fractional Schrödinger equation can be used to describe those diffusion processes in which the dependence of the mean squared displacement on time is non linear ([7]). Many results were obtained for instance in turbulent fluid dynamics ([29], [31]), ecology ([57], [96], [110]), image processing ([48]), mathematical finance ([3], [90], [127]), quantum mechanics [87, 88], elasticity ([128]) and physics in general ([37], [39], [47], [87], [100], [142]), among many others ([125, 115, 7, 113, 15]).

Another application of the fractional Calderón problem is for indirectly detecting corrosion. This kind of problem can be formulated by means of the Robin inverse problem ([67, 68, 126, 11]), which in turn can be related to the inverse problem for the fractional Schrödinger equation via the Caffarelli-Silvestre extension ([18], (D)).

2. Calderón type problems

2.1. **The classical Calderón problem.** Being a prototypical elliptic inverse problem, the Calderón problem has received large attention since its formulation ([20]). In this section we will recall the main known results and open problems, while referring to the surveys [138, 124] for greater detail.

2.1.1. *Boundary determination.* The first and most natural question is whether the conductivity γ and its normal derivatives can be recovered at the boundary. Kohn and Vogelius showed that this can be done and obtained uniqueness results for real-analytic ([72]) and piecewise real-analytic ([74]) conductivities. These results are *local*, in the sense that the DN map needs to be known just in an open set of the boundary in order to determine γ in that open set. A stability result based on a microlocal technique ([133]) then extends the uniqueness to continuous conductivities. In all the above methods, the trick always is to test the DN map against functions oscillating rapidly at the boundary point where the conductivity is to be determined.

2.1.2. *CGO solutions.* Complex geometrical optics (CGO) solutions to the conductivity equation were first devised by Sylvester and Uhlmann ([132, 131]) with the goal of emulating the behaviour of Calderón's exponential solutions ([20]) at high frequencies. These are functions of the form

$$u(x) = e^{x\cdot\rho}(1 + \psi(x,\rho)) ,$$

where the error ψ is small when $|\rho|$ is large and vanishes when $|\rho| \to \infty$. The construction in the cited papers is suited for C^2 conductivities, but it has been upgraded to different regularity assumptions ([104, 123, 134]) and even to the case of the magnetic Schrödinger equation ([80]). The significance of these CGO solutions is that they can be used as test functions for the reconstruction of the conductivity from the DN map.

2.1.3. *Results in dimension $n \geq 3$.* Using specific CGO solutions to test an integral identity derived from the assumption that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ via a reduction to the classical Schrödinger equation, it is possible to show the uniqueness result $\gamma_1 = \gamma_2$ for strictly positive C^2 conductivities ([132]). This requires the boundary determination from [72] as well. In the following years this result has been improved on various occasions. In [51] Haberman and Tataru showed uniqueness for C^1 conductivities, in [23] Caro and Rogers extended the result to Lipschitz conductivities, and in [50] Haberman treated the case of conductivities belonging to $W^{1,n} \cap L^\infty(\Omega)$, $n = 3, 4$, thus showing that the gradient of the conductivity does not need to be bounded. Whether uniqueness still holds for less regular conductivities in higher dimension is an open problem at the time of writing.

The main stability result in dimension $n \geq 3$ was proved by Alessandrini in [4], where a logarithmic modulus of continuity was shown to appear

$$\|\gamma_1 - \gamma_2\|_{L^\infty} \leq C \left(|log\|\Lambda_1 - \Lambda_2\|_{\frac{1}{2},-\frac{1}{2}}|^{-\sigma} + \|\Lambda_1 - \Lambda_2\|_{\frac{1}{2},-\frac{1}{2}} \right) , \quad \sigma \in (0,1)$$

for smooth conductivities. The optimality of this estimate was later proved by Mandache ([95]), showing that the Calderón problem is severely *ill-posed*. However, better results were obtained by adding a-priori information about the conductivity ([5, 112]). It is thought ([103]) that the stability estimates get better closer to the boundary. It was also proved ([65]) that in the case $n = 3$ the inverse problem for the Helmoltz equation shows increased stability at high frequencies.

A reconstruction result for the Schrödinger equation was obtained by Nachman and Novikov ([102, 108]). Using the CGO construction, they showed that the potential q can be determined from the associated DN map Λ_q. Using the Liouville reduction and the boundary determination results cited above, they were able to reconstruct the corresponding conductivity γ as well.

2.1.4. *Partial data.* It is often impossible to perform measurements on the whole boundary of the domain, as some parts of it might be inaccessible. Whenever the DN map is known only on part of the boundary, we are dealing with a *partial data* problem. The question of whether it is possible to determine the potential q from measurements performed only on an arbitrary open subset of the boundary is an open problem (see [36] for a result in the linearized case). However, Isakov proved uniqueness ([64]) when the remaining part of the boundary belongs to a plane or a sphere, making use of a reflection trick. This technique was later generalized to the Maxwell system ([22]; see also the related paper [109] for the general inverse problem). In the case in which the domain is only known to be strictly convex, [70] grants global identifiability for a DN map measured on any open subset of the boundary for functions supported in a neighbourhood of its complement. The method in [70] extends that of [132], as it requires a new kind of CGO solutions of the form

$$u = e^{-\frac{1}{h}(\phi + i\psi)}(a + r) ,$$

where ψ solves an eikonal equation, a is a smooth solution of a complex transport equation, h is a small parameter and r is an error function whose L^2 norm vanishes as $h \to 0$. It is also essential that ϕ is a *limiting Carleman weight*, and the existence of such function is granted by Carleman estimates. This uniqueness result was then improved with a

reconstruction method ([101]) and extended to both the magnetic case ([35, 71, 79, 136]) and the Maxwell system ([26]).
An extreme partial data case was studied in [107], where just one voltage-current measurement was shown to suffice for the estimation of the size of an inclusion embedded in a two-dimensional body with discontinuous conductivity.

2.1.5. *Results in dimension* $n = 2$. In two dimensions also methods from complex analysis are available. While the Calderón problem in 2D is formally determined by variable counting, one can also construct a larger class of CGO solutions in this case. This has led to better results in the case $n = 2$, first and foremost the one by Astala and Päivärinta ([9]) which shows uniqueness for an L^∞ conductivity. Their technique has also been used for numerical reconstruction procedures ([62, 63]). Bukhgeim ([16]) proved that a potential $q \in C^1$ can be uniquely determined starting from Cauchy data. This result was later improved by Blåsten, Imanuvilov and Yamamoto for $q \in L^p$, $p > 2$ ([13]), and again by Blåsten, Tzou and Wang for $p > 4/3$ ([14]). Uniqueness was also proved in [30] for an unbounded conductivity with a specific a-priori estimate depending on the domain Ω. Further, in [60] Imanuvilov, Uhlmann and Yamamoto solved the partial data problem in two dimensions for the Schrödinger equation, and thus for the conductivity equation as well, by showing that the potential is uniquely determined in a bounded domain by the Cauchy data on an arbitrary open subset of the boundary.

2.1.6. *Anisotropic conductivities.* In some materials, such as crystals or muscle tissue, the electrical properties in a point depend on direction. In these cases, the conductivity is better represented by a positive definite, smooth, symmetric matrix, and it is said to be *anisotropic*. The version of the Calderón problem which asks to recover such matrix conductivity from the associated DN map has been shown by Tartar not to be solvable in general because of a natural gauge ([73]). However, uniqueness up to the gauge class for $n = 2$ has been proved in [8] for L^∞ conductivities, using a change of variable (the *isothermal coordinates*, [2, 130]) to reduce the problem to the isotropic one. In the case $n \geq 3$ the problem is of geometric nature and better discussed on manifolds ([89]). We refer to [138] and the references therein for more details.

2.1.7. *Inaccurately known boundary.* The accuracy of the recovered conductivities can be affected by multiple factors, among which the exact knowledge of the boundary and the contact impedances. In a typical application, the experimenter may not have precise data about the shape of the boundary of the domain of interest. In a series of papers ([75, 76, 77, 78]) various aspects of this problem were addressed and some new computational methods were proposed.

2.2. **The fractional Calderón problem.** The fractional version of the Calderón problem has been object of intensive study in the years following its formulation in the seminal paper [46] by Ghosh, Salo and Uhlmann. This problem enjoys several properties that distinguish it from its classical counterpart and make it somehow more manageable ([33, 34, 118, 116, 43]). Already from a heuristic point of view, a simple variable count shows that the problem is overdetermined in any dimension. In this section we list some of the main results and techniques, referring to [125] for more details and references.

2.2.1. *Uniqueness.* The main uniqueness result was achieved in the paper [46] itself for L^∞ potentials. The proof is based on a strong approximation property enjoyed by the fractional Schrödinger equation, the *Runge approximation property*, which itself depends on the *unique continuation property* for the fractional Laplacian. It is interesting to note that the principal uniqueness result is already formulated for all dimensions and for partial data, while the corresponding problem is open in the classical case for $n \geq 3$ and requires a different technique in dimension 2. Low regularity was investigated in [118], where the proof of uniqueness was extended to potentials in $L^{n/2s}(\Omega)$ and $W^{-s,n/s}(\Omega)$.

2.2.2. *Stability.* Similarly to its classical counterpart, the fractional Calderón problem was shown to be severely ill-posed, due to the presence of a logarithmic modulus of continuity. In a series of papers from year 2017 ([120, 118, 119, 117]), Rüland and Salo showed that one has

$$\|q_1 - q_2\|_{L^{n/2s}(\Omega)} \leq C|log\|\Lambda_{q_1} - \Lambda_{q_2}\|_*|^{-\sigma} , \quad \sigma \in (0,1) ,$$

and moreover that this type of stability is optimal. This result was made possible by a careful analysis of the quantitative aspects of the estimates contained in the uniqueness proof for low regularity potentials.

2.2.3. *Reconstruction and single measurement.* The possibility of recovering and even reconstructing a low regularity potential q from its associated DN map Λ_q was shown in [45], even in the case of a single measurement. Ths kind of result is specific to the fractional case, and sets it strongly apart from the classical problem. The methods involved require a unique continuation property from sets of vanishing measure, as well as various regularization schemes. More related results for full-data reconstruction were obtained using monotonicity methods ([53, 54]).

2.2.4. *Related problems.* Perturbed versions of the fractional Calderón problem have been studied in several recent papers. Variable coefficients were considered in two different settings in our paper (A) about the fractional conductivity equation and in [44], where an anisotropic case was studied. Different versions of a fractional magnetic Schrödinger equation have been the object of several works, among which our papers (B) and (C) (see also [91, 93, 24]). A lower order nonlocal perturbation was introduced in [12], while our paper (E) considers general local perturbations. Other variants include semilinear equations ([83, 84]), the fractional heat equation ([85, 118]) and nonlocal Schrödinger-type elliptic operators ([21]), among many others (see for instance [94, 121, 42]).

At this stage there are of course many problems left open in the field. Some of them were outlined in our article (C).

3. Preliminaries: function spaces and the fractional Laplacian

In this section we recall the main function spaces used in the included articles, as well as the definition of the omnipresent fractional Laplace operator. We follow [1, 46, 99, 97, 135, 140] as references.

3.1. **Inhomogeneous fractional L^2-based Sobolev spaces.** Let $r \in \mathbb{R}$. If $u \in \mathscr{S}(\mathbb{R}^n)$ is a Schwartz function, let

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \mathrm{d}x$$

indicate the Fourier transform of u. The Fourier transform can be extended to act as an isomorphism $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ on tempered distributions. By $\mathcal{F}^{-1}(u)$ we indicate the inverse Fourier transform of u. The inhomogeneous fractional L^2-based Sobolev space of order $r \in \mathbb{R}$ is

$$H^r(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^2(\mathbb{R}^n)\} ,$$

and its norm is defined as

$$\|u\|_{H^r(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^2(\mathbb{R}^n)} ,$$

where $\langle x \rangle := (1+|x|^2)^{1/2}$. If $\Omega, F \subset \mathbb{R}^n$ respectively are an open and a closed set, then define

$$\begin{aligned} H^r_F(\mathbb{R}^n) &= \{u \in H^r(\mathbb{R}^n) : \operatorname{spt}(u) \subset F\} \\ \widetilde{H}^r(\Omega) &= \text{ closure of } C^\infty_c(\Omega) \text{ in } H^r(\mathbb{R}^n) \\ H^r(\Omega) &= \{u|_\Omega : u \in H^r(\mathbb{R}^n)\} \\ H^r_0(\Omega) &= \text{ closure of } C^\infty_c(\Omega) \text{ in } H^r(\Omega) , \end{aligned}$$

where we use the symbol $\operatorname{spt}(u)$ to indicate the support of u. To the space $H^r(\Omega)$ we associate the quotient norm

$$\|v\|_{H^r(\Omega)} = \inf\{\|w\|_{H^r(\mathbb{R}^n)} : w \in H^r(\mathbb{R}^n),\ w|_\Omega = v\} .$$

The following inclusions among the above spaces hold for any open set Ω and $r \in \mathbb{R}$:

$$\widetilde{H}^r(\Omega) \subset H^r_0(\Omega), \quad \widetilde{H}^r(\Omega) \subset H^r_{\overline{\Omega}}(\mathbb{R}^n), \quad (\widetilde{H}^r(\Omega))^* = H^{-r}(\Omega), \quad (H^r(\Omega))^* = \widetilde{H}^{-r}(\Omega) .$$

If in particular Ω is a Lipschitz domain, we also have

$$\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n), \quad \text{for all } r \in \mathbb{R} ,$$

$$H^r_0(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n), \quad \text{if } r > -1/2 \text{ and } r \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \dots\right\} .$$

3.2. **Bessel potential spaces.** More in general, if $1 \leq p \leq \infty$ and $r \in \mathbb{R}$ we can define the Bessel potential space

$$H^{r,p}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^p(\mathbb{R}^n)\}$$

and its norm

$$\|u\|_{H^{r,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^p(\mathbb{R}^n)} .$$

The name is due to the fact that $J := (\mathrm{Id} - \Delta)^{1/2}$ is called the Bessel potential, and thus $\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) = J^r u$. Similarly to what we did in 3.1, we define the spaces $H^{r,p}_F(\mathbb{R}^n), \widetilde{H}^{r,p}(\Omega), H^{r,p}(\Omega)$ and $H^{r,p}_0(\Omega)$ for $\Omega, F \subset \mathbb{R}^n$ an open and a closed set. As before we get the inclusions

$$\widetilde{H}^{r,p}(\Omega) \subset H^{r,p}_0(\Omega), \quad \widetilde{H}^{r,p}(\Omega) \subset H^{r,p}_{\overline{\Omega}}(\mathbb{R}^n)$$

for all $r \in \mathbb{R}$, $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ open. Moreover, if Ω is a bounded C^∞-domain and $1 < p < \infty$ by [135, Section 4.3.2, Theorem 1] we have

$$\begin{aligned} \widetilde{H}^{r,p}(\Omega) &= H^{r,p}_{\overline{\Omega}}(\mathbb{R}^n), \quad \text{for all } r \in \mathbb{R} , \\ H^{r,p}_0(\Omega) &= H^{r,p}(\Omega), \quad \text{if } r \leq \frac{1}{p} . \end{aligned}$$

3.3. **Homogeneous fractional L^2-based Sobolev spaces.** The norm of the fractional Sobolev space $H^r(\mathbb{R}^n)$ is not homogeneous with respect to the scaling $\xi \to \lambda\xi$. It is also possible to define a variety of fractional Sobolev space for which this homogeneity holds: we let

$$\dot{H}^r(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \hat{u} \in L^1_{loc}(\mathbb{R}^n) \text{ and } |\cdot|^r \hat{u} \in L^2(\mathbb{R}^n)\}$$

and define

$$\|u\|_{\dot{H}^r(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2}$$

to be its norm. For negative r we have the inclusion $\dot{H}^r(\mathbb{R}^n) \subsetneq H^r(\mathbb{R}^n)$, while for positive r we have $H^r(\mathbb{R}^n) \subsetneq \dot{H}^r(\mathbb{R}^n)$. When $r < n/2$, we have that $\dot{H}^r(\mathbb{R}^n)$ is a Hilbert space; in this case we also have that $\mathscr{S}_0(\mathbb{R}^n)$ is dense in $\dot{H}^r(\mathbb{R}^n)$, where $\mathscr{S}_0(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ is defined as

$$\mathscr{S}_0(\mathbb{R}^n) = \{\varphi \in \mathscr{S}(\mathbb{R}^n) : \hat{\varphi}|_{B(0,\epsilon)} = 0 \text{ for some } \epsilon > 0\} .$$

3.4. **Semiclassical Sobolev spaces.** Let $h \in (0,1)$ and $r \in \mathbb{R}$. If $u \in L^2(\mathbb{R}^n)$, we define the semiclassical Fourier transform ([143]) as

$$\mathcal{F}_{sc}u(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}x\cdot\xi} u(x) dx .$$

Correspondingly, the semiclassical Sobolev norm will be

$$\|u\|^2_{H^r_{sc}(\mathbb{R}^n)} := (2\pi h)^{-n} \, \|\langle\cdot\rangle^r \mathcal{F}_{sc}u\|^2_{L^2(\mathbb{R}^n)} .$$

The semiclassical Sobolev spaces $L^2_{sc}(\mathbb{R}^n)$ and $H^1_{sc}(\mathbb{R}^n)$ are then defined as those subspaces of $L^2(\mathbb{R}^n)$ where the semiclassical norms $\|\cdot\|^2_{L^2_{sc}(\mathbb{R}^n)}$, $\|\cdot\|^2_{H^1_{sc}(\mathbb{R}^n)}$ are finite. Observe that in these two special cases we have

$$\|u\|_{L^2_{sc}(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \quad \text{ and } \quad \|u\|_{H^1_{sc}(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + h^{-1}\|\nabla u\|_{L^2(\mathbb{R}^n)} .$$

3.5. **Sobolev multiplier spaces.** Let $r, t \in \mathbb{R}$. Following [97, Ch. 3], we say that a distribution f belongs to $M(H^r \to H^t)$ if and only if the norm

$$\|f\|_{r,t} := \sup\{|\langle f, uv\rangle| \; ; \, u, v \in C^\infty_c(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} = \|v\|_{H^{-t}(\mathbb{R}^n)} = 1\}$$

is finite. Since it holds that $|\langle f, uv\rangle| \leq \|f\|_{r,t} \, \|u\|_{H^r(\mathbb{R}^n)} \, \|v\|_{H^{-t}(\mathbb{R}^n)}$, by density f acts as a multiplier between $H^r(\mathbb{R}^n)$ and $H^{-t}(\mathbb{R}^n)$.

One can prove many interesting properties of these multiplier spaces (see for instance [97]). We have

$$M(H^r \to H^t) = M(H^{-t} \to H^{-r}) , \qquad \text{for all } r, t \in \mathbb{R} ,$$

$$M(H^r \to H^t) = \{0\} , \qquad \text{if } r < t ,$$

and if $\lambda, \mu \geq 0$ then also

$$M(H^{r-\lambda} \to H^{t+\mu}) \hookrightarrow M(H^r \to H^t) .$$

Let $M_0(H^r \to H^t)$ be the closure of $C^\infty_c(\mathbb{R}^n)$ in $M(H^r \to H^t) \subset \mathcal{D}'(\mathbb{R}^n)$. For this space we have

$$M_0(H^{r-\lambda} \to H^{t+\mu}) \subseteq M_0(H^r \to H^t) .$$

3.6. **The fractional Laplacian.** The fractional Laplacian $(-\Delta)^s$ is the main operator studied in the included articles. It is a nonlocal operator of order $2s$, and for this reason its behaviour is quite different from that of the classical Laplacian, which can be described as a local differential operator of order 2. It is however true that at the limit $s \to 1^-$ we recover the classical behaviour from the nonlocal operator [32].

One may define the fractional Laplacian in many different equivalent ways [81] in the most typical regime $s \in (0,1)$. We use the definition $(-\Delta)^s\varphi := \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{\varphi})$, which is valid for $\varphi \in \mathscr{S}(\mathbb{R}^n)$. In this case, a simple computation shows that $(-\Delta)^s \colon \mathscr{S}(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is linear and continuous. Therefore, it is possible to uniquely extend it to act on $H^r(\mathbb{R}^n)$ for every $r \in \mathbb{R}$, in which case we have

$$(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)\ .$$

We can do something similar for the homogeneous fractional Sobolev spaces. In this case we define $(-\Delta)^s\varphi := \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{\varphi})$ for $\varphi \in \mathscr{S}_0(\mathbb{R}^n)$, observe that $(-\Delta)^s \colon \mathscr{S}_0(\mathbb{R}^n) \to \dot{H}^{r-2s}(\mathbb{R}^n)$ is an isometry with respect to $\|\cdot\|_{\dot{H}^r(\mathbb{R}^n)}$, and eventually extend the operator to a continuous map

$$(-\Delta)^s \colon \dot{H}^r(\mathbb{R}^n) \to \dot{H}^{r-2s}(\mathbb{R}^n)$$

by density, whenever $r < n/2$.

The fractional Laplacian can be studied more generally for $s > -n/4$ and $u \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$. In this case we see that $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{u}) \in \mathscr{S}'(\mathbb{R}^n)$, that is, $(-\Delta)^s u$ makes sense as a tempered distribution (see for instance section 2.2 in our paper (C)).

4. Main results

In this section we will review the results achieved in the included articles. Each of the following subsections is dedicated to one of the articles (A) to (E). For each one of them we will give some context, the relevant definitions, the statements of the theorems and a sketch of their proofs.

4.1. **Uniqueness for the inverse problem for the fractional conductivity equation, (A).** The main goal of article (A) is to define and study a fractional counterpart of the classical Calderón problem. In light of the recent paper [46], it was expected that we could achieve better results than the classical ones employing the intrinsic nonlocality of the fractional operators. We have proved in (A) that this is indeed the case.

Fix $s \in (0,1)$ and consider the *fractional gradient* operator $\nabla^s : C_c^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$

$$\nabla^s u(x,y) := -\frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}}\frac{u(y)-u(x)}{|y-x|^{n/2+s+1}}(y-x)\ .$$

Since one sees that $\|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n})} \leq \|u\|^2_{H^s(\mathbb{R}^n)}$, this operator can be extended by density to act on $H^s(\mathbb{R}^n)$. We also define the *fractional divergence* operator $(\nabla\cdot)^s : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$ in such a way that it is the adjoint of ∇^s. These operators were firstly introduced in the more general framework of [38]. They should be thought of as nonlocal counterparts of the standard divergence and gradient; just as in the classical case, they have the interesting property that $(-\Delta)^s u = (\nabla\cdot)^s\nabla^s u$ ([(A), Lemma 2.1]).

We set up the Dirichlet problem for the fractional conductivity equation as

$$\begin{cases} \mathrm{C}^s_\gamma u := (\nabla\cdot)^s(\gamma(x)^{1/2}\gamma(y)^{1/2}\nabla^s u) = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases}\ .$$

One shows that this problem is well-posed ([(A), Theorem 3.1]), and thus the DN map $\Lambda^s_\gamma : H^s(\Omega_e) \to (H^s(\Omega_e))^*$ can be defined in a weak sense ([(A), Lemma 3.3]). The inverse problem we are interested in now asks to recover γ knowing Λ^s_γ.

The main results in paper (A) are the two following theorems. Theorem 4.1 gives uniqueness for the inverse problem for the nonlocal conductivity equation, while theorem 4.2 gives a uniqueness result and a reconstruction procedure in the case of a single measurement.

Theorem 4.1. ((A), Theorem 1.1) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set, $s \in (0,1)$, and for $j = 1,2$ let $\gamma_j : \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\begin{cases} \text{for some } \underline{\gamma_j}, \overline{\gamma_j} \in \mathbb{R}, \quad 0 < \underline{\gamma_j} \leq \gamma_j(x) \leq \overline{\gamma_j} < \infty, \text{ for a.e. } x \in \mathbb{R}^n \\ \gamma_j^{1/2}(x) - 1 := m_j(x) \in W_c^{2s, n/2s}(\Omega) \end{cases} .$$

Suppose $W_1, W_2 \subset \Omega_e$ are open sets, and that the DN maps for the conductivity equations in Ω relative to γ_1 and γ_2 satisfy

$$\Lambda^s_{\gamma_1}[f]|_{W_2} = \Lambda^s_{\gamma_2}[f]|_{W_2}, \qquad \forall f \in C^\infty_c(W_1) .$$

Then $\gamma_1 = \gamma_2$.

Theorem 4.2. ((A), Theorem 1.2) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded open set, $s \in (0,1)$, $\epsilon > 0$, and let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\begin{cases} \text{for some } \underline{\gamma}, \overline{\gamma} \in \mathbb{R}, \quad 0 < \underline{\gamma} \leq \gamma(x) \leq \overline{\gamma} < \infty, \text{ for a.e. } x \in \mathbb{R}^n \\ \gamma^{1/2}(x) - 1 := m(x) \in W_c^{2s+\epsilon, p}(\Omega), \quad \text{for } p > n/\epsilon \end{cases} .$$

Suppose $W_1, W_2 \subset \Omega_e$ are open sets, with $\overline{\Omega} \cap \overline{W_1} = \emptyset$. Given any fixed function $g \in \widetilde{H}^s(W_1) \setminus \{0\}$, γ is uniquely determined and can be reconstructed from the knowledge of $\Lambda^s_\gamma[g]|_{W_2}$.

The proofs of the two above theorems are achieved by reduction. The plan is to express the inverse problem for the fractional conductivity equation as an inverse problem for the fractional Schrödinger equation, which is in turn well understood thanks to the previous results ([120], [45]).

Thus our first step is to show that the fractional conductivity equation can be rephrased as a special case of the fractional Schrödinger equation for an appropriate choice of the potential q, namely $q = \frac{(-\Delta)^s(1-\gamma^{1/2})}{\gamma^{1/2}}$. In fact, as shown in [(A), Theorem 3.1] we have that for all $u \in H^s(\mathbb{R}^n)$

$$\mathrm{C}^s_\gamma u = \gamma^{1/2}((-\Delta)^s + q)(u\gamma^{1/2})$$

holds, which entails that for all $g \in H^s(\Omega_e)$

$$\begin{cases} \mathrm{C}^s_\gamma u = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e \end{cases} \quad \Leftrightarrow \quad \begin{cases} \Big((-\Delta)^s + q\Big) w = 0 & \text{in } \Omega \\ w = \gamma^{1/2} g & \text{in } \Omega_e \end{cases} ,$$

with $w = \gamma^{1/2} u$. This is reminiscent to one of the strategies used to study the classical conductivity equation, the so called Liouville reduction ([124]).

This is of course not enough, as one still needs to show that the DN map for the new Schrödinger problem can be deduced from the DN map of the original fractional conductivity problem. This issue is dealt with in [(A), Lemma 3.4] by means of the following integral identity, holding for all $f, v \in H^s(\mathbb{R}^n)$ with support in Ω_e:

$$\Lambda^s_{q_\gamma}[f]([v]) - \Lambda^s_\gamma[f]([v]) = \int_{\Omega_e} fv(-\Delta)^s m \, dx .$$

Once the reduction procedure is complete, one can apply the results [120], [45] cited above, which respectively have the effect of either proving uniqueness for the potential q or even reconstructing it from a single measurement. The key points in these works are the strong uniqueness and approximation results obtained in [34].

In order to complete our proof, we will need to show that the information we have obtained about q is enough to show the uniqueness and reconstruction results for γ. This last step makes use of the uniqueness of solutions of the fractional Schrödinger equation, which was proved in [46].

The last section of article (A) shows how the fractional conductivity equation naturally arises as continuous limit of a long jump random walk with weights, as it is to expect for an equation concerning anomalous diffusion [139].

4.2. **Uniqueness for the inverse problem for the fractional Schrödinger equation in a magnetic field, (B).** In article (B) our main goal is to define and study a fractional version of the classical inverse problem for the magnetic Schrödinger equation. This was, in a sense, previously studied in the paper [24], whose authours find that no gauge exists for a certain magnetic Schrödinger equation in which all the lower order terms are local. This turned out to be the case also in the following related works [92, 93, 91]. In contrast, we have proved in (B) that our version of the fractional magnetic Schrödinger equation (FMSE), which is in a sense completely nonlocal, does indeed posses a natural gauge.

Fix $s \in (0,1)$ and a vector potential A. Here we assume that $A = A(x,y)$ depends on two spatial variables $x, y \in \mathbb{R}^n$, in order to account for the nonlocality of the problem. We define the magnetic versions ∇^s_A and $(\nabla\cdot)^s_A$ of the fractional gradient and divergence operators weakly as

$$\langle \nabla^s_A u, v\rangle := \langle \nabla^s u + Au, v\rangle$$

and

$$\langle (\nabla\cdot)^s_A v, u\rangle := \langle v, \nabla^s_A u\rangle \ ,$$

for all $u \in H^s(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^{2n})$. These respectively act as operators $\nabla^s_A : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ and $(\nabla\cdot)^s_A : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$. Observe that this way of constructing magnetic divergence and gradient resembles the one used in the classical case [104]. The magnetic fractional Laplacian will be the combination of the two, namely $(-\Delta)^s_A := (\nabla\cdot)^s_A(\nabla^s_A)$ acting from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$. One sees immediately that in the case $A \equiv 0$ this reduces back to the fractional Laplacian (see (B)).
Next we set up the Dirichlet problem for the fractional magnetic Schrödinger equation as

$$\begin{cases} (-\Delta)^s_A u + qu = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases} ,$$

and define the DN map $\Lambda^s_{A,q} : H^s(\Omega_e) \to (H^s(\Omega_e))^*$. Again, the inverse problem is to recover A and q in Ω from $\Lambda^s_{A,q}$. This turns out to be impossible in general, because of the natural gauge associated to the equation. We say that the couples of potentials (A_1, q_1) and (A_2, q_2) *are in gauge* when it happens that the corresponding operators $(-\Delta)^s_{A_j} + q_j$ coincide, and we indicate this eventuality with $(A_1, q_1) \sim (A_2, q_2)$. As we have proved in [(B), Lemmas 3.8, 3.9], for all couples (A_1, q_1) it is possible to find a *different* couple (A_2, q_2) such that $(A_1, q_1) \sim (A_2, q_2)$. Thus, we say that the fractional magnetic Schrödinger equation *enjoys the gauge* $\sim$.

Observe that the gauge holding for MSE, which we indicate with $\approx$, is quite different from $\sim$. One may define $\approx$ as

$$(A_1, q_1) \approx (A_2, q_2) \quad \Leftrightarrow \quad \exists \phi \in G : (-\Delta)^s_{A_1}(u\phi) + q_1 u\phi = \phi((-\Delta)^s_{A_2} u + q_2 u) \ ,$$

for all $u \in H^s(\mathbb{R}^n)$, where $G := \{\phi \in C^\infty(\mathbb{R}^n) : \phi > 0, \phi|_{\Omega_e} = 1\}$. In lemmas 3.9 and 3.10 of (B) we proved that FMSE enjoys only $\sim$, while MSE only enjoys $\approx$. The reason of this difference emerges from the nonlocality of FMSE: as shown in formula (10) in (B), the coefficient of the gradient term in FMSE is related only to the antisymmetric part A_a of the vector potential A, and such antisymmetry requirement does not allow FMSE to enjoy $\approx$. It follows that, in contrast to the classical case, the scalar potential q can not be in general uniquely determined for FMSE.

It is clear from the discussion above that we can only hope to recover (A, q) up to $\sim$; this is what we prove in our main theorem:

Theorem 4.3. ((B), Theorem 1.1) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded open set, $s \in (0,1)$, and let $(A_i, q_i) \in \mathcal{P}$ for $i = 1, 2$. Suppose $W_1, W_2 \subset \Omega_e$ are open sets, and that the DN maps for the fractional magnetic Schrödinger equations in Ω relative to (A_1, q_1) and (A_2, q_2) satisfy*

$$\Lambda^s_{A_1,q_1}[f]|_{W_2} = \Lambda^s_{A_2,q_2}[f]|_{W_2}, \qquad \forall f \in C^\infty_c(W_1) \ .$$

Then $(A_1, q_1) \sim (A_2, q_2)$, that is, the potentials coincide up to the gauge $\sim$.

Here $\mathcal{P}$ is a class of potentials verifying certain technical regularity assumptions (see section 3 in (B)). The proof of the above theorem is based on a technique initially developed for the fractional case with $A \equiv 0$ in [46]. The first step is to show that the fractional magnetic Schrödinger operator enjoys the so called *weak unique continuation property* ([(B), Lemmas 3.4, 4.1]), a very nonlocal property which states that any $u \in H^s(\mathbb{R}^n)$ such that $u = (-\Delta)^s_A u = 0$ in some open set W must vanish identically everywhere. This is easily achieved thanks to our assumptions on $\mathcal{P}$ and the previous work [114].

Next, we prove the *Runge approximation property* ([(B), Lemma 3.15]) for the fractional magnetic Schrödinger operator. This property states that any $L^2(\Omega)$ function may be approximated arbitrarily well by the restriction to Ω of a solution to the fractional magnetic Schrödinger equation with some exterior value $f \in C^\infty_c(W)$, where W is any open subset of Ω_e. For this proof we use the Hahn-Banach theorem and the previously cited weak unique continuation property.

We also need an *Alessandrini identity*, that is an integral identity relating the difference of the DN maps corresponding to potentials (A_1, q_1), (A_2, q_2) to the differences of the potentials themselves. This is obtained in [(B), Lemma 3.13]. In order to extract useful information from this identity, we test it with some aptly shaped solutions to the fractional magnetic Schrödinger equation, which in turn are cooked up using the Runge approximation property. Eventually, this lets us reconstruct the gauge class to which our couples of potentials (A_j, q_j) must belong.

Article (B) also contains a discussion of how our fractional magnetic Schrödinger equation naturally arises as a continuous limit of a long jump random walk with weights depending on position. This feels like a natural generalization of both [139] and our article (A). In the last section of the paper, we briefly entertain the idea of a hybrid fractional conductivity-magnetic equation and show that for it we can get similar results as for the purely magnetic case.

4.3. **The higher order fractional Laplacian: unique continuation property, Poincaré inequality and higher order fractional magnetic Schrödinger equation, (C).** The third included paper deals with some properties of the high order fractional Laplacian, i.e. of the nonlocal operator $(-\Delta)^s$, with $s \in (-n/2, \infty) \setminus \mathbb{Z}$. In particular, we investigate the *unique continuation property* and the *Poincaré inequality*, achieving quite satisfactory results in both cases.

We are interested in the unique continuation property for the fractional Laplacian because it has been by now extensively employed in showing uniqueness results for fractional Schrödinger equations [45, 46, 120]. It dates back to at least Riesz [111]; subsequently it has been used in [59] for Riesz potentials I_α. In the case $s \in (0,1)$, the unique continuation property of $(-\Delta)^s$ for functions in $H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, was proved in [46] with a technique based on Carleman estimates and Caffarelli-Silvestre extensions ([114, 17, 18]). The unique continuation property for the fractional Schrödinger equation is also strictly related to the *fractional Landis conjecture*, which asks to determine the maximal vanishing rate at infinity of solutions of $(-\Delta)^s u + qu = 0$ ([122]). Our result generalizes the unique continuation property to all $s \in (-n/2, \infty) \setminus \mathbb{Z}$:

Theorem 4.4. ((C), Theorem 1.1) *Let $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $u \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. The claim holds also for $s \in (-n/2, -n/4] \setminus \mathbb{Z}$ if $u \in H^{r,1}(\mathbb{R}^n)$ or $u \in \mathscr{O}'_C(\mathbb{R}^n)$.*

We propose a proof of the above theorem by reduction: using the decomposition $(-\Delta)^s u = (-\Delta)^{s-k}(-\Delta)^k u$, with $k \in \mathbb{Z}$ and $s \in (0,1)$, we can achieve the desired result by invoking [46]. Of course this trick will only work for u belonging to aptly chosen function spaces. In the corollaries [(C), Corollaries 4.4, 4.5, 4.6] we obtain related results for the case of Bessel potential spaces and homogeneous Sobolev spaces, while in [(C), Corollary 4.2] we study Riesz potentials and in [(C), Corollary 4.3] we consider a slightly stronger result in the case of compact support.

The second property of the higher order fractional Laplacian $(-\Delta)^s$, $s \geq 0$, which we study in article (C) is the Poincaré inequality. It will be needed in the proof of the well-posedness of the inverse problem for the fractional Schrödinger equation. The higher order fractional Poincaré inequality has already appeared in [141] for smooth functions in a bounded Lipschitz domain, and in [10] for homogeneous Sobolev norms. Our contribution is to have extended some known results, given alternative proofs, and studied the connection between the fractional and the classical Poincaré constants.

Theorem 4.5. ((C), Theorem 1.4) *Let $s \geq t \geq 0$, $K \subset \mathbb{R}^n$ a compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $\tilde{c} = \tilde{c}(n, K, s) > 0$ such that*

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} \leq \tilde{c} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)} .$$

For the sake of illustrating some possibly unnoticed connections between methods, in our paper we present five different proofs for the fractional Poincaré inequality. The first of the proofs is very direct, and consists in splitting low and high frequencies in the Fourier side of the L^2 norm of the fractional Laplacian; this has the pleasant effect of giving an estimate for the Poincaré constant. The second proof, which is quite technical, derives from the approach considered in [46] and is based on several estimates, most notably the Hardy-Littlewood-Sobolev inequality. The third proof extends the result obtained in [24] by means of a reduction argument. Using the interpolation of homogeneous Sobolev spaces, we obtain a fourth proof and also an explicit constant in terms of the classical Poincaré constant. Finally, the fifth proof uses some uncertainty inequalities from [41].

Eventually, with all the previous results in mind, we consider the higher order fractional Schrödinger equation. We have achieved uniqueness results for the associated inverse problem at first in the case of a singular electric potential [(C), Theorems 1.5, 1.6], which generalizes the results obtained in [46, 120], and then in the case of non vanishing magnetic potential, which in turn generalizes our paper (B).

The first step towards these results consists in defining the *higher order fractional gradient* in a way that is reminiscent of the one we used in our first paper (A), keeping in mind that we now expect to get a tensor of order $\lfloor s \rfloor + 1$. We assume the following definition

$$\nabla^s u(x,y) := \frac{\mathcal{C}^{1/2}_{n,\{s\}}}{\sqrt{2}} \frac{\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)}{|y-x|^{n/2+\{s\}+1}} \otimes (y-x)$$

to hold for u smooth and compactly supported. We shall then extend this to $u \in H^s(\mathbb{R}^n)$ by density. Next, we define the *higher order fractional divergence* $(\nabla\cdot)^s$ by duality, and the magnetic counterparts of the fractional gradient and divergence operators as in (B). Their composition $(-\Delta)^s_A = (\nabla\cdot)^s_A \nabla^s_A$ is our *higher order magnetic fractional Laplacian*, which reduces to the magnetic fractional Laplacian considered in (B) as soon as $s \in (0,1)$, and eventually to the fractional Laplacian $(-\Delta)^s$ itself if A vanishes.

Thanks to [(C), Lemma 7.4], we can express the corresponding *fractional magnetic Schrödinger equation* in a more convenient form, which highlights the fractional Laplacian and the perturbation components of the equation. Using this and our higher order Poincaré inequality, we can prove the coericivity estimate for the bilinear form associated to the fractional magnetic Schrödinger equation ([(C), Lemma 7.5]), which eventually leads to the proof of well-posedness for the corresponding Dirichlet problem and the definition of the DN map ([(C), Lemma 7.6]). This is enough to state the inverse problem, for which we prove uniqueness in our main theorem:

Theorem 4.6. ((C), Theorem 1.7) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and let A_i, q_i verify assumptions (1)-(5) for $i = 1, 2$. Let $W_1, W_2 \subset \Omega_e$ be open sets. If the DN maps for the fractional magnetic Schrödinger equations in Ω relative to (A_1, q_1) and (A_2, q_2) satisfy*

$$\Lambda^s_{A_1,q_1}[f]|_{W_2} = \Lambda^s_{A_2,q_2}[f]|_{W_2}, \quad \text{for all } f \in C^\infty_c(W_1) ,$$

then $(A_1, q_1) \sim (A_2, q_2)$, that is, the potentials coincide up to gauge.

The assumptions (1)-(5) are purely technical, and coincide with the ones required by the previous results in our paper (B) and in [46] when $s \in (0,1)$ and $A = 0$. Observe that, just as in our previous paper (B), we obtain here that the problem has a natural gauge $\sim$: we will say that (A_1, q_1) and (A_2, q_2) are in gauge if and only if they give rise to the same equation, that is $(-\Delta)^s_{A_1} + q_1 = (-\Delta)^s_{A_2} + q_2$ as operators. It is thus clear that recovery may only be possible within the limits prescribed by the gauge, which is exactly what we prove.

The proof itself is based on the weak unique continuation property and the Runge approximation property, which hold for the higher order fractional magnetic Schrödinger equation as a consequence of [(C), Remark 7.7]. We can also write an integral identity for the equation; testing it with some aptly shaped exponential functions eventually produces the wanted result.

Part of article (C) is dedicated to the Radon transform and region of interest tomography. We have proved that a unique continuation property holds for the normal operator of the d-plane transform for odd d ([(C), Corollary 4.8]), and as a consequence that the X-ray transform enjoys a uniqueness property ([(C), Corollary 4.9]). These interesting results are however auxiliary to the topic of the present work, and thus we will not discuss them any further.

4.4. **The classical Calderón problem with mixed boundary conditions, (D).** This article provides a different point of view on fractional, nonlocal inverse problems by adopting a local "Caffarelli-Silvestre perspective". This is interesting in the reconstruction of non-directly measurable potentials on the boundary in addition to electric and magnetic potentials in the interior of a medium. In order to clarify the connection to the fractional Calderón problem, we shall first describe the set-up of the problem at hand.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, smooth domain, and assume that $\Sigma_1, \Sigma_2 \subset \partial\Omega$ are two disjoint, relatively open, smooth non-empty sets. In this setting we consider the following magnetic Schrödinger equation with mixed boundary conditions

$$
\begin{aligned}
-\Delta u - iA\cdot\nabla u - i\nabla\cdot(Au) + (|A|^2+V)u &= 0 \text{ in } \Omega,\\
\partial_\nu u + qu &= 0 \text{ on } \Sigma_1,\\
u &= f \text{ on } \Sigma_2,\\
u &= 0 \text{ on } \partial\Omega\setminus(\Sigma_1\cup\Sigma_2),
\end{aligned}
\tag{1}
$$

where the coefficients are supposed to be smooth and $\nu\cdot A = 0$ on $\partial\Omega$. Here the set Σ_1 represents an inaccessible part of the boundary where an unknown Robin coefficent q is present. The inverse problem consists in recovering the potentials A, V and q from the usual measurements encoded in the partial DN map $\Lambda_{A,V,q} : \widetilde{H}^{\frac12}(\Sigma_2) \mapsto H^{-\frac12}(\Sigma_2),\ f|_{\Sigma_2} \mapsto \partial_\nu u|_{\Sigma_2}$. We thus combine a classical Calderón problem with a Robin inverse problem, which arises for instance in the study of corrosion detection ([61]). In particular, we aim at a *simultaneous* recovery of the potentials; see the survey [69] for some partial results. The following is the result we achieved in (D) for the simple model described above:

Theorem 4.7. ((D), Theorem 1) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open, bounded and C^2-regular domain. Assume $\Omega_1 \Subset \Omega$ is an open, bounded set with $\Omega\setminus\Omega_1$ simply connected and that $\Sigma_1, \Sigma_2 \subset \partial\Omega$ are two disjoint, relatively open sets. If the potentials $q_1, q_2 \in L^\infty(\Sigma_1)$, $A_1, A_2 \in C^1(\Omega_1, \mathbb{R}^n)$ and $V_1, V_2 \in L^\infty(\Omega_1)$ in the equation* (1) *are such that*

$$\Lambda_1 := \Lambda_{A_1,V_1,q_1} = \Lambda_{A_2,V_2,q_2} =: \Lambda_2\ ,$$

then $q_1 = q_2$, $V_1 = V_2$ and $dA_1 = dA_2$.

Observe that in theorem 4.7 we have allowed some "safety distance" between the compact set Ω_1 in which the interior potentials are defined and the sets Σ_1, Σ_2 on the boundary. Also notice that the magnetic potential is only recovered in the sense that $dA_1 = dA_2$; the existence of this gauge is however expected and is reminiscent of [104].

Our proof is based on the Runge approximation ideas from [6, 119], which allow the approximation of full data CGO solutions in Ω_1 by partial data solutions in the whole domain Ω. We of course have to deal with the additional challenge due to the potential q on the piece Σ_1 of the boundary. However, we have proved that simultaneous density results *both in the bulk and on the boundary* are possible in [(D), Lemmas 1.1, 4.2]: for instance, if

$$\tilde{S}_{V,q} := \{u \in H^1(\Omega_1) :\ u \text{ is a weak solution to (1) in } \Omega\} \subset L^2(\Omega_1),$$

we prove the following simultaneous boundary and bulk approximation result:

Lemma 4.8. *Assume the consuete geometrical setting holds for Ω, Ω_1 and Σ_1, Σ_2. Let $V \in L^\infty(\Omega)$, $q \in L^\infty(\partial\Omega)$. Then the set*

$$\mathcal{R}_{bb} := \{(u|_{\Sigma_1}, u|_{\Omega_1}) :\ u|_{\Sigma_1} = Pf|_{\Sigma_1} \text{ and } u|_{\Omega_1} = Pf|_{\Omega_1} \text{ with } f \in C^\infty_c(\Sigma_2)\} \subset L^2(\Sigma_1)\times L^2(\Omega_1)$$

is dense in $L^2(\Sigma_1)\times\tilde{S}_{V,q}$ with the $L^2(\Sigma_1)\times L^2(\Omega_1)$ topology. Here P denotes the Poisson operator.

The rest of the proof of theorem 4.7 is an application of the above Runge approximation property and of an Alessandrini identity, similar to what was done in the previous articles (A)-(C).

The above problem can be made more interesting by introducing operators whose conductivities or potentials depend on the distance to the boundary. Let $d : \Omega \to [0, \infty)$ be a smooth function coinciding with the distance to the boundary in a neighbourhood of $\partial\Omega$, and let $s \in (0, 1)$. Consider the problem

(2)
$$\begin{aligned} -\nabla \cdot d^{1-2s}\nabla u - iAd^{1-2s} \cdot \nabla u - i\nabla \cdot (d^{1-2s}Au) + d^{1-2s}(|A|^2 + V)u &= 0 \text{ in } \Omega, \\ \lim_{d(x)\to 0} d^{1-2s}\partial_\nu u + qu &= 0 \text{ on } \Sigma_1, \\ u &= f \text{ on } \Sigma_2, \\ u &= 0 \text{ on } \partial\Omega \setminus (\Sigma_1 \cup \Sigma_2). \end{aligned}$$

The associated DN map will be

$$\Lambda_{s,A,V,q} : \widetilde{H}^s(\Sigma_2) \to H^{-s}(\Sigma_2),\ f|_{\Sigma_2} \mapsto \lim_{d(x)\to 0} d(x)^{1-2s}\partial_\nu u|_{\Sigma_2}.$$

We now wish to clarify the relation between this problem (and the previous one, which corresponds to the case $s = 1/2$) and the fractional Calderón problem. This is done by means of the so called *Caffarelli-Silvestre extension* [18]. Given a function $u \in H^s(\mathbb{R}^n)$, we study the degenerate elliptic problem

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s}\nabla \tilde{u} &= 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \tilde{u} &= u \text{ on } \mathbb{R}^n \times \{0\}. \end{aligned} \tag{3}$$

It is possible to prove that the degenerate DN operator associated to this equation is $(-\Delta)^s$, the fractional Laplacian. More exactly we have

$$(-\Delta)^s u := c_s \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}(x)\ .$$

This idea has been explored also in [129, 19]. In this sense, it is possible to understand equation (2) as a localized version of the inverse problem consisting in recovering the potentials $\tilde{A}$, $\tilde{V}$ and $\tilde{q}$ in the fractional Schrödinger equation

$$\begin{aligned} (-(\nabla + i\tilde{A})^2 + \tilde{V})^s u + \tilde{q}u &= 0 \text{ in } \Sigma_1 \subset \mathbb{R}^n, \\ u &= f \text{ on } \mathbb{R}^n \setminus \overline{\Sigma_1}, \end{aligned}$$

$\mathrm{supp}(f) \subseteq \Sigma_2$, from an associated DN map. In (2), the bounded domain $\Omega \subset \mathbb{R}^n$ plays the same role as $\mathbb{R}^{n+1}_+$ in (3).

We study problem (2) in the simplified assumptions that $\overline{\Sigma_1} := \overline{\Omega} \cap \{x_{n+1} = 0\}$, $\Sigma_2 = \partial\Omega \setminus \overline{\Sigma_1}$ and $A = 0$. Such geometric assumptions are not uncommon in partial data problems. Thus we consider

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s}\nabla u + Vx_{n+1}^{1-2s}u &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \Sigma_2, \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}u + qu &= 0 \text{ on } \Sigma_1, \end{aligned} \tag{4}$$

for $q \in L^\infty(\Sigma_1)$, $V \in L^\infty(\Omega)$ and, for instance, $f \in C^\infty_c(\Sigma_2)$. After having shown that the direct problem is well-posed, one can consider the associated DN map

$$\Lambda_{V,q} : f \mapsto \lim_{x_{n+1}\to\partial\Omega} x_{n+1}^{1-2s}\partial_\nu u|_{\Sigma_2}$$

and ask the relative inverse problem of simultaneously recovering q and V knowing $\Lambda_{V,q}$. For this question we achieved the following result in the regime $s \in (1/2, 1)$:

Theorem 4.9. ((D), Theorem 2) *Let $\Omega \subset \mathbb{R}^{n+1}_+$, $n \geq 3$, be an open, bounded and smooth domain. Assume that $\Sigma_1 := \partial\Omega \cap \{x_{n+1} = 0\}$ and $\Sigma_2 \subset \partial\Omega \setminus \Sigma_1$ are two relatively open, non-empty subsets of the boundary such that $\overline{\Sigma_1 \cup \Sigma_2} = \partial\Omega$. Let $s \in (1/2, 1)$. If the potentials $q_1, q_2 \in L^\infty(\Sigma_1)$ and $V_1, V_2 \in L^\infty(\Omega)$ relative to problem (4) are such that*

$$\Lambda_1 := \Lambda_{s,V_1,q_1} = \Lambda_{s,V_2,q_2} =: \Lambda_2 ,$$

then $q_1 = q_2$ and $V_1 = V_2$.

Since now V may be supported up to the sets Σ_1, Σ_2, the Runge approximation technique can not be applied anymore in Ω. We thus resort to CGO solutions to test the Alessandrini identity deriving from the assumption that $\Lambda_1 = \Lambda_2$. However, because of the additional Robin boundary condition on Σ_1, we can not directly apply the CGO solutions for the magnetic Schrödinger equation known to the literature. There has been previous work in this respect in [27, 28] for mixed boundary condition, but in our case we also have the additional challenge posed by the *unknown* potential q. In the next theorem, we construct a new family of CGO solutions suited for unknown bulk and boundary potentials:

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^{n+1}_+$, $n \geq 3$, be an open, bounded smooth domain. Assume that $\Sigma_1 = \partial\Omega \cap (\mathbb{R}^n \times \{0\})$ is a relatively open, non-empty subset of the boundary, and that $\Sigma_2 = \partial\Omega \setminus \overline{\Sigma_1}$. Let $s \in [1/2, 1)$ and let $V \in L^\infty(\Omega)$ and $q \in L^\infty(\Sigma_1)$. Then there exists a non-trivial solution $u \in H^1(\Omega, x_{n+1}^{1-2s})$ of the problem*

$$\nabla \cdot x_{n+1}^{1-2s}\nabla u + x_{n+1}^{1-2s} V u = 0 \text{ in } \Omega,$$
$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1} u + qu = 0 \text{ on } \Sigma_1,$$

of the form $u(x) = e^{\xi'\cdot x'}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r(x))$, where $k \in \mathbb{R}^{n+1}$, $\xi' \in \mathbb{C}^n$ is such that $\xi' \cdot \xi' = 0$, $k \cdot \xi' = 0$, and

- *if $s = 1/2$, then $\|r\|_{L^2(\Omega)} = O(|\xi'|^{-\frac{1}{2}})$, $\|r\|_{H^1(\Omega)} = O(|\xi'|^{\frac{1}{2}})$ and $\|r\|_{L^2(\Sigma_1)} = O(1)$;*
- *if $s > 1/2$, then $\|r\|_{L^2(\Omega, x_{n+1}^{1-2s})} = O(|\xi'|^{-s})$, $\|r\|_{H^1(\Omega, x_{n+1}^{1-2s})} = O(|\xi'|^{1-s})$ and $\|r\|_{L^2(\Sigma_1)} = O(|\xi'|^{1-2s})$.*

This is proved by duality relying on new Carleman estimates for a Caffarelli-Silvestre type extension problem, as shown in the quite technical proofs of [(D), Proposition 6.1] and [(D), Corollary 6.4]. Using the CGO solutions from theorem 4.10 we are then able to completely prove theorem 4.9.

4.5. **Uniqueness for the higher order fractional Calderón problem with local perturbations, (E).** Firstly introduced in [46] as a fractional counterpart to the classical Calderón problem ([137, 138]), the fractional Calderón problem was later studied in the cases of “rough" potentials ([120]) and first order perturbations ([24]). Our article (C) introduced and studied the higher order case $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. This framework motivates the study of higher order perturbations to the fractional Laplacian, which was proposed as an open problem in [(C), Question 2.5] and is the main focus of our article (E).

Consider a linear partial differential operator

$$P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

of order $m \in \mathbb{N}$, where the coefficients a_α are functions defined in a bounded open set $\Omega \subset \mathbb{R}^n$ and $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. The Dirichlet problem for the perturbed fractional Schrödinger equation then is

$$\begin{cases} (-\Delta)^s u + P(x, D)u = 0 \text{ in } \Omega \\ u = f \text{ in } \Omega_e \end{cases}.$$

We assume that the order m of the local perturbation $P(x, D)$ is such that the fractional, nonlocal part governs the equation, that is we let $2s > m$.

As our first step towards the formulation of the inverse problem, we need to prove the well-posedness of the direct problem. We achieve this in [(E), Lemmas 3.3, 4.3] for two different classes of coefficients a_α, namely for Fourier multipliers and bounded Sobolev spaces. We can then define the DN map $\Lambda_P \colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$ encoding our data for the inverse problem, which in turn can be formulated in the following way: does the DN map Λ_P determine uniquely the coefficients a_α in Ω? In other words, does $\Lambda_{P_1} = \Lambda_{P_2}$ imply that $a_{1,\alpha} = a_{2,\alpha}$ in Ω for all $|\alpha| \leq m$?

Our main theorems 4.11 and 4.12 prove that this is indeed the case for both Fourier multipliers and bounded coefficients. In particular, our first theorem generalizes the results obtained in [120, Theorem 1.1] for the case $m = 0, s \in (0, 1)$ and in [(C), Theorem 1.5] for the case $m = 0, s \in \mathbb{R}^+ \setminus \mathbb{Z}$.

Theorem 4.11. ((E), Theorem 1.1) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set where $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j = \sum_{|\alpha|\leq m} a_{j,\alpha} D^\alpha, \quad j = 1, 2,$$

be linear PDOs of order m with coefficients $a_{j,\alpha} \in M_0(H^{s-|\alpha|} \to H^{-s})$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations $((-\Delta)^s + P_j)u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1|_\Omega = P_2|_\Omega$.

On the other hand, our second theorem is a generalization of both [24, Theorem 1.1], [46, Theorem 1.1], which studied the cases $m \in \{0, 1\}, s \in (0, 1)$, and of [(C), Theorem 1.5], in which the case $m = 0, s \in \mathbb{R}^+ \setminus \mathbb{Z}$ was considered.

Theorem 4.12. ((E), Theorem 1.2) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain where $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j(x, D) = \sum_{|\alpha|\leq m} a_{j,\alpha}(x) D^\alpha, \quad j = 1, 2,$$

be linear PDOs of order m with coefficients $a_{j,\alpha} \in H^{r_\alpha,\infty}(\Omega)$ where

$$r_\alpha := \begin{cases} 0 & \text{if} \quad |\alpha| - s < 0, \\ |\alpha| - s + \delta & \text{if} \quad |\alpha| - s \in \{1/2, 3/2, ...\}, \\ |\alpha| - s & \text{if} \quad \text{otherwise} \end{cases}$$

for any fixed $\delta > 0$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations $((-\Delta)^s + P_j(x,D))u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1(x,D) = P_2(x,D)$.

The proofs of theorems 4.11 and 4.12 are structurally similar, but differ in many technical details. In particular, the boundedness of the bilinear forms associated to the equations and the well-posedness of the direct problem are achieved in the second case by using the assumption that $\partial\Omega$ is Lipschitz and the Kato-Ponce inequality. Both for Fourier multipliers and for bounded coefficients, such proofs involve the Riesz representation theorem and some ad hoc estimates.

Since $P(x,D)$ is by assumption a local operator, the *weak unique continuation property* is easily shown to hold true. This opens the way to the proof of the *Runge approximation property*, which is given in [(E), Lemmas 3.7, 4.6]; observe that in this case we obtain the density in $\widetilde{H}^s(\Omega)$, and not just $L^2(\Omega)$, of the set of restrictions to Ω of the solutions to our equation. We then find in [(E), Lemmas 3.6, 4.5] that an *Alessandrini identity* holds for our equation, namely

$$\langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2] \rangle = \sum_{|\alpha| \le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_1 u_2^* \rangle \ ,$$

where $u_1, u_2^* \in H^s(\mathbb{R}^n)$ respectively solve

$$(-\Delta)^s u_1 + \sum_{|\alpha|\le m} a_{1,\alpha} D^\alpha u_1 = 0 \quad \text{in } \Omega, \quad u_1 - f_1 \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_2^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha} u_2^*) = 0 \quad \text{in } \Omega, \quad u_2^* - f_2 \in \widetilde{H}^s(\Omega)$$

for some $f_1, f_2 \in H^s(\mathbb{R}^n)$. Next, we test the above integral identity with appropriate solutions to the equation, cooked up by means of the Runge approximation property. This is done using the principle of complete induction: at each step we test the identity with a different solution and deduce that $a_{1,\alpha} = a_{2,\alpha}$ for one of the multi-indices in the sum, thus making it shorter by one term. After a finite amount of steps, the proof is complete.

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Inverse problems for a fractional conductivity equation

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ABSTRACT

This paper shows global uniqueness in two inverse problems for a fractional conductivity equation: an unknown conductivity in a bounded domain is uniquely determined by measurements of solutions taken in arbitrary open, possibly disjoint subsets of the exterior. Both the cases of infinitely many measurements and a single measurement are addressed. The results are based on a reduction from the fractional conductivity equation to the fractional Schrödinger equation, and as such represent extensions of previous works. Moreover, a simple application is shown in which the fractional conductivity equation is put into relation with a long jump random walk with weights.

1. Introduction

This paper introduces and studies a fractional conductivity equation, and establishes uniqueness and reconstruction results for related inverse problems. The main point of interest is a fractional version of the standard Calderón problem [5], which requires to find the electrical conductivity of a medium from voltage and current measurements on its boundary.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a regular enough boundary (e.g., let $\partial\Omega$ be Lipschitz), representing a medium whose electrical properties must be studied. The Dirichlet problem for the conductivity equation asks to find a function u satisfying

$$\begin{cases} \nabla\cdot(\gamma\nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases},$$

where f is some prescribed boundary value and γ is the electrical conductivity of the medium. The boundary measurements are given by the Dirichlet-to-Neumann (or DN) map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, which is defined weakly using the bilinear form of the equation. The inverse problem consists in finding the function γ in Ω from the knowledge of Λ_γ.

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The classical Calderón problem we stated above has general mathematical interest, as it serves as a model case for the study of inverse problems for elliptic equations, and is of course useful in the applied fields of medical, seismic and industrial imaging. The survey [19] provides many more details on this topic. The main physical motivation, and actually Calderón's original one, comes from electrical mineral prospecting. In this application, the electrical properties of a patch of soil are measured by an array of electrodes distributed on the ground, with the goal of determining whether any economically interesting mineral source is present underneath.
On the other hand, fractional mathematical models are nowadays widely used in many fields of science. It is known for example that they arise in the study of turbulent fluids such as the atmosphere. They also appear in probability theory as generators of certain Levy processes, and because of this they are used in mathematical finance. For the many modern applications of fractional models, check e.g. [4].
It is therefore very promising to study a fractional extension of the Calderón problem, in view of its many potential applications. This is the model we introduce below.

Fix $s \in (0,1)$ and consider the new operators $(\nabla\cdot)^s$ and ∇^s, which in this paper are called *fractional divergence* and *fractional gradient*. Their rigorous definitions will be given in Section 2 following [9], but for now they can be thought of as non-local counterparts of the standard divergence and gradient. They are "nonlocal" because they do not preserve supports, in the sense that $\nabla^s u|_\Omega$ can only be computed knowing u over all of $\mathbb{R}^n$. Later on we will show that, just as in the local case, the combination of these operators gives the fractional Laplacian, that is $(-\Delta)^s u = (\nabla\cdot)^s \nabla^s u$.

Remark. It is worth noticing at this point that our choice for the names of the non-local operators, which has been guided by the similarity with the local case, is not universal. In [9], for example, our fractional gradient is called *adjoint of the fractional divergence*, while the name *fractional gradient* is assigned to a completely different operator which does not play any role in this paper.

We set up the Dirichlet problem for the fractional conductivity equation as

$$\begin{cases} (\nabla\cdot)^s(\Theta \cdot \nabla^s u) = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases},$$

where Θ is an interaction matrix depending on γ. Because of the non-local nature of the operators, the exterior value is given over all of $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$. In Section 3 it will be shown that the bilinear form associated to the conductivity equation is positive definite; this assures that 0 is not an eigenvalue of $(\nabla\cdot)^s(\gamma\nabla^s)$, and therefore the problem above is well-posed. Consequently, the DN map $\Lambda^s_\gamma : H^s(\Omega_e) \to (H^s(\Omega_e))^*$ can be defined in a weak sense starting from the bilinear form of the equation. The inverse problem asks to recover γ in Ω from Λ^s_γ.

The following theorems are the main results in this paper. The first one solves the injectivity question relative to the inverse problem for the non-local conductivity equation in any dimension $n \geq 1$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set, $s \in (0,1)$, and for $j = 1,2$ let $\gamma_j : \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\begin{cases} \text{for some } \underline{\gamma_j}, \overline{\gamma_j} \in \mathbb{R}, \;\; 0 < \underline{\gamma_j} \leq \gamma_j(x) \leq \overline{\gamma_j} < \infty, \text{ for a.e. } x \in \mathbb{R}^n \\ \gamma_j^{1/2}(x) - 1 := m_j(x) \in W_c^{2s,n/2s}(\Omega) \end{cases}.$$

Suppose $W_1, W_2 \subset \Omega_e$ are open sets, and that the DN maps for the conductivity equations in Ω relative to γ_1 and γ_2 satisfy

$$\Lambda^s_{\gamma_1}[f]|_{W_2} = \Lambda^s_{\gamma_2}[f]|_{W_2}, \qquad \forall f \in C_c^\infty(W_1)\ .$$

Then $\gamma_1 = \gamma_2$.

The second theorem gives a uniqueness result and even a reconstruction procedure for the same inverse problem with a single measurement.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded open set, $s \in (0,1)$, $\epsilon > 0$, and let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\begin{cases} \text{for some } \underline{\gamma}, \overline{\gamma} \in \mathbb{R}, \;\; 0 < \underline{\gamma} \leq \gamma(x) \leq \overline{\gamma} < \infty, \text{ for a.e. } x \in \mathbb{R}^n \\ \gamma^{1/2}(x) - 1 := m(x) \in W_c^{2s+\epsilon,p}(\Omega), \;\; \text{for } p > n/\epsilon \end{cases}.$$

Suppose $W_1, W_2 \subset \Omega_e$ are open sets, with $\overline{\Omega} \cap \overline{W_1} = \emptyset$. Given any fixed function $g \in \tilde{H}^s(W_1) \setminus \{0\}$, γ is uniquely determined and can be reconstructed from the knowledge of $\Lambda^s_\gamma[g]|_{W_2}$.

Remark. In the theorems above we make some regularity assumptions on m: namely, it is required to belong to Sobolev spaces of the form $W_c^{k,p}(\Omega)$, which are defined in Section 2. Such assumptions are needed in order to be able to apply the previous results [10,16], which are recalled in Section 3 and constitute the core of the proofs of our theorems.

A tool that is often used for treating the second order conductivity equation is Liouville's reduction, which consists in rephrasing the problem in terms of the function $w = \gamma^{1/2}u$ and the potential $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$. It is easily shown that the resulting equation is $-\Delta w + qw = 0$, i.e. Schrödinger's equation. The idea behind the proofs of Theorems 1.1 and 1.2 is to use a reduction similar to Liouville's, but suited for a non-local setting: as it will be shown in Section 3, the potential will be $q = -\frac{(-\Delta)^s m}{\gamma^{1/2}}$. The problems considered are thus transformed into special cases of inverse problems for the fractional Schrödinger equation. These are in turn well understood and dealt with thanks to the previous results [10,16]. The key points in these works are the strong uniqueness and approximation results obtained in [7]. For an overview of the fractional Calderón problem and many more references, see the survey [17].

This paper is organized as follows. Section 1 is the introduction. Section 2 is devoted to preliminaries and definitions, including Sobolev spaces and non-local operators. Section 3 first defines the conductivity equation and the DN map, then proves the main theorems. Section 4 contains an analysis of the limit case $s \to 1^-$, which is expected to give the local problem. The last part, Section 5, is devoted to a simple model for a random walk with long jumps from which the fractional conductivity equation naturally arises.

2. Preliminaries

In this section the main function spaces, operators and notations of the paper will be introduced. For the Sobolev spaces, the notation will be the usual one (check, e.g., [11]). The non-local operators are based on the theoretical framework presented in [8].

Sobolev spaces. If $k \in \mathbb{R}$, $p \in (1,\infty)$ and $n \in \mathbb{N} \setminus \{0\}$, the symbols $W^{k,p} = W^{k,p}(\mathbb{R}^n)$ indicate the usual L^p-based Sobolev space. If $\Omega \subset \mathbb{R}^n$ is an open set, the symbol $W_c^{k,p}(\Omega)$ indicates that subset of $W^{k,p}$ whose elements can be approximated in the Sobolev norm by functions belonging to $C_c^\infty(\Omega)$.
In particular, given $s \in (0,1)$ and $n \in \mathbb{N} \setminus \{0\}$, the symbols $H^s = H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ indicate the standard L^2-based Sobolev space with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u})\|_{L^2(\mathbb{R}^n)} ,$$

where $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$. The notation for the Fourier transform is $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx$. If $U, F \subset \mathbb{R}^n$ are an open and a closed set, define

$$H^s(U) = \{u|_U, u \in H^s(\mathbb{R}^n)\} ,$$

$$\tilde{H}^s(U) = \text{closure of } C_c^\infty(U) \text{ in } H^s(\mathbb{R}^n)\ , \text{ and}$$
$$H_F^s(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) : \mathrm{supp}(u) \subset F\}\ .$$

The set $H^s(U)$ is equipped with the norm $\|u\|_{H^s(U)} = \inf\{\|w\|_{H^s(\mathbb{R}^n)}; w \in H^s(\mathbb{R}^n), w|_U = u\}$. If U is a Lipschitz domain, the Sobolev spaces $\tilde{H}^s(U)$ and $H^s_{\overline{U}}(\mathbb{R}^n)$ can be naturally identified for all real s. For more details on this topic, check [11].

If $U \subset \mathbb{R}^n$ is a bounded open set and $s \in (0,1)$, let $X = H^s(\mathbb{R}^n)/\tilde{H}^s(U)$ be the *abstract trace space*. If U is a Lipschitz domain, X is the quotient $H^s(\mathbb{R}^n)/H^s_{\overline{U}}(\mathbb{R}^n)$, in which two functions $u, v \in H^s(\mathbb{R}^n)$ are equivalent if and only if $u|_{U_e} = v|_{U_e}$.

Remark. There exist several other definitions of Sobolev spaces. In fact ([6], prop. 3.4), given $s \in (0,1)$ and an open set $U \subset \mathbb{R}^n$ whose boundary is regular enough (in the sense of [6], prop. 2.2), $H^s(U)$ might just as well be defined as

$$\check{H}^s(U) = \left\{ u \in L^2(U) : \frac{|u(x) - u(y)|}{|x-y|^{n/2+s}} \in L^2(U^2) \right\}\ ,$$

with the natural norm

$$\begin{aligned} \|u\|_{\check{H}^s(U)} &= \left(\|u\|^2_{L^2(U)} + [u]^2_{\check{H}^s(U)} \right)^{1/2}\ , \\ [u]_{\check{H}^s(U)} &:= \left(\int_U \int_U \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx\, dy \right)^{1/2}\ . \end{aligned} \tag{1}$$

Non-local operators. If $u \in \mathcal{S}(\mathbb{R}^n)$, its fractional Laplacian is

$$(-\Delta)^s u(x) := \mathcal{C}_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(y) - u(x)}{|y-x|^{n+2s}} dy\ ,$$

where $\mathcal{C}_{n,s} := \frac{4^s \Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|}$ is a constant satisfying (see [6])

$$\lim_{s \to 1^-} \frac{\mathcal{C}_{n,s}}{s(1-s)} = \frac{4n}{\omega_{n-1}}\ . \tag{2}$$

This choice assures that the Fourier symbol of the fractional Laplacian is $|\xi|^{2s}$, i.e. the equality $(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\hat{u}(\xi))$ holds. If $k \in \mathbb{R}$ and $p \in (1,\infty)$, $(-\Delta)^s$ extends as a bounded map [14], Chapter 4 and [18]

$$(-\Delta)^s : W^{k,p}(\mathbb{R}^n) \to W^{k-2s,p}(\mathbb{R}^n)\ . \tag{3}$$

For the sake of completeness, it should be added that there exist many equivalent definitions for the fractional Laplacian [15]. As shown by change of variables in [6], one of them is

$$(-\Delta)^s v(x) = -\frac{\mathcal{C}_{n,s}}{2}\ PV \int_{\mathbb{R}^n} \frac{\delta v(x,y)}{|y|^{n+2s}} dy\ , \tag{4}$$

which holds if v is a Schwartz function. The symbol $\delta v(x,y)$, which is quite recurrent in this paper, is defined as follows:

$$\delta v(x,y) := v(x+y) + v(x-y) - 2v(x)\ . \tag{5}$$

This way of writing the fractional Laplacian is very useful for removing the singularity at the origin: in fact, if v is a smooth function, by means of a Taylor expansion one gets

$$\frac{v(x+y) + v(x-y) - 2v(x)}{|y|^{n+2s}} \leq \frac{\|D^2 v\|_{L^\infty}}{|y|^{n+2s-2}}\ ,$$

which is integrable near 0.

Motivated by the elementary decomposition $\Delta u = \nabla\cdot(\nabla u)$, the next step will be to define two fractional counterparts of such differential operators, following [8]. These will share the non-local nature of the fractional Laplacian.

Let $u \in C_c^\infty(\mathbb{R}^n)$, $x, y \in \mathbb{R}^n$. The *fractional gradient of u at points x and y* is the vector

$$\nabla^s u(x,y) := -\frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}} \frac{u(y)-u(x)}{|y-x|^{n/2+s+1}}(y-x)\ . \tag{6}$$

Using the result (Proposition 3.6, [6]), formula (1) and the fact that $0 \le |\xi|/\langle\xi\rangle \le 1$, it is easy to find the following inequality:

$$\begin{aligned}\|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n})} &= \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\,dy = \frac{C_{n,s}}{2}[u]^2_{\check{H}^s(\mathbb{R}^n)} \\ &= \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^s\hat{u}(\xi)\|^2_{L^2(\mathbb{R}^n)} = \left\|\frac{|\xi|^s}{\langle\xi\rangle^s}\langle\xi\rangle^s\hat{u}(\xi)\right\|^2_{L^2(\mathbb{R}^n)} \\ &\le \|\langle\xi\rangle^s\hat{u}(\xi)\|^2_{L^2(\mathbb{R}^n)} = \|u\|^2_{H^s(\mathbb{R}^n)}\ .\end{aligned} \tag{7}$$

Thus the linear operator ∇^s maps $C_c^\infty(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$. What is more, since $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ one can extend ∇^s so that it is defined in $H^s(\mathbb{R}^n)$ and formula (7) still holds.

The next operator is defined by duality. Let $u \in H^s(\mathbb{R}^n)$, $v \in L^2(\mathbb{R}^{2n})$; the *fractional divergence* is that operator $(\nabla\cdot)^s : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$ such that the following formula holds:

$$\langle(\nabla\cdot)^s v, u\rangle_{L^2(\mathbb{R}^n)} = \langle v, \nabla^s u\rangle_{L^2(\mathbb{R}^{2n})}\ . \tag{8}$$

The next simple lemma allows the composition of the fractional divergence and its adjoint into the fractional Laplacian.

Lemma 2.1. *Let $u \in H^s(\mathbb{R}^n)$. Then the equality $(\nabla\cdot)^s(\nabla^s u)(x) = (-\Delta)^s u(x)$ holds in weak sense, with $(\nabla\cdot)^s(\nabla^s u) \in H^{-s}(\mathbb{R}^n)$.*

Proof. Let $u, \phi \in H^s(\mathbb{R}^n)$, and by density for all $i \in \mathbb{N}$ let u_i, ϕ_i be smooth, compactly supported functions such that $\|u-u_i\|_{H^s(\mathbb{R}^n)} \le 1/i$ and $\|\phi-\phi_i\|_{H^s(\mathbb{R}^n)} \le 1/i$. By Cauchy–Schwarz inequality and formula (7), it is seen that

$$\langle\nabla^s u, \nabla^s\phi\rangle = \lim_{i\to\infty}\left(\langle\nabla^s(u-u_i),\nabla^s\phi\rangle + \langle\nabla^s u_i,\nabla^s\phi\rangle\right) = \lim_{i\to\infty}\langle\nabla^s u_i,\nabla^s\phi\rangle\ ,$$

and thus $\langle\nabla^s u, \nabla^s\phi\rangle = \lim_{i\to\infty}\langle\nabla^s u_i, \nabla^s\phi_i\rangle$. Now compute

$$\begin{aligned}\langle\nabla^s u_i,\nabla^s\phi_i\rangle &= \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{u_i(y)-u_i(x)}{|y-x|^{n+2s}}(\phi_i(y)-\phi_i(x))\,dydx \\ &= \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{u_i(x\pm z)-u_i(x)}{|z|^{n+2s}}(\phi_i(x\pm z)-\phi_i(x))\,dzdx \\ &= \frac{C_{n,s}}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|z|^{n+2s}}\Big\{-\phi_i(x)\delta u_i(x,z) + (u_i\phi_i)(x+z) + (u_i\phi_i)(x-z) \\ &\qquad - u_i(x)(\phi_i(x+z)+\phi_i(x-z))\Big\}\,dzdx\ .\end{aligned}$$

By adding and subtracting the term $2(u_i\phi_i)(x)$ we then get

$$\langle\nabla^s u_i,\nabla^s\phi_i\rangle = \frac{C_{n,s}}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{-\phi_i(x)\delta u_i(x,z) + \delta(u_i\phi_i)(x,z) - u_i(x)\delta\phi_i(x,z)}{|z|^{n+2s}}\,dzdx.$$

This integral can be split in three parts, which are all well defined because of the above consideration about the removal of the singularity at the origin:

$$\begin{aligned}\langle \nabla^s u_i, \nabla^s \phi_i \rangle &= \frac{1}{2}\Big(\langle \phi_i, (-\Delta)^s u_i \rangle - \langle 1, (-\Delta)^s (u_i\phi_i) \rangle + \langle u_i, (-\Delta)^s \phi_i \rangle \Big) \\ &= \langle \phi_i, (-\Delta)^s u_i \rangle \ .\end{aligned}$$

The last line follows from the fact that $u_i, \phi_i \in C_c^\infty(\mathbb{R}^n)$, which means that the first and last terms are equal. Moreover, the second term vanishes because, by Fubini's theorem,

$$\begin{aligned}\langle 1, (-\Delta)^s (u_i\phi_i) \rangle &= -\frac{\mathcal{C}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_i\phi_i)(x+y) + (u_i\phi_i)(x-y) - 2(u_i\phi_i)(x)}{|y|^{n+2s}} \, dydx \\ &= -\frac{\mathcal{C}_{n,s}}{2} \int_{\mathbb{R}^n} \frac{1}{|y|^{n+2s}} \int_{\mathbb{R}^n} ((u_i\phi_i)(x+y) + (u_i\phi_i)(x-y) - 2(u_i\phi_i)(x)) \, dxdy \ ,\end{aligned}$$

and the integral in dx is of course independent of y and equal to 0. Therefore $\langle \nabla^s u_i, \nabla^s \phi_i \rangle = \langle (-\Delta)^s u_i, \phi_i \rangle$, and eventually

$$\begin{aligned}\langle (\nabla\cdot)^s (\nabla^s) u, \phi \rangle &:= \langle \nabla^s u, \nabla^s \phi \rangle = \lim_{i\to\infty} \langle \nabla^s u_i, \nabla^s \phi_i \rangle = \lim_{i\to\infty} \langle (-\Delta)^s u_i, \phi_i \rangle \\ &= \lim_{i\to\infty} \Big(\langle (-\Delta)^s (u_i - u), \phi_i \rangle + \langle (-\Delta)^s u, \phi_i - \phi \rangle \Big) + \langle (-\Delta)^s u, \phi \rangle \\ &= \langle (-\Delta)^s u, \phi \rangle \ ,\end{aligned}$$

just as wanted. Notice that the limit vanishes because $\|(-\Delta)^s w\|_{H^{-s}} \leq \|w\|_{H^s}$. This proves the first statement; the second one now follows from the previous remark about the extensions of the fractional Laplacian. □

Remark. ∇^s and $(\nabla\cdot)^s$ can be respectively identified with the operators $\mathcal{D}^*$ and $\mathcal{D}$ from [8], where the antisymmetric vector mapping $\alpha(x,y) : \mathbb{R}^{2n} \to \mathbb{R}^n$ is chosen as

$$\alpha(x,y) = \frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}} \frac{y-x}{|y-x|^{n/2+s+1}} \ . \tag{9}$$

The choice of α comes from the fact that we want to have $(\nabla\cdot)^s(\nabla^s u) = (-\Delta)^s u$, which at least for $u \in \mathcal{S}$ means

$$(-\Delta)^s u(x) = 2 \int_{\mathbb{R}^n} (u(x) - u(y)) |\alpha(x,y)|^2 dy \ .$$

Thus the most natural choice would be to have $|\alpha(x,y)| = \frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}|y-x|^{n/2+s}}$, which motivates our choice of α. In this case we also have, for $u \in C_c^\infty(\mathbb{R}^n)$,

$$|\nabla^s u| = \frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}} \frac{|u(y) - u(x)|}{|y-x|^{n/2+s}} \ . \tag{10}$$

Anyway, different choices of α could in principle be considered.

3. Main results

Non-local conductivity equation. Let $\Omega \subset \mathbb{R}^n$ be an open set; we call $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ the *exterior domain.*

Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that there exist $\underline{\gamma}, \overline{\gamma} \in \mathbb{R}$ such that $0 < \underline{\gamma} \leq \gamma(x) \leq \overline{\gamma} < \infty$ for all $x \in \mathbb{R}^n$, and let $m(x) := \gamma^{1/2}(x) - 1$ belong to $W_c^{2s,n/2s}(\Omega)$. The assumptions for the *conductivity* γ

are similar to the ones that are typically made in the second order case. The values of $\gamma(x)$ for $x \in$ supp m represent the conductivity in the object of study. Outside of this region $\gamma(x) \equiv 1$, because the electrical properties of the surroundings are thought of as constant.

Let $\Theta : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the variable matrix $\Theta(x,y) := \gamma(x)^{1/2}\gamma(y)^{1/2}\mathrm{Id}$. The *interaction matrix* Θ represents how readily an electron will jump from x to y. We assume the material to be isotropic, meaning that the interaction does not depend on direction; therefore, $\Theta(x,y)$ is a symmetrical scalar multiple of the identity matrix.

Remark. According to formula (3), it makes sense to compute $(-\Delta)^s m$, and it belongs to $W^{0,n/2s}(\mathbb{R}^n) = L^{n/2s}(\mathbb{R}^n)$.

By using the boundedness of γ and Lemma 2.1 it is seen that if $u \in H^s(\mathbb{R}^n)$, then $\Theta \cdot \nabla^s u \in L^2(\mathbb{R}^{2n})$:

$$\|\Theta \cdot \nabla^s u\|^2_{L^2(\mathbb{R}^{2n})} = \int_{\mathbb{R}^{2n}} \gamma(x)\gamma(y)\nabla^s u \cdot \nabla^s u \, dx \, dy \le \overline{\gamma}^2 \|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n})} < \infty.$$

Let $u \in H^s(\mathbb{R}^n)$. The *non-local conductivity operator* is $\mathrm{C}^s_\gamma u := (\nabla \cdot)^s(\Theta \cdot \nabla^s u)$, while the *non-local conductivity equation* is the statement $\mathrm{C}^s_\gamma u = 0$ in Ω.

The next theorem reduces the conductivity equation to Schrödinger's.

Theorem 3.1. *Let $u \in H^s(\mathbb{R}^n)$, $g \in H^s(\Omega_e)$, $w = \gamma^{1/2}u$, $f = \gamma^{1/2}g$ and $q = -\frac{(-\Delta)^s m}{\gamma^{1/2}}$. u solves the conductivity equation with exterior value g if and only if w solves Schrödinger's equation with exterior value f, that is*

$$\begin{cases} (\nabla \cdot)^s(\Theta \cdot \nabla^s u) = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e \end{cases} \quad \Leftrightarrow \quad \begin{cases} \Big((-\Delta)^s + q\Big) w = 0 & \text{in } \Omega \\ w = f & \text{in } \Omega_e \end{cases} .$$

Moreover, the following formula holds for all $w \in H^s(\mathbb{R}^n)$:

$$\mathrm{C}^s_\gamma(\gamma^{-1/2}w) = \gamma^{1/2}((-\Delta)^s + q)w \, .$$

Proof. Start by observing that m is a Fourier multiplier on H^s, because we have the embedding $(W^{2s,n/2s} \cap L^\infty) \times H^s \hookrightarrow H^s$ (check Lemma 6, [3]). This of course means that also $\gamma^{1/2} = 1 + m$ is a Fourier multiplier on H^s, which in turn implies that $w \in H^s$ and $f \in H^s(\Omega_e)$. Moreover, the computation

$$qw = -\frac{(-\Delta)^s m}{\gamma^{1/2}} \gamma^{1/2} u = -u(-\Delta)^s m$$

and the observation that, by Theorem 6.1 in [1] and Sobolev embedding theorem,

$$L^{n/2s} \times H^s \hookrightarrow L^{2n/(n+2s)} \hookrightarrow H^{-s}$$

imply that $((-\Delta)^s + q)w \in H^{-s}$.

Our proof will be very similar to the one of the previous Lemma 2.1. Take $\phi \in H^s$, and for all $i \in \mathbb{N}$ let $\phi_i, u_i \in C_c^\infty(\mathbb{R}^n)$ be such that $\|\phi - \phi_i\|_{H^s} < 1/i$ and $\|u - u_i\|_{H^s} < 1/i$. By definition, Cauchy–Schwarz inequality and formula (7) we get

$$\begin{aligned} \langle \mathrm{C}^s_\gamma u, \phi\rangle &= \langle (\nabla \cdot)^s(\Theta \cdot \nabla^s u), \phi\rangle = \langle \Theta \cdot \nabla^s u, \nabla^s \phi\rangle \\ &= \lim_{i\to\infty} \Big(\langle \Theta \cdot \nabla^s u, \nabla^s \phi_i\rangle + \langle \Theta \cdot \nabla^s u, \nabla^s(\phi - \phi_i)\rangle \Big) \\ &= \lim_{i\to\infty} \langle \Theta \cdot \nabla^s u, \nabla^s \phi_i\rangle = \lim_{i\to\infty} \langle \Theta \cdot \nabla^s u_i, \nabla^s \phi_i\rangle \, . \end{aligned} \tag{11}$$

By change of variables,

$$\begin{aligned}\langle \Theta \cdot \nabla^s u_i, \nabla^s \phi_i \rangle &= \frac{\mathcal{C}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \gamma(x)^{1/2} \gamma(y)^{1/2} \frac{(u_i(y) - u_i(x))(\phi_i(y) - \phi_i(x))}{|y-x|^{n+2s}} dy dx \\ &= \frac{\mathcal{C}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \gamma(x)^{1/2} \gamma(x \pm z)^{1/2} \frac{(u_i(x \pm z) - u_i(x))(\phi_i(x \pm z) - \phi_i(x))}{|z|^{n+2s}} dz dx \\ &= \frac{\mathcal{C}_{n,s}}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big\{ \gamma(x)^{1/2} \gamma(x+z)^{1/2} \frac{(u_i(x+z) - u_i(x))(\phi_i(x+z) - \phi_i(x))}{|z|^{n+2s}} \\ &\qquad + \gamma(x)^{1/2} \gamma(x-z)^{1/2} \frac{(u_i(x-z) - u_i(x))(\phi_i(x-z) - \phi_i(x))}{|z|^{n+2s}} \Big\} dz dx \,.\end{aligned}$$

Now consider the integrand function. By defining $w_i := \gamma^{1/2} u_i$ it can be rewritten as

$$\begin{aligned}\frac{\gamma(x)^{1/2}}{|z|^{n+2s}} \Big\{ &- \phi_i(x) \Big(w_i(x+z) + w_i(x-z) - u_i(x)(\gamma^{1/2}(x+z) + \gamma^{1/2}(x-z)) \Big) + \\ &(w_i\phi_i)(x+z) + (w_i\phi_i)(x-z) - u_i(x) \Big((\gamma^{1/2}\phi_i)(x+z) + (\gamma^{1/2}\phi_i)(x-z) \Big) \Big\} ,\end{aligned}$$

so that, if we add and subtract the term $2w_i(x)$ from the first line and the term $2(w_i\phi_i)(x)$ from the second one, by formula (5) we get

$$\frac{\gamma(x)^{1/2}}{|z|^{n+2s}} \Big\{ \delta(w_i\phi_i)(x,z) - u_i(x)\delta(\gamma^{1/2}\phi_i)(x,z) - \phi_i(x) \Big(\delta w_i(x,z) - u_i(x)\delta(\gamma^{1/2} - 1)(x,z) \Big) \Big\} .$$

Therefore

$$\begin{aligned}\langle \Theta \cdot \nabla^s u_i, \nabla^s \phi_i \rangle = \frac{\mathcal{C}_{n,s}}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\gamma(x)^{1/2}}{|z|^{n+2s}} \Big\{ &\delta(w_i\phi_i)(x,z) - u_i(x)\delta(\gamma^{1/2}\phi_i)(x,z) \\ &- \phi_i(x) \Big(\delta w_i(x,z) - u_i(x)\delta(\gamma^{1/2} - 1)(x,z) \Big) \Big\} ,\end{aligned}$$

and the interior integral can be split in the following four parts by Lemma 2.1, since the δ's make each of them integrable at the origin:

$$\begin{aligned}\langle \Theta \cdot \nabla^s u_i, \nabla^s \phi_i \rangle &= \frac{1}{2} \int_{\mathbb{R}^n} \Big\{ -\gamma^{1/2}(-\Delta)^s(w_i\phi_i) + w_i(-\Delta)^s(\gamma^{1/2}\phi_i) \\ &\qquad + \phi_i\gamma^{1/2}(-\Delta)^s w_i - \phi_i\gamma^{1/2} u_i(-\Delta)^s(\gamma^{1/2} - 1) \Big\} \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \Big\{ (1 - \gamma^{1/2})(-\Delta)^s(w_i\phi_i) + w_i(-\Delta)^s(\gamma^{1/2}\phi_i) \\ &\qquad + \phi_i\gamma^{1/2}(-\Delta)^s w_i - \phi_i\gamma^{1/2} w_i \frac{(-\Delta)^s(\gamma^{1/2} - 1)}{\gamma^{1/2}} \Big\} .\end{aligned}$$

In the last line, we have added the term $\frac{1}{2}\int_{\mathbb{R}^n}(-\Delta)^s(w_i\phi_i)$, which equals 0. Now by the first part of the proof we can compute

$$\begin{aligned}\langle \Theta \cdot \nabla^s u_i, \nabla^s \phi_i \rangle &= \frac{\langle \gamma^{1/2}\phi_i, ((-\Delta)^s + q)w_i \rangle}{2} + \frac{\langle -(\gamma^{1/2} - 1), (-\Delta)^s(w_i\phi_i) \rangle + \langle w_i, (-\Delta)^s(\gamma^{1/2}\phi_i) \rangle}{2} \\ &= \frac{\langle \gamma^{1/2}\phi_i, ((-\Delta)^s + q)w_i \rangle}{2} + \frac{\langle -((-\Delta)^s m)u_i, \gamma^{1/2}\phi_i \rangle + \langle (-\Delta)^s w_i, \gamma^{1/2}\phi_i \rangle}{2} \\ &= \frac{\langle \gamma^{1/2}\phi_i, ((-\Delta)^s + q)w_i + qw_i + (-\Delta)^s w_i \rangle}{2} \\ &= \langle \gamma^{1/2}\phi_i, ((-\Delta)^s + q)w_i \rangle \,.\end{aligned}$$

Eventually, by using this and (11),

$$\langle \mathrm{C}^s_\gamma u, \phi\rangle = \lim_{i\to\infty} \langle \gamma^{1/2}\phi_i, ((-\Delta)^s + q)w_i\rangle = \langle \phi, \gamma^{1/2}((-\Delta)^s + q)w\rangle \ .$$

This last step holds true because

$$\lim_{i\to\infty} |\langle \gamma^{1/2}(\phi_i - \phi), ((-\Delta)^s + q)w_i\rangle| \le c \lim_{i\to\infty} \|\phi_i - \phi\|_{H^s} \|((-\Delta)^s + q)w_i\|_{H^{-s}} = 0$$

and

$$\begin{aligned}\lim_{i\to\infty} |\langle \gamma^{1/2}\phi, ((-\Delta)^s + q)(w_i - w)\rangle| &\le c\|\phi\|_{H^s} \lim_{i\to\infty} \|((-\Delta)^s + q)(w_i - w)\|_{H^{-s}} \\ &\le c\|\phi\|_{H^s}(1 + \|q\|_{L^{n/2s}}) \lim_{i\to\infty} \|w_i - w\|_{H^s} = 0 \ .\end{aligned}$$

□

Bilinear form. Let $s \in (0,1)$, $u, v \in H^s(\mathbb{R}^n)$, and define the *bilinear form* $B^s_\gamma : H^s \times H^s \to \mathbb{R}$ as follows

$$B^s_\gamma[u,v] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s v \cdot (\Theta \cdot \nabla^s u) dy\, dx \ . \tag{12}$$

B^s_γ is a useful instrument to show the well-posedness of the direct problem for the conductivity equation. In [8], Theorem 4.9, it is proved that for all $F \in (\tilde{H}^s(\Omega))^*$ there exists a unique solution $u_F \in \tilde{H}^s(\Omega)$ to $B^s_\gamma[u,v] = F(v)$, $\forall v \in \tilde{H}^s(\Omega)$. This is equivalent to saying that for all $F \in (\tilde{H}^s(\Omega))^*$ there exists one and only one $u_F \in H^s(\Omega)$ such that $\mathrm{C}^s_\gamma u = F$ in Ω, $u_F|_{\Omega_e} = 0$. To treat the case of non-zero exterior value, suppose $f \in H^s(\mathbb{R}^n)$ and let $u_f = \overline{u} + f$, where $\overline{u} \in H^s(\Omega)$ is the unique solution to the problem

$$\begin{cases} \mathrm{C}^s_\gamma u = F - B^s_\gamma[f,\cdot] & \text{in } \Omega \\ u = 0 & \text{in } \Omega_e \end{cases} . \quad \text{Then} \quad \begin{cases} \mathrm{C}^s_\gamma u = F & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases}$$

has $u_f \in H^s(\mathbb{R}^n)$ as its unique solution. Moreover, it follows from [11] that

$$\|u_f\|_{H^s(\mathbb{R}^n)} \le c(\|F\|_{(\tilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)}) \ . \tag{13}$$

The next lemma collects some properties of B^s_γ.

Lemma 3.2. *Let $v, w \in H^s(\mathbb{R}^n)$, $f, g \in H^s(\Omega_e)$ and $u_f, u_g \in H^s(\mathbb{R}^n)$ be such that $\mathrm{C}^s_\gamma u_f = \mathrm{C}^s_\gamma u_g = 0$ in Ω, $u_f|_{\Omega_e} = f$ and $u_g|_{\Omega_e} = g$. Then*

1. $B^s_\gamma[v,w] = B^s_\gamma[w,v]$ (symmetry),
2. $|B^s_\gamma[v,w]| \le \overline{\gamma}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}$,
3. $B^s_\gamma[u_f, e_g] = B^s_\gamma[u_g, e_f]$,

where $e_g, e_f \in H^s(\mathbb{R}^n)$ are extensions of g, f respectively.

Proof. Symmetry is showed by using (6) in (12),

$$B^s_\gamma[v,w] = \frac{\mathcal{C}_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \gamma(x)^{1/2}\gamma(y)^{1/2}\frac{(v(y)-v(x))\,(w(y)-w(x))}{|y-x|^{n+2s}}\, dy\, dx \ .$$

For the second point, using Hölder's inequality and the known estimate for the L^2 norm of the fractional gradient,

$$\begin{aligned}|B^s_\gamma[v,w]| &\le \|\nabla^s v\|_{L^2(\mathbb{R}^{2n})}\|\Theta \cdot \nabla^s w\|_{L^2(\mathbb{R}^{2n})} \le \overline{\gamma}\|\nabla^s v\|_{L^2(\mathbb{R}^{2n})}\|\nabla^s w\|_{L^2(\mathbb{R}^{2n})} \\ &\le \overline{\gamma}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)} \ .\end{aligned}$$

In order to prove the last point, use the definition of fractional divergence (8)

$$B^s_\gamma[u_f, u_g] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s u_g \cdot (\Theta \cdot \nabla^s u_f) dy\, dx = \int_{\mathbb{R}^n} u_g \mathrm{C}^s_\gamma u_f\, dx \ ,$$

then observe that $C^s_\gamma u_f = 0$ in Ω and $u_g = g$ in Ω_e, so that actually

$$B^s_\gamma[u_f, u_g] = \int_{\Omega_e} u_g \mathrm{C}^s_\gamma u_f \, dx = \int_{\Omega_e} g \mathrm{C}^s_\gamma u_f \, dx = \int_{\mathbb{R}^n} e_g \mathrm{C}^s_\gamma u_f \, dx = B^s_\gamma[u_f, e_g] \ .$$

This completes the proof, since by symmetry

$$B^s_\gamma[u_f, e_g] = B^s_\gamma[u_f, u_g] = B^s_\gamma[u_g, u_f] = B^s_\gamma[u_g, e_f] \ . \quad \square$$

DN map. The main use of the bilinear form in this paper is the definition of the DN map. In the case of the fractional Calderón problem for the Schrödinger equation with an unknown potential q, such map is $\Lambda_q : X \to X^*$,

$$\Lambda_q[f]([v]) = \int_{\mathbb{R}^n} v(-\Delta)^s w_f \, dx + \int_\Omega q v w_f \, dx \ ,$$

as defined in [11]. In the above formula, $f, v \in H^s(\mathbb{R}^n)$ and $w_f \in H^s(\mathbb{R}^n)$ is the unique solution to $(-\Delta)^s w + qw = 0$ in Ω with $w - f \in \tilde{H}^s(\Omega)$.

Lemma 3.3. *There exists a bounded, linear, self-adjoint map $\Lambda^s_\gamma : X \to X^*$ defined by*

$$\langle \Lambda^s_\gamma[f], [g] \rangle = B^s_\gamma[u_f, g], \qquad \forall f, g \in H^s(\mathbb{R}^n) \ ,$$

where X is the abstract quotient space $H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ and $u_f \in H^s(\mathbb{R}^n)$ solves $\mathrm{C}^s_\gamma u = 0$ in Ω with $u - f \in \tilde{H}^s(\Omega)$.

Proof. The DN map needs to be well defined, that is for all $\phi, \psi \in \tilde{H}^s(\Omega)$ and $f, g \in H^s(\mathbb{R}^n)$ the equality $B^s_\gamma[u_f, g] = B^s_\gamma[u_{f+\phi}, g + \psi]$ must hold. By Lemma 3.2,

$$\begin{aligned} B^s_\gamma[u_{f+\phi}, g+\psi] &= B^s_\gamma[u_{f+\phi}, g] + B^s_\gamma[u_{f+\phi}, \psi] = B^s_\gamma[f+\phi, u_g] + \int \psi \mathrm{C}^s_\gamma u_{f+\phi} \, dx \\ &= B^s_\gamma[f, u_g] + B^s_\gamma[\phi, u_g] = B^s_\gamma[u_f, g] + \int \phi \mathrm{C}^s_\gamma u_g \, dx = B^s_\gamma[u_f, g] \ , \end{aligned}$$

since $u_{f+\phi}, u_g$ are solutions to the conductivity equation, and ϕ, ψ are supported in Ω. The boundedness of Λ^s_γ follows from the second point of Lemma 3.2 and Eq. (13). In fact,

$$\begin{aligned} |\langle \Lambda^s_\gamma[f], [g] \rangle| &= |B^s_\gamma[u_f, g]| \leq c\|u_f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)} \\ &\leq c\|f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)} \ , \qquad \forall f \in [f], \ \forall g \in [g] \ , \end{aligned}$$

which implies

$$|\langle \Lambda^s_\gamma[f], [g] \rangle| \leq c \inf_{f \in [f]} \|f\|_{H^s(\mathbb{R}^n)} \inf_{g \in [g]} \|g\|_{H^s(\mathbb{R}^n)} = c\|[f]\|_X \|[g]\|_X \ .$$

Self-adjointness is trivial, in light of point (3) of Lemma 3.2:

$$\langle \Lambda^s_\gamma[f], [g] \rangle = B^s_\gamma[u_f, g] = B^s_\gamma[u_g, f] = \langle \Lambda^s_\gamma[g], [f] \rangle = \langle [f], \Lambda^s_\gamma[g] \rangle \ . \quad \square$$

Lemma 3.4. *Let $f, v \in H^s(\mathbb{R}^n)$ be such that supp(f), supp(v) $\subset \Omega_e$. The DN maps for the conductivity equation Λ^s_γ and for the corresponding Schrödinger equation Λ_{q_γ} satisfy*

$$\Lambda_{q_\gamma}[f]([v]) - \Lambda^s_\gamma[f]([v]) = \int_{\Omega_e} f v (-\Delta)^s m \, dx \ .$$

Proof. First of all observe that we have $\gamma^{1/2}f = f$ and $\gamma^{1/2}v = v$, because $\text{supp}(f)\cap\text{supp}(m) = \emptyset$ and $\text{supp}(v)\cap\text{supp}(m) = \emptyset$. With this in mind and making use of Theorem 3.1 it is easy to compute

$$\begin{aligned}\Lambda^s_\gamma[f]([v]) = B^s_\gamma[u_f, v] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s v \cdot (\Theta \cdot \nabla^s u_f)\, dydx \\ &= \int_{\mathbb{R}^n} v\mathrm{C}^s_\gamma u_f\, dx = \int_{\mathbb{R}^n} v\gamma^{1/2}\left((-\Delta)^s + q_\gamma\right)w_f\, dx \\ &= \int_{\mathbb{R}^n} \gamma^{1/2}v(-\Delta)^s w_f\, dx + \int_{\mathbb{R}^n} \gamma^{1/2} v q_\gamma w_f\, dx \\ &= \int_{\mathbb{R}^n} v(-\Delta)^s w_f\, dx - \int_{\Omega_e} vf(-\Delta)^s m\, dx\,.\end{aligned}$$

Moreover, recalling the assumptions about the supports,

$$\Lambda_{q_\gamma}[f]([v]) = \int_{\mathbb{R}^n} v(-\Delta)^s w_f\, dx + \int_{\Omega} q_\gamma v w_f\, dx = \int_{\mathbb{R}^n} v(-\Delta)^s w_f\, dx\,.$$

The statement of the Lemma is thus proved by taking the difference of the last two formulas. □

The definition of the DN map given above, which is abstract in nature, lets us formulate and solve the inverse problems completely. Nonetheless, in the next theorem we will give a more concrete definition of the DN map under stronger assumptions.

Theorem 3.5. *Let Ω be a bounded open set with C^∞ boundary, let $s \in (0,1)$ and let $\gamma^{1/2} = 1 + m$, with $m \in C^\infty_c(\Omega)$ and $0 < \underline{\gamma} \leq \gamma(x)$, for all $x \in \mathbb{R}^n$. For any $\beta \geq 0$ such that $\beta \in (s - 1/2, 1/2)$ the restriction of Λ^s_γ to $H^{s+\beta}(\Omega_e)$ is the map*

$$\Lambda^s_\gamma : H^{s+\beta}(\Omega_e) \to H^{-s+\beta}(\Omega_e), \quad \Lambda^s_\gamma f = \mathrm{C}^s_\gamma u_f|_{\Omega_e}\,,$$

where $u_f \in H^{s+\beta}(\mathbb{R}^n)$ solves $\mathrm{C}^s_\gamma u = 0$ in Ω with $u|_{\Omega_e} = f$, $f \in H^{s+\beta}(\Omega_e)$.

Proof. Start by observing that the embedding $H^a \times H^c \hookrightarrow H^c$ can be made to work for any $c \in \mathbb{R}$, if a is taken accordingly large enough: in the case $c < 0$, use Theorem 8.1 from [1] with $a > n/2$, while if $c \geq 0$ use Theorem 7.3 with $a > \max\{n/2, c\}$. Since now $m \in C^\infty_c(\Omega) \subset H^a(\mathbb{R}^n)$ for all $a \geq 0$, and consequently $(-\Delta)^s m \in H^{a-2s}$ for all $a \geq 0$, we have that $h \in H^c$ implies $mh, (-\Delta)^s m\, h \in H^c$. It also easily follows that $\gamma^{1/2}h, \gamma^{-1/2}h \in H^c$.

Now take $f \in H^{s+\beta}(\Omega_e)$; by the above observations, $g := \gamma^{1/2} f \in H^{s+\beta}(\Omega_e)$, and so there exists a unique $w_g \in H^{s+\beta}$ satisfying $((-\Delta)^s + q_\gamma)w = 0$ in Ω, $w|_{\Omega_e} = g$. This was proved in [11], Lemma 3.1, making use of earliest results found in [12,21] and [13]. Now let $u_f := \gamma^{-1/2} w_g$. Again by the above observations we have $u_f \in H^{s+\beta}(\Omega_e)$, and by Theorem 3.1 u_f is the unique solution of $\mathrm{C}^s_\gamma u = 0, u|_{\Omega_e} = f$. We also have

$$\begin{aligned}\|\mathrm{C}^s_\gamma u_f\|_{H^{\beta-s}} &= \|\gamma^{1/2}((-\Delta)^s + q_\gamma)w_g\|_{H^{\beta-s}} \\ &\leq \|\gamma^{1/2}(-\Delta)^s w_g\|_{H^{\beta-s}} + \|w_g(-\Delta)^s m\|_{H^{\beta-s}} < \infty\,,\end{aligned}$$

and moreover, if $e_h \in H^{s+\beta}(\mathbb{R}^n)$ is any extension of a given $h \in H^{s+\beta}(\Omega_e)$,

$$\langle \Lambda^s_\gamma f, h\rangle = B^s_\gamma[u_f, e_h] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s e_h \cdot (\Theta \cdot \nabla^s u_f)\, dydx = \langle \mathrm{C}^s_\gamma u_f, e_h\rangle\,.$$

Given an open set U and a function u, let $r_U u := u|_U$. The statement would be proved if we could decompose

$$\langle \mathrm{C}^s_\gamma u_f, e_h\rangle = \langle \mathrm{r}_\Omega C^s_\gamma u_f, r_\Omega e_h\rangle_\Omega + \langle r_{\Omega_e} \mathrm{C}^s_\gamma u_f, r_{\Omega_e} e_h\rangle_{\Omega_e}\,,$$

because then since u_f solves the fractional conductivity equation in Ω we would be able to conclude $\langle \Lambda^s_\gamma f, h\rangle = \langle r_{\Omega_e} \mathrm{C}^s_\gamma u_f, h\rangle_{\Omega_e}$. In order to use the above decomposition we need to find an $\alpha \in (-1/2, 1/2)$ such that $C^s_\gamma u_f \in H^\alpha$ and $e_h \in H^{-\alpha}$, as in the proof of Lemma 3.1 in [11]; this task is easily accomplished by taking $\alpha = \beta - s$. $\square$

Two inverse problems. The two main uniqueness results about the Calderón problem for the fractional Schrödinger equation are [16], Theorem 1.1, and [10], Theorem 1:

Injectivity (*Infinitely Many Measurements*). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded open, let $s \in (0,1)$, and let $q_1, q_2 \in L^{n/2s}(\mathbb{R}^n)$ be such that 0 is not an eigenvalue of $(-\Delta)^s + q_j$. Let also $W_1, W_2 \subset \Omega_e$ be open. If the DN maps for the equations $((-\Delta)^s + q_j)u = 0$ in Ω satisfy*

$$\Lambda_{q_1}[f]|_{W_2} = \Lambda_{q_2}[f]|_{W_2}, \quad \forall f \in C^\infty_c(W_1) ,$$

then $q_1 = q_2$ in Ω.

Uniqueness and reconstruction (*Single Measurement*). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded open, let $s \in (0,1)$, and suppose that 0 is not an eigenvalue of $(-\Delta)^s + q$. Let also $W_1, W_2 \subset \Omega_e$ be open, with $\overline{\Omega} \cap \overline{W_1} = \emptyset$. Assume that either*

- *$s \in [\frac{1}{4}, 1)$ and $q \in L^\infty(\Omega)$, or*
- *$q \in C^0(\overline{\Omega})$.*

Given any fixed function $g \in \tilde{H}^s(W_1) \setminus \{0\}$, the potential q is uniquely determined and can be reconstructed from the knowledge of $\Lambda_q[g]|_{W_2}$.

By using the results stated above, one can prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. If $W_1 \cap W_2 \neq \emptyset$, there still exist two open sets $W_1' \subset W_1$ and $W_2' \subset W_2$ such that $W_1' \cap W_2' = \emptyset$; so without loss of generality assume that W_1, W_2 and Ω are three pairwise disjoint open sets.

Let $v \in C^\infty_c(W_2)$; the hypothesis of the theorem then reads

$$\Lambda^s_{\gamma_1}[f]([v]) = \Lambda^s_{\gamma_2}[f]([v]), \qquad \text{for } f \in C^\infty_c(W_1) .$$

Since $\gamma_1 = \gamma_2 = 1$ in Ω_e, one has $\gamma_1^{-1/2} f = \gamma_2^{-1/2} f = f$ in all of $\mathbb{R}^n$. Therefore, from the previous equality and from Lemma 3.4

$$\begin{aligned}\Lambda_{q_{\gamma_1}}[f]([v]) &= \Lambda^s_{\gamma_1}[f]|([v]) + \int_{\Omega_e} f v (-\Delta)^s m_1 \, dx \\ &= \Lambda^s_{\gamma_1}[f]([v]) = \Lambda^s_{\gamma_2}[f]([v]) = \Lambda_{q_{\gamma_2}}[f]([v]) ,\end{aligned}$$

where the integral disappears because $\mathrm{supp}(f) \cap \mathrm{supp}(v) = \emptyset$. Hence

$$\Lambda_{q_{\gamma_1}}[f]|_{W_2} = \Lambda_{q_{\gamma_2}}[f]|_{W_2}, \qquad \text{for } f \in C^\infty_c(W_1) . \tag{14}$$

It is known that $(-\Delta)^s m_j \in L^{n/2s}(\mathbb{R}^n)$. Therefore,

$$\|q_{\gamma_j}\|^{n/2s}_{L^{n/2s}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left| \frac{(-\Delta)^s m_j}{\gamma_j^{1/2}} \right|^{n/2s} dx \leq \underline{\gamma}_j^{-n/4s} \|(-\Delta)^s m_j\|^{n/2s}_{L^{n/2s}(\mathbb{R}^n)} < \infty .$$

Using this and condition (14), one gets $q_{\gamma_1} = q_{\gamma_2}$ in Ω by the previously stated injectivity result with infinitely many measurements.

Now let $\bar{m} = m_2 - m_1$; of course $\mathrm{supp}(\bar{m}) \subset \Omega$, and in Ω

$$\begin{aligned} 0 &= \gamma_1^{1/2}\gamma_2^{1/2}(q_{\gamma_1} - q_{\gamma_2}) = \gamma_1^{1/2}(-\Delta)^s m_2 - \gamma_2^{1/2}(-\Delta)^s m_1 \\ &= (-\Delta)^s m_2 - (-\Delta)^s m_1 + m_1(-\Delta)^s m_2 - m_2(-\Delta)^s m_1 \\ &= (1+m_1)(-\Delta)^s \bar{m} - \bar{m}(-\Delta)^s m_1 \; . \end{aligned} \tag{15}$$

Formula (15) can be written as $(-\Delta)^s\bar{m} - \frac{(-\Delta)^s m_1}{1+m_1}\bar{m} = 0$, which shows that $\bar{m}$ solves the following Dirichlet problem for the fractional Schrödinger equation:

$$\begin{cases} (-\Delta)^s u - \frac{(-\Delta)^s m_1}{1+m_1} u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega_e \end{cases} .$$

Observe that the equation that u must satisfy in Ω is the fractional conductivity equation with conductivity γ_1, by Theorem 3.1. Thus the problem above is well-posed, and so $\bar{m} = 0$ in Ω. This in turn implies $m_1 = m_2$, which is the same as saying $\gamma_1 = \gamma_2$ in Ω. □

Proof of Theorem 1.2. By reasoning as before, W_1 and W_2 can be again supposed to be disjoint. If $v \in H^s(W_2)$, by Lemma 3.4

$$\Lambda_{q_\gamma}[f]([v]) = \int_{\Omega_e} fv(-\Delta)^s m \, dx + \Lambda^s_\gamma[f]([v]), \quad \forall f \in H^s(\mathbb{R}^n) \; ,$$

so that, by taking $f = \gamma^{1/2} g$,

$$\Lambda_{q_\gamma}[\gamma^{1/2}g]([v]) = \Lambda^s_\gamma[g]|_{W_2}([v]) \; .$$

Hence $\Lambda_{q_\gamma}[\gamma^{1/2}g]|_{W_2}$ is completely known from $\Lambda^s_\gamma[g]|_{W_2}$. Fix $\epsilon > 0$ and observe that the condition $m \in W^{2s+\epsilon,p}_c(\Omega), \forall p > n/\epsilon$ implies $m \in W^{2s,n/2s}_c(\Omega)$ and $(-\Delta)^s m \in C^0(\mathbb{R}^n)$ by Sobolev embedding theorem. Therefore $q_\gamma \in C^0(\overline{\Omega})$, and by the previously stated result concerning uniqueness and reconstruction with a single measurement, q_γ can be reconstructed uniquely. By the definition of q_γ, m solves

$$\begin{cases} (-\Delta)^s m - q_\gamma m = -q_\gamma & \text{in } \Omega \\ m = 0 & \text{in } \Omega_e \end{cases} ,$$

and thus m can be recovered by solving the above problem for Schrödinger's equation. □

4. A limit case

Now the previous considerations will be extended to the case $s \to 1^-$. Since for the fractional Laplacian one has $\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u$ [6], it is logical to expect something similar for the other non-local operators. The following holds:

Lemma 4.1. *Let $u \in H^1(\mathbb{R}^n)$. Then $\lim_{s\to 1^-} \|\nabla^s u\|_{L^2(\mathbb{R}^{2n})} = \|\nabla u\|_{L^2(\mathbb{R}^n)}$.*

Remark. This result is a special case of the one given in [2], namely when $p = 2$. However, since our proof is much easier than the one of the general case, we will still include it for completeness.

Proof. Given $i \in \mathbb{N}$, let $u_i \in C^\infty_c(\mathbb{R}^n)$ be such that $\|u - u_i\|_{H^1(\mathbb{R}^n)} \leq 1/i$. By the definition of fractional divergence and Lemma 2.1,

$$\begin{aligned} \lim_{s\to 1^-} \|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n})} &= \lim_{s\to 1^-} \int_{\mathbb{R}^n} u(-\Delta)^s u \, dx \\ &= \lim_{i\to\infty} \lim_{s\to 1^-} \Big(\int_{\mathbb{R}^n} u(-\Delta)^s(u - u_i)\, dx + \int_{\mathbb{R}^n} u(-\Delta)^s u_i \, dx \Big) \; . \end{aligned} \tag{16}$$

Since the following estimates hold [11],

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} u(-\Delta)^s(u-u_i)\,dx\Big| &= \Big|\int_{\mathbb{R}^n} (-\Delta)^{s/2}u(-\Delta)^{s/2}(u-u_i)\,dx\Big| \\ &\leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|\,|(-\Delta)^{s/2}(u-u_i)|\,dx \\ &\leq \|(-\Delta)^{s/2}u\|_{L^2}\,\|(-\Delta)^{s/2}(u-u_i)\|_{L^2} \\ &\leq \|u\|_{H^s}\,\|u-u_i\|_{H^s} \leq \|u\|_{H^1}\,\|u-u_i\|_{H^1} \leq c/i\ ,\end{aligned} \tag{17}$$

one gets that $\int_{\mathbb{R}^n} u(-\Delta)^s(u-u_i)\,dx \to 0$ upon taking the limits. Moreover $(-\Delta)^s u_i \in \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, and so the second integral in (16) is finite by Hölder. Hence

$$\begin{aligned}\lim_{i\to\infty}\lim_{s\to1^-}\int_{\mathbb{R}^n} u(-\Delta)^s u_i\,dx &= \lim_{i\to\infty}\int_{\mathbb{R}^n} u \lim_{s\to1^-}(-\Delta)^s u_i\,dx \\ &= -\lim_{i\to\infty}\int_{\mathbb{R}^n} u\Delta u_i\,dx = \lim_{i\to\infty}\int_{\mathbb{R}^n} \nabla u\nabla u_i\,dx \\ &= \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \lim_{i\to\infty}\int_{\mathbb{R}^n}\nabla u\nabla(u_i-u)\,dx = \|\nabla u\|^2_{L^2(\mathbb{R}^n)}\ ,\end{aligned} \tag{18}$$

since the last limit is easily shown to equal 0 by means of Hölder's inequality. The result is obtained by combining (16)–(18). □

Remark. It is not always true that $\nabla^s u(x,y) \to \nabla u(x)\delta(x-y)$ in distributional sense; quite counter-intuitively, $\lim_{s\to1^-}\nabla^s u = 0$ in distributional sense for all $u \in C_c^\infty(\mathbb{R}^n)$. In fact, if $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^{2n})$, then for some n-dimensional balls B_1, B_2, B_3 centered at the origin,

$$\begin{aligned}|\langle\nabla^s u,\phi\rangle| &\leq \int_{\mathbb{R}^{2n}} |\phi(x,y)|\,|\nabla^s u(x,y)|\,dxdy = \int_{\mathbb{R}^{2n}} |\phi(x,y)|\,\frac{\mathcal{C}^{1/2}_{n,s}}{\sqrt{2}}\frac{|u(y)-u(x)|}{|y-x|^{n/2+s}}\,dxdy \\ &\leq c\,\mathcal{C}^{1/2}_{n,s}\int_{B_1}\int_{B_2}\frac{|u(y)-u(x)|}{|y-x|^{n/2+s}}\,dxdy \leq c\,\mathcal{C}^{1/2}_{n,s}\int_{B_1}\int_{B_2}\frac{1}{|y-x|^{n/2+s-1}}\,dxdy \\ &\leq c\,\mathcal{C}^{1/2}_{n,s}\int_{B_1}\int_{B_3}\frac{1}{|z|^{n/2+s-1}}\,dzdy \leq c\,\mathcal{C}^{1/2}_{n,s}\ .\end{aligned}$$

Since $\mathcal{C}^{1/2}_{n,s}$ is bounded by a constant which is independent of s and also $\lim_{s\to1^-}\mathcal{C}^{1/2}_{n,s} = 0$, by dominated convergence the computation above implies

$$\langle \lim_{s\to1^-}\nabla^s u,\phi\rangle = \lim_{s\to1^-}\langle\nabla^s u,\phi\rangle = 0\ .$$

Observe that this computation is valid also for a more general definition of the fractional gradient, namely one in which α is naturally chosen in such a way that (10) still holds.

Next, some limit results for the non-local conductivity operator and its DN map. In the rest of this section, the function m will be taken from $W^{2,n/2s}_c(\Omega)$, which embeds into the usual $W^{2s,n/2s}_c(\Omega)$.

Lemma 4.2. *If $u \in H^2(\mathbb{R}^n)$, $\lim_{s\to1^-}\mathrm{C}^s_\gamma u = \nabla\cdot(\gamma\nabla u)$ in distributional sense.*

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. By reducing the conductivity operator to Schrödinger's, one is able to write

$$\begin{aligned}\lim_{s\to1^-}\int_{\mathbb{R}^n}\phi(x)(\nabla\cdot)^s(\Theta\cdot\nabla^s u)(x)\,dx &= \lim_{s\to1^-}\int_{\mathbb{R}^n}\phi\,\mathrm{C}^s_\gamma u\,dx \\ &= \lim_{s\to1^-}\int_{\mathbb{R}^n}\left(\phi\gamma^{1/2}(-\Delta)^s w - \phi\gamma^{1/2}u(-\Delta)^s m\right)\,dx\ .\end{aligned} \tag{19}$$

Observe now that since $\phi \in C_c^\infty(\mathbb{R}^n)$ and $u \in H^2(\mathbb{R}^n)$, we have $\phi u \in H^2(\mathbb{R}^n)$ as well. Moreover, since $s < 1$, we certainly have $m \in W_c^{2,n/2s}(\Omega) \cap L^\infty(\mathbb{R}^n) \subset W_c^{2,n/2}(\Omega) \cap L^\infty(\mathbb{R}^n)$; this means that $\gamma^{1/2}$ is a Fourier multiplier on $H^2(\mathbb{R}^n)$, and therefore $w, \gamma^{1/2}u\phi$ and $\gamma^{1/2}\phi$ all belong to $H^2(\mathbb{R}^n)$. We can compute

$$\|(-\Delta)^s m\|_{H^{-2}} = \left\| \mathcal{F}^{-1}\left(\frac{|\xi|^{2s}}{1+|\xi|^2}\hat{m}(\xi)\right)\right\|_{L^2} \leq c\|\mathcal{F}^{-1}\hat{m}(\xi)\|_{L^2} = c\|m\|_{L^2} \ . \tag{20}$$

In fact, it is easily seen that the function $h_s(x) := \frac{x^{2s}}{1+x^2}$ takes values in $[0,1)$ for all non-negative x and for all $s \in (0,1)$, which makes h_s a Fourier multiplier on L^2. Since m belongs to $L^\infty(\mathbb{R}^n)$ and has compact support, we see that $\|(-\Delta)^s m\|_{H^{-2}} \leq c\|m\|_{L^2} < \infty$, i.e. $(-\Delta)^s m \in H^{-2}(\mathbb{R}^n)$. By using again (20) with m replaced by w, we get $\|(-\Delta)^s w\|_{H^{-2}} \leq c\|w\|_{L^2}$; since $w \in H^2(\mathbb{R}^n)$, this leads to $(-\Delta)^s w \in H^{-2}(\mathbb{R}^n)$.
The above discussion lets us rewrite Eq. (19) in the form

$$\lim_{s\to 1^-} \langle \phi, (\nabla\cdot)^s(\Theta\cdot\nabla^s u)\rangle = \lim_{s\to 1^-}\langle \phi\gamma^{1/2}, (-\Delta)^s w\rangle - \lim_{s\to 1^-}\langle \phi\gamma^{1/2}u, (-\Delta)^s m\rangle \ . \tag{21}$$

Trivially, $|h_1(x) - h_s(x)| \leq 2$ for all non-negative x and for all $s \in (0,1)$. With this in mind we can compute

$$\begin{aligned}\|(-\Delta)m - (-\Delta)^s m\|_{H^{-2}} &= \left\| \mathcal{F}^{-1}\left(\frac{|\xi|^2 - |\xi|^{2s}}{1+|\xi|^2}\hat{m}(\xi)\right)\right\|_{L^2} \\ &\leq c\|\mathcal{F}^{-1}\hat{m}(\xi)\|_{L^2} = c\|m\|_{L^2} < \infty \ ,\end{aligned}$$

which means that

$$\begin{aligned}\lim_{s\to 1^-} \|-\Delta m - (-\Delta)^s m\|_{H^{-2}} &= \lim_{s\to 1^-} \left\|\mathcal{F}^{-1}\left((h_1(x) - h_s(x))\hat{m}(\xi)\right)\right\|_{L^2} \\ &= \left\| \lim_{s\to 1^-}(h_1(x) - h_s(x))\hat{m}(\xi)\right\|_{L^2} = 0 \ .\end{aligned}$$

Thus $(-\Delta)^s m \to -\Delta m$ in $H^{-2}(\mathbb{R}^n)$ as $s \to 1^-$, and the same proof can be used to show the analogous result for $(-\Delta)^s w$ as well. We can now deduce from Eq. (21) that

$$\lim_{s\to 1^-}\langle \phi, (\nabla\cdot)^s(\Theta\cdot\nabla^s u)\rangle = \langle \phi\gamma^{1/2}, -\Delta w\rangle - \langle \phi\gamma^{1/2}u, -\Delta m\rangle \ .$$

Performing some elementary vector calculus computation on this last formula the desired result is immediately obtained:

$$\lim_{s\to 1^-}\int_{\mathbb{R}^n} \phi\, \mathrm{C}_\gamma^s u\, dx = \int_{\mathbb{R}^n} \phi\nabla\cdot(\gamma\nabla u)\, dx \ . \quad \square$$

Lemma 4.3. *Let $u, v \in H^1(\mathbb{R}^n)$. Then $\lim_{s\to 1} B_\gamma^s[u,v] = \int_{\mathbb{R}^n}\gamma\nabla u\cdot\nabla v\, dx$.*

Proof. For all $i \in \mathbb{N}$, let $u_i, v_i \in C_c^\infty(\mathbb{R}^n)$ be such that $\|u - u_i\|_{H^1(\mathbb{R}^n)} \leq 1/i$ and $\|v - v_i\|_{H^1(\mathbb{R}^n)} \leq 1/i$. Then we can compute

$$\lim_{s\to 1^-} B_\gamma^s[u,v] = \lim_{i\to\infty}\lim_{s\to 1^-}\left(B_\gamma^s[u-u_i, v] + B_\gamma^s[u_i, v - v_i] + B_\gamma^s[u_i, v_i]\right) \ . \tag{22}$$

By Hölder's inequality we see that

$$\begin{aligned}|B_\gamma^s[u-u_i, v]| &= |\langle \nabla^s(u-u_i), \Theta\cdot\nabla^s v\rangle| \leq \|\nabla^s(u-u_i)\|_{L^2}\|\Theta\cdot\nabla^s v\|_{L^2} \\ &\leq \overline{\gamma}\|u-u_i\|_{H^s}\|v\|_{H^s} \leq \overline{\gamma}\|u-u_i\|_{H^1}\|v\|_{H^1} \ ,\end{aligned}$$

so that the first term on the right hand side of (22) vanishes upon taking the limits. The second term behaves similarly, and so we are left with $\lim_{s\to 1^-} B^s_\gamma[u,v] = \lim_{i\to\infty}\lim_{s\to 1^-} B^s_\gamma[u_i,v_i]$. Now apply Lemma 4.2 to deduce that

$$\begin{aligned}\lim_{s\to 1^-} B^s_\gamma[u,v] &= \lim_{i\to\infty}\lim_{s\to 1^-} B^s_\gamma[u_i,v_i] = \lim_{i\to\infty}\lim_{s\to 1^-} \langle \nabla^s u_i, \Theta\cdot\nabla^s v_i\rangle \\ &= \lim_{i\to\infty}\lim_{s\to 1^-} \langle u_i, \mathrm{C}^s_\gamma v_i\rangle = \lim_{i\to\infty} \langle u_i, \nabla\cdot(\gamma\nabla v_i)\rangle \\ &= \lim_{i\to\infty} \langle \nabla u_i, \gamma\nabla v_i\rangle\,.\end{aligned}$$

The result is now recovered by decomposing this term as in (22) and then applying again Hölder's inequality. □

Corollary 4.4. *Let $f,g\in H^1(\mathbb{R}^n)$. Then $\lim_{s\to 1^-}\langle \Lambda^s_\gamma[f],[g]\rangle = \int_{\mathbb{R}^n} \gamma\nabla u_f\cdot\nabla g\,dx$.*

Proof. The result immediately follows from the previous Lemma and from the definition $\langle \Lambda^s_\gamma[f],[g]\rangle = B^s_\gamma[u_f,g]$. □

5. A simple model: the random walk

This section shows how the non-local conductivity equation naturally arises from weighted long jump random walks. This is an extension of [20], where the fractional Laplacian is related to unweighted long jump random walks.

Let $h>0$, $\tau=h^{2s}$, $k\in\mathbb{Z}^n$, $x\in h\mathbb{Z}^n$ and $t\in\tau\mathbb{Z}$. Consider a random walk on the lattice $h\mathbb{Z}^n$, subject to discrete time steps belonging to $\tau\mathbb{Z}$. Define

$$f(x,k) := \begin{cases}\gamma^{1/2}(x+hk)|k|^{-n-2s} & \text{if}\quad k\neq 0\\ 0 & \text{if}\quad k=0\end{cases}.$$

Observe that, $\forall x\in h\mathbb{Z}^n$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} f(x,k) &= \sum_{k\in\mathbb{Z}^n\setminus\{0\}} f(x,k) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \gamma^{1/2}(x+hk)|k|^{-n-2s}\\ &\le \overline{\gamma}^{1/2}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{-n-2s} < \infty,\end{aligned}\tag{23}$$

and therefore it makes sense to define a normalized version of $f(x,k)$, namely

$$P(x,k) := \begin{cases}\left(\sum_{j\in\mathbb{Z}^n} f(x,j)\right)^{-1}\gamma^{1/2}(x+hk)|k|^{-n-2s} & \text{if}\quad k\neq 0\\ 0 & \text{if}\quad k=0\end{cases}.$$

Of course one has $0\le P(x,k)\le 1$, and from the definition it follows that

$$\sum_{k\in\mathbb{Z}^n} P(x,k) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} P(x,k) = \frac{\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\gamma^{1/2}(x+hk)|k|^{-n-2s}}{\sum_{j\in\mathbb{Z}^n\setminus\{0\}}\gamma^{1/2}(x+hj)|j|^{-n-2s}} = 1\,.\tag{24}$$

$P(x,k)$ is the probability that a particle found at point $x+hk$ will jump to x in the next discrete step. With $\gamma\equiv 1$ one recovers the case [20], where the probability only depends on the distance between the two points. A non constant function γ can instead account for spatially changing properties of the medium, so that the jumping probability is higher from a point whose conductivity is large, while still decreasing with distance.

Let $u(x,t)$ be the probability that at some instant t the particle is found at point x. It is clearly related to the previous state of the particle by the equation

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} P(x,k)u(x+hk,t) \ .$$

Now compute the time derivative of $u(x,t)$:

$$\begin{aligned}\partial_t u(x,t) &= \lim_{\tau\to 0} \frac{u(x,t+\tau)-u(x,t)}{\tau} \\ &= \lim_{h\to 0} \frac{1}{h^{2s}} \left(\sum_{k\in\mathbb{Z}^n\setminus\{0\}} P(x,k)u(x+hk,t) - u(x,t) \right) \\ &= \lim_{h\to 0} \frac{1}{h^{2s}} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} P(x,k)\left(u(x+hk,t)-u(x,t)\right) \ ,\end{aligned}$$

where the last line is due to the normalization property (24) of $P(x,k)$. So,

$$\partial_t u(x,t) = \lim_{h\to 0} \frac{\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \left[\gamma^{1/2}(x+hk)|k|^{-n-2s}\left(u(x+hk,t)-u(x,t)\right)\right]}{h^{2s}\ \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \gamma^{1/2}(x+hj)|j|^{-n-2s}} \ . \tag{25}$$

The denominator is finite, as observed in (23), and also bounded away from 0:

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \gamma^{1/2}(x+hk)|k|^{-n-2s} \geq \underline{\gamma}^{1/2} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{-n-2s} > 0 \ . \tag{26}$$

By using (26) in Eq. (25), one can compute

$$\begin{aligned}\partial_t u(x,t) &= \lim_{h\to 0} \frac{\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \left[h^n\gamma^{1/2}(x+hk)|hk|^{-n-2s}\left(u(x+hk,t)-u(x,t)\right)\right]}{\sum_{j\in\mathbb{Z}^n\setminus\{0\}} \gamma^{1/2}(x+hj)|j|^{-n-2s}} \\ &= C\int_{\mathbb{R}^n} \frac{\gamma^{1/2}(x+z)}{|z|^{n+2s}}\left(u(x+z,t)-u(x,t)\right)dz \\ &= \frac{C}{\gamma(x)^{1/2}} \int_{\mathbb{R}^n} \frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|y-x|^{n+2s}}\left(u(y,t)-u(x,t)\right)dy \ ,\end{aligned}$$

because the sum approximates the Riemannian integral. Eventually, $\partial_t u(x,t) = \frac{C}{\gamma(x)^{1/2}} \mathrm{C}^s_\gamma u$. If $u(x,t)$ is independent of t, the fractional conductivity equation $\mathrm{C}^s_\gamma u = 0$ is recovered.

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(B)

An inverse problem for the fractional Schrödinger equation in a magnetic field

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An inverse problem for the fractional Schrödinger equation in a magnetic field

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Abstract

This paper shows global uniqueness in an inverse problem for a fractional magnetic Schrödinger equation (FMSE): an unknown electromagnetic field in a bounded domain is uniquely determined up to a natural gauge by infinitely many measurements of solutions taken in arbitrary open subsets of the exterior. The proof is based on Alessandrini's identity and the Runge approximation property, thus generalizing some previous works on the fractional Laplacian. Moreover, we show with a simple model that the FMSE relates to a long jump random walk with weights.

Keywords: Fractional magnetic Schrödinger equation, Non-local operators, Inverse problems, Calderón problem

2010 MSC: 35R11, 35R30

1. Introduction

This paper studies a fractional version of the Schrödinger equation in a magnetic field, or a fractional magnetic Schrödinger equation (FMSE), establishing a uniqueness result for a related inverse problem. We thus deal with a non-local counterpart of the classical magnetic Schrödinger equation (MSE) (see [33]), which requires to find up to gauge the scalar and vector potentials existing in a medium from voltage and current measurements on its boundary.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, representing a medium containing an unknown electromagnetic field. The solution of the Dirichlet problem for the MSE is a function u satisfying

$$\begin{cases} (-\Delta)_A u + qu := -\Delta u - i\nabla \cdot (Au) - iA \cdot \nabla u + (|A|^2 + q)u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases},$$

where f is the prescribed boundary value and A, q are the vector and scalar potentials in the medium. The boundary measurements are encoded in $\Lambda_{A,q}$: $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, the Dirichlet-to-Neumann (or DN) map. The inverse problem consists in finding A, q in Ω up to gauge by knowing $\Lambda_{A,q}$.

The study of the local MSE has both mathematical and practical interest, since it constitutes a substantial generalization of the Calderón problem (see [6]). This problem first arose for the prospection of the ground in search of valuable minerals. In the method known as Electrical Impedance Tomography (EIT), electrodes are placed on the ground in order to deliver voltage and measure current flow; the resulting data carries information about the conductivity of the materials underground, allowing deductions about their composition ([42]). A similar method is also used in medical imaging. Since the tissues of a body have different electrical conductivities ([26]), using the same setup harmless currents can be allowed to flow in the body of a patient, thus collecting information about its internal structure. This technique can be applied to cancer detection ([20]), monitoring of vital functions ([8]) and more (see e.g. [23]). Various engineering applications have also been proposed. A recent one (see [21]) describes a sensing skin consisting of a thin layer of conductive copper paint applied on concrete. In case of cracking of the block, the rupture of the surface would result in a local decrease in conductivity, which would in turn be detected by EIT, allowing the timely substitution of the failing block. The version of the problem with non-vanishing magnetic field is interesting on its own, as it is related to the inverse scattering problem with a fixed energy (see [33]). First order terms also arise by reduction in the study of numerous other inverse problems, among which isotropic elasticity ([35]), special cases of Maxwell and Schrödinger equations ([31], [16]), Dirac equations ([34]) and the Stokes system ([22]). The survey [39] contains more references on inverse boundary value problems for the MSE.

Below we introduce a fractional extension of the local problem. This is motivated by the connection between anomalous diffusion and random walks, as explained in the end of the Introduction and in Section 5. Fix $s \in (0, 1)$, and consider the fractional divergence and gradient operators $(\nabla\cdot)^s$ and ∇^s. These are based on the theoretical framework laid down in [13], [14], and were introduced in [10] as non-local counterparts of the classical divergence and gradient. They are defined to be the adjoint of each other, and also they have the expected property that $(\nabla\cdot)^s\nabla^s = (-\Delta)^s$, the fractional Laplacian. Fix now a vector potential A, and consider the magnetic versions $(\nabla\cdot)^s_A$ and ∇^s_A of the above operators. These correspond to $(-i\nabla + A)\cdot$ and $(-i\nabla + A)$, whose combination results in the local magnetic Laplacian $(-\Delta)_A$. Analogously, we will show how $(\nabla\cdot)^s_A$ and ∇^s_A can be combined in a fractional magnetic Laplacian $(-\Delta)^s_A$. As expected, this operator will reduce to the known $(-\Delta)^s$ if $A = 0$. The next step will be setting up the Dirichlet problem for FMSE as

$$\begin{cases} (-\Delta)^s_A u + qu = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases} .$$

Since our operators are non-local, the exterior values are taken over $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$. The well-posedness of the direct problem is granted by the assumption that 0 is not an eigenvalue for the left hand side of FMSE (see e.g. [38]). We can therefore define the DN map $\Lambda^s_{A,q} : H^s(\Omega_e) \to (H^s(\Omega_e))^*$ from the bilinear form associated to the equation. The inverse problem is to recover A and q in Ω from $\Lambda^s_{A,q}$. Because of a natural gauge $\sim$ enjoyed by FMSE, solving the inverse problem completely is impossible; however, the gauge class of the solving potentials can be fully recovered:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded open set, $s \in (0,1)$, and let $(A_i, q_i) \in \mathcal{P}$ for $i = 1, 2$. Suppose $W_1, W_2 \subset \Omega_e$ are non empty open sets, and that the DN maps for the FMSEs in Ω relative to (A_1, q_1) and (A_2, q_2) satisfy*

$$\Lambda^s_{A_1,q_1}[f]|_{W_2} = \Lambda^s_{A_2,q_2}[f]|_{W_2}, \quad \forall f \in C^\infty_c(W_1) .$$

Then $(A_1, q_1) \sim (A_2, q_2)$, that is, the potentials coincide up to the gauge $\sim$.

The set $\mathcal{P}$ of potentials and the gauge $\sim$ are defined in Section 3. $\mathcal{P}$ contains all potentials (A, q) satisfying certain properties, among which *(p5)*: $\text{supp}(A) \subseteq \Omega^2$. We suspect this assumption to be unnecessary, but we nonetheless prove our Theorem in this easier case, and highlight the occasions when *(p5)* is used.

The proof is based on three preliminary results: the integral identity for the DN map, the weak unique continuation property (WUCP) and the Runge approximation property (RAP). The WUCP is easily proved by reducing our case to that of the fractional Laplacian $(-\Delta)^s$, for which the result is already known (see e.g. [37], [17]). For this we use *(p5)*. The proof of the RAP then comes from the WUCP and the Hahn-Banach theorem. Eventually, we use this result, the integral identity and *(p5)* to complete the proof by means of Alessandrini's identity. This technique generalizes the one studied in [17].

We consider Theorem 1.1 to be very satisfactory, as gauges show up in the local case $s = 1$ as well (again, see [33]). For comparison see [7], where it is shown that no gauge exists for a fractional Schrödinger equation with a local first-order perturbation of the form $b(x) \cdot \nabla u(x)$. As observed in Section 3, our operator can be regarded as a fractional Schrödinger equation with a non-local perturbation of the kind

$$\int_{\mathbb{R}^n} b(x,y) \cdot \nabla^s u(x,y) dy \ ,$$

and thus our results extend the investigation in [7] in a rather natural way. One may also compare our operator with the one studied in [3]. In such work the authors consider non-local lower-order perturbations of the fractional Schrödinger equation of the form $(-\Delta)^{t/2}_\Omega b(x)(-\Delta)^{t/2}_\Omega u(x)$, where the symbol $(-\Delta)^{t/2}_\Omega$ denotes the regional fractional Laplacian. In the case of complete data, [3] shows that the perturbation b and the potential q can be completely recovered; however, in the case of a single measurement, the authors interestingly find that there exist natural obstacles to the full recovery of both b and q.

Besides the purely mathematical appeal, we believe that the problem we are considering may also be interesting from a practical point of view. As a matter of fact, fractional mathematical models are nowadays quite common in many different fields of science, including image processing ([19]), physics ([13], [15], [18], [28], [32], [44]), ecology ([25], [30], [36]), turbulent fluid dynamics ([9], [11]) and mathematical finance ([1], [29], [40]). For more references, see [5]. All these applications involve anomalous diffusion, i.e. a diffusion process in which events that are quite far from the mean are still allowed to happen with a relatively high probability. As a consequence, one can model such phenomena with anomalous diffusion random walks. These are "anomalous" in the sense that the variance of the length of the jumps is not finite as in the classical diffusion case. The authors of [43] have proved how the fractional Laplacian corresponds to a long jump random walk of this kind. In Section 5 we extend their line of reasoning to our magnetic fractional operator, showing that its leading term corresponds to a long jump random walk with weights. We also prove that this is an anomalous diffusion random walk.

2. Preliminaries

Operators on bivariate vector functions.

Definition 2.1. *Let $A \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^n)$. The* symmetric, antisymmetric, parallel and perpendicular parts of A *at points x, y are*

$$A_s(x,y) := \frac{A(x,y)+A(y,x)}{2}, \qquad A_a(x,y) := A(x,y) - A_s(x,y),$$

$$A_\parallel(x,y) := \begin{cases} \frac{A(x,y)\cdot(x-y)}{|x-y|^2}(x-y) & \text{if } x \neq y \\ A(x,y) & \text{if } x = y \end{cases}, \quad A_\perp(x,y) := A(x,y) - A_\parallel(x,y).$$

The L^2 norms of A with respect to the first *and* second variable *at point x are*

$$\mathcal{J}_1 A(x) := \left(\int_{\mathbb{R}^n} |A(y,x)|^2\, dy\right)^{1/2}, \quad \mathcal{J}_2 A(x) := \left(\int_{\mathbb{R}^n} |A(x,y)|^2\, dy\right)^{1/2}.$$

Remark 2.2. *Being $A \in C_c^\infty$, these two integrals are finite and the definitions make sense. Moreover, since $A_a \cdot A_s$ is an antisymmetric scalar function and $A_\parallel \cdot A_\perp = 0$, by the following computations*

$$\begin{aligned}\|A\|_{L^2}^2 &= \|A_a + A_s\|_{L^2}^2 = \|A_a\|_{L^2}^2 + \|A_s\|_{L^2}^2 + 2\langle A_a, A_s\rangle \\ &= \|A_a\|_{L^2}^2 + \|A_s\|_{L^2}^2 + 2\int_{\mathbb{R}^{2n}} A_a \cdot A_s\, dx\, dy = \|A_a\|_{L^2}^2 + \|A_s\|_{L^2}^2,\end{aligned} \tag{1}$$

$$\begin{aligned}\|A\|_{L^2}^2 &= \|A_\parallel + A_\perp\|_{L^2}^2 = \|A_\parallel\|_{L^2}^2 + \|A_\perp\|_{L^2}^2 + 2\langle A_\parallel, A_\perp\rangle \\ &= \|A_\parallel\|_{L^2}^2 + \|A_\perp\|_{L^2}^2 + 2\int_{\mathbb{R}^{2n}} A_\parallel \cdot A_\perp\, dx\, dy = \|A_\parallel\|_{L^2}^2 + \|A_\perp\|_{L^2}^2\end{aligned} \tag{2}$$

the four operators $(\cdot)_s, (\cdot)_a, (\cdot)_\parallel, (\cdot)_\perp$ can be extended to act from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^{2n})$. This is true of $\mathcal{J}_1 A$ and $\mathcal{J}_2 A$ as well:

$$\|\mathcal{J}_1 A\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |(\mathcal{J}_1 A)(x)|^2\, dx = \int_{\mathbb{R}^{2n}} |A(y,x)|^2\, dy\, dx = \|A\|_{L^2(\mathbb{R}^{2n})}^2. \tag{3}$$

Lemma 2.3. *The equalities defining $(\cdot)_s, (\cdot)_a, (\cdot)_\parallel, (\cdot)_\perp$ in Definition 2.1 for $A \in C_c^\infty$ still hold a.e. for $A \in L^2(\mathbb{R}^{2n})$.*

Proof. We prove the Lemma only for $(\cdot)_s$, as the other cases are similar. For all $i \in \mathbb{N}$, let $A^i \in C_c^\infty(\mathbb{R}^{2n}, \mathbb{C}^n)$ such that $\|A - A^i\|_{L^2} \leq 1/i$. By (1),

$$\left\| A_s - \frac{A(x,y)+A(y,x)}{2} \right\|_{L^2} \leq$$

$$
\begin{aligned}
&\leq \|(A-A^i)_s\|_{L^2} + \left\| A^i_s - \frac{A^i(x,y)+A^i(y,x)}{2} \right\|_{L^2} \\
&\quad + \left\| \frac{(A(x,y)-A^i(x,y)) + (A(y,x)-A^i(y,x))}{2} \right\|_{L^2} \\
&= \|(A-A^i)_s\|_{L^2} + \left\| \frac{(A(x,y)-A^i(x,y)) + (A(y,x)-A^i(y,x))}{2} \right\|_{L^2} \\
&\leq 2\|A-A^i\|_{L^2} \leq 2/i \ . \qquad \square
\end{aligned}
$$

Remark 2.4. *If $A \in C_c^\infty$, the operators $(\cdot)_s, (\cdot)_a, (\cdot)_\|, (\cdot)_\perp$ commute with each other; because of Lemma 2.3, this still holds a.e. for $A \in L^2(\mathbb{R}^{2n})$. Thus in the following we use e.g. the symbol $A_{s\|}$ for both $(A_s)_\|$ and $(A_\|)_s$.*

Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be open and $r \in \mathbb{R}$, $p \in (1,\infty)$, $n \in \mathbb{N} \setminus \{0\}$. By the symbols $W^{r,p} = W^{r,p}(\mathbb{R}^n)$ and $W^{r,p}_c(\Omega)$ we denote the usual L^p-based Sobolev spaces. We also let $H^s = H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ be the standard L^2-based Sobolev space with norm $\|u\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(\langle\xi\rangle^s\hat{u})\|_{L^2(\mathbb{R}^n)}$, where $s \in \mathbb{R}$, $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and the Fourier transform is

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx \ .$$

One should note that there exist many equivalent definitions of fractional Sobolev spaces (see e.g. [12]). Using the Sobolev embedding and multiplication theorems (see e.g. [4], [2]), these spaces can often be embedded into each other:

Lemma 2.5. *Let $s \in (0,1)$, $p :=\max\{2, n/2s\}$ and $h \geq 0$. Then the embeddings*

(e1). $H^s \times H^s \hookrightarrow L^{n/(n/2+sp-2s)}$,

(e2). $H^s \times L^p \hookrightarrow L^{2n/(n+2s)}$,

(e3). $L^{2p} \times L^2 \hookrightarrow L^{2n/(n+2s)}$,

(e4). $L^{2p} \times H^s \hookrightarrow L^2$,

(e5). $L^{2p} \times L^{2p} \hookrightarrow L^p$,

(e6). $H^{sp-2s} \hookrightarrow L^p$,

(e7). $L^{2n/(n+2h)} \hookrightarrow H^{-h}$

hold, where $\times$ indicates the pointwise product. $\square$

Let $U, F \subset \mathbb{R}^n$ be an open and a closed set. We define the spaces

$$H^s(U) = \{u|_U, u \in H^s(\mathbb{R}^n)\} \ ,$$

$$\tilde{H}^s(U) = \text{closure of } C_c^\infty(U) \text{ in } H^s(\mathbb{R}^n) \ , \text{ and}$$

$$H^s_F(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) : \text{supp}(u) \subset F\} \,,$$

where $\|u\|_{H^s(U)} = \inf\{\|w\|_{H^s(\mathbb{R}^n)}; w \in H^s(\mathbb{R}^n), w|_U = u\}$. For $s \in (0,1)$ and a bounded open set $U \subset \mathbb{R}^n$, let $X := H^s(\mathbb{R}^n)/\tilde{H}^s(U)$. If U is a Lipschitz domain, then then $\tilde{H}^s(U)$ and $H^s_{\bar{U}}(\mathbb{R}^n)$ can be identified for all $s \in \mathbb{R}$ (see [17]); therefore, $X = H^s(\mathbb{R}^n)/H^s_{\bar{U}}(\mathbb{R}^n)$, and its elements are equivalence classes of functions from $H^s(\mathbb{R}^n)$ coinciding on U_e. X is called *abstract trace space*.

Non-local operators. If $u \in \mathcal{S}(\mathbb{R}^n)$, its fractional Laplacian is (see [27], [12])

$$(-\Delta)^s u(x) := \mathcal{C}_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|y-x|^{n+2s}} dy \,,$$

for a constant $\mathcal{C}_{n,s}$. Its Fourier symbol is $|\xi|^{2s}$, i.e. $(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\hat{u}(\xi))$. By [24], Ch. 4 and [41], $(-\Delta)^s$ extends as a bounded map $(-\Delta)^s : W^{r,p}(\mathbb{R}^n) \to W^{r-2s,p}(\mathbb{R}^n)$ for $r \in \mathbb{R}$ and $p \in (1,\infty)$. Let $\alpha(x,y) : \mathbb{R}^{2n} \to \mathbb{R}^n$ be the map

$$\alpha(x,y) = \frac{\mathcal{C}^{1/2}_{n,s}}{\sqrt{2}} \frac{y-x}{|y-x|^{n/2+s+1}} \,.$$

If $u \in C^\infty_c(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$, the *fractional gradient of* u *at points* x *and* y is

$$\nabla^s u(x,y) := (u(x) - u(y))\alpha(x,y) \,, \tag{4}$$

and is thus a symmetric and parallel vector function of x and y. Since it was proved in [10] that $\|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n})} \leq \|u\|^2_{H^s(\mathbb{R}^n)}$, and thus that the linear operator ∇^s maps $C^\infty_c(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$, we see that ∇^s can be extended to $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$. Using a proof by density similar to the one for Lemma 2.3, one sees that (4) still holds a.e. for $u \in H^s(\mathbb{R}^n)$.
If $u \in H^s(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^{2n})$, the *fractional divergence* is defined as that operator $(\nabla\cdot)^s : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$ satisfying

$$\langle (\nabla\cdot)^s v, u \rangle_{L^2(\mathbb{R}^n)} = \langle v, \nabla^s u \rangle_{L^2(\mathbb{R}^{2n})} \,, \tag{5}$$

i.e. it is by definition the adjoint of the fractional gradient. As observed in [10], Lemma 2.1, if $u \in H^s(\mathbb{R}^n)$ the equality $(\nabla\cdot)^s(\nabla^s u)(x) = (-\Delta)^s u(x)$ holds in weak sense, and $(\nabla\cdot)^s(\nabla^s u) \in H^{-s}(\mathbb{R}^n)$.

Lemma 2.6. *Let* $u \in C^\infty_c(\mathbb{R}^n)$*. There exists a constant* $k_{n,s}$ *such that*

$$\mathcal{F}(\nabla^s u)(\xi,\eta) = k_{n,s} \left(\frac{\xi}{|\xi|^{n/2+1-s}} + \frac{\eta}{|\eta|^{n/2+1-s}} \right) \mathcal{F}u(\xi+\eta) \,.$$

Proof. As $u \in C_c^\infty(\mathbb{R}^n)$, we know that $\nabla^s u \in L^2(\mathbb{R}^{2n})$, and we can compute its Fourier transform in the variables ξ, η. By a change of variables,

$$\begin{aligned}
\mathcal{F}(\nabla^s u)(\xi,\eta) &= \frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-iy\cdot\eta} \frac{u(x)-u(y)}{|y-x|^{n/2+s+1}}(y-x)\,dx\,dy \\
&= k'_{n,s} \int_{\mathbb{R}^n} \frac{e^{-iz\cdot\eta}}{|z|^{n/2+s+1}} z \int_{\mathbb{R}^n} e^{-ix\cdot(\xi+\eta)}(u(x)-u(x+z))\,dx\,dz \\
&= k'_{n,s} \int_{\mathbb{R}^n} \frac{z}{|z|^{n/2+s+1}} e^{-iz\cdot\eta}\,\mathcal{F}u(\xi+\eta)(1-e^{iz\cdot(\xi+\eta)})\,dz \\
&= k''_{n,s}\,\mathcal{F}u(\xi+\eta)\int_{\mathbb{R}^n}(e^{-iz\cdot\eta}-e^{iz\cdot\xi})\nabla_z(|z|^{1-n/2-s})\,dz \\
&= k''_{n,s}\,\mathcal{F}u(\xi+\eta)\left(\eta\mathcal{F}(|z|^{1-n/2-s})(\eta)+\xi\mathcal{F}(|z|^{1-n/2-s})(-\xi)\right) \\
&= k_{n,s}\left(\frac{\xi}{|\xi|^{n/2+1-s}}+\frac{\eta}{|\eta|^{n/2+1-s}}\right)\mathcal{F}u(\xi+\eta)\,.
\end{aligned}$$

□

Lemma 2.7. *The fractional gradient extends as a bounded map*

$$\nabla^s : H^r(\mathbb{R}^n) \to \langle D_x + D_y\rangle^{r-s} L^2(\mathbb{R}^{2n})\,,$$

and if $r \le s$ *then also* $\nabla^s : H^r(\mathbb{R}^n) \to H^{r-s}(\mathbb{R}^{2n})$.

Proof. Start with $u \in C_c^\infty(\mathbb{R}^n)$, and let $r \in \mathbb{R}$. Then

$$\begin{aligned}
\|\nabla^s u\|^2_{\langle D_x+D_y\rangle^{r-s}L^2} &= (\langle D_x+D_y\rangle^{r-s}\nabla^s u, \langle D_x+D_y\rangle^{r-s}\nabla^s u)_{L^2} \\
&= (\langle D_x+D_y\rangle^{2(r-s)}\nabla^s u, \nabla^s u)_{L^2} \\
&= (\mathcal{F}(\langle D_x+D_y\rangle^{2(r-s)}\nabla^s u), \mathcal{F}(\nabla^s u))_{L^2}\,.
\end{aligned} \tag{6}$$

From the previous Lemma we can deduce that

$$\begin{aligned}
\mathcal{F}(\langle D_x+D_y\rangle^{2(r-s)}\nabla^s u) &= (1+|\xi+\eta|^2)^{r-s}\mathcal{F}(\nabla^s u) \\
&= (1+|\xi+\eta|^2)^{r-s} k_{n,s}\left(\frac{\xi}{|\xi|^{n/2+1-s}}+\frac{\eta}{|\eta|^{n/2+1-s}}\right)\mathcal{F}u(\xi+\eta) \\
&= k_{n,s}\left(\frac{\xi}{|\xi|^{n/2+1-s}}+\frac{\eta}{|\eta|^{n/2+1-s}}\right)\mathcal{F}(\langle D_x\rangle^{2(r-s)}u)(\xi+\eta) \\
&= \mathcal{F}(\nabla^s(\langle D_x\rangle^{2(r-s)}u))\,.
\end{aligned}$$

Using the properties of the fractional gradient and (6),

$$\begin{aligned}
\|\nabla^s u\|^2_{\langle D_x+D_y\rangle^{r-s}L^2} &= (\mathcal{F}(\nabla^s(\langle D_x\rangle^{2(r-s)}u)), \mathcal{F}(\nabla^s u))_{L^2} \\
&= (\nabla^s(\langle D_x\rangle^{2(r-s)}u), \nabla^s u)_{L^2} = (\langle D_x\rangle^{2(r-s)}u, (-\Delta)^s u)_{L^2} \\
&= (\langle D_x\rangle^{r-s}(-\Delta)^{s/2}u, \langle D_x\rangle^{r-s}(-\Delta)^{s/2}u)_{L^2} \\
&= \|(-\Delta)^{s/2}u\|^2_{H^{r-s}} \le c\|u\|^2_{H^r}\,.
\end{aligned}$$

An argument by density completes the proof of the first part of the statement. For the second one, observe that $r \leq s$ implies

$$\begin{aligned}\|v\|^2_{H^{r-s}} &= (\langle D_{x,y}\rangle^{r-s}v, \langle D_{x,y}\rangle^{r-s}v)_{L^2} = (\langle D_{x,y}\rangle^{2(r-s)}v, v)_{L^2} \\ &= ((1+|\xi|^2+|\eta|^2)^{r-s}\hat{v}, \hat{v})_{L^2} \leq c((1+|\xi+\eta|^2)^{r-s}\hat{v}, \hat{v})_{L^2} \\ &= c(\langle D_x + D_y\rangle^{2(r-s)}v, v)_{L^2} = c\|v\|^2_{\langle D_x+D_y\rangle^{r-s}L^2} ,\end{aligned}$$

and so $\langle D_x + D_y\rangle^{r-s}L^2(\mathbb{R}^{2n}) \subseteq H^{r-s}(\mathbb{R}^{2n})$. □

As a consequence of the above Lemma, the fractional divergence can be similarly extended as $(\nabla\cdot)^s : H^t(\mathbb{R}^{2n}) \to H^{t-s}(\mathbb{R}^n)$ for all $t \geq s$.

3. Definition and properties of FMSE

Fractional magnetic Schrödinger equation. Let $\Omega \subset \mathbb{R}^n$ be open, $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ be the *exterior domain*, and also recall that $p :=\max\{2, n/2s\}$. The *vector potential* and *scalar potential* are two functions $A : \mathbb{R}^{2n} \mapsto \mathbb{C}^n$ and $q : \mathbb{R}^n \mapsto \mathbb{R}$. The following properties are of interest:

(p1). $\mathcal{J}_1 A,\ \mathcal{J}_2 A \in L^{2p}(\mathbb{R}^n)$,

(p2). $A_{s\|} \in H^{sp-s}(\mathbb{R}^{2n}, \mathbb{C}^n)$,

(p3). $A_{a\|}(x,y)\cdot(y-x) \geq 0$, for all $x, y \in \mathbb{R}^n$,

(p4). $q \in L^p(\Omega)$,

(p5). $A \in L^2(\mathbb{R}^{2n}),\quad \mathrm{supp}(A) \subseteq \Omega^2$.

With respect to the above properties, we define four sets of potentials:

$$\begin{aligned}\mathcal{A}_0 &:= \{\text{vector potentials } A \text{ verifying } (p1)-(p3)\}, \\ \mathcal{A} &:= \{\text{vector potentials } A \text{ verifying } (p1)-(p3) \text{ and } (p5)\}, \\ \mathcal{P}_0 &:= \{\text{pairs of potentials } (A,q) \text{ verifying } (p1)-(p4)\}, \\ \mathcal{P} &:= \{\text{pairs of potentials } (A,q) \text{ verifying } (p1)-(p5)\}.\end{aligned}$$

Remark 3.1. *The peculiar definitions for the spaces in* (p1), (p2) *and* (p4) *are due to computational necessities: they make the following quantities*

$$\|qu\|_{H^{-s}}, \quad \|(\nabla\cdot)^s A_{s\|}\|_{L^p}, \quad \|(\mathcal{J}_2 A)^2\|_{L^p}, \quad \|u\mathcal{J}_2 A\|_{L^2}$$

finite for $u \in H^s$, as needed in Remark 3.6, Lemma 3.3 and (7). This is easily proved by using Lemma 2.5. However, if $n \geq 4$, then $p = n/2s$, and so in this case $L^{2p} = L^{n/s}$ and $H^{sp-s} = H^{n/2-s}$; this simplifies the assumptions for n large enough.

Let $A \in \mathcal{A}_0$ and $u \in H^s(\mathbb{R}^n)$. By *(p1)* and *(e4)*,

$$\begin{aligned} \|A(x,y)u(x)\|_{L^2(\mathbb{R}^{2n})} &= \left(\int_{\mathbb{R}^n} u(x)^2 \int_{\mathbb{R}^n} |A(x,y)|^2 dy\, dx\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^n} u(x)^2\, \mathcal{J}_2 A(x)^2\, dx\right)^{1/2} = \|u\, \mathcal{J}_2 A\|_{L^2(\mathbb{R}^n)} \\ &\leq k\|u\|_{H^s}\|\mathcal{J}_2 A\|_{L^{2p}} < \infty\,, \end{aligned} \tag{7}$$

and thus the *magnetic fractional gradient* of u can be defined as the function $\nabla^s_A u : \mathbb{R}^{2n} \to \mathbb{C}^n$ such that

$$\langle \nabla^s_A u, v\rangle := \langle \nabla^s u + A(x,y)u(x), v\rangle\,, \quad \text{for all } v \in L^2(\mathbb{R}^{2n})\,. \tag{8}$$

By the same computation, ∇^s_A acts as an operator $\nabla^s_A : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$. Let $A \in \mathcal{A}_0$, $u \in H^s(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^{2n})$. The *magnetic fractional divergence* is defined by duality as that operator $(\nabla\cdot)^s_A : L^2(\mathbb{R}^{2n}) \to H^{-s}(\mathbb{R}^n)$ such that

$$\langle (\nabla\cdot)^s_A v, u\rangle := \langle v, \nabla^s_A u\rangle\,.$$

By construction, the magnetic fractional divergence and gradient can be combined; we call *magnetic fractional Laplacian* $(-\Delta)^s_A := (\nabla\cdot)^s_A(\nabla^s_A)$ that operator from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$ such that, for all $u, v \in H^s(\mathbb{R}^n)$,

$$\langle (-\Delta)^s_A u, v\rangle = \langle \nabla^s_A u, \nabla^s_A v\rangle\,. \tag{9}$$

Remark 3.2. *If $A \equiv 0$, the magnetic fractional Laplacian $(-\Delta)^s_A$ is reduced to its non-magnetic counterpart $(-\Delta)^s$, as expected. Since the fractional Laplacian is well understood (see e.g. [17]), from now on we assume $A \not\equiv 0$.*

Lemma 3.3. *Let $A \in L^2(\mathbb{R}^{2n}) \cap \mathcal{A}_0$ and $u \in H^s(\mathbb{R}^n)$. The equation*

$$(-\Delta)^s_A u = (-\Delta)^s u + 2\int_{\mathbb{R}^n} \left(A_{a\|} \cdot \nabla^s u\right) dy + \left((\nabla\cdot)^s A_{s\|} + \int_{\mathbb{R}^n} |A|^2\, dy\right) u \tag{10}$$

holds in weak sense.

Proof. By (9), $(-\Delta)^s_A u \in H^{-s}(\mathbb{R}^n)$, and in order to prove (10) in weak sense one needs to compute $\langle(-\Delta)^s_A u, v\rangle$ for $v \in H^s(\mathbb{R}^n)$. By (9) and (8),

$$\begin{aligned}\langle(-\Delta)^s_A u, v\rangle &= \langle\nabla^s u + A(x,y)u(x), \nabla^s v + A(x,y)v(x)\rangle \\ &= \langle\nabla^s u, \nabla^s v\rangle + \langle Au, Av\rangle + \langle\nabla^s u, Av\rangle + \langle\nabla^s v, Au\rangle \ ,\end{aligned}$$

where all the above terms make sense, since by formula (7) $\nabla^s u, \nabla^s v, Au$ and Av all belong to $L^2(\mathbb{R}^{2n})$. The new term $\langle\nabla^s u, A(y,x)v(x)\rangle$ is also finite, so

$$\begin{aligned}\langle(-\Delta)^s_A u, v\rangle = &\langle\nabla^s u, \nabla^s v\rangle + \langle Au, Av\rangle + \\ &+ \langle\nabla^s u, A(x,y)v(x)\rangle - \langle\nabla^s u, A(y,x)v(x)\rangle + \\ &+ \langle\nabla^s u, A(y,x)v(x)\rangle + \langle\nabla^s v, A(x,y)u\rangle \ .\end{aligned} \tag{11}$$

For the first term on the right hand side of (11), by definition,

$$\langle\nabla^s u, \nabla^s v\rangle = \langle(\nabla\cdot)^s\nabla^s u, v\rangle = \langle(-\Delta)^s u, v\rangle \ . \tag{12}$$

For the second one, by the embeddings *(e5)*, *(e2)* and *(e7)*,

$$\langle Au, Av\rangle = \left\langle u(x)\int_{\mathbb{R}^n}|A(x,y)|^2 dy, v\right\rangle = \langle u(\mathcal{J}_2 A)^2, v\rangle \ . \tag{13}$$

Since $u \in H^s(\mathbb{R}^n)$, by (3) we deduce $\mathcal{J}_2(\nabla^s u) \in L^2(\mathbb{R}^n)$. Now *(e3)* implies that $\mathcal{J}_2(\nabla^s u)\mathcal{J}_2 A \in L^{\frac{2n}{n+2s}}$. On the other hand, by Cauchy-Schwarz

$$\begin{aligned}\left\|\int_{\mathbb{R}^n}\nabla^s u\cdot A dy\right\|^{\frac{2n}{n+2s}}_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\nabla^s u\cdot A dy\right|^{\frac{2n}{n+2s}} dx \\ &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\nabla^s u|\,|A| dy\right)^{\frac{2n}{n+2s}} dx \le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\nabla^s u|^2 dy\int_{\mathbb{R}^n}|A|^2 dy\right)^{\frac{n}{n+2s}} dx \\ &= \int_{\mathbb{R}^n}|\mathcal{J}_2(\nabla^s u)\ \mathcal{J}_2 A|^{\frac{2n}{n+2s}} dx = \|\mathcal{J}_2(\nabla^s u)\ \mathcal{J}_2 A\|^{\frac{2n}{n+2s}}_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} \ ,\end{aligned}$$

and so $\int_{\mathbb{R}^n}\nabla^s u\cdot A dy \in L^{\frac{2n}{n+2s}}$. Now $\langle\int_{\mathbb{R}^n}\nabla^s u\cdot A dy, v\rangle$ is finite by *(e7)*, and

$$\begin{aligned}\langle\nabla^s u, A(x,y)\ v(x)\rangle - \langle\nabla^s u, A(y,x)v(x)\rangle &= \\ &= \left\langle\int_{\mathbb{R}^n}\nabla^s u\cdot A(x,y)dy, v\right\rangle - \left\langle\int_{\mathbb{R}^n}\nabla^s u\cdot A(y,x)dy, v\right\rangle \\ &= \left\langle\int_{\mathbb{R}^n}\nabla^s u\cdot(A(x,y)-A(y,x))dy, v\right\rangle \\ &= \left\langle 2\int_{\mathbb{R}^n}\nabla^s u\cdot A_a\, dy, v\right\rangle = \left\langle 2\int_{\mathbb{R}^n}\nabla^s u\cdot A_{a\|}\, dy, v\right\rangle \ .\end{aligned} \tag{14}$$

The last steps use Lemma 2.3 to write A_a for $A \in L^2$ and to see that $\nabla^s u$ is a.e. a parallel vector for $u \in H^s(\mathbb{R}^n)$, which implies $\nabla^s u \cdot A_{a\perp} = 0$ *a.e.*. This computes the third and fourth terms on the right hand side of (11). For the last two terms observe that, since $A(y,x)v(x) - A(x,y)v(y)$ is antisymmetric, by Lemma 2.3 we have $\langle \nabla^s u, A(y,x)v(x) - A(x,y)v(y)\rangle = 0$, and so

$$\begin{aligned}
\langle \nabla^s u, A(y,x)\ v(x)\rangle + \langle \nabla^s v, Au\rangle \\
&= \int_{\mathbb{R}^{2n}} A(x,y)\cdot (v(y)\nabla^s u + u(x)\nabla^s v)\, dx\, dy \\
&= \int_{\mathbb{R}^{2n}} A\cdot \alpha\Big(v(y)(u(x)-u(y)) + u(x)(v(x)-v(y))\Big)\, dx\, dy \quad (15)\\
&= \int_{\mathbb{R}^{2n}} A_{s\|}\cdot \alpha\Big(u(x)v(x) - u(y)v(y)\Big)\, dx\, dy \\
&= \langle A_{s\|}, \nabla^s(uv)\rangle = \langle u(\nabla\cdot)^s A_{s\|}, v\rangle\ .
\end{aligned}$$

On the third line of (15) the integrand is the product of a symmetric, parallel vector and A; this reduces A to $A_{s\|}$. From *(e1)*, *(e7)* and Lemma 2.7 one sees that $\nabla^s(uv) \in H^{s-sp}$, and now $\langle A_{s\|}, \nabla^s(uv)\rangle$ makes sense by *(p2)*. Eventually, (5), *(e6)*, *(e2)* and *(e7)* explain the last step. Equation (10) follows from (11), (12), (13), (14) and (15). □

Lemma 3.4. *Let $A \in L^2(\mathbb{R}^{2n}) \cap \mathcal{A}_0$. There exists a symmetric distribution $\sigma \in \mathcal{D}'(\mathbb{R}^{2n})$ such that $\sigma \geq 1$ and $A_{a\|} = \alpha(\sigma - 1)$ a.e..*

Proof. Because of Lemma 2.3, $A_{a\|}$ is a parallel vector almost everywhere, and thus $\|A_{a\|} - (A_{a\|})_\|\|_{L^2} = 0$. Again by Lemma 2.3,

$$\begin{aligned}
0 = \|A_{a\|} - (A_{a\|})_\|\|_{L^2} &= \left\| A_{a\|} - \frac{A_{a\|}\cdot(x-y)}{|x-y|^2}(x-y)\right\|_{L^2} \\
&= \left\| A_{a\|} - \left(-\frac{\sqrt{2}}{\mathcal{C}_{n,s}^{1/2}}\frac{A_{a\|}\cdot(x-y)}{|x-y|^{1-n/2-s}}\right)\frac{\mathcal{C}_{n,s}^{1/2}}{\sqrt{2}}\frac{y-x}{|y-x|^{n/2+s+1}}\right\|_{L^2} \\
&= \left\| A_{a\|} - \left(\left(1 + \frac{\sqrt{2}}{\mathcal{C}_{n,s}^{1/2}}\frac{A_{a\|}\cdot(y-x)}{|x-y|^{1-n/2-s}}\right) - 1\right)\alpha\right\|_{L^2}\ .
\end{aligned}$$

Moreover, if $\phi \in C_c^\infty(\mathbb{R}^{2n})$ and B_{r_1}, B_{r_2} are balls in $\mathbb{R}^n$ centered at the origin such that $\text{supp}(\phi) \subset B_{r_1} \times B_{r_2}$, then by (1), (2) and Cauchy-Schwarz inequality

$$\left|\left\langle 1 + \frac{\sqrt{2}}{C_{n,s}^{1/2}} \; |y-x|^{n/2+s}\left(A_{a\|}\cdot\frac{y-x}{|y-x|}\right), \phi\right\rangle\right| =$$

$$= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1 + \frac{\sqrt{2}}{C_{n,s}^{1/2}}|y-x|^{n/2+s}\left(A_{a\|}\cdot\frac{y-x}{|y-x|}\right)\right)\phi\, dy\, dx\right|$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|1 + \frac{\sqrt{2}}{C_{n,s}^{1/2}}|y-x|^{n/2+s}\left(A_{a\|}\cdot\frac{y-x}{|y-x|}\right)\right||\phi|\, dy\, dx$$

$$\leq \|\phi\|_{L^\infty}\int_{B_{r_1}}\int_{B_{r_2}}\left|1 + \frac{\sqrt{2}}{C_{n,s}^{1/2}}|y-x|^{n/2+s}\left(A_{a\|}\cdot\frac{y-x}{|y-x|}\right)\right|\, dy\, dx$$

$$\leq k\|\phi\|_{L^\infty}\left(1 + \int_{B_{r_1}}\int_{B_{r_2}}|y-x|^{n/2+s}\left|A_{a\|}\cdot\frac{y-x}{|y-x|}\right|\, dy\, dx\right)$$

$$\leq k\|\phi\|_{L^\infty}\left(1 + \int_{B_{r_1}}\int_{B_{r_2}}(|x|+|y|)^{n/2+s}|A_{a\|}|\, dy\, dx\right)$$

$$\leq k'\|\phi\|_{L^\infty}\left(1 + \int_{B_{r_1}}\int_{B_{r_2}}|A_{a\|}|\, dy\, dx\right)$$

$$\leq k'\|\phi\|_{L^\infty}\left(1 + \|A_{a\|}\|^2_{L^2(\mathbb{R}^{2n})}\right) \leq k'\|\phi\|_{L^\infty}\left(1 + \|A\|^2_{L^2(\mathbb{R}^{2n})}\right) < \infty\,.$$

Thus it makes sense to define a distribution $\sigma \in \mathcal{D}'(\mathbb{R}^{2n})$ such that

$$\langle \sigma, \phi\rangle = \left\langle 1 + \frac{\sqrt{2}}{C_{n,s}^{1/2}}|y-x|^{n/2+s}\left(A_{a\|}\cdot\frac{y-x}{|y-x|}\right), \phi\right\rangle$$

holds for all $\phi \in C_c^\infty(\mathbb{R}^{2n})$. Given that $A_{a\|}$ is antisymmetric, it is clear that σ is symmetric; moreover, property *(p3)* assures that $\sigma \geq 1$. □

Remark 3.5. *If $u \in \mathcal{S}(\mathbb{R}^n)$, by the previous Lemma we can rewrite the leading term of $(-\Delta)^s_A$ as*

$$\mathcal{C}_{n,s}\; PV\int_{\mathbb{R}^n}\sigma(x,y)\frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy\,.$$

This shows the connection between the magnetic and classical fractional Laplacians: if $\sigma(x,y) \equiv 1$, i.e. if $A_{a\|} \equiv 0$, the formula above defines $(-\Delta)^s u$. Moreover, if $\sigma(x,y)$ is separable *(i.e. there are functions $\sigma_1, \sigma_2 : \mathbb{R}^n \to \mathbb{R}$ such that $\sigma(x,y) = \sigma_1(x)\sigma_2(y)$) we get the fractional conductivity operator (see [10]).*

Consider $(A, q) \in \mathcal{P}_0$ and $f \in H^s(\Omega_e)$. We say that $u \in H^s(\mathbb{R}^n)$ solves

FMSE with exterior value f if and only if

$$\begin{cases} (-\Delta)^s_A u + qu = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases}$$

holds in weak sense, that is if and only if $u - f \in \tilde{H}^s(\Omega)$ and, for all $v \in H^s(\mathbb{R}^n)$,

$$\langle (-\Delta)^s_A u, v\rangle + \langle qu, v\rangle = 0 \,. \tag{16}$$

Remark 3.6. *By* (p1), (p2) *and* (p4), *formula* (16) *makes sense. This was already partially shown in the above discussion about the magnetic fractional Laplacian. For the last term, just use* (p4), (e2) *and* (e7).

Old gauges, new gauges. Let $(G, \cdot)$ be the abelian group of all strictly positive functions $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi|_{\Omega_e} = 1$. For $(A, q), (A', q') \in \mathcal{P}_0$, define

$$(A, q) \sim (A', q') \quad \Leftrightarrow \quad (-\Delta)^s_A u + qu = (-\Delta)^s_{A'} u + q'u \,, \tag{17}$$

$$(A, q) \approx (A', q') \quad \Leftrightarrow \quad \exists \phi \in G : (-\Delta)^s_A (u\phi) + qu\phi = \phi((-\Delta)^s_{A'} u + q'u) \tag{18}$$

for all $u \in H^s(\mathbb{R}^n)$. Both $\sim$ and $\approx$ are equivalence relations on $\mathcal{P}_0$, and thus we can consider the quotient spaces $\mathcal{P}_0/\sim$ and $\mathcal{P}_0/\approx$. Moreover, since $\phi \equiv 1 \in G$, we have $(A, q) \sim (A', q') \Rightarrow (A, q) \approx (A', q')$.

We say that FMSE *has the gauge* $\sim$ if for each $(A, q) \in \mathcal{P}_0$ there exists $(A', q') \in \mathcal{P}_0$ such that $(A', q') \neq (A, q)$ and $(A, q) \sim (A', q')$. Similarly, we say that FMSE *has the gauge* $\approx$ if for each $(A, q) \in \mathcal{P}_0$ there exist $\phi \in G$, $(A', q') \in \mathcal{P}_0$ such that $\phi \not\equiv 1$, $(A', q') \neq (A, q)$ and $(A, q) \approx (A', q')$.

Remark 3.7. *The definitions* (17) *and* (18), *which have been given for FMSE, can be extended to the local case in the natural way.*

If $s = 1$, it is known that $(-\Delta)_A(u\phi) + qu\phi = \phi\left((-\Delta)_{A+\frac{\nabla\phi}{\phi}} u + qu\right)$ for all $\phi \in G$ and $u \in H^1(\mathbb{R}^n)$. If we choose $\phi \not\equiv 1$, we have $\left(A + \frac{\nabla\phi}{\phi}, q\right) \neq (A, q)$ and $(A, q) \approx \left(A + \frac{\nabla\phi}{\phi}, q\right)$, which shows that MSE has the gauge $\approx$. On the other hand, if $(A, q) \sim (A', q')$ then necessarily $A = A'$ and $q = q'$: thus, MSE does not enjoy the gauge $\sim$. We now treat the case $s \in (0, 1)$.

Lemma 3.8. *Let* $(A, q), (A', q') \in \mathcal{P}_0$. *Then* $(A, q) \sim (A', q')$ *if and only if* $A_{a\|} = A'_{a\|}$ *and* $Q = Q'$, *where*

$$Q := q + \int_{\mathbb{R}^n} |A|^2 \, dy + (\nabla\cdot)^s A_{s\|} \,, \qquad Q' := q' + \int_{\mathbb{R}^n} |A'|^2 \, dy + (\nabla\cdot)^s A'_{s\|} \,.$$

Proof. One direction of the implication is trivial: by (10) and the definition, it is clear that if $A_{a\|} = A'_{a\|}$ and $Q = Q'$ then $(-\Delta)^s_A u + qu = (-\Delta)^s_{A'} u + q'u$. For the other one, use Lemmas 3.3 and 3.4 to write $(-\Delta)^s_A u+qu = (-\Delta)^s_{A'} u+q'u$ as

$$\begin{aligned} 0 &= 2\int_{\mathbb{R}^n} |\alpha|^2(\sigma'-\sigma)(u(y)-u(x))\,dy + u(x)(Q-Q') \\ &= \mathcal{C}_{n,s}\int_{\mathbb{R}^n}(\sigma'-\sigma)\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy + u(x)(Q-Q')\,. \end{aligned} \tag{19}$$

Fix $\psi \in C^\infty_c(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $u(y) := \psi(y)e^{-1/|x-y|}|x-y|^{n+2s}$; one sees that $u \in \mathcal{S}$, since it is compactly supported and all the derivatives of the smooth function $e^{-1/|x-y|}$ vanish at x. Thus $u \in H^s$, and we can substitute it in (19):

$$0 = \int_{\mathbb{R}^n}(\sigma(x,y)-\sigma'(x,y))e^{-1/|x-y|}\psi(y)\,dy = \langle(\sigma(x,\cdot)-\sigma'(x,\cdot))e^{-1/|x-y|},\psi\rangle\,.$$

Being ψ arbitrary and $e^{-1/|x-y|}$ non-negative, we deduce that $y \mapsto \sigma(x,y) - \sigma'(x,y)$ is zero for any fixed x, that is, $\sigma = \sigma'$. Then $A_{a\|} = A'_{a\|}$ by Lemma 3.4, and also $Q = Q'$ by (19). □

Lemma 3.9. *Let $A \not\equiv 0$. Then FMSE has the gauge $\sim$.*

Proof. If $(A, q) \in \mathcal{P}_0$ and $A' \in \mathcal{A}_0$ is such that $A_{a\|} = A'_{a\|}$, then by the previous Lemma $(A, q) \sim (A', q')$ if and only if $Q = Q'$, that is

$$q' = q + \int_{\mathbb{R}^n}|A|^2\,dy + (\nabla\cdot)^s A_{s\|} - \int_{\mathbb{R}^n}|A'|^2\,dy - (\nabla\cdot)^s A'_{s\|}\,.$$

Since $A, A' \in \mathcal{A}_0$, we have $A_{s\|}, A'_{s\|} \in H^{sp-s}$ and $\mathcal{J}_2A, \mathcal{J}_2A' \in L^{2p}$. By the former fact, $(\nabla\cdot)^s A_{s\|}, (\nabla\cdot)^s A'_{s\|}$ belong to H^{sp-2s} and eventually to L^p because of *(e6)*. By the latter fact and *(e5)*, $\int_{\mathbb{R}^n}|A|^2\,dy, \int_{\mathbb{R}^n}|A'|^2\,dy \in L^p$. Also, $q \in L^p$ because $(A, q) \in \mathcal{P}_0$. This implies that *(p4)* holds for the q' computed above. Hence, if we find $A' \in \mathcal{A}_0$ such that $A_{a\|} = A'_{a\|}$, and then take q' as above, we get a $(A', q') \in \mathcal{P}_0$ in gauge $\sim$ with a given $(A, q) \in \mathcal{P}_0$. We now show how to do this with $A \neq A'$, which implies that FMSE enjoys $\sim$.

Fix $(A, q) \in \mathcal{P}_0$, and for the case $A_\perp \not\equiv 0$ let $A' := A_\| - A_\perp$. Then $A \neq A'$, because $A_\perp \neq A'_\perp$; moreover, from $A_\| = A'_\|$ we get $A_{a\|} = A'_{a\|}$ and $A'_{s\|} = A_{s\|} \in H^{sp-s}$. Eventually, $|A'|^2 = |A'_\||^2 + |A'_\perp|^2 = |A_\||^2 + |-A_\perp|^2 = |A_\||^2 + |A_\perp|^2 = |A|^2$ implies $\mathcal{J}_2A' = \mathcal{J}_2A$, and A' verifies *(p1)*. If instead we have $A_\perp \equiv 0$, let $A' = A_\| + RA_\|$, where R is any $\pi/2$ rotation. Then as before $A_{a\|} = A'_{a\|}$

and $A'_{s\|} = A_{s\|} \in H^{sp-s}$, because $A_\| = A'_\|$. We also have $A \neq A'$, because $A_\perp = 0 \neq RA_\| = A'_\perp$. Finally, since $\mathcal{J}_2 A \in L^p$, A' verifies *(p1)*:

$$\mathcal{J}_2 A' = \left(\int_{\mathbb{R}^n} |A'|^2 dy\right)^{1/2} = \left(\int_{\mathbb{R}^n} |A'_\||^2 + |A'_\perp|^2 dy\right)^{1/2}$$
$$= \left(\int_{\mathbb{R}^n} |A_\||^2 + |RA_\||^2 dy\right)^{1/2} = \left(\int_{\mathbb{R}^n} 2|A_\||^2 dy\right)^{1/2} = \sqrt{2}\,\mathcal{J}_2 A\,. \quad \square$$

Lemma 3.10. *FMSE does not have the gauge $\approx$.*

Proof. Let $(A,q),(A',q') \in \mathcal{P}_0$ such that $(A,q) \approx (A',q')$. Then there exists $\phi \in G$ such that $(-\Delta)^s_A(u\phi) + qu\phi = \phi((-\Delta)^s_{A'}u + q'u)$ for all $u \in H^s$. Fix $\psi \in C^\infty_c(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $u(y) := \psi(y)e^{-1/|x-y|}|x-y|^{n+2s}$ as in Lemma 3.8. Then $u \in \mathcal{S}$, and by Lemma 3.3 and Remark 3.5,

$$0 = \mathcal{C}_{n,s}\ PV \int_{\mathbb{R}^n} \left(\sigma(x,y)\frac{u(x)\phi(x) - u(y)\phi(y)}{|x-y|^{n+2s}} - \sigma'(x,y)\frac{u(x)\phi(x) - u(y)\phi(x)}{|x-y|^{n+2s}}\right) dy$$
$$+ u(x)\phi(x)(Q - Q')$$
$$= \mathcal{C}_{n,s}\ PV \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+2s}} \left(\sigma'(x,y)\phi(x) - \sigma(x,y)\phi(y)\right) dy$$
$$= \mathcal{C}_{n,s} \int_{\mathbb{R}^n} \psi(y)e^{-1/|x-y|} \left(\sigma'(x,y)\phi(x) - \sigma(x,y)\phi(y)\right) dy\,.$$

Here the principal value disappears because the integral is not singular. Given the arbitrarity of ψ and the non negativity of the exponential, we deduce $\sigma(x,y)\phi(y) = \sigma'(x,y)\phi(x)$ for all $y \neq x$. On the other hand, since σ, σ' are symmetric and $\phi > 0$, by taking the symmetric part of each side

$$\sigma(x,y)\frac{\phi(x)+\phi(y)}{2} = (\sigma(x,y)\phi(y))_s = (\sigma'(x,y)\phi(x))_s = \sigma'(x,y)\frac{\phi(x)+\phi(y)}{2}\,.$$

This implies $\sigma = \sigma'$, and the equation can be rewritten as $\sigma(x,y)(\phi(y)-\phi(x)) = 0$. Being $\sigma > 0$, it is clear that ϕ must be constant, and therefore equal to 1. This means that whenever $(A,q),(A',q') \in \mathcal{P}_0$ are such that $(A,q) \approx (A',q')$ with some $\phi \in G$, then $\phi \equiv 1$, i.e. FMSE does not have the gauge $\approx$. $\square$

By the last two Lemmas, FMSE enjoys $\sim$, but not $\approx$. Observe that the reverse is true for the classical magnetic Schrödinger equation. This surprising difference is due to the non-local nature of the operators involved: FMSE has $\sim$ because the coefficient of its gradient term is not the whole vector potential A, as in the classical case, but just a part of it. On the other hand, the restriction imposed by the antisymmetry of such part motivates the absence of $\approx$.

Bilinear form. Let $s \in (0,1)$, $u, v \in H^s(\mathbb{R}^n)$, and define the *bilinear form* $B^s_{A,q} : H^s \times H^s \to \mathbb{R}$ as follows:

$$B^s_{A,q}[u,v] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \nabla^s_A u \cdot \nabla^s_A v \, dydx + \int_{\mathbb{R}^n} quv \, dx \, .$$

Observe that by Fubini's theorem and Lemmas 3.3, 3.4

$$\begin{aligned} B^s_{A,q}[u,u] &= \langle (-\Delta)^s u, u\rangle + 2\langle \nabla^s u, A_{a\|} u\rangle + \langle Qu, u\rangle \\ &= \langle \nabla^s u, \nabla^s u\rangle + 2\langle \nabla^s u, \alpha(\sigma-1)u\rangle + \langle Qu, u\rangle \\ &= \langle \nabla^s u, \nabla^s u + (\sigma-1)\alpha(u(x)-u(y))\rangle + \langle Qu, u\rangle \\ &= \langle \nabla^s u, \sigma\nabla^s u\rangle + \langle Qu, u\rangle \, . \end{aligned}$$

Since again by Lemma 3.4 we have $\sigma \geq 1$, for the first term

$$\langle \nabla^s u, \sigma \nabla^s u\rangle = \int_{\mathbb{R}^{2n}} \sigma |\nabla^s u|^2 \, dydx \geq \int_{\mathbb{R}^{2n}} |\nabla^s u|^2 \, dydx = \langle (-\Delta)^s u, u\rangle \, ,$$

and thus $B^s_{A,q}[u,u] \geq B^s_{0,Q}[u,u]$. Now Lemma 2.6 from [38] gives the well-posedness of the direct problem for FMSE, in the assumption that 0 is not an eigenvalue for the equation: if $F \in (\tilde{H}^s(\Omega))^*$ then there exists a unique solution $u_F \in H^s(\Omega)$ to $B^s_{A,q}[u,v] = F(v)$, $\forall v \in \tilde{H}^s(\Omega)$, that is a unique $u_F \in H^s(\Omega)$ such that $(-\Delta)^s_A u + qu = F$ in Ω, $u_F|_{\Omega_e} = 0$. For non-zero exterior value, see e.g. [10] and [17]; one also gets the estimate

$$\|u_f\|_{H^s(\mathbb{R}^n)} \leq c(\|F\|_{(\tilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)}) \, . \tag{20}$$

Lemma 3.11. *Let $v, w \in H^s(\mathbb{R}^n)$, $f, g \in H^s(\Omega_e)$ and $u_f, u_g \in H^s(\mathbb{R}^n)$ be such that $((-\Delta)^s_A + q)u_f = ((-\Delta)^s_A + q)u_g = 0$ in Ω, $u_f|_{\Omega_e} = f$ and $u_g|_{\Omega_e} = g$. Then*

1. *$B^s_{A,q}[v,w] = B^s_{A,q}[w,v]$ (symmetry),*
2. *$|B^s_{A,q}[v,w]| \leq k\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}$,*
3. *$B^s_{A,q}[u_f, e_g] = B^s_{A,q}[u_g, e_f]$,*

where $e_g, e_f \in H^s(\mathbb{R}^n)$ are extensions of g, f respectively.

Proof. Symmetry follows immediately from the definition. For the second point, use *(e2)*, *(e7)* and the definition of magnetic fractional gradient to write

$$\begin{aligned} |B^s_{A,q}[v,w]| &= |\langle \nabla^s_A v, \nabla^s_A w\rangle + \langle qv, w\rangle| \leq |\langle \nabla^s_A v, \nabla^s_A w\rangle| + |\langle qv, w\rangle| \\ &\leq \|\nabla^s_A v\|_{L^2}\|\nabla^s_A w\|_{L^2} + \|qv\|_{H^{-s}}\|w\|_{H^s} \\ &\leq k'\|v\|_{H^s}\|w\|_{H^s} + k''\|q\|_{L^p}\|v\|_{H^s}\|w\|_{H^s} \leq k\|v\|_{H^s}\|w\|_{H^s} \, . \end{aligned}$$

For the third point, first compute

$$B^s_{A,q}[u_f,u_g] = \int_{\mathbb{R}^n} ((-\Delta)^s_A u_f + qu_f)u_g\,dx = \int_{\Omega_e} ((-\Delta)^s_A u_f + qu_f)u_g\,dx$$
$$= \int_{\Omega_e} ((-\Delta)^s_A u_f + qu_f)e_g\,dx = B^s_{A,q}[u_f,e_g]\ ,$$

and then $B^s_{A,q}[u_f,e_g] = B^s_{A,q}[u_f,u_g] = B^s_{A,q}[u_g,u_f] = B^s_{A,q}[u_g,e_f]$. □

The DN-map and the integral identity.

Lemma 3.12. *There exists a bounded, linear, self-adjoint map* $\Lambda^s_{A,q} : X \to X^*$ *defined by*

$$\langle \Lambda^s_{A,q}[f],[g]\rangle = B^s_{A,q}[u_f,g], \qquad \forall f,g \in H^s(\mathbb{R}^n)\ ,$$

where X *is the abstract quotient space* $H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ *and* $u_f \in H^s(\mathbb{R}^n)$ *solves* $(-\Delta)^s_A u_f + qu_f = 0$ *in* Ω *with* $u - f \in \tilde{H}^s(\Omega)$.

Proof. We first prove that the tentative definition of the DN-map does not depend on the representatives of the equivalence classes involved. Let $\phi,\psi \in \tilde{H}^s(\Omega)$ and compute by Lemma 3.11

$$B^s_{A,q}[u_{f+\phi},g+\psi] = \int_{\Omega_e} (g+\psi)((-\Delta)^s_A + q)u_{f+\phi}\,dx$$
$$= \int_{\Omega_e} g((-\Delta)^s_A + q)u_f\,dx = B^s_{A,q}[u_f,g]\ .$$

The ψ disappears because it vanishes in Ω_e, while the ϕ plays actually no role, since $f = f+\phi$ over Ω_e implies $u_{f+\phi} = u_f$. The boundedness of $\Lambda^s_{A,q}$ follows from 3.11 and (20): first compute

$$|\langle \Lambda^s_{A,q}[f],[g]\rangle| = |B^s_{A,q}[u_f,g]| \leq k\|u_f\|_{H^s}\|g\|_{H^s} \leq c\|f\|_{H^s}\|g\|_{H^s}\ ,$$

for all $f \in [f]$, $g \in [g]$, and then observe that this implies

$$|\langle \Lambda^s_{A,q}[f],[g]\rangle| \leq k \inf_{f\in[f]} \|f\|_{H^s} \inf_{g\in[g]} \|g\|_{H^s} = k\|[f]\|_X\|[g]\|_X\ .$$

Finally, we prove the self-adjointness from Lemma 3.11:

$$\langle \Lambda^s_{A,q}[f],[g]\rangle = B^s_{A,q}[u_f,e_g] = B^s_{A,q}[u_g,e_f] = \langle \Lambda^s_{A,q}[g],[f]\rangle = \langle [f],\Lambda^s_{A,q}[g]\rangle\ . \quad □$$

The DN-map will now be used to prove an integral identity.

Lemma 3.13. *Let $(A_1, q_1), (A_2, q_2) \in \mathcal{P}$, f_1, f_2 be exterior data belonging to $H^s(\mathbb{R}^n)$ and $u_i \in H^s(\mathbb{R}^n)$ be the solution of $(-\Delta)^s_{A_i} u_i + q_i u_i = 0$ with $u_i - f_i \in \tilde{H}^s(\Omega)$ for $i = 1, 2$. The following integral identity holds:*

$$\begin{aligned} &\langle(\Lambda^s_{A_1,q_1} - \Lambda^s_{A_2,q_2})f_1, f_2\rangle = \\ &\quad = 2\Big\langle \int_{\mathbb{R}^n} ((A_1)_{a\|} - (A_2)_{a\|}) \cdot \nabla^s u_1 \, dy, u_2 \Big\rangle + \langle (Q_1 - Q_2)u_1, u_2\rangle . \end{aligned} \tag{21}$$

Proof. The proof is a computation based on the results of Lemmas 3.12 and 3.3:

$$\begin{aligned} \langle(\Lambda^s_{A_1,q_1} - \Lambda^s_{A_2,q_2})f_1, f_2\rangle &= B^s_{A_1,q_1}[u_1, u_2] - B^s_{A_2,q_2}[u_1, u_2] \\ &= \langle \nabla^s u_1, \nabla^s u_2\rangle + 2\Big\langle \int_{\mathbb{R}^n} (A_1)_{a\|} \cdot \nabla^s u_1 \, dy, u_2 \Big\rangle + \langle Q_1 u_1, u_2\rangle - \\ &\quad - \langle \nabla^s u_1, \nabla^s u_2\rangle - 2\Big\langle \int_{\mathbb{R}^n} (A_2)_{a\|} \cdot \nabla^s u_1 \, dy, u_2 \Big\rangle - \langle Q_2 u_1, u_2\rangle \\ &= 2\Big\langle \int_{\mathbb{R}^n} ((A_1)_{a\|} - (A_2)_{a\|}) \cdot \nabla^s u_1 \, dy, u_2 \Big\rangle + \langle (Q_1 - Q_2)u_1, u_2\rangle . \quad \square \end{aligned}$$

The WUCP and the RAP. Let $W \subseteq \Omega_e$ be open and $u \in H^s(\mathbb{R}^n)$ be such that $u = 0$ and $(-\Delta)^s_A u + qu = 0$ in W. If this implies that $u = 0$ in Ω as well, we say that FMSE has got the WUCP. It is known that WUCP holds if both A and q vanish, that is, in the case of the fractional Laplace equation (see [38]).

Let $\mathcal{R} = \{u_f|_\Omega, f \in C^\infty_c(W)\} \subset L^2(\Omega)$ be the set of the restrictions to Ω of those functions u_f solving FMSE for some smooth exterior value f supported in W. If $\mathcal{R}$ is dense in $L^2(\Omega)$, we say that FMSE has got the RAP.

Remark 3.14. *The WUCP and the RAP are non-local properties. For example, the RAP shows a certain freedom of the solutions to fractional PDEs, since it states that they can approximate any L^2 function. This is not the case for a local operator, e.g. the classical Laplacian, whose solutions are much more rigid.*

Lemma 3.15. *The WUCP implies the RAP in the case of FMSE.*

Proof. We follow the spirit of the analogous Lemma of [17]. Let $v \in L^2(\Omega)$, and assume that $\langle v, w\rangle = 0$ for all $w \in \mathcal{R}$. Then if $f \in C^\infty_c(W)$ and $\phi \in \tilde{H}^s(\Omega)$

solves $(-\Delta)^s_A\phi + q\phi = v$ in Ω, we have

$$\begin{aligned}0 &= \langle v, u_f|_\Omega\rangle = \langle v, u_f - f\rangle = \int_{\mathbb{R}^n} v(u_f - f)\,dx \\ &= \int_\Omega v(u_f - f)\,dx = \int_\Omega ((-\Delta)^s_A\phi + q\phi)(u_f - f)\,dx \\ &= \int_{\mathbb{R}^n} ((-\Delta)^s_A\phi + q\phi)(u_f - f)\,dx \\ &= B^s_{A,q}[\phi, u_f] - \int_{\mathbb{R}^n} ((-\Delta)^s_A\phi + q\phi) f\,dx\;.\end{aligned}$$

However, $B^s_{A,q}[\phi, u_f] = \int_{\mathbb{R}^n}((-\Delta)^s_A u_f + qu_f)\phi\,dx = 0$, and so $\int_{\mathbb{R}^n}((-\Delta)^s_A\phi + q\phi)f\,dx = 0$. Given the arbitrarity of $f \in C^\infty_c(W)$, this implies that $(-\Delta)^s_A\phi + q\phi = 0$ in W. Now we use the WUCP: from $(-\Delta)^s_A\phi + q\phi = 0$ and $\phi = 0$ in W, an open subset of Ω_e, we deduce that $\phi = 0$ in Ω as well. By the definition of ϕ and the fact that $v \in L^2(\Omega)$ it now follows that $v \equiv 0$. Thus if $\langle v, w\rangle = 0$ holds for all $w \in \mathcal{R}$, then $v \in L^2(\Omega)$ must vanish; by the Hahn-Banach theorem this implies that $\mathcal{R}$ is dense in $L^2(\Omega)$. □

4. Main results

The inverse problem. We prove Theorem 1.1 under the assumption $(A, q) \in \mathcal{P}$, while for all the previous results we only required $(A, q) \in \mathcal{P}_0$. We find that *(p5)* makes physical sense, as the random walk interpretation of FMSE suggests; however, we move the consideration of the general case to future work.
By *(p5)* and Lemma 3.4 we easily deduce that $\sigma(x, y) \equiv 1$ whenever $(x, y) \notin \Omega^2$, since in this case $A_{a\|}(x, y) = 0$. Another consequence of *(p5)* is:

Lemma 4.1. *Let $(A, q) \in \mathcal{P}$. Then FMSE enjoys the WUCP.*

Proof. Suppose that for all $x \in W \subseteq \Omega_e$ we have $u(x) = 0$, $(-\Delta)^s_A u(x) + q(x)u(x) = 0$. This in particular implies that $(-\Delta)^s_A u(x) = 0$. Since $x \notin \Omega$, for almost every $y \in \mathbb{R}^n$ we must have $A(x, y) = A(y, x) = 0$ by property *(p5)*, which means that $A_{a\|}(x, y) = 0$. It is now an easy consequence of Lemma 3.3 that $(-\Delta)^s u(x) = 0$ for all $x \in W$. The known WUCP for the fractional Laplacian ([17]) gives the wanted result. □

We are ready to solve the inverse problem, which we restate here:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded open set, $s \in (0,1)$, and let $(A_i, q_i) \in \mathcal{P}$ for $i = 1, 2$. Suppose $W_1, W_2 \subset \Omega_e$ are non empty open sets, and that the DN maps for the FMSEs in Ω relative to (A_1, q_1) and (A_2, q_2) satisfy*

$$\Lambda^s_{A_1,q_1}[f]|_{W_2} = \Lambda^s_{A_2,q_2}[f]|_{W_2}, \qquad \forall f \in C_c^\infty(W_1) .$$

Then $(A_1, q_1) \sim (A_2, q_2)$, that is, the potentials coincide up to the gauge $\sim$.

Proof. Without loss of generality, let $W_1 \cap W_2 = \emptyset$. Let $f_i \in C_c^\infty(W_i)$, and let $u_i \in H^s(\mathbb{R}^n)$ solve $(-\Delta)^s_{A_i} u_i + q_i u_i = 0$ with $u_i - f_i \in \tilde{H}^s(\Omega)$ for $i = 1, 2$. Knowing that the DN maps computed on $f \in C_c^\infty(W_1)$ coincide when restricted to W_2 and the integral identity (21), we write *Alessandrini's identity*:

$$\begin{aligned} 0 &= \langle (\Lambda^s_{A_1,q_1} - \Lambda^s_{A_2,q_2}) f_1, f_2 \rangle \\ &= 2\Big\langle \int_{\mathbb{R}^n} ((A_1)_{a\|} - (A_2)_{a\|}) \cdot \nabla^s u_1 \, dy, u_2 \Big\rangle + \langle (Q_1 - Q_2) u_1, u_2 \rangle \, . \end{aligned} \tag{22}$$

We can refine (22) by substituting every instance of u_i with $u_i|_\Omega$. In fact, since u_i is supported in $\Omega \cup W_i$ and $(\Omega \cup W_1) \cap (\Omega \cup W_2) = \Omega$,

$$\begin{aligned} \langle (Q_1 - Q_2) u_1, u_2 \rangle &= \int_{\mathbb{R}^n} u_1 u_2 (Q_1 - Q_2) \, dx = \int_\Omega u_1 u_2 (Q_1 - Q_2) \, dx \\ &= \int_\Omega u_1|_\Omega u_2|_\Omega (Q_1 - Q_2) \, dx = \int_{\mathbb{R}^n} u_1|_\Omega u_2|_\Omega (Q_1 - Q_2) \, dx. \end{aligned}$$

Moreover, by property *(p5)*,

$$\begin{aligned} &\Big\langle \int_{\mathbb{R}^n} \nabla^s u_1 \cdot ((A_1)_{a\|} - (A_2)_{a\|}) \, dy, u_2 \Big\rangle = \\ &\quad = \int_{\mathbb{R}^n} u_2 \int_{\mathbb{R}^n} ((A_1)_{a\|} - (A_2)_{a\|}) \cdot \nabla^s u_1 \, dy \, dx \\ &\quad = \int_{\mathbb{R}^n} u_2(x) \int_{\mathbb{R}^n} (\sigma_1(x,y) - \sigma_2(x,y)) \, |\alpha|^2 (u_1(x) - u_1(y)) \, dy \, dx \\ &\quad = \int_\Omega (u_2|_\Omega)(x) \int_\Omega (\sigma_1(x,y) - \sigma_2(x,y)) \, |\alpha|^2 \Big((u_1|_\Omega)(x) - (u_1|_\Omega)(y) \Big) \, dy \, dx \, . \end{aligned}$$

Eventually we get

$$\begin{aligned} 0 = 2 &\int_{\mathbb{R}^n} (u_2|_\Omega)(x) \int_{\mathbb{R}^n} (\sigma_1(x,y) - \sigma_2(x,y)) \, |\alpha|^2 \Big((u_1|_\Omega)(x) - (u_1|_\Omega)(y) \Big) \, dy \, dx + \\ &+ \int_{\mathbb{R}^n} u_1|_\Omega u_2|_\Omega (Q_1 - Q_2) \, dx \, . \end{aligned} \tag{23}$$

The RAP holds by Lemmas 3.15 and 4.1. Fix any $f \in L^2(\Omega)$, and let $f_i^{(k)} \in C_c^\infty(W_i)$ for $i = 1, 2$ and $k \in \mathbb{N}$ be such that $u_1^{(k)}|_\Omega \to 1$, $u_2^{(k)}|_\Omega \to f$ in L^2. Inserting these solutions in (23) and taking the limit as $k \to \infty$ implies that $\int_{\mathbb{R}^n} f(Q_1 - Q_2)\, dx = 0$, so that, given that $f \in L^2(\Omega)$ is arbitrary, we deduce $Q_1(x) = Q_2(x)$ for $x \in \Omega$. Coming back to (23), we can write

$$\int_{\mathbb{R}^n} (u_2|_\Omega)(x) \int_{\mathbb{R}^n} (\sigma_1(x,y) - \sigma_2(x,y)) \frac{(u_1|_\Omega)(x) - (u_1|_\Omega)(y)}{|x-y|^{n+2s}}\, dy\, dx = 0,$$

where $u_i \in H^s(\mathbb{R}^n)$ once again solves $(-\Delta)^s_{A_i} u_i + q_i u_i = 0$ with $u_i - f_i \in \tilde{H}^s(\Omega)$ for some $f_i \in C_c^\infty(W_i)$ and $i = 1, 2$. Choosing $u_2^{(k)}|_\Omega \to f$ in L^2 for some arbitrary $f \in L^2$, by the same argument

$$\int_{\mathbb{R}^n} (\sigma_1(x,y) - \sigma_2(x,y)) \frac{(u_1|_\Omega)(x) - (u_1|_\Omega)(y)}{|x-y|^{n+2s}}\, dy = 0$$

for $x \in \Omega$. Fix now some $x \in \Omega$ and an arbitrary $\psi \in C_c^\infty(\Omega)$. Since $g(y) := \psi(y) e^{-1/|x-y|} |x-y|^{n+2s} \in \mathcal{S} \subset L^2(\Omega)$ as in Lemma 3.8, by the RAP we find a sequence $u_1^{(k)}|_\Omega \to g$. Substituting these solutions and taking the limit,

$$\int_{\mathbb{R}^n} (\sigma_1(x,y) - \sigma_2(x,y)) \psi(y) e^{-1/|x-y|}\, dy = 0\, .$$

Thus we conclude that for all $x \in \Omega$ it must be $\sigma_1(x,y) = \sigma_2(x,y)$ for all $y \in \Omega$, i.e. $\sigma_1 = \sigma_2$ over Ω^2. But then σ_1 and σ_2 coincide everywhere, because they are both 1 in $\mathbb{R}^{2n} \setminus \Omega^2$. This means that $(A_1)_{a\|} = (A_2)_{a\|}$. Moreover, since by *(p2)*, *(p4)* and *(p5)* we have $Q_1 = 0 = Q_2$ over Ω_e, by the argument above $Q_1 = Q_2$ everywhere. It thus follows from Lemma 3.8 that $(A_1, q_1) \sim (A_2, q_2)$. □

5. A random walk interpretation for FMSE

Diffusion phenomena can often be seen as continuous limits of random walks. The classical result for the Laplacian was extended in [43] to the fractional one by considering long jumps. Similarly, the fractional conductivity equation was shown in [10] to arise from a long jump random walk with weight $\gamma^{1/2}$, where γ is the conductivity. We now show how the leading term in FMSE is itself the limit of a long jump random walk with weights. For simplicity, here we take σ as smooth and regular as needed. Let $h > 0$, $\tau = h^{2s}$, $k \in \mathbb{Z}^n$, $x \in h\mathbb{Z}^n$ and $t \in \tau\mathbb{Z}$. We consider a random walk on $h\mathbb{Z}^n$ with time steps from $\tau\mathbb{Z}$. Define

$$f(x,k) := \begin{cases} \sigma(x, x+hk)|k|^{-n-2s} & \text{if} \quad k \neq 0 \\ 0 & \text{if} \quad k = 0 \end{cases},$$

and then observe that $\forall x \in h\mathbb{Z}^n$

$$\sum_{k\in\mathbb{Z}^n} f(x,k) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} f(x,k) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \sigma(x,x+hk)|k|^{-n-2s}$$
$$\leq \|\sigma\|_{L^\infty} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{-n-2s} < \infty \, .$$

Thus we can normalize $f(x,k)$, and get the new function $P(x,k)$

$$P(x,k) := \begin{cases} \left(\sum_{j\in\mathbb{Z}^n} f(x,j)\right)^{-1} \sigma(x,x+hk)|k|^{-n-2s} & \text{if} \quad k\neq 0 \\ 0 & \text{if} \quad k = 0 \end{cases} . \qquad (24)$$

$P(x,k)$ takes values in $[0,1]$ and verifies $\sum_{k\in\mathbb{Z}^n} P(x,k) = 1$; we interpret it as the probability that a particle will jump from $x+hk$ to x in the next step.

Remark 5.1. *Let us compare $P(x,k)$ for the fractional Laplacian, conductivity and magnetic Laplacian operators. $P(x,k)$ always decreases when k increases; the fractional Laplacian, which has $\sigma(x,y)\equiv 1$, treats all the points of $\mathbb{R}^n$ equally: no point is intrinsically more likely to be reached at the next jump; the fractional conductivity operator, which has $\sigma(x,y)=\sqrt{\gamma(x)\gamma(y)}$, distinguishes the points of $\mathbb{R}^n$: those with high conductivity are more likely to be reached. However, the conductivity field is independent from the current position of the particle. The magnetic fractional Laplacian operator has no special $\sigma(x,y)$ and it distinguishes the points of $\mathbb{R}^n$ in a more subtle way, as the conductivity field depends on the position of the particle: the same point may have high conductivity if the particle is at x and a low one if it is at y.*

Remark 5.2. *We now see why $\sigma>0$ and $\sigma(x,y)=1$ if $(x,y)\notin\Omega^2$: these are needed for $y\mapsto\sigma(x,y)$ to be a conductivity as in [10] for all $x\in\mathbb{R}^n$.*

Let $u(x,t)$ be the probability that the particle is at point x at time t. Then

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} P(x,k)u(x+hk,t) \ .$$

We can compute $\partial_t u(x,t)$ as the limit for $\tau\to 0$ of the difference quotients, and then substitute the above formula (see [10]). As the resulting sum approximates the Riemannian integral, we eventually get that for some constant $C>0$

$$\partial_t u(x,t) = C\int_{\mathbb{R}^n} \sigma(x,y)\frac{u(y,t)-u(x,y)}{|x-y|^{n+2s}}\,dy \, .$$

If $u(x,t)$ is independent of t, the leading term of FMSE is recovered.

We shall now prove that the random walk we are considering represents anomalous diffusion. In a classical diffusion scenario, very long jumps should happen with very low probability. This is mathematically reflected in the property that the variance of the length of the jumps is finite. However, this is not the case for our random walk:

Lemma 5.3. *If $s \in (0,1)$, then the second moment of the length of the jumps of the random walk (24) is infinity. Moreover, if $s \in (1/2,1)$ the first moment is finite, and if $s \in (0,1/2]$ it is infinity.*

Proof. Fix any point $x \in \mathbb{R}^n$, and let $s \in (0,1)$. The second moment of the length of the jumps of the random walk (24) is proportional to

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^2 \frac{\sigma(x,x+hk)}{|k|^{n+2s}} \geq \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|k|^{n+2s-2}} \ .$$

By the integral test for the convergence of a series, we deduce that the above series diverges because

$$\int_{\mathbb{R}^n\setminus B_1} \frac{dx}{|x|^{n+2s-2}} = c\int_1^\infty \rho^{1-2s} d\rho = c\,\rho^{2-2s}\big|_{\rho=1}^\infty = \infty \ .$$

Thus the second moment is infinity. For the first moment, let $M > \mathrm{diam}(\Omega)/h$. Then $(x, x+hk) \notin \Omega^2$ for $|k| > M$, either because $x \notin \Omega$ or because, if $x \in \Omega$, then $|hk| >\mathrm{diam}(\Omega)$ and therefore $x + hk \notin \Omega$. Thus by *(p5)* we know that $\sigma(x, x+hk) = 1$ if $|k| > M$. Of course we have

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k| \frac{\sigma(x,x+hk)}{|k|^{n+2s}} = \sum_{\substack{|k|>M\\ k\in\mathbb{Z}^n\setminus\{0\}}} \frac{1}{|k|^{n+2s-1}} + \sum_{\substack{|k|\leq M\\ k\in\mathbb{Z}^n\setminus\{0\}}} \frac{\sigma(x,x+hk)}{|k|^{n+2s-1}} \ ,$$

because the second sum in the right hand side has only a finite amount of finite terms, and is therefore finite itself. For the other sum in the right hand side, we use again the integral test: the first moment will be finite if and only if the integral

$$\int_{\mathbb{R}^n\setminus B_M} \frac{dx}{|x|^{n+2s-1}} = c\int_M^\infty \rho^{-2s} d\rho$$

is itself finite. We see that this happens if and only if $s \in (1/2, 1)$, which concludes the proof. □

Remark 5.4. *If $s \in (1/2, 1)$, the variance of the length of the jumps of the random walk (24) is infinity because of the above Lemma. Therefore, in this case the random walk represents anomalous diffusion.*

6. One slight generalization

We now briefly consider a fractional magnetic *conductivity* equation (FMCE) and show that it shares similar features as FMSE. Let $(A, q) \in \mathcal{P}$ and let γ be a conductivity in the sense of [10]. Consider $u \in H^s(\mathbb{R}^n)$. Since $\nabla^s_A : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$, if $\Theta(x, y) := \sqrt{\gamma(x)\gamma(y)}\mathrm{Id}$ by the properties of γ we know that $\Theta \cdot \nabla^s_A u \in L^2(\mathbb{R}^{2n})$. Thus we define the *fractional magnetic conductivity operator*

$$\mathrm{C}^s_{\gamma,A} u(x) := (\nabla\cdot)^s_A (\Theta \cdot \nabla^s_A u)(x) \ , \qquad \mathrm{C}^s_{\gamma,A} : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n) \ .$$

We say that $u \in H^s(\mathbb{R}^n)$ solves the FMCE with exterior value $f \in H^s(\Omega_e)$ if

$$\begin{cases} \mathrm{C}^s_{\gamma,A} u(x) + q(x)u(x) = 0 & \text{in } \Omega \\ u = f & \text{in } \Omega_e \end{cases}$$

holds in weak sense.

Lemma 6.1. *Let $u \in H^s(\mathbb{R}^n)$, $g \in H^s(\Omega_e)$, $w = \gamma^{1/2} u$ and $f = \gamma^{1/2} g$. Moreover, let $(A, q) \in \mathcal{P}$ and*

$$q' := q'_{\gamma,A,q} = \frac{q}{\gamma} - (\nabla\cdot)^s A_{s\|} + \frac{(\nabla\cdot)^s (A\gamma^{1/2}(y))}{\gamma^{1/2}(x)} - \frac{(-\Delta)^s(\gamma^{1/2})}{\gamma^{1/2}(x)} + \\ + \int_{\mathbb{R}^n} \Big(- \frac{\nabla^s(\gamma^{1/2}) \cdot A}{\gamma^{1/2}(x)} + |A|^2 \Big(\frac{\gamma^{1/2}(y)}{\gamma^{1/2}(x)} - 1 \Big) \Big) \, dy \ .$$

FMCE with potentials (A, q), conductivity γ and exterior value g is solved by u if and only if w solves FMSE with potentials (A, q') and exterior value f, i.e.

$$\begin{cases} \mathrm{C}^s_{\gamma,A} u + qu = 0 & \text{in } \Omega \\ u = g & \text{in } \Omega_e \end{cases} \quad \Leftrightarrow \quad \begin{cases} (-\Delta)^s_A w + q' w = 0 & \text{in } \Omega \\ w = f & \text{in } \Omega_e \end{cases} .$$

Moreover, the following formula holds for all $w \in H^s(\mathbb{R}^n)$:

$$\mathrm{C}^s_{\gamma,A}(\gamma^{-1/2} w) + q\gamma^{-1/2} w = \gamma^{1/2} \Big((-\Delta)^s_A + q' \Big) w \, .$$

Proof. Let us start from some preliminary computations. One sees that

$$\begin{aligned}\nabla^s w &= \nabla^s(\gamma^{1/2}u) = \gamma^{1/2}(y)\nabla^s u + u(x)\nabla^s(\gamma^{1/2}) \\ &= \gamma^{1/2}(y)\nabla^s u + w(x)\frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)}\ ,\end{aligned}$$

from which $\nabla^s u = \frac{\nabla^s w}{\gamma^{1/2}(y)} - w(x)\frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)\gamma^{1/2}(y)}$, and eventually

$$\nabla^s_A u = \frac{\nabla^s w}{\gamma^{1/2}(y)} - w(x)\frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)\gamma^{1/2}(y)} + A(x,y)\frac{w(x)}{\gamma^{1/2}(x)}\ . \tag{25}$$

By the definition of magnetic fractional divergence, if $v \in H^s(\mathbb{R}^n)$,

$$\begin{aligned}\langle(\nabla\cdot)^s_A(\Theta\cdot\nabla^s_A u), v\rangle &= \langle\gamma^{1/2}(x)\gamma^{1/2}(y)\nabla^s_A u, \nabla^s_A v\rangle \\ &= \langle\gamma^{1/2}(x)\gamma^{1/2}(y)\nabla^s_A u, \nabla^s v\rangle + \langle\gamma^{1/2}(x)\gamma^{1/2}(y)\nabla^s_A u, Av\rangle \\ &= \langle\gamma^{1/2}(x)\gamma^{1/2}(y)\nabla^s_A u, \nabla^s v\rangle + \Big\langle \int_{\mathbb{R}^n}\gamma^{1/2}(y)\nabla^s_A u\cdot A\,dy, \gamma^{1/2}v\Big\rangle\ .\end{aligned}$$

Applying formula (25), we get

$$\begin{aligned}\langle(\nabla\cdot)^s_A(\Theta\cdot\nabla^s_A u), v\rangle &= \langle\gamma^{1/2}(x)\nabla^s w, \nabla^s v\rangle + \langle w(x)(A(x,y)\gamma^{1/2}(y) - \nabla^s(\gamma^{1/2})), \nabla^s v\rangle \\ &\quad + \Big\langle\int_{\mathbb{R}^n}\gamma^{1/2}(y)\Big(\frac{\nabla^s w}{\gamma^{1/2}(y)} - w(x)\frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)\gamma^{1/2}(y)} + A(x,y)\frac{w(x)}{\gamma^{1/2}(x)}\Big)\cdot A\,dy, \gamma^{1/2}v\Big\rangle \\ &= \langle\gamma^{1/2}(x)\nabla^s w, \nabla^s v\rangle + \langle w(x)(A(x,y)\gamma^{1/2}(y) - \nabla^s(\gamma^{1/2})), \nabla^s v\rangle \qquad (26) \\ &\quad + \Big\langle\int_{\mathbb{R}^n}\Big(\nabla^s w\cdot A - w(x)\frac{\nabla^s(\gamma^{1/2})\cdot A}{\gamma^{1/2}(x)} + |A|^2 w(x)\frac{\gamma^{1/2}(y)}{\gamma^{1/2}(x)}\Big)\,dy, \gamma^{1/2}v\Big\rangle\ .\end{aligned}$$

We treat the resulting terms separately. For the first one, by symmetry,

$$\begin{aligned}\langle\gamma^{1/2}(x)\nabla^s w, \nabla^s v\rangle &= \langle\nabla^s w, \gamma^{1/2}(x)\nabla^s v\rangle = \langle\nabla^s w, \nabla^s(v\gamma^{1/2}) - v(y)\nabla^s(\gamma^{1/2})\rangle \\ &= \langle(-\Delta)^s w, v\gamma^{1/2}\rangle - \langle\nabla^s w, v(y)\nabla^s(\gamma^{1/2})\rangle = \langle(-\Delta)^s w, v\gamma^{1/2}\rangle - \langle\nabla^s w, v(x)\nabla^s(\gamma^{1/2})\rangle \\ &= \langle(-\Delta)^s w, v\gamma^{1/2}\rangle - \Big\langle\int_{\mathbb{R}^n}\nabla^s w\cdot\frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)}\,dy, \gamma^{1/2}v\Big\rangle\ . \qquad (27)\end{aligned}$$

For the second part of (26), we will compute as follows:

$$\begin{aligned}&\langle A(x,y)\gamma^{1/2}(y) - \nabla^s(\gamma^{1/2}), w(x)\nabla^s v\rangle = \\ &= \langle A(x,y)\gamma^{1/2}(y) - \nabla^s(\gamma^{1/2}), \nabla^s(vw) - v(y)\nabla^s w\rangle \\ &= \Big\langle(\nabla\cdot)^s\Big(A(x,y)\gamma^{1/2}(y) - \nabla^s(\gamma^{1/2})\Big), vw\Big\rangle - \Big\langle\Big(A(x,y)\gamma^{1/2}(y) - \nabla^s(\gamma^{1/2})\Big)v(y), \nabla^s w\Big\rangle \\ &= \Big\langle\Big(\frac{(\nabla\cdot)^s(A\gamma^{1/2}(y))}{\gamma^{1/2}(x)} - \frac{(-\Delta)^s(\gamma^{1/2})}{\gamma^{1/2}(x)}\Big)w(x), v\gamma^{1/2}\Big\rangle - \Big\langle\Big(A(y,x)\gamma^{1/2}(x) - \nabla^s(\gamma^{1/2})\Big)v(x), \nabla^s w\Big\rangle\end{aligned}$$

$$= \Big\langle \Big(\frac{(\nabla\cdot)^s (A\gamma^{1/2}(y))}{\gamma^{1/2}(x)} - \frac{(-\Delta)^s(\gamma^{1/2})}{\gamma^{1/2}(x)} \Big) w(x), v\gamma^{1/2} \Big\rangle - \\ - \Big\langle \int_{\mathbb{R}^n} A(y,x)\cdot\nabla^s w\, dy, v\gamma^{1/2} \Big\rangle + \Big\langle \int_{\mathbb{R}^n} \frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)} \cdot \nabla^s w\, dy, v\gamma^{1/2} \Big\rangle . \tag{28}$$

Substituting (27) and (28) into (26), we conclude the proof:

$$\begin{aligned} \langle (\nabla\cdot)^s_A(\Theta\cdot\nabla^s_A u), v\rangle &= \langle (-\Delta)^s w, v\gamma^{1/2}\rangle - \Big\langle \int_{\mathbb{R}^n} \nabla^s w \cdot \frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)}\, dy, \gamma^{1/2} v \Big\rangle + \\ &+ \Big\langle \Big(\frac{(\nabla\cdot)^s (A\gamma^{1/2}(y))}{\gamma^{1/2}(x)} - \frac{(-\Delta)^s(\gamma^{1/2})}{\gamma^{1/2}(x)} \Big) w(x), v\gamma^{1/2} \Big\rangle - \\ &- \Big\langle \int_{\mathbb{R}^n} A(y,x)\cdot\nabla^s w\, dy, v\gamma^{1/2} \Big\rangle + \Big\langle \int_{\mathbb{R}^n} \frac{\nabla^s(\gamma^{1/2})}{\gamma^{1/2}(x)} \cdot \nabla^s w\, dy, v\gamma^{1/2} \Big\rangle + \\ &+ \Big\langle \int_{\mathbb{R}^n} \Big(\nabla^s w\cdot A - w(x)\frac{\nabla^s(\gamma^{1/2})\cdot A}{\gamma^{1/2}(x)} + |A|^2 w(x) \frac{\gamma^{1/2}(y)}{\gamma^{1/2}(x)} \Big)\, dy, \gamma^{1/2} v \Big\rangle \\ &= \Big\langle (-\Delta)^s w + 2\int_{\mathbb{R}^n} A_{a\|}\cdot\nabla^s w\, dy + w(x)\Big(\int_{\mathbb{R}^n} |A|^2\, dy + (\nabla\cdot)^s A_{s\|} \Big), v\gamma^{1/2} \Big\rangle + \\ &+ \Big\langle \Big\{ -(\nabla\cdot)^s A_{s\|} + \frac{(\nabla\cdot)^s(A\gamma^{1/2}(y))}{\gamma^{1/2}(x)} - \frac{(-\Delta)^s(\gamma^{1/2})}{\gamma^{1/2}(x)} + \\ &\quad + \int_{\mathbb{R}^n} \Big(-\frac{\nabla^s(\gamma^{1/2})\cdot A}{\gamma^{1/2}(x)} + |A|^2 \Big(\frac{\gamma^{1/2}(y)}{\gamma^{1/2}(x)} - 1 \Big) \Big)\, dy \Big\} w(x), v\gamma^{1/2} \Big\rangle \\ &= \langle (-\Delta)^s_A w + (q' - q/\gamma) w, v\gamma^{1/2} \rangle . \qquad \square \end{aligned}$$

Thus the FMCEs can be reduced to FMSEs; hence, we know that FMCE enjoys the same gauges as FMSE, and most importantly we can consider and solve an analogous inverse problem.

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Unique continuation property and Poincaré inequality for higher order fractional Laplacians with applications in inverse problems

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UNIQUE CONTINUATION PROPERTY AND POINCARÉ INEQUALITY FOR HIGHER ORDER FRACTIONAL LAPLACIANS WITH APPLICATIONS IN INVERSE PROBLEMS

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Abstract. We prove a unique continuation property for the fractional Laplacian $(-\Delta)^s$ when $s \in (-n/2, \infty) \setminus \mathbb{Z}$ where $n \geq 1$. In addition, we study Poincaré-type inequalities for the operator $(-\Delta)^s$ when $s \geq 0$. We apply the results to show that one can uniquely recover, up to a gauge, electric and magnetic potentials from the Dirichlet-to-Neumann map associated to the higher order fractional magnetic Schrödinger equation. We also study the higher order fractional Schrödinger equation with singular electric potential. In both cases, we obtain a Runge approximation property for the equation. Furthermore, we prove a uniqueness result for a partial data problem of the d-plane Radon transform in low regularity. Our work extends some recent results in inverse problems for more general operators.

1. Introduction

The fractional Laplacian $(-\Delta)^s$, $s \in (-n/2, \infty) \setminus \mathbb{Z}$, is a non-local operator by definition and thus differs substantially from the ordinary Laplacian $(-\Delta)$. The non-local behaviour can be exploited when solving fractional inverse problems. In section 3.1, we prove that $(-\Delta)^s$ admits a unique continuation property (UCP) for open sets, that is, if u and $(-\Delta)^s u$ both vanish in a nonempty open set, then u vanishes everywhere. Clearly this property cannot hold for local operators. We give many other versions of UCPs as well.

We have also included a quite comprehensive discussion of the Poincaré inequality for the higher order fractional Laplacian $(-\Delta)^s$, $s \geq 0$, in section 3.2. We give many proofs for the higher order fractional Poincaré inequality based on various different methods in the literature. The higher order fractional Poincaré inequality appears earlier at least in [84] for functions in $C_c^\infty(\Omega)$ where Ω is a bounded Lipschitz domain. Also similar inequalities are proved in the book [4] for homogeneous Sobolev norms but without referring to the fractional Laplacian. However, we have extended some known results, given alternative proofs, and studied a connection between the fractional and the classical Poincaré constants. We believe that section 3.2 will serve as a helpful reference on fractional Poincaré inequalities in the future.

Our main applications are fractional Schrödinger equations with and without a magnetic potential, and the d-plane Radon transforms with partial data. We apply the UCP result and the Poincaré inequality for higher order fractional Laplacians to show uniqueness for the associated fractional Schrödinger equation and the Runge approximation properties. UCPs have also applications in integral geometry since certain partial data inverse problems for the Radon transforms can be reduced to unique continuation problems of the normal operators. We remark that the normal operators of the Radon transforms are negative order fractional Laplacians (Riesz potentials) up to constant coefficients.

In this section, we introduce our models, discuss some related results and present our main theorems and corollaries. We start with the classical Calderón problem as a motivation.

1.1. **The Calderón problem.**

We will study a non-local version of the famous Calderón problem called the fractional Calderón problem. A survey of the fractional Calderón problem is given in [79]. The Calderón problem is a classical inverse problem where one wants to determine the

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electrical conductivity on some sufficiently smooth domain by boundary measurements [77, 83]. Suppose that $\Omega \subset \mathbb{R}^n$ is a domain with regular enough boundary $\partial\Omega$. The electrical conductivity is usually represented as a bounded positive function γ, and the conductivity equation is

$$\begin{cases} \nabla \cdot (\gamma \nabla u) & = 0 \text{ in } \Omega \\ u|_{\partial\Omega} & = f \end{cases} \tag{1}$$

where f is the potential on the boundary $\partial\Omega$ and u is the induced potential in Ω. The data in this problem is the Dirichlet-to-Neumann (DN) map $\Lambda_\gamma(f) = (\gamma\partial_\nu u)|_{\partial\Omega}$, where ν is the outer unit normal on the boundary. The DN map basically tells how the applied voltage on the boundary induces normal currents on the boundary by the electrical properties of the interior. The inverse problem is to determine γ from the DN map Λ_γ. One of the associated basic questions is the uniqueness problem, that is, whether $\gamma_1 = \gamma_2$ follows from $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$.

Equation (1) can be reduced to a Schrödinger equation

$$\begin{cases} (-\Delta + q)u & = 0 \text{ in } \Omega \\ u|_{\partial\Omega} & = f \end{cases} \tag{2}$$

where $q = (\Delta\sqrt{\gamma})/\sqrt{\gamma}$ now represents the electric potential in Ω. One typically assumes that 0 is not a Dirichlet eigenvalue of the operator $(-\Delta+q)$ to obtain unique solutions to equation (2). The inverse problem then is to know whether one can determine the electric potential q uniquely from the DN map Λ_q, which can be expressed in terms of the normal derivative $\Lambda_q f = \partial_\nu u|_{\partial\Omega}$ for regular enough boundaries. For more details on the classical Calderón problem and its applications to medical, seismic and industrial imaging, see [77, 83].

1.2. **Fractional Schrödinger equation.** In this article, we focus on the fractional Schrödinger equation and its generalization, the fractional magnetic Schrödinger equation. The main difference between the classical and fractional Schrödinger operators is that the first one is local and the second one is non-local. This can be seen since the Laplacian $(-\Delta)$ is local as a differential operator while the fractional counterpart $(-\Delta)^s$, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, is a non-local Fourier integral operator. In other words, the value $(-\Delta)^s u(x)$, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, depends on the values of u everywhere, not just in a small neighbourhood of $x \in \mathbb{R}^n$. Fractional Laplacians have a close connection to Levý processes and have been used in many areas of mathematics and physics, for example to model anomalous and nonlocal diffusion, and also in the formulation of fractional quantum mechanics where the fractional Schrödinger equation arises naturally as a generalization of the ordinary Schrödinger equation [3, 7, 18, 28, 50, 51, 58, 71].

Since the fractional Laplacian is a non-local operator, it is more natural to fix exterior values for the solutions of the equation instead of just boundary values. This motivates the study of the following exterior value problem, first introduced in [28],

$$\begin{cases} ((-\Delta)^s + q)u & = 0 \text{ in } \Omega \\ u|_{\Omega_e} & = f \end{cases} \tag{3}$$

where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior of Ω. The associated DN map for equation (3) is a bounded linear operator $\Lambda_q \colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$ which, under stronger assumptions, has an expression $\Lambda_q f = (-\Delta)^s u|_{\Omega_e}$ [28]. We assume that the potential q is such that the following holds:

$$\text{If } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + q)u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u = 0. \tag{4}$$

In other words, condition (4) requires that 0 is not a Dirichlet eigenvalue of the operator $((-\Delta)^s + q)$.

In section 5, we will prove that, under certain assumptions, one can uniquely determine the potential q in equation (3) from exterior measurements when $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and we also prove a Runge approximation property for equation (3) (see also section 1.5). These generalize the results in [28, 75] to higher fractional powers of s. The proofs basically reduce to the fact that the operator $(-\Delta)^s$ has the following UCP: if $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$ everywhere. This reflects the fact that $(-\Delta)^s$ is a non-local operator since such UCP can never hold for local operators.

Unique continuation of the fractional Laplacian has been extensively studied and used to show uniqueness results for fractional Schrödinger equations [14, 27, 28, 75]. One version was already proved by Riesz [28, 70] and similar methods were used in [41] to show a UCP of Riesz potentials I_α which can be seen as fractional Laplacians with negative exponents. See also [45] for a unique continuation result of Riesz potentials. UCP of $(-\Delta)^s$ for functions in $H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, was proved in [28] when $s \in (0,1)$. The proof is based on Carleman estimates from [72] and on Caffarelli-Silvestre extension [8, 9]. Using the known result for $s \in (0,1)$, we provide an elementary proof which generalizes the UCP for all $s \in (-n/2, \infty) \setminus \mathbb{Z}$. With the same trick we obtain several other unique continuation results. There are also strong unique continuation results for $s \in (0,1)$ if one assumes more regularity from the function [22, 72]. In the strong UCP, one replaces the condition $u|_V = 0$ by the requirement that u vanishes to infinite order at some point $x_0 \in V$. The higher order case $s \in \mathbb{R}^+ \setminus (\mathbb{Z} \cup (0,1))$ has been studied recently by several authors [23, 26, 86]. These results however assume some special conditions on the function u, i.e. they require that u is in a Sobolev space which depends on the power s of the fractional Laplacian $(-\Delta)^s$. We only require that u is in some Sobolev space $H^r(\mathbb{R}^n)$ where $r \in \mathbb{R}$ can be an arbitrarily small (negative) number.

See also [45] where the author proves a higher order Runge approximation property by s-harmonic functions in the unit ball when $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ (compare to theorem 1.7). Here s-harmonicity simply means that $(-\Delta)^s u = 0$ in some domain Ω. The s-harmonic approximation in the case $s \in (0,1)$ was already studied in [17]; similar higher regularity approximation results are proved in [11, 28] for the fractional Schrödinger equation.

1.3. **Fractional magnetic Schrödinger equation.** Section 6 of this paper extends the study of the fractional magnetic Schrödinger equation (FMSE) begun in [14], expanding the uniqueness result for the related inverse problem to the cases when $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. The direct problem for the classical magnetic Schrödinger equation (MSE) consists in finding a function u satisfying

$$\begin{cases} (-\Delta)_A u + qu &= -\Delta u - i\nabla \cdot (Au) - iA \cdot \nabla u + (|A|^2 + q)u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is some bounded open set with Lipschitz boundary representing a medium, f is the boundary value for the solution u, and A, q are the vector and scalar potentials of the equation. In the associated inverse problem, we are given measurements on the boundary in the form of a DN map $\Lambda_{A,q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, and we are asked to recover A, q in Ω using this information. It was shown in [60] that this is only possible up to a natural gauge: one can uniquely determine the potential q and the magnetic *field* curl$A$, but the magnetic *potential* A can not be determined in greater detail. The inverse problem for MSE is of great interest, because it generalizes the non-magnetic case by adding some first order terms, and shows a quite different behavior. It also possesses multiple applications in the sciences: the papers [60, 62, 56, 20, 61] and [35] give some examples of this, treating the inverse scattering problem with a fixed energy, isotropic elasticity, the Maxwell, Schrödinger and Dirac equations and the Stokes system. We refer to the survey [76] for many more references on inverse boundary value problems related to MSE.

We are interested in the study of a high order fractional version of the MSE. There have been many studies in this direction (see for instance [54, 52, 53]). In our work, we will build upon the results from [14] and generalize them to higher order. Thus, for us the direct problem for FMSE asks to find a function u which satisfies

$$\begin{cases} (-\Delta)^s_A u + qu &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases}$$

where Ω, f, A and q play a similar role as in the local case, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $(-\Delta)^s_A$ is the magnetic fractional Laplacian. This is a fractional version of $(-i\nabla + A) \cdot (-i\nabla + A)$, the magnetic Laplacian from which MSE arises. In section 6, we will construct the fractional magnetic Laplacian based on the fractional gradient operator ∇^s. The fractional gradient is based on the framework laid down in [18, 19], and has been studied in the papers [15, 14]. One should keep in mind that for $s > 1$ the fractional gradient is a tensor of order $\lfloor s \rfloor$ rather than a vector. In the corresponding inverse problem, we assume to know the DN map $\Lambda^s_{A,q} : H^s(\Omega_e) \to (H^s(\Omega_e))^*$, and we wish to recover A, q in Ω. In the cases when $s \in (0,1)$, it has been shown that the pair A, q can only be recovered up to a natural gauge [14]. We generalize this result to the case $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. This is achieved by first proving a weak UCP and the Runge approximation property for FMSE, and then testing the Alessandrini identity for the equation with suitably chosen functions.

Remark 1.1. *The case of the high order magnetic Schrödinger equation, that is the one in which $s \in \mathbb{N}$, $s \neq 1$, is still open at the time of writing to the best of the authors' knowledge. Our methods are purely nonlocal, and thus cannot be applied to the integer case. It was however showed in* [60]*, as cited above, that a uniqueness result up to a natural gauge holds when $s = 1$.*

1.4. **Radon transforms and region of interest tomography.** Unique continuation results have also applications in integral geometry. It was proved in [41] that the normal operator of the X-ray transform admits a UCP in the class of compactly supported distributions. This was done by considering the normal operator as a Riesz potential. We generalize the result for the normal operator of the d-plane transform R_d where $d \in \mathbb{N}$ is odd such that $0 < d < n$. In the case $d = 1$ the transform R_d corresponds to the X-ray transform and in the case $d = n-1$ to the Radon transform. The UCP of the normal operator $N_d = R_d^* R_d$ implies uniqueness for the following partial data problem: if f integrates to zero over all d-planes which intersect some nonempty open set V and $f|_V = 0$, then $f = 0$. This can be seen as a complementary result to the Helgason support theorem for the d-plane transform [36]. Helgason's theorem says that if f integrates to zero over all d-planes not intersecting a convex and compact set K and $f|_K = 0$, then $f = 0$. The d-plane transform R_d is injective on continuous functions which decay rapidly enough at infinity and also on compactly supported distributions [36]. The d-plane transform has been recently studied in the periodic case on the flat torus [2, 40, 67] but also in other settings [16, 37, 69]. Weighted and limited data Radon transforms ($d = n-1$) have been studied recently for example in [25, 29, 30, 31].

When $d = 1$, partial data problems as discussed above arise for example in seismology and medical imaging. In [41], it is explained how one can use shear wave splitting data to uniquely determine the difference of the anisotropic perturbations in the S-wave speeds, and also how one can use local measurements of travel times of seismic waves to uniquely determine the conformal factor in the linearization. Both of these problems reduce to the following partial data result: if f integrates to zero over all lines which intersect some nonempty open set V and $f|_V = 0$, then $f = 0$. In medical imaging, one typically wants to reconstruct a specific part of the human body. Can this be done by using only X-rays which go though our region of interest (ROI)? Generally this is not possible even for C_c^∞-functions [43, 63, 81], but if we know some information of f in the ROI, then the reconstruction can be done. For example, if the function f is piecewice constant, piecewice polynomial or analytic in the ROI, then f can be uniquely determined from the X-ray data [42, 43, 85]. Also, if we know the X-ray data through the ROI and the values of f in an arbitrarily small open set inside the ROI, then f is uniquely determined everywhere [13, 41]. For practical applications of ROI tomography in medical imaging, see for example [87, 88]. See also [44, 65, 66] for a discussion of the difficulties of obtaining stable reconstruction in partial data problems for the X-ray transform (visible and invisible singularities).

1.5. **Main results.** We briefly introduce the basic notation; more details can be found in sections 2, 4, 5 and 6. Let $H^r(\mathbb{R}^n)$ be the L^2 Sobolev space of order $r \in \mathbb{R}$ and $\widetilde{H}^r(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^r(\mathbb{R}^n)$ when Ω is an open set. The L^1 Bessel potential space is denoted by $H^{r,1}(\mathbb{R}^n)$. We define $H^r_K(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ to be those Sobolev functions which have support in the compact set K. The fractional Laplacian is defined via the Fourier transform $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{u})$. Then $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is a continuous operator when

$s \in \mathbb{R}^+ \setminus \mathbb{Z}$. The d-plane transform R_d takes a function which decreases rapidly enough at infinity and integrates it over d-dimensional planes where $0 < d < n$. The normal operator of the d-plane transform is defined as $N_d = R_d^* R_d$ where R_d^* is the adjoint operator. Further, we denote by $\mathcal{D}'(\mathbb{R}^n)$ the space of all distributions, $\mathcal{E}'(\mathbb{R}^n)$ the space of compactly supported distributions, $\mathscr{O}'_C(\mathbb{R}^n)$ the space of rapidly decreasing distributions and $C_\infty(\mathbb{R}^n)$ the set of rapidly decreasing continuous functions. The space of singular potentials $Z_0^{-s}(\mathbb{R}^n)$ is a certain subset of distributions $\mathcal{D}'(\mathbb{R}^n)$ and can be interpreted as a set of bounded multipliers from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$.

The following theorem extends a result in [28] and has a central role in this article. We call it the UCP of the operator $(-\Delta)^s$.

Theorem 1.2. *Let $n \geq 1$, $s \in (-n/4, \infty) \setminus \mathbb{Z}$ and $u \in H^r(\mathbb{R}^n)$ where $r \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$. The claim holds also for $s \in (-n/2, -n/4] \setminus \mathbb{Z}$ if $u \in H^{r,1}(\mathbb{R}^n)$ or $u \in \mathscr{O}'_C(\mathbb{R}^n)$.*

Theorem 1.2 is proved in section 3.1. The UCP of $(-\Delta)^s$ implies corresponding UCP for Riesz potentials (see corollary 3.2 and [41, Theorem 5.2]). This in turn implies the following UCP for the normal operator of the d-plane transform N_d when d is odd; the case $d = 1$ was already studied in [41].

Corollary 1.3. *Let $n \geq 2$ and let f belong to either $\mathcal{E}'(\mathbb{R}^n)$ or $C_\infty(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ be odd such that $0 < d < n$. If $N_d f|_V = 0$ and $f|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $f = 0$.*

From the UCP of N_d we obtain the next result which is in a sense complementary to the Helgason support theorem for the d-plane transform [36, Theorem 6.1]. It extends a result in [41] where the authors prove a similar uniqueness property for the X-ray transform.

Corollary 1.4. *Let $n \geq 2$, $V \subset \mathbb{R}^n$ a nonempty open set and $f \in C_\infty(\mathbb{R}^n)$. Let $d \in \mathbb{N}$ be odd such that $0 < d < n$. If $f|_V = 0$ and $R_d f = 0$ for all d-planes intersecting V, then $f = 0$. The claim holds also for $f \in \mathcal{E}'(\mathbb{R}^n)$ when the assumption $R_d f = 0$ for all d-planes intersecting V is understood in the sense of distributions.*

If d is even, then f is uniquely determined in V by its integrals over d-planes which intersect V, i.e. $R_d f = 0$ for all d-planes intersecting V implies $f|_V = 0$ (see remark 4.2). The authors do not know if the result of corollary 1.4 holds when d is even. However, if d is even, then the result of corollary 1.3 cannot be true as the normal operator N_d is the inverse of a local operator. See section 4 for the proofs and the definition of the d-plane transform of distributions.

The following result is a general version of the Poincaré inequality which we need for the well-posedness of the inverse problem for the fractional Schrödinger equation.

Theorem 1.5. *Let $n \geq 1$, $s \geq t \geq 0$, $K \subset \mathbb{R}^n$ a compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $\tilde{c} = \tilde{c}(n, K, s) > 0$ such that*

$$\left\| (-\Delta)^{t/2} u \right\|_{L^2(\mathbb{R}^n)} \leq \tilde{c} \left\| (-\Delta)^{s/2} u \right\|_{L^2(\mathbb{R}^n)}.$$

The constant $\tilde{c}$ can be expressed in terms of the classical Poincaré constant when $s \geq 1$ (see theorem 3.17. See section 3.2 for several proofs of the Poincaré inequality. From the unique continuation of $(-\Delta)^s$ we obtain results for the higher order fractional Schrödinger equation with singular electric potential. The following theorems generalize the results in [28, 75] for higher exponents $s \in \mathbb{R}^+ \setminus (\mathbb{Z} \cup (0, 1))$.

Theorem 1.6. *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and $q_1, q_2 \in Z_0^{-s}(\mathbb{R}^n)$ which satisfy condition (4). Let $W_1, W_2 \subset \Omega_e$ be open sets. If the DN maps for the equations $(-\Delta)^s u + m_{q_i}(u) = 0$ in Ω satisfy $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$, then $q_1|_\Omega = q_2|_\Omega$.*

Theorem 1.7. *Let $n \geq 1$ and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\Omega_1 \supset \Omega$ any open set such that $int(\Omega_1 \setminus \Omega) \neq \varnothing$. If $q \in Z_0^{-s}(\mathbb{R}^n)$ satisfies condition (4), then any $g \in \widetilde{H}^s(\Omega)$ can be approximated arbitrarily well in $\widetilde{H}^s(\Omega)$ by solutions $u \in H^s(\mathbb{R}^n)$ to the equation $(-\Delta)^s u + m_q(u) = 0$ in Ω such that $\mathrm{spt}(u) \subset \overline{\Omega}_1$.*

We remark that the approximation property in theorem 1.7 also holds in $L^2(\Omega)$ when one takes restrictions of the solutions (see [28, Theorem 1.3]). In [17, 45] the authors prove similar approximation results: C^k-functions can be approximated (in the C^k-norm) in the unit ball by s-harmonic functions, i.e. functions u which satisfy $(-\Delta)^s u = 0$ in $B_1(0)$ (see also [28, Remark 7.3]). Theorems 1.6 and 1.7 are proved in section 5. The proofs are almost identical to those in [28, 75] and only slight changes need to be done. We will present the main ideas of the proofs for clarity and in order to make a comparison to the more complicated case of FMSE.

We have achieved the following result on the Calderón problem for FMSE:

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, and let A_i, q_i verify assumptions (a1)-(a5) in section 6 for $i = 1, 2$. Let $W_1, W_2 \subset \Omega_e$ be open sets. If the DN maps for the FMSEs in Ω relative to (A_1, q_1) and (A_2, q_2) satisfy*

$$\Lambda^s_{A_1,q_1}[f]|_{W_2} = \Lambda^s_{A_2,q_2}[f]|_{W_2} \quad \text{for all } f \in C_c^\infty(W_1),$$

then $(A_1, q_1) \sim (A_2, q_2)$, that is, the potentials coincide up to gauge.

An in-depth clarification of the assumptions and the definition of the gauge involved in the proof are presented in section 6.

1.6. **Organization of the article.** This article is organized as follows. Section 2 is devoted to preliminaries. We introduce our notation and definitions of relevant quantities. In sections 3.1 and 3.2 we prove the unique continuation property of $(-\Delta)^s$ for $s \in (-n/2, \infty) \setminus \mathbb{Z}$ and give several proofs for the fractional Poincaré inequality. We introduce some applications in integral geometry and partial data problems of the d-plane transform in section 4. In section 5, we show the uniqueness and the Runge approximation results for the higher order fractional Schrödinger equation with singular electric potential. We prove the uniqueness result up to a gauge for the higher order fractional magnetic Schrödinger equation in section 6. Finally, in section 7, we discuss other problems that would now naturally continue our work. There are many potential recent results in inverse problems which perhaps can be generalized to higher order fractional Laplacians using our unique continuation result and fractional Poincaré inequality.

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2. Preliminaries

In this section, we will go through our basic notations and definitions. The following theory of distributions, Fourier analysis and Sobolev spaces can be found in many books (see for example [1, 4, 6, 38, 39, 57, 59, 78, 82]). We write $|\cdot|$ for both the Euclidean norm of vectors and the absolute value of complex numbers. We denote by $\mathbb{N}_0$ the set of natural numbers including zero.

2.1. **Distributions and Fourier transform.** We denote by $\mathcal{E}(\mathbb{R}^n)$ the set of smooth functions equipped with the topology of uniform convergence of derivatives of all order on compact sets. We also denote by $\mathcal{D}(\mathbb{R}^n)$ the set of compactly supported smooth functions with the topology of uniform convergence of derivatives of all order in a fixed compact set. The topological duals of these spaces are denoted by $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$. Elements in the space $\mathcal{E}'(\mathbb{R}^n)$ can be identified as distributions in $\mathcal{D}'(\mathbb{R}^n)$ with compact support.

We also use the space of rapidly decreasing smooth functions, i.e. Schwartz functions. Define the Schwartz space as

$$\mathscr{S}(\mathbb{R}^n) = \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \left\| \langle \cdot \rangle^N \partial^\beta \varphi \right\|_{L^\infty(\mathbb{R}^n)} < \infty \text{ for all } N \in \mathbb{N} \text{ and } \beta \in \mathbb{N}_0^n \right\},$$

where $\langle x \rangle = (1+|x|^2)^{1/2}$, equipped with the topology induced by the seminorms $\left\| \langle \cdot \rangle^N \partial^\beta \varphi \right\|_{L^\infty(\mathbb{R}^n)}$. The continuous dual of $\mathscr{S}(\mathbb{R}^n)$ is denoted by $\mathscr{S}'(\mathbb{R}^n)$ and its elements are called tempered distributions. We have the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. The Fourier transform of $u \in L^1(\mathbb{R}^n)$ is defined as

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \mathrm{d}x$$

and it is an isomorphism $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. By duality the Fourier transform is also an isomorphism $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. By density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ the Fourier transform can be extended to an isomorphism $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The following subset of Schwartz space

$$\mathscr{S}_0(\mathbb{R}^n) = \{ \varphi \in \mathscr{S}(\mathbb{R}^n) : \hat{\varphi}|_{B(0,\epsilon)} = 0 \text{ for some } \epsilon > 0 \}$$

is used to define fractional Laplacians on homogeneous Sobolev spaces.

Finally, we denote by $\mathscr{O}_C'(\mathbb{R}^n)$ the space of rapidly decreasing distributions. One has that $T \in \mathscr{O}_C'(\mathbb{R}^n)$ if and only if for any $N \in \mathbb{N}$ there exist $M(N) \in \mathbb{N}$ and continuous functions g_β such that

$$T = \sum_{|\beta| \le M(N)} \partial^\beta g_\beta \,,$$

where $\langle \cdot \rangle^N g_\beta$ is a bounded function for every $|\beta| \le M(N)$. Alternatively one can characterize $\mathscr{O}_C'(\mathbb{R}^n)$ via the Fourier transform: it holds that $\mathcal{F}\colon \mathscr{O}_C'(\mathbb{R}^n) \to \mathscr{O}_M(\mathbb{R}^n)$ is a bijective map where $\mathscr{O}_M(\mathbb{R}^n)$ is the space of smooth functions with polynomially bounded derivatives of all orders. We have the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. For example $C_\infty(\mathbb{R}^n) \subset \mathscr{O}_C'(\mathbb{R}^n)$, where $f \in C_\infty(\mathbb{R}^n)$ if and only if f is continuous and $\langle \cdot \rangle^N f$ is bounded for every $N \in \mathbb{N}$. The convolution formula for the Fourier transform $\widehat{f * g} = \hat{f}\hat{g}$ holds in the sense of distributions when $f \in \mathscr{O}_C'(\mathbb{R}^n)$ and $g \in \mathscr{S}'(\mathbb{R}^n)$. For more details on distributions, see the classic books [38, 39, 82].

2.2. **Fractional Laplacian on Sobolev spaces.** Let $r \in \mathbb{R}$. We define the inhomogeneous fractional L^2 Sobolev space of order r to be the set

$$H^r(\mathbb{R}^n) = \{ u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \in L^2(\mathbb{R}^n) \}$$

equipped with the norm

$$\|u\|_{H^r(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \right\|_{L^2(\mathbb{R}^n)} .$$

The spaces $H^r(\mathbb{R}^n)$ are Hilbert spaces for all $r \in \mathbb{R}$. It follows that both $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}_0(\mathbb{R}^n)$ are dense in $H^r(\mathbb{R}^n)$ for all $r \in \mathbb{R}$. Note that

$$\mathscr{O}_C'(\mathbb{R}^n) \subset \bigcup_{r \in \mathbb{R}} H^r(\mathbb{R}^n).$$

If $s \in (0,1)$, the fractional Laplacian can be defined in several equivalent ways [46]. We will take the Fourier transform approach which allows us to define it as a continuous map on Sobolev spaces for all $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. Define the fractional Laplacian of order $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ as $(-\Delta)^s \varphi = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{\varphi})$ for $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then $(-\Delta)^s \colon \mathscr{S}(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is linear and continuous with respect to the norm $\|\cdot\|_{H^r(\mathbb{R}^n)}$ by a simple calculation. Thus we can uniquely extend it to a continuous linear operator $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ as $(-\Delta)^s u = \lim_{k\to\infty} (-\Delta)^s \varphi_k$, where $\varphi_k \in \mathscr{S}(\mathbb{R}^n)$ is such that $\varphi_k \to u$ in $H^r(\mathbb{R}^n)$.

On the other hand, if $s > -n/4$, one can always define $(-\Delta)^s u$ for $u \in H^r(\mathbb{R}^n)$ as the tempered distribution $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{u})$, note that we also allow integer values of s here. This can be seen in the following way: let $\varphi_k \in \mathscr{S}(\mathbb{R}^n)$ such that $\varphi_k \to 0$ in $\mathscr{S}(\mathbb{R}^n)$. It holds that

$|\cdot|^{-\beta} \in L^1_{loc}(\mathbb{R}^n)$ if and only if $\beta < n$. Taking $N \in \mathbb{N}$ large enough and using Cauchy-Schwartz we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^{2s} |\hat{u}(x)| |\varphi_k(x)| \,\mathrm{d}x &\leq \left(\int_{\mathbb{R}^n} \langle x \rangle^{2r} |\hat{u}(x)|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^n} |x|^{4s} \langle x \rangle^{-2r} |\varphi_k(x)|^2 \,\mathrm{d}x \right)^{1/2} \\ &\leq C \left(\int_{\mathbb{R}^n} \frac{|x|^{4s}}{\langle x \rangle^{2N}} \mathrm{d}x \right)^{1/2} \left\| \langle \cdot \rangle^{N-r} \varphi_k \right\|_{L^\infty(\mathbb{R}^n)} \to 0.\end{aligned}$$

Hence $|\cdot|^{2s} \hat{u} \in \mathscr{S}'(\mathbb{R}^n)$ and also $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{u}) \in \mathscr{S}'(\mathbb{R}^n)$. The definition can be relaxed to $s > -n/2$ if we assume that $\langle \cdot \rangle^t \hat{u} \in L^\infty(\mathbb{R}^n)$ for some $t \in \mathbb{R}$. This holds for example if $u \in \mathscr{O}'_C(\mathbb{R}^n)$ or $u \in H^{r,1}(\mathbb{R}^n)$ (see the definition of Bessel potential spaces below). When $s \geq 0$, we again obtain that $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ is continuous. It follows from the properties of the Fourier transform that $(-\Delta)^k(-\Delta)^s = (-\Delta)^{k+s}$ when $s > -n/2$ and $k \in \mathbb{N}$. This relation will be used many times.

Fractional Laplacians with negative powers s have a connection to Riesz potentials. Let $\alpha \in \mathbb{R}$ such that $0 < \alpha < n$. We define the Riesz potential $I_\alpha \colon \mathscr{O}'_C(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ as $I_\alpha f = f * h_\alpha$, where the kernel is $h_\alpha(x) = |x|^{-\alpha}$. It follows that I_α is continuous in the distributional sense and $I_\alpha = (-\Delta)^{-s}$, up to a constant factor, where $s = (n - \alpha)/2$. On the other hand, if $-n/2 < s < 0$, then one can write $(-\Delta)^s f = f * |\cdot|^{-2s-n} = I_{2s+n} f$, also up to a constant factor. Hence fractional Laplacians with negative powers correspond to Riesz potentials and vice versa.

Following [4], one can define fractional Laplacians and Riesz potentials on homogeneous Sobolev spaces. Let us define

$$\dot{H}^r(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \hat{u} \in L^1_{loc}(\mathbb{R}^n) \text{ and } |\cdot|^r \hat{u} \in L^2(\mathbb{R}^n)\}$$

and equip it with the norm

$$\|u\|_{\dot{H}^r(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \right)^{1/2}.$$

The norm $\|u\|_{\dot{H}^r(\mathbb{R}^n)}$ is homogeneous with respect to scaling $\xi \to \lambda\xi$ in contrast to the norm $\|u\|_{H^r(\mathbb{R}^n)}$. We have the inclusions $\dot{H}^r(\mathbb{R}^n) \subsetneq H^r(\mathbb{R}^n)$ for $r < 0$ and $H^r(\mathbb{R}^n) \subsetneq \dot{H}^r(\mathbb{R}^n)$ for $r > 0$. If $r < n/2$, then $\dot{H}^r(\mathbb{R}^n)$ is a Hilbert space and $\mathscr{S}_0(\mathbb{R}^n)$ is dense in $\dot{H}^r(\mathbb{R}^n)$. Let $s \geq 0$ and define $(-\Delta)^s \varphi = \mathcal{F}^{-1}(|\cdot|^{2s} \hat{\varphi})$ for $\varphi \in \mathscr{S}_0(\mathbb{R}^n)$. Then $(-\Delta)^s \colon \mathscr{S}_0(\mathbb{R}^n) \to \dot{H}^{r-2s}(\mathbb{R}^n)$ is an isometry with respect to the norm $\|\cdot\|_{\dot{H}^r(\mathbb{R}^n)}$ and by density can be extended to a continuous map $(-\Delta)^s \colon \dot{H}^r(\mathbb{R}^n) \to \dot{H}^{r-2s}(\mathbb{R}^n)$ when $r < n/2$. Similarly one obtains that $I_\alpha \colon \dot{H}^r(\mathbb{R}^n) \to \dot{H}^{r+n-\alpha}(\mathbb{R}^n)$ is a continuous map for $r < \alpha - n/2$ and corresponds to fractional Laplacians with negative powers, up to a constant factor.

The fractional Laplacian can also be defined on Bessel potential spaces. Let $1 \leq p < \infty$. We define

$$H^{r,p}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \in L^p(\mathbb{R}^n)\}$$

and equip it with the norm

$$\|u\|_{H^{r,p}(\mathbb{R}^n)} = \left\| \mathcal{F}^{-1}(\langle \cdot \rangle^r \hat{u}) \right\|_{L^p(\mathbb{R}^n)}.$$

It follows that $H^{r,p}(\mathbb{R}^n)$ is a Banach space and $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^{r,p}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$. By the Mikhlin multiplier theorem, one obtains that the operator $(-\Delta)^s \colon H^{r,p}(\mathbb{R}^n) \to H^{r-2s,p}(\mathbb{R}^n)$ is continuous for $s \geq 0$ and $1 < p < \infty$. The fractional Laplacian is also defined in the space $H^{r,1}(\mathbb{R}^n)$ since $H^{r,1}(\mathbb{R}^n) \hookrightarrow H^{\frac{2r-n-\epsilon}{2}}(\mathbb{R}^n)$ for any $\epsilon > 0$ by the continuity of the Fourier transform $\mathcal{F} \colon L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

One can define fractional Laplacians on more general spaces. It follows that if $s \in (-n/2, 1]$, then $(-\Delta)^s \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}_s(\mathbb{R}^n)$ is continuous where $\mathscr{S}_s(\mathbb{R}^n)$ is the set

$$\mathscr{S}_s(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \langle \cdot \rangle^{n+2s} \partial^\beta \varphi \in L^\infty(\mathbb{R}^n) \text{ for all } \beta \in \mathbb{N}_0^n\}$$

equipped with the topology induced by the seminorms $\left\|\langle\cdot\rangle^{n+2s}\partial^\beta\varphi\right\|_{L^\infty(\mathbb{R}^n)}$. One can then extend $(-\Delta)^s$ by duality to a continuous map $(-\Delta)^s\colon (\mathscr{S}_s(\mathbb{R}^n))^* \to \mathscr{S}'(\mathbb{R}^n)$. See [28, 80] for more details and a characterization of the dual $(\mathscr{S}_s(\mathbb{R}^n))^*$.

2.3. **Trace spaces and singular potentials.** Let U, $F \subset \mathbb{R}^n$ be an open and a closed set. We define the following Sobolev spaces

$$\begin{aligned} H^r(U) &= \{u|_U : u \in H^r(\mathbb{R}^n)\} \\ \widetilde{H}^r(U) &= \text{ closure of } C_c^\infty(U) \text{ in } H^r(\mathbb{R}^n) \\ H_0^r(U) &= \text{ closure of } C_c^\infty(U) \text{ in } H^r(U) \\ H_F^r(\mathbb{R}^n) &= \{u \in H^r(\mathbb{R}^n) : \operatorname{spt}(u) \subset F\}. \end{aligned}$$

It is obvious that $\widetilde{H}^r(U) \subset H^r_{\overline{U}}(\mathbb{R}^n)$ and $\widetilde{H}^r(U) \subset H^r_0(U)$. In nonlocal problems, we impose exterior values for the equation instead of boundary values. Therefore exterior values are considered to be the same if their difference is in the space $\widetilde{H}^r(U)$. For example, in equation (3) the condition $u|_{\Omega_e} = f$ means that $u - f \in \widetilde{H}^s(\Omega)$, i.e. u and f are equal outside $\overline{\Omega}$, where Ω is bounded open set. This motivates the definition of the abstract trace space $X = H^r(\mathbb{R}^n)/\widetilde{H}^r(\Omega)$ which identifies functions in Ω_e. If Ω is a Lipschitz domain, then we have $H^r_0(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$ when $r > -1/2$, $r \notin \{1/2, 3/2, \dots\}$, $\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$, $X = H^r(\Omega_e)$ and $X^* = H^{-r}_{\overline{\Omega}_e}(\mathbb{R}^n)$. Thus for more regular domains it could be more convenient to work with the spaces $H^r_{\overline{\Omega}}(\mathbb{R}^n)$, but in this article we do not assume any regularity of the set Ω. For more theory of Sobolev spaces on (non-Lipschitz) domains and their properties, see [12, 57].

We also use some properties of singular potentials which were introduced in [75]. Let $t \geq 0$ and define $Z^{-t}(U)$ as a subspace of distributions $\mathcal{D}'(U)$ equipped with the norm

$$\|f\|_{Z^{-t}(U)} = \sup\{|\langle f, u_1u_2\rangle_U| : u_i \in C_c^\infty(U),\ \|u_i\|_{H^t(\mathbb{R}^n)} = 1\}\ ,$$

where $\langle\cdot,\cdot\rangle_U$ is the dual pairing. We denote by $Z_0^{-t}(U)$ the closure of $C_c^\infty(U)$ in $Z^{-t}(U)$. Elements in $Z^{-t}(\mathbb{R}^n)$ can be seen as multipliers: every $f \in Z^{-t}(\mathbb{R}^n)$ induces a map $m_f\colon H^t(\mathbb{R}^n) \to H^{-t}(\mathbb{R}^n)$ defined as $\langle m_f(u), v\rangle_{\mathbb{R}^n} = \langle f, uv\rangle_{\mathbb{R}^n}$. Also $|\langle f, uv\rangle_{\mathbb{R}^n}| \leq \|f\|_{Z^{-t}(\mathbb{R}^n)}\|u\|_{H^t(\mathbb{R}^n)}\|v\|_{H^t(\mathbb{R}^n)}$, and this inequality can be seen as a motivation for the definition of the space $Z^{-t}(\mathbb{R}^n)$. Clearly we have $Z_0^{-t}(\mathbb{R}^n) \subset Z^{-t}(\mathbb{R}^n)$. If U is bounded, then $L^{\frac{n}{2t}}(U) \subset Z_0^{-t}(\mathbb{R}^n)$ for $0 < t < n/2$ and $L^\infty(U) \subset Z_0^{-t}(\mathbb{R}^n)$ in the sense of zero extensions. Further, it holds that $L^p(U) \subset Z_0^{-t}(\mathbb{R}^n)$ when $p > \max\{1, n/2t\}$ (see section 6). We will only need these basic inclusions. For a more detailed treatment of the space of singular potentials $Z^{-t}(U)$, see [55, 75].

3. Unique continuation property and Poincaré inequality

3.1. **Unique continuation results.** In this section, we prove theorem 1.2 and give several other unique continuation results for fractional Laplacians and Riesz potentials in inhomogeneous and homogeneous Sobolev spaces. Even though we do not need all the results to solve the inverse problems considered in this article, we still state those variants since they are not given in earlier literature to the best of our knowledge. The strategy to prove results in this chapter is straightforward: if something is true for $(-\Delta)^s$ when $s \in (0, 1)$, then by the splitting $(-\Delta)^s = (-\Delta)^k(-\Delta)^{s-k}$ it should also be true for all powers s whenever the operations and claims are meaningful.

First we need a basic lemma for polyharmonic distributions, i.e. distributions which satisfy $(-\Delta)^k g = 0$ for some integer $k \in \mathbb{N}$. We sketch the proof since it reflects the method of reduction we repeatedly use in this section.

Lemma 3.1. *Let $V \subset \mathbb{R}^n$ be any nonempty open set. If $g \in \mathcal{D}'(\mathbb{R}^n)$ satisfies $(-\Delta)^k g = 0$ and $g|_V = 0$ for some $k \in \mathbb{N}$, then $g = 0$.*

Proof. The proof is by induction. The case $k = 1$ is true since harmonic distributions are harmonic functions and therefore analytic [59]. Assume that the lemma holds for some $k =$

$m \in \mathbb{N}$. If $(-\Delta)^{m+1}g = 0$ and $g|_V = 0$, then $(-\Delta)^m((-\Delta)g) = 0$ and $(-\Delta)g|_V = 0$ since $(-\Delta)$ is a local operator. The induction assumption implies $(-\Delta)g = 0$, and since also $g|_V = 0$, we obtain $g = 0$ by harmonicity. This implies the claim. Alternatively one could use the fact that polyharmonic distributions are analytic [59, Theorem 7.30]. □

Now we can prove theorem 1.2. The idea is to reduce the general case back to the one where $s \in (0,1)$ and use the UCP proved in [28]. Note that the corresponding UCP cannot hold for local operators such as $(-\Delta)^k$ when $k \in \mathbb{N}$. Therefore we have to assume that $s \in \mathbb{R} \setminus \mathbb{Z}$. For the proof of the case $s \in (0,1)$, see [28, Theorem 1.2].

Proof of theorem 1.2. Because of our assumptions for u, the fractional Laplacian $(-\Delta)^s u$ for $s \in (-n/2, \infty) \setminus \mathbb{Z}$ is well-defined, see section 2.2. Assume that $k-1 < s < k$ for some $k \in \mathbb{N}$. Now we can split $(-\Delta)^s u = (-\Delta)^{s-(k-1)}((-\Delta)^{k-1}u)$ where $s-(k-1) \in (0,1)$. Since the operator $(-\Delta)^{k-1}$ is local, we obtain $(-\Delta)^{s-(k-1)}((-\Delta)^{k-1}u)|_V = 0$ and $(-\Delta)^{k-1}u|_V = 0$ where $(-\Delta)^{k-1}u \in H^{r-2(k-1)}(\mathbb{R}^n)$. By the UCP of $(-\Delta)^{s-(k-1)}$, we have $(-\Delta)^{k-1}u = 0$. Since u is polyharmonic and $u|_V = 0$, lemma 3.1 implies $u = 0$.

If $-n/2 < s < 0$, $s \notin \mathbb{Z}$, choose $k \in \mathbb{N}$ such that $k + s > 0$. Then by the locality of $(-\Delta)^k$ we obtain $(-\Delta)^{k+s}u|_V = 0$ and $u|_V = 0$. The first part of the proof implies the claim. □

Note that theorem 1.2 implies UCP for equations of the type $(-\Delta)^s u + Lu = 0$ where L is any local operator. Especially, this holds if $L = P(x,D)$ where

$$P(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

is a differential operator of order m.

The following unique continuation result of Riesz potentials was presented in [41]. We use it to show uniqueness for partial data problems of the d-plane transform in section 4. We recall the short proof since it relies on the UCP of the fractional Laplacian.

Corollary 3.2. *Let $\alpha \in \mathbb{R}$ such that $0 < \alpha < n$ and $(\alpha - n)/2 \in \mathbb{R} \setminus \mathbb{Z}$. Let $f \in \mathscr{O}_C'(\mathbb{R}^n)$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $I_\alpha f|_V = 0$ and $f|_V = 0$, then $f = 0$.*

Proof. Recall that $f \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. We can write $I_\alpha f = (-\Delta)^{-s} f$ where $s = (n-\alpha)/2$. Choose $k \in \mathbb{N}$ such that $k - s > 0$. By locality of $(-\Delta)^k$ we obtain the conditions $(-\Delta)^{k-s} f|_V = 0$ and $f|_V = 0$. Theorem 1.2 implies $f = 0$. □

It is also independently proved in [41], without using the UCP of $(-\Delta)^s$, that if $f \in \mathcal{E}'(\mathbb{R}^n)$, then one can replace the condition $I_\alpha f|_V = 0$ by the requirement $\partial^\beta (I_\alpha f)(x_0) = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}_0^n$. In fact, this can be used to prove a slightly stronger result for $(-\Delta)^s$ in the case of compact support.

Corollary 3.3. *Let $u \in \mathcal{E}'(\mathbb{R}^n)$, $V \subset \mathbb{R}^n$ some nonempty open set and $s \in (-n/2, \infty) \setminus \mathbb{Z}$. If $\partial^\beta((-\Delta)^s u)(x_0) = 0$ and $u|_V = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}_0^n$, then $u = 0$.*

Proof. Let $k-1 < s < k$ where $k \in \mathbb{N}$. Now $(-\Delta)^s = (-\Delta)^k(-\Delta)^{s-k} = (-\Delta)^k I_\alpha$ where $\alpha = n + 2s - 2k \in (n-2, n)$. Furthermore, $\partial^\beta(-\Delta)^s u = \partial^\beta I_\alpha (-\Delta)^k u$ since the Riesz potential commutes with derivatives. By the locality of $(-\Delta)^k$ we obtain the conditions $\partial^\beta(I_\alpha(-\Delta)^k u)(x_0) = 0$ and $(-\Delta)^k u|_V = 0$ where $(-\Delta)^k u \in \mathcal{E}'(\mathbb{R}^n)$. By [41, Theorem 1.1], we must have $(-\Delta)^k u = 0$. Since also $u|_V = 0$, we obtain $u = 0$ by lemma 3.1.

Let then $s \in (-n/2, 0)$, $s \notin \mathbb{Z}$, and pick $k \in \mathbb{N}$ such that $s + k > 0$. All the derivatives $\partial^\beta((-\Delta)^s u)(x_0)$ vanish, and hence $((-\Delta)^k \partial^\beta)((-\Delta)^s u)(x_0) = 0$. Now $((-\Delta)^k \partial^\beta)((-\Delta)^s u) = \partial^\beta((-\Delta)^{s+k} u)$ and we get the conditions $\partial^\beta((-\Delta)^{s+k}u)(x_0) = 0$ and $u|_V = 0$. The first part of the proof gives the claim. □

The UCP of $(-\Delta)^s$ also extends to homogeneous Sobolev spaces. The following result is a simple consequence of theorem 1.2. See [22, 23] for related results (strong UCP and measurable UCP in some special cases).

Corollary 3.4. *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $u \in \dot{H}^r(\mathbb{R}^n)$, $r < n/2$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$.*

Proof. If $r < 0$, then $u \in H^r(\mathbb{R}^n)$ and the claim follows from theorem 1.2. Let $r > 0$ and choose $k \in \mathbb{N}$ such that $r - 2k < 0$. Now $(-\Delta)^k(-\Delta)^s = (-\Delta)^s(-\Delta)^k$ holds in $\mathscr{S}_0(\mathbb{R}^n)$ so by the density of $\mathscr{S}_0(\mathbb{R}^n)$ and the locality of $(-\Delta)^k$ we obtain $(-\Delta)^s((-\Delta)^k u)|_V = 0$ and $(-\Delta)^k u|_V = 0$, where $(-\Delta)^k u \in \dot{H}^{r-2k}(\mathbb{R}^n) \subset H^{r-2k}(\mathbb{R}^n)$. Hence $(-\Delta)^k u = 0$ by theorem 1.2 and since $u|_V = 0$ we obtain $u = 0$ by lemma 3.1. □

Since $(-\Delta)^k(-\Delta)^{-s} = (-\Delta)^{k-s}$ also holds by the density of $\mathscr{S}_0(\mathbb{R}^n)$, one can reduce the case of negative exponents to the case of positive exponents. Thus one obtains the corresponding UCP for the Riesz potential I_α in $\dot{H}^r(\mathbb{R}^n)$ where $r < \alpha - n/2$. By the Sobolev embedding theorem we obtain the following unique continuation result for Bessel potential spaces when $1 \leq p \leq 2$.

Corollary 3.5. *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $1 \leq p \leq 2$ and $u \in H^{r,p}(\mathbb{R}^n)$, $r \in \mathbb{R}$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$.*

Proof. If $p = 1$, then $\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^1(\mathbb{R}^n)$ which implies $\langle\cdot\rangle^r \hat{u} \in L^\infty(\mathbb{R}^n)$ since $\mathcal{F}\colon L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is continuous. Hence $u \in H^t(\mathbb{R}^n)$ for some $t \in \mathbb{R}$ and the claim follows from theorem 1.2. Let then $1 < p \leq 2$. By the Sobolev embedding theorem (see e.g. [6, Theorem 6.5.1]) $H^{r,p}(\mathbb{R}^n) \hookrightarrow H^{r_1,p_1}(\mathbb{R}^n)$ when $r_1 \leq r$, $1 < p \leq p_1 < \infty$ and

$$r - \frac{n}{p} = r_1 - \frac{n}{p_1}.$$

Choose $p_1 = 2$. Then for any $1 < p \leq 2$ the previous equality holds when

$$r_1 = \frac{2rp + n(p-2)}{2p} \leq r.$$

Hence $u \in H^{r_1,2}(\mathbb{R}^n) = H^{r_1}(\mathbb{R}^n)$ and by theorem 1.2 we obtain $u = 0$. □

For higher exponents p, we can prove the following version of unique continuation considering the Fourier transform.

Corollary 3.6. *Let $r \geq 0$, $2 \leq p < \infty$ and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. Let $u \in H^{r,p}(\mathbb{R}^n)$ and $V \subset \mathbb{R}^n$ some nonempty open set. If $(-\Delta)^s \hat{u}|_V = 0$ and $\hat{u}|_V = 0$, then $u = 0$.*

Proof. By the inclusion $H^{r,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $r \geq 0$, we can assume $u \in L^p(\mathbb{R}^n)$. If $p = 2$, then $\hat{u} \in L^2(\mathbb{R}^n)$. By theorem 1.2, we obtain $\hat{u} = 0$ and hence $u = 0$. If $2 < p < \infty$, then we have that $\hat{u} \in H^{-t}(\mathbb{R}^n)$ where $t > n(1/2 - 1/p)$ by [38, Theorem 7.9.3]. Again we obtain $\hat{u} = 0$ by theorem 1.2 and eventually $u = 0$. □

Note that if u has compact support, then by the Paley-Wiener theorem the condition $\hat{u}|_V = 0$ already implies that $u = 0$.

3.2. **The fractional Poincaré inequality.** This subsection is dedicated to the proofs of a fractional Poincaré inequality. It serves the goal of estimating the L^2-norm of $u \in \widetilde{H}^s(\Omega)$ with that of its fractional Laplacian $(-\Delta)^{s/2}u$. We give five possible proofs for the fractional Poincaré inequality. We believe that giving several proofs will be helpful in subsequent works. This also illustrates some connections between methods which might have been unnoticed before.

The first proof is the most direct one and is based on splitting of frequencies on the Fourier side. The second proof utilizes several estimates (most importantly Hardy-Littlewood-Sobolev inequalities). This proof is motivated by the approach taken in [28]. Third proof uses a reduction argument to extend the inequality proved in [11] for all powers $s \geq 0$. Fourth proof is based on interpolation of homogeneous Sobolev spaces and it also gives an explicit constant in terms of the classical Poincaré constant. Fifth proof uses uncertainty inequalities which are treated in [24].

We begin our first proof by dividing the Fourier side into high and low frequencies. We only use simple estimates in the proof. In this approach we also get a control on the Poincaré constant. The result is basically the same as [4, Proposition 1.55].

Theorem 3.7 (Poincaré inequality)**.** *Let $s \geq 0$, $K \subset \mathbb{R}^n$ compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $c = c(n, K, s) > 0$ such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}.$$

Proof. We divide the integration into high and low frequencies

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \int_{|\xi|\leq\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi + \int_{|\xi|>\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi$$

where $\epsilon > 0$ is determined later on. Let us analyze the first part. Since $u \in L^2(\mathbb{R}^n)$ and has support in K, Hölder's inequality implies

$$|\hat{u}(\xi)| \leq \|u\|_{L^1(\mathbb{R}^n)} \leq |K|^{1/2} \|u\|_{L^2(\mathbb{R}^n)}.$$

Thus we have

$$\int_{|\xi|\leq\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \leq \int_{|\xi|\leq\epsilon} |K| \|u\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}\xi = \epsilon^n |K| |B(0,1)| \|u\|^2_{L^2(\mathbb{R}^n)}$$

where $|K|$ and $|B(0,1)|$ are the measures of K and the unit ball $B(0,1)$. For high frequencies we can do the following trick

$$\int_{|\xi|>\epsilon} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi = \int_{|\xi|>\epsilon} \frac{|\xi|^{2s} |\hat{u}(\xi)|^2}{|\xi|^{2s}} \mathrm{d}\xi \leq \epsilon^{-2s} \left\|(-\Delta)^{s/2} u\right\|^2_{L^2(\mathbb{R}^n)}.$$

Now choose $0 < \epsilon < (|K| |B(0,1)|)^{-1/n}$. Then one obtains the inequality

$$\|u\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon^{-s}}{\sqrt{1 - \epsilon^n |K| |B(0,1)|}} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}. \quad \square$$

Remark 3.8. *Choosing $\epsilon = (2 |K| |B(0,1)|)^{-1/n}$ one obtains the following inequality in theorem 3.7*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq \sqrt{2}(2 |K| |B(0,1)|)^{s/n} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}.$$

If K is a ball, the constant in this inequality has the same scaling with respect to the diameter of the set as in theorem 3.17, i.e. $c \approx (diam(K))^s$. Further, one can use similar method of proof as in theorem 3.7 to show Poincaré inequalities for more general pseudodifferential operators on certain manifolds. See [84] *for details.*

Provided we have the Poincaré inequality, we can prove the generalized version of it. See also [4, Corollary 1.56] for a similar inequality when K is a ball. In that case one can take $\tilde{c} \approx (\mathrm{diam}(K))^{s-t}$. The cases $s \geq t \geq 1$ and $s \geq 1 \geq t \geq 0$ are also proved for $u \in \widetilde{H}^s(\Omega)$ in theorem 3.17.

Proof of theorem 1.5. Since $s \geq t \geq 0$ we have the continuous embeddings $H^t(\mathbb{R}^n) \hookrightarrow \dot{H}^t(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$. Using the Poincaré inequality in theorem 3.7 we obtain

$$\begin{aligned}
\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} &= \|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq \|u\|_{H^t(\mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)} \leq 2^{\frac{s+1}{2}} \Big(\|u\|_{L^2(\mathbb{R}^n)} + \|u\|_{\dot{H}^s(\mathbb{R}^n)} \Big) \\
&\leq 2^{\frac{s+1}{2}} \left(c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)} \right) \\
&= \tilde{c} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

where the constants c and $\tilde{c}$ do not depend on u. In the fourth step we used the elementary inequality $(a + b)^r \leq 2^r (a^r + b^r)$ for $a, b \geq 0$. This concludes the proof. $\square$

We then start preparation for our second proof by stating some known lemmas:

- lemma 3.9 is the continuity of Riesz potentials,
- lemma 3.10 is the L^2 boundedness of inverse of elliptic second order operators,
- lemma 3.11 is a convolution L^p estimate from below by an inhomogeneous Hölder norm,
- lemma 3.12 is a specific form of the Poincaré inequality for fractional Laplacians, and
- lemma 3.13 is a simple commutation property for the gradient and a Fourier multiplier.

Lemma 3.9 (Theorem 4.5.3 in [38])**.** *Let $t \geq 0$, $1 < p < \infty$ be such that $n > tp$, and define $q = \frac{np}{n-tp}$. Then the Riesz potential $(-\Delta)^{-t/2} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous.*

Lemma 3.10 (Section 6 in [21])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in L^2(\Omega)$. If $w \in H_0^1(\Omega)$ is the unique solution of the problem*

$$\begin{cases} (-\Delta)w & = f \quad in\,\Omega \\ w|_{\partial\Omega} & = 0 \end{cases},$$

then there exists a constant $C = C(\Omega)$ such that

$$\|w\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \,. \tag{5}$$

Lemma 3.11 (Theorem 4.5.10 in [38])**.** *Let $\psi \in C^1(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree $-n/a$, $p \in [1,\infty]$ and $\gamma = n(1 - 1/a - 1/p)$ be such that $\gamma \in (0,1)$. Then if $v \in L^p(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ we have*

$$\sup_{x \neq y} \left\{ \frac{|(\psi * v)(x) - (\psi * v)(y)|}{|x-y|^\gamma} \right\} \leq C\|v\|_{L^p(\mathbb{R}^n)},$$

where C does not depend on w.

Lemma 3.12 (Formula (1.3) in [64])**.** *Let $1 < p \leq q < \infty$ and $f \in W^{n/p,p}(\mathbb{R}^n)$. There is a constant $C = C(n,p)$ such that*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq Cq^{1-1/p}\|(-\Delta)^{n/2p}f\|_{L^p(\mathbb{R}^n)}^{1-p/q}\|f\|_{L^p(\mathbb{R}^n)}^{p/q} \,. \tag{6}$$

This estimate is proved using sharp Hardy-Littlewood-Sobolev inequalities.

Lemma 3.13. *Let $t \geq 0$ and $f \in H^{1+2t}(\mathbb{R}^n)$. Then $[\nabla, (-\Delta)^t]f = 0$, that is, the gradient and the fractional Laplacian of exponent t commute.*

Proof. The proof is just a trivial computation with Fourier symbols:

$$\mathcal{F}(\nabla(-\Delta)^t f) = i\xi|\xi|^{2t}\hat{f}(\xi) = |\xi|^{2t}i\xi\hat{f}(\xi) = \mathcal{F}((-\Delta)^t(\nabla f)) \,.$$

□

We are now ready to state and prove the fractional Poincaré inequality.

Theorem 3.14 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $s \in [0,\infty)$ and $u \in \widetilde{H}^s(\Omega)$. There exists a constant $c = c(n,\Omega,s)$ such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)} \,.$$

Proof. In the inequalities the constants (usually denoted by c, C, etc.) do not depend on the function which is being estimated and can change from line to line. We let the symbol $s' = s - \lfloor s \rfloor$ indicate the fractional part of the exponent s, with the convention that $s' \in [0,1)$. First observe that by using lemma 3.9 with $p = 2$ and Hölder's inequality we get the following useful estimate

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_\Omega\|u\|_{L^q(\mathbb{R}^n)} \leq c\|(-\Delta)^{t/2}u\|_{L^2(\mathbb{R}^n)} \tag{7}$$

when $u \in \widetilde{H}^t(\Omega)$ where q and t are as in lemma 3.9. Our proof is divided in several cases.

Case 1: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' = 0$.

Recall that $\widetilde{H}^{2h}(\Omega) \subset H_0^{2h}(\Omega)$. We show that if $u \in H_0^{2h}(\Omega)$ and $h \in \mathbb{N}$ then there exists a constant $c = c(n,\Omega,h)$ such that

$$\|(-\Delta)^h u\|_{L^2(\mathbb{R}^n)} \geq c\|u\|_{L^2(\mathbb{R}^n)} \,. \tag{8}$$

The estimate (8) holds trivially if $h = 0$, while if $h = 1$ then (8) follows from the boundedness of the inverse lemma 3.10. Assume now that $h \geq 2$, and by induction that (8) holds for $h-1$. Then $(-\Delta)u \in H_0^{2h-2}(\Omega)$, so we can apply (8) and (5) to get

$$\|(-\Delta)^h u\|_{L^2(\mathbb{R}^n)} = \|(-\Delta)^{h-1}(-\Delta u)\|_{L^2(\mathbb{R}^n)} \geq c\|(-\Delta)u\|_{L^2(\mathbb{R}^n)} \geq c'\|u\|_{L^2(\mathbb{R}^n)} \, .$$

In the next steps we consider $s \notin \mathbb{N}$.

Case 2: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' \in (0, 1/2)$ or $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' \in [1/2, 1)$, $n \geq 2$.
Now it holds that $n > 2s'$, and there exists $k \in \mathbb{N}$ such that $s \in (2k, 2k+1)$ and we can write $(-\Delta)^{s/2}u = (-\Delta)^{s'/2}(-\Delta)^k u$. Since $(-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega) = \widetilde{H}^{s'}(\Omega)$, we can apply formula (7)

$$\|(-\Delta)^k u\|_{L^2(\mathbb{R}^n)} \leq c\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)} \, .$$

Since $u \in H_0^s(\Omega) \subset H_0^{2k}(\Omega)$, we can get the result using formula (8).

Case 3: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' \in (1/2, 1)$, $n = 1$.
As in the second case, there exists $k \in \mathbb{N}$ such that $s \in (2k, 2k+1)$ and we can write $(-\Delta)^{s/2}u = (-\Delta)^{s'/2}(-\Delta)^k u$. However, since now $n < 2s'$, we cannot directly use formula (7).
Assume first that $w \in C_c^\infty(\Omega)$. Then we can take $y_0 \in \Omega$ such that $w(y_0) = 0$ and $x_0 \in \Omega$ such that $w(x_0) = \|w\|_{L^\infty(\Omega)}$. With these choices and for any $\gamma > 0$ we can write

$$\|w\|_{L^2(\mathbb{R}^n)} \leq C\|w\|_{L^\infty(\Omega)} \leq C\frac{w(x_0) - w(y_0)}{|x_0 - y_0|^\gamma} \, . \tag{9}$$

We now let $\gamma = s' - n/2 = s' - 1/2 \in (0, 1/2)$, and define $\psi = |x|^{s'-1}$, $v = (-\Delta)^{s'/2}w$. By the mapping properties of the fractional Laplacian and the Mikhlin theorem, we can observe that $v \in L^p(\mathbb{R})$ for all $1 < p < \infty$ (see [1, Theorem 7.2]). Using the continuity of the Riesz potential in lemma 3.9, we see that for a constant $c = c(n, s)$ the following holds almost everywhere:

$$w = (-\Delta)^{-s'/2}((-\Delta)^{s'/2}w) = (-\Delta)^{-s'/2}v = cI_{1-s'}v = c|x|^{s'-1} * v = c(\psi * v) \, .$$

Let χ_R be the characteristic function of the ball B_R of radius $R > 0$, and define $w_R = c(\psi * (\chi_R v))$, with c as above. We see that

$$w_R(x) = c(\psi * (\chi_R v))(x) = c\int_{\mathbb{R}} \psi(x-y)\chi_R(y)v(y)dy \, ,$$

and the integrand is dominated by $|\psi(x-y)v(y)|$. This is an integrable function, since

$$\int_{\mathbb{R}} |\psi(x-y)v(y)|dy = \int_{\mathbb{R}} \psi(x-y)|v(y)|dy = I_{1-s'}(|v|)(x) \, ,$$

and the Riesz potential is well defined almost everywhere on $L^p(\mathbb{R})$ for any $1 < p < 1/s'$. Now the dominated convergence theorem gives that $w_R(x) \to w(x)$ as $R \to \infty$ for almost every fixed $x \in \mathbb{R}$.

Let $\epsilon > 0$ and $x_0', y_0' \in \mathbb{R}$ be such that $|x_0 - x_0'| < \epsilon$, $|y_0 - y_0'| < \epsilon$ and $w_R(x_0')$, $w_R(y_0')$ converge to $w(x_0')$, $w(y_0')$ as $R \to \infty$. Applying lemma 3.11 with $p = 2$, $n = 1$ and $a = 1 - s'$, we see that

$$\begin{aligned}
\frac{w_R(x_0') - w_R(y_0')}{|x_0 - y_0|^\gamma} &\leq \sup_{x \neq y}\left\{\frac{w_R(x) - w_R(y)}{|x-y|^\gamma}\right\} \\
&= c\sup_{x \neq y}\left\{\frac{(\psi * (\chi_R v))(x) - (\psi * (\chi_R v))(y)}{|x-y|^\gamma}\right\} \\
&\leq C\|\chi_R v\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})} = C\|(-\Delta)^{s'/2}w\|_{L^2(\mathbb{R})} \, .
\end{aligned}$$

We now first take the limit for $R \to \infty$ and then the one for $\epsilon \to 0$. By the smoothness of w, this gives

$$\frac{w(x_0) - w(y_0)}{|x_0 - y_0|^\gamma} \leq C\|(-\Delta)^{s'/2} w\|_{L^2(\mathbb{R})} \, . \tag{10}$$

Combining formulas (9) and (10) we get $\|w\|_{L^2(\mathbb{R}^n)} \leq C\|(-\Delta)^{s'/2} w\|_{L^2(\mathbb{R}^n)}$, and the same inequality holds for $w \in \widetilde{H}^{s'}(\Omega)$ by density. Let now $w := (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega) = \widetilde{H}^{s'}(\Omega)$. The result is then obtained applying formula (8).

Case 4: $\lfloor s \rfloor \in 2\mathbb{Z}$, $s' = 1/2$, $n = 1$.
Let $w := (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega) = \widetilde{H}^{s'}(\Omega)$. Here we make use of formula (6) with $p = 2$, $q = 3$ in order to estimate

$$\|w\|_{L^2(\mathbb{R}^n)} = \|w\|^3_{L^2(\mathbb{R}^n)} \|w\|^{-2}_{L^2(\mathbb{R}^n)} \leq \|w\|^3_{L^3(\mathbb{R}^n)} \|w\|^{-2}_{L^2(\mathbb{R}^n)} \leq C\|(-\Delta)^{n/4} w\|_{L^2(\mathbb{R}^n)} \, . \tag{11}$$

Since $n/4$ equals $s'/2$ for $n = 1$, the results follows from (11) and (8).

Case 5: $\lfloor s \rfloor \notin 2\mathbb{Z}$.
Let $u \in C_c^\infty(\Omega)$. In this case $s = s' + 2k + 1$ for some $k \in \mathbb{N}$, therefore we can calculate

$$\begin{aligned} \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)} &= \|(-\Delta)^{1/2}(-\Delta)^{(s'+2k)/2} u\|_{L^2(\mathbb{R}^n)} \\ &= \|\nabla(-\Delta)^{(s'+2k)/2} u\|_{L^2(\mathbb{R}^n)} \\ &= \|(-\Delta)^{(s'+2k)/2} \nabla u\|_{L^2(\mathbb{R}^n)} \\ &\geq C\|\nabla u\|_{L^2(\mathbb{R}^n)} \geq C\|u\|_{L^2(\mathbb{R}^n)} \, . \end{aligned} \tag{12}$$

The second equality in (12) is just an L^2 property of the gradient and the $(-\Delta)^{1/2}$ operator. The third equality in (12) follows from lemma 3.13. The first inequality in (12) follows from the even cases, given that $\lfloor s' + 2k \rfloor \in 2\mathbb{Z}$ and $\nabla u \in \widetilde{H}^{s'+2k}(\Omega)$ componentwise. The last inequality follows from the classical Poincaré inequality. The rest follows by approximation. □

Remark 3.15. *Third way to prove the Poincaré inequality is using the known result in the case $n \geq 1$ and $s \in (0,1)$* [11, Lemma 2.2]. *This result is proved using Caffarelli-Silvestre extension. Then one can use similar reduction argument to prove it for all $s \geq 0$ and $u \in C_c^\infty(\Omega)$. Namely, one shows using the classical Poincaré inequality that the claim holds for all $s \in [0,2)$. The higher order fractional cases are obtained by splitting the fractional Laplacian as $(-\Delta)^s = (-\Delta)^k(-\Delta)^{t/2}$ where $t \in (0,2)$. Boundedness of the inverse and the fractional Poincaré inequality for $t \in (0,2)$ imply the claim for fractional exponents. Integer order exponents are obtained from the boundedness of the inverse as before. The inequality for $u \in \widetilde{H}^s(\Omega)$ follows by approximation.*

For the fourth proof we use the following interpolation lemma of homogeneous Sobolev spaces which is a simple consequence of Hölder's inequality, see [4, Proposition 1.32].

Lemma 3.16. *Let $s_0 \leq r \leq s_1$ and $f \in \dot{H}^{s_0}(\mathbb{R}^n) \cap \dot{H}^{s_1}(\mathbb{R}^n)$. Then $f \in \dot{H}^r(\mathbb{R}^n)$ and*

$$\|f\|_{\dot{H}^r(\mathbb{R}^n)} \leq \|f\|^{1-\theta}_{\dot{H}^{s_0}(\mathbb{R}^n)} \|f\|^\theta_{\dot{H}^{s_1}(\mathbb{R}^n)} \, , \quad r = (1-\theta)s_0 + \theta s_1.$$

Using the interpolation lemma and the usual Poincaré inequality we can easily prove the following theorem. Note that we also obtain explicit constant from the proof.

Theorem 3.17 (Poincaré inequality)**.** *Let $s \geq t \geq 1$ or $s \geq 1 \geq t \geq 0$, $\Omega \subset \mathbb{R}^n$ bounded open set and $u \in \widetilde{H}^s(\Omega)$. The following inequality holds*

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} \leq C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}$$

where $C = C(n,\Omega)$ is the classical Poincaré constant.

Proof. Let $s \geq t \geq 1$ and $u \in C_c^\infty(\Omega)$. The usual Poincaré inequality can be written in terms of the homogeneous Sobolev norm as

$$\|u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} = C \|\nabla u\|_{L^2(\mathbb{R}^n)} = C \|u\|_{\dot{H}^1(\mathbb{R}^n)}$$

where $C = C(n, \Omega)$. We use the interpolation lemma 3.16 twice. First choose $s_0 = 0$, $r = 1$ and $s_1 = t \geq 1$. Now $\theta = 1/t$ and by the classical Poincaré inequality we obtain

$$\|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{\theta} \leq C^{1-\theta} \|u\|_{\dot{H}^1(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{\theta}.$$

From this we get the following inequality

$$\|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq C^{\frac{1-\theta}{\theta}} \|u\|_{\dot{H}^t(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\Omega)$. Hence

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C \|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq C^t \|u\|_{\dot{H}^t(\mathbb{R}^n)}.$$

Then choose $s_0 = 0$, $r = t$ and $s_1 = s \geq t$ in lemma 3.16. Now $\theta = t/s$ and by the previous inequality

$$\|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta} \leq C^{t(1-\theta)} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta}.$$

From this we obtain

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} \leq C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}$$

for $u \in C_c^\infty(\Omega)$.

Let then $s \geq 1 \geq t \geq 0$ and $u \in C_c^\infty(\Omega)$. First interpolate for $s \geq 1 \geq t$ to obtain

$$\|u\|_{\dot{H}^1(\mathbb{R}^n)} \leq \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta}, \quad \theta = \frac{1-t}{s-t}.$$

Second, interpolate for $1 \geq t \geq 0$ and use the previous inequality and the classical Poincaré inequality to get

$$\|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}^{1-\tilde{\theta}} \|u\|_{\dot{H}^1(\mathbb{R}^n)}^{\tilde{\theta}} \leq C^{1-\tilde{\theta}} \|u\|_{\dot{H}^t(\mathbb{R}^n)}^{1-\theta} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\theta}, \quad \tilde{\theta} = t,$$

which eventually implies the inequality

$$\left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} = \|u\|_{\dot{H}^t(\mathbb{R}^n)} \leq C^{s-t} \|u\|_{\dot{H}^s(\mathbb{R}^n)} = C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\Omega)$.

Then let $u \in \widetilde{H}^s(\Omega)$. By definition there is a sequence $\varphi_k \in C_c^\infty(\Omega)$ such that

$$\varphi_k \to u \quad \text{in } H^s(\mathbb{R}^n).$$

The continuity of $(-\Delta)^{t/2}$ implies that

$$(-\Delta)^{t/2} \varphi_k \to (-\Delta)^{t/2} u \quad \text{in } H^{s-t}(\mathbb{R}^n).$$

The embedding $H^{s-t}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous and thus

$$(-\Delta)^{t/2} \varphi_k \to (-\Delta)^{t/2} u \quad \text{in } L^2(\mathbb{R}^n).$$

By the continuity of the norm and $(-\Delta)^{s/2}$ we finally obtain

$$\begin{aligned} \left\|(-\Delta)^{t/2} u\right\|_{L^2(\mathbb{R}^n)} = \lim_k \left\|(-\Delta)^{t/2} \varphi_k\right\|_{L^2(\mathbb{R}^n)} &\leq C^{s-t} \lim_k \left\|(-\Delta)^{s/2} \varphi_k\right\|_{L^2(\mathbb{R}^n)} \\ &= C^{s-t} \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

□

We remark that the case $t = 0$ and $s = 1$ corresponds to the classical Poincaré inequality since $\|\nabla u\|_{L^2(\mathbb{R}^n)} = \left\|(-\Delta)^{1/2} u\right\|_{L^2(\mathbb{R}^n)}$. Also the constant C^{s-t} is the expected one. In the usual Poincaré inequality we take one derivative and the constant is C. In the higher order version we take t and s derivatives and the constant naturally becomes C^{s-t}. The constant C can be taken to be proportional to the diameter of the set, $C \approx \operatorname{diam}(\Omega)$.

Remark 3.18. *Fifth way to prove the Poincaré inequality is using uncertainty inequalities. If $u \in L^2(\mathbb{R}^n)$, then there is a constant $c = c(n,s)$ such that*

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leq c \, \||\cdot|^s u\|_{L^2(\mathbb{R}^n)} \, \||\cdot|^s \hat{u}\|_{L^2(\mathbb{R}^n)}, \tag{13}$$

see the discussion after theorem 4.1 in [24]*. We can interpret this inequality as*

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2}(\mathcal{F}^{-1}(u))\right\|_{L^2(\mathbb{R}^n)} \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}$$

whenever the terms on the right hand side of equation (13) *are finite. If u is supported in some fixed compact set K, then one obtains similar inequality as in theorem 3.7, i.e.*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c' \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)}$$

holds for all $u \in H^s_K(\mathbb{R}^n)$ and for some constant $c' = c'(n,K,s)$.

Remark 3.19. *The Poincaré inequality for the operator $(-\Delta)^{s/2}$ implies also Poincaré inequality for the fractional gradient $\nabla^s \colon H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s \rfloor + 1})$ which is defined as*

$$\nabla^s u(x,y) := \frac{\mathcal{C}^{1/2}_{n,s'}}{\sqrt{2}} \frac{\nabla^{\lfloor s \rfloor}u(x) - \nabla^{\lfloor s \rfloor}u(y)}{|y-x|^{n/2+s'+1}} \otimes (y-x),$$

see section 6 for more details. If $s \geq t \geq 0$ and $u \in C^\infty_c(\Omega)$, then

$$\left\|\nabla^t u\right\|_{L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s \rfloor+1})} = \left\|(-\Delta)^{t/2}u\right\|_{L^2(\mathbb{R}^n)} \leq \tilde{c} \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)} = \tilde{c} \left\|\nabla^s u\right\|_{L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s \rfloor+1})},$$

where the constant $\tilde{c}$ does not depend on u. By approximation and the continuity of ∇^s the previous inequality is also true for $u \in \widetilde{H}^s(\Omega)$.

4. Applications to integral geometry

In this section we discuss how the UCP of Riesz potentials can be used in partial data problems in integral geometry. We follow [36] for the treatment of the d-plane transform, theory of X-ray transform and Radon transform can also be found in [63, 68, 81]. Let $d \in \{1, \dots, n-1\}$ and denote by $\mathbf{P}^d$ the space of all d-dimensional affine planes in $\mathbb{R}^n$. We define the d-plane transform of a function f to be

$$R_d f(A) = \int_{x \in A} f(x) \mathrm{d}m(x)$$

where $A \in \mathbf{P}^d$ and m is the Hausdorff measure on A. The adjoint of R_d is defined as

$$R^*_d g(x) = \int_{A \ni x} g(A) \mathrm{d}\mu(A)$$

where g is a function on $\mathbf{P}^d$ and μ is the associated measure. These transforms are defined for functions such that the integrals exist. The case $d = 1$ corresponds to the usual X-ray transform and $d = n-1$ to the Radon transform. The normal operator of the d-plane transform $N_d = R^*_d R_d$ has an expression $N_d f = c_{n,d}(f * |\cdot|^{-(n-d)})$ where $c_{n,d}$ is a constant depending on n and d. The normal operator is well defined if f is a function that decreases rapidly enough at infinity [36]. This holds for example if $f \in C_\infty(\mathbb{R}^n)$ where $C_\infty(\mathbb{R}^n)$ is the space of continuous functions which decrease faster than any polynomial at infinity (see section 2.1 for a precise definition). Thus, up to a constant factor, N_d can be represented as a Riesz potential $N_d = I_\alpha = (-\Delta)^{-d/2}$ where $\alpha = n - d \in \{1, \dots, n-1\}$.

The transforms R_d and R^*_d can be extended to distributions by duality. Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $g \in \mathcal{D}'(\mathbb{R}^n)$. Since $R_d \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbf{P}^d)$ and $R^*_d \colon \mathcal{E}(\mathbf{P}^d) \to \mathcal{E}(\mathbb{R}^n)$ are continuous [32], we can define the following operations

$$\langle R_d f, \psi \rangle = \langle f, R^*_d \psi \rangle, \quad \psi \in \mathcal{E}(\mathbf{P}^d)$$
$$\langle R^*_d g, \varphi \rangle = \langle g, R_d \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Therefore $R_d f \in \mathcal{E}'(\mathbf{P}^d)$ and $R_d^* g \in \mathcal{D}'(\mathbb{R}^n)$. This shows that the normal operator $N_d = R_d^* R_d\colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is always defined and $N_d f = c_{n,d}(f * |\cdot|^{-(n-d)})$ holds in the sense of distributions. Let $V \subset \mathbb{R}^n$ be a nonempty open set and $f \in \mathcal{E}'(\mathbb{R}^n)$. We say that $R_d f$ vanishes on all d-planes intersecting V, if $\langle R_d f, \varphi \rangle = 0$ for all $\varphi \in C_c^\infty(\mathbf{P}_V^d)$ where $\mathbf{P}_V^d$ is the set of all d-planes intersecting V. If $V = B(0, R)$ is a ball, $\varphi \in C_c^\infty(\mathbf{P}_V^d)$ means that φ is smooth and $\varphi(A) = 0$ for all d-planes A for which $d(0, A) > r$ for some $r < R$. For more details on the range of the d-plane transform and duality in integral geometry, see [32] and [36, Chapter II].

Remark 4.1. *The UCP of Riesz potentials (corollary 3.2) immediately implies the UCP of the normal operator of the d-plane transform when d is odd (corollary 1.3) since* $N_d \approx I_{n-d}$ *and* $d/2 \notin \mathbb{Z}$. *However, such UCP cannot hold if d is even, which can be shown by contradiction. Assume that corollary 1.3 holds when d is even. Take any nonzero* $f \in C_c^\infty(\mathbb{R}^n)$. *By the properties of the Fourier transform and Riesz potentials we have* $(-\Delta)^{d/2} f = (-\Delta)^{-d/2}((-\Delta)^d f) = N_d(-\Delta)^d f$ *up to a constant factor. Since d is even* $(-\Delta)^{d/2}$ *is a local operator and we obtain* $N_d(-\Delta)^d f|_V = (-\Delta)^d f|_V = 0$ *where* $V \subset \mathbb{R}^n$ *is an open set outside the support of f and* $(-\Delta)^d f \in C_c^\infty(\mathbb{R}^n)$. *By the assumption we would get that* $(-\Delta)^d f = 0$, *i.e. f is polyharmonic. But this implies* $f = 0$ *by lemma 3.1, which is a contradiction. Hence the UCP cannot hold for* N_d *when d even.*

Using the UCP of N_d we can now prove corollary 1.4.

Proof of corollary 1.3. Consider first $f \in C_\infty(\mathbb{R}^n)$. Taking the adjoint, we get the conditions $N_d f|_V = 0$ and $f|_V = 0$. By corollary 1.3 we obtain $f = 0$ whenever d is odd. Then let $f \in \mathcal{E}'(\mathbb{R}^n)$. We can assume that $V = B(0, R)$ is a ball of radius R centered at the origin. As in [36] we define the "convolution"

$$(g \times \varphi)(A) = \int_{\mathbb{R}^n} g(y)\varphi(A - y)\mathrm{d}y$$

where $g \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbf{P}^d)$, $A \in \mathbf{P}^d$ and $A - y$ is a d-plane shifted by $y \in \mathbb{R}^n$. Then one can calculate that $R_d^*(g \times \varphi) = g * R_d^* \varphi$ (see [36, Proof of theorem 5.4]). Let $j_\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and consider the mollifications $f * j_\epsilon \in C_c^\infty(\mathbb{R}^n)$. By the properties of the convolutions

$$\langle R_d(f * j_\epsilon), \varphi \rangle = \langle f * j_\epsilon, R_d^* \varphi \rangle = \langle f, j_\epsilon * R_d^* \varphi \rangle = \langle f, R_d^*(j_\epsilon \times \varphi) \rangle = \langle R_d f, j_\epsilon \times \varphi \rangle . \tag{14}$$

Take $r > 0$ and $\epsilon > 0$ small enough so that $r + \epsilon < R$. Let $\varphi \in C_c^\infty(\mathbf{P}^d)$ such that $\varphi(A) = 0$ for all d-planes which satisfy $d(0, A) > r$. Then $(j_\epsilon \times \varphi)(A) = 0$ for all d-planes for which $d(0, A) > r + \epsilon$. Thus $j_\epsilon \times \varphi \in C_c^\infty(\mathbf{P}_V^d)$ and by (14) it follows that $R_d(f * j_\epsilon) = 0$ for all d-planes intersecting $B(0, r)$. We also have $(f * j_\epsilon)|_{B(0,r)} = 0$ and the first part of the proof implies the claim for $f * j_\epsilon$ for small $\epsilon > 0$. Since $f * j_\epsilon \to f$ in $\mathcal{E}'(\mathbb{R}^n)$ when $\epsilon \to 0$, we obtain the claim for f. □

Remark 4.2. *When d is even, corollary 1.4 does not say that the result is false. It only indicates that we cannot use the UCP of the normal operator in the proof. This boils down to the fact that* $(-\Delta)^s$ *does not admit the UCP for* $s \in \mathbb{Z}$. *However, if d is even, then the function f is determined uniquely in V by its integrals over d-planes which intersect V. Namely, if* $R_d f = 0$ *on all d-planes intersecting V, then* $N_d f|_V = 0$. *Since* $N_d \approx (-\Delta)^{-d/2}$, *one can invert* $N_d f$ *by the local operator* $(-\Delta)^{d/2}$ *to obtain* $f|_V = 0$. *Hence the ROI problem is uniquely solvable in this case without the additional knowledge of f in an open set inside the ROI.*

Remark 4.3. *We also note that unlike in the global data case lower dimensional data does not determine higher dimensional data. In other words,* $R_k f = 0$ *for all k-planes intersecting V does not necessarily imply that* $R_d f = 0$ *for all d-planes which intersect V where* $0 < k < d < n$. *Thus one cannot reduce the partial data problem for k-planes to the partial data problem for d-planes.*

5. Higher order fractional Schrödinger equation with singular potential

In this section, we study the fractional Schrödinger equation with higher order fractional Laplacian and singular potential. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and consider the equation

$$\begin{cases} ((-\Delta)^s + q)u &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f \end{cases} \tag{15}$$

where $u \in H^s(\mathbb{R}^n)$, $f \in H^s(\mathbb{R}^n)$ is the exterior value of u and $q \in L^\infty(\Omega)$ represents the electric potential. When the potential q is more singular one has to interpret the product qu in a suitable way. If $q \in Z_0^{-s}(\mathbb{R}^n)$, then q acts as a multiplier and induces a map $m_q\colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ defined by $\langle m_q(u), v\rangle_{\mathbb{R}^n} = \langle q, uv\rangle_{\mathbb{R}^n}$. Then equation (15) becomes

$$\begin{cases} (-\Delta)^s u + m_q(u) &= 0 \text{ in } \Omega \\ u|_{\Omega_e} &= f. \end{cases} \tag{16}$$

We will prove that the generalized DN map Λ_q for equation (16) determines the restriction of the potential $q \in Z_0^{-s}(\mathbb{R}^n)$ to Ω uniquely from exterior measurements. We also obtain the Runge approximation property for equation (16): any function $g \in \widetilde{H}^s(\Omega)$ can be approximated arbitrarily well by solutions of (16).

Similar results were proved in [75] when $0 < s < 1$. Our theorems generalize those results for higher order fractional Laplacians. The proofs rely essentially on the same thing: the UCP of the operator $(-\Delta)^s$ which was proved for $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ in section 3.1. Also the higher order Poincaré inequality is needed for the well-posedness of the inverse problem. In this section, we provide the basic ideas for the proofs of the lemmas, which are reminescent of the ones in [75] and [28]. We will mainly follow the same notation as in those articles.

The strategy to prove theorems 1.6 and 1.7 is the following. First one constructs a bilinear form and proves that unique weak solutions are obtained in the complement of a countable set of eigenvalues. One also proves that 0 is not an eigenvalue when (4) holds. Then one defines the abstract DN map and proves the Alessandrini identity using it. Using the UCP of $(-\Delta)^s$ one obtains the Runge approximation property for equation (16). From the Runge approximation and the Alessandrini identity, one can prove the uniqueness result for the inverse problem.

If $U \subset \mathbb{R}^n$ is open and $u, v \in L^2(U)$, we denote the inner product by

$$\langle u, v\rangle_U = \int_U uv\mathrm{d}x.$$

We also use the same notation $\langle\cdot,\cdot\rangle_U$ for dual pairing.

The following lemma guarantees the existence of unique weak solutions (see [75, Lemma 2.6]).

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q \in Z_0^{-s}(\mathbb{R}^n)$. For $v, w \in H^s(\mathbb{R}^n)$ define the bilinear form B_q as*

$$B_q(v, w) = \left\langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w \right\rangle_{\mathbb{R}^n} + \langle m_q(v), w\rangle_{\mathbb{R}^n}.$$

The following claims hold:

(a) There is a countable set $\Sigma = \{\lambda_i\}_{i=1}^\infty \subset \mathbb{R}$, $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$, with the following property: if $\lambda \notin \Sigma$, then for any $F \in (\widetilde{H}^s(\Omega))^*$ and $f \in H^s(\mathbb{R}^n)$ there is unique $u \in H^s(\mathbb{R}^n)$ satisfying

$$B_q(u, w) - \lambda \langle u, w\rangle_{\mathbb{R}^n} = F(w) \text{ for } w \in \widetilde{H}^s(\Omega), \quad u - f \in \widetilde{H}^s(\Omega)$$

with the norm estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\Big(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)} \Big)$$

where C is independent of F and f.

(b) *The function u in (a) is the unique $u \in H^s(\mathbb{R}^n)$ satisfying*

$$((-\Delta)^s u + m_q(u) - \lambda u)|_\Omega = F$$

in the sense of distributions with $u - f \in \widetilde{H}^s(\Omega)$.

(c) *One has $0 \notin \Sigma$ if (4) holds. If $q \in L^\infty(\Omega)$ and $q \geq 0$, then $\Sigma \subset (0, \infty)$ and (4) always holds.*

Proof. The constants in the inequalities do not depend on the function v in the proof. It is enough to solve the problem in (a) for $u - f = v \in \widetilde{H}^s(\Omega)$. Using the fractional Poincaré inequality (theorem 1.5) we obtain

$$\|v\|^2_{H^s(\mathbb{R}^n)} \leq C \left(\left\| (-\Delta)^{s/2} v \right\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)} \right) \leq C' \left\| (-\Delta)^{s/2} v \right\|^2_{L^2(\mathbb{R}^n)}.$$

Let $0 < \epsilon < 1/C'$ where the constant $C' > 0$ comes from the previous inequality. Since $q \in Z_0^{-s}(\mathbb{R}^n)$, we can find $q_s \in C_c^\infty(\mathbb{R}^n)$ and $q_r \in Z^{-s}(\mathbb{R}^n)$ such that $q = q_s + q_r$ and $\|q_r\|_{Z^{-s}(\mathbb{R}^n)} < \epsilon$. When we take $\mu = \|q_s^-\|_{L^\infty(\mathbb{R}^n)}$ where $q_s^- = -\min\{0, q_s(x)\}$, we obtain the coercivity condition

$$B_q(v, v) + \mu \langle v, v \rangle_{\mathbb{R}^n} \geq \left\| (-\Delta)^{s/2} v \right\|^2_{L^2(\mathbb{R}^n)} + \langle q_r, vv \rangle_{\mathbb{R}^n} \geq \frac{1}{C'} \|v\|^2_{H^s(\mathbb{R}^n)} - \epsilon \|v\|^2_{H^s(\mathbb{R}^n)}.$$

Hence $v, w \mapsto B_q(v, w) + \mu \langle v, w \rangle_{\mathbb{R}^n}$ defines an equivalent inner product in $\widetilde{H}^s(\Omega)$. The proof is then completed as in [28]: the Riesz representation theorem implies that for every $\widetilde{F} \in (\widetilde{H}^s(\Omega))^*$ there is unique $v = G_\mu \widetilde{F} \in \widetilde{H}^s(\Omega)$ such that $B_q(v, w) + \mu \langle v, w \rangle_{\mathbb{R}^n} = \widetilde{F}(w)$ for all $w \in \widetilde{H}^s(\Omega)$. The map $G_\mu \colon (\widetilde{H}^s(\Omega))^* \to \widetilde{H}^s(\Omega)$ induces a compact, self-adjoint and positive definite operator $\widetilde{G}_\mu \colon L^2(\Omega) \to L^2(\Omega)$ by the compact Sobolev embedding theorem. The spectral theorem for the self-adjoint compact operator $\widetilde{G}_\mu$ implies the claim in (a). Part (b) holds since $C_c^\infty(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$. The first claim in (c) follows from the Fredholm alternative. The second claim in (c) is essentially the same as in [28, Lemma 2.3] and is proved by replacing $\widetilde{H}^s(\Omega)$ with $H_{\overline{\Omega}}(\mathbb{R}^n)$ in the proof of (a). $\square$

Recall the definition of the abstract trace space $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ which we equip with the quotient norm

$$\|[f]\|_X = \inf_{\varphi \in \widetilde{H}^s(\Omega)} \|f - \varphi\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s(\mathbb{R}^n).$$

The following lemma implies that the DN map is well-defined and continuous. The result follows immediately from the definition of the bilinear form $B_q(\cdot, \cdot)$ and from the continuity of $(-\Delta)^{s/2} \colon H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (see [28, Lemma 2.4]).

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q \in Z_0^{-s}(\mathbb{R}^n)$ which satisfies (4). Then the map $\Lambda_q \colon X \to X^*$, $\langle \Lambda_q[f], [g] \rangle = B_q(u_f, g)$, is linear and continuous, where $u_f \in H^s(\mathbb{R}^n)$ solves $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$. One also has the self-adjointness property $\langle \Lambda_q[f], [g]) \rangle = \langle [f], \Lambda_q[g] \rangle$ for $f, g \in H^s(\mathbb{R}^n)$.*

Proof. Since u_f is a solution to $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u_f - f \in \widetilde{H}^s(\Omega)$ and solutions are unique, we obtain $B_q(u_{f+\varphi}, g + \psi) = B_q(u_f, g)$ for $\varphi, \psi \in \widetilde{H}^s(\Omega)$. This implies that Λ_q is well-defined. Further, using continuity of $(-\Delta)^{s/2}$ and the norm estimate for solution u_f from lemma 5.1, we obtain

$$\begin{aligned} |\langle \Lambda_q[f], [g] \rangle| &\leq \left\| (-\Delta)^{s/2} u_f \right\|_{L^2(\mathbb{R}^n)} \left\| (-\Delta)^{s/2} g \right\|_{L^2(\mathbb{R}^n)} + \|q\|_{Z^{-s}(\mathbb{R}^n)} \|u_f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)} \\ &\leq C \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)}, \end{aligned}$$

where C does not depend on f and g. By the definition of the quotient norm $|\langle \Lambda_q[f], [g] \rangle| \leq C \|[f]\|_X \|[g]\|_X$, so Λ_q is continuous. Choosing $g = u_g$ we obtain by symmetry of $B_q(\cdot, \cdot)$

$$\langle \Lambda_q[f], [g] \rangle = B_q(u_f, u_g) = \langle \Lambda_q[g], [u_f] \rangle = \langle [f], \Lambda_q[g] \rangle$$

where we used the fact that $u_f - f \in \widetilde{H}^s(\Omega)$. $\square$

We immediately obtain the Alessandrini identity from lemma 5.2 (see [75, Lemma 2.7]). We use some abuse of notation and write f instead of $[f]$.

Lemma 5.3 (Alessandrini identity). *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q_1, q_2 \in Z_0^{-s}(\mathbb{R}^n)$ which satisfy (4). For any $f_1, f_2 \in X$ one has*

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle = \langle m_{q_1 - q_2}(u_1), u_2 \rangle_{\mathbb{R}^n}$$

where $u_i \in H^s(\mathbb{R}^n)$ solves $(-\Delta)^s u_i + m_{q_i}(u_i) = 0$ in Ω with $u_i - f_i \in \widetilde{H}^s(\Omega)$.

Proof. Using the self-adjointness of Λ_q and the property $B_q(u_i, g+\psi) = B_q(u_i, g)$ for $\psi \in \widetilde{H}^s(\Omega)$, we obtain

$$\begin{aligned}\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle &= \langle \Lambda_{q_1} f_1, f_2 \rangle - \langle f_1, \Lambda_{q_2} f_2 \rangle = B_{q_1}(u_1, f_2) - B_{q_2}(u_2, f_1) \\ &= B_{q_1}(u_1, f_2 + (u_2 - f_2)) - B_{q_2}(u_2, f_1 + (u_1 - f_1)) \\ &= B_{q_1}(u_1, u_2) - B_{q_2}(u_1, u_2) = \langle m_{q_1 - q_2}(u_1), u_2 \rangle_{\mathbb{R}^n}\end{aligned}$$

which gives the claim. $\square$

From the UCP of $(-\Delta)^s$ (theorem 1.2), we obtain the following approximation result (see [75, Lemma 8.1]).

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^n$ be bounded open set, $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $q \in Z_0^{-s}(\mathbb{R}^n)$ which satisfies (4). Denote by $P_q \colon X \to H^s(\mathbb{R}^n)$, $P_q f = u_f$, where $u_f \in H^s(\mathbb{R}^n)$ is the unique solution to $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$ given by lemma 5.1. Let $W \subset \Omega_e$ be any open set and define the set*

$$\mathcal{R} = \{P_q f - f : f \in C_c^\infty(W)\}.$$

Then $\mathcal{R}$ is dense in $\widetilde{H}^s(\Omega)$.

Proof. By the Hahn-Banach theorem it is enough to show that if $F \in (\widetilde{H}^s(\Omega))^*$ and $\langle F, v \rangle = 0$ for all $v \in \mathcal{R}$, then $F = 0$. Let $F \in (\widetilde{H}^s(\Omega))^*$ and assume that

$$\langle F, P_q f - f \rangle = 0, \quad f \in C_c^\infty(W).$$

Let $\varphi \in \widetilde{H}^s(\Omega)$ be the solution to

$$(-\Delta)^s \varphi + m_q(\varphi) = F \text{ in } \Omega, \quad \varphi|_{\Omega_e} = 0$$

which exists by lemma 5.1. This means that $B_q(\varphi, w) = \langle F, w \rangle$ for all $w \in \widetilde{H}^s(\Omega)$. Let $u_f = P_q f \in H^s(\mathbb{R}^n)$ where $u_f - f \in \widetilde{H}^s(\Omega)$. Now

$$\langle F, P_q f - f \rangle = B_q(\varphi, u_f - f) = -B_q(\varphi, f)$$

since u_f is a solution to $(-\Delta)^s u + m_q(u) = 0$ in Ω and $\varphi \in \widetilde{H}^s(\Omega)$. Thus $B_q(\varphi, f) = 0$ for all $f \in C_c^\infty(W)$. Using the fact that $\mathrm{spt}(\varphi)$ and $\mathrm{spt}(f)$ are disjoint, we obtain

$$0 = \left\langle (-\Delta)^{s/2} \varphi, (-\Delta)^{s/2} f \right\rangle_{\mathbb{R}^n} = \langle (-\Delta)^s \varphi, f \rangle_{\mathbb{R}^n}.$$

Here we used that $\left\langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v \right\rangle_{\mathbb{R}^n} = \langle (-\Delta)^s u, v \rangle_{\mathbb{R}^n}$ for $u, v \in \mathscr{S}(\mathbb{R}^n)$ and the equality holds also in $H^s(\mathbb{R}^n)$ by density. Hence $\varphi|_W = (-\Delta)^s \varphi|_W = 0$ and theorem 1.2 implies $\varphi = 0$ and eventually $F = 0$. $\square$

We remark that exactly the same proof gives the density of $r_\Omega \mathcal{R}$ in $L^2(\Omega)$ where r_Ω is the restriction to Ω (see [28, Lemma 5.1]). Now it is easy to prove theorems 1.6 and 1.7.

Proof of theorem 1.6. Since we can always shrink the sets W_i, we can assume without loss of generality that $\overline{W}_1 \cap \overline{W}_2 = \varnothing$ and $(\overline{W}_1 \cup \overline{W}_2) \cap \overline{\Omega} = \varnothing$. Using the Alessandrini identity (lemma 5.3), we obtain

$$\langle m_{q_1-q_2}(u_1), u_2 \rangle_{\mathbb{R}^n} = 0$$

for any $u_i \in H^s(\mathbb{R}^n)$ which solves $(-\Delta)^s u_i + m_{q_i}(u_i) = 0$ in Ω with exterior values in $C_c^\infty(W_i)$. Let $v_1, v_2 \in \widetilde{H}^s(\Omega)$. By lemma 5.4 there are sequences of exterior values $f_1^k \in C_c^\infty(W_1)$ and $f_2^k \in C_c^\infty(W_2)$ with sequences of solutions $u_1^k, u_2^k \in H^s(\mathbb{R}^n)$ such that

- $(-\Delta)^s u_i^k + m_{q_i}(u_i^k) = 0$ in Ω
- $u_i^k - f_i^k \in \widetilde{H}^s(\Omega)$
- $u_i^k = f_i^k + v_i + r_i^k$ where $r_i^k \xrightarrow{k\to\infty} 0$ in $\widetilde{H}^s(\Omega)$.

When we insert the solutions u_i^k into the Alessandrini identity, use the support conditions and take the limit $k \to \infty$, we obtain

$$\langle m_{q_1-q_2}(v_1), v_2 \rangle_{\mathbb{R}^n} = 0.$$

Let $\varphi \in C_c^\infty(\Omega)$. Choose $v_1 = \varphi$ and $v_2 \in C_c^\infty(\Omega)$ such that $v_2 = 1$ in a neighborhood of $\mathrm{spt}(\varphi)$. This implies

$$0 = \langle m_{q_1-q_2}(v_1), v_2 \rangle_{\mathbb{R}^n} = \langle q_1 - q_2, v_1 v_2 \rangle_{\mathbb{R}^n} = \langle q_1 - q_2, \varphi \rangle_{\mathbb{R}^n}$$

which is equivalent to that $q_1|_\Omega = q_2|_\Omega$ as distributions. □

Proof of theorem 1.7. Since $\mathrm{int}(\Omega_1 \setminus \Omega) \neq \varnothing$, there is open set $W \subset \Omega_e$ such that $\overline{W} \subset \Omega_1 \setminus \overline{\Omega}$. By lemma 5.4, the set $\mathcal{R}$ is dense in $\widetilde{H}^s(\Omega)$. Hence, we can approximate any $g \in \widetilde{H}^s(\Omega)$ arbitrarily well by solutions $u \in H^s(\mathbb{R}^n)$ to the equation $(-\Delta)^s u + m_q(u) = 0$ in Ω with $u - f \in \widetilde{H}^s(\Omega)$. Since $f \in C_c^\infty(W)$ we especially have $\mathrm{spt}(u) \subset \overline{\Omega}_1$. □

6. Higher order fractional magnetic Schrödinger equation

In this section we will be dealing with the definition of the FMSE, as well as with the proof of the injectivity result for the corresponding inverse problem. For the sake of simplicity, let us fix the convention throughout this section that the symbol $\langle \cdot, \cdot \rangle$ indicates both the scalar product (duality pairing) on $L^2(\mathbb{R}^n)$ and the one on $L^2(\mathbb{R}^{2n})$, the distinction between the two being always possible by checking the number of free variables inside the brackets. We also let the norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^s}$ etc. to denote the norms over the whole $\mathbb{R}^n$ or $\mathbb{R}^{2n}$ when the base set is not specified.

6.1. **High order bivariate functions.**

Let $l, n \in \mathbb{N}$, and consider a family A of scalar two-point functions indexed over the set $\{1, ..., n\}^l$. A generic member of the family is determined by a vector $(i_1, ..., i_l)$ such that $i_j \in \{1, ..., n\}$ for all $j \in \{1, ..., l\}$, and it is a function $A_{i_1,...,i_l} : \mathbb{R}^{2n} \to \mathbb{R}$. We call such family A a *bivariate function of order* l. One can see the family A as a function $A : \mathbb{R}^{2n} \to \mathbb{M}^l$, where $\mathbb{M}^l$ is the set of all $n \times ... \times n = n^l$ arrays of real numbers, i.e. tensors of order l.

Let $a, b \in \mathbb{N}$, and let A, B be bivariate functions of orders a and b respectively, in the variables x_1, x_2. The *tensor product* of A and B is the bivariate function of order $a + b$ given by

$$(A \otimes B)_{i_1,...,i_a,j_1,...,j_b}(x_1, x_2) := A_{i_1,...,i_a}(x_1, x_2) B_{j_1,...,j_b}(x_1, x_2) \ .$$

In particular, for a vector $\xi \in \mathbb{R}^n$ we let $\xi^{\otimes 0} = 0$, $\xi^{\otimes 1} = \xi$ and recursively $\xi^{\otimes m} = \xi^{\otimes(m-1)} \otimes \xi$. Let A, B as before, but assume now that $a \geq b$. The *contraction* of A and B is the bivariate function of order $a - b$ given by

$$(A \cdot B)_{i_1,...,i_{a-b}}(x_1, x_2) := \sum_{j_1,...,j_b=1}^{n} A_{i_1,...,i_{a-b},j_1,...,j_b}(x_1, x_2) B_{j_1,...,j_b}(x_1, x_2) \ .$$

If $A = B$, then of course $a = b$, so that $A \cdot A$ is a scalar function of the variables (x_1, x_2). One sees that $|A| := (A \cdot A)^{1/2}$ defines a norm for fixed x_1 and x_2, and that this coincides with the usual one when A is a vector function.

Lemma 6.1. *Let $a, b \in \mathbb{N}$, and let A, B, v be bivariate functions of orders a, b and 1 respectively, in the variables x_1, x_2. Assume that $a \geq b+1$; then*

$$A \cdot (B \otimes v) = (A \cdot v) \cdot B \,.$$

Proof. The proof is just a simple computation:

$$\begin{aligned}
[A \cdot (B \otimes v)]_{i_1,\dots,i_{a-b-1}} &= \sum_{j_1,\dots,j_{b+1}=1}^{n} A_{i_1,\dots,i_{a-b-1},j_1,\dots,j_{b+1}} (B \otimes v)_{j_1,\dots,j_{b+1}} \\
&= \sum_{j_1,\dots,j_{b+1}=1}^{n} A_{i_1,\dots,i_{a-b-1},j_1,\dots,j_{b+1}} B_{j_1,\dots,j_b} v_{j_{b+1}} \\
&= \sum_{j_1,\dots,j_b=1}^{n} B_{j_1,\dots,j_b} \sum_{j_{b+1}=1}^{n} A_{i_1,\dots,i_{a-b-1},j_1,\dots,j_{b+1}} v_{j_{b+1}} \\
&= \sum_{j_1,\dots,j_b=1}^{n} B_{j_1,\dots,j_b} (A \cdot v)_{i_1,\dots,i_{a-b-1},j_1,\dots,j_b} \\
&= [(A \cdot v) \cdot B]_{i_1,\dots,i_{a-b-1}} \,.
\end{aligned}$$

□

Let A be a bivariate function of any order. Following [14], we recall the definitions of the *symmetric* and *antisymmetric parts of A with respect to the variables x and y* and the *L^2 norms of A with respect to the first and second variable at point x*:

$$A_s(x,y) := \frac{A(x,y) + A(y,x)}{2} \,, \qquad A_a(x,y) := A(x,y) - A_s(x,y) \,,$$

$$\mathcal{J}_1 A(x) := \left(\int_{\mathbb{R}^n} |A(y,x)|^2 \, dy \right)^{1/2} \,, \quad \mathcal{J}_2 A(x) := \left(\int_{\mathbb{R}^n} |A(x,y)|^2 \, dy \right)^{1/2} \,.$$

It is easily seen that $A \in L^2$ implies $A_s, A_a \in L^2$, since

$$(17) \quad \|A_s\|_{L^2} = \left\| \frac{A(x,y) + A(y,x)}{2} \right\|_{L^2} \leq \|A\|_{L^2} \,, \quad \|A_a\|_{L^2} = \left\| \frac{A(x,y) - A(y,x)}{2} \right\|_{L^2} \leq \|A\|_{L^2} \,.$$

A bivariate function A of any order will be called *symmetric* if $A = A_s$ almost everywhere, and *antisymmetric* if $A = A_a$ almost everywhere.

Lemma 6.2. *Let $A \in L^1(\mathbb{R}^{2n}, \mathbb{M}^l)$ be an antisymmetric bivariate function of order l for some $l \in \mathbb{N}$. Then $\int_{\mathbb{R}^{2n}} A(x,y)\, dydx = 0$.*

Proof. Let D^+, D^- be the sets respectively above and under the diagonal $D := \{(x,y) \in \mathbb{R}^{2n} : x = y\}$ of $\mathbb{R}^{2n}$. Since $\int_{D^\pm} A(x,y)\, dydx \leq \int_{D^\pm} |A(x,y)|\, dydx \leq \|A\|_{L^1} < \infty$, we can decompose the integral as $\int_{\mathbb{R}^{2n}} A(x,y)\, dydx = \int_{D^+} A(x,y)\, dydx + \int_{D^-} A(x,y)\, dydx$. Given the symmetry of the sets D^+ and D^-, this can be rewritten as $\int_{\mathbb{R}^{2n}} A(x,y)\, dydx = \int_{D^+} (A(x,y) + A(y,x))\, dydx$, which vanishes by virtue of the antisymmetry of A. □

6.2. **Fractional operators.** Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $u \in C_c^\infty(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$. Let $\lfloor s \rfloor := \sup\{n \in \mathbb{N} : n < s\}$ and $s' := s - \lfloor s \rfloor$, so that by definition $s' \in (0,1)$. The *fractional gradient of order s of u at points x and y* is the following symmetric bivariate function of order $\lfloor s \rfloor + 1$:

$$\nabla^s u(x,y) := \frac{\mathcal{C}_{n,s'}^{1/2}}{\sqrt{2}} \frac{\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)}{|y-x|^{n/2+s'+1}} \otimes (y-x) \,.$$

Observe that this definition coincides with the usual one for $s \in (0,1)$, since in this case $\lfloor s \rfloor = 0$ and $s' = s$. One can compute

$$\begin{aligned}\|\nabla^s u\|^2_{L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1})} &= \frac{\mathcal{C}_{n,s'}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\nabla^{\lfloor s\rfloor}u(x)-\nabla^{\lfloor s\rfloor}u(y)|^2}{|x-y|^{n+2s'}}dx\,dy\\ &= \frac{\mathcal{C}_{n,s'}}{2}[\nabla^{\lfloor s\rfloor}u]^2_{\dot H^{s'}(\mathbb{R}^n)} = \left\|(-\Delta)^{s'/2}\nabla^{\lfloor s\rfloor}u\right\|^2_{L^2(\mathbb{R}^n)}\\ &= \||\xi|^{s'}\xi^{\otimes\lfloor s\rfloor}\hat u(\xi)\|^2_{L^2(\mathbb{R}^n)} = \||\xi|^s\hat u(\xi)\|^2_{L^2(\mathbb{R}^n)}\\ &= \left\|(-\Delta)^{s/2}u\right\|^2_{L^2(\mathbb{R}^n)} \le \|u\|^2_{H^s(\mathbb{R}^n)}\ .\end{aligned}$$

Thus, by the density of C_c^∞ in H^s, ∇^s can be extended to a continuous operator $\nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s\rfloor+1})$. One sees by density that the formula given for $\nabla^s u$ in the case $u \in C_c^\infty(\mathbb{R}^n)$ still holds almost everywhere for $u \in H^s(\mathbb{R}^n)$. Thus if $u, v \in H^s$, by the above computation,

$$\langle\nabla^s u, \nabla^s u\rangle = \|(-\Delta)^{s/2}u\|^2_{L^2} = \langle(-\Delta)^{s/2}u, (-\Delta)^{s/2}u\rangle = \langle(-\Delta)^s u, u\rangle\ ,$$

so that by the polarization identity and the self-adjointness of $(-\Delta)^s$,

$$\begin{aligned}\langle\nabla^s u, \nabla^s v\rangle &= \frac{\langle\nabla^s(u+v), \nabla^s(u+v)\rangle - \langle\nabla^s u, \nabla^s u\rangle - \langle\nabla^s v, \nabla^s v\rangle}{2}\\ &= \frac{\langle(-\Delta)^s(u+v), u+v\rangle - \langle(-\Delta)^s u, u\rangle - \langle(-\Delta)^s v, v\rangle}{2}\\ &= \frac{\langle(-\Delta)^s u, v\rangle + \langle(-\Delta)^s v, u\rangle}{2} = \langle(-\Delta)^s u, v\rangle\ .\end{aligned}$$

This proves that if the *fractional divergence* $(\nabla\cdot)^s : L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s\rfloor+1}) \to H^{-s}(\mathbb{R}^n)$ is defined as the adjoint of ∇^s, then weakly $(\nabla\cdot)^s\nabla^s = (-\Delta)^s$ for $s \in \mathbb{R}^+ \setminus \mathbb{Z}$. This result was already proved in [15], but only for the case $s \in (0,1)$. If we define the antisymmetric bivariate vector function

$$\alpha(x,y) := \frac{\mathcal{C}^{1/2}_{n,s'}}{\sqrt{2}}\frac{y-x}{|y-x|^{n/2+s'+1}}$$

then for $u \in H^s$ the identity

$$\nabla^s u(x,y) = (\nabla^{\lfloor s\rfloor}u(x) - \nabla^{\lfloor s\rfloor}u(y)) \otimes \alpha(x,y)$$

holds almost everywhere.

We now define the magnetic versions of the above operators. Fix $p > \max\{1, n/2s\}$, and let A be a bivariate function of order $\lfloor s\rfloor + 1$ such that

(a1) $\mathcal{J}_2 A \in L^{2p}(\mathbb{R}^n)$

(a2) $\mathrm{spt}(A) \subset \Omega \times \Omega$.

With such choice of p, the embedding $H^s \times L^{2p} \hookrightarrow L^2$ always holds by [5, Theorem 6.1], recall that $W^r(\mathbb{R}^n) = H^r(\mathbb{R}^n)$ with equivalent norms when $r \in \mathbb{R}$ and $W^r(\mathbb{R}^n)$ is the L^2 Sobolev-Slobodecki space [5, 57]. Therefore, if $u \in H^s$,

$$\begin{aligned}\|A(x,y)u(x)\|_{L^2(\mathbb{R}^{2n},\mathbb{M}^{\lfloor s\rfloor+1})} &= \left(\int_{\mathbb{R}^n}|u(x)|^2\int_{\mathbb{R}^n}|A(x,y)|^2 dy\,dx\right)^{1/2}\\ &= \left(\int_{\mathbb{R}^n}|u(x)|^2\ |\mathcal{J}_2 A(x)|^2\ dx\right)^{1/2} = \|u\,\mathcal{J}_2 A\|_{L^2(\mathbb{R}^n)}\\ &\le c\|u\|_{H^s}\|\mathcal{J}_2 A\|_{L^{2p}} < \infty,\end{aligned}$$

where c does not depend on u and A. This allows the definition of $\nabla^s_A u(x,y) := \nabla^s u(x,y) + A(x,y)u(x)$ and its adjoint $(\nabla\cdot)^s_A$ just as in [14], in such a way that $\nabla^s_A : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s\rfloor+1})$ and $(\nabla\cdot)^s_A : L^2(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s\rfloor+1}) \to H^{-s}(\mathbb{R}^n)$. By definition, the *magnetic fractional Laplacian* $(-\Delta)^s_A : H^s \to H^{-s}$ will be the composition $(\nabla\cdot)^s_A\nabla^s_A$. Let now q be a scalar field such that (a3) $q \in L^p(\Omega)$.

By [5, Theorem 8.3] we have the embedding $H^s \times L^p \hookrightarrow H^{-s}$ and hence $qu \in H^{-s}$ holds for $u \in H^s$. We can thus define the magnetic Schrödinger operator $(-\Delta)^s_A + q : H^s \to H^{-s}$ and the fractional magnetic Schrödinger equation (FMSE)

$$(-\Delta)^s_A u + qu = 0 \ .$$

In the next Lemma we write $(-\Delta)^s_A$ in a more convenient form. To this scope, we introduce the bivariate function of order $\lfloor s \rfloor$ given by $S(x,y) := A(x,y) \cdot \alpha(x,y)$, for which we assume that

(a4) $|S(x,y)| \leq \tilde{S}(y)$ for a.e. $x, y \in \mathbb{R}^n$, with $\tilde{S} \in L^2$,
(a5) $S(x,y) \in H^{\lfloor s \rfloor}(\mathbb{R}^{2n}, \mathbb{M}^{\lfloor s \rfloor})$.

Remark 6.3. *Assumption (a4) is really relevant only when $\lfloor s \rfloor \neq 0$, as it will be clear from the proof; in the case $s \in (0,1)$, this assumption can be reduced. We refer to [14] for a set of sufficient conditions in that regime. Moreover, with a more careful analysis, one could reduce the exponent of the space to which $\tilde{S}$ belongs. However, we decided to keep L^2 for the sake of simplicity.*

Lemma 6.4. *Let A be a bivariate function of order $\lfloor s \rfloor + 1$ satisfying conditions (a1), (a2) (a4), (a5), and let $u \in H^s$. There exist linear operators $\mathfrak{N}, \mathfrak{M}_\beta$ acting on bivariate functions of order $\lfloor s \rfloor$, with β a multi-index of length $|\beta| \leq \lfloor s \rfloor$, such that the equation*

$$\begin{aligned}(-\Delta)^s_A u(x) =& (-\Delta)^s u(x) + \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta u(x) (\mathfrak{M}_\beta(S))(x) + \\ &+ \int_{\mathbb{R}^n} u(y) (\mathfrak{N}(S))(x,y)\, dy + u(x) \int_{\mathbb{R}^n} |A(x,y)|^2 dy\end{aligned}$$

holds in weak sense.

Proof. If $v \in H^s$, then in weak sense

$$\langle (-\Delta)^s_A u, v \rangle = \langle \nabla^s u, \nabla^s v \rangle + \langle \nabla^s u, Av \rangle + \langle \nabla^s v, Au \rangle + \langle Au, Av \rangle \ , \tag{18}$$

where all the terms on the right hand side are finite, as observed above.

Step 1. Let us start by computing the third term on the right hand side of (18). The bivariate function $\nabla^s v(x,y)[A(x,y)u(x)]_a$ is antisymmetric, and by Cauchy-Schwartz and formula (17) we have $\|\nabla^s v\, (Au)_a\|_{L^1} \leq \|\nabla^s v\|_{L^2} \|(Au)_a\|_{L^2} \leq \|v\|_{H^s} \|Au\|_{L^2} < \infty$. Therefore Lemma 6.2 gives $\langle \nabla^s v, (Au)_a \rangle = 0$, and we can use Lemma 6.1 to write

$$\begin{aligned}\langle \nabla^s v, Au \rangle &= \langle \nabla^s v, Au \rangle - \langle \nabla^s v, (Au)_a \rangle = \langle \nabla^s v, (Au)_s \rangle \\ &= \langle (\nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y)) \otimes \alpha, (Au)_s \rangle \\ &= \langle \nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y), (Au)_s \cdot \alpha \rangle \\ &= \langle \nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y), (A \cdot \alpha u)_a \rangle \\ &= \langle \nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y), (Su)_a \rangle \ .\end{aligned} \tag{19}$$

The bivariate function $[\nabla^{\lfloor s \rfloor} v(x) + \nabla^{\lfloor s \rfloor} v(y)][S(x,y)u(x)]_a$ is antisymmetric, and we can estimate its L^1 norm by means of the triangle inequality as

$$\|(\nabla^{\lfloor s \rfloor} v(x) + \nabla^{\lfloor s \rfloor} v(y))(Su)_a\|_{L^1} \leq \|(\nabla^{\lfloor s \rfloor} v(x) - \nabla^{\lfloor s \rfloor} v(y))(Su)_a\|_{L^1} + \|2\nabla^{\lfloor s \rfloor} v(x)(Su)_a\|_{L^1} \ .$$

The first term on the right hand side equals $\|\nabla^s v\, (Au)_s\|_{L^1}$ by computation (19), so that it is finite by $\|\nabla^s v\, (Au)_s\|_{L^1} \leq \|\nabla^s v\|_{L^2} \|(Au)_s\|_{L^2} \leq \|v\|_{H^s} \|Au\|_{L^2} < \infty$. We estimate the other term again by triangular inequality as

$$\|2\nabla^{\lfloor s \rfloor} v(x)(Su)_a\|_{L^1} \leq \|\nabla^{\lfloor s \rfloor} v(x) S(x,y) u(x)\|_{L^1} + \|\nabla^{\lfloor s \rfloor} v(x) S(y,x) u(y)\|_{L^1} \ . \tag{20}$$

The estimation of the second term on the right hand side of (20) can be done as follows, and similarly for the other one:

$$
\begin{aligned}
\|\nabla^{\lfloor s\rfloor}v(x)S(y,x)u(y)\|_{L^1} &= \int_{\mathbb{R}^n}|\nabla^{\lfloor s\rfloor}v(x)|\int_{\mathbb{R}^n}|S(y,x)|\,|u(y)|\,dydx \\
&\leq \int_{\mathbb{R}^n}|\nabla^{\lfloor s\rfloor}v(x)|\tilde{S}(x)\int_{\Omega}|u(y)|\,dydx \\
&\leq c\|u\|_{L^2}\int_{\mathbb{R}^n}|\nabla^{\lfloor s\rfloor}v(x)|\tilde{S}(x)dx \\
&\leq c\|u\|_{L^2}\|\nabla^{\lfloor s\rfloor}v(x)\|_{L^2}\|\tilde{S}\|_{L^2} \\
&\leq c\|u\|_{H^s}\|v\|_{H^s}\|\tilde{S}\|_{L^2}<\infty,
\end{aligned} \tag{21}
$$

where the constant c can change from line to line and does not depend on v, u and S.
Thus we have proved that $\|2\nabla^{\lfloor s\rfloor}v(x)(Su)_a\|_{L^1}<\infty$, which in turn implies that $\|(\nabla^{\lfloor s\rfloor}v(x)+\nabla^{\lfloor s\rfloor}v(y))(Su)_a\|_{L^1}<\infty$. Now we can use again Lemma 6.2 to conclude that $\langle\nabla^{\lfloor s\rfloor}v(x)+\nabla^{\lfloor s\rfloor}v(y),(Su)_a\rangle=0$. From this fact and formula (19), integrating by parts,

$$
\begin{aligned}
\langle\nabla^s v, Au\rangle &= \langle\nabla^{\lfloor s\rfloor}v(x)-\nabla^{\lfloor s\rfloor}v(y),(Su)_a\rangle+\langle\nabla^{\lfloor s\rfloor}v(x)+\nabla^{\lfloor s\rfloor}v(y),(Su)_a\rangle \\
&= 2\langle\nabla^{\lfloor s\rfloor}v(x),(Su)_a\rangle=\langle\nabla^{\lfloor s\rfloor}v(x),S(x,y)u(x)-S(y,x)u(y)\rangle \\
&= \langle\nabla^{\lfloor s\rfloor}v(x),S(x,y)u(x)\rangle-\langle\nabla^{\lfloor s\rfloor}v(x),S(y,x)u(y)\rangle \\
&= (-1)^{\lfloor s\rfloor}\left\langle v,(\nabla\cdot)_x^{\lfloor s\rfloor}\left(u(x)\int_{\mathbb{R}^n}S(x,y)dy\right)\right\rangle \\
&\quad -(-1)^{\lfloor s\rfloor}\left\langle v,(\nabla\cdot)_x^{\lfloor s\rfloor}\int_{\mathbb{R}^n}S(y,x)u(y)dy\right\rangle\,.
\end{aligned}
$$

In the last term the derivatives can pass under the integral sign by means of the dominated convergence theorem, since $|S(x,y)u(y)|\leq\tilde{S}(y)|u(y)|$, and $\int_{\mathbb{R}^n}\tilde{S}(y)|u(y)|dy\leq\|\tilde{S}\|_{L^2}\|u\|_{L^2}<\infty$. Eventually,

$$
\begin{aligned}
\langle\nabla^s v, Au\rangle &= (-1)^{\lfloor s\rfloor}\left\langle v,(\nabla\cdot)_x^{\lfloor s\rfloor}\left(u(x)\int_{\mathbb{R}^n}S(x,y)dy\right)\right\rangle \\
&\quad +(-1)^{\lfloor s\rfloor+1}\left\langle v,\int_{\mathbb{R}^n}u(y)(\nabla\cdot)_x^{\lfloor s\rfloor}S(y,x)dy\right\rangle\,.
\end{aligned} \tag{22}
$$

Step 2. Next we compute the second term on the right hand side of (18). With a computation similar to (19), we obtain $\langle\nabla^s u, Av\rangle=\langle\nabla^{\lfloor s\rfloor}u(x)-\nabla^{\lfloor s\rfloor}u(y),S(x,y)v(x)\rangle$; moreover, we have estimates similar to the ones in (21), and so we can split the integral. Eventually, we integrate by parts and get

$$
\begin{aligned}
\langle\nabla^s u, Av\rangle &= \langle\nabla^{\lfloor s\rfloor}u(x),S(x,y)v(x)\rangle-\langle\nabla^{\lfloor s\rfloor}u(y),S(x,y)v(x)\rangle \\
&= \left\langle v(x),\nabla^{\lfloor s\rfloor}u(x)\cdot\int_{\mathbb{R}^n}S(x,y)dy\right\rangle-\left\langle v(x),\int_{\mathbb{R}^n}\nabla^{\lfloor s\rfloor}u(y)\cdot S(x,y)dy\right\rangle \\
&= \left\langle v(x),\nabla^{\lfloor s\rfloor}u(x)\cdot\int_{\mathbb{R}^n}S(x,y)dy\right\rangle+(-1)^{\lfloor s\rfloor+1}\left\langle v(x),\int_{\mathbb{R}^n}u(y)(\nabla\cdot)_y^{\lfloor s\rfloor}S(x,y)dy\right\rangle\,.
\end{aligned} \tag{23}
$$

Step 3. The properties $\langle(-\Delta)^s u,v\rangle=\langle\nabla^s u,\nabla^s v\rangle$ and $\langle Au,Av\rangle=\left\langle v,u\int_{\mathbb{R}^n}|A(x,y)|^2dy\right\rangle$ hold, as proved in [14]. Using this information and formulas (22), (23) we can write the fractional magnetic Schrödinger operator as

$$
\begin{aligned}
&\langle(-\Delta)^s u,v\rangle+\left\langle\nabla^{\lfloor s\rfloor}u(x)\cdot\int_{\mathbb{R}^n}S(x,y)dy+(-1)^{\lfloor s\rfloor}(\nabla\cdot)_x^{\lfloor s\rfloor}\left(u(x)\int_{\mathbb{R}^n}S(x,y)dy\right),v\right\rangle+ \\
&+(-1)^{\lfloor s\rfloor+1}\left\langle\int_{\mathbb{R}^n}u(y)\left((\nabla\cdot)_x^{\lfloor s\rfloor}S(y,x)+(\nabla\cdot)_y^{\lfloor s\rfloor}S(x,y)\right)dy,v\right\rangle+\left\langle u\int_{\mathbb{R}^n}|A|^2dy,v\right\rangle.
\end{aligned}
$$

Let us compute the left hand side of the second bracket and collect the resulting terms according to the order of their derivatives of u. For every multi-index β such that $|\beta| \leq \lfloor s \rfloor$ we can find a linear operator $\mathfrak{M}_\beta$ such that

$$\nabla^{\lfloor s \rfloor} u(x) \cdot \int_{\mathbb{R}^n} S(x,y)dy + (-1)^{\lfloor s \rfloor} (\nabla\cdot)_x^{\lfloor s \rfloor} \left(u(x) \int_{\mathbb{R}^n} S(x,y)dy \right) = \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta u(x) \mathfrak{M}_\beta(S) \ .$$

We can also define the following linear operator:

$$\mathfrak{N}(S) = (-1)^{\lfloor s \rfloor + 1} \left((\nabla\cdot)_x^{\lfloor s \rfloor} S(y,x) + (\nabla\cdot)_y^{\lfloor s \rfloor} S(x,y) \right) \ .$$

With these new definitions, we can rewrite the fractional magnetic Schrödinger operator as in the statement of the Lemma. □

6.3. **The bilinear form and the DN map.** For every $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $u, v \in H^s$ we define the bilinear form $B^s_{A,q} : H^s \times H^s \to \mathbb{R}$ as in [14]:

$$B^s_{A,q}(u,v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla^s_A u \cdot \nabla^s_A v \, dy dx + \int_{\mathbb{R}^n} quv \, dx \ .$$

Lemma 6.5. *There are constants $\mu', k' > 0$ such that, for all $u \in H^s$,*

$$B^s_{A,q}(u,u) + \mu' \langle u, u \rangle \geq k' \|u\|^2_{H^s} \ .$$

Proof. The formula we want to prove is called *coercivity estimate*. Using (18), we can write

$$\begin{aligned} B^s_{A,q}(u,u) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla^s_A u \cdot \nabla^s_A u \, dy dx + \int_{\mathbb{R}^n} qu^2 \, dx \\ &= \int_{\mathbb{R}^n} u(-\Delta)^s_A u \, dx + \int_{\mathbb{R}^n} qu^2 \, dx = \langle (-\Delta)^s_A u, u \rangle + \langle qu, u \rangle \\ &= \langle (-\Delta)^s u, u \rangle + 2 \langle \nabla^s u, Au \rangle + \left\langle \left(q + \int_{\mathbb{R}^n} |A(x,y)|^2 dy \right) u, u \right\rangle \\ &= \langle (-\Delta)^s u, u \rangle + 2 \left\langle \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy, u \right\rangle + \langle Qu, u \rangle \ , \end{aligned} \tag{24}$$

where $Q(x) := q(x) + \int_{\mathbb{R}^n} |A(x,y)|^2 dy$ belongs to L^p since Cauchy-Schwartz and assumptions (a1) and (a3) imply the embedding $L^{2p} \times L^{2p} \hookrightarrow L^p$. Since we always have $L^p \times H^s \hookrightarrow H^{-s}$, we get $\langle Qu, u \rangle \leq \|u\|_{H^s} \|Qu\|_{H^{-s}} \leq \|Q\|_{L^p} \|u\|^2_{H^s}$. For the second term on the right hand side of (24) we first perform an estimate by means of the Young inequality

$$2 \left\langle \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy, u \right\rangle \leq \epsilon^{-1} \|u\|^2_{L^2} + \epsilon \left\| \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy \right\|^2_{L^2} \ ,$$

then estimate the second term with the Cauchy-Schwartz inequality, in light of (a4):

$$\begin{aligned} \epsilon \left\| \int_{\mathbb{R}^n} \nabla^s u \cdot A \, dy \right\|^2_{L^2} &= \epsilon \left\| \int_{\mathbb{R}^n} \left((\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)) \otimes \alpha \right) \cdot A \, dy \right\|^2_{L^2} \\ &= \epsilon \left\| \int_{\mathbb{R}^n} (\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)) \cdot (A \cdot \alpha) \, dy \right\|^2_{L^2} \\ &= \epsilon \left\| \int_\Omega (\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)) \cdot S(x,y) \, dy \right\|^2_{L^2(\Omega)} \\ &\leq \epsilon \left\| \left(\int_\Omega |\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)|^2 dy \right)^{1/2} \left(\int_\Omega |S(x,y)|^2 \, dy \right)^{1/2} \right\|^2_{L^2(\Omega)} \\ &= \epsilon \int_\Omega \left(\int_\Omega |\nabla^{\lfloor s \rfloor} u(x) - \nabla^{\lfloor s \rfloor} u(y)|^2 dy \int_\Omega |S(x,y)|^2 \, dy \right) dx \end{aligned}$$

$$\leq \epsilon \int_\Omega \left(\int_\Omega (|\nabla^{\lfloor s \rfloor} u(x)| + |\nabla^{\lfloor s \rfloor} u(y)|)^2 dy \int_\Omega \tilde{S}^2(y)\, dy \right) dx$$
$$= \epsilon \|\tilde{S}\|^2_{L^2(\Omega)} \int_\Omega \int_\Omega (|\nabla^{\lfloor s \rfloor} u(x)| + |\nabla^{\lfloor s \rfloor} u(y)|)^2 dy dx$$
$$\leq 2\epsilon \|\tilde{S}\|^2_{L^2(\Omega)} \int_\Omega \int_\Omega (|\nabla^{\lfloor s \rfloor} u(x)|^2 + |\nabla^{\lfloor s \rfloor} u(y)|^2) dy dx$$
$$\leq 4|\Omega|\epsilon \|\tilde{S}\|^2_{L^2(\Omega)} \|\nabla^{\lfloor s \rfloor} u\|^2_{L^2} \leq c\,\epsilon \|\nabla^{\lfloor s \rfloor} u\|^2_{H^{s'}} \leq c\,\epsilon \|u\|^2_{H^s},$$

where the constant c can change from line to line and does not depend on u.
Eventually

$$2 \left\langle \int_{\mathbb{R}^n} \nabla^s u \cdot A\, dy, u \right\rangle \leq \epsilon^{-1} \|u\|^2_{L^2} + c\,\epsilon \|u\|^2_{H^s},$$

which leads to

$$B^s_{A,q}(u,u) \geq B^s_{0,Q}(u,u) - \epsilon^{-1}\|u\|^2_{L^2} - c\,\epsilon \|u\|^2_{H^s}\ . \tag{25}$$

Since $C^\infty_c(\Omega)$ is dense in $L^p(\Omega)$, for every $\delta > 0$ we can find functions Q_s, Q_r such that $Q_s \in C^\infty_c(\Omega)$, $\|Q_r\|_{L^p(\Omega)} \leq \delta$ and $Q = Q_s + Q_r$. Also, if $\phi_j \in C^\infty_c(\Omega)$ and $\|\phi_j\|_{H^s} = 1$ for $j = 1,2$, then $|\langle Q_r \phi_1, \phi_2 \rangle| \leq c\|\phi_1\|_{H^s}\|\phi_2\|_{H^s}\|Q_r\|_{L^p} \leq c\delta$ by the embedding $L^p \times H^s \hookrightarrow H^{-s}$. Therefore,

$$\|Q_r\|_{Z^{-s}} = \sup_{\|\phi_j\|_{H^s}=1} \{|\langle Q_r \phi_1, \phi_2\rangle|\} \leq c\delta\ ,$$

which means that Q belongs to the closure of $C^\infty_c(\Omega)$ in $Z^{-s}(\mathbb{R}^n)$, that is $Q \in Z^{-s}_0(\mathbb{R}^n)$. Now by Lemma 5.1 we know the coercivity estimate for the non-magnetic high exponent case; this lets us write (25) as

$$B^s_{A,q}(u,u) + (\mu + \epsilon^{-1})\langle u, u\rangle \geq (k - c\,\epsilon)\|u\|^2_{H^s}\ ,$$

which is the coercivity estimate for $B^s_{A,q}$ as soon as ϵ is fixed small enough and $\mu' := \mu + \epsilon^{-1}$, $k' := k - c\,\epsilon$ are defined. □

By means of the lemma above, if we assume 0 is not an eigenvalue for the equation, we can proceed as in the proof of Lemma 2.6 from [75] and get the well-posedness of the direct problem for FMSE. This can be stated as follows: if $F \in (\widetilde{H}^s(\Omega))^*$, there exists unique solution $u \in H^s(\mathbb{R}^n)$ to $B^s_{A,q}(u,v) = F(v)$ for all $v \in \widetilde{H}^s(\Omega)$, i.e. unique $u \in H^s(\mathbb{R}^n)$ such that $(-\Delta)^s_A u + qu = F$ in Ω, $u|_{\Omega_e} = 0$. This is also true for non-vanishing exterior value $f \in H^s(\mathbb{R}^n)$ (see [15] and [28]), and the following estimate holds:

$$\|u\|_{H^s(\mathbb{R}^n)} \leq c(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)}), \tag{26}$$

where c does not depend on F and f.
One can prove (see Lemma 3.11 from [14]) that $B^s_{A,q}$ also enjoys these properties:

1. $B^s_{A,q}(v,w) = B^s_{A,q}(w,v)$, for all $v, w \in H^s$,
2. $|B^s_{A,q}(v,w)| \leq c\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}$ for all $v, w \in H^s$, where c does not depend on v and w.
3. $B^s_{A,q}(u_1, e_2) = B^s_{A,q}(u_2, e_1)$, for $u_j \in H^s$ solution to the direct problem for FMSE with exterior value $f_j \in H^s(\Omega_e)$ and e_j any extension of f_j to H^s, $j = 1,2$.

Lemma 6.6. *Let $X = H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ be the abstract quotient space, and let $u_1 \in H^s$ be the solution to the direct problem for FMSE with exterior value $f_1 \in H^s(\Omega_e)$. Then*

$$\langle \Lambda^s_{A,q}[f_1], [f_2]\rangle = B^s_{A,q}(u_1, f_2), \qquad f_j \in H^s,\ j = 1,2$$

defines a bounded, linear, self-adjoint map $\Lambda^s_{A,q} : X \to X^*$. We call $\Lambda^s_{A,q}$ the DN map.

Proof. The proof follows trivially from properties (1)-(3) of $B^s_{A,q}$ and (26). □

6.4. **The gauge.** Consider two couples of potentials (A_1, q_1) and (A_2, q_2). We say that $(A_1, q_1) \sim (A_2, q_2)$ if and only if the following conditions are met:

- $\mathfrak{N}(S_1 - S_2) = 0$ for almost every $x, y \in \mathbb{R}^n$
- $\mathfrak{M}_{(0,\dots,0)}(S_1 - S_2) + \int_{\mathbb{R}^n}(|A_1|^2 - |A_2|^2)dy + (q_1 - q_2) = 0$ for almost every $x \in \mathbb{R}^n$
- $\mathfrak{M}_\beta(S_1 - S_2) = 0$ for all $1 \le |\beta| \le \lfloor s \rfloor$ and almost every $x \in \mathbb{R}^n$.

It is clear from the linearity of $\mathfrak{N}$ and $\mathfrak{M}_\alpha$ that $\sim$ is an equivalence relation, and so the set of all couples of potentials is divided into equivalence classes by $\sim$. We call these *gauge classes*, and if $(A_1, q_1) \sim (A_2, q_2)$ we say that (A_1, q_1) and (A_2, q_2) *are in gauge*.
Observe that this gauge $\sim$ coincides with the one defined in [14] if $s \in (0, 1)$, although it looks quite different. Since in this case $\lfloor s \rfloor = 0$, there is no third condition. In the language of that paper, the first condition reads

$$\begin{aligned}
0 = -\mathfrak{N}(S_1 - S_2) &= S_1(y, x) + S_1(x, y) - S_2(y, x) - S_2(x, y) \\
&= (A_1(x, y) - A_2(x, y)) \cdot \alpha(x, y) + (A_1(y, x) - A_2(y, x)) \cdot \alpha(y, x) \\
&= (A_1(x, y) - A_1(y, x) - A_2(x, y) + A_2(y, x)) \cdot \alpha(x, y) \\
&= 2(A_1 - A_2)_a \cdot \alpha = 2(A_1 - A_2)_{a\|} \cdot \alpha ,
\end{aligned}$$

which is equivalent to $(A_1)_{a\|} = (A_2)_{a\|}$, since the two vectors in the last scalar product have the same direction. Given this fact, for any $v \in H^s$ the first term in the second condition weakly is

$$\begin{aligned}
\langle \mathfrak{M}_{(0,\dots,0)}(S_1 - S_2), v\rangle &= 2\langle S_1 - S_2, v\rangle = 2\langle \alpha \cdot (A_1 - A_2), v\rangle = 2\langle \alpha \cdot (A_1 - A_2)_{\|}, v\rangle \\
&= 2\langle \alpha \cdot (A_1 - A_2)_{s\|}, v\rangle = 2\langle \alpha v, (A_1 - A_2)_{s\|}\rangle = 2\langle (\alpha v)_s, (A_1 - A_2)_{s\|}\rangle \\
&= \langle \alpha(x, y)v(x) + \alpha(y, x)v(y), (A_1 - A_2)_{s\|}\rangle \\
&= \langle \alpha(x, y)(v(x) - v(y)), (A_1 - A_2)_{s\|}\rangle \\
&= \langle \nabla^s v, (A_1 - A_2)_{s\|}\rangle = \langle v, (\nabla\cdot)^s((A_1 - A_2)_{s\|})\rangle ,
\end{aligned}$$

which lets us rewrite the second condition as

$$(\nabla\cdot)^s(A_1)_{s\|} + \int_{\mathbb{R}^n} |A_1|^2 dy + q_1 = (\nabla\cdot)^s(A_2)_{s\|} + \int_{\mathbb{R}^n} |A_2|^2 dy + q_2 .$$

Remark 6.7. *Observe that the gauge enjoyed by the FMSE is quite different from the one holding for the MSE. For the sake of simplicity, we shall compare the classical case with the fractional one in the regime $s \in (0, 1)$, following section 3 in* [14].

Given lemma 6.4, one sees that the following is an equivalent definition for the gauge $\sim$ above:

$$(A_1, q_1) \sim (A_2, q_2) \quad \Leftrightarrow \quad (-\Delta)^s_{A_1} u + q_1 u = (-\Delta)^s_{A_2} u + q_2 u ,$$

for all $u \in H^s(\mathbb{R}^n)$. One may also define the accessory gauge $\approx$ as

$$(A_1, q_1) \approx (A_2, q_2) \quad \Leftrightarrow \quad \exists \phi \in G : (-\Delta)^s_{A_1}(u\phi) + q_1 u\phi = \phi((-\Delta)^s_{A_2} u + q_2 u) ,$$

for all $u \in H^s(\mathbb{R}^n)$, where $G := \{\phi \in C^\infty(\mathbb{R}^n) : \phi > 0, \phi|_{\Omega_e} = 1\}$. These definitions can be extended to the MSE in the natural way. It was proved in lemmas 3.9 and 3.10 of [14] *that the FMSE enjoys the gauge $\sim$, but not $\approx$. In the same discussion, it was argued that the opposite holds for MSE. The reason for this surprising discrepancy should be looked for in the nonlocal structure of the FMSE. As apparent in formula (10) in* [14], *the coefficient of the gradient term in FMSE is not related to the whole vector potential A itself, but only to its antisymmetric part A_a. It is such antisymmetry requirement what eventually does not allow the FMSE to enjoy $\approx$ as the MSE. As a result, the scalar potential q can not be in general uniquely determined as in the classical case.*

6.5. **Main result.**

Remark 6.8. *Assume $W \subseteq \Omega_e$ is an open set and $u \in H^s$ satisfies $u = 0$ and $(-\Delta)^s_A u + qu = 0$ in W. We say that the fractional magnetic Schrödinger operator enjoys the weak unique continuation property (WUCP) if we can deduce that $u = 0$ in Ω. This was proved in* [14] *by using the UCP of the fractional Laplacian for $s \in (0, 1)$; since we know by Theorem 1.2 that*

UCP still holds for $(-\Delta)^s$ in the regime $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, we can deduce WUCP for $(-\Delta)^s_A + q$ by the same proof.

Proof of theorem 1.8. **Step 1.** Without loss of generality, let $W_1 \cap W_2 = \emptyset$. Let $f_i \in C^\infty_c(W_i)$, and let $u_i \in H^s(\mathbb{R}^n)$ solve $(-\Delta)^s_{A_i} u_i + q_i u_i = 0$ with $u_i - f_i \in \widetilde{H}^s(\Omega)$ for $i = 1, 2$. Knowing that the DN maps computed on $f \in C^\infty_c(W_1)$ coincide when restricted to W_2, using Lemmas 6.4 and 6.6 we write this integral identity

$$\begin{aligned}0 &= \langle (\Lambda^s_{A_1,q_1} - \Lambda^s_{A_2,q_2}) f_1, f_2 \rangle = B^s_{A_1,q_1}(u_1, u_2) - B^s_{A_2,q_2}(u_1, u_2) \\ &= \left\langle u_2, \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta u_1 \mathfrak{M}_\beta (S_1 - S_2) \right\rangle + \left\langle u_2, \int_{\mathbb{R}^n} u_1(y) \mathfrak{N}(S_1 - S_2)\, dy \right\rangle + \\ &\quad + \left\langle u_2, u_1 \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right) \right\rangle .\end{aligned}$$

Since if $x \notin \Omega$ or $y \notin \Omega$ we have $A_1(x, y) = A_2(x, y)$ and $q_1(x) = q_2(x)$, we can restrict u_1, u_2 and $\partial^\beta u_1$ over Ω in the previous formula; it is also true that $(\partial^\beta u_1)|_\Omega = \partial^\beta (u_1|_\Omega)$, and therefore

$$\begin{aligned}0 = &\left\langle u_2|_\Omega, \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) \right\rangle + \left\langle u_2|_\Omega, \int_{\mathbb{R}^n} u_1|_\Omega(y) \mathfrak{N}(S_1 - S_2)\, dy \right\rangle + \\ &+ \left\langle u_2|_\Omega, u_1|_\Omega \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right) \right\rangle .\end{aligned}$$

This is the Alessandrini identity, which now we will test with certain solutions in order to obtain information about the potentials. The appropriate test solutions will be produced by means of the Runge approximation property (RAP), which holds for the FMSE because of Remark 6.8 and Lemma 3.15 in [14]. This property says that the set $\mathcal{R} = \{u_f|_\Omega : f \in C^\infty_c(W)\} \subset L^2(\Omega)$ of the restrictions to Ω of those functions u_f solving FMSE for some smooth exterior value f supported in W is dense in $L^2(\Omega)$.

Step 2. Given any $f \in L^2(\Omega)$, by the RAP we can find a sequence of solutions $(u_2)_k \to f$ in L^2 sense as $k \to \infty$. Substituting these in the Alessandrini identity and taking limits, by the arbitrarity of f we can deduce that

$$\begin{aligned}0 = &\sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) + \int_{\mathbb{R}^n} u_1|_\Omega(y) \mathfrak{N}(S_1 - S_2)\, dy + \\ &+ u_1|_\Omega \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right)\end{aligned}$$

holds for every solution $u_1 \in H^s$ and almost every point $x \in \Omega$. Fix $x \in \Omega$. Consider now any $\psi \in C^\infty_c(\Omega)$ and let $g(y) := e^{-1/|x-y|}\psi(y)$, $g(x) = 0$. Since $e^{-1/|x-y|}$ is smooth, it is easy to see that $g \in C^\infty_c(\Omega) \subset L^2(\Omega)$; also, by the properties of $e^{-1/|x-y|}$ one has that $\partial^\beta g(x) = 0$ for all multi-indices β. By the RAP we can find a sequence of solutions $(u_1)_k \to g$ in L^2 sense as $k \to \infty$. Substituting these in the above identity and taking limits, we get

$$\int_{\mathbb{R}^n} e^{-1/|x-y|} \psi(y) \mathfrak{N}(S_1 - S_2)\, dy = 0 \,,$$

which by the arbitrarity of ψ and the positivity of the exponential now implies $\mathfrak{N}(S_1 - S_2) = 0$ for almost all $x, y \in \Omega$. We can now return to the above equation with this new information: for every solution $u_1 \in H^s$ and almost every $x \in \Omega$,

$$0 = \sum_{|\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) + u_1|_\Omega \left(\int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) \right) .$$

For every multi-index β we can consider the function $h_\beta(x) = x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$, which belongs to $L^2(\Omega)$. Let $(h_\beta)_k$ be a sequence of solutions approximating h_β in L^2, which exists by the RAP. We will first substitute $(h_{(0,\dots,0)})_k$ into the last formula, take limits and deduce

$$\mathfrak{M}_{(0,\dots,0)}(S_1 - S_2) + \int_{\mathbb{R}^n} (|A_1|^2 - |A_2|^2) dy + (q_1 - q_2) = 0 \ ,$$

which has the effect of reducing the equation to

$$\sum_{1 \leq |\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) = 0.$$

If $\lfloor s \rfloor \geq 1$, we will repeat the last steps with each h_β such that $|\beta| = 1$, deducing $\mathfrak{M}_\beta(S_1 - S_2) = 0$ for every such β, and subsequently

$$\sum_{2 \leq |\beta| \leq \lfloor s \rfloor} \partial^\beta (u_1|_\Omega) \mathfrak{M}_\beta (S_1 - S_2) = 0.$$

Repeating this process for a total of $\lfloor s \rfloor$ times eventually leads to

$$\mathfrak{M}_\beta(S_1 - S_2) = 0 \quad \forall \ 1 \leq |\beta| \leq \lfloor s \rfloor \ ,$$

which proves the theorem by the definition of the gauge $\sim$. □

7. Possible generalizations and applications beyond this article

We discuss some possible directions for the future research on higher order fractional inverse problems, fractional Poincaré inequalities and unique continuation properties. It seems that now it would be the most natural to reconsider many of the recent developments in fractional inverse problems for higher order operators. We outline here some problems which we would like to see solved in the future.

We have split this section in three in order to emphasize some open problems which we find especially interesting. We do not claim that answers to all questions are positive and it would be interesting to see why and where the greatest difficulties, or even counterexamples, would show up. We first list the most natural directions to continue our work on higher order fractional Calderón problems. One could study for example the following cases:

(i) Is reconstruction from a single measurement [15, 27] possible also in the higher order cases?
(ii) Is there stability [75] in the higher order cases?
(iii) Is there exponential instability [73] in the higher order cases?
(iv) Is there uniqueness for the Calderón problem for fractional semilinear Schrödinger equations [47, 48] in the higher order cases?
(v) Do the monotonicity methods [33, 34] extend to the higher order cases?
(vi) Is there uniqueness for the conductivity type fractional Calderón problems [10, 15] in the higher order cases?
(vii) Could recent results on fractional heat equations [49, 74] be generalized to the higher order cases?
(viii) Does the higher regularity Runge approximation in [11, 28] generalize to higher order cases?

7.1. Unique continuation problems.

We state here some unique continuation problems, which do not follow directly from the earlier results and the techniques that we have developed for this article.

Question 7.1 (UCP for Bessel potentials). *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $p \in [1, \infty)$ and $r \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be an open set. Suppose that $f \in H^{r,p}(\mathbb{R}^n)$, $f|_V = 0$ and $(-\Delta)^s f|_V = 0$. Show that $f \equiv 0$ or give a counterexample.*

The positive answer to question 7.1 is known when $p \in [1, 2]$ (see corollary 3.5). If f has compact support, then the answer is positive for all $p \in [1, \infty)$ (see corollary 3.3). Question 7.1 is also open for the exponents $s \in (0, 1)$ when $p > 2$. See section 3.1 for details.

Question 7.2 (Measurable UCP)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $r \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be an open set and $E \subset V$ a measurable set with positive measure. Suppose that $f \in H^r(\mathbb{R}^n)$, $f|_E = 0$ and $(-\Delta)^s f|_V = 0$. Show that $f \equiv 0$ or give a counterexample.*

The positive answer to question 7.2 is known when $s \in (0, 1)$ [27]. Question 7.2 with a potential q from a suitable class of functions is also an interesting and more challenging problem. See [27, Proposition 5.1] for more details.

Question 7.3 (Alternative strong UCP)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $r \in \mathbb{R}$. Let $V \subset \mathbb{R}^n$ be an open set. Suppose that $f \in H^r(\mathbb{R}^n)$, $f|_V = 0$ and $\partial^\beta((-\Delta)^s f)(x_0) = 0$ for some $x_0 \in V$ and all $\beta \in \mathbb{N}_0^n$. Show that $f \equiv 0$ or give a counterexample.*

Question 7.3 can be seen as a version of the strong unique continuation property (see e.g. [22, 26, 72]) with interchanged decay conditions. When f has compact support, the answer to question 7.3 is positive for $s \in (-n/2, \infty) \setminus \mathbb{Z}$ (see corollary 3.3).

The problems posed in questions 7.1–7.3 for the fractional Laplacian are interesting mathematical problems on their own right, but they also have important applications in inverse problems. The UCPs can be used to show Runge approximation properties for nonlocal equations such as the fractional Schrödinger equation (see theorem 1.7), which in turn can be used to show uniqueness for the corresponding nonlocal inverse problem (see theorem 1.6). The UCPs have also applications in integral geometry, where the uniqueness of the ROI problem for the d-plane transform can be reduced to a unique continuation problem for the fractional Laplacian (see remark 4.1 and corollaries 1.3 and 1.4).

7.2. **Fractional Poincaré inequality for L^p-norms.** In section 3.2 we prove the fractional Poincaré inequality for L^2-norms in multiple ways. The inequality is needed for the well-posedness of the inverse problem for the fractional Schrödinger equation. One could try to extend the Poincaré inequality for general L^p-norms. This suggests the following natural question which is also interesting from the pure mathematical point of view.

Question 7.4. *Let $s \geq 0$, $1 \leq p < \infty$, $K \subset \mathbb{R}^n$ compact set and $u \in H^{s,p}(\mathbb{R}^n)$ such that $\mathrm{spt}(u) \subset K$. Show that there exists a constant $c = c(n, K, s, p)$ such that*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq c \left\| (-\Delta)^{s/2} u \right\|_{L^p(\mathbb{R}^n)} \tag{27}$$

or give a counterexample.

Since we have presented several proofs for the Poincaré inequality in the case $p = 2$, one could try some of our methods to solve question 7.4. However, some of our proofs are heavily based on Fourier analysis and those approaches might be difficult to generalize to the L^p-case when $p \neq 2$. Like in theorem 1.5 and in theorem 3.17, another interesting question is whether one can replace u in the left-hand side of equation (27) with $(-\Delta)^{t/2} u$ when $0 \leq t \leq s$, and whether the constant c in equation (27) can be expressed in terms of the classical Poincaré constant when $s \geq 1$.

7.3. **The Calderón problem for determining a higher order PDO.** In this discussion, we try to make as simple assumptions as possible. The whole point is to introduce a new inverse problem that we think is a very natural and interesting one, at least from a pure mathematical point of view. Therefore the optimal regularity in the statement of the problem is not as important. Let Ω be a domain with smooth boundary. Suppose that $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ is a partial differential operator (PDO) of order m with smooth coefficients on Ω. We argue in section 3.1 that the operator $(-\Delta)^s + P(x, D)$ admits the UCP (in open sets).

It is shown in the seminal work of Ghosh, Uhlmann and Salo [28] that if $P(x, D)$ is of order $m = 0$, then one can determine the zeroth order coefficient (i.e. the potential q) from the associated DN map. It was then later shown in [11] that if $P(x,D)$ is of order $m=1$, then one can also determine the coefficients (i.e. the potential q and the magnetic drift b) from the associated DN map whenever the order of $(-\Delta)^s$ is large enough, namely when $2s>1$. This and our work on higher order Calderón problems motivate the following inverse problem.

Question 7.5. *Suppose that $\Omega\subset\mathbb{R}^n$ is a bounded open domain with smooth boundary. Let $P_j(x,D)$, $j=1,2$, be smooth PDOs of order $m\in\mathbb{N}$ in Ω. Let $s\in\mathbb{R}^+\setminus\mathbb{Z}$ be such that $2s>m$. Given any two open sets $W_1,W_2\subset\Omega_e$, suppose that the DN maps Λ_{P_i} for the equations*

$$((-\Delta)^s+P_j(x,D))u_j=0 \ \ in\ \Omega$$

satisfy $\Lambda_{P_1}f|_{W_2}=\Lambda_{P_2}f|_{W_2}$ for all $f\in C_c^\infty(W_1)$. Show that $P_1(x,D)=P_2(x,D)$ or give a counterexample.

Another interesting question is whether the strong UCP [26] can be extended to higher order PDOs.

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(D)

On some partial data Calderón type problems with mixed boundary conditions

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ON SOME PARTIAL DATA CALDERÓN TYPE PROBLEMS WITH MIXED BOUNDARY CONDITIONS

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Abstract. In this article we consider the simultaneous recovery of bulk and boundary potentials in (degenerate) elliptic equations modelling (degenerate) conducting media with inaccessible boundaries. This connects local and nonlocal Calderón type problems. We prove two main results on these type of problems: On the one hand, we derive simultaneous bulk and boundary Runge approximation results. Building on these, we deduce uniqueness for localized bulk and boundary potentials. On the other hand, we construct a family of CGO solutions associated with the corresponding equations. These allow us to deduce uniqueness results for arbitrary bounded, not necessarily localized bulk and boundary potentials. The CGO solutions are constructed by duality to a new Carleman estimate.

1. Introduction

There has been a substantial amount of work on nonlocal inverse problems in the last years (see for instance the survey articles [Sal17, Rül18] and the references cited below). These nonlocal equations arise naturally in many problems from applications including, for instance, finance [AB88, Sch03, Lev04], ecology [RR09, H+10, MV17], image processing [GO08], turbulent fluid mechanics [Con06], quantum mechanics [Las00, Las18] and elasticity [Sch89] as well as many other fields [GL97, MK00, Eri02, DGLZ12, AVMRTM10, DZ10, DGV13, RO15, BV16]. In this article, we provide yet another point of view on these non-local inverse problems by adopting a local "Caffarelli-Silvestre perspective". The resulting equations and the associated inverse problems are of interest in their own right, modelling for instance situations in which there are unknown, not-directly measurable *fluxes* or *potentials on the boundary* of an electric device in addition to *electric and/or magnetic potentials in the interior* of it. Moreover, we also include situations in which the conducting property of the (electric) medium may deteriorate or improve towards the boundary. In this setting of unknown and not directly accessible *boundary and bulk potentials* at possibly *degenerate* conductivities, we are interested in the reconstruction of both of these boundary and bulk potentials which are coupled through possibly degenerate, linear elliptic equations.

1.1. **A model setting.** As a model case, we consider the following problem set-up with non-degenerate conductivities: Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, C^2-regular (or smooth) domain, modelling the conducting body. Assume that $\Sigma_1, \Sigma_2 \subset \partial\Omega$ are two disjoint, relatively open, smooth non-empty sets. Consider the following magnetic Schrödinger equation with mixed boundary conditions

$$(1)\qquad \begin{aligned} -\Delta u - iA\cdot\nabla u - i\nabla\cdot(Au) + (|A|^2+V)u &= 0 \text{ in } \Omega,\\ \partial_\nu u + qu &= 0 \text{ on } \Sigma_1,\\ u &= f \text{ on } \Sigma_2,\\ u &= 0 \text{ on } \partial\Omega\setminus(\Sigma_1\cup\Sigma_2), \end{aligned}$$

where, for simplicity, the coefficients are supposed to satisfy the conditions that

$$V \in L^\infty(\Omega,\mathbb{R}),\ A \in L^\infty(\Omega,\mathbb{R}^n),\ q \in L^\infty(\partial\Omega,\mathbb{R}), \tag{2}$$

and that

$$\nu \cdot A = 0 \text{ on } \partial\Omega. \tag{3}$$

In analogy to the setting of the Schrödinger version of the partial data Calderón problem we seek to recover the potentials A, V and q from boundary measurements encoded in the (partial) Dirichlet-to-Neumann map

$$\Lambda_{A,V,q} : \widetilde{H}^{\frac{1}{2}}(\Sigma_2) \mapsto H^{-\frac{1}{2}}(\Sigma_2),\ f|_{\Sigma_2} \mapsto \partial_\nu u|_{\Sigma_2}.$$

In a formally correct way this will be defined by means of the bilinear form

$$B_{A,V,q}(u,v) := \int\limits_\Omega \nabla u \cdot \overline{\nabla v} + i\overline{v}A\cdot\nabla u - iAu\cdot\overline{\nabla v} + Vu\overline{v}dx + \int\limits_{\Sigma_1} qu\overline{v}d\mathcal{H}^{n-1} \text{ for } u,v \in H^1(\Omega,\mathbb{C}), \tag{4}$$

in Definition 3.9 in Section 3 below. Here $\overline{u}$ denotes the complex conjugate of u.

We remark that in contrast to the "usual" partial data, magnetic Schrödinger version of the Calderón problem, in (1) the first boundary condition yields a new ingredient: Besides the partial data character of the problem which is encoded in the measurement of data on Σ_2 only, we now also consider a setting in which a part of the domain, Σ_1, is modelled as inaccessible and on which we also seek to recover an unknown boundary flux/potential. This is closely related to the so-called inverse Robin problem which arises, for instance, in corrosion detection (see [Ing97] and the references below). We thus combine a Calderón with a Robin inverse problem, studying a setting in which in addition to the bulk potentials in the interior of the domain Ω also unknown boundary potentials and mixed-type boundary conditions are present.

In this framework it is our objective to investigate the following questions:

(Q1) Let us assume that A, V, q and $\Lambda_{A,V,q}$ are as above. Can we then simultaneously recover the boundary potential q, the magnetic potential A and the bulk potential V, if the bulk (gradient) potentials A and V are supported in a set $\Omega_1 \Subset \Omega$ which is open and bounded?

(Q2) Is this recovery still possible – at least for V and q – if the bulk potentials are not compactly supported in Ω? In particular, is this possible, if there is no longer some safety distance between Ω_1 and the boundary parts given by Σ_1 and Σ_2?

Let us comment on these questions: Both of these are *partial* data problems with the objective of reconstructing unknown potentials *simultaneously* on the boundary and in the bulk (see [KS14] for a survey on the known partial data results). As explained in the sequel, the effect of the boundary and bulk potentials however is expected to differ quite substantially in the context of the inverse problem.

On the one hand, the magnetic and scalar potentials A and V are *local, interior* potentials. The dimension counting heuristics on the recovery of these follow from the ones for the classical Calderón problem: One seeks to recover unknown objects of n degrees of freedom from the (partial) Dirichlet-to-Neumann map, an operator which encodes $2(n-1)$ degrees of freedom. Building on the seminal result [SU87], a canonical tool to address the associated uniqueness question for the "local" potentials A, V are complex geometric optics (CGO) solutions. It is further well-known that the presence of the magnetic potential creates additional difficulties due to the resulting gauge invariances. In spite of this, both in the full and the partial data settings, CGO solutions have been constructed starting with the works [NSU95, Sun93], see also [Chu14, CT16]. These however do not cover our mixed-data set-up in which additional unknowns are present on the boundary.

On the other hand, the heuristics on the recovery of the boundary potential give hope for substantially stronger *boundary* uniqueness results: Indeed, recalling from the argument above that the Dirichlet-to-Neumann operator formally contains $2(n-1)$ degrees of freedom, we note that the recovery of q which is a function of $n-1$ degrees of freedom is always overdetermined. Hence, in analogy to [GRSU20], even single measurement results for the uniqueness of the boundary data can be expected (see [CJ99, ADPR03] for results of this type for the Robin inverse problem). We view this as a "non-local" reconstruction problem at the boundary; a connection to the fractional Calderón problem is explained below.

In dealing with the questions (Q1) and (Q2) we thus combine ideas from "local" and "non-local" inverse problems. Here in our analysis of the question (Q1) the softer "non-local" effects dominate, while in our approach towards the problem (Q2), the "local" interior effects prevail. In particular, we thus

- address question (Q1) using simultaneous Runge approximation results in the bulk and on the boundary (see Sections 4-5),
- deal with question (Q2) by constructing suitable CGO solutions (see Sections 6-7).

Indeed, in (1) we view the boundary data on Σ_1 as a local formulation à la Caffarelli-Silvestre [CS07] of a Schrödinger equation for the half-Laplacian on Σ_1. Then, using the fact that in question (Q1) the local interior potentials A, V are only supported in a compact subset of Ω which has some safety distance to Σ_1, Σ_2, this indicates that the problem can be reduced to a full data type problem by means of Runge approximation results. In order to deal with the interior potentials, we recall the Runge approximation ideas developed in [AU04] and quantified in [RS19b]. These allow one to approximate full data CGO solutions in Ω_1 by partial data solutions in the whole domain Ω. Compared to [AU04] in our setting of (1), we have to deal with the additional challenge that also on the boundary of Ω an unknown potential is present. However, due to the disjointness of the domains Σ_1 and Σ_2 and motivated by the interpretation of the equation on Σ_1 as a fractional Schrödinger equation, it is possible to prove corresponding simultaneous density results *both in the bulk and on the boundary* (see Proposition 5.1).

In contrast to the setting of the question (Q1), the question (Q2) is dominated by "local" effects. Since now V may be supported in the whole domain Ω and may in particular be supported up to the sets Σ_1, Σ_2, the Runge approximation techniques are no longer applicable in Ω. In order to nevertheless address the uniqueness question, we thus construct CGO solutions. Here we can however not directly make use of the known full/partial data CGO solutions from the magnetic Schrödinger problem, due to the presence of the additional boundary condition on Σ_1 in (1). A related difficulty had earlier been addressed in [Chu14, Chu15] in the context of partial data problems. However with respect to the setting in [Chu15] our equation on the boundary imposes an additional challenge in that the potential q is assumed to be *unknown* and the problem is of *mixed-data* type. Thus, aiming at uniqueness results by means of CGO solutions, we construct a new family of CGO solutions which takes into account *both* the unknown bulk and boundary potentials. This relies on new Carleman estimates for a Caffarelli-Silvestre type extension problem (see Proposition 6.1 and Corollary 6.4).

1.2. **A family of (degenerate) boundary-bulk partial data Schrödinger problems.** Before discussing our main results, let us present a variation of the problem outlined above in which we also study operators whose conductivities or potentials depend on the distance to the boundary. More precisely, for $s \in (0,1)$ and for the potentials A, V, q satisfying the conditions in

(3) and (2), we consider the following equation

$$
\begin{aligned}
-\nabla \cdot d^{1-2s}\nabla u - iAd^{1-2s}\cdot \nabla u - i\nabla\cdot (d^{1-2s}Au) + d^{1-2s}(|A|^2+V)u &= 0 \text{ in } \Omega,\\
\lim_{d(x)\to 0} d^{1-2s}\partial_\nu u + qu &= 0 \text{ on } \Sigma_1,\\
u &= f \text{ on } \Sigma_2,\\
u &= 0 \text{ on } \partial\Omega \setminus (\Sigma_1\cup\Sigma_2).
\end{aligned} \tag{5}
$$

Here $d:\Omega\to[0,\infty)$ denotes a smooth function which is equal to the distance to the boundary in a neighbourhood of the boundary. If not otherwise explained, all the functions and in particular u,v in the sequel will be complex-valued. As in the case $s=\frac{1}{2}$ we define an associated (partial) Dirichlet-to-Neumann map as

$$\Lambda_{s,A,V,q} : \widetilde{H}^s(\Sigma_2)\to H^{-s}(\Sigma_2),\ f|_{\Sigma_2}\mapsto \lim_{d(x)\to 0} d(x)^{1-2s}\partial_\nu u|_{\Sigma_2}.$$

Again, in a formally precise way it is defined by means of the bilinear form

$$
\begin{aligned}
B_{s,A,V,q}(u,v) = &\int_\Omega d^{1-2s}\nabla u\cdot\overline{\nabla v} - d^{1-2s} i\overline{v}A\cdot\nabla u + d^{1-2s} iAu\cdot\overline{\nabla v} + d^{1-2s}(V+|A|^2)u\overline{v}dx\\
&+\int_{\Sigma_1} qu\overline{v}d\mathcal{H}^{n-1} \text{ for } u,v\in H^1(\Omega,d^{1-2s}).
\end{aligned} \tag{6}
$$

For the equation (5) and the Dirichlet-to-Neumann map (6) (and a slight variant of it, see (9) below) we seek to investigate the analogues of the questions (Q1) and (Q2) for $s\in(0,1)$, i.e. the reconstruction of the scalar, magnetic and boundary potentials from the generalized Dirichlet-to-Neumann map in the cases that the interior potentials are either supported away from the boundary or reach up to the boundary.

These questions share the same type of local and nonlocal features as explained above. However, the relation to the fractional Laplacian may become more transparent. To illustrate this, we recall the Caffarelli-Silvestre extension [CS07] which allows one to compute the fractional Laplacian through a problem of the type (5) in the unbounded domain $\mathbb{R}^{n+1}_+$. To this end, given a function $u\in H^s(\mathbb{R}^n)$ one considers the degenerate elliptic problem

$$
\begin{aligned}
\nabla\cdot x_{n+1}^{1-2s}\nabla\tilde{u} &= 0 \text{ in } \mathbb{R}^{n+1}_+,\\
\tilde{u} &= u \text{ on } \mathbb{R}^n\times\{0\}.
\end{aligned} \tag{7}
$$

The fractional Laplacian then turns into the generalized Dirichlet-to-Neumann operator associated with this equation; $(-\Delta)^s u := c_s \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}(x)$. The idea of realizing the fractional Laplacian as a (degenerate) Dirichlet-to-Neumann operator of a local, degenerate elliptic equation has been further extended to rather general variable coefficient settings, see for instance [ST10, CS16]. In this sense, we view the equation (5) and also (1) as a localized proxy for the inverse problem of recovering the potentials $\tilde{A}$, $\tilde{V}$ and $\tilde{q}$ in the fractional Schrödinger equation

$$
\begin{aligned}
(-(\nabla+i\tilde{A})^2+\tilde{V})^s u + \tilde{q}u &= 0 \text{ in } \tilde{\Omega}\subset\mathbb{R}^{n-1},\\
u &= f \text{ on } \tilde{W}\subset\mathbb{R}^{n-1}\setminus\overline{\tilde{\Omega}},
\end{aligned} \tag{8}
$$

from an associated Dirichlet-to-Neumann map. We note that in (5) the set $\Omega\subset\mathbb{R}^n$ plays the role of the extended space $\mathbb{R}^{n+1}_+$ in (7). As a word of caution we however remark that, following the classical formulation of the Caffarelli-Silvestre extension (7) as an equation in $n+1$ dimensions, the formulation of the problem (7) is shifted by one dimension with respect to our setting in (5). In contrast to the Caffarelli-Silvestre extension problem associated with (8), (5) has the advantage that we can work in a bounded domain Ω. This allows us to circumvent the discussion of various issues which arise in the inverse problem for the full Caffarelli-Silvestre extension of (8). We emphasize that just as (5) the problem (8) has a natural gauge invariance. In particular it represents yet another nonlocal model with gauge invariances besides the ones which had been introduced and analysed in [BGU18, Cov20a, CLR20, Li20a].

1.3. **Main results.** As one of the main results of this article we provide a complete answer (at L^∞ regularity) for the uniqueness question in (Q1) in the case $s = \frac{1}{2}$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open, bounded and C^2-regular domain. Assume $\Omega_1 \Subset \Omega$ is an open, bounded set with $\Omega \setminus \Omega_1$ simply connected and that $\Sigma_1, \Sigma_2 \subset \partial\Omega$ are two disjoint, relatively open sets. If the potentials $q_1, q_2 \in L^\infty(\Sigma_1)$, $A_1, A_2 \in C^1(\Omega_1, \mathbb{R}^n)$ and $V_1, V_2 \in L^\infty(\Omega_1)$ in the equation (1) are such that*

$$\Lambda_1 := \Lambda_{A_1,V_1,q_1} = \Lambda_{A_2,V_2,q_2} =: \Lambda_2 \ ,$$

then $q_1 = q_2$, $V_1 = V_2$ and $dA_1 = dA_2$.

This relies on simultaneous approximation results for the bulk and boundary measurements. For instance, restricting first to the case in which $A = 0$ and considering the sets

$$S_{V,q} := \{u \in L^2(\Omega) : \ u \text{ is a weak solution to (1) in } \Omega\},$$
$$\tilde{S}_{V,q} := \{u \in H^1(\Omega_1) : \ u \text{ is a weak solution to (1) in } \Omega\} \subset L^2(\Omega_1),$$

we prove the following simultaneous boundary and bulk approximation result.

Lemma 1.1. *Assume that the conditions from Section 2.3 hold for Ω, Ω_1 and Σ_1, Σ_2. Let $V \in L^\infty(\Omega)$, $q \in L^\infty(\partial\Omega)$. Then the set*

$$\mathcal{R}_{bb} := \{(u|_{\Sigma_1}, u|_{\Omega_1}) : \ u|_{\Sigma_1} = Pf|_{\Sigma_1} \text{ and } u|_{\Omega_1} = Pf|_{\Omega_1} \text{ with } f \in C_c^\infty(\Sigma_2)\} \subset L^2(\Sigma_1) \times L^2(\Omega_1)$$

is dense in $L^2(\Sigma_1) \times \tilde{S}_{V,q}$ with the $L^2(\Sigma_1) \times L^2(\Omega_1)$ topology. Here P denotes the Poisson operator from Definition 3.3.

We remark that substantial generalizations are possible for these type of approximation results. These involve both approximations in stronger topologies and more general Schrödinger type operators. We refer to Lemma 4.2 and the discussion in Sections 4 and 5 for more on this.

Similar approximation results also hold in the setting of the problem (5), see for instance Proposition 5.1. Furthermore, an analogous uniqueness result as in Theorem 1 can also proved in this situation, see Theorem 3. In spite of the degenerate character of the equation (5) this is reduced to the construction of CGO solutions to a non-degenerate Schrödinger type problem and an application of the Runge approximation result.

We next turn to a variant of the problem (5) and investigate the question (Q2) for this model. Here we follow the usual notation from the Caffarelli-Silvestre extension which was also already used in (7) and assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set. We emphasise that we thus increase the dimension of the problem under consideration by one with respect to our discussion of the question (Q1). Here, in order to simplify the geometric setting which in partial data problems is not uncommon, we assume that $\overline{\Sigma_1} := \overline{\Omega} \cap \{x_{n+1} = 0\}$ and that $\Sigma_2 = \partial\Omega \setminus \overline{\Sigma}_1$. In contrast of considering (5) we study a slight variation of it. For $q \in L^\infty(\Sigma_1)$, $V \in L^\infty(\Omega)$ we investigate solutions to

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s} \nabla u + V x_{n+1}^{1-2s} u &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \Sigma_2, \\ \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} u + qu &= 0 \text{ on } \Sigma_1. \end{aligned} \tag{9}$$

Here, for instance, $f \in C_c^\infty(\Sigma_2)$. Using the same ideas as in Section 3, it can be shown that this problem is well-posed if zero is not an eigenvalue with respect to our mixed data setting. Thus, an associated Dirichlet-to-Neumann map can be (formally) defined as the map,

$$\Lambda_{V,q} : f \mapsto \lim_{x_{n+1}\to\partial\Omega} x_{n+1}^{1-2s}\partial_\nu u|_{\Sigma_2}.$$

We refer to Section 7 for a more detailed discussion of the Dirichlet-to-Neumann map associated with (9) and the function spaces it acts on. Now no longer imposing conditions on the support of V, we seek to recover both V and q. Since this implies that Runge approximation methods are no longer applicable in the interior of Ω, we instead rely on a new Carleman inequality for the equation (9) (see Proposition 6.1 and Corollary 6.4) and by duality construct CGO solutions from it:

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^{n+1}_+$, $n \geq 3$, be an open, bounded smooth domain. Assume that $\Sigma_1 = \partial\Omega \cap (\mathbb{R}^n \times \{0\})$ is a relatively open, non-empty subset of the boundary, and that $\Sigma_2 = \partial\Omega \setminus \overline{\Sigma_1}$. Let $s \in [1/2, 1)$ and let $V \in L^\infty(\Omega)$ and $q \in L^\infty(\Sigma_1)$. Then there exists a non-trivial solution $u \in H^1(\Omega, x_{n+1}^{1-2s})$ of the problem*

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s}\nabla u + x_{n+1}^{1-2s} V u &= 0 \text{ in } \Omega, \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1} u + qu &= 0 \text{ on } \Sigma_1, \end{aligned} \tag{10}$$

of the form $u(x) = e^{\xi'\cdot x'}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r(x))$, where $k \in \mathbb{R}^{n+1}$, $\xi' \in \mathbb{C}^n$ is such that $\xi'\cdot\xi' = 0$, $k \cdot \xi' = 0$, and

- *if $s = 1/2$, then $\|r\|_{L^2(\Omega)} = O(|\xi'|^{-\frac{1}{2}})$, $\|r\|_{H^1(\Omega)} = O(|\xi'|^{\frac{1}{2}})$ and $\|r\|_{L^2(\Sigma_1)} = O(1)$;*
- *if $s > 1/2$, then $\|r\|_{L^2(\Omega, x_{n+1}^{1-2s})} = O(|\xi'|^{-s})$, $\|r\|_{H^1(\Omega, x_{n+1}^{1-2s})} = O(|\xi'|^{1-s})$ and $\|r\|_{L^2(\Sigma_1)} = O(|\xi'|^{1-2s})$.*

Remark 1.3. *We remark that by inspection of the proof given in Section 7 below, one observes that for $s = \frac{1}{2}$ one only needs to assume that $n \geq 2$ and may work with $\xi' \in \mathbb{R}^{n+1}$ instead of $\xi' \in \mathbb{R}^n$.*

Remark 1.4. *Instead of considering CGOs of the form*

$$u(x) = e^{\xi'\cdot x'}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r(x)),$$

by the same arguments we can also construct CGOs of the form

$$u(x) = e^{\xi'\cdot x'}(e^{ik'\cdot x' - k_{n+1}x_{n+1}^{2s}} + r(x))$$

for $k_{n+1} > 0$ which thus have some decay behaviour in the x_{n+1}-direction in the amplitude.

We emphasize that the CGOs here contain new ingredients compared to the classical CGOs in that the amplitude contains the normal contribution $k_{n+1}x_{n+1}^{2s}$ instead of a linear phase. Also, in order to avoid dealing with the non-degeneracy of the equation, with respect to the classical CGOs, we loose one dimension in the case $s \in (\frac{1}{2}, 1)$, having to restrict ourselves to $n \geq 3$ (and thus $n + 1 \geq 4$).

Relying on this new family of CGO solutions for $s \in (\frac{1}{2}, 1)$, we give a complete answer to the question (Q2) for $n \geq 3$:

Theorem 2. *Let $\Omega \subset \mathbb{R}^{n+1}_+$, $n \geq 3$, be an open, bounded and smooth domain. Assume that $\Sigma_1 := \partial\Omega \cap \{x_{n+1} = 0\}$ and $\Sigma_2 \subset \partial\Omega \setminus \Sigma_1$ are two relatively open, non-empty subsets of the boundary such that $\overline{\Sigma_1 \cup \Sigma_2} = \partial\Omega$. Let $s \in (1/2, 1)$. If the potentials $q_1, q_2 \in L^\infty(\Sigma_1)$ and $V_1, V_2 \in L^\infty(\Omega)$ relative to problem (9) are such that*

$$\Lambda_1 := \Lambda_{s,V_1,q_1} = \Lambda_{s,V_2,q_2} =: \Lambda_2 \ ,$$

then $q_1 = q_2$ *and* $V_1 = V_2$.

This provides the first uniqueness result combining both *local* and *nonlocal* features of the described form in Calderón type problems. We hope that these ideas could also be of interest in the study of (8).

In the case $s = \frac{1}{2}$, the lack of sufficiently strong boundary decay estimates only allows us to recover V given q (see Proposition 7.1 and Remark 6.3). We seek to study improvements of this in future work.

1.4. **Connection to the literature.** The study of nonlocal fractional Calderón type problems has been very active in the last years: After its formulation and the study of its uniqueness properties in [GSU20], optimal stability and uniqueness in scaling critical spaces have been addressed in [RS20a, RS18]. In [GRSU20] single measurement reconstruction results have been proved, see also [HL19, HL20] for full-data reconstruction results by monotonicity methods. Further, variable coefficient versions were studied in [GLX17, Cov20b] and magnetic potentials were introduced in [BGU18, Cov20a, CLR20, Li20a]. We refer to the articles [LLR19, Lin20, Li20b, CMR20, RS19c, GFR19] for further variants of related nonlocal problems. Reviews for the fractional Calderón problem with additional literature can be found in [Sal17, Rül18].

In all these works, a striking flexibility property of nonlocal equations is used, see also [DSV17, DSV19, RS19a, Rül19, GFR20]: As a consequence of the antilocality of the fractional Laplacian (see [Ver93]), one obtains that the set of solutions to a given fractional Schrödinger problem with scaling-critical or subcritical potential in Ω is already dense in $L^2(\Omega)$. This allows one to prove uniqueness and reconstruction results by means of Runge approximation properties. These often lead to substantially stronger results for the nonlocal inverse problems than the known ones (e.g. partial data, low regularity) for the classical local case. Apart from the intrinsic interest in the described effects of anti- and nonlocality, these nonlocal inverse problems are also of relevance in various applications and in order to obtain an improved understanding of the classical, local Calderón problem.

By virtue of the Caffarelli-Silvestre extension, the described fractional Schrödinger inverse problems are also closely connected to (degenerate) versions of the Robin inverse problem as proposed and formulated for instance in [KS95, KSV96, SVX98, BCC08]. These problems arise in the indirect detection of corrosion through electrostatic measurements and in thermal imaging techniques. Mathematically, under sufficiently strong regularity conditions on the potentials and the measurement sets, these can be addressed using ideas on unique continuation, see for instance [CFJL03, Sin07, AS06, Cho04, BBL16, BCH11, HM19] for uniqueness, stability and reconstruction results on the Robin inverse problem. In contrast to our setting which combines unknown potentials on the boundary and in the bulk, the literature on the inverse Robin boundary problem however typically does *not* consider a combination of these two challenges. Typically, in works on the inverse Robin problem, a setting complementary to the classical Calderón problem is studied, where it is assumed that the bulk properties of the material are known, while reconstruction at inaccessible boundaries is explored.

The classical, local Calderón problem is a prototypical and well-studied elliptic inverse problem. It had originally been formulated and studied in its linearized version by Calderón, see [Cal06]. For $n \geq 3$ the uniqueness question for the full, nonlinear problem had been solved in the seminal work [SU87] by introducing CGO solutions. For recent, low regularity contributions on uniqueness, we refer to [CR16, HT13, Hab15]. Also stability [Ale88], reconstruction [Nac88] and partial data [KS14] problems have been addressed. We refer to [Uhl09] for a more detailed survey on the results for the Calderón problem.

In this article, we seek to combine both effects, local and nonlocal, with the objective of connecting these and providing new perspectives on them. Studying boundary and bulk potentials simultaneously, we thus combine both the local (bulk) effects and the nonlocal (boundary) effects of the two classes of inverse problems described above. By studying the questions (Q1) and (Q2) outlined above, we illustrate that either effect can dominate. Combining the two settings we investigate an interesting model problem in its own right and hope to to derive ideas and results connecting the local and nonlocal realms.

1.5. **Organization of the remainder of the article.** The remainder of the article is organized as follows. In the next section, we introduce our notation and recall some results on weighted Sobolev spaces. Next, we discuss the well-posedness of problems (1) and (5). Building on this, in Sections 4 and 5 we address the question (Q1). Here we also provide the proofs of Theorem 1 and Lemma 1.1. In Section 6 we prove a new Carleman estimate for the generalized Caffarelli-Silvestre extension in (9). Arguing by duality, we derive the existence of CGO solutions for these in Section 7 and thus present the proof of Proposition 1.2. Building on this, we provide the proof of Theorem 2 there. Last but not least, we provide a proof of the density result of Proposition 2.3 in the appendix.

2. Notation and Auxiliary Results

2.1. **Function spaces.** In the following we will make use a number of function spaces. Unless explicitly stated, all function spaces consist of complex valued functions.

2.1.1. *Weighted Sobolev spaces.* We will fix $s \in (0,1)$ and assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded, C^2-regular domain. We let $d : \Omega \to [0,\infty)$ denote a C^1-regular function which close to the boundary $\partial\Omega$ measures the distance to $\partial\Omega$ and is extended to Ω in a C^1-regular way. Then we set:

$$\begin{aligned} L^2(\Omega, d^{1-2s}) &:= \{u : \Omega \to \mathbb{C} \text{ measureable}: \ \|d^{\frac{1-2s}{2}} u\|_{L^2(\Omega)} < \infty\}, \\ H^1(\Omega, d^{1-2s}) &:= \{u : \Omega \to \mathbb{C} \text{ measureable}: \ \|d^{\frac{1-2s}{2}} u\|_{L^2(\Omega)} + \|d^{\frac{1-2s}{2}} \nabla u\|_{L^2(\Omega)} < \infty\}. \end{aligned}$$

We further use the following notation for fractional Sobolev spaces:

$$H^s(\Omega) := \{u|_\Omega : \ u \in H^s(\mathbb{R}^n)\},$$

and equip it with the quotient topology

$$\|u\|_{H^s(\Omega)} := \inf\{\|U\|_{H^s(\mathbb{R}^n)} : \ U|_\Omega = u\}.$$

It will also be convenient to work with functions obtained by completion of smooth functions with compact support:

$$\widetilde{H}^s(\Omega) := \text{ closure of } C_c^\infty(\mathbb{R}^n) \text{ in } H^s(\mathbb{R}^n).$$

We remark that in our setting of sufficiently regular domains, we have that

$$\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}},$$

where $H^s_{\overline{\Omega}} := \{u \in H^s(\mathbb{R}^n) : \ \text{supp}(u) \subset \overline{\Omega}\}$. Working in charts, similar definitions hold for functions on (sub)manifolds.

We recall the following extension and trace estimates which we will be using for the weighted $H^1(\Omega, d^{1-2s})$ spaces. We remark that both Lemmas 2.1 and 2.2 are not new and had first been proved in [Nek93]. We only provide a (rough) argument for these for completeness and the convenience of the reader.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, C^2-regular set and let $u \in H^1(\Omega, d^{1-2s})$. Then there exists a continuous trace operator into $H^s(\partial\Omega)$, i.e. $u|_{\partial\Omega}$ exists in a weak sense, coincides with $u|_{\partial\Omega}$ if $u \in C^\infty(\Omega)$ and*

$$\|u|_{\partial\Omega}\|_{H^s(\partial\Omega)} \leq C\|u\|_{H^1(\Omega,d^{1-2s})}.$$

Proof. The claim follows from the flat result (see for instance [RS20a, Lemma 4.4] for this) and a partition of unity. Indeed, using boundary normal coordinates and a partition of unity $\{\eta_k\}$ whose elements have a sufficiently small support we obtain with $u_k = u\eta_k$ and $\tilde{u}_k(x) := u_k \circ \phi_k(x)$ where ϕ_k locally maps the boundary of Ω to the flat boundary $\{x_{n+1} = 0\}$

$$\begin{aligned}
&C(\|d^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)} + \|d^{\frac{1-2s}{2}}u\|_{L^2(\Omega)}) \geq \sum_{k=1}^{M}\left(\|d^{\frac{1-2s}{2}}\nabla u_k\|_{L^2(\Omega)} + \|d^{\frac{1-2s}{2}}u_k\|_{L^2(\Omega)}\right)\\
&\geq C^{-1}\sum_{k=1}^{M}\left(\|x_{n+1}^{\frac{1-2s}{2}}\nabla \tilde{u}_k\|_{L^2(\mathbb{R}^{n+1}_+)} + \|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}_k\|_{L^2(\mathbb{R}^{n+1}_+)}\right)\\
&\geq C^{-1}\sum_{k=1}^{M}\|\tilde{u}_k\|_{H^s(\{x_{n+1}=0\})}\\
&\geq C^{-1}\sum_{k=1}^{M}\|u_k\|_{H^s(\partial\Omega)} \geq C^{-1}\|u\|_{H^s(\partial\Omega)}.
\end{aligned}$$

Here $C > 1$ is a generic constant which may change from line to line. In the estimates, we have used that $|\nabla\phi_k|$ can be chosen as small as desired in the support of u_k (by possibly enlarging $M \in \mathbb{N}$) and that $\|\tilde{u}_k\|_{H^s(\{x_{n+1}=0\})} \sim \|u_k\|_{H^s(\partial\Omega)}$ (see for instance [McL00, Theorem 3.23]). □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, C^2-regular set. For $f \in H^s(\partial\Omega)$ there exists a continuous extension operator $E_s(f)$ into $H^1(\Omega, d^{1-2s})$, i.e. $E(f)|_{\partial\Omega} = f$ and*

$$\|E_s f\|_{H^1(\Omega,d^{1-2s})} \leq C\|f\|_{H^s(\partial\Omega)}.$$

Proof. Again, this follows by relying on a partition of unity $\{\eta_k\}_{k\in\mathbb{N}}$ and a flatting argument. Flattening $\partial\Omega$ by local diffeomorphisms ϕ_k with small C^1 norm, we consider $\tilde{f}_k := (f\eta_k) \circ \phi_k$. As $\tilde{f}_k$ may be assumed to be compactly supported in $\{x_{n+1} = 0\}$, we obtain an extension $\tilde{u}_k$ satisfying the bound

$$\|\tilde{u}_k\|_{H^1(\mathbb{R}^n\times[0,4],x_{n+1}^{1-2s})} \leq C\|\tilde{f}_k\|_{H^s(\mathbb{R}^n)}. \tag{11}$$

One possibility of achieving this is by choosing $\tilde{u}_k$ to be the solution to

$$\begin{aligned}
\nabla \cdot x_{n+1}^{1-2s}\nabla u &= 0 \text{ in } \mathbb{R}^{n+1}_+,\\
u &= \tilde{f}_k \text{ on } \mathbb{R}^n \times \{0\}.
\end{aligned}$$

We, for instance, refer to the Appendix in [GFR19] for the derivation of the associated estimates of the form (11). Finally, using the local diffeomorphisms ϕ_k and the behaviour of the $H^s(\partial\Omega)$ and $H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ norms under diffeomorphisms, the estimate (11) turns into a corresponding estimate in Ω. Defining $u := \sum_{k=1}^{M} \eta_k \tilde{u}_k \circ \phi_k^{-1}$ then concludes the proof. □

With the trace estimates in hand, we further define the following spaces including boundary data. To this end, let $\Sigma \subset \partial\Omega$ be a C^2-regular, relatively open set. Then,

$$H^1_{\Sigma,0}(\Omega, d^{1-2s}) := \{u : \Omega \to \mathbb{C} : \ \|d^{\frac{1-2s}{2}}u\|_{L^2(\Omega)} + \|d^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)} < \infty, \ u|_\Sigma = 0\}.$$

2.1.2. *Test functions for the CGO construction.* In addition to the weighted Sobolev spaces from above, in our construction of complex geometric optics solutions, we will make use of the following function spaces: For $\Omega \subset \mathbb{R}^{n+1}_+$ with $\overline{\Omega} \cap \{x_{n+1} = 0\} := \overline{\Sigma}$, we set

$$x_{n+1}^{2s} C_c^\infty(\overline{\Omega}) := \{f : \Omega \to \mathbb{C} : f(x) = x_{n+1}^{2s} h(x', x_{n+1}) \text{ with } h \in C_c^\infty(\overline{\Omega})\},$$

where we use the notation

$$C_c^\infty(\overline{\Omega}) := \{h : \Omega \to \mathbb{C} : \ h \text{ is infinitely often continuously differentiable and } \operatorname{supp}(h) \subset \Omega \cup \Sigma\}.$$

We stress that this in particular enforces that $h, \partial_\nu h = 0$ on $\partial\Omega \backslash \overline{\Sigma}$ but that h does *not* necessarily vanish on $\overline{\Sigma}$.

For $\Sigma_1 \subset \partial\Omega$ a smooth, n-dimensional, star-shaped set, we further consider

$$\begin{aligned}\tilde{\mathcal{C}} := \{f : \Omega \to \mathbb{R} : \ f \in C^\infty(\overline{\Omega}) \text{ with } f|_{N(\Sigma_2,\epsilon)} = 0, \ \partial_\nu f|_{N(\Sigma_2,\epsilon)} = 0, \\ \partial_{n+1} f(x) = 0 \text{ for } x \in N(\Sigma_1, \epsilon) \times [0, t] \text{ for some } \epsilon > 0, t > 0\}.\end{aligned}$$

For simplicity of notation, we have set $\Sigma_2 := \partial\Omega \setminus \overline{\Sigma_1}$ and denote by $N(\Sigma_2, \epsilon)$, $N(\Sigma_1, \epsilon)$ an ϵ-neighbourhood of Σ_1, Σ_2 on $\partial\Omega$.

As an important property which we will make use of in our construction of CGO solutions, we state a density result for the space $\mathcal{C}$:

Proposition 2.3. *Assume that the conditions from above hold. Then the set* $\tilde{\mathcal{C}} \subset H^1_{\Sigma_2,0}(\Omega, x_{n+1}^{1-2s})$ *is dense.*

We postpone the proof of Proposition 2.3 to the appendix.

Finally, for $s \in (0, 1)$ we define

$$\mathcal{C} := \tilde{\mathcal{C}} + x_{n+1}^{2s} C_c^\infty(\overline{\Omega}). \tag{12}$$

We will use this space extensively in Section 7.

2.1.3. *Semiclassical spaces and the Fourier transform.* In our construction of CGOs it will be useful to work with semiclassical Sobolev spaces. To this end, we use the following notation for the Fourier transform

$$\hat{u}(y) = \mathcal{F}u(y) = \int_{\mathbb{R}^{n+1}} e^{-ix\cdot y} u(x) dx \ .$$

We introduce the following definitions for the semiclassical Sobolev spaces. Let $\xi' \in \mathbb{C}^n$. Eventually we will consider the limit case $|\xi'| \to \infty$, and thus for us $|\xi'|^{-1}$ constitutes a small parameter. Following [Zwo12], we define the semiclassical Fourier transform as

$$\mathcal{F}_{sc} u(y) := \int_{\mathbb{R}^{n+1}} e^{-i|\xi'| x\cdot y} u(x) dx \ ,$$

and then use it in order to define the semiclassical Sobolev norm

$$\|u\|^2_{H^s_{sc}(\mathbb{R}^{n+1})} := \left(\frac{|\xi'|}{2\pi}\right)^{n+1} \|\langle y\rangle^s |\mathcal{F}_{sc} u(y)|\|^2_{L^2(\mathbb{R}^{n+1})} \ ,$$

where $s \in \mathbb{R}$, $u \in L^2(\mathbb{R}^{n+1})$ and $\langle y\rangle := (1 + |y|^2)^{1/2}$ for $y \in \mathbb{R}^{n+1}$. The cases of interest for us are $s = 0$ and $s = 1$, for which we have

$$\|u\|_{L^2_{sc}(\mathbb{R}^{n+1})} = \|u\|_{L^2(\mathbb{R}^{n+1})} \quad \text{ and } \quad \|u\|_{H^1_{sc}(\mathbb{R}^{n+1})} = \|u\|_{L^2(\mathbb{R}^{n+1})} + |\xi'|^{-1} \|\nabla u\|_{L^2(\mathbb{R}^{n+1})} \ .$$

The semiclassical Sobolev spaces $L^2_{sc}(\mathbb{R}^{n+1})$ and $H^1_{sc}(\mathbb{R}^{n+1})$ are then defined as the subspaces of $L^2(\mathbb{R}^{n+1})$ where the corresponding semiclassical norms are finite. Moreover, if Ω is some open subset of $\mathbb{R}^{n+1}$ and $w(x)$ is a weight function, we define the weighted semiclassical Sobolev space $H^s_{sc}(\Omega, w)$ as the subspace of $L^2(\mathbb{R}^{n+1})$ where the norm

$$\|u\|^2_{H^s_{sc}(\Omega,w)} := \left(\frac{|\xi'|}{2\pi}\right)^{n+1} \|\langle y\rangle^s |\mathcal{F}_{sc} u(y)|\|^2_{L^2(\Omega,w)}$$

is finite. In the special cases $s = 0$ and $s = 1$ this of course gives

$$\|u\|_{L^2_{sc}(\Omega,w)} = \|u\|_{L^2(\Omega,w)} \quad \text{and} \quad \|u\|_{H^1_{sc}(\Omega,w)} = \|u\|_{L^2(\Omega,w)} + |\xi'|^{-1}\|\nabla u\|_{L^2(\Omega,w)} .$$

2.2. **Trace estimates.** In this section we collect a number of (weighted) trace estimates. These are not new and have already been used in for instance [Rül15, Rül17, RW19].

We begin with the case $s = \frac{1}{2}$:

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, C^2-regular domain. Then there exist constants $C = C(\Omega, \partial\Omega) > 1$, $c_0 = c_0(\Omega) > 1$ such that for all $u \in H^1(\Omega)$ and $\mu \geq c_0$ it holds*

$$\|u\|_{L^2(\partial\Omega)} \leq C(\mu^{-1}\|\nabla u\|_{L^2(\Omega)} + \mu\|u\|_{L^2(\Omega)}).$$

Proof. By density, we may without loss of generality assume that u is smooth. We work in boundary normal coordinates and denote the coordinates by $x = (x', t)$, where $x = x' + \tau\nu(x')$, $x' \in \partial\Omega$ and $\nu(x')$ denotes the inner unit normal to $\partial\Omega$ at x'. By the fundamental theorem, we thus write for some $t > 0$

$$u(x', 0) = \int_0^t \partial_s u(x', s) ds + u(x', t).$$

As a consequence, by Hölder,

$$|u(x', 0)|^2 \leq C(t\int_0^t |\partial_s u(x', s)|^2 ds + |u(x', t)|^2).$$

Integrating over $x' \in \partial\Omega$ thus yields

$$\begin{aligned}\|u\|^2_{L^2(\partial\Omega)} &\leq Ct \int_{\partial\Omega}\int_0^t |\partial_s u(x', s)|^2 ds dx' + \|u(\cdot, t)\|^2_{L^2(\partial\Omega)} \\ &\leq Ct\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega_t)}.\end{aligned}$$

Integrating in $t \in (0, \mu^{-1})$ with $\mu \geq C_0(\Omega) > 0$ leads to

$$\mu^{-1}\|u\|^2_{L^2(\partial\Omega)} \leq C\mu^{-2}\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}.$$

Multiplying by $\mu > 0$ implies the desired result. □

More generally, also a weighted trace estimate holds for $s \in (0, 1)$:

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, C^2-regular domain. Let $d : \Omega \to [0, \infty)$ be a C^1 regular function which close to the boundary $\partial\Omega$ coincides with the distance function to $\partial\Omega$. Then there exist constants $C = C(\Omega, \partial\Omega) > 1$, $c_0 = c_0(\Omega) > 1$ such that for all $u \in H^1(\Omega)$ and $\mu \geq c_0$ it holds*

$$\|u\|_{L^2(\partial\Omega)} \leq C(\mu^{-s}\|d^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)} + \mu^{1-s}\|d^{\frac{1-2s}{2}} u\|_{L^2(\Omega)}).$$

Proof. As above, we only prove the result for u smooth and work in boundary normal coordinates $x = (x', t)$ as in the proof of Lemma 2.4. Again the fundamental theorem in combination with Hölder's inequality, yields

$$\begin{aligned} |u(x',0)| &\leq \int\limits_0^t |\partial_r u(x',r)| dr + |u(x',r)| \\ &\leq C_s t^s \left(\int_0^r r^{1-2s} |\partial_r u(x',r)|^2 dr \right)^{\frac{1}{2}} + |u(x',r)|. \end{aligned}$$

Squaring this and integrating in the tangential coordinates yields for $t = t(\Omega) > 0$ sufficiently small that

$$\|u\|^2_{L^2(\partial\Omega)} \leq C t^{2s} \|t^{\frac{1-2s}{2}} \partial_t u\|^2_{L^2(\Omega)} + C\|u\|^2_{L^2(\partial\Omega_t)}.$$

Integrating this in $t \in [r, 2r]$ for $r \in (0, r_0)$ and $r_0 = r_0(\Omega) > 0$ entails that

$$\begin{aligned} r\|u\|^2_{L^2(\partial\Omega)} &\leq C r^{2s+1} \|t^{\frac{1-2s}{2}} \partial_t u\|^2_{L^2(\Omega)} + C\|u\|^2_{L^2(\Omega)} \\ &\leq C r^{2s+1} \|t^{\frac{1-2s}{2}} \partial_t u\|^2_{L^2(\Omega)} + C r^{2s-1} \|t^{\frac{1-2s}{2}} u\|^2_{L^2(\Omega)} \\ &\leq C r^{2s+1} \|d^{\frac{1-2s}{2}} \nabla u\|^2_{L^2(\Omega)} + C r^{2s-1} \|d^{\frac{1-2s}{2}} u\|^2_{L^2(\Omega)}. \end{aligned}$$

Dividing by $r > 0$ and defining $\mu^{-1} = r$, we obtain the desired result for $\mu \geq r_0^{-1}$. □

2.3. **Notation for sets.** In the following we will work with Calderón type problems with mixed boundary conditions. To this end, we will use the following notation in the remainder of the article. In Sections 3-5 we will always assume that $\Omega \subset \mathbb{R}^{n+1}_+$ is a relatively open, C^2-regular set. Furthermore, the sets $\Sigma_1, \Sigma_2 \subset \partial\Omega$ are C^2-regular and satisfy $\Sigma_1 \cap \Sigma_2 = \emptyset$. For the sake of simplicity, in the sequel we will always assume that $\Omega_1 \Subset \Omega$ is a bounded, open set such that $\Omega \setminus \overline{\Omega_1}$ is simply connected. In Sections 6 and 7 we will in addition assume that all sets are smooth and that Σ_1 is star-shaped.

Working with sets in the neighbourhood of $\partial\Omega$ or with some distance to $\partial\Omega$, we further define for $\delta \in (0, 1)$ sufficiently small

$$\begin{aligned} \Omega_\delta &:= \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) \geq \delta\}, \\ \partial\Omega_\delta &:= \{x + t\nu(x) : \ x \in \partial\Omega, \ t \in (0, \delta)\} \subset \Omega. \end{aligned} \tag{13}$$

Here for $x \in \partial\Omega \subset \mathbb{R}^n$ the vector $\nu(x) \in S^{n-1}$ denotes the inner unit normal at the point x. For a subset $\Sigma \subset \partial\Omega$ we further set

$$N(\Sigma, \delta) := \{x \in \partial\Omega : \ \mathrm{dist}(x, \Sigma) \leq \delta\}.$$

3. Well-Posedness of the Mixed Boundary Value Problems (1) and (5)

In this section, we discuss the well-posedness of the (weak) forms of the equations (1) and (5) in the associated energy spaces. Based on this, we define the associated Dirichlet-to-Neumann maps and derive the central Alessandrini identities which we will use in the following sections when dealing with the associated inverse problems.

We begin by discussing the well-posedness of the problem (1).

Proposition 3.1 (Well-posedness, $s = \frac{1}{2}$)**.** *Let $B_{A,V,q}$ denote the bilinear form from (4) and let $\Omega, \Sigma_1, \Sigma_2$ be as above. Then, there exists a countable set $M \subset \mathbb{C}$ such that if $\lambda \in \mathbb{C} \setminus M$, for all*

$F \in (H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega))^*$, $f_2 \in H^{\frac{1}{2}}_{\overline{\Sigma_2}}$ and $f_1 \in H^{-\frac{1}{2}}(\Sigma_1)$, there is $u \in H^1_{\partial\Omega\setminus(\Sigma_1\cup\Sigma_2),0}(\Omega)$ with $u|_{\Sigma_2} = f_2$ and with

$$B_{A,V,q}(u,v) - \lambda(u,v)_{L^2(\Omega)} = \langle F, v\rangle + (f_1, v)_{L^2(\Sigma_1)}, \tag{14}$$

for all $v \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$. Here $\langle\cdot,\cdot\rangle$ denotes the $(H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega))^*$, $H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$ duality pairing.

If $\lambda \notin M$, there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq C(\|F\|_{(H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega))^*} + \|f_1\|_{H^{-\frac{1}{2}}(\Sigma_1)} + \|f_2\|_{H^{\frac{1}{2}}(\Sigma_2)}). \tag{15}$$

Remark 3.2. *We remark that in Proposition 3.1, compared to the problem in (1), we consider the slightly more general setting of constructing (weak) solutions to*

$$\begin{aligned} L_\lambda u := -\Delta u - iA\cdot\nabla u - i\nabla\cdot(Au) + (|A|^2 + V + \lambda)u &= F \text{ in } \Omega,\\ \partial_\nu u + qu &= f_1 \text{ on } \Sigma_1,\\ u &= f_2 \text{ on } \Sigma_2,\\ u &= 0 \text{ on } \partial\Omega\setminus(\Sigma_1\cup\Sigma_2). \end{aligned} \tag{16}$$

This will be convenient when discussing density properties by studying the adjoint equation (see Section 4).

For $\lambda \notin M$, we will refer to solutions of (14) with the described properties as weak solutions *to (16). It is this notion of a solution that we will work with in the sequel.*

Proof. We argue in several steps.

Step 1: Reduction. We first reduce the problem to the case of $f_2 = 0$ by considering $u = u_1 + E(f_2)$ where $E(f_2)$ is an $H^1_{\partial\Omega\setminus\Sigma_2,0}(\Omega)$ extension of f_2 satisfying the bound $\|E(f_2)\|_{H^1(\Omega)} \leq C\|f_2\|_{H^{\frac{1}{2}}(\Sigma_2)}$. This is possible by for instance defining $E(f_2)$ to be the harmonic extension of f_2 into Ω. The function u_1 thus solves a similar problem as the original function u with a new functional $\tilde{F} := F - L_\lambda(E(f_2)) \in (H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega))^*$, but now in addition satisfies $\tilde{f}_2 := u_1|_{\Sigma_2} = 0$. Here the expression $L_\lambda(E(f_2))$ is understood in the weak sense, i.e. as the functional $H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega) \ni v \mapsto -B_{A,V,q}(E(f_2), v)$. With slight abuse of notation, in the following we will only work with the function u_1 and drop the subindex in the notation for u_1 and the tildas in the data.

Step 2: Continuity.
We observe that for $v \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$ as above, we have (using the trace inequality)

$$|B_{A,V,q}(u,v)| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Here the constant $C > 0$ depends on λ, $\|q\|_{L^\infty}$, $\|A\|_{L^\infty}$, $\|V\|_{L^\infty}$. This proves the continuity of the bilinear form.

Step 3: Coercivity. We next study the coercivity properties of the bilinear form. By Cauchy-Schwarz

$$\left|\int_{\partial\Omega} qu\bar{v}\, d\mathcal{H}^{n-1}\right| \leq \|q\|_{L^\infty(\partial\Omega)}\|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}.$$

Thus, by the trace inequality from Lemma 2.4 we infer that

$$\left|\int_{\partial\Omega} q|u|^2 d\mathcal{H}^{n-1}\right| \leq C\|q\|_{L^\infty(\partial\Omega)}\|u\|^2_{L^2(\partial\Omega)} \leq C\|q\|_{L^\infty(\partial\Omega)}(\mu^{-2}\|\nabla u\|^2_{L^2(\Omega)} + \mu^2\|u\|^2_{L^2(\Omega)}).$$

Choosing $\mu > 1$ such that $C\|q\|_{L^\infty(\partial\Omega)}\mu^{-2} \leq \frac{1}{10}$, we thus obtain

$$\left| \int_{\partial\Omega} q|u|^2 d\mathcal{H}^{n-1} \right| \leq \frac{1}{10}\|\nabla u\|^2_{L^2(\Omega)} + C\|q\|_{L^2(\partial\Omega)}\|u\|^2_{L^2(\Omega)}.$$

Moreover, by Young's inequality,

$$\left| \int_{\Omega} uA \cdot \overline{\nabla u} dx \right| \leq \frac{1}{10}\|\nabla u\|^2_{L^2(\Omega)} + C\|A\|^2_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)}.$$

Noting that by Poincaré's inequality there exists a constant $C > 0$ such that for all $u \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$ we have

$$C\|\nabla u\|_{L^2(\Omega)} \geq \|u\|_{H^1(\Omega)},$$

and combining this with the previous estimates for the lower order bulk and boundary contributions, we thus obtain that for $\mu = \|V_-\|_{L^\infty(\Omega)} + C_1\|A\|_{L^\infty(\Omega)} + C_1\|q\|_{L^\infty(\partial\Omega)}$ with suitable constants $C_1, C_2 > 0$, we have

$$B_\mu(u,u) := B_{A,V,q}(u,u) + \mu(u,u)_{L^2(\Omega)} \geq C_2\|u\|^2_{H^1(\Omega)}.$$

Step 4: Conclusion. By the discussion in Steps 2 and 3 above, $B_\mu(\cdot,\cdot)$ is a scalar product and the Riesz representation theorem is applicable. Since $F \in (H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega))^*$ and also for $f_1 \in H^{-\frac{1}{2}}(\Sigma_1)$ the map

$$H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega) \ni v \mapsto (v, f_1)_{L^2(\Sigma_1)},$$

is a bounded linear functional on $H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$, this yields the existence of a unique function $u := G_\mu(F, f_1)$ such that

$$B_\mu(u,v) = \langle F, v\rangle + (f_1, v)_{L^2(\Sigma_1)} \text{ for all } v \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega).$$

Moreover, the operator

$$G_\mu : (H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega))^* \times H^{-\frac{1}{2}}(\Sigma_1) \to H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$$

is bounded. Now, the equation

$$B_{A,V,q}(u,v) - \lambda(u,v) = \tilde{F}(v)$$

with v as above and $\tilde{F}$ a functional on this space, is equivalent to

$$u = G_\mu((\mu + \lambda)u + \tilde{F}). \tag{17}$$

As $G_\mu : L^2(\Omega) \times L^2(\Sigma_1) \to L^2(\Omega)$ is compact and self-adjoint, the spectral theorem for compact, self-adjoint operators yields the existence of a set $\tilde{M}$ such that for $\lambda \notin \tilde{M}$ (17) is (uniquely) solvable. Hence, the original equation is (uniquely) solvable outside of the set $M := \left\{\frac{1}{\lambda_j + \mu}\right\}_{j=1}^{\infty}$.
□

With the well-posedness result available, it is possible to define the Poisson operator associated with the equation (1).

Definition 3.3. *Let $M \subset \mathbb{C}$ be as in Proposition 3.1 and assume that $0 \notin M$. Let $f \in H^{\frac{1}{2}}_{\overline{\Sigma_2}}$ and let $u \in H^1(\Omega)$ be the solution constructed in Proposition 3.1 with $F = 0, f_1 = 0$ and $f_2 = f$. Then, we define the* Poisson operator

$$P : H^{\frac{1}{2}}_{\overline{\Sigma_2}} \to H^1(\Omega),\ f \mapsto u.$$

We remark that by the apriori estimates from Proposition 3.1 the Poisson operator is bounded.

With the well-posedness of (1) at our disposal, we proceed to the well-posedness of the equation (5).

Proposition 3.4 (Well-posedness, $s \in (0,1)$)**.** *Let $B_{s,A,V,q}$ denote the bilinear form from (6). Then, there exists a countable set $M \subset \mathbb{C}$ such that if $\lambda \in \mathbb{C}\backslash M$ for any $F \in (H^1_{\partial\Omega\backslash\Sigma_1,0}(\Omega, d^{1-2s}))^*$, $f_2 \in H^s_{\overline{\Sigma_2}}$ and $f_1 \in H^{-s}(\Sigma_1)$ there is $u \in H^1_{\partial\Omega\backslash(\Sigma_1\cup\Sigma_2),0}(\Omega, d^{1-2s})$ with $u|_{\Sigma_2} = f_2$ and with*

$$B_{s,A,V,q}(u,v) - \lambda(u,v)_{L^2(\Omega)} = \langle F, v\rangle + (f_1, v)_{L^2(\Sigma_1)}$$

for all $v \in H^1_{\partial\Omega\backslash\Sigma_1,0}(\Omega, d^{1-2s})$. Here $\langle\cdot,\cdot\rangle$ denotes the $(H^1(\Omega, d^{1-2s}))^*$, $H^1(\Omega, d^{1-2s})$ duality pairing. If $\lambda \notin M$, there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega,d^{1-2s})} \leq C(\|F\|_{(H^1_{\partial\Omega\backslash\Sigma_1,0}(\Omega,d^{1-2s}))^*} + \|f\|_{H^s(\Sigma_2)}). \tag{18}$$

Remark 3.5. *As in Proposition 3.1, compared to the problem in (5), we here consider the slightly more general setting of constructing (weak) solutions to*

$$\begin{aligned} -\nabla\cdot d^{1-2s}\nabla u - iAd^{1-2s}\cdot\nabla u - i\nabla\cdot(d^{1-2s}Au) + d^{1-2s}(|A|^2+V)u &= F \text{ in } \Omega,\\ \lim_{d(x)\to 0} d^{1-2s}\partial_\nu u + qu &= f_1 \text{ on } \Sigma_1,\\ u &= f_2 \text{ on } \Sigma_2,\\ u &= 0 \text{ on } \partial\Omega\setminus(\Sigma_1\cup\Sigma_2). \end{aligned} \tag{19}$$

Again this will be convenient when discussing density properties by means of studying the adjoint equation. For convenience of notation, we define

$$L_{\lambda,s}u := -\nabla\cdot d^{1-2s}\nabla u - iAd^{1-2s}\cdot\nabla u - i\nabla\cdot(d^{1-2s}Au) + d^{1-2s}(|A|^2+V)u.$$

As in the case $s = \frac{1}{2}$, for $\lambda \notin M$ we define a weak solution *to (19) to be the corresponding function $u \in H^1_{\partial\Omega\backslash(\Sigma_1\cup\Sigma_2),0}(\Omega, d^{1-2s})$ from Proposition 3.4.*

The proof of Proposition 3.1 follows along similar lines as the proof of Proposition 3.4. Due to the presence of the weights we however need to rely on suitable modifications of the boundary-bulk inequalities as recalled in Section 2.1.

Proof. Step 1: Reduction. As in the proof of Proposition 3.1 we first reduce the setting to $f_2 = 0$ by considering $u = u_1 + E_s(f_2)$, where $E_s(f_2) \in H^1_{\partial\Omega\backslash\Sigma_2,0}(\Omega, d^{1-2s})$ is obtained from Lemma 2.2 and has the property that $\|E_s(f_2)\|_{H^1_{\partial\Omega\backslash\Sigma_2,0}(\Omega,d^{1-2s})} \leq C\|f_2\|_{H^s(\Sigma_2)}$.

Working with the equation for u_1 yields an equation of the desired form with $u_1|_{\Sigma_2} = \tilde{f}_2 = 0$ and a new inhomogeneity $\tilde{F} := F - L_{\lambda,s}(E_s(f_2)) \in (H^1_{\partial\Omega\backslash\Sigma_1,0}(\Omega, d^{1-2s}))^*$. As in the case $s = \frac{1}{2}$, the functional $L_{\lambda,s}(E(f_2))$ is interpreted weakly, in that it is given by

$$H^1_{\partial\Omega\backslash\Sigma_1,0}(\Omega, d^{1-2s}) \ni v \mapsto B_{s,A,V,q}(E_s(f_2), v).$$

With a slight abuse of notation, we drop the subscript in the notation for u_1 and the tildas in the notation for the data in the following.

Step 3: Continuity. The continuity of the bilinear form then is a consequence of the following observations and estimates:

(i) *Continuity of the boundary terms.* We observe that by Lemma 2.1

$$\begin{aligned}\left|\int_{\partial\Omega} qu\overline{v}dx\right| &\leq \|q\|_{L^\infty(\partial\Omega)}\|u\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)}\\ &\leq \|q\|_{L^\infty(\partial\Omega)}\|u\|_{H^s(\partial\Omega)}\|v\|_{H^s(\partial\Omega)}\\ &\leq C\|q\|_{L^\infty(\partial\Omega)}\|u\|_{H^1(\Omega,d^{1-2s})}\|v\|_{H^1(\Omega,d^{1-2s})}.\end{aligned}$$

(ii) *Continuity of the bulk terms.* As the continuity of all the bulk terms follows analogously, we only discuss the first magnetic term: In this case, by Cauchy-Schwarz, we obtain

$$\left|\int_{\Omega} d^{1-2s}vA_1\cdot\overline{\nabla u}dx\right| \leq \|A_1\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega,d^{1-2s})}\|u\|_{H^1(\Omega,d^{1-2s})}.$$

(iii) *Boundedness of the right hand side.* The mapping

$$H^{-s}(\Sigma_1)\ni f_1\mapsto (f_1,v)_{L^2(\Sigma_1)}$$

for $v\in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s})$ satisfies the bound

$$\left|(f_1,v)_{L^2(\Sigma_1)}\right| \leq \|f_1\|_{H^{-s}(\Sigma_1)}\|v\|_{H^s(\partial\Omega)} \leq \|f_1\|_{H^{-s}(\Sigma_1)}\|v\|_{H^1(\Omega,d^{1-2s})}.$$

It is thus a bounded linear functional on $(H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s}))^*$. Similarly, for $F\in(H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s}))^*$, by definition, the map $v\mapsto\langle F,v\rangle$ is a bounded linear functional on $H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s})$.

Step 4: Coercivity. For the coercivity estimate, we need to bound $B_{s,A,V,q}(u,u)$ from below. Again the bulk estimates follow from the Cauchy-Schwarz and Young's inequality. The main point is to consider the boundary contributions and to prove their coercivity. This is a consequence of the trace estimate from Lemma 2.5. Indeed, we deduce that for $\mu>1$ to be chosen below

$$\begin{aligned}\left|\int_{\partial\Omega} q|u|^2dx\right| &\leq \|q\|_{L^\infty(\partial\Omega)}\|u\|^2_{L^2(\partial\Omega)}\\ &\leq C\|q\|_{L^\infty(\partial\Omega)}(\mu^{-2s}\|d^{\frac{1-2s}{2}}\nabla u\|^2_{L^2(\Omega)}+\mu^{2-2s}\|d^{\frac{1-2s}{2}}u\|^2_{L^2(\Omega)}).\end{aligned}$$

Choosing $\mu>1$ such that

$$C\|q\|_{L^\infty(\partial\Omega)}\mu^{-2s}\leq\frac{1}{10},$$

we thus infer that

$$\left|\int_{\partial\Omega} q|u|^2dx\right| \leq \frac{1}{10}\|u\|^2_{H^1(\Omega,d^{1-2s})}+C\|q\|^{\frac{1}{s}-1}_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega,d^{1-2s})}.$$

Next we note that there exists a constant $C>0$ such that for all $u\in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$ it holds that

$$C\|\nabla u\|_{L^2(\Omega,d^{1-2s})}\geq\|u\|_{H^1(\Omega,d^{1-2s})}.$$

This follows from the fact that $u|_{\partial\Omega\setminus\Sigma_1}=0$ and an application of Poincaré's inequality, see for instance [RS20a, Lemma 4.7] and the proof of Lemma 2.5. Combining all these observations and also invoking the estimate in Step 3(ii) for the bulk contributions (to which we still apply Young's inequality), we hence infer that the bilinear form $B_{s,A,V,q}(\cdot,\cdot)$ is coercive, i.e. that

$$B_{s,A,V,q}(u,u) \geq \frac{1}{2}\|u\|^2_{H^1(\Omega,d^{1-2s})} - C_{low}\|u\|^2_{L^2(\Omega,d^{1-2s})},$$

where the constant $C_{low} > 0$ depends on $s, \|A\|_{L^\infty(\Omega)}, \|V\|_{L^\infty(\Omega)}, \|q\|_{L^\infty(\partial\Omega)}, n$.

Step 5: Conclusion. With the available upper and lower bounds, we conclude as in the proof of Proposition 3.1. More precisely, for $\mu > C_{low}$ the bilinear form $B_{s,\mu}(u,v) := B_{s,A,V,q}(u,v) + \mu(u,v)_{L^2(\Omega,d^{1-2s})}$ is a scalar product. Hence, in combination with the third estimate in Step 3, the Riesz representation theorem is applicable and yields a unique solution $u = G(\bar{F}) \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s})$ with $\bar{F} = (F,f_1)$ to the equation

$$B_{s,\mu}(u,v) = (F,v)_{L^2(\Omega)} + (f_1,v)_{L^2(\Sigma_1)} \text{ for all } v \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s}).$$

Using the compactness of the space $L^2(\Omega,d^{1-2s}) \subset H^1(\Omega,d^{1-2s})$ and the fact that

$$\begin{aligned} G_\mu : L^2(\Omega,d^{2s-1}) \times L^2(\Sigma_1) &\subset (H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s}))^* \times H^{-s}(\Sigma_1) \\ &\to H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega,d^{1-2s}) \subset L^2(\Omega,d^{1-2s}), \end{aligned}$$

the claim on the set $M \subset \mathbb{R}$ follows from the spectral theorem for self-adjoint, compact operators in the same way as in Proposition 3.1 (see for instance [McL00, Theorem 2.37 and Corollary 2.39]). ☐

As in the case $s = \frac{1}{2}$ the well-posedness result allows us to define the Poisson operator associated with the Schrödinger equation (5).

Definition 3.6. *Let $M \subset \mathbb{C}$ be as in Proposition 3.4 and assume that $0 \notin M$. Let $f \in H^s_{\overline{\Sigma_2}}$ and let $u \in H^1(\Omega,d^{1-2s})$ be the solution constructed in Proposition 3.1 with $F = 0, f_1 = 0$ and $f_2 = f$. Then, we define the* Poisson operator

$$P_s : H^s_{\overline{\Sigma_2}} \to H^1(\Omega,d^{1-2s}),\ f \mapsto u.$$

Again this operator is bounded by the apriori estimates from the well-posedness result in Proposition 3.4.

In order to simplify our discussion, for convenience we will, for the remainder of the article, always make the following assumption:

Assumption 3.7. *For the remainder of the article we will assume that zero is not an eigenvalue of the Schrödinger operators (1) and (5), i.e. we will assume that $\lambda = 0 \notin M$, where M denotes the sets constructed in Propositions 3.1 and 3.4, respectively.*

We remark that as a consequence of Proposition 3.4, we also obtain the following regularity result for the weighted normal derivative:

Lemma 3.8. *Let u be a weak solution to (5) possibly also with a bulk inhomogeneity $F \in L^2(\Omega,d^{2s-1})$. Then, there exists a constant $C > 0$ such that for each $\delta > 0$ sufficiently small we have that $d^{1-2s}\partial_\nu u \in H^{-s}(\partial\Omega_\delta)$ with*

$$\|d^{1-2s}\partial_\nu u\|_{H^{-s}(\partial\Omega_\delta)} \leq C\left(\|F\|_{L^2(\Omega,d^{2s-1})} + \|u\|_{H^s(\partial\Omega)}\right), \tag{20}$$

where $\partial\Omega_\delta := \{x + t\nu(x) :\ x \in \partial\Omega,\ t \in (0,\delta)\}$ and where $\nu(x)$ denotes the inward pointing unit normal at a point $x \in \partial\Omega$. Moreover,

$$\|\lim_{d\to 0} d^{1-2s}\partial_\nu u - d^{1-2s}\partial_\nu u\|_{H^{-s}(\partial\Omega_\delta)} \to 0 \text{ as } \delta \to 0. \tag{21}$$

Proof. The fact that for any $\delta \in (0,1)$ sufficiently small $d^{1-2s}\partial_\nu u|_{\partial\Omega_\delta} \in H^{-s}(\partial\Omega_\delta)$ with a uniform estimate (in $\delta > 0$) follows by duality and the weak form of the equation: Indeed, due to the validity of the equation (5) and the fact that this is a uniformly elliptic equation away from $\partial\Omega$, we have that $d^{1-2s}\partial_\nu u \in L^2(\partial\Omega_\delta)$ for any $\delta > 0$. Integrating by parts, we further observe that for any $w \in H^s(\partial\Omega_\delta)$ and an associated extension $E_s(w) \in H^1(\Omega_\delta, d^{1-2s})$, we obtain

$$(w, d^{1-2s}\partial_\nu u)_{L^2(\partial\Omega_\delta)} = B_{s,A,V,q}(E_s(w), u). \tag{22}$$

Using (22) and the boundary estimate from Lemma 2.2, we hence estimate

$$\begin{aligned}|(w, d^{1-2s}\partial_\nu u)_{L^2(\partial\Omega_\delta)}| &\leq |(E_s(w), F)_{L^2(\Omega_\delta)}| + |(\nabla E_s(w), d^{1-2s}\nabla u)_{L^2(\Omega_\delta)}|\\ &\leq C\|w\|_{H^s(\partial\Omega_\delta)}(\|u\|_{H^1(\Omega,d^{1-2s})} + \|F\|_{L^2(\Omega,d^{2s-1})})\\ &\leq C\|w\|_{H^s(\partial\Omega_\delta)}(\|u\|_{H^s(\partial\Omega)} + \|F\|_{L^2(\Omega,d^{2s-1})}).\end{aligned}$$

Thus, taking the supremum in $w \in H^s(\partial\Omega_\delta)$ with $\|w\|_{H^s(\partial\Omega_\delta)} = 1$ implies the claim (20).

Moreover, by the definition of the normal derivative by means of the bilinear form as in (22) for $\delta_1, \delta_2 > 0$ small,

$$\begin{aligned}&\|(d^{1-2s}\partial_\nu u)(\cdot + \delta_1\nu) - (d^{1-2s}\partial_\nu u)(\cdot + \delta_2\nu)\|_{H^{-s}(\partial\Omega)}\\ &= \sup_{\|w\|_{H^s(\partial\Omega)}\leq 1} (w, (d^{1-2s}\partial_\nu u)(\cdot + \delta_1\nu) - (d^{1-2s}\partial_\nu u)(\cdot + \delta_1\nu))_{L^2(\partial\Omega)}\\ &= \sup_{\|w\|_{H^s(\partial\Omega)}\leq 1} \big((E_s(w)\chi_{\Omega_{\delta_1}} - E_s(w)\chi_{\Omega_{\delta_2}}, F)_{L^2(\Omega)}\\ &\quad + (d^{1-2s}(\chi_{\Omega_{\delta_1}}\nabla E_s(w) - \chi_{\Omega_{\delta_2}}\nabla E_s(w)), \nabla u)_{L^2(\Omega)}\big)\\ &\leq C\|w\|_{H^s(\partial\Omega)}\big(\|(\chi_{\Omega_{\delta_1}} - \chi_{\Omega_{\delta_2}})F\|_{L^2(\Omega,d^{2s-1})} + \|d^{1-2s}(\chi_{\Omega_{\delta_1}} - \chi_{\Omega_{\delta_2}})\nabla u\|_{L^2(\Omega)}\big) \to 0\\ &\qquad \text{as } \delta_1, \delta_2 \to 0.\end{aligned}$$

Here we used that $\nabla u \in L^2(\Omega, d^{1-2s})$ by the apriori estimates from the well-posedness results and have set $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \delta\}$ for $\delta > 0$ sufficiently small. This proves that $\{(d^{1-2s}\partial_\nu u)(\cdot + n^{-1}\nu)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^{-s}(\partial\Omega)$, that $\lim\limits_{n\to\infty}(d^{1-2s}\partial_\nu u)(\cdot + n^{-1}\nu)$ exists in $H^{-s}(\partial\Omega)$ as $n \to \infty$ and that (21) holds. □

With the well-posedness results of Propositions 3.1, 3.4 and the global Assumption 3.7 in hand, we can now also define the (partial data) Dirichlet-to-Neumann maps which we will study in the sequel.

Definition 3.9 (Partial Dirichlet-to-Neumann maps)**.** *Let $s \in (0,1)$ and let $B_{A,V,q}(\cdot,\cdot)$ and $B_{s,A,V,q}(\cdot,\cdot)$ denote the bilinear forms from (4), (6). We then define the (partial) Dirichlet-to-Neumann maps $\Lambda_{A,V,q} : \widetilde{H}^{\frac{1}{2}}(\Sigma_2) \to H^{-\frac{1}{2}}(\Sigma_2)$ and $\Lambda_{s,A,V,q} : \widetilde{H}^s(\Sigma_2) \to H^{-s}(\Sigma_2)$ weakly as*

$$\begin{aligned}\langle \Lambda_{A,V,q} f, g\rangle_* &:= B_{A,V,q}(u_f, E(g)),\\ \langle \Lambda_{s,A,V,q} f, g\rangle_{*_s} &:= B_{s,A,V,q}(u_f, E_s(g)),\end{aligned}$$

where $E(g)$ denotes an $H^1_{\partial\Omega\setminus(\Sigma_1\cup\Sigma_2),0}(\Omega)$ extension of g into Ω and u_f denotes a weak solution (in the sense of Proposition 3.1 of (1)). Similarly, $E_s(g)$ is an $H^1_{\partial\Omega\setminus(\Sigma_1\cup\Sigma_2),0}(\Omega, d^{1-2s})$ extension of g into Ω and u_f denotes a weak solution (in the sense of Proposition 3.4) of (5). Here the notation $\langle\cdot,\cdot\rangle_*$ and $\langle\cdot,\cdot\rangle_{*_s}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Sigma_2)$ and $H^{\frac{1}{2}}(\Sigma_2)$ and between $H^{-s}(\Sigma_2)$ and $H^s(\Sigma_2)$, respectively.

Remark 3.10. *By definition we of course have that $\Lambda_{A,V,q} = \Lambda_{\frac{1}{2},A,V,q}$.*

As in the standard (partial data) setting, these Dirichlet-to-Neumann maps are well-defined and do not depend on the choice of the extension.

Lemma 3.11. *Let $\Lambda_{A,V,q} : \widetilde{H}^{\frac{1}{2}}(\Sigma_2) \to H^{-\frac{1}{2}}(\Sigma_2)$ and $\Lambda_{s,A,V,q} : \widetilde{H}^{s}(\Sigma_2) \to H^{-s}(\Sigma_2)$ be as in Definition 3.9. Then these maps are well-defined, i.e. they do not depend on the choice of the extension $E(g)$ and $E_s(g)$. Moreover, both maps are linear and bounded.*

Proof. The independence of the choice of the extension follows from the well-posedness theory. Indeed, considering two extensions $E(g)$ and $\tilde{E}(g)$ of $g \in \widetilde{H}^{1/2}(\Sigma_2)$, we deduce that $E(g) - \tilde{E}(g) \in H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega)$. Hence, we obtain that $B_{A,V,q}(u_f, E(g) - \tilde{E}(g)) = 0$ since u_f is a weak solution to the equation (1). A similar argument holds for the weighted operator. The linearity of the map follows from the linearity of the Schrödinger equations (1) and (5). The boundedness follows from the apriori estimates (15) and (18). □

As in the classical setting, the (partial data) Dirichlet-to-Neumann maps are self-adjoint operators:

Lemma 3.12 (Symmetry)**.** *Let $\Lambda_{A,V,q}$ and $\Lambda_{s,A,V,q}$ be as in (3.9). Then, we have*

$$\langle \Lambda_{A,V,q} f, g\rangle_* = \langle f, \Lambda_{A,V,q} g\rangle_*,$$
$$\langle \Lambda_{s,A,V,q} f, g\rangle_{*_s} = \langle g, \Lambda_{s,A,V,q} g\rangle_{*_s}.$$

Proof. The claim follows from the fact that two solutions u_f and u_g associated with the data f, g in (1) or (5) are particular extensions of $f, g \in \widetilde{H}^s(\Sigma_2)$. Since the bilinear forms $B_{A,V,q}(\cdot,\cdot)$ and $B_{s,A,V,q}(\cdot,\cdot)$ are symmetric (with respect to the complex scalar product) the claim follows. □

Furthermore, a central Alessandrini identity involving all potentials holds true:

Lemma 3.13 (Alessandrini)**.** *Let A_j, V_j, q_j and Λ_{s,A_j,V_j,q_j} with $j \in \{1,2\}$ be as above. Then, for two solutions u_1, u_2 of (5) associated with the respective boundary data and potentials,*

$$\begin{aligned}\langle (\Lambda_{s,A_1,V_1,q_1} - \Lambda_{s,A_2,V_2,q_2}) f_1, f_2\rangle_{*_s} = &\int_\Omega (V_1 - V_2 + A_1^2 - A_2^2) u_1 \overline{u_2} d^{1-2s} dx \\ &+ i \int_\Omega d^{1-2s}(A_1 - A_2)\cdot(u_1 \overline{\nabla u_2} - u_2 \cdot \overline{\nabla u_1}) dx \\ &+ \int_{\Sigma_1} (q_1 - q_2) u_1 \overline{u_2} d\mathcal{H}^{n-1}.\end{aligned}$$

Proof. This follows by using the symmetry result of Lemma 3.12 in combination with the structure of $B_{s,A,V,q}$ and the fact that all (bulk, boundary and gradient) potentials are real valued:

$$\begin{aligned}\langle (\Lambda_{s,A_1,V_1,q_1} - \Lambda_{s,A_2,V_2,q_2}) f_1, f_2\rangle_{*_s} &= \langle \Lambda_{s,A_1,V_1,q_1} f_1, f_2\rangle_{*_s} - \langle f_1, \Lambda_{s,A_2,V_2,q_2} f_2\rangle_{*_s} \\ &= B_{s,A_1,V_1,q_1}(u_1,u_2) - B_{s,A_2,V_2,q_2}(u_1,u_2) \\ &= \int_\Omega (V_1 - V_2 + A_1^2 - A_2^2) u_1 \overline{u_2} d^{1-2s} dx + \int_{\Sigma_1} (q_1 - q_2) u_1 \overline{u}_2 d\mathcal{H}^{n-1} \\ &\quad + i \int_\Omega d^{1-2s}(A_1 - A_2)\cdot(u_1 \overline{\nabla u_2} - u_2 \cdot \overline{\nabla u_1}) dx.\end{aligned}$$

This proves the claim. □

4. Simultaneous Runge Approximation in the Bulk and on the Boundary – Resolution of the Question (Q1) for $s = \frac{1}{2}$

In this section we discuss the resolution of the question (Q1) for the case $s = \frac{1}{2}$ by proving simultaneous Runge approximation results. This requires a certain "safety distance" between Ω_1 and the sets Σ_1, Σ_2 and a topological condition on the connectedness of $\Omega \setminus \Omega_1$. We refer to the set-up which had been layed out in Section 2.3 for the precise conditions. Although our setting could have been generalized to allow for Ω_1 including some boundary portions (see for instance [RS20b]), for clarity of exposition, we do not address this in the present article.

Let

$$\begin{aligned} S_{A,V,q} &:= \{u \in L^2(\Omega) : u \text{ is a weak solution to (1) in } \Omega\}, \\ \tilde{S}_{A,V,q} &:= \{u \in H^1(\Omega_1) : u \text{ is a weak solution to (1) in } \Omega_1\} \subset L^2(\Omega_1). \end{aligned} \tag{23}$$

Here by a weak solution we mean a solution as obtained in our well-posedness discussion in Section 3. For simplicity, we also simply set $S_{V,q} := S_{0,V,q}$ and $\tilde{S}_{V,q} := \tilde{S}_{0,V,q}$.

As a first step towards answering the question (Q1), we prove the simultaneous Runge approximation result (in the absence of magnetic potentials) from Lemma 1.1.

Remark 4.1. *Together with the (known) existence results of whole space CGO solutions, this approximation result allows us to recover the potentials $V \in L^\infty(\Omega_1)$ and $q \in L^\infty(\partial\Omega)$ simultaneously in the inverse problem for (1). Instead of explaining this at this point already, we refer to the proof of Theorem 1, where this is deduced even in the presence of magnetic potentials.*

Proof of Lemma 1.1. By the Hahn-Banach theorem, it suffices to prove that if $v = (v_1, v_2) \in L^2(\Sigma_1) \times L^2(\Omega_1)$ satisfies $v \perp \mathcal{R}_{bb}$ (with respect to the scalar product in $L^2(\Sigma_1) \times L^2(\Omega_1)$), then we have

$$v \perp (L^2(\Sigma_1) \times \tilde{S}_{V,q}).$$

To this end, let $f \in C_c^\infty(\Sigma_2)$ and define $u := Pf$. Moreover, let w be a solution to the associated adjoint problem

$$\begin{aligned} -\Delta w + Vw &= v_2 \chi_{\Omega_1} \text{ in } \Omega, \\ \partial_\nu w + qw &= v_1 \text{ on } \Sigma_1, \\ w &= 0 \text{ on } \partial\Omega \setminus \Sigma_1. \end{aligned} \tag{24}$$

Here χ_{Ω_1} denotes the characteristic function of the set Ω_1. By the assumption $v \perp \mathcal{R}_{bb}$ and the definitions of u and w, we have

$$\begin{aligned} 0 &= (v_1, u|_{\Sigma_1})_{L^2(\Sigma_1)} + (v_2, u|_{\Omega_1})_{L^2(\Omega_1)} \\ &= \langle \partial_\nu w + qw, u - f \rangle_* + (-\Delta w + Vw, u)_{L^2(\Omega)} \\ &= \langle \partial_\nu w + qw, u - f \rangle_* + (-\Delta u + Vu, w)_{L^2(\Omega)} + \langle \partial_\nu u, w \rangle_* - \langle u, \partial_\nu w \rangle_* \\ &= \langle qw, u \rangle_* - \langle \partial_\nu w + qw, f \rangle_* + \langle \partial_\nu u, w \rangle_* \\ &= -\langle \partial_\nu w + qw, f \rangle_* \ , \end{aligned} \tag{25}$$

where we integrated by parts twice and where $\langle \cdot, \cdot \rangle_*$ denotes the $H^{-\frac{1}{2}}(\Sigma_1)$, $H^{\frac{1}{2}}(\Sigma_1)$ duality pairing. We remark that this computation which – a priori is formal, since due to the mixed boundary conditions w, u may not be in $H^2(\Omega)$ – can be justified by considering the identities in a smaller domain Ω_ϵ for $\epsilon > 0$ sufficiently small first and then passing to the limit $\epsilon \to 0$. More precisely, by standard regularity theory, we obtain that $w, u \in H^2(\Omega_\epsilon)$ which allows us to justify the following manipulations:

$$\begin{aligned}
&(\partial_\nu w + qw, u - u|_{\partial\Omega_\epsilon\setminus\Sigma_{1,\epsilon}})_{L^2(\partial\Omega_\epsilon)} + (-\Delta w + Vw, u)_{L^2(\Omega_\epsilon)} \\
&= (\partial_\nu w + qw, u - u|_{\partial\Omega_\epsilon\setminus\Sigma_{1,\epsilon}})_{L^2(\partial\Omega_\epsilon)} + (-\Delta u + Vu, w)_{L^2(\Omega_\epsilon)} + (\partial_\nu u, w)_{L^2(\partial\Omega_\epsilon)} - (u, \partial_\nu w)_{L^2(\partial\Omega_\epsilon)} \\
&= (\partial_\nu w + qw, u - u|_{\partial\Omega_\epsilon\setminus\Sigma_{1,\epsilon}})_{L^2(\partial\Omega_\epsilon)} + (\partial_\nu u, w)_{L^2(\partial\Omega_\epsilon)} - (u, \partial_\nu w)_{L^2(\partial\Omega_\epsilon)}.
\end{aligned}$$

Here $\partial\Omega_\epsilon$ is defined as in (13) and $\Sigma_{1,\epsilon} := \{x \in \Omega :\ x = y + \epsilon\nu(y),\ y \in \Sigma_1\}$. Then passing to the limit $\epsilon \to 0$ and using the observations from Lemma 3.8 allows us to recover the first and fourth lines in (25), i.e.

$$\begin{aligned}
&(\partial_\nu w + qw, u - u|_{\partial\Omega_\epsilon\setminus\Sigma_{1,\epsilon}})_{L^2(\partial\Omega_\epsilon)} + (-\Delta w + Vw, u)_{L^2(\Omega_\epsilon)} \\
&\to \langle \partial_\nu w + qw, u - f\rangle_* + (v_2, u|_{\Omega_1})_{L^2(\Omega_1)}, \\
&(\partial_\nu w + qw, u - u|_{\partial\Omega_\epsilon\setminus\Sigma_{1,\epsilon}})_{L^2(\partial\Omega_\epsilon)} + (\partial_\nu u, w)_{L^2(\partial\Omega_\epsilon)} - (u, \partial_\nu w)_{L^2(\partial\Omega_\epsilon)} \\
&\to \langle qw, u\rangle_* - \langle \partial_\nu w + qw, f\rangle_* + \langle \partial_\nu u, w\rangle_*.
\end{aligned}$$

This then allows us to conclude the identity $\langle \partial_\nu w + qw, f\rangle_* = 0$ as in the formal argument from (25).

By the arbitrary choice of $f \in C_c^\infty(\Sigma_2)$, (25) yields that $\partial_\nu w + qw = 0$ in Σ_2, which in turn by the defining property of w gives $\partial_\nu w|_{\Sigma_2} = w|_{\Sigma_2} = 0$. Thus now the unique continuation property (see for instance [ARRV09]) implies that $w = 0$ in $\Omega \setminus \overline{\Omega_1}$, and therefore

$$w|_{\Sigma_1} = \nabla w|_{\Sigma_1} = 0 \quad \text{and} \quad w|_{\partial\Omega_1} = \nabla w|_{\partial\Omega_1} = 0\,. \tag{26}$$

The first part of (26) implies $v_1 = 0$ by the definition of the associated dual problem (24). In particular, $(v_1, \psi_1)_{L^2(\Sigma_1)} = 0$ for all $\psi_1 \in L^2(\Sigma_1)$. If now $\psi_2 \in \tilde{S}_{V,q}$, denoting the $H^{-\frac{1}{2}}(\partial\Omega_1)$, $H^{\frac{1}{2}}(\partial\Omega_1)$ duality pairing by $\langle\cdot,\cdot\rangle_{*,\partial\Omega_1}$ and integrating by parts we get

$$\begin{aligned}
(v_2, \psi_2)_{L^2(\Omega_1)} &= (-\Delta w + Vw, \psi_2)_{L^2(\Omega_1)} \\
&= (-\Delta\psi_2 + V\psi_2, w)_{L^2(\Omega_1)} + \langle \partial_\nu\psi_2, w\rangle_{*,\partial\Omega_1} - \langle \psi_2, \partial_\nu w\rangle_{*,\partial\Omega_1}\,,
\end{aligned}$$

which vanishes because of the second part of formula (26) and because $\psi_2 \in \tilde{S}_{V,q}$ (which is also true in the weak form of the equation by definition). Hence,

$$(v, \psi)_{L^2(\Sigma_1)\times L^2(\Omega_1)} = 0 \quad \text{for all} \quad \psi = (\psi_1, \psi_2) \in (L^2(\Sigma_1) \times \tilde{S}_{V,q})\,,$$

that is, $v \perp (L^2(\Sigma_1) \times \tilde{S}_{V,q})$ with respect to the $L^2(\Sigma_1) \times L^2(\Omega_1)$ scalar product as desired. □

For our next step towards the solution of question (Q1), we shall consider a generalization of equation (1), namely

$$\begin{aligned}
Lu := -\nabla\cdot(g\nabla u) - iA_1\cdot\nabla u - i\nabla\cdot(A_2 u) + Vu &= 0 \text{ in } \Omega, \\
\nu\cdot(g\nabla u) + qu &= 0 \text{ on } \Sigma_1, \\
u &= f \text{ on } \Sigma_2, \\
u &= 0 \text{ on } \partial\Omega\setminus(\Sigma_1\cup\Sigma_2),
\end{aligned} \tag{27}$$

where $g = (g_{ij})_{i,j=1,\dots,n}$ is a C^2 metric, i.e. a symmetric, positive definite, elliptic, C^2-regular matrix valued function on Ω, and the magnetic potentials A_1 and A_2 do not necessarily coincide. We avoid discussing the well-posedness for this and refer to [McL00] and [GT15] for a discussion of it. In the sequel, we will assume the well-posedness of this problem and its associated dual problem.

In connection to the problem (27) we define the sets

$$\begin{aligned}
S_{g,A_1,A_2,V,q} &:= \{u \in L^2(\Omega) : u \text{ is a weak solution to (27) in } \Omega\}, \\
\tilde{S}_{g,A_1,A_2,V,q} &:= \{u \in H^1(\Omega_1) : u \text{ is a weak solution to (27) in } \Omega_1\} \subset L^2(\Omega_1).
\end{aligned}$$

The next Lemma 4.2 shows that the result of Lemma 1.1 still holds for equation (27), and the approximation can even be given in $H^1(\Omega_1)$ instead of $L^2(\Omega_1)$.

Lemma 4.2. *Assume that the set-up is as above. Then, the set*

$$\mathcal{R}_{bb} := \{(u|_{\Sigma_1}, u|_{\Omega_1}) : \ u|_{\Sigma_1} = Pf|_{\Sigma_1} \text{ and } u|_{\Omega_1} = Pf|_{\Omega_1} \text{ with } f \in C_c^\infty(\Sigma_2)\} \subset L^2(\Sigma_1) \times H^1(\Omega_1)$$

is dense in $L^2(\Sigma_1) \times \tilde{S}_{g,A_1,A_2,V,q}$ equipped with the $L^2(\Sigma_1) \times H^1(\Omega_1)$ topology. Here P denotes the Poisson operator which is defined in analogy to Definition 3.3 and in particular maps data $f \in C_c^\infty(\Sigma_2)$ into the associated (weak) solution u to the equation (27).

Proof. We use the same strategy as in the proof of the previous Lemma. Let $(v_1, v_2^*) \in L^2(\Sigma_1) \times (H^1(\Omega_1))^*$, and consider the unique Riesz representative $v_2 \in H^1(\Omega_1)$ of the functional $v_2^* \in (H^1(\Omega_1))^*$. By the Hahn-Banach theorem, it suffices to prove that if $v = (v_1, v_2) \in L^2(\Sigma_1) \times H^1(\Omega_1)$ satisfies $v \perp \mathcal{R}_{bb}$ with respect to the scalar product in $L^2(\Sigma_1) \times H^1(\Omega_1)$, then we have

$$v \perp (L^2(\Sigma_1) \times \tilde{S}_{g,A_1,A_2,V,q}).$$

To this end, let $f \in C_c^\infty(\Sigma_2)$ and define $u := Pf$. Moreover, let w be a solution to the associated adjoint problem

$$\begin{aligned} L^* w &= \tilde{v}_2^* \text{ in } \Omega, \\ \nu \cdot (g\nabla w) + (q - i\nu \cdot A_1 - i\nu \cdot A_2) w &= v_1 \text{ on } \Sigma_1, \\ w &= 0 \text{ on } \partial\Omega \setminus \Sigma_1, \end{aligned} \tag{28}$$

where $L^* := -\nabla \cdot (g\nabla) + iA_2 \cdot \nabla + i\nabla \cdot A_1 + V$ and $\tilde{v}_2^*(\cdot) := v_2^*(\cdot|_{\Omega_1})$. First we observe that $\tilde{v}_2^* \in (H_b^1(\Omega))^* := (H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega) + \tilde{H}^{\frac{1}{2}}(\Sigma_2 \cup \Sigma_1))^*$. The associated bound is easily proved, since for $u \in H_b^1(\Omega)$

$$|\tilde{v}_2^*(u)| = |{v_2}^*(u|_{\Omega_1})| \leq \|{v_2}^*\| \, \|u|_{\Omega_1}\|_{H^1(\Omega_1)} \leq \|v_2^*\| \, \|u\|_{H^1(\Omega)} \, .$$

Now we have

$$(v_2, u|_{\Omega_1})_{H^1(\Omega_1)} = v_2^*(u|_{\Omega_1}) = \tilde{v}_2^*(u) \, ,$$

which by the assumption $v \perp \mathcal{R}_{bb}$ leads to

$$\begin{aligned} 0 &= (v_1, u|_{\Sigma_1})_{L^2(\Sigma_1)} + (v_2, u|_{\Omega_1})_{H^1(\Omega_1)} \\ &= \langle \nu \cdot (g\nabla w) + (q - i\nu \cdot A_1 - i\nu \cdot A_2) w, u|_{\Sigma_1} \rangle_* + \tilde{v}_2^*(u) \\ &= \langle \nu \cdot (g\nabla w) + (q - i\nu \cdot A_1 - i\nu \cdot A_2) w, u - f \rangle_* + (L^* w, u)_{L^2(\Omega)} \, . \end{aligned} \tag{29}$$

As in the previous proof, $\langle \cdot, \cdot \rangle_*$ denotes the $H^{\frac{1}{2}}(\partial\Omega)$, $H^{-\frac{1}{2}}(\partial\Omega)$ duality pairing.

Integrating by parts twice (which can be justified in the same way as in the previous section), we obtain the following formula linking the operators L and L^*:

$$\begin{aligned} (Lu, w)_\Omega - (u, L^* w)_\Omega &= -(\nabla \cdot (g\nabla u), w)_\Omega - i(A_1 \cdot \nabla u, w)_\Omega - i(\nabla \cdot (A_2 u), w)_\Omega \\ &\quad + (\nabla \cdot (g\nabla w), u)_\Omega - i(A_2 \cdot \nabla w, u)_\Omega - i(\nabla \cdot (A_1 w), u)_\Omega \\ &= -\langle \nu \cdot (g\nabla u), w \rangle_* - i\langle \nu \cdot (A_1 + A_2) u, w \rangle_* + \langle u, \nu \cdot (g\nabla w) \rangle_* \, . \end{aligned} \tag{30}$$

Here we have used $(\cdot, \cdot)_\Omega$ as a shorthand notation for $(\cdot, \cdot)_{L^2(\Omega)}$. Combining formulas (29) and (30), we infer

$$\langle \nu \cdot (g\nabla w) + (q - i\nu \cdot A_1 - i\nu \cdot A_2) w, f \rangle_* = 0 \, ,$$

which by the arbitrary choice of $f \in C_c^\infty(\Sigma_2)$ gives

$$\nu \cdot (g\nabla w) + (q - i\nu \cdot A_1 - i\nu \cdot A_2) w = 0 \quad \text{on } \ \Sigma_2 \, .$$

Thus, by definition of the adjoint equation (28), we are left with

$$\begin{aligned} L^* w &= 0 \text{ in } \Omega \setminus \overline{\Omega_1}, \\ \nu \cdot (g \nabla w) &= 0 \text{ on } \Sigma_2, \\ w &= 0 \text{ on } \Sigma_2, \end{aligned}$$

and now the UCP (see for instance [ARRV09]) leads to $w = 0$ in $\Omega \setminus \overline{\Omega_1}$. As a consequence of this fact, we obtain $w|_{\partial(\Omega\setminus\overline{\Omega_1})} = 0$ and $\nabla w|_{\partial(\Omega\setminus\overline{\Omega_1})} = 0$, which in particular implies

$$w|_{\Sigma_1} = \nabla w|_{\Sigma_1} = 0 \quad \text{and} \quad w|_{\partial\Omega_1} = \nabla w|_{\partial\Omega_1} = 0\,. \tag{31}$$

The first part of (31) implies $v_1 = 0$ by the associated dual problem (28). In particular, $\langle v_1, \psi_1\rangle_* = 0$ for all $\psi_1 \in L^2(\Sigma_1)$.

Let now $\psi_2 \in \tilde{S}_{g,A_1,A_2,V,q}$. If $E : H^1(\Omega_1) \to H^1(\Omega)$ is any extension operator, we have

$$\begin{aligned} (v_2, \psi_2)_{H^1(\Omega_1)} &= (v_2, (E\psi_2)|_{\Omega_1})_{H^1(\Omega_1)} = v_2^*((E\psi_2)|_{\Omega_1}) \\ &= \tilde{v}_2^*(E\psi_2) = (L^* w, E\psi_2)_{L^2(\Omega)} = (L^* w, \psi_2)_{L^2(\Omega_1)}\,. \end{aligned}$$

Using the integration by parts formula (30) with Ω_1 instead of Ω, we infer

$$\begin{aligned} (v_2, \psi_2)_{H^1(\Omega_1)} &= (L^* w, \psi_2)_{L^2(\Omega_1)} \\ &= (L\psi_2, w)_{\Omega_1} + \langle \nu \cdot (g\nabla\psi_2), w\rangle_{*,\partial\Omega_1} \\ &\quad + i\langle \nu \cdot (A_1 + A_2)\psi_2, w\rangle_{*,\partial\Omega_1} - \langle \psi_2, \nu \cdot (g\nabla w)\rangle_{*,\partial\Omega_1}\,. \end{aligned}$$

Here we have denoted the $H^{-\frac{1}{2}}(\partial\Omega_1)$, $H^{\frac{1}{2}}(\partial\Omega_1)$ duality pairing by $\langle\cdot,\cdot\rangle_{*,\partial\Omega_1}$.

The right hand side of the above equation vanishes because of the second part of formula (31) and because $\psi_2 \in \tilde{S}_{g,A_1,A_2,V,q}$. Thus we have obtained that

$$(v, \psi)_{L^2(\Sigma_1)\times H^1(\Omega_1)} = 0 \quad \text{for all} \quad \psi = (\psi_1, \psi_2) \in (L^2(\Sigma_1) \times \tilde{S}_{g,A_1,A_2,V,q})\,,$$

that is, $v \perp (L^2(\Sigma_1) \times \tilde{S}_{g,A_1,A_2,V,q})$. □

The desired uniqueness result of Theorem 1 now follows from Alessandrini's identity.

Proof of Theorem 1. Using the assumption that the DN maps coincide and Lemma 3.13 with $s = 1/2$, we see that

$$\begin{aligned} 0 &= \langle (\Lambda_1 - \Lambda_2) f_1, f_2\rangle_* \\ &= \int_{\Omega_1} (U_1 - U_2) u_1 \overline{u_2} dx + i \int_{\Omega_1} (A_1 - A_2) \cdot (u_1 \overline{\nabla u_2} - u_2 \cdot \overline{\nabla u_1}) dx + \int_{\Sigma_1} (q_1 - q_2) u_1 \overline{u_2} d\mathcal{H}^{n-1} \end{aligned} \tag{32}$$

holds for every $f_1, f_2 \in C_c^\infty(\Sigma_2)$, where u_1, u_2 are the solutions of (1) associated with the respective boundary data and potentials. For the sake of simplicity, here we set $U_j := V_j + |A_j|^2$ and $\Lambda_j := \Lambda_{A_j,V_j,q_j}$.

Let $\phi_j \in L^2(\Sigma_1)$ and $\psi_j \in \tilde{S}_{Id,A_j,A_j,V_j,q_j}$ for $j = 1, 2$. By Lemma 4.2, for every $k \in \mathbb{N}$ we can find $f_1^{(k)}, f_2^{(k)} \in C_c^\infty(\Sigma_2)$ such that

$$\|\phi_j - u_j^{(k)}|_{\Sigma_1}\|_{L^2(\Sigma_1)} < k^{-1} \quad \text{and} \quad \|\psi_j - u_j^{(k)}|_{\Omega_1}\|_{H^1(\Omega_1)} < k^{-1}\,, \qquad j = 1, 2\,,$$

where $u_j^{(k)}$ solves (1) with boundary value $f_j^{(k)}$ and potentials A_j, V_j, q_j. We now substitute these solutions $u_1^{(k)}$, $u_2^{(k)}$ into formula (32) and send $k \to \infty$. Given the approximations above, by Cauchy-Schwarz the limits can be moved inside the integrals. In fact,

$$\begin{aligned}\int_{\Sigma_1} |(q_1-q_2)u_1^{(k)}\overline{u_2^{(k)}}|d\mathcal{H}^{n-1} &\leq \|q_1-q_2\|_{L^\infty(\Sigma_1)}\|u_1^{(k)}|_{\Sigma_1}\|_{L^2(\Sigma_1)}\|u_2^{(k)}|_{\Sigma_1}\|_{L^2(\Sigma_1)} \\ &\leq c(\|\phi_1-u_1^{(k)}|_{\Sigma_1}\|_{L^2(\Sigma_1)}+\|\phi_1\|_{L^2(\Sigma_1)})(\|\phi_2-u_2^{(k)}|_{\Sigma_1}\|_{L^2(\Sigma_1)}+\|\phi_2\|_{L^2(\Sigma_1)}) \\ &\leq c(1+\|\phi_1\|_{L^2(\Sigma_1)})(1+\|\phi_2\|_{L^2(\Sigma_1)}) < c\,,\end{aligned}$$

$$\begin{aligned}\int_{\Omega_1} |u_1^{(k)}(A_1-A_2)\cdot\overline{\nabla u_2^{(k)}}|dx &\leq \|A_1-A_2\|_{L^\infty(\Omega_1)}\|u_1^{(k)}|_{\Omega_1}\|_{L^2(\Omega_1)}\|\nabla u_2^{(k)}|_{\Omega_1}\|_{L^2(\Omega_1)} \\ &\leq c(\|\psi_1-u_1^{(k)}|_{\Omega_1}\|_{L^2(\Omega_1)}+\|\psi_1\|_{L^2(\Omega_1)})(\|\nabla\psi_2-\nabla u_2^{(k)}|_{\Omega_1}\|_{L^2(\Omega_1)}+\|\nabla\psi_2\|_{L^2(\Omega_1)}) \\ &\leq c(\|\psi_1-u_1^{(k)}|_{\Omega_1}\|_{H^1(\Omega_1)}+\|\psi_1\|_{H^1(\Omega_1)})(\|\psi_2-u_2^{(k)}|_{\Omega_1}\|_{H^1(\Omega_1)}+\|\psi_2\|_{H^1(\Omega_1)}) \\ &\leq c(1+\|\psi_1\|_{H^1(\Omega_1)})(1+\|\psi_2\|_{H^1(\Omega_1)}) < c\,,\end{aligned}$$

and similarly for the other terms. Eventually, we have proved that the following formula holds for every $\phi_j \in L^2(\Sigma_1)$ and $\psi_j \in \tilde{S}_{Id,A_j,A_j,V_j,q_j}$ for $j=1,2$:

(33)
$$\int_{\Omega_1}(A_1-A_2)\cdot(\psi_2\overline{\nabla\psi_1}-\psi_1\overline{\nabla\psi_2})\,dx+\int_{\Omega_1}(U_1-U_2)\psi_1\overline{\psi_2}dx+\int_{\Sigma_1}(q_1-q_2)\phi_1\overline{\phi_2}d\mathcal{H}^{n-1}=0\,.$$

If we substitute $\psi_1=\psi_2=0$ and $\phi_2=1$ into (33), we are left with only

$$\int_{\Sigma_1}(q_1-q_2)\phi_1 d\mathcal{H}^{n-1}=0\,,$$

which by the arbitrary choice of $\phi_1 \in L^2(\Sigma_1)$ implies $q_1=q_2$ in Σ_1. In light of this, formula (33) is reduced to

$$\int_{\Omega_1}(A_1-A_2)\cdot(\psi_2\overline{\nabla\psi_1}-\psi_1\overline{\nabla\psi_2})\,dx+\int_{\Omega_1}(U_1-U_2)\psi_1\overline{\psi_2}dx=0\,. \tag{34}$$

The problem of deducing information about the magnetic and electric potentials from the above equation has been studied e.g. in [NSU95, Tol98, FKSU07], see also the survey [Sal06]. In all these uniqueness results the key step consists in the construction of suitable complex geometrical optics solutions of the form

$$u(x)=e^{\frac{\phi+i\psi}{h}}(a(x)+hr(x,h))\,,$$

for appropriate phase functions ϕ,ψ, amplitudes a and decaying errors r. Substituting such a special solution into equation (34) and using our Runge approximation results from Lemma 4.2 allows us to deduce that $V_1=V_2$ and $dA_1=dA_2$ as in the cited references, which concludes the proof of Theorem 1. □

5. Simultaneous Runge Approximation $s\in(0,1)$

Similarly as in deriving the results in the previous section, we can also deduce simultaneous Runge approximation results for the "Caffarelli-Silvestre extension" for general $s\in(0,1)$.

In analogy to the setting in the previous section we thus set

$$\begin{aligned}S_{s,A,V,q} &:= \{u\in L^2(\Omega,d^{1-2s}):\ u \text{ is a weak solution to (5) in } \Omega\}, \\ \tilde{S}_{s,A,V,q} &:= \{u\in H^1(\Omega,d^{1-2s}):\ u \text{ is a weak solution to (5) in } \Omega_1\}\subset L^2(\Omega_1,d^{1-2s}).\end{aligned}$$

In order to illustrate these ideas we only discuss the $L^2(\Sigma_1)\times L^2(\Omega_1,d^{1-2s})$ approximation result in the case that $A=0$.

Proposition 5.1. *Assume that $A = 0$ and that V, q satisfy the conditions from (3) and (2) and let P_s be the associated Poisson operator. Then the set*

$$\mathcal{R}_{bbs} := \{(u|_{\Sigma_1}, u|_{\Omega_1}) : u|_{\Sigma_1} = P_s f|_{\Sigma_1} \text{ and } u|_{\Omega_1} = P_s f|_{\Omega_1} \text{ with } f \in C_c^\infty(\Sigma_2)\} \subset L^2(\Sigma_1) \times S_{s,0,V,q}$$

is dense in $L^2(\Sigma_1) \times \tilde{S}_{s,0,V,q}$ equipped with the $L^2(\Sigma_1) \times L^2(\Omega_1, d^{1-2s})$ topology. The operator P_s denotes the Poisson operator from Definition 3.6.

Proof. Step 1: Set-up. The argument is similar as in the one for Lemma 1.1. To this end, we first note that with respect to the $L^2(\Omega)$ scalar product, we have that $(L^2(\Omega, d^{1-2s}))^* \sim L^2(\Omega, d^{2s-1})$. As a consequence, as above, we seek to prove that if $(v_1, v_2) \in L^2(\Sigma_1) \times L^2(\Omega, d^{2s-1})$ satisfies $(v_1, v_2) \perp (u|_{\Sigma_1}, u|_{\Omega})$ with $u = P_s(f)$ with $f \in C_c^\infty(\Sigma_2)$ (with orthogonality with respect to the $L^2(\Sigma_1) \times L^2(\Omega_1)$ scalar product), then also $(v_1, v_2) \perp L^2(\Sigma_1) \times \tilde{S}_{s,0,V,q}$ holds. To this end, we consider weak solutions to the adjoint problem

$$\begin{aligned} -\nabla \cdot d^{1-2s} \nabla w + d^{1-2s} V w &= v_2 \chi_{\Omega_1} \text{ in } \Omega, \\ \lim_{d(x) \to 0} d^{1-2s} \partial_\nu w + q w &= -v_1 \text{ on } \Sigma_1, \\ w &= 0 \text{ on } \partial\Omega \setminus \Sigma_1. \end{aligned} \tag{35}$$

Let us thus assume that $(v_1, v_2) \in L^2(\Sigma_1) \times L^2(\Omega, d^{2s-1})$ are such that for all $u = P_s(f)$ with $f \in C_c^\infty(\Sigma_1)$ we have

$$0 = (v_1, u)_{L^2(\Sigma_2)} + (v_2, u)_{L^2(\Omega)}. \tag{36}$$

We remark that due to the assumptions that $v_2 \in L^2(\Omega, d^{2s-1})$ and $u \in L^2(\Omega, d^{1-2s})$ the bulk $L^2(\Omega)$ scalar product is well-defined.

Step 2: Orthogonality. We argue on the level of the strong equation. This can be justified as in the proof of Lemma 1.1 using the boundedness and convergence results from Lemma 3.8. Beginning with the bulk contribution and using the dual equation, we then obtain

$$\begin{aligned} (u, v_2)_{L^2(\Omega_1)} &= (u, -\nabla \cdot d^{1-2s} \nabla w + d^{1-2s} V w)_{L^2(\Omega)} \\ &= \langle u, \lim_{d \to 0} d^{1-2s} \partial_\nu w \rangle_{*,s} - \langle \lim_{d \to 0} d^{1-2s} \partial_\nu u, w \rangle_{*,s} \\ &= (u, qw)_{L^2(\Sigma_1)} - (u, v_1)_{L^2(\Sigma_1)} - (qu, w)_{L^2(\Sigma_1)} + \langle f, \lim_{d \to 0} d^{1-2s} \partial_\nu w \rangle_{*,s} \\ &= -(u, v_1)_{L^2(\Sigma_1)} + \langle f, \lim_{d \to 0} d^{1-2s} \partial_\nu w \rangle_{*,s}, \end{aligned}$$

where $u = P_s(f)$ and $f \in C_c^\infty(\Sigma_2)$ and $\langle \cdot, \cdot \rangle_{*,s}$ denotes the $H^{-s}(\partial\Omega)$, $H^s(\partial\Omega)$ duality pairing. Combining this with (36), we obtain that

$$0 = \langle f, \lim_{d \to 0} d^{1-2s} \partial_\nu w \rangle_{*,s} \text{ for all } f \in C_c^\infty(\Sigma_2).$$

Hence, $\lim_{d \to 0} d^{1-2s} \partial_\nu w = 0$ on Σ_2. Since moreover also $w|_{\Sigma_2} = 0$, boundary unique continuation for the fractional Schrödinger equation (35) implies that $w \equiv 0$ in $\Omega \setminus \Omega_1$. Indeed, it is possible to flatten the boundary $\partial\Omega$ by a suitable diffeomorphism and then invoke the unique continuation results from for instance [Rül15, FF14] or [Yu17]. Now, by definition of w (see (35)), this however implies that $v_1 \equiv 0$.

Further, for $h \in \tilde{S}_{s,0,V,q}$, by the vanishing of $w|_{\partial\Omega_1}$ and $\lim_{d \to 0} d^{1-2s} \partial_\nu w|_{\partial\Omega_1}$, we infer that

$$\begin{aligned} (h, v_2)_{L^2(\Omega_1)} &= (h, -\nabla \cdot d^{1-2s} \nabla w + V d^{1-2s} w)_{L^2(\Omega_1)} \\ &= (-\nabla \cdot d^{1-2s} \nabla h + d^{1-2s} V h, w)_{L^2(\Omega_1)} = 0. \end{aligned}$$

Here the last equality follows from the fact that $h \in \tilde{S}_{s,0,V,q}$. Thus, in particular, $v_2 \perp \tilde{S}_{s,0,V,q}$, which concludes the argument. ∎

Using the simultaneous bulk and boundary approximation result of Proposition 5.1, it is possible to recover V and q simultaneously also in this weighted setting:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an open, bounded and C^3-regular domain. Suppose that $d \in C^2(\Omega)$. Assume $\Omega_1 \Subset \Omega$ is an open, bounded set with $\Omega \setminus \Omega_1$ simply connected and that $\Sigma_1, \Sigma_2 \subset \partial\Omega$ are two disjoint, relatively open sets. If the potentials $q_1, q_2 \in L^\infty(\Sigma_1)$ and $V_1, V_2 \in C_c^\infty(\Omega_1)$ in the equation (5) are such that*

$$\Lambda_{s,1} := \Lambda_{s,0,V_1,q_1} = \Lambda_{s,0,V_2,q_2} =: \Lambda_{s,2}\ ,$$

then $q_1 = q_2$ and $V_1 = V_2$.

Proof. By virtue of the Alessandrini identity we obtain

$$0 = \int_{\Omega_1} d^{1-2s}(V_1 - V_2)u_1\overline{u_2}dx + \int_{\Sigma_1}(q_1 - q_2)u_1\overline{u_2}d\mathcal{H}^{n-1},$$

for u_1, u_2 weak solutions to (5). Now an approximation argument as in the proof of Theorem 1 implies that for every $\phi_j \in L^2(\Sigma_1)$ and $\psi_j \in \tilde{S}_{s,0,V_j,q_j}$ and $j \in \{1,2\}$ we obtain

$$0 = \int_{\Omega_1} d^{1-2s}(V_1 - V_2)\psi_1\overline{\psi_2}dx + \int_{\Sigma_1}(q_1 - q_2)\phi_1\overline{\phi_2}d\mathcal{H}^{n-1}.$$

With this in hand, the proof that $q_1 = q_2$ is immediate by choosing $\psi_1 = \psi_2 = 0$, $\phi_1 \in C_c^\infty(\Sigma_1)$ arbitrary and $\phi_2 = 1$. The uniqueness $V_1 = V_2$ follows by a reduction of the problem in Ω_1 to a Schrödinger type problem. Carrying out a Liouville transform (see for instance [Sal08]), the equation

$$-\nabla \cdot d^{1-2s}\nabla u + Vd^{1-2s}u = 0 \text{ in } \Omega_1$$

is transferred to the Schrödinger type problem

$$-\Delta w + (Q + Vd^{\frac{1-2s}{2}})w = 0 \text{ in } \Omega_1\ ,$$

where $Q := \frac{\Delta d^{\frac{1-2s}{2}}}{d^{\frac{1-2s}{2}}}$ and $w := d^{\frac{1-2s}{2}}u$. We note that $Q \in L^\infty(\Omega_1)$ since $d \in C^2(\Omega)$ and $\text{dist}(\partial\Omega_1, \partial\Omega) > 0$. Now, standard CGO constructions allow to obtain solutions of the form

$$w_1 = e^{i\xi\cdot x}(e^{ik\cdot x} + r_1),\ w_2 = e^{i\xi'\cdot x}(e^{-ik\cdot x} + r_2),$$

with $\xi, \xi' \in \mathbb{C}^n$, $k \in \mathbb{R}^n$, $\xi\cdot\xi = k\cdot\xi = 0$, $\xi' = -\text{Re}(\xi) + i\text{Im}(\xi)$ and $\|r_j\|_{L^2(\Omega_1)} \to 0$ as $|\xi'| \to \infty$. Then the functions

$$u_j := d^{\frac{2s-1}{2}}w_j,\ j \in \{1,2\}$$

however solve the equation

$$d^{\frac{2s-1}{2}}(-\nabla \cdot d^{1-2s}\nabla w_j + Vd^{1-2s}w_j) = 0 \text{ in } \Omega_1$$

in a weak sense. Due to the assumed regularity of d, they also satisfy

$$-\nabla \cdot d^{1-2s}\nabla w_j + Vd^{1-2s}w_j = 0 \text{ in } \Omega_1$$

in a weak sense. By virtue of the result from Proposition 5.1 we may thus approximate these functions by functions $\psi_j \in S_{s,0,V,q}$. Inserting these into the Alessandrini identity, recalling that $q_1 = q_2$ and passing to the limit in the approximation parameter then implies

$$0 = \int_{\Omega_1}(V_1 - V_2)e^{2ik\cdot x}dx.$$

As a consequence, also $V_1 = V_2$. □

Remark 5.2. *While in the study of the question (Q1) the situation in which $\Omega_1 \Subset \Omega$, the construction of CGOs to the degenerate equation (5) can essentially be avoided by using the non-degeneracy of the equation in Ω_1, this can no longer be circumvented in the setting of question (Q2).*

We refer to the next two sections for the construction of a new family of CGO type solutions for a closely related equation. These will be used to answer the question (Q2) in the case $s \in (\frac{1}{2}, 1)$ and will also provide a partial answer in the case $s = \frac{1}{2}$.

6. On a Carleman Estimate for the "Caffarelli-Silvestre Extension"

In this and the next section we address the question (Q2) for $s \geq \frac{1}{2}$ in the absence of magnetic potentials. As a major ingredient, we here construct CGO solutions to the equation

$$\begin{aligned} &\nabla \cdot x_{n+1}^{1-2s} \nabla u + x_{n+1}^{1-2s} V u = 0 \text{ in } \Omega, \\ &\lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} u + qu = 0 \text{ on } \Sigma_1, \end{aligned} \tag{37}$$

where $\Sigma_1 = \overline{\Omega} \cap \{x_{n+1} = 0\}$ is assumed to be a smooth, n-dimensional set and $\partial\Omega$ is C^∞ regular (the arguments from below show that C^m-regular with $m = m(s) > 0$ would suffice). The CGO construction is achieved by virtue of a duality argument and a suitable Carleman estimate.

The degenerate behaviour of the equation is reflected in the form of the CGOs. In order to avoid issues with the Muckenhoupt weight in the equation at $x_{n+1} = 0$, using the notation $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}_+$, we only consider wave vectors $\xi' \in \mathbb{C}^n$ with $\xi' \cdot \xi' = 0$ which are orthogonal to e_{n+1}. More precisely, we seek to construct solutions of the form

$$u(x) = e^{\xi' \cdot x'}(a(x) + r(x)).$$

with amplitudes $a(x) = e^{ik' \cdot x' + ik_{n+1} x_{n+1}^{2s}}$, $k \in \mathbb{R}^{n+1}$, and errors $r : \Omega \to \mathbb{R}$. We emphasize that the nonlinear (in x_{n+1}) phase dependence $ik_{n+1} x_{n+1}^{2s}$ is also a consequence of the degenerate elliptic character of the equation (see the estimate for $\tilde{L}^s_{-\xi', V}$ in (68) in the proof of Proposition 1.2). The function $r : \Omega \to \mathbb{R}$ is an error for which we seek to produce decay estimates as $|\xi'| \to \infty$ by means of a suitable Carleman estimate.

We begin by a discussion of the Carleman estimate which underlies our CGO construction.

Proposition 6.1 (Carleman estimate)**.** *Let $s \in [\frac{1}{2}, 1)$ and let $\xi' \in \mathbb{C}^n$ be such that $\xi' \cdot \xi' = 0$. Assume that $\Omega \subset \overline{\mathbb{R}}^{n+1}_+$ is a smooth domain and that $\overline{\Omega} \cap \{x_{n+1} = 0\} =: \Sigma_1$ is a smooth, n-dimensional set. If $s = \frac{1}{2}$, further assume that $\|q\|_{L^\infty(\Sigma_1)}$ is sufficiently small. Let $f \in (H^1(\Omega, x_{n+1}^{1-2s}))^*$ with $\operatorname{supp}(f) \subset \Omega \cup (\overline{\Omega} \cap \{x_{n+1} = 0\})$ and $g \in L^2(\Sigma_1)$. Then, for $u \in H^1_{\partial\Omega \setminus \overline{\Sigma_1}, 0}(\Omega, x_{n+1}^{1-2s}) \cap \mathcal{C}$ with $u = 0$ and $\lim\limits_{x \to \partial\Omega} x_{n+1}^{1-2s} \partial_\nu u = 0$ on $\partial\Omega \setminus \overline{\Sigma_1}$ being a weak solution to*

$$\begin{aligned} &\nabla \cdot x_{n+1}^{1-2s} \nabla u = f \text{ in } \Omega, \\ &\lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} u + qu = g \text{ on } \Sigma_1, \end{aligned} \tag{38}$$

we have

$$\begin{aligned} &|\xi'|^s \|e^{\xi' \cdot x'} u\|_{L^2(\Sigma_1)} + |\xi'| \|e^{\xi' \cdot x'} x_{n+1}^{\frac{1-2s}{2}} u\|_{L^2(\Omega)} + \|e^{\xi' \cdot x'} x_{n+1}^{\frac{1-2s}{2}} \nabla u\|_{L^2(\Omega)} \\ &\leq C(|\xi'| \|e^{\xi' \cdot x'} x_{n+1}^{\frac{1-2s}{2}} \tilde{F}\|_{L^2(\Omega)} + \|e^{\xi' \cdot x'} x_{n+1}^{\frac{1-2s}{2}} F_0\|_{L^2(\Omega)} + |\xi'|^{1-s} \|e^{\xi' \cdot x'} g\|_{L^2(\Sigma_1)}). \end{aligned} \tag{39}$$

Here the constant $C > 0$ depends on $\|q\|_{L^\infty(\Sigma_1)}$ and $F = (F_0, \tilde{F}) \in L^2(\mathbb{R}^{n+1}_+, \mathbb{R}^{n+2})$ is the Riesz representation of f, i.e., it is such that

$$f(v) = (v, x_{n+1}^{1-2s} F_0)_{L^2(\Omega)} + (\nabla v, x_{n+1}^{1-2s} \tilde{F})_{L^2(\Omega)} \text{ for all } v \in H^1(\Omega, x_{n+1}^{1-2s}).$$

We remark that $\|F\|_{L^2(\Omega,x_{n+1}^{1-2s})} = \|f\|_{(H^1(\Omega,x_{n+1}^{1-2s}))^*}$.

Remark 6.2. *We remark that as $\partial\Omega$ is smooth and as $x_{n+1} = 0$ on Σ_1, we have that x_{n+1} vanishes to infinite order at $\partial\Sigma_1$, i.e. that the domain is arbitrarily flat in a neighbourhood of $\partial\Sigma_1$.*

Proof. We argue in three steps using a splitting strategy. More precisely, we write $u = u_1 + u_2$ where u_1 (weakly) solves the problem

$$\begin{aligned} -\nabla\cdot x_{n+1}^{1-2s}\nabla u_1 + K|\xi'|^2 x_{n+1}^{1-2s}u_1 &= -f \text{ in } \Omega,\\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}u_1 &= -qu + g \text{ on } \Sigma_1,\\ x_{n+1}^{1-2s}\partial_\nu u_1 &= 0 \text{ on } \partial\Omega\setminus\overline{\Sigma_1}. \end{aligned}$$

By the Lax-Milgram theorem, a unique (weak) solution to this problem exists in $H^1(\Omega, x_{n+1}^{1-2s})$ if $K > 0$ is sufficiently large. It satisfies

$$(x_{n+1}^{1-2s}\nabla u_1, \nabla\varphi)_\Omega + K|\xi'|^2(x_{n+1}^{1-2s}u_1,\varphi)_\Omega = (F_0, x_{n+1}^{1-2s}\varphi)_\Omega + (\tilde{F}, x_{n+1}^{1-2s}\nabla\varphi)_\Omega + (-qu+g,\varphi)_{\Sigma_1}$$

for any $\varphi \in H^1(\Omega, x_{n+1}^{1-2s})$. Here the notation $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\Sigma_1}$ refer to the $L^2(\Omega)$ and $L^2(\Sigma_1)$ scalar products respectively. The function $u_2 = u - u_1$ is defined accordingly.

Step 1: Estimate for u_1. We first estimate u_1. To this end, we test the equation for u_1 with $\varphi := |\xi'|^2 e^{2x'\cdot\xi'}u_1$. This yields

$$\begin{aligned} &|\xi'|^2(x_{n+1}^{1-2s}\nabla u_1, \nabla(e^{2x'\cdot\xi'}u_1))_\Omega + K|\xi'|^4(x_{n+1}^{1-2s}u_1, e^{2x'\cdot\xi'}u_1)_\Omega = |\xi'|^2(-qu+g, e^{2x'\cdot\xi'}u_1)_{\Sigma_1}\\ &\quad - |\xi'|^2(F_0, x_{n+1}^{1-2s}e^{2x'\cdot\xi'}u_1)_\Omega - |\xi'|^2(\tilde{F}, x_{n+1}^{1-2s}\nabla(e^{2\xi'\cdot x'}u_1))_\Omega. \end{aligned}$$

Using Young's inequality and choosing $K > 0$ sufficiently large this implies that

$$\begin{aligned} &\frac{K}{2}|\xi'|^4\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}u_1\|^2_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\nabla u_1\|^2_{L^2(\Omega)}\\ &\leq C|\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\tilde{F}\|^2_{L^2(\Omega)} + C\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}F_0\|^2_{L^2(\Omega)} + \epsilon|\xi'|^{2+2s}\|e^{x'\cdot\xi'}u_1\|^2_{L^2(\Sigma_1)}\\ &\quad + C_\epsilon|\xi'|^{2-2s}(\|e^{x'\cdot\xi'}g\|^2_{L^2(\Sigma_1)} + \|q\|^2_{L^\infty(\Sigma_1)}\|e^{x'\cdot\xi'}u\|^2_{L^2(\Sigma_1)}). \end{aligned} \tag{40}$$

Now the boundary-bulk interpolation estimate from Lemma 2.5 allows us to further add a boundary contribution to the left hand side of this:

$$\begin{aligned} &|\xi'|^{2+2s}\|e^{x'\cdot\xi'}u_1\|^2_{L^2(\Sigma_1)} + \frac{K}{2}|\xi'|^4\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}u_1\|^2_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\nabla u_1\|^2_{L^2(\Omega)}\\ &\leq C|\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\tilde{F}\|^2_{L^2(\Omega)} + C\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}F_0\|^2_{L^2(\Omega)} + \epsilon|\xi'|^{2+2s}\|e^{x'\cdot\xi'}u_1\|^2_{L^2(\Sigma_1)}\\ &\quad + C_\epsilon|\xi'|^{2-2s}(\|e^{x'\cdot\xi'}g\|^2_{L^2(\Sigma_1)} + \|q\|^2_{L^\infty(\Sigma_1)}\|e^{x'\cdot\xi'}u\|^2_{L^2(\Sigma_1)}). \end{aligned} \tag{41}$$

In particular, this allows us to absorb the boundary contributions involving u_1 from the right hand side of (41) into the left hand side of this inequality. As a consequence, we obtain the bound

$$\begin{aligned} &|\xi'|^{2+2s}\|e^{x'\cdot\xi'}u_1\|^2_{L^2(\Sigma_1)} + \frac{K}{2}|\xi'|^4\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}u_1\|^2_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\nabla u_1\|^2_{L^2(\Omega)}\\ &\leq C|\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\tilde{F}\|^2_{L^2(\Omega)} + C\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}F_0\|^2_{L^2(\Omega)} +\\ &\quad + C_\epsilon|\xi'|^{2-2s}(\|e^{x'\cdot\xi'}g\|^2_{L^2(\Sigma_1)} + \|q\|^2_{L^\infty(\Sigma_1)}\|e^{x'\cdot\xi'}u\|^2_{L^2(\Sigma_1)}). \end{aligned} \tag{42}$$

Step 2: Estimate for u_2. Next we estimate the contribution from u_2 which (weakly) solves the equation

$$
\begin{aligned}
\nabla \cdot x_{n+1}^{1-2s} \nabla u_2 &= -K|\xi'|^2 x_{n+1}^{1-2s} u_1 \text{ in } \Omega,\\
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} u_2 &= 0 \text{ on } \Sigma_1,\\
x_{n+1}^{1-2s} \partial_\nu u_2 &= 0 \text{ on } \partial\Omega \setminus \Sigma_1.
\end{aligned}
\tag{43}
$$

In order to estimate u_2, we first assume that $u_1 \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. With only slight modifications it is then possible to invoke the regularity results from [KRS19, Appendix A]. Indeed, the regularity estimates from [KRS19, Proposition 8.2] yield $C^{2,\alpha}$ regularity up to the boundary in $\text{int}(\Sigma_1)$. Classical, uniformly elliptic regularity estimates in turn yield $C^{2,\alpha}$ regularity in a neighbourhood of $\partial\Omega \setminus \overline{\Sigma_1}$ up to the boundary. Thus, it remains to discuss the regularity in a neighbourhood of $\partial\Sigma_1$ up to the boundary. This however follows from the C^2 regularity of the boundary which implies that the approximation by the flat problem at that point is still valid. Combining these results yields the global $C^{2,\alpha}(\Omega)$ regularity of u_2.

In order to estimate u_2, we conjugate the operator $L_s := \nabla \cdot x_{n+1}^{1-2s} \nabla$ with the weight $e^{x'\cdot\xi'}$. This yields the conjugated operator

$$
\tilde{L}_{s,\phi} := \nabla \cdot x_{n+1}^{1-2s} \nabla - 2x_{n+1}^{1-2s} \xi' \cdot \nabla'.
$$

Next, we define $u_2 = x_{n+1}^{\frac{2s-1}{2}} e^{-x'\cdot\xi'} w$ and multiply the operator $\tilde{L}_{s,\phi}$ by $x_{n+1}^{\frac{2s-1}{2}}$. As a consequence, the operator acting on w turns into

$$
L_{s,\phi} := x_{n+1}^{\frac{2s-1}{2}} \nabla \cdot x_{n+1}^{1-2s} \nabla x_{n+1}^{\frac{2s-1}{2}} - 2\xi' \cdot \nabla',
$$

and since $\xi' \perp e_{n+1}$ the boundary condition on Σ_1 correspondingly becomes

$$
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} (x_{n+1}^{\frac{2s-1}{2}} w) = 0. \tag{44}
$$

On $\partial\Omega \setminus \overline{\Sigma_1}$ the boundary contributions however is non-trivial and turns into

$$
\lim_{d\to 0} x_{n+1}^{1-2s} \partial_\nu (x_{n+1}^{\frac{2s-1}{2}} w) = \lim_{d\to 0} x_{n+1}^{1-2s} (\nu \cdot \xi')(x_{n+1}^{\frac{2s-1}{2}} w). \tag{45}
$$

Up to boundary contributions the bulk part of the operator can be split into its symmetric and antisymmetric parts:

$$
\begin{aligned}
S_\phi &= x_{n+1}^{\frac{2s-1}{2}} \nabla \cdot x_{n+1}^{1-2s} \nabla x_{n+1}^{\frac{2s-1}{2}},\\
A_\phi &= -2\xi' \cdot \nabla'.
\end{aligned}
$$

Expanding the norm, computing the boundary terms (BC) and using the regularity of u_2, we thus infer

$$
\|L_{s,\phi} w\|_{L^2(\Omega)}^2 = \|S_\phi w\|_{L^2(\Omega)}^2 + \|A_\phi w\|_{L^2(\Omega)}^2 + \text{ (BC)}. \tag{46}
$$

We emphasise that the $C^{2,\alpha}$ regularity of u_2 allows us to carry out the expansion of $L_{s,\phi} w$ as classically differentiable functions away from the boundary and that the resulting boundary contributions are given as classical boundary integrals. Using the observations from (44) and (45) these are of the form

$$
(BC) = (BC)_1 + (BC)_2,
$$

where the contributions from $(BC)_1$ come from shifting $(S_\phi w, A_\phi w)_{L^2(\Omega)} = (w, S_\phi A_\phi w)_{L^2(\Omega)} + (BC)_1$ and the ones from $(BC)_2$ from $(S_\phi w, A_\phi w)_{L^2(\Omega)} = -(A_\phi S_\phi w, w)_{L^2(\Omega)} + (BC)_2$.

We next estimate these boundary contributions individually.

Step 2a: $(BC)_1$. For the boundary contribution $(BC)_1$ we obtain

$$
\begin{aligned}
(BC)_1 :=& -2(x_{n+1}^{1-2s}\partial_\nu(x_{n+1}^{\frac{2s-1}{2}}w), \xi'\cdot\nabla'(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&+ 2(x_{n+1}^{\frac{2s-1}{2}}w, x_{n+1}^{1-2s}\partial_\nu((\xi'\cdot\nabla')(x_{n+1}^{\frac{2s-1}{2}}w)))_{L^2(\partial\Omega)}\\
=& -2(x_{n+1}^{1-2s}\partial_\nu(x_{n+1}^{\frac{2s-1}{2}}w), \xi'\cdot\nabla'(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&- 2(x_{n+1}^{\frac{2s-1}{2}}w, x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu]\cdot\nabla(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&+ 2(x_{n+1}^{\frac{2s-1}{2}}w, x_{n+1}^{1-2s}(\xi'\cdot\nabla')\partial_\nu(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
=& -2(x_{n+1}^{1-2s}(\nu\cdot\xi')(x_{n+1}^{\frac{2s-1}{2}}w), \xi'\cdot\nabla'(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&- 2(x_{n+1}^{\frac{2s-1}{2}}w, x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu]\cdot\nabla(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&+ 2((\xi'\cdot\nabla')[x_{n+1}^{1-2s}(\nu\cdot\xi')(x_{n+1}^{\frac{2s-1}{2}}w)], x_{n+1}^{\frac{2s-1}{2}}w)_{L^2(\partial\Omega)}.
\end{aligned}
\tag{47}
$$

Here we have used (44) and (45) in the third equality. We now discuss these contributions separately. We split the derivative $\xi'\cdot\nabla'$ into a tangential and a normal contribution. If $\tau_j(x)$, $j=1,...,n$ are unit vectors depending smoothly on x and forming with the addition of $\nu(x)$ an orthonormal basis of $\mathbb{R}^{n+1}$, then we can write

$$\nabla = \nu(x)\partial_\nu + \sum_{j=1}^n \tau_j(x)(\tau_j(x)\cdot\nabla)\ ,$$

and therefore

$$\xi'\cdot\nabla' = |\xi'|[(e_{\xi'}\cdot\nu(x))\partial_\nu + \sum_{j=1}^n (e_{\xi'}\cdot\tau_j(x))(\tau_j(x)\cdot\nabla)] = |\xi'|[(e_{\xi'}\cdot\nu(x))\partial_\nu + \beta(x)\cdot\nabla_\tau]\ , \tag{48}$$

where $e_{\xi'} := \frac{1}{|\xi'|}\xi'$, β is a smooth vector function whose norm is bounded uniformly, independently of $|\xi'|$ and whose j-th component is $e_{\xi'}\cdot\tau_j(x)$, and the operator ∇_τ represents the tangential derivatives $\tau_j(x)\cdot\nabla$.

For the first contribution in (47), we use the splitting (48) in combination with (44), (45) for the normal derivatives and integrate by parts in the tangential directions:

$$
\begin{aligned}
&2(x_{n+1}^{1-2s}(\nu\cdot\xi')(x_{n+1}^{\frac{2s-1}{2}}w), \xi'\cdot\nabla'(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&= 2|\xi'|^2(x_{n+1}^{1-2s}(\nu\cdot e_{\xi'})(x_{n+1}^{\frac{2s-1}{2}}w), (e_{\xi'}\cdot\nu)\partial_\nu(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&\quad + 2|\xi'|^2(x_{n+1}^{1-2s}(\nu\cdot e_{\xi'})(x_{n+1}^{\frac{2s-1}{2}}w), \beta(x)\cdot\nabla_\tau(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&= 2|\xi'|^3([x_{n+1}^{1-2s}(\nu\cdot e_{\xi'})^3](x_{n+1}^{\frac{2s-1}{2}}w), (x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}\\
&\quad + |\xi'|^2(x_{n+1}^{1-2s}(\nu\cdot e_{\xi'})\beta(x), \nabla_\tau(x_{n+1}^{2s-1}|w|^2))_{L^2(\partial\Omega)}\\
&= -|\xi'|^2((x_{n+1}^{\frac{2s-1}{2}}w)[\mathrm{div}_{\partial\Omega}(\beta(x)x_{n+1}^{1-2s}(\nu\cdot e_{\xi'}))], x_{n+1}^{\frac{2s-1}{2}}w)_{L^2(\partial\Omega)}\\
&\quad + 2|\xi'|^3([x_{n+1}^{1-2s}(\nu\cdot e_{\xi'})^3](x_{n+1}^{\frac{2s-1}{2}}w), (x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}.
\end{aligned}
\tag{49}
$$

We remark that both boundary terms are controlled by

$$|\xi'|^3\|x_{n+1}^{\frac{2s-1}{2}}w\|^2_{L^2(\partial\Omega\setminus\Sigma_1)}. \tag{50}$$

Indeed, to observe this, it suffices to prove that for $x \in \partial\Omega$ with $x_{n+1} \rightarrow 0$ we have that for the weights

$$[x_{n+1}^{1-2s}(\nu(x)\cdot e_{\xi'})^3] \rightarrow 0, \ [\mathrm{div}_{\partial\Omega}(\beta(x)x_{n+1}^{1-2s}(\nu(x)\cdot e_{\xi'}))] \rightarrow 0 \text{ as } x_{n+1} \rightarrow 0. \tag{51}$$

Parametrizing the boundary $\partial\Omega$ in a neighbourhood of $\partial\Sigma_1$, we obtain that if $\partial\Omega$ is sufficiently smooth and thus sufficiently flat at $\partial\Omega$ the claim of (51) can always be ensured. Indeed, in this case the boundary can be locally parametrized by $\psi(x) = (x', |x'-\gamma(x')|^m)$, where $\gamma(x')$ is a smooth function describing $\partial\Sigma_1$. Thus, expressing x_{n+1} and $\nu(x')\cdot e_{\xi'}$ in terms of x', for instance yields

$$|x_{n+1}^{1-2s}(\nu(x')\cdot e_{\xi'})| \leq C_{\gamma,|\nabla'\gamma|}|x'-\gamma(x')|^{m(1-2s)}|x'-\gamma(x')|^{m-1} \rightarrow 0,$$

as $x' \rightarrow \gamma(x')$ and thus $x_{n+1} \rightarrow 0$ by choosing $m = m(s) > 0$ sufficiently large (which is ensured by the boundary smoothness, see Remark 6.2). Since in local coordinates the expression for the divergence only involves derivatives in the tangential directions, the same argument applies to the second expression in (49). Together with the boundedness of $\overline{\Omega}$ this proves the bound (50).

The third term in (47) can be treated analogously as the first term in (47) . To this end, we first note that

$$\begin{aligned} &2((\xi'\cdot\nabla')[x_{n+1}^{1-2s}(\nu\cdot\xi')(x_{n+1}^{\frac{2s-1}{2}}w)], x_{n+1}^{\frac{2s-1}{2}}w)_{L^2(\partial\Omega)} \\ &= 2(x_{n+1}^{1-2s}(\nu\cdot\xi')(\xi'\cdot\nabla')(x_{n+1}^{\frac{2s-1}{2}}w), x_{n+1}^{\frac{2s-1}{2}}w)_{L^2(\partial\Omega)} \\ &\quad + 2((x_{n+1}^{\frac{2s-1}{2}}w)[(\xi'\cdot\nabla')(x_{n+1}^{1-2s}(\nu\cdot\xi'))], x_{n+1}^{\frac{2s-1}{2}}w)_{L^2(\partial\Omega)}. \end{aligned} \tag{52}$$

Hence, the first contribution is of the same form as the term from (49). It suffices to deal with the second one and to prove that

$$[(\xi'\cdot\nabla')(x_{n+1}^{1-2s}(\nu\cdot\xi'))] \rightarrow 0$$

for $x \in \partial\Omega$ with $x_{n+1} \rightarrow 0$. This however follows in the same way as in (51) and implies that the contributions in (52) are also controlled by terms of the form (50).

Finally, it remains to deal with the second contribution in (47). For this we observe that $(\xi'\cdot\nabla')\nu$ does not have any normal component. Thus, an integration by parts yields

$$\begin{aligned} &-2(x_{n+1}^{\frac{2s-1}{2}}w, x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu]\cdot\nabla(x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)} \\ &= -(x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu], \nabla(x_{n+1}^{2s-1}|w|^2))_{L^2(\partial\Omega)} \\ &= ([\mathrm{div}_{\partial\Omega}(x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu])](x_{n+1}^{\frac{2s-1}{2}}w), (x_{n+1}^{\frac{2s-1}{2}}w))_{L^2(\partial\Omega)}. \end{aligned} \tag{53}$$

It remains to prove that

$$[\mathrm{div}_{\partial\Omega}(x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu])] \rightarrow 0$$

for $x \in \partial\Omega$ with $x_{n+1} \rightarrow 0$, as this then ensures that also the boundary contribution in (53) is controlled by (50). The desired estimate however follows from the explicit parametrization $\psi(x) = (x', |x'-\gamma(x')|^m)$, which yields that

$$|[\mathrm{div}_{\partial\Omega}(x_{n+1}^{1-2s}[(\xi'\cdot\nabla')\nu])]| \leq C|x'-\gamma(x')|^{m(1-2s)+m-2}|\xi'|.$$

Thus, for $m = m(s) > 0$ sufficiently large, the claim follows.

Inspecting the quantities in (49)-(53) and recalling that $\xi' \perp e_{n+1}$, we note that all right hand side contributions in (49)-(53) are really only integrals over $\partial\Omega \setminus \overline{\Sigma_1}$. Thus, due to the assumed boundary regularity of Ω and the boundedness of Ω, all of the contributions on the right hand side of (47) are bounded in terms of (50).

Last but not least, we seek to estimate the quantity (50) by bulk contributions of u_1. Rewriting (50) in terms of u_2, recalling that $u_2 = u - u_1$ and that $u|_{\partial\Omega\setminus\overline{\Sigma_1}} = 0$, we infer that all boundary contributions in $(BC)_1$ are controlled by

$$|\xi'|^3 \|e^{\xi'\cdot x'} u_2\|^2_{L^2(\partial\Omega\setminus\overline{\Sigma_1})} \leq |\xi'|^3 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\partial\Omega\setminus\overline{\Sigma_1})}. \tag{54}$$

Using the trace estimate from Lemma 2.4 and the fact that $s \geq \frac{1}{2}$, we deduce that

$$\begin{aligned}
|\xi'|^3 \|e^{\xi'\cdot x'} u_2\|^2_{L^2(\partial\Omega\setminus\overline{\Sigma_1})} &\leq |\xi'|^3 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\partial\Omega\setminus\overline{\Sigma_1})} \\
&\leq C(|\xi'|^4 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\Omega)} + |\xi'|^2 \|\nabla(e^{\xi'\cdot x'} u_1)\|^2_{L^2(\Omega)}) \\
&\leq C(|\xi'|^4 \|e^{\xi'\cdot x'} x_{n+1}^{\frac{1-2s}{2}} u_1\|^2_{L^2(\Omega)} + |\xi'|^2 \|e^{\xi'\cdot x'} x_{n+1}^{\frac{1-2s}{2}} \nabla u_1\|^2_{L^2(\Omega)}).
\end{aligned} \tag{55}$$

Step 2b: $(BC)_2$. Next we deal with the contributions in $(BC)_2$. These are of the form

$$\begin{aligned}
(x_{n+1}^{\frac{2s-1}{2}} \nabla \cdot x_{n+1}^{1-2s} \nabla (x_{n+1}^{\frac{2s-1}{2}} w), (\xi'\cdot\nu) w)_{L^2(\partial\Omega)} &= -K|\xi'|^2 (x_{n+1}^{\frac{1-2s}{2}} e^{\xi'\cdot x'} u_1, (\xi'\cdot\nu) w)_{L^2(\partial\Omega)} \\
&\quad + 2(\xi'\cdot\nabla' w, (\xi'\cdot\nu) w)_{L^2(\partial\Omega)}.
\end{aligned} \tag{56}$$

Here we have used the bulk equation for w which, due to the regularity of w, is continuous up to the boundary.

Splitting $\xi'\cdot\nabla'$ into tangential and normal components as in (48), the second term can be dealt with similarly as in the argument for (53): Indeed,

$$\begin{aligned}
2(\xi'\cdot\nabla' w, (\xi'\cdot\nu) w)_{L^2(\partial\Omega)} &= 2|\xi'|^2 (e_{\xi'}\cdot\nabla'(x_{n+1}^{\frac{2s-1}{2}} w), (e_{\xi'}\cdot\nu) x_{n+1}^{1-2s} x_{n+1}^{\frac{2s-1}{2}} w)_{L^2(\partial\Omega)} \\
&= 2|\xi'|^2 (\partial_\nu (x_{n+1}^{\frac{2s-1}{2}} w), (e_{\xi'}\cdot\nu)^2 x_{n+1}^{1-2s} (x_{n+1}^{\frac{2s-1}{2}} w))_{L^2(\partial\Omega)} \\
&\quad - |\xi'|^2 (x_{n+1}^{\frac{2s-1}{2}} w, (x_{n+1}^{\frac{2s-1}{2}} w)[\operatorname{div}_{\partial\Omega}(\beta(e_{\xi'}\cdot\nu) x_{n+1}^{1-2s})])_{L^2(\partial\Omega)} \\
&= 2|\xi'|^3 ((x_{n+1}^{\frac{2s-1}{2}} w), (e_{\xi'}\cdot\nu)^3 x_{n+1}^{1-2s} (x_{n+1}^{\frac{2s-1}{2}} w))_{L^2(\partial\Omega)} \\
&\quad - |\xi'|^2 (x_{n+1}^{\frac{2s-1}{2}} w, (x_{n+1}^{\frac{2s-1}{2}} w)[\operatorname{div}_{\partial\Omega}(\beta(e_{\xi'}\cdot\nu) x_{n+1}^{1-2s})])_{L^2(\partial\Omega)}.
\end{aligned}$$

Using the regularity of $\partial\Omega$, both terms can be estimates by a contribution of the form (50).

For the first term on the right hand side of (56), we note that

$$-K|\xi'|^2 (x_{n+1}^{\frac{1-2s}{2}} e^{\xi'\cdot x'} u_1, (\xi'\cdot\nu) w)_{L^2(\partial\Omega)} = -K|\xi'|^2 (e^{\xi'\cdot x'} u_1, x_{n+1}^{1-2s} (\xi'\cdot\nu) x_{n+1}^{\frac{2s-1}{2}} w)_{L^2(\partial\Omega)}.$$

Since $x_{n+1}^{1-2s}(\xi'\cdot\nu) \rightarrow 0$ for $x \in \partial\Omega$ with $x_{n+1} \rightarrow 0$ and since $x_{n+1}^{\frac{2s-1}{2}} w = e^{\xi'\cdot x'} u_2$, it is only active at the boundary $\partial\Omega\setminus\overline{\Sigma_1}$. Rewriting $w = e^{\xi'\cdot x'} x_{n+1}^{\frac{1-2s}{2}} u_2 = e^{\xi'\cdot x'} x_{n+1}^{\frac{1-2s}{2}} (u - u_1)$ and using the boundary conditions for u_1, the first term in (56) hence turns into

$$K|\xi'|^2 (e^{\xi'\cdot x'} u_1, x_{n+1}^{1-2s} (\xi'\cdot\nu) e^{\xi'\cdot x'} u_1)_{L^2(\partial\Omega\setminus\overline{\Sigma_1})}.$$

Due to the boundary regularity, we observe that this contribution is bounded by

$$CK|\xi'|^3 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\partial\Omega\setminus\overline{\Sigma_1})}, \tag{57}$$

where $C = C(\Omega) > 1$. Using the boundary trace estimate from Lemma 2.4 (with $\mu = |\xi'|^{\frac{1}{2}}$) we may control this by bulk contributions:

$$\begin{aligned}
K|\xi'|^3 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\partial\Omega\setminus\overline{\Sigma_1})} &\leq CK(|\xi'|^2 \|\nabla(e^{\xi'\cdot x'} u_1)\|^2_{L^2(\Omega)} + |\xi'|^4 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\Omega)}) \\
&\leq CK(|\xi'|^2 \|e^{\xi'\cdot x'} \nabla u_1\|^2_{L^2(\Omega)} + |\xi'|^4 \|e^{\xi'\cdot x'} u_1\|^2_{L^2(\Omega)}) \\
&\leq CK(|\xi'|^2 \|e^{\xi'\cdot x'} x_{n+1}^{\frac{1-2s}{2}} \nabla u_1\|^2_{L^2(\Omega)} + |\xi'|^4 \|e^{\xi'\cdot x'} x_{n+1}^{\frac{1-2s}{2}} u_1\|^2_{L^2(\Omega)}).
\end{aligned} \tag{58}$$

Step 2c: Antisymmetric and symmetric terms. Next, we invoke the compact support of u to deduce a lower bound for A_ϕ: Rewriting $w = e^{\xi'\cdot x'}u_2 = e^{\xi'\cdot x'}(u-u_1)$, then the compact support of u in the tangential slices yields by virtue of Poincaré's inequality that

$$\begin{aligned}
\|A_\phi w\|_{L^2(\Omega)} &\geq \|\xi'\cdot\nabla'(x_{n+1}^{\frac{1-2s}{2}} e^{x'\cdot\xi'} u)\|_{L^2(\Omega)} - |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} \nabla(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}\\
&\geq C^{-1}|\xi'|\|x_{n+1}^{\frac{1-2s}{2}} e^{x'\cdot\xi'} u\|_{L^2(\Omega)} - |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} \nabla(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}\\
&\geq C^{-1}|\xi'|\|x_{n+1}^{\frac{1-2s}{2}} e^{x'\cdot\xi'} u_2\|_{L^2(\Omega)} - |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} \nabla(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}\\
&\quad - |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} (e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}\\
&= C^{-1}|\xi'|\|w\|_{L^2(\Omega)} - |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} \nabla(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)} - |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} (e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}\\
&\geq C^{-1}|\xi'|\|w\|_{L^2(\Omega)} - |\xi'|\|e^{x'\cdot\xi'} x_{n+1}^{\frac{1-2s}{2}} \nabla u_1\|_{L^2(\Omega)} - |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}} (e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}.
\end{aligned} \tag{59}$$

Testing the symmetric part of the operator with w itself, we further obtain that

$$\begin{aligned}
\|x_{n+1}^{\frac{1-2s}{2}} \nabla(x_{n+1}^{\frac{2s-1}{2}} w)\|_{L^2(\Omega)} &\leq \|S_\phi w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\
&\quad + (\lim_{x\rightarrow\partial\Omega} x_{n+1}^{1-2s}\partial_\nu(x_{n+1}^{\frac{2s-1}{2}} w), x_{n+1}^{\frac{2s-1}{2}} w)_{L^2(\partial\Omega)}\\
&\leq \|S_\phi w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\
&\quad + (x_{n+1}^{1-2s}(\xi'\cdot\nu)(x_{n+1}^{\frac{2s-1}{2}} w), x_{n+1}^{\frac{2s-1}{2}} w)_{L^2(\partial\Omega)}\\
&= \|S_\phi w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\
&\quad + (x_{n+1}^{1-2s}(\xi'\cdot\nu)(x_{n+1}^{\frac{2s-1}{2}} w), x_{n+1}^{\frac{2s-1}{2}} w)_{L^2(\partial\Omega\setminus\overline{\Sigma_1})}.
\end{aligned} \tag{60}$$

We may now estimate the boundary contribution arising in these estimates as above (see (44), (45)), as it is controlled by (50).

Step 2d: Conclusion of the estimate for u_2.
Thus, for $|\xi'|\geq 1$, combining the estimates (46)-(60), in total, the Carleman estimate turns into

$$\begin{aligned}
&|\xi'|\|w\|_{L^2(\Omega)} + \|x_{n+1}^{\frac{1-2s}{2}} \nabla(x_{n+1}^{\frac{2s-1}{2}} w)\|_{L^2(\Omega)}\\
&\leq C(\|L_{s,\phi}w\|_{L^2(\Omega)} + |\xi'|\|x_{n+1}^{\frac{1-2s}{2}} \nabla(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}} (e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}).
\end{aligned} \tag{61}$$

Next we seek to complement (61) with a boundary contribution on the left hand side of the Carleman inequality. To this end, we use the boundary-bulk-interpolation estimate from Lemma 2.5. This implies that

$$|\xi'|^s\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\Sigma_1)} \leq C|\xi'|\|x_{n+1}^{\frac{1-2s}{2}} (x_{n+1}^{\frac{2s-1}{2}} w)\|_{L^2(\Omega)} + \|x_{n+1}^{\frac{1-2s}{2}} \nabla(x_{n+1}^{\frac{2s-1}{2}} w)\|_{L^2(\Omega)}.$$

As a consequence, the estimate (61) becomes

$$\begin{aligned}
&|\xi'|^s\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\Sigma_1)} + |\xi'|\|w\|_{L^2(\Omega)} + \|x_{n+1}^{\frac{1-2s}{2}} \nabla(x_{n+1}^{\frac{2s-1}{2}} w)\|_{L^2(\Omega)}\\
&\leq C(\|L_{s,\phi}w\|_{L^2(\Omega)} + |\xi'|\|e^{x'\cdot\xi'} x_{n+1}^{\frac{1-2s}{2}} \nabla u_1\|_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}} (e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)}).
\end{aligned} \tag{62}$$

Returning to u_2 then yields the bound

$$
\begin{aligned}
&|\xi'|^s\|e^{x'\cdot\xi'}u_2\|_{L^2(\Sigma_1)} + |\xi'|\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}u_2\|_{L^2(\Omega)} + \|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}\nabla u_2\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\leq C(\|L_{s,\phi}(e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}u_2)\|_{L^2(\mathbb{R}^{n+1}_+)} + |\xi'|\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}\nabla u_1\|_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)})\\
&= C(K|\xi'|^2\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}u_1\|_{L^2(\Omega)} + |\xi'|\|x_{n+1}^{\frac{1-2s}{2}}\nabla(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)} + |\xi'|^2\|x_{n+1}^{\frac{1-2s}{2}}(e^{x'\cdot\xi'}u_1)\|_{L^2(\Omega)})\\
&\leq C(K|\xi'|^2\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}u_1\|_{L^2(\Omega)} + |\xi'|\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}\nabla u_1\|_{L^2(\Omega)}).
\end{aligned} \tag{63}
$$

Now, if $u_1 \in H^1(\Omega, x_{n+1}^{1-2s})$ is not $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, we simply replace u_1 by $u_{1,\epsilon} := (u_1\chi_\Omega) * \varphi_\epsilon \in C^{0,\alpha}(\Omega)$ (where χ_Ω is the characteristic function of Ω and φ_ϵ is a standard mollifier) and consider the equation (43) with u_1 replaced by $u_{1,\epsilon}$. We denote the corresponding solution by $u_{2,\epsilon}$. This allows us to derive all estimates including (63) with u_1, u_2 replaced by $u_{1,\epsilon}$ and $u_{2,\epsilon}$. Combining the estimate (63), weak lower semi-continuity and the $H^1(\Omega, x_{n+1}^{1-2s})$ regularity of u_1 then allows us to pass to the limit $\epsilon \to 0$. This then also yields (63) with the functions u_1, u_2 (instead of $u_{1,\epsilon}, u_{2,\epsilon}$).

Step 3: Conclusion. Combining the estimates from (42) and (63), by the triangle inequality, we obtain that

$$
\begin{aligned}
&|\xi'|^s\|e^{x'\cdot\xi'}u\|_{L^2(\Sigma_1)} + |\xi'|\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\Omega)} + \|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)}\\
&\leq CK|\xi'|^2\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}u_1\|_{L^2(\Omega)} + |\xi'|\|e^{x'\cdot\xi'}x_{n+1}^{\frac{1-2s}{2}}\nabla u_1\|_{L^2(\Omega)}\\
&\quad + C\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}\tilde{F}\|_{L^2(\Omega)} + C|\xi'|^{-1}\|x_{n+1}^{\frac{1-2s}{2}}e^{x'\cdot\xi'}F_0\|_{L^2(\Omega)}+\\
&\quad + C_\epsilon|\xi'|^{-s}\left(\|e^{x'\cdot\xi'}g\|_{L^2(\Sigma_1)} + \|q\|_{L^\infty(\Sigma_1)}\|e^{x'\cdot\xi'}u\|_{L^2(\Sigma_1)}\right)\\
&\leq CK|\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{F}\|_{L^2(\Omega)} + CK\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}F_0\|_{L^2(\Omega)}\\
&\quad + C_\epsilon K|\xi'|^{1-s}(\|e^{x'\cdot\xi'}g\|_{L^2(\Sigma_1)} + \|q\|_{L^\infty(\Sigma_1)}\|e^{x'\cdot\xi'}u\|_{L^2(\Sigma_1)}).
\end{aligned} \tag{64}
$$

Now, if $s > \frac{1}{2}$ and $|\xi'| \gg 1$ is sufficiently large (depending on $\|q\|_{L^\infty(\Sigma_1)}$), it is possible to absorb the boundary term involving q on the right hand side into the left hand side of (64). If $s = \frac{1}{2}$, the absorption is still possible if we assume that $\|q\|_{L^\infty(\Sigma_1)}$ is sufficiently small. Under these assumptions, (64) thus turns into the desired estimate (39). □

Remark 6.3. *We expect that for $s = \frac{1}{2}$ it might be possible to improve the Carleman estimate by relying on the Lopatinskii condition. For $s \in (\frac{1}{2}, 1)$ this is less clear. We postpone this to a future project.*

As a corollary to Proposition 6.1 we note that the estimate (39) remains true if in (38) we consider the bulk equation

$$\nabla \cdot x_{n+1}^{1-2s}\nabla u + Vx_{n+1}^{1-2s}u = f \text{ in } \Omega,$$

with $f \in (H^1(\Omega, x_{n+1}^{1-2s}))^*$.

Corollary 6.4. *Let $s \in [\frac{1}{2}, 1)$, $\xi' \in \mathbb{C}^n$ such that $\xi'\cdot\xi' = 0$. Assume that the same conditions as in Proposition 6.1 hold for Ω, q, f and g. Let $V \in L^\infty(\Omega)$ and assume that $u \in H^1_{\partial\Omega\setminus\overline{\Sigma_1},0}(\overline{\Omega}, x_{n+1}^{1-2s})$*

with $u=0$ *and* $\lim_{x\to\partial\Omega} x_{n+1}^{1-2s}\partial_\nu u = 0$ *on* $\partial\Omega\setminus\overline{\Sigma_1}$ *is a weak solution to*

$$\begin{aligned} &\nabla\cdot x_{n+1}^{1-2s}\nabla u + V x_{n+1}^{1-2s} u = f \ \textit{in}\ \Omega,\\ &\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}u + qu = g \ \textit{on}\ \Sigma_1. \end{aligned} \tag{65}$$

Then, we have

$$\begin{aligned} &|\xi'|^s\|e^{\xi'\cdot x'}u\|_{L^2(\Sigma_1)} + |\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)}\\ &\leq C(|\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{F}\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}F_0\|_{L^2(\Omega)} + |\xi'|^{1-s}\|e^{\xi'\cdot x'}g\|_{L^2(\Sigma_1)}). \end{aligned} \tag{66}$$

Here the constant $C>0$ *depends on* $\|q\|_{L^\infty(\Sigma_1)}$ *and* $\|V\|_{L^\infty(\Omega)}$, *while* $F=(F_0,\tilde{F})\in L^2(\mathbb{R}^{n+1}_+,\mathbb{R}^{n+2})$ *is the Riesz representation of* f, *i.e., it is such that*

$$f(v) = (v, x_{n+1}^{1-2s}F_0)_{L^2(\Omega)} + (\nabla v, x_{n+1}^{1-2s}\tilde{F})_{L^2(\Omega)} \ \textit{for all}\ v\in H^1(\Omega, x_{n+1}^{1-2s}).$$

Proof. The proof follows directly by a reduction to the setting of Proposition 6.1. Indeed, we interpret (65) as an equation of the form (38) with $\tilde{f} = f - x_{n+1}^{1-2s}Vu$. If the Riesz representative of f had been given by $F=(F_0,\tilde{F})$, the one for $\tilde{f}$ is now given by $\bar{F} = (F_0 - Vu, \tilde{F})$. As a consequence, (39) turns into

$$\begin{aligned} &|\xi'|^s\|e^{\xi'\cdot x'}u\|_{L^2(\Sigma_1)} + |\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)}\\ &\leq C(|\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{F}\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}(F_0 - Vu)\|_{L^2(\Omega)} + |\xi'|^{1-s}\|e^{\xi'\cdot x'}g\|_{L^2(\Sigma_1)}). \end{aligned}$$

Applying the triangle inequality, we obtain

$$\begin{aligned} &|\xi'|^s\|e^{\xi'\cdot x'}u\|_{L^2(\Sigma_1)} + |\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)}\\ &\leq C(|\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{F}\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}F_0\|_{L^2(\Omega)} + \|V\|_{L^\infty(\Omega)}\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\Omega)}\\ &\quad + |\xi'|^{1-s}\|e^{\xi'\cdot x'}g\|_{L^2(\Sigma_1)}). \end{aligned}$$

Now choosing $|\xi'|>1$ so large that $C\|V\|_{L^\infty(\Omega)} \leq \frac{1}{2}|\xi'|$, it is possible to absorb the contribution involving V from the right hand side into the left hand side of the Carleman estimate. This implies the desired bound. ☐

7. Construction of CGOs for the Generalized Caffarelli-Silvestre Extension

We shall now use estimate (39) in order to prove the result of Proposition 1.2 and to thus deduce the existence of CGOs (associated with the weak form of the equation (9)) by means of a duality argument.

Proof of Proposition 1.2. Fix $k\in\mathbb{R}^{n+1}$ and consider two vectors $\zeta_1,\zeta_2\in(k^\perp\cap e_{n+1}^\perp)$ such that $|\zeta_1|=|\zeta_2|$ and $\zeta_1\cdot\zeta_2=0$. This is possible by the assumption $n\geq 3$, since then $\dim(k^\perp\cap e_{n+1}^\perp)\geq (n+1)-2 = n-1\geq 2$. If now we let $\xi' := \zeta_1 + i\zeta_2$, we can observe that the condition $\xi'\cdot\xi'=0$ is satisfied. One also has $\xi'\cdot k' = \xi'\cdot k = 0$, the two equalities being respectively consequences of $\xi'\in e_{n+1}^\perp$ and $\xi'\in k^\perp$.

Substituting the required solution $u(x) = e^{\xi'\cdot x'}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r(x))$ into problem (37), we are left with an equivalent problem for the function $r(x)$:

$$\begin{aligned} \tilde{L}^s_{-\xi',V}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r) = 0 \text{ in } \Omega,\\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r) + q(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + r) = 0 \text{ on } \Sigma_1. \end{aligned} \tag{67}$$

Here $\tilde{L}^s_{-\xi',V} = \nabla \cdot x_{n+1}^{1-2s}\nabla + x_{n+1}^{1-2s}V + 2x_{n+1}^{1-2s}\xi' \cdot \nabla'$.

We shall first study the following norm:

(68)
$$\begin{aligned}
&\|\tilde{L}^s_{-\xi',V} e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}}\|_{L^2(\Omega, x_{n+1}^{2s-1})} = \|(\tilde{L}^s_{-\xi'} + x_{n+1}^{1-2s}V)(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}})\|_{L^2(\Omega, x_{n+1}^{2s-1})}\\
&\quad \leq \|x_{n+1}^{\frac{1-2s}{2}} V e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}}\|_{L^2(\Omega)} + \|\tilde{L}^s_{-\xi'}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}})\|_{L^2(\Omega, x_{n+1}^{2s-1})}\\
&\quad \leq \|V\|_{L^\infty(\Omega)}\|x_{n+1}^{1/2-s}\|_{L^2(\Omega)} + \|(\nabla \cdot x_{n+1}^{1-2s}\nabla + 2x_{n+1}^{1-2s}\xi' \cdot \nabla')e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}}\|_{L^2(\Omega, x_{n+1}^{2s-1})}\\
&\quad = \|V\|_{L^\infty(\Omega)}\|x_{n+1}^{1/2-s}\|_{L^2(\Omega)} + \|(x_{n+1}^{1/2-s}|k'|^2 + (2s)^2 x_{n+1}^{3s-3/2}k_{n+1}^2)e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}}\|_{L^2(\Omega)}\\
&\quad \leq (\|V\|_{L^\infty(\Omega)} + |k'|^2)\|x_{n+1}^{1/2-s}\|_{L^2(\Omega)} + 4s^2k_{n+1}^2\|x_{n+1}^{3s-3/2}\|_{L^2(\Omega)}\\
&\quad \leq C_{\Omega,V,k,s} < \infty.
\end{aligned}$$

In the last step we have used our assumption that $s \geq 1/2$ and that $\xi' \cdot k = 0$. If we define

$$f(x) := -\tilde{L}^s_{-\xi',V}(e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}}),$$

then by (68) we have proved that $\|f\|_{L^2(\Omega, x_{n+1}^{2s-1})} = O(1)$ with respect to $|\xi'| \to \infty$. Next, we compute that for almost every $x' \in \Sigma_1$

$$\begin{aligned}
&\left|\lim_{x_{n+1}\to 0}(x_{n+1}^{1-2s}\partial_{n+1}e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}} + q(x')e^{ik'\cdot x' + ik_{n+1}x_{n+1}^{2s}})\right|\\
&= |e^{ik'\cdot x'}||q(x') + 2si\, k_{n+1}| \leq C_{q,k} < \infty\ .
\end{aligned}$$

Thus, we define

$$g(x') := -e^{ik'\cdot x'}(2si\, k_{n+1} + q(x')),$$

and obtain that $\|g\|_{L^2(\Sigma_1)} \leq C_{q,k}|\Sigma_1|^{1/2} = O(1)$ with respect to $|\xi'| \to \infty$.

In light of the above computations, we can rewrite (67) as an inhomogeneous problem for r:

(69)
$$\begin{aligned}
\tilde{L}^s_{-\xi',V} r &= f \text{ in } \Omega,\\
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}r + qr &= g \text{ on } \Sigma_1.
\end{aligned}$$

We will construct a solution to the problem (69) with the claimed decay properties by using a duality argument and the Carleman estimate (66).

To this end, we first recall the function space $\mathcal{C}$ from (12) in Section 2.1.2 which is a subvector space of $L^2(\Omega, x_{n+1}^{2s-1})$ and has the property that

$$\lim_{x_{n+1}\to 0}(x_{n+1}^{1-2s}\partial_{n+1}w + qw) \in L^2(\Sigma_1) \text{ and } \tilde{L}^s_{\xi',V}w \in L^2(\Omega, x_{n+1}^{2s-1}) \subset (H^1(\Omega, x_{n+1}^{1-2s}))^*$$

and $\operatorname{supp}(\tilde{L}^s_{\xi',V}w) \subset \Omega \cup (\Omega \cap \{x_{n+1} = 0\})$.

We define the operator $\mathcal{B}_s : \mathcal{C} \to L^2(\Sigma_1),\ w \mapsto \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}w + qw$.

We now seek to study a suitable functional which builds on the injectivity of the following mapping: For $u \in \mathcal{C}$ consider

(70)
$$(\tilde{L}^s_{\xi',V}u, \mathcal{B}_s u) \mapsto u.$$

In order to derive the injectivity of the map in (70), we invoke the Carleman estimates from Proposition 6.1 and Corollary 6.4. To this end, we rephrase the Carleman estimate from Proposition 6.1 and Corollary 6.4 in terms of an estimate for the operators $\tilde{L}^s_{\xi',V}$ and $\mathcal{B}_s$. For $u \in \mathcal{C}$ we consider the Carleman estimate of Corollary 6.4 for the function $\tilde{u} := e^{-x'\cdot\xi'}u$. This function clearly satisfies the boundary conditions stated in Corollary 6.4 on $\partial\Omega\backslash\overline{\Sigma_1}$. Now, if u is a solution to the equation

$$\begin{aligned}\tilde{L}^s_{\xi',V}u &= f \text{ in } \Omega,\\ \mathcal{B}_s(u) &= g \text{ on } \Sigma_1,\end{aligned}$$

for some $f \in (H^1(\Omega, x_{n+1}^{1-2s}))^*$ and $g \in L^2(\Sigma_1)$, then the function $\tilde{u}$ satisfies an equation of the form (65) with a bulk inhomogeneity $\tilde{f} = e^{-\xi'\cdot x'}f$ and a boundary inhomogeneity $\tilde{g} = e^{-\xi'\cdot x'}g$. If $(F_0, \bar{F})$ was the Riesz representative of f in $(H^1(\Omega, x_{n+1}^{1-2s}))^*$, then the Riesz representative of $\tilde{f}$ is given by $(\tilde{F}_0, \tilde{\bar{F}}) := (e^{-x'\cdot\xi'}F_0 - e^{-x'\cdot\xi'}\bar{F}_{n+1}, e^{-x'\cdot\xi'}\bar{F})$. The Carleman estimate from Corollary 6.4 for $\tilde{u}$ is thus applicable and yields

$$\begin{aligned}&|\xi'|^s\|e^{\xi'\cdot x'}\tilde{u}\|_{L^2(\Sigma_1)} + |\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(\Omega)}\\ &\leq C(|\xi'|\|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{\bar{F}}\|_{L^2(\Omega)} + \|e^{\xi'\cdot x'}x_{n+1}^{\frac{1-2s}{2}}\tilde{F}_0\|_{L^2(\Omega)} + |\xi'|^{1-s}\|e^{x'\cdot\xi'}\tilde{g}\|_{L^2(\Sigma_1)}).\end{aligned}$$

Using the triangle inequality, this can now be rewritten in terms of u, the operators $\tilde{L}^s_{\xi',V}$ and $\mathcal{B}_s$ and then becomes

$$\begin{aligned}&|\xi'|^s\|u\|_{L^2(\Sigma_1)} + |\xi'|\|x_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\Omega)} + \|x_{n+1}^{\frac{1-2s}{2}}\nabla u\|_{L^2(\Omega)}\\ &\leq C(|\xi'|\|x_{n+1}^{\frac{1-2s}{2}}\bar{F}\|_{L^2(\Omega+)} + \|x_{n+1}^{\frac{1-2s}{2}}F_0\|_{L^2(\Omega)} + |\xi'|^{1-s}\|g\|_{L^2(\Sigma_1)})\\ &\leq C(|\xi'|\|\tilde{L}^s_{\xi',V}u\|_{(H^1(\Omega,x_{n+1}^{1-2s})^*)} + |\xi'|^{1-s}\|\mathcal{B}_s(u)\|_{L^2(\Sigma_1)}).\end{aligned} \tag{71}$$

As a result, we infer that the map (70) is injective.

Building on this observation, we obtain that the linear functional

$$T: \tilde{L}^s_{\xi',V}(\mathcal{C}) \times \mathcal{B}_s(\mathcal{C}) \to \mathbb{R},\ (\tilde{L}^s_{\xi',V}u, \mathcal{B}_s u) \mapsto (u,f)_{L^2(\Omega)} + (u,g)_{L^2(\Sigma_1)}$$

is well defined.

Moreover, using (71), the bound

$$\begin{aligned}&|(u,f)_{L^2(\Omega)} + (u,g)_{L^2(\Sigma_1)}| \leq \|u\|_{L^2(\Omega,x_{n+1}^{1-2s})}\|f\|_{L^2(\Omega,x_{n+1}^{2s-1})} + \|u\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_1)}\\ &\leq C_{\Omega,V,k,s}\|u\|_{L^2(\Omega,x_{n+1}^{1-2s})} + C_{q,k,\Sigma_1}\|u\|_{L^2(\Sigma_1)}\\ &\leq (C_{\Omega,V,k,s}|\xi'|^{-1} + C_{q,k,\Sigma_1}|\xi'|^{-s})(\||\xi'|\tilde{F}\|_{L^2(\Omega,x_{n+1}^{1-2s})} + \|F_0\|_{L^2(\Omega,x_{n+1}^{1-2s})} + \||\xi'|^{1-s}\mathcal{B}_s(u)\|_{L^2(\Sigma_1)})\\ &\leq c(|\xi'|^{-1} + |\xi'|^{-s})(\|\tilde{L}^s_{\xi',V}u\|_{(H^1_{sc}(\Omega,x_{n+1}^{1-2s}))^*} + \|\mathcal{B}_s u\|_{L^2_{sc}(\Sigma_1)}).\end{aligned}$$

holds for a constant $c = c_{\Omega,\Sigma_1,k,V,q}$. Here $\tilde{L}^s_{\xi',V}u = \nabla\cdot\tilde{F} + F_0$ in the sense of distributions. The subscript denotes the use of semiclassical norms with $|\xi'|^{-1}$ as a small parameter, i.e.

$$\begin{aligned}\|\tilde{L}^s_{\xi',V}u\|_{(H^1_{sc}(\Omega,x_{n+1}^{1-2s}))^*} &:= \||\xi'|\tilde{F}\|_{L^2(\Omega,x_{n+1}^{1-2s})} + \|F_0\|_{L^2(\Omega,x_{n+1}^{1-2s})},\\ \|\mathcal{B}_s u\|_{L^2_{sc}(\Sigma_1)} &:= \||\xi'|^{1-s}\mathcal{B}_s u\|_{L^2(\Sigma_1)}.\end{aligned}$$

As a consequence, as a functional on a subset of $(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^* \times L^2_{sc}(\Sigma_1)$, we have $\|T\| = O(|\xi'|^{-s})$ for $|\xi'| \to \infty$. Since for $s \in [\frac{1}{2}, 1)$ the vector space $\tilde{L}^s_{\xi',V}(\mathcal{C}) \times \mathcal{B}_s(\mathcal{C})$ is a subvector space of $(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^* \times L^2_{sc}(\Sigma_1)$, by the Hahn-Banach theorem, the functional T can be extended to act on all of $(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^* \times L^2_{sc}(\Sigma_1)$ while maintaining the same norm.

Making use of the Riesz representation theorem, we find some $\tilde{r}_1 \in (H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*$ and $\tilde{r}_2 \in L^2_{sc}(\Sigma_1)$ such that for every choice of $v = (v_1, v_2) \in (H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^* \times L^2_{sc}(\Sigma_1)$ it holds that

$$T(v_1, v_2) = (v_1, \tilde{r}_1)_{(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*} + (v_2, \tilde{r}_2)_{L^2_{sc}(\Sigma_1)} \ ,$$
$$\|\tilde{r}_1\|_{(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*} + \|\tilde{r}_2\|_{L^2_{sc}(\Sigma_1)} = \|T\| = O(|\xi'|^{-s}).$$

However, if we let r_1 be the Riesz representative of $\tilde{r}_1$ in $H^1_{sc}(\Omega, x_{n+1}^{1-2s})$ and define $r_2 := |\xi'|^{2-2s}\tilde{r}_2$, we can compute

$$T(v_1, v_2) = (v_1, \tilde{r}_1)_{(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*} + (v_2, |\xi'|^{2-2s}\tilde{r}_2)_{L^2(\Sigma_1)} = \langle v_1, r_1\rangle + (v_2, r_2)_{L^2(\Sigma_1)} \ ,$$
$$|\xi'|^{s-1}\|r_2\|_{L^2(\Sigma_1)} = |\xi'|^{s-1}\||\xi'|^{2-2s}\tilde{r}_2\|_{L^2(\Sigma_1)} = \||\xi'|^{1-s}\tilde{r}_2\|_{L^2(\Sigma_1)} = \|\tilde{r}_2\|_{L^2_{sc}(\Sigma_1)} \ ,$$
$$\|r_1\|_{H^1_{sc}(\Omega, x_{n+1}^{1-2s})} = \|\tilde{r}_1\|_{(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*} \ ,$$

where $\langle \cdot, \cdot\rangle$ denotes the $(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*$, $H^1_{sc}(\Omega, x_{n+1}^{1-2s})$ duality pairing. This eventually gives

$$\begin{aligned} &T(v_1, v_2) = \langle v_1, r_1\rangle + (v_2, r_2)_{L^2(\Sigma_1)} \ , \\ &\|r_1\|_{L^2(\Omega, x_{n+1}^{1-2s})} + |\xi'|^{-1}\|\nabla r_1\|_{L^2(\Omega, x_{n+1}^{1-2s})} + |\xi'|^{s-1}\|r_2\|_{L^2(\Sigma_1)} = \\ &\qquad = \|r_1\|_{H^1_{sc}(\Omega, x_{n+1}^{1-2s})} + |\xi'|^{s-1}\|r_2\|_{L^2(\Sigma_1)} \\ &\qquad = \|\tilde{r}_1\|_{(H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*} + \|\tilde{r}_2\|_{L^2_{sc}(\Sigma_1)} = O(|\xi'|^{-s}) \ . \end{aligned} \tag{72}$$

Using that $L^2_{sc}(\Omega, x_{n+1}^{2s-1}) \subset (H^1_{sc}(\Omega, x_{n+1}^{1-2s}))^*$ with the identification that the functional ℓ_{v_1} associated with $v_1 \in L^2_{sc}(\Omega, x_{n+1}^{2s-1})$ is given by

$$\ell_{v_1}(f) := (v_1, f)_{L^2(\Omega)} \text{ for } f \in L^2_{sc}(\Omega, x_{n+1}^{1-2s}),$$

we have that for $v_1 \in L^2_{sc}(\Omega, x_{n+1}^{2s-1})$

$$\langle v_1, r_1\rangle := \langle \ell_{v_1}, r_1\rangle = (v_1, r_1)_{L^2(\Omega)}. \tag{73}$$

Integrating by parts, we next deduce the equations satisfied by r_1, r_2. Formally this follows by integrating the equations by parts twice and then inserting suitable test functions. Since a priori no weighted second derivatives of r_1, r_2 are given, we need to argue more carefully. To this end, recalling (73), we compute for $u \in \mathcal{C}$ with $u = \partial_\nu u = 0$ on Σ_2

$$\begin{aligned} (u, f)_{L^2(\Omega)} + (u, g)_{L^2(\Sigma_1)} &= T(\tilde{L}^s_{\xi', V} u, \mathcal{B}_s(u)) \\ &= (\tilde{L}^s_{\xi'} u, r_1)_{L^2(\Omega)} + (x_{n+1}^{1-2s} V u, r_1)_{L^2(\Omega)} + (\mathcal{B}_s(u), r_2)_{L^2(\Sigma_1)} \\ &= (x_{n+1}^{1-2s}\nabla u, \nabla r_1)_{L^2(\Omega)} - 2(x_{n+1}^{1-2s}\xi' \cdot \nabla' u, r_1)_{L^2(\Omega)} + (x_{n+1}^{1-2s} V u, r_1)_{L^2(\Omega)} \\ &\quad + (\mathcal{B}_s(u), r_2 - r_1)_{L^2(\Sigma_1)} + (qu, r_1)_{L^2(\Sigma_1)}. \end{aligned} \tag{74}$$

As a consequence, considering $u \in C^\infty_c(\Omega)$ we infer that the function r_1 is a weak solution to the bulk equation

$$\tilde{L}^s_{\xi', V} r_1 = f \text{ in } \Omega$$

and

$$(u, f)_{L^2(\Omega)} = (x_{n+1}^{1-2s}\nabla u, \nabla r_1)_{L^2(\Omega)} - 2(x_{n+1}^{1-2s}\xi' \cdot \nabla' u, r_1)_{L^2(\Omega)} + (x_{n+1}^{1-2s} V u, r_1)_{L^2(\Omega)} \tag{75}$$

for all $u \in C^\infty_c(\Omega)$. Next, by an approximation result which uses the fact that $s \geq \frac{1}{2}$, we obtain that the identity (75), which a priori only holds for $u \in C^\infty_c(\Omega)$, also remains true for $u \in x_{n+1}^{2s} C^\infty_c(\overline{\Omega})$. Combining this with (74), thus implies in turn that for $u \in x_{n+1}^{2s} C^\infty_c(\overline{\Omega})$ we have the following boundary equation

$$(u, g)_{L^2(\Sigma_1)} = (\mathcal{B}_s(u), r_2 - r_1)_{L^2(\Sigma_1)} + (qu, r_1)_{L^2(\Sigma_1)}. \tag{76}$$

Using this observation, we now consider a suitable test function to deduce further information from (76): Let $h \in C_c^\infty(\Sigma_1)$ and consider an open set $\overline{\Sigma}$ such that $\text{supp}(h) \subset \overline{\Sigma} \subset \Sigma_1$. Let $\epsilon > 0$ be so small that $\overline{\Sigma} \times (0,\epsilon) \subset\subset \Omega$ and consider $\psi \in C_c^\infty(\bar{\Omega})$ such that $\psi(x) = 1$ if $x \in \text{supp}(h) \times [0,\epsilon/2)$ and $\psi(x) = 0$ if $x \notin \overline{\Sigma} \times [0,\epsilon]$. Finally, let $u(x) = x_{n+1}^{2s}\psi(x)h(x')$.

Observe that since $\text{supp}(u) \subset \Sigma_1 \times (0,\epsilon) \subset\subset \Omega$ we have $u = \partial_\nu u = 0$ on Σ_2. Moreover, since $\psi h \in C_c^\infty(\overline{\Omega})$, we have $u \in x_{n+1}^{2s} C_c^\infty(\overline{\Omega})$. Thus, u is a valid test function. We can compute

$$\mathcal{B}_s u = \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}u + qu = h(x')\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}(\psi(x)x_{n+1}^{2s}) = 2s\, h(x')$$

by the properties of ψ. Also, $u(x) = 0$ if $x \in \Sigma_1$. Thus, (76) is reduced to

$$0 = (\mathcal{B}_s u, r_2 - r_1)_{L^2(\Sigma_1)} = 2s(h, r_2 - r_1)_{L^2(\Sigma_1)} ,$$

which implies $r_1 = r_2$ in Σ_1 by the arbitrary choice of h.
As a consequence, this implies that r_1 satisfies the equation

$$\begin{aligned}(u,f)_{L^2(\Omega)} + (u,g)_{L^2(\Sigma_1)} &= -(x_{n+1}^{1-2s}\nabla u, \nabla r_1)_{L^2(\Omega)} + 2(x_{n+1}^{1-2s}\xi'\cdot\nabla' u, r_1)_{L^2(\Omega)} \\ &\quad + (x_{n+1}^{1-2s}Vu, r_1)_{L^2(\Omega)} + (qu, r_1)_{L^2(\Sigma_1)},\end{aligned}$$

for all $u \in \mathcal{C}$. Now by density of $\mathcal{C}$ in $H^1(\Omega, x_{n+1}^{1-2s})$ (see Proposition 2.3), this exactly corresponds to r_1 being a weak solution of the equation

$$\begin{aligned}\tilde{L}^s_{-\xi,V} r_1 &= f \text{ in } \Omega, \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}r_1 + qr_1 &= g \text{ on } \Sigma_1.\end{aligned}$$

Finally, we recall that since we proved that $r_1 = r_2$ in Σ_1, formula (72) now reads

$$\|r_1\|_{L^2(\Omega, x_{n+1}^{1-2s})} + |\xi'|^{-1}\|\nabla r_1\|_{L^2(\Omega, x_{n+1}^{1-2s})} + |\xi'|^{s-1}\|r_1\|_{L^2(\Sigma_1)} = \|T\| = O(|\xi'|^{-s}),$$

which yields the desired correction function $r := r_1$ and the claimed estimates. □

With the construction of CGO solutions to (9) in hand, we now turn to the associated inverse problem. Arguing as in Section 3, it is possible to prove the well-posedness of the weak formulation of the problem (9) outside of a discrete set of eigenvalues. More precisely, to obtain this we consider the associated bilinear form

$$\tilde{B}_{q,V}(u,v) := \int_\Omega x_{n+1}^{1-2s}\nabla u\cdot\nabla v dx + \int_\Omega V x_{n+1}^{1-2s} uv dx + \int_{\Sigma_1} quv dx',$$

for $u, v \in H^1(\Omega, x_{n+1}^{1-2s})$. Further we investigate the Dirichlet problem (9) for data f belonging to the abstract space

$$R := H^1(\Omega, x_{n+1}^{1-2s})/H^1_{\Sigma_2,0}(\Omega, x_{n+1}^{1-2s}),$$

endowed with the usual quotient topology

$$\|f\|_R := \inf_{u\in f}\left\{\|u\|_{H^1(\Omega, x_{n+1}^{1-2s})}\right\}.$$

This choice is motivated by the the observation that for all $u, v \in H^1(\Omega, x_{n+1}^{1-2s})$ we have for the corresponding remainder classes $[u], [v] \in R$

$$[u] = [v] \quad \Leftrightarrow \quad u|_{\Sigma_2} = v|_{\Sigma_2} ,$$

and thus the equivalence classes of R can be interpreted as restrictions on Σ_2 of functions belonging to $H^1(\Omega, x_{n+1}^{1-2s})$. In view of this interpretation, one can make sense of the assertion $u|_{\Sigma_2} = f$, with $u \in H^1(\Omega, x_{n+1}^{1-2s})$ and $f \in R$, as equivalent to $u \in f$. Moreover, by the properties of the infimum for all $f \in R$ with $\|f\|_R > 0$ and $\epsilon > 0$, we can find $u \in H^1(\Omega, x_{n+1}^{1-2s})$ with $u|_{\Sigma_2} = f$ such that

$$\|u\|_{H^1(\Omega, x_{n+1}^{1-2s})} \leq \|f\|_R + \epsilon.$$

By just choosing $\epsilon \leq \|f\|_R$ we deduce that for all boundary data f on Σ_2 there exists an extension $E_s(f) \in H^1(\Omega, x_{n+1}^{1-2s})$ such that

$$\|E_s(f)\|_{H^1(\Omega, x_{n+1}^{1-2s})} \leq 2\|f\|_R.$$

This lets us argue similarly as in Section 3, and we obtain analogous well-posedness results.

We denote the dual space of R by R^*. In the following we assume that zero is not a Dirichlet eigenvalue and thus define for $f \in R$ a Dirichlet-to-Neumann operator $\tilde{\Lambda}_{q,V} : R \to R^*$ by setting

$$\langle \tilde{\Lambda}_{s,q,V} f, g \rangle_{R^*,R} = B_{q,V}(u_f, E_s g).$$

Here $E_s g$ denotes a $H^1(\Omega, x_{n+1}^{1-2s})$ extension of the function $g \in R$. Relying on similar arguments as for the Dirichlet-to-Neumann maps studied in Section 3, the map $\tilde{\Lambda}_{s,q,V}$ is continuous from R into R^*.

With the CGO solutions available, we can now address the proof of Theorem 2. Indeed, with the given special solutions, the solution to our inverse problem now follows from the Alessandrini identity.

Proof of Theorem 2. Let $V := V_1 - V_2$ and $q := q_1 - q_2$. The assumption that $\Lambda_1 = \Lambda_2$ and the Alessandrini identity from Lemma 3.13 allow us to write that, for any solutions u_1, u_2 to (1),

$$\int_{\mathbb{R}^{n+1}} \chi_\Omega V u_1 \overline{u_2} x_{n+1}^{1-2s} dx + \int_{\mathbb{R}^n} \chi_{\Sigma_1} q u_1 \overline{u_2} dx' = 0\,.$$

We shall test this identity using our special CGO solutions. Fix ξ, k as in Proposition 1.2 and let

$$\begin{aligned} u_1(x) &:= e^{\xi' \cdot x'} (e^{(ik' \cdot x' + ik_{n+1} x_{n+1}^{2s})/2} + r_1(x))\,, \\ u_2(x) &:= e^{\tilde{\xi}' \cdot x'} (e^{-(ik' \cdot x' + ik_{n+1} x_{n+1}^{2s})/2} + r_2(x))\,. \end{aligned}$$

Here if $\xi' = \zeta_1 + i\zeta_2$, we set $\tilde{\xi}' := -\zeta_1 + i\zeta_2$. Substituting these into the above identity gives rise to

$$\begin{aligned} 0 = &\int_{\mathbb{R}^{n+1}} \chi_\Omega V x_{n+1}^{1-2s} \left(r_1 r_2 + (r_1 + r_2) e^{(ik' \cdot x' + ik_{n+1} x_{n+1}^{2s})/2} + e^{ik' \cdot x' + ik_{n+1} x_{n+1}^{2s}} \right) dx + \\ &+ \int_{\mathbb{R}^n} \chi_{\Sigma_1} q \left(r_1 r_2 + (r_1 + r_2) e^{ik' \cdot x'/2} + e^{ik' \cdot x'} \right) dx'\,. \end{aligned}$$

We now aim to estimate the terms involving r_1 and r_2, showing that they can be dropped in the limit $|\xi'| \to \infty$. Recall from Proposition 1.2 that

$$\|r_j\|_{L^2(\Omega, x_{n+1}^{1-2s})} + \|r_j\|_{L^2(\Sigma_1)} = O(|\xi'|^{1-2s})$$

for $j=1,2$, and thus since $s\in(1/2,1)$ we have both $\|r_j\|_{L^2(\Omega,x_{n+1}^{1-2s})}\to 0$ and $\|r_j\|_{L^2(\Sigma_1)}\to 0$ as $|\xi'|\to\infty$. Therefore,

$$\int_{\mathbb{R}^{n+1}} |\chi_\Omega V x_{n+1}^{1-2s} r_1 r_2| dx \leq \|V\|_{L^\infty(\Omega)} \|r_1\|_{L^2(\Omega,x_{n+1}^{1-2s})} \|r_2\|_{L^2(\Omega,x_{n+1}^{1-2s})} \to 0\ ,$$

$$\int_{\mathbb{R}^{n+1}} |e^{(ik'\cdot x'+ik_{n+1}x_{n+1}^{2s})/2}\chi_\Omega V x_{n+1}^{1-2s} r_1| dx \leq \|V\|_{L^\infty(\Omega)} \|r_1\|_{L^2(\Omega,x_{n+1}^{1-2s})} \|x_{n+1}^{1/2-s}\|_{L^2(\Omega)} \to 0\ ,$$

$$\int_{\mathbb{R}^{n}} |\chi_{\Sigma_1} q r_1 r_2| dx' \leq \|q\|_{L^\infty(\Sigma_1)} \|r_1\|_{L^2(\Sigma_1)} \|r_2\|_{L^2(\Sigma_1)} \to 0\ ,$$

$$\int_{\mathbb{R}^{n}} |e^{ik'\cdot x'/2}\chi_{\Sigma_1} q r_1| dx' \leq \|q\|_{L^\infty(\Sigma_1)} \|r_1\|_{L^2(\Sigma_1)} |\Sigma_1|^{1/2} \to 0,$$

as $|\xi'|\to\infty$, and similarly for the remaining terms. The Alessandrini identity is thus reduced to

$$\int_{\mathbb{R}^{n+1}} \chi_\Omega V x_{n+1}^{1-2s} e^{ik'\cdot x'+ik_{n+1}x_{n+1}^{2s}} dx + \int_{\mathbb{R}^n} \chi_{\Sigma_1} q e^{ik'\cdot x'} dx' = 0\ ,$$

which after the change of variables $(y', y_{n+1}) = (x', x_{n+1}^{2s})$ in the first integral takes the form

$$\int_{\mathbb{R}^{n+1}} \left(\frac{\chi_\Omega V}{2s}\right)(y', y_{n+1}^{1/2s}) y_{n+1}^{1/s-2} e^{ik\cdot y} dy + \int_{\mathbb{R}^n} \chi_{\Sigma_1} q e^{ik'\cdot x'} dx' = 0\ . \tag{77}$$

Let $\mathcal{S}(\mathbb{R}^{n+1})$ and $\mathcal{S}'(\mathbb{R}^{n+1})$ respectively be the sets of Schwartz functions and tempered distributions over $\mathbb{R}^{n+1}$. Consider $\delta_{x_{n+1}}(0)\in\mathcal{S}'(\mathbb{R}^{n+1})$ defined by

$$\langle \delta_{x_{n+1}}(0), \phi\rangle = \int_{\mathbb{R}^n} \phi((x',0)) dx'$$

for all $\phi\in\mathcal{S}(\mathbb{R}^{n+1})$. Then

$$f(x) := \left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s}) x_{n+1}^{1/s-2} \chi_{[0,\infty)}(x_{n+1}) + \delta_{x_{n+1}}(0)(\chi_{\Sigma_1} q)(x')$$

where $\chi_{[0,\infty)}(x_{n+1})$ denotes the characteristic function of $[0,\infty)$ is also a tempered distribution, since for all $\phi\in\mathcal{S}(\mathbb{R}^{n+1})$ we have

$$\begin{aligned}|\langle f,\phi\rangle| &= \left|\int_{\mathbb{R}^{n+1}} \left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s}) x_{n+1}^{1/s-2}\chi_{[0,\infty)}(x_{n+1})\phi(x) dx + \int_{\mathbb{R}^n} (\chi_{\Sigma_1} q)(x')\phi((x',0)) dx'\right| \\ &\leq \frac{\|V\|_{L^\infty(\Omega)}}{2s}\int_\Omega x_{n+1}^{1/s-2}|\phi(x)| dx + \|q\|_{L^\infty(\Sigma_1)}\int_{\Sigma_1} |\phi((x',0))| dx' \\ &\leq \|\phi\|_{L^\infty}\left(\frac{\|V\|_{L^\infty(\Omega)}}{2s}\int_\Omega x_{n+1}^{1/s-2} dx + \|q\|_{L^\infty(\Sigma_1)}|\Sigma_1|\right) < \infty\ .\end{aligned}$$

The Fourier transform of f belongs to $\mathcal{S}'(\mathbb{R}^{n+1})$ as well, and by definition it is the tempered distribution given by

$$\begin{aligned}\langle \hat{f}, \phi\rangle &= \langle \mathcal{F}\left[\left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s})x_{n+1}^{1/s-2}\chi_{[0,\infty)}(x_{n+1})\right](k), \phi(k)\rangle + \langle \delta_{x_{n+1}}(0)(\chi_{\Sigma_1} q)(x'), \hat{\phi}(x)\rangle \\ &= \langle \mathcal{F}\left[\left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s})x_{n+1}^{1/s-2}\chi_{[0,\infty)}(x_{n+1})\right](k), \phi(k)\rangle + \int_{\mathbb{R}^n} (\chi_{\Sigma_1} q)(x')\hat{\phi}((x',0))dx' \\ &= \int_{\mathbb{R}^{n+1}} \phi(k) \int_{\mathbb{R}^{n+1}} \left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s})x_{n+1}^{1/s-2}\chi_{[0,\infty)}(x_{n+1})e^{ix\cdot k}dxdk \\ &\quad + \int_{\mathbb{R}^n} (\chi_{\Sigma_1} q)(x') \int_{\mathbb{R}^{n+1}} \phi(k)e^{ik'\cdot x'}dkdx' \\ &= \int_{\mathbb{R}^{n+1}} \phi(k) \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s})x_{n+1}^{1/s-2}e^{ik\cdot x}dx + \int_{\mathbb{R}^n} (\chi_{\Sigma_1} q)(x')e^{ik'\cdot x'}dx'\right)dk\end{aligned}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$, where for convenience of notation, we both use the notation $\hat{f}$ and $\mathcal{F}f$ to denote the Fourier transform. By (77) the last expression vanishes, which proves that $\hat{f} = 0$. Now the Fourier inversion theorem for tempered distributions allows us to deduce that $\langle f, \phi\rangle = 0$ for every $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$. Testing this equality with an arbitrary function $\phi \in C_c^\infty(\Omega)$ we get

$$\begin{aligned}0 &= \langle \left(\frac{\chi_\Omega V}{2s}\right)(x', x_{n+1}^{1/2s})x_{n+1}^{1/s-2}\chi_{[0,\infty)}(x_{n+1}) + \delta_{x_{n+1}}(0)(\chi_{\Sigma_1} q)(x'), \phi\rangle \\ &= \int_{\mathbb{R}^{n+1}_+} \left(\frac{\chi_\Omega V}{2s}\right) x_{n+1}^{1/s-2}\phi\, dx\ ,\end{aligned}$$

which by the arbitrary choice of ϕ implies $V = 0$ in Ω, and we are left with $f(x) = \delta_{x_{n+1}}(0)(\chi_{\Sigma_1} q)(x')$.

Let now $\psi \in C_c^\infty(\Sigma_1)$, and consider $\eta \in C^\infty(\mathbb{R})$ such that $\eta(x) = 1$ if $x \in (-1,1)$ and $\eta(x) = 0$ if $x \notin (-2,2)$. Since it belongs to $C_c^\infty(\mathbb{R}^{n+1})$, the function $\phi(x) := \psi(x')\eta(x_{n+1})$ is a suitable test function for $\langle f, \phi\rangle = 0$, and by using it we obtain

$$0 = \langle f, \phi\rangle = \langle \delta_{x_{n+1}}(0)(\chi_{\Sigma_1} q)(x'), \psi(x')\eta(x_{n+1})\rangle = \int_{\mathbb{R}^n} \chi_{\Sigma_1} q\psi\, dx'\ .$$

Eventually, by the arbitrary choice of ψ we conclude that $q = 0$ in Σ_1. □

As a corollary of this argument we remark that while for $s = \frac{1}{2}$ with the described method we cannot simultaneously prove uniqueness for the potentials q and V (due to the lack of the decay of r on the boundary), this method still allows us to prove uniqueness for V given a fixed potential q:

Corollary 7.1. *Let $\Omega \subset \mathbb{R}^{n+1}_+$, $n \geq 2$, be an open, bounded and smooth domain. Assume that $\Sigma_1 := \partial\Omega \cap \{x_{n+1} = 0\}$ and $\Sigma_2 \subset \partial\Omega \setminus \Sigma_1$ are two relatively open, non-empty subsets of the boundary such that $\overline{\Sigma_1 \cup \Sigma_2} = \partial\Omega$. Let $s = \frac{1}{2}$. If the potentials $q \in L^\infty(\Sigma_1)$ and $V_1, V_2 \in L^\infty(\Omega)$ relative to problem (9) are such that*

$$\Lambda_1 := \Lambda_{s,V_1,q} = \Lambda_{s,V_2,q} =: \Lambda_2\ ,$$

then $V_1 = V_2$.

Proof. The proof follows that of Theorem 2, but it is significantly easier due to the lack of boundary terms. Again we let $V := V_1 - V_2$, but this time the Alessandrini identity from Lemma 3.13 reduces to simply

$$\int_{\mathbb{R}^{n+1}} \chi_\Omega V u_1 \overline{u_2} dx = 0\,,$$

where u_1, u_2 solve (1). Fix $\xi', k \in \mathbb{R}^{n+1}$ as in Lemma 1.2 with the modification from Remark 5.2, and for $\xi' = \zeta_1 + i\zeta_2$ set $\tilde{\xi}' := -\zeta_1 + i\zeta_2$. Testing the equation above with the following CGOs

$$u_1(x) := e^{\xi'\cdot x'}(e^{ik\cdot x/2} + r_1(x)) , \quad u_2(x) := e^{\tilde{\xi}'\cdot x'}(e^{-ik\cdot x/2} + r_2(x)),$$

leads to

$$0 = \int_{\mathbb{R}^{n+1}} \chi_\Omega V \left(r_1 r_2 + (r_1 + r_2)e^{ik\cdot x/2} + e^{ik\cdot x}\right) dx \, .$$

In our current case $s = 1/2$, Proposition 1.2 does not grant any decay for the correction functions r_j on the boundary; however, we will make use only of their decay estimate in the bulk. Given that $\|r_j\|_{L^2(\Omega)} = O(|\xi'|^{-1/2})$, by Cauchy-Schwarz

$$\int_{\mathbb{R}^{n+1}} |\chi_\Omega V r_1 r_2| dx \leq \|V\|_{L^\infty(\Omega)} \|r_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)} = O(|\xi'|^{-1}),$$

$$\int_{\mathbb{R}^{n+1}} |e^{ik\cdot x/2} \chi_\Omega V r_j| dx \leq \|V\|_{L^\infty(\Omega)} |\Omega|^{1/2} \|r_j\|_{L^2(\Omega)} = O(|\xi'|^{-1/2}) \, .$$

Therefore, by finding the limit $|\xi'| \to \infty$ of the tested equation we obtain

$$0 = \int_{\mathbb{R}^{n+1}} \chi_\Omega V e^{ik\cdot x} dx = \mathcal{F}[\chi_\Omega V](k)$$

for all $k \in \mathbb{R}^{n+1}$. It now follows from the Fourier inversion theorem that $V = 0$ on Ω, that is, the potentials V_1 and V_2 must coincide. $\square$

Appendix A. Proof of Proposition 2.3

In this section, we provide the proof of Proposition 2.3. To this end, we begin by showing the following auxiliary result:

Lemma A.1. *The set $C^\infty(\overline{\mathbb{R}^{n+1}_+})$ is dense in $H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$.*

Proof of Lemma A.1. We consider $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\text{supp}(\varphi) \subset B_1^-(0)$, $\varphi \geq 0$ and $\int_{\mathbb{R}^{n+1}} \varphi dx = 1$. Set $\varphi_\epsilon(x) = \epsilon^{-n-1}\varphi(\frac{x}{\epsilon})$. Further let $f \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$. We construct a smooth sequence f_ϵ such that $f_\epsilon \to f$ in $H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$. To this end define $f_\epsilon(x) := (f * \varphi_\epsilon)(x)$. Then, since $f \in L^1_{loc}(\mathbb{R}^{n+1}_+)$, we obtain that f_ϵ is smooth. Moreover, as a consequence of the maximal function estimate for weights in the Muckenhoupt class (see for instance Theorem 1.2 in [Kil94] with the difference of working with half-balls instead of balls) f_ϵ, $\nabla f_\epsilon = (\nabla f)_\epsilon \in L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$. In order to prove the convergence, we only show $f_\epsilon \to f$ in $L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ (the statement for the gradient is then analogous) and begin by collecting a number of auxiliary observations. We note that for each $\Omega \subset \overline{\mathbb{R}^{n+1}_+}$, by the maximal function estimates we also have that

$$\|f_\epsilon\|_{L^2(\Omega, x_{n+1}^{1-2s})} \leq C\|f\|_{L^2(N(\Omega,\epsilon), x_{n+1}^{1-2s})}. \tag{78}$$

Here $N(\Omega, \epsilon)$ denotes an ϵ neighbourhood of Ω in $\overline{\mathbb{R}^{n+1}_+}$. Now, since $f \in L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$, for each $\delta > 0$ there exists $R > 1$ such that $\|f\|_{L^2(\mathbb{R}^{n+1}_+ \setminus \overline{B_R}, x_{n+1}^{1-2s})} \leq \delta$ and thus by (78) also $\|f_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+ \setminus \overline{B_R}, x_{n+1}^{1-2s})} \leq \delta$. Moreover, again by the integrability of f, there exists $\tilde{\delta} > 0$ such that

$$\|f\|_{L^2(B_R \cap \{x_{n+1} \leq \tilde{\delta}\}, x_{n+1}^{1-2s})} + \|f_\epsilon\|_{L^2(B_R \cap \{x_{n+1} \leq \tilde{\delta}\}, x_{n+1}^{1-2s})} \leq \delta.$$

Finally in $B_{2R}\cap\{x_{n+1}>\tilde{\delta}/2\}$ there exists a sequence $f_k\in C^\infty(\overline{B_{2R}\cap\{x_{n+1}>\tilde{\delta}/2\}})$ such that $f_k\to f$ in $L^2(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$. Since $(f_k)_\epsilon := f_k*\varphi_\epsilon\to f_k$ uniformly on compact sets, we may thus also conclude that

$$\|f_k-(f_k)_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}\leq\delta.$$

Also, by (78)

$$\begin{aligned}\|f_\epsilon-(f_k)_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})} &= \|(f-f_k)_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}\\ &\leq C\|f-f_k\|_{L^2(B_{2R}\cap\{x_{n+1}\geq\tilde{\delta}/2\},x_{n+1}^{1-2s})}.\end{aligned}$$

Combining the above observations, infer that

$$\begin{aligned}\|f-f_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})} &\leq \|f\|_{L^2(\mathbb{R}^{n+1}_+\setminus\overline{B_R},x_{n+1}^{1-2s})}+\|f_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+\setminus\overline{B_R},x_{n+1}^{1-2s})}\\ &\quad+\|f\|_{L^2(B_R\cap\{x_{n+1}\leq\tilde{\delta}\},x_{n+1}^{1-2s})}+\|f_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\leq\tilde{\delta}\},x_{n+1}^{1-2s})}\\ &\quad+\|f-f_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}\\ &\leq 3\delta+\|f-f_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}\\ &\leq 3\delta+\|f_k-(f_k)_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}+\|f-f_k\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}\\ &\quad+\|f_\epsilon-(f_k)_\epsilon\|_{L^2(B_R\cap\{x_{n+1}\geq\tilde{\delta}\},x_{n+1}^{1-2s})}\\ &\leq 6\delta.\end{aligned}$$

Arguing analogously on the level of the derivative implies the claim. □

Next we define the following auxiliary set

$$C^\infty_{\Sigma_2}(\Omega) := \{f\in C^\infty(\overline{\Omega}):\ \exists\delta>0 \text{ s.t. } f|_{N(\Sigma_2,\delta)}=0\}.$$

Using this, we turn to the proof of the approximation result.

Proof of Proposition 2.3. Using Lemma (A.1), we argue in three steps.

Step 1: Density of $\bigcup_{\delta\in(0,\delta_0)} H^1_{N(\mathbb{R}^n\setminus\Sigma_1,\delta),0}(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})\subset H^1_{\mathbb{R}^n\setminus\Sigma_1,0}(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$.

This follows by rescaling: Indeed, by translation we may assume that $x=0$ is a center of the star-shaped set Σ_1. Now let $u\in H^1_{\mathbb{R}^n\setminus\Sigma_1,0}(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$. Then, as $C^\infty(\overline{\mathbb{R}^{n+1}_+})$ is dense in $H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$, there exists $(u_k)_{k\in\mathbb{N}}\subset C^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that $u_k\to u$ in $H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$. Since Σ_1 is star-shaped, if we define $d:=\mathrm{dist}(0,\partial\Sigma_1)$ and $u_\delta(x):=u\left(\frac{d}{d-\delta}x\right)$ for $\delta\in(0,d)$, then we have that $u_\delta\in H^1_{N(\mathbb{R}^n\setminus\Sigma_1,\delta),0}(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})$ and

(79)

$$\begin{aligned}\|u_\delta-u\|_{H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})} &\leq \|u_\delta-u_k\left(\frac{d}{d-\delta}\cdot\right)\|_{H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})}+\|u_k\left(\frac{d}{d-\delta}\cdot\right)-u_k\|_{H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})}\\ &\quad+\|u-u_k\|_{H^1(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})}.\end{aligned}$$

Now, the first and third contributions in (79) converge to zero by definition of u_k as approximations to u. The middle right hand side contribution converges to zero by the assumed regularity of u_k and a Taylor approximation up to order one.

Using a partition of unity and straightening out the boundary by a suitable diffeomorphism this also implies that $\bigcup_{\delta\in(0,\delta_0)} H^1_{N(\partial\Omega\setminus\Sigma_1,\delta),0}(\Omega,x_{n+1}^{1-2s})\subset H^1_{\Sigma_2,0}(\Omega,x_{n+1}^{1-2s})$ is dense.

Step 2: Density of $C^\infty_{\Sigma_2}(\Omega) \subset H^1_{\partial\Omega\setminus\Sigma_1,0}(\Omega, x_{n+1}^{1-2s})$.

By Step 1 it suffices to prove that $\bigcup\limits_{\epsilon\in(0,\delta/2)} C^\infty_{N(\Sigma_2,\delta+\epsilon)}(\Omega) \subset H^1_{N(\Sigma_2,\delta/2),0}(\Omega, x_{n+1}^{1-2s})$ is dense in $H^1_{N(\Sigma_2,\delta),0}(\Omega, x_{n+1}^{1-2s})$ for all sufficiently small $\delta > 0$.

By virtue of a partition of unity and by straightening out the boundary, it suffices to consider $u \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ satisfying one of the following two cases:

(i) $u|_{\mathbb{R}^n}$ has compact, but non-trivial support in Σ_1,
(ii) $u|_{\mathbb{R}^n} = 0$,

and to prove a corresponding approximation result in these cases. The first case arises when working with a patch of the partition of unity which includes $N(\Sigma_1, \epsilon)$, the second occurs for any other patch (we remark that without loss of generality, it is possible to arrange for this).

Step 2a: Case (i). For case (i) we in turn argue in two steps.

Step 2a, part 1; constant modification at $x_{n+1} = 0$. First we define the function $\tilde{u}_\epsilon$ such that $\tilde{u}_\epsilon(x', x_{n+1}) = u(x', 0)$ for $x_{n+1} \in [0, 2\epsilon]$ and $\tilde{u}_\epsilon(x', x_{n+1}) = 0$ for $x_{n+1} > 2\epsilon$. We observe that $\tilde{u}_\epsilon = 0$ in $\mathbb{R}^{n+1}_+ \setminus (\Sigma_1 \times [0, 2\epsilon])$. We further consider $\eta : [0,\infty) \to [0,1]$ with $\eta \in C^\infty([0,\infty))$, $\eta(t) = 1$ on $[0, \frac{1}{2}]$, $\operatorname{supp}(\eta) \subset [0, 2]$ and $|\nabla \eta| \leq C$. Based on this we define $\eta_\epsilon(t) := \eta(\frac{t}{\epsilon})$ and $u_\epsilon := \eta_\epsilon(x_{n+1})\tilde{u}_\epsilon(x) + (1 - \eta_\epsilon(x_{n+1}))u(x)$. We claim that $u_\epsilon \to u$ in $H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$.

To this end, we observe that

$$\begin{aligned}
\|u_\epsilon - u\|_{L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} &= \|\eta_\epsilon(\tilde{u}_\epsilon - u)\|_{L^2(\mathbb{R}^n \times [0,2\epsilon], x_{n+1}^{1-2s})} \\
&\leq \|u(x',0) - u(x)\|_{L^2(\mathbb{R}^n \times [0,2\epsilon], x_{n+1}^{1-2s})} \to 0,
\end{aligned}$$

by the integrability of $u(x', 0) - u(x)$. For the derivative we note that

$$\nabla u - \nabla u_\epsilon = (u - \tilde{u}_\epsilon)\nabla \eta_\epsilon + \eta_\epsilon \nabla (u - \tilde{u}_\epsilon). \tag{80}$$

Due to the support conditions for η_ϵ and by the fact that $\nabla(u - \tilde{u}_\epsilon) \in L^2(\mathbb{R}^n \times [0, 2\epsilon_0), x_{n+1}^{1-2s})$ for some fixed $\epsilon_0 > 0$, we have that

$$\|\eta_\epsilon \nabla(u - \tilde{u}_\epsilon)\|_{L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} \to 0$$

as $\epsilon \to 0$.

For the first contribution in the expression for the gradient (80), we use the fundamental theorem (which makes use of the approximation statement from Lemma A.1): We have that

$$|(u - \tilde{u}_\epsilon)(x)| = |u(x', x_{n+1}) - u(x', 0)| \leq \int\limits_0^{x_{n+1}} |\partial_{n+1} u(x', t)| dt.$$

Thus, using Hölder's inequality, we obtain

$$|(u - \tilde{u}_\epsilon)(x)|^2 \leq x_{n+1}^{2s} \int\limits_0^{x_{n+1}} t^{1-2s} |\partial_{n+1} u(x', t)|^2 dt.$$

As a consequence, an integration yields

$$\begin{aligned}
\|(u-\tilde{u}_\epsilon)\nabla\eta_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+,x_{n+1}^{1-2s})} &\leq C\epsilon^{-1}\|u-\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,\epsilon],x_{n+1}^{1-2s})}\\
&\leq C\epsilon^{s-1}\left\|\left(\int\limits_0^{x_{n+1}} t^{1-2s}|\partial_{n+1}u(x',t)|^2dt\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n\times[0,\epsilon],x_{n+1}^{1-2s})}\\
&\leq C\epsilon^{s-1}\left(\int_0^\epsilon x_{n+1}^{1-2s}dx_{n+1}\right)^{1/2}\left\|\left(\int\limits_0^{\epsilon} t^{1-2s}|\partial_{n+1}u(x',t)|^2dt\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^n)}\\
&\leq C_s\epsilon^{s-1}\epsilon^{1-s}\|\nabla u\|_{L^2(\mathbb{R}^n\times[0,\epsilon],x_{n+1}^{1-2s})}\\
&= C_s\|\nabla u\|_{L^2(\mathbb{R}^n\times[0,\epsilon],x_{n+1}^{1-2s})} \to 0 \text{ as } \epsilon\to 0,
\end{aligned}$$

since $\nabla u \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$. This proves the claimed convergence $u_\epsilon \to u$.

Step 2a, part 2, mollification. As a second step, we start with a function u_ϵ as obtained in Step 2a, part 1 which by a slight abuse of notation (by dropping the index) we denote by u. For this function, we now consider $u_\delta(x) := u * \varphi_\delta(x)$, where $\delta \in (0,\epsilon)$ and $\varphi_\delta(x) := \delta^{-n-1}\varphi(\frac{x}{\delta})$ with $\int\limits_{\mathbb{R}^{n+1}_+}\varphi(y)dy = 1$, $\varphi \in C^\infty(\mathbb{R}^{n+1}_+)$ is a mollifier supported in B_1^-. By the properties of the function u (in particular, recall that $u = 0$ in $(\mathbb{R}^n \setminus \Sigma_1)\times[0,\epsilon]$), for $\delta > 0$ sufficiently small, the function $u_\delta \in C^\infty(\mathbb{R}^{n+1}_+) \cap H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ then satisfies that $\text{supp}(u_\delta|_{\mathbb{R}^n}) \subset N(\Sigma_1,\delta)$ and $u_\delta \to u$ in $H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$.

Combining both steps from Steps 2a by means of a diagonal argument then implies the claim for case (i).

Step 2b: The case (ii). Now for case (ii) we argue as in the classical case, but replace the trace inequalities by correspondingly weighted ones; we refer to [Eva10, Chapter 5.5, Theorem 2]. We present some of the details for completeness. First by the density of $C^\infty(\overline{\mathbb{R}^{n+1}_+}) \subset H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ there exists a sequence $(u_m)_{m\in\mathbb{N}} \subset C^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that $u_m \to u$ in $H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$. Due to trace estimates similarly as in Lemma 2.5 and the fact that $u|_{\mathbb{R}^n} = 0$ we have $u_m|_{\mathbb{R}^n} \to 0$. Now by the fundamental theorem we obtain

$$|u_m(x',x_{n+1})| \leq |u_m(x',0)| + \int\limits_0^{x_{n+1}} |Du_m(x',t)|dt.$$

Integrating and applying Hölder's inequality implies that

$$\|u_m(\cdot,x_{n+1})\|^2_{L^2(\mathbb{R}^n)} \leq C(\|u_m(\cdot,0)\|^2_{L^2(\mathbb{R}^n)} + x_{n+1}^{2s}\int\limits_0^{x_{n+1}} t^{1-2s}\|\nabla u_m(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}dt).$$

In particular, for $m \to \infty$, by the vanishing of the trace of u, we arrive at

$$\|u(\cdot,x_{n+1})\|^2_{L^2(\mathbb{R}^n)} \leq Cx_{n+1}^{2s}\int\limits_0^{x_{n+1}} t^{1-2s}\|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}dt. \tag{81}$$

We now define

$$w_m := u(1-\zeta_m),$$

where $\zeta_m(x) := \zeta(mx_{n+1})$ and $\zeta \in C^\infty(\mathbb{R})$ is such that $\zeta(t) = 1$ for $t \in [0,1]$ and $\zeta = 0$ on $(2,\infty)$ and $0 \leq \zeta \leq 1$. We obtain

$$\begin{aligned}\partial_{n+1} w_m &= (1-\zeta_m)\partial_{n+1} u - mu\zeta'|_{mx_{n+1}},\\ \partial_j w_m &= (1-\zeta_m)\partial_j u \text{ for all } j \in \{1,\dots,n\}.\end{aligned}$$

Thus,

$$\|\nabla(w_m - u_m)\|^2_{L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} \leq C\|\zeta_m \nabla u\|^2_{L^2(\mathbb{R}^{n+1}, x_{n+1}^{1-2s})} + Cm^2 \int_0^{2/m}\int_{\mathbb{R}^n} x_{n+1}^{1-2s}|u|^2 dx' dx_{n+1}.$$

By construction, the first term converges to zero, as $\zeta_m \neq 0$ only for $x_{n+1} \in (0, 2/m)$. For the second contribution we use (81). This yields

$$\begin{aligned}m^2 \int_0^{2/m}\int_{\mathbb{R}^n} x_{n+1}^{1-2s}|u|^2 dx' dx_{n+1} &= m^2 \int_0^{2/m} x_{n+1}^{1-2s}\|u(\cdot, x_{n+1})\|^2_{L^2(\mathbb{R}^n)} dx_{n+1}\\ &\leq Cm^2 \int_0^{2/m} x_{n+1} \int_0^{x_{n+1}} t^{1-2s}\|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} dt dx_{n+1}\\ &\leq Cm^2 \left(\int_0^{2/m} x_{n+1} dx_{n+1}\right) \int_0^{2/m} t^{1-2s}\|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} dt\\ &\leq C\|\nabla u\|_{L^2(\mathbb{R}^n\times[0,2/m], x_{n+1}^{1-2s})} \to 0 \text{ as } m \to \infty.\end{aligned}$$

Step 3: Density of $\tilde{\mathcal{C}} \subset H^1_{\Sigma_1,0}(\Omega, x_{n+1}^{1-2s})$.

Let $u \in C^\infty_{\Sigma_2}(\Omega)$. We now approximate this function by a function of the desired structure. Working in boundary normal coordinates $x = x' + t\nu(x')$ we define $\tilde{u}_\epsilon(x) := u(x')$ for $x \in \partial\Omega_{2\epsilon}$. Let now η_ϵ be a smooth cut-off function which is equal to one in $\partial\Omega_\epsilon$ supported in $\partial\Omega_{2\epsilon}$ with $|\nabla'\eta_\epsilon| \leq C$ and $|\partial_\nu \eta_\epsilon| \leq \frac{C}{\epsilon}$. We then set $u_\epsilon(x) := \eta_\epsilon(x)\tilde{u}_\epsilon(x) + (1-\eta_\epsilon)u(x)$. Then,

$$\|u - u_\epsilon\|_{L^2(\Omega, x_{n+1}^{1-2s})} = \|\eta_\epsilon(u - \tilde{u}_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})}.$$

Since $u \in C^\infty(\Omega)$, we have $|\eta_\epsilon(x)||u(x) - \tilde{u}_\epsilon(x)| \leq C \sup\limits_{x\in \text{supp}(\eta_\epsilon)} |\partial_t u(x)|t \leq C\epsilon$. Thus,

$$\|u - u_\epsilon\|_{L^2(\Omega, x_{n+1}^{1-2s})} \leq C_s \epsilon \epsilon^{1-s}.$$

For the derivative we note that

$$\begin{aligned}\|\nabla(u - u_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})} &= \|\nabla[\eta_\epsilon(u-\tilde{u}_\epsilon)]\|_{L^2(\Omega, x_{n+1}^{1-2s})}\\ &\leq \|(u-\tilde{u}_\epsilon)(\nabla\eta_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})} + \|\eta_\epsilon\nabla(u-\tilde{u}_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})}.\end{aligned}$$

Now using that $|\nabla\eta_\epsilon| \leq C\epsilon^{-1}$, $|u - \tilde{u}_\epsilon| \leq C\epsilon$ and $|\nabla(u-\tilde{u}_\epsilon)| \leq C$, we obtain

$$\begin{aligned}\|\nabla(u - u_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})} &\leq \|(u-\tilde{u}_\epsilon)(\nabla\eta_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})} + \|\eta_\epsilon\nabla(u-\tilde{u}_\epsilon)\|_{L^2(\Omega, x_{n+1}^{1-2s})}\\ &\leq C\omega_s(\text{supp}(\eta_\epsilon))^{\frac{1}{2}} \leq C_s\epsilon^{1-s},\end{aligned}$$

where for $\Omega' \subset \mathbb{R}^{n+1}_+$ measurable $\omega_s(\Omega') := \int\limits_{\Omega'} x_{n+1}^{1-2s} dx$. We note that the function u_ϵ has the desired property defining $\tilde{\mathcal{C}}$. Indeed, by the construction of $\tilde{u}_\epsilon$ we have $u_\epsilon = 0$ on Σ_2 and by construction of $\tilde{u}_\epsilon$ and of η_ϵ we also have $\partial_\nu \tilde{u}_\epsilon = 0$ on $\partial\Omega$. It remains to argue that $\partial_{n+1} u_\epsilon = 0$

in $N(\Sigma_1, \delta) \times [0, \delta)$ for some $\delta > 0$ small. This on the one hand follows from the fact that in $\Sigma_1 \times [0, \epsilon/2]$ the boundary normal coordinates are simply Euclidean coordinates $x = (x', x_{n+1})$ and that the function $\tilde{u}_\epsilon$ does not depend on the x_{n+1} variable there by definition. On the other hand, we also have that in $N(\Sigma_2, \tilde{\epsilon})$ for some $\tilde{\epsilon} > 0$ the function $u \in C^\infty_{\Sigma_2}(\Omega)$ satisfies $u = 0$. As a consequence, the function $\tilde{u}_\epsilon(x) = 0$ in a set $\{x \in \Omega : x = x' + t\nu,\ x' \in N(\Sigma_1, \delta) \setminus \Sigma_1,\ t \in [0, 2\delta]\}$ for some small $\delta > 0$. This however implies that $\nabla \tilde{u}_\epsilon = 0$ on this set, which entails that $\partial_{n+1} u_\epsilon = 0$ also in a set $N(\Sigma_1, \delta) \times [0, \delta)$.

Combining all the steps from above by a diagonal argument concludes the proof. □

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The higher order fractional Calderón problem for linear local operators: uniqueness

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THE HIGHER ORDER FRACTIONAL CALDERÓN PROBLEM FOR LINEAR LOCAL OPERATORS: UNIQUENESS

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Abstract. We study an inverse problem for the fractional Schrödinger equation (FSE) with a local perturbation by a linear partial differential operator (PDO) of the order smaller than the order of the fractional Laplacian. We show that one can uniquely recover the coefficients of the PDO from the Dirichlet-to-Neumann (DN) map associated to the perturbed FSE. This is proved for two classes of coefficients: coefficients which belong to certain spaces of Sobolev multipliers and coefficients which belong to fractional Sobolev spaces with bounded derivatives. Our study generalizes recent results for the zeroth and first order perturbations to higher order perturbations.

1. Introduction

Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $\Omega \subset \mathbb{R}^n$ a bounded open set where $n \geq 1$, $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ its exterior and $P(x, D)$ a linear partial differential operator (PDO) of order $m \in \mathbb{N}$

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

where the coefficients $a_\alpha = a_\alpha(x)$ are functions defined in Ω. We study a nonlocal inverse problem for the perturbed fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + P(x, D)u = 0 \text{ in } \Omega \\ u = f \text{ in } \Omega_e \end{cases} \tag{1}$$

where $(-\Delta)^s$ is a nonlocal pseudo-differential operator $(-\Delta)^s u = \mathcal{F}^{-1}(|\cdot|^{2s}\,\hat{u})$ in contrast to the local operator $P(x, D)$. In the inverse problem, one aims to recover the local operator P from the associated Dirichlet-to-Neumann map.

We always assume that 0 is not a Dirichlet eigenvalue of the operator $((-\Delta)^s + P(x, D))$, i.e.

$$\text{If } u \in H^s(\mathbb{R}^n) \text{ solves } ((-\Delta)^s + P(x, D))u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \text{ then } u = 0.$$

Our data for the inverse problem is the Dirichlet-to-Neumann (DN) map $\Lambda_P \colon H^s(\Omega_e) \to (H^s(\Omega_e))^*$ which maps Dirichlet exterior values to a nonlocal version of the Neumann boundary value (see section 2 and 3.1). The main question that we study in this article is whether the DN map Λ_P determines uniquely the coefficients a_α in Ω. In other words, does $\Lambda_{P_1} = \Lambda_{P_2}$ imply that $a_{1,\alpha} = a_{2,\alpha}$ in Ω for all $|\alpha| \leq m$? We prove that the answer is positive under certain restrictions on the coefficients a_α and the order of the PDOs.

This gives positive answer to the uniqueness problem [10, Question 2.5] posed by the first three authors in a previous work. The precise statement in [10] asks to prove uniqueness for the higher order fractional Calderón problem in the case of a bounded domain with smooth boundary and PDOs with smooth coefficients (up to the boundary). The positive answer to this question follows from theorem 1.2. The study of the fractional Calderón problem was initiated by Ghosh, Salo and Uhlmann in the work [15] where the uniqueness for the associated inverse problem is proved when $m = 0$, $s \in (0, 1)$ and $a_0 \in L^\infty(\Omega)$.

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We briefly note that by Peetre's theorem any linear operator $L\colon C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ which does not increase supports, i.e. $\mathrm{spt}(Lf) \subset \mathrm{spt}(f)$ for all $f \in C_c^\infty(\Omega)$, is in fact a differential operator [30] (see also the original work [32]). Therefore our results apply to any local operator satisfying such properties and it is enough to study PDOs only. For a more general formulation of Peetre's theorem on the level of vector bundles, see [31].

1.1. **Main results.** We denote by $M(H^{s-|\alpha|} \to H^{-s})$ the space of all bounded Sobolev multipliers between the Sobolev spaces $H^{s-|\alpha|}(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$. We denote by $M_0(H^{s-|\alpha|} \to H^{-s}) \subset M(H^{s-|\alpha|} \to H^{-s})$ the space of bounded Sobolev multipliers that can be approximated with smooth compactly supported functions in the multiplier norm of $M(H^{s-|\alpha|} \to H^{-s})$. We also write $H^{r,\infty}(\Omega)$ for the local Bessel potential space with bounded derivatives. See section 2 for more detailed definitions.

Our first theorem is a generalization of [36, Theorem 1.1] which considered the case $m = 0$ with $s \in (0, 1)$. It also generalizes [10, Theorem 1.5] which considered the higher order cases $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ when $m = 0$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set where $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j = \sum_{|\alpha| \leq m} a_{j,\alpha} D^\alpha, \quad j = 1, 2,$$

be linear PDOs of order m with coefficients $a_{j,\alpha} \in M_0(H^{s-|\alpha|} \to H^{-s})$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations $((-\Delta)^s + P_j)u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1|_\Omega = P_2|_\Omega$.

In theorem 1.1 one can pick the lower order coefficients ($|\alpha| < s$) from $L^p(\Omega)$ for high enough p (especially from $L^\infty(\Omega)$) and higher order coefficients ($s < |\alpha| < 2s$) from the closure of $C_c^\infty(\Omega)$ in $H^{r,\infty}(\Omega)$ for certain values of $r \in \mathbb{R}$. Lemmas 2.8 and 2.9 give more examples of Sobolev spaces which belong to the space of multipliers $M_0(H^{s-|\alpha|} \to H^{-s})$. We also note that when $|\alpha| = 0$, then the space of multipliers $M_0(H^s \to H^{-s})$ coincides with the one studied in [36].

It follows that the space of multipliers is trivial for higher order operators, i.e. $M(H^{s-|\alpha|} \to H^{-s}) = \{0\}$ when $s - |\alpha| < -s$. It would be possible to state theorem 1.1 for higher order PDOs, but that forces $a_\alpha = 0$ for all $|\alpha| > 2s$. For this reason we only consider PDOs whose order is $m < 2s$. See lemma 2.5 and the related remarks for more details.

Our second theorem generalizes [7, Theorem 1.1] and [15, Theorem 1.1] where similar results are proved when $m = 0, 1$ and $s \in (0, 1)$. It also generalizes [10, Theorem 1.5] where the case $m = 0$ and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ was studied.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain where $n \geq 1$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$. Let*

$$P_j(x, D) = \sum_{|\alpha| \leq m} a_{j,\alpha}(x) D^\alpha, \quad j = 1, 2,$$

be linear PDOs of order m with coefficients $a_{j,\alpha} \in H^{r_\alpha,\infty}(\Omega)$ where

$$r_\alpha := \begin{cases} 0 & \text{if} \quad |\alpha| - s < 0, \\ |\alpha| - s + \delta & \text{if} \quad |\alpha| - s \in \{1/2, 3/2, ...\}, \\ |\alpha| - s & \text{if} \quad \text{otherwise} \end{cases} \tag{2}$$

for any fixed $\delta > 0$. Given any two open sets $W_1, W_2 \subset \Omega_e$, suppose that the DN maps Λ_{P_i} for the equations $((-\Delta)^s + P_j(x, D))u = 0$ in Ω satisfy

$$\Lambda_{P_1} f|_{W_2} = \Lambda_{P_2} f|_{W_2}$$

for all $f \in C_c^\infty(W_1)$. Then $P_1(x, D) = P_2(x, D)$.

Our first theorem is formulated for general bounded open sets and the second theorem for Lipschitz domains. The difference arises in the proof of the well-posedness of the inverse problem. We note that theorem 1.2 holds for coefficients a_α which are smooth up to the boundary ($a_\alpha = g|_\Omega$ where $g \in C^\infty(\mathbb{R}^n)$). The conditions (1.2) imply that one can choose $a_\alpha \in L^\infty(\Omega)$ for every α such that $|\alpha| < s$. The case $|\alpha| = s$ never happens, as s is assumed not to be an integer. If $|\alpha| > s$, we have $a_\alpha \in H^{|\alpha|-s,\infty}(\Omega)$ when $|\alpha| - s \notin \{1/2, 3/2, ...\}$. Thus the conditions (1.2) coincide with [7, 15] when $m = 0, 1$ and $s \in (0, 1)$.

Our article is roughly divided into two parts. The first part of the article (theorem 1.1 and section 3) generalizes the study of the uniqueness problem for singular potentials in [36] and the second part (theorem 1.2 and section 4) generalizes the uniqueness problem for bounded first order perturbations in [7].

The approach to prove theorems 1.1 and 1.2 is the following. First one shows that the inverse problem is well-posed and the corresponding bilinear forms are bounded. This leads to the boundedness of the DN maps and an Alessandrini identity. By a unique continuation property of the higher order fractional Laplacian one obtains a Runge approximation property for equation (1). Using the Runge approximation and the Alessandrini identity for suitable test functions one proves the uniqueness of the inverse problem.

1.2. **On the earlier literature.** Equation (1) and theorems 1.1 and 1.2 are related to the Calderón problem for the fractional Schrödinger equation first introduced in [15]. There one tries to uniquely recover the potential q in Ω by doing measurements in the exterior Ω_e. This is a nonlocal (fractional) counterpart of the classical Calderón problem arising in electrical impedance tomography where one obtains information about the electrical properties of some bounded domain by doing voltage and current measurements on the boundary [39, 40]. In [36] the study of the fractional Calderón problem is extended for "rough" potentials q, i.e. potentials which are in general bounded Sobolev multipliers. First order perturbations were studied in [7] assuming that the fractional part dominates the equation, i.e. $s \in (1/2, 1)$, and that the perturbations have bounded fractional derivatives. A higher order version ($s \in \mathbb{R}^+ \setminus \mathbb{Z}$) of the fractional Calderón problem was introduced and studied in [10]. These three articles [7, 10, 36] motivate the study of higher order (rough) perturbations to the fractional Laplacian $(-\Delta)^s$ in equation (1). The natural restriction for the order of $P(x, D)$ in theorems 1.1 and 1.2 is then $2s > m$ so that the fractional part governs the equation (1).

The fractional Calderón problem for $s \in (0, 1)$ has been studied in many settings. We refer to the survey [37] for a more detailed treatment. In the work [36] stability was proved for singular potentials, and in [34] the related exponential instability was shown. The fractional Calderón problem has also been solved under single measurement [14]. The perturbed equation is related to the fractional magnetic Schrödinger equation which is studied in [9, 24, 25, 26]. See also [4] for a fractional Schrödinger equation with a lower order nonlocal perturbation. Other variants of the fractional Calderón problem include semilinear fractional (magnetic) Schrödinger equation [19, 20, 24, 25], fractional heat equation [21, 35] and fractional conductivity equation [8] (see also [6, 13] for equations arising from a nonlocal Schrödinger-type elliptic operator). In the recent work [10], the first three authors of this article studied higher order versions ($s \in \mathbb{R}^+ \setminus \mathbb{Z}$) of the fractional Calderón problem and proved uniqueness for the Calderón problem for the fractional magnetic Schrödinger equation (up to a gauge). This article continues these studies by showing uniqueness for the fractional Schrödinger equation with higher order perturbations and gives positive answer to the question 2.5 posed in [10].

1.3. **Examples of fractional models in the sciences.** Equations involving fractional Laplacians like (1) have applications in mathematics and natural sciences. Fractional Laplacians appear in the study of anomalous and nonlocal diffusion, and these diffusion phenomena can be used in many areas such as continuum mechanics, graph theory and ecology just to mention a few [2, 5, 12, 27, 33]. Another place where the fractional counterpart of the classical Laplacian naturally shows up is the formulation of fractional quantum mechanics [22, 23]. For more applications of fractional mathematical models, see [5] and the references therein.

1.4. **The organization of the article.** In section 2 we introduce the notation and give preliminaries on Sobolev spaces and fractional Laplacians. We also define the spaces of rough coefficients (Sobolev multipliers) and discuss some of the basic properties. In section 3 we prove theorem 1.1 in detail. Finally, in section 4 we prove theorem 1.2 but as the proofs of both theorems are very similar we do not repeat all identical steps and we keep our focus in the differences of the proofs.

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2. Preliminaries

In this section we recall some basic theory of Sobolev spaces, Fourier analysis and fractional Laplacians on $\mathbb{R}^n$. We also introduce the spaces of Sobolev multipliers and prove a few properties for them. Some auxiliary lemmas which are needed in the proofs of our main theorems are given as well. We follow the references [1, 15, 29, 28, 38, 41] (see also section 3 in [10]).

2.1. **Sobolev spaces.** The (inhomogeneous) fractional L^2-based Sobolev space of order $r \in \mathbb{R}$ is defined to be

$$H^r(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^2(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{H^r(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^2(\mathbb{R}^n)}.$$

Here $\hat{u} = \mathcal{F}(u)$ is the Fourier transform of a tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n)$, $\mathcal{F}^{-1}$ is the inverse Fourier transform and $\langle x\rangle = (1+|x|^2)^{1/2}$. We define the fractional Laplacian of order $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ as $(-\Delta)^s\varphi = \mathcal{F}^{-1}(|\cdot|^{2s}\hat{\varphi})$ where $\varphi \in \mathscr{S}(\mathbb{R}^n)$ is a Schwartz function. Then $(-\Delta)^s$ extends to a bounded operator $(-\Delta)^s \colon H^r(\mathbb{R}^n) \to H^{r-2s}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ by density of $\mathscr{S}(\mathbb{R}^n)$ in $H^r(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $F \subset \mathbb{R}^n$ a closed set. We define the following Sobolev spaces

$$\begin{aligned} H^r_F(\mathbb{R}^n) &= \{u \in H^r(\mathbb{R}^n) : \mathrm{spt}(u) \subset F\} \\ \widetilde{H}^r(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^r(\mathbb{R}^n) \\ H^r(\Omega) &= \{u|_\Omega : u \in H^r(\mathbb{R}^n)\} \\ H^r_0(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^r(\Omega). \end{aligned}$$

It follows that $\widetilde{H}^r(\Omega) \subset H^r_0(\Omega)$, $\widetilde{H}^r(\Omega) \subset H^r_{\overline{\Omega}}(\mathbb{R}^n)$, $(\widetilde{H}^r(\Omega))^* = H^{-r}(\Omega)$ and $(H^r(\Omega))^* = \widetilde{H}^{-r}(\Omega)$ for any open set Ω and $r \in \mathbb{R}$. If Ω is in addition a Lipschitz domain, then we have $\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ and $H^r_0(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n)$ when $r > -1/2$ such that $r \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \dots\}$.

More generally, let $1 \leq p \leq \infty$ and $r \in \mathbb{R}$. We define the Bessel potential space

$$H^{r,p}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) \in L^p(\mathbb{R}^n)\}$$

equipped with the norm

$$\|u\|_{H^{r,p}(\mathbb{R}^n)} = \left\|\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u})\right\|_{L^p(\mathbb{R}^n)}.$$

We also write $\mathcal{F}^{-1}(\langle\cdot\rangle^r \hat{u}) =: J^r u$ where the Fourier multiplier $J = (\mathrm{Id}-\Delta)^{1/2}$ is called the Bessel potential. We have the continuous inclusions $H^{r,p}(\mathbb{R}^n) \hookrightarrow H^{t,p}(\mathbb{R}^n)$ whenever $r \geq t$ [41]. By the

Mikhlin multiplier theorem one can show that $(-\Delta)^s\colon H^{r,p}(\mathbb{R}^n) \to H^{r-2s,p}(\mathbb{R}^n)$ is continuous whenever $s \geq 0$ and $1 < p < \infty$. The local version of the space $H^{r,p}(\mathbb{R}^n)$ is defined as earlier by the restrictions

$$H^{r,p}(\Omega) = \{u|_\Omega : u \in H^{r,p}(\mathbb{R}^n)\}$$

where $\Omega \subset \mathbb{R}^n$ is any open set. This space is equipped with the quotient norm

$$\|v\|_{H^{r,p}(\Omega)} = \inf\{\|w\|_{H^{r,p}(\mathbb{R}^n)} : w \in H^{r,p}(\mathbb{R}^n),\ w|_\Omega = v\}.$$

We have the continuous inclusions $H^{r,p}(\Omega) \hookrightarrow H^{t,p}(\Omega)$ whenever $r \geq t$ by the definition of the quotient norm.

We also define the spaces

$$\begin{aligned} H^{r,p}_F(\mathbb{R}^n) &= \{u \in H^{r,p}(\mathbb{R}^n) : \mathrm{spt}(u) \subset F\} \\ \widetilde{H}^{r,p}(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^{r,p}(\mathbb{R}^n) \\ H^{r,p}_0(\Omega) &= \text{ closure of } C_c^\infty(\Omega) \text{ in } H^{r,p}(\Omega) \end{aligned}$$

where $F \subset \mathbb{R}^n$ is a closed set. Note that $\widetilde{H}^{r,p}(\Omega) \subset H^{r,p}_0(\Omega)$ since the restriction map $|_\Omega\colon H^{r,p}(\mathbb{R}^n) \to H^{r,p}(\Omega)$ is by definition continuous. One can also see that $\widetilde{H}^{r,p}(\Omega) \subset H^{r,p}_{\overline{\Omega}}(\mathbb{R}^n)$. If Ω is a bounded C^∞-domain and $1 < p < \infty$, then we have [38, Theorem 1 in section 4.3.2]

$$\begin{aligned} \widetilde{H}^{r,p}(\Omega) &= H^{r,p}_{\overline{\Omega}}(\mathbb{R}^n), \quad s \in \mathbb{R} \\ H^{r,p}_0(\Omega) &= H^{r,p}(\Omega), \quad s \leq \frac{1}{p}. \end{aligned}$$

Some authors (especially in [7, 36]) use the notation $W^{r,p}(\Omega)$ for Bessel potential spaces. We have decided to use the notation $H^{r,p}(\Omega)$ so that these spaces are not confused with the Sobolev-Slobodeckij spaces which are in general different from the Bessel potential spaces [11].

The equation (1) we study is nonlocal. Instead of putting boundary conditions we impose exterior values for the equation. This can be done by saying that $u = f$ in Ω_e if $u - f \in \widetilde{H}^s(\Omega)$. Motivated by this we define the (abstract) trace space $X = H^r(\mathbb{R}^n)/\widetilde{H}^r(\Omega)$, i.e. functions in X are the same (have the same trace) if they agree in Ω_e. If Ω is a Lipschitz domain, then we have $X = H^r(\Omega_e)$ and $X^* = H^{-r}_{\overline{\Omega}_e}(\mathbb{R}^n)$.

2.2. **Properties of the fractional Laplacian.** The fractional Laplacian admits two important properties which we need in our proofs. The first one is unique continuation property (UCP) which is used in proving the Runge approximation property.

Lemma 2.1 (UCP)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $r \in \mathbb{R}$ and $u \in H^r(\mathbb{R}^n)$. If $(-\Delta)^s u|_V = 0$ and $u|_V = 0$ for some nonempty open set $V \subset \mathbb{R}^n$, then $u = 0$.*

Lemma 2.1 is proved in [10] for $s > 1$ by reducing the problem to the UCP result for $s \in (0, 1)$ in [15]. Note that such property is not true for local operators like the classical Laplacian $(-\Delta)$. The second property we need is the Poincaré inequality, which is used in showing that the fractional Calderón problem is well-posed.

Lemma 2.2 (Poincaré inequality)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $K \subset \mathbb{R}^n$ compact set and $u \in H^s_K(\mathbb{R}^n)$. There exists a constant $c = c(n, K, s) > 0$ such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq c \left\|(-\Delta)^{s/2} u\right\|_{L^2(\mathbb{R}^n)}.$$

Many different proofs for lemma 2.2 are given in [10]. We note that in the literature, the fractional Poincaré inequality is typically considered only when $s \in (0, 1)$.

Finally, we recall the fractional Leibniz rule, also known as the Kato-Ponce inequality. It is used to show the boundedness of the bilinear forms associated to the perturbed fractional Schrödinger equation in the case when the coefficients of the PDO have bounded fractional derivatives.

Lemma 2.3 (Kato-Ponce inequality)**.** *Let $s \geq 0$, $1 < r < \infty$, $1 < q_1 \leq \infty$ and $1 < p_2 \leq \infty$ such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. If $f \in L^{p_2}(\mathbb{R}^n)$, $J^s f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{q_1}(\mathbb{R}^n)$ and $J^s g \in L^{q_2}(\mathbb{R}^n)$, then $J^s(fg) \in L^r(\mathbb{R}^n)$ and*

$$\|J^s(fg)\|_{L^r(\mathbb{R}^n)} \leq C(\|J^s f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{q_1}(\mathbb{R}^n)} + \|f\|_{L^{p_2}(\mathbb{R}^n)} \|J^s g\|_{L^{q_2}(\mathbb{R}^n)})$$

where J^s is the Bessel potential of order s and $C = C(s, n, r, p_1, p_2, q_1, q_2)$.

The proof of lemma 2.3 can be found in [17] (see also [16, 18]).

2.3. **Spaces of rough coefficients.** Following [28, Ch. 3], we introduce the space of multipliers $M(H^r \to H^t)$ between pairs of Sobolev spaces. Here we are assuming that $r, t \in \mathbb{R}$. The coefficients of $P(x, D)$ in theorem 1.1 will be picked from such spaces of multipliers.

If $f \in \mathcal{D}'(\mathbb{R}^n)$ is a distribution, we say that $f \in M(H^r \to H^t)$ whenever the norm

$$\|f\|_{r,t} := \sup\{|\langle f, uv\rangle| \ ; \ u, v \in C_c^\infty(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} = \|v\|_{H^{-t}(\mathbb{R}^n)} = 1\}$$

is finite. Here $\langle \cdot, \cdot \rangle$ is the duality pairing. By $M_0(H^r \to H^t)$ we indicate the closure of $C_c^\infty(\mathbb{R}^n)$ in $M(H^r \to H^t) \subset \mathcal{D}'(\mathbb{R}^n)$. If $f \in M(H^r \to H^t)$ and $u, v \in C_c^\infty(\mathbb{R}^n)$ are both non-vanishing, we have the multiplier inequality

(3)

$$|\langle f, uv\rangle| = \left|\left\langle f, \frac{u}{\|u\|_{H^r(\mathbb{R}^n)}} \frac{v}{\|v\|_{H^{-t}(\mathbb{R}^n)}} \right\rangle\right| \|u\|_{H^r(\mathbb{R}^n)} \|v\|_{H^{-t}(\mathbb{R}^n)} \leq \|f\|_{r,t} \|u\|_{H^r(\mathbb{R}^n)} \|v\|_{H^{-t}(\mathbb{R}^n)}.$$

By density (2.3) can be extended to act over $u \in H^r(\mathbb{R}^n), v \in H^{-t}(\mathbb{R}^n)$. Moreover, each $f \in M(H^r \to H^t)$ gives rise to a multiplication map $m_f : H^r(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ defined as

$$\langle m_f(u), v\rangle := \langle f, uv\rangle \quad \text{for all} \quad u \in H^r(\mathbb{R}^n), v \in H^{-t}(\mathbb{R}^n).$$

We have as well the unique adjoint multiplication map $m_f^* : H^{-t}(\mathbb{R}^n) \to H^{-r}(\mathbb{R}^n)$ such that

$$\langle m_f^*(v), u\rangle := \langle f, uv\rangle \quad \text{for all} \quad u \in H^r(\mathbb{R}^n), v \in H^{-t}(\mathbb{R}^n).$$

Since one sees that the adjoint of m_f is m_f^*, the chosen notation is justified. For convenience, in the rest of the paper we will just write fu for both $m_f(u)$ and $m_f^*(u)$.

Remark 2.4. *The spaces of rough coefficients we use are generalizations of the ones considered in* [36]*. In fact, the space $Z^{-s}(\mathbb{R}^n)$ used there coincides with our space $M(H^s \to H^{-s})$.*

In the next lemma we state some elementary properties of the spaces of multipliers. Other interesting properties may be found in [28].

Lemma 2.5. *Let $\lambda, \mu \geq 0$ and $r, t \in \mathbb{R}$. Then*

- *(i) $M(H^r \to H^t) = M(H^{-t} \to H^{-r})$, and the norms associated to the two spaces also coincide.*
- *(ii) $M(H^{r-\lambda} \to H^{t+\mu}) \hookrightarrow M(H^r \to H^t)$ continuously.*
- *(iii) $M(H^r \to H^t) = \{0\}$ whenever $r < t$.*

Proof. (i) Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. Then by just using the definition we see that

$$\begin{aligned}\|f\|_{r,t} &= \sup\{|\langle f, uv\rangle| \ ; \ u, v \in C_c^\infty(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} = \|v\|_{H^{-t}(\mathbb{R}^n)} = 1\} \\ &= \sup\{|\langle f, vu\rangle| \ ; \ v, u \in C_c^\infty(\mathbb{R}^n), \|v\|_{H^{-t}(\mathbb{R}^n)} = \|u\|_{H^{-(-r)}(\mathbb{R}^n)} = 1\} = \|f\|_{-t,-r}.\end{aligned}$$

(ii) Observe that the given definition of $\|f\|_{r,t}$ is equivalent to the following:

$$\|f\|_{r,t} = \sup\{|\langle f, uv\rangle| \ ; \ u, v \in C_c^\infty(\mathbb{R}^n), \|u\|_{H^r(\mathbb{R}^n)} \leq 1, \|v\|_{H^{-t}(\mathbb{R}^n)} \leq 1\}.$$

Since $\lambda, \mu \geq 0$, we also have

$$\|u\|_{H^{r-\lambda}(\mathbb{R}^n)} \leq \|u\|_{H^r(\mathbb{R}^n)}, \quad \|v\|_{H^{-(t+\mu)}(\mathbb{R}^n)} \leq \|v\|_{H^{-t}(\mathbb{R}^n)}.$$

This implies $\|f\|_{r,t} \leq \|f\|_{r-\lambda,t+\mu}$, which in turn gives the wanted inclusion.

(iii) If $0 \le r < t$, then this was considered in [28, Ch. 3]. The proof given there recalls the easier one for Sobolev spaces ([28, Sec. 2.1]), which is based on the explicit computation of derivatives of aptly chosen exponential functions.

If $r < t \le 0$, then by point (i) we have $M(H^r \to H^t) = M(H^{-t} \to H^{-r})$. We need to show that $M(H^{-t} \to H^{-r}) = \{0\}$ whenever $0 \le -t < -r$. This reduces the problem back to the case of non-negative Sobolev scales.

If $r \le 0 < t$, then $-r \ge 0$. Now by point (ii), we have $M(H^r \to H^t) \subseteq M(H^{r+(-r)} \to H^t) = M(L^2 \to H^t)$. It is therefore enough to show that this last space is trivial, which again immediately follows from the case of non-negative Sobolev scales.

If $r < 0 \le t$, then the problem can be reduced again to the earlier cases. ☐

Remark 2.6. *We also have* $M_0(H^{r-\lambda} \to H^{t+\mu}) \subseteq M_0(H^r \to H^t)$ *whenever* $\lambda, \mu \ge 0$, *since the inclusion in (ii) is continuous.*

Remark 2.7. *In light of lemma 2.5 (ii) we are only interested in* $M(H^r \to H^t)$ *in the case* $r \ge t$, *the case* $r < t$ *being trivial. For our theorem 1.1, this translates into the condition* $m \le 2s$. *We decided not to consider the limit case* $m = 2s$ *in this work, as our machinery (in particular, the coercivity estimate* (4.1)*) breaks down in this case. However, it should be noted that since by assumption we have* $m \in \mathbb{Z}$ *and* $s \notin \mathbb{Z}$, *the equality* $m = 2s$ *can only arise if* m *is odd, which forces* $s = 1/2 + k$ *with* $k \in \mathbb{Z}$. *This case was excluded in* [7, 15] *as well.*

The next lemmas relate our spaces of multipliers with some special Bessel potential spaces. This is interesting since in the coming section 3 we will consider the inverse problem for coefficients coming from such spaces. We start with a general result.

Lemma 2.8. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $t \in \mathbb{R}$ *and* $r \in \mathbb{R}$ *be such that* $t > \max\{0, r\}$. *The following inclusions hold:*

(i) $\widetilde{H}^{r',\infty}(\Omega) \subset M_0(H^{-r} \to H^{-t})$ *whenever* $r' \ge \max\{0, r\}$.

(ii) $H_0^{r',\infty}(\Omega) \subset M_0(H^{-r} \to H^{-t})$ *whenever* $r' \ge \max\{0, r\}$ *such that* $r' \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$ *and* Ω *is a Lipschitz domain.*

(iii) $\widetilde{H}^{r'}(\Omega) \subset M_0(H^{-r} \to H^{-t})$ *whenever* $r' \ge t$ *and* $r' > n/2$. *The same holds true for* $H^{r'}_{\overline{\Omega}}(\mathbb{R}^n)$ *if* Ω *is a Lipschitz domain, and for* $H_0^{r'}(\Omega)$ *when* Ω *is a Lipschitz domain and* $r' \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$.

Proof. Throughout the proof we assume that $u, v \in C_c^\infty(\mathbb{R}^n)$ such that $\|u\|_{H^{-r}(\mathbb{R}^n)} = \|v\|_{H^t(\mathbb{R}^n)} = 1$. In parts (i) and (ii) we can assume that $r' < t$ since if $r' \ge t$, then we have the continuous inclusion $H^{r',\infty}(\Omega) \hookrightarrow H^{r'',\infty}(\Omega)$ where $\max\{0, r\} \le r'' < t$ (such r'' always exists since $t > \max\{0, r\}$).

(i) Let $f \in \widetilde{H}^{r',\infty}(\Omega)$. Now $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\|f_2\|_{H^{r',\infty}(\mathbb{R}^n)} \le \epsilon$. Then

$$\begin{aligned}|\langle f_2, uv\rangle| &\le \|f_2 v\|_{H^{r'}(\mathbb{R}^n)} \|u\|_{H^{-r'}(\mathbb{R}^n)} \le C \|f_2\|_{H^{r',\infty}(\mathbb{R}^n)} \|v\|_{H^{r'}(\mathbb{R}^n)} \|u\|_{H^{-r}(\mathbb{R}^n)} \\ &\le C\epsilon \|v\|_{H^t(\mathbb{R}^n)} = C\epsilon.\end{aligned}$$

Here we used the Kato-Ponce inequality (lemma 2.3)

$$\begin{aligned}\left\|J^{r'}(f_2 v)\right\|_{L^2(\mathbb{R}^n)} &\le C(\|f_2\|_{L^\infty(\mathbb{R}^n)} \left\|J^{r'} v\right\|_{L^2(\mathbb{R}^n)} + \left\|J^{r'} f_2\right\|_{L^\infty(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}) \\ &\le C \|f_2\|_{H^{r',\infty}(\mathbb{R}^n)} \|v\|_{H^{r'}(\mathbb{R}^n)}\end{aligned}$$

and the assumption $\max\{0, r\} \le r' < t$. Therefore $\|f - f_1\|_{-r,-t} = \|f_2\|_{-r,-t} \le C\epsilon$ which shows that $f \in M_0(H^{-r} \to H^{-t})$.

(ii) Let $f \in H_0^{r',\infty}(\Omega)$. Now $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\|f_2\|_{H^{r',\infty}(\Omega)} \le \epsilon$. By the definition of the quotient norm $\|\cdot\|_{H^{r',\infty}(\Omega)}$ we can take $F \in H^{r',\infty}(\mathbb{R}^n)$ such that $F|_\Omega = f_2$ and $\|F\|_{H^{r',\infty}(\mathbb{R}^n)} \le 2\|f_2\|_{H^{r',\infty}(\Omega)}$. The assumptions imply the duality $(H^{-r'}(\Omega))^* = H_0^{r'}(\Omega) \subset$

$H^{r'}(\Omega)$. Using the Kato-Ponce inequality for the extension F we obtain as in the proof of part (i) that

$$\left\| J^{r'}(Fv) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| F \right\|_{H^{r',\infty}(\mathbb{R}^n)} \left\| v \right\|_{H^{r'}(\mathbb{R}^n)} \leq 2C \left\| f_2 \right\|_{H^{r',\infty}(\Omega)} \left\| v \right\|_{H^t(\mathbb{R}^n)} \leq 2C\epsilon$$

and hence

$$\begin{aligned} |\langle f_2, uv \rangle| &\leq \left\| f_2 v \right\|_{(H^{-r'}(\Omega))^*} \left\| u \right\|_{H^{-r'}(\Omega)} \leq \left\| f_2 v \right\|_{H^{r'}(\Omega)} \left\| u \right\|_{H^{-r}(\mathbb{R}^n)} \\ &\leq \left\| J^{r'}(Fv) \right\|_{L^2(\mathbb{R}^n)} \leq 2C\epsilon. \end{aligned}$$

This shows that $f \in M_0(H^{-r} \to H^{-t})$.

(iii) Let $f \in \widetilde{H}^{r'}(\Omega)$. Now $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\left\| f_2 \right\|_{H^{r'}(\mathbb{R}^n)} \leq \epsilon$. Now [3, Theorem 7.3] implies the continuity of the multiplication $H^{r'}(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ when $r' \geq t$ and $r' > n/2$. We obtain

$$|\langle f_2, uv \rangle| \leq \left\| f_2 v \right\|_{H^t(\mathbb{R}^n)} \left\| u \right\|_{H^{-t}(\mathbb{R}^n)} \leq C \left\| f_2 \right\|_{H^{r'}(\mathbb{R}^n)} \left\| v \right\|_{H^t(\mathbb{R}^n)} \left\| u \right\|_{H^{-r}(\mathbb{R}^n)} \leq C\epsilon.$$

Hence $f \in M_0(H^{-r} \to H^{-t})$. If Ω is a Lipschitz domain, then $H^{r'}_{\overline{\Omega}}(\mathbb{R}^n) = \widetilde{H}^{r'}(\Omega)$. If in addition $r' \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$, we also have $H_0^{r'}(\Omega) = \widetilde{H}^{r'}(\Omega)$. □

Note that the assumptions in theorem 1.1 satisfy the conditions of the previous lemma since then $r = |\alpha| - s$ and $t = s$. The following lemma gives examples of spaces of lower order coefficients ($|\alpha| \leq s$).

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $t > 0$. The following inclusions hold:*

(i) $L^p(\Omega) \subset M_0(H^0 \to H^{-t})$ whenever $2 \leq p < \infty$ and $p > n/t$. Especially, if Ω is bounded, then $L^\infty(\Omega) \subset M_0(H^0 \to H^{-t})$.

(ii) $\widetilde{H}^r(\Omega) \subset M_0(H^0 \to H^{-t})$ whenever $r \geq 0$ and $r > n/2 - t$. The same holds true for $H^r_{\overline{\Omega}}(\mathbb{R}^n)$ if Ω is a Lipschitz domain, and for $H_0^r(\Omega)$ when Ω is Lipschitz domain and $r \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$.

Proof. Throughout the proof we assume that $u, v \in C_c^\infty(\mathbb{R}^n)$ such that $\left\| u \right\|_{L^2(\mathbb{R}^n)} = \left\| v \right\|_{H^t(\mathbb{R}^n)} = 1$.

(i) Let $f \in L^p(\Omega)$. By density of $C_c^\infty(\Omega)$ in $L^p(\Omega)$ we have $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\left\| \widetilde{f_2} \right\|_{L^p(\mathbb{R}^n)} \leq \epsilon$ where $\widetilde{f_2}$ is the zero extension of $f_2 \in L^p(\Omega)$. The assumptions on p imply the continuity of the multiplication $L^p(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ ([3, Theorem 7.3]) and we have

$$\left| \left\langle \widetilde{f_2}, uv \right\rangle \right| \leq \left\| \widetilde{f_2} v \right\|_{L^2(\mathbb{R}^n)} \left\| u \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \widetilde{f_2} \right\|_{L^p(\mathbb{R}^n)} \left\| v \right\|_{H^t(\mathbb{R}^n)} \leq C\epsilon.$$

This gives that $f \in M_0(H^0 \to H^{-t})$. If Ω is bounded, we have $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$, giving the second claim.

(ii) Let $f \in \widetilde{H}^r(\Omega)$. Now we have $f = f_1 + f_2$ where $f_1 \in C_c^\infty(\Omega)$ and $\left\| f_2 \right\|_{H^r(\mathbb{R}^n)} \leq \epsilon$. The assumptions on r imply that the multiplication $H^r(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous ([3, Theorem 7.3]). We obtain

$$|\langle f_2, uv \rangle| \leq \left\| f_2 v \right\|_{L^2(\mathbb{R}^n)} \left\| u \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| f_2 \right\|_{H^r(\mathbb{R}^n)} \left\| v \right\|_{H^t(\mathbb{R}^n)} \leq C\epsilon$$

and therefore $f \in M_0(H^0 \to H^{-t})$. The claims for $H^r_{\overline{\Omega}}(\mathbb{R}^n)$ and $H_0^r(\Omega)$ follow as in the proof of part (iii) of lemma 2.8 from the usual identifications for Lipschtiz domains. □

As mentioned above we put $t = s > 0$ in theorem 1.1 and the condition in lemma 2.9 is satisfied. Note that under the assumption $|\alpha| \leq s$ we have $M_0(H^0 \to H^{-s}) \subset M_0(H^{s-|\alpha|} \to H^{-s})$. Hence we can choose the lower order coefficients from a less regular space in theorem 1.1 (compare to lemma 2.8).

3. Main theorem for singular coefficients

In this section, to shorten the notation, we will write $\|\cdot\|_{H^s}$, $\|\cdot\|_{L^2}$ and so on for the global norms in $\mathbb{R}^n$ when the base set is not written explicitly.

3.1. Well-posedness of the inverse problem.

Consider the problem

$$(-\Delta)^s u + \sum_{|\alpha|\le m} a_\alpha(D^\alpha u) = F \quad \text{in } \Omega, \qquad u = f \quad \text{in } \Omega_e \tag{4}$$

and the corresponding *adjoint-problem*

$$(-\Delta)^s u^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha(a_\alpha u^*) = F^* \quad \text{in } \Omega, \qquad u^* = f^* \quad \text{in } \Omega_e. \tag{5}$$

Note that if $u, u^* \in H^s(\mathbb{R}^n)$ and $a_\alpha \in M(H^{s-|\alpha|} \to H^{-s}) = M(H^s \to H^{|\alpha|-s})$, then $a_\alpha(D^\alpha u) \in H^{-s}(\mathbb{R}^n)$ and $D^\alpha(a_\alpha u^*) \in H^{-s}(\mathbb{R}^n)$ matching with $(-\Delta)^s u, (-\Delta)^s u^* \in H^{-s}(\mathbb{R}^n)$.

The problems (3.1) and (3.1) are associated to the bilinear forms

$$B_P(v, w) := \langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w\rangle_{\mathbb{R}^n} + \sum_{|\alpha|\le m} \langle a_\alpha, (D^\alpha v) w\rangle_{\mathbb{R}^n} \tag{6}$$

and

$$B_P^*(v, w) := \langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w\rangle_{\mathbb{R}^n} + \sum_{|\alpha|\le m} \langle a_\alpha, v(D^\alpha w)\rangle_{\mathbb{R}^n}, \tag{7}$$

defined on $v, w \in C_c^\infty(\mathbb{R}^n)$. In the latter terms of the bilinear forms we have written the dual pairing as $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ since a_α is now a distribution in the whole space $\mathbb{R}^n$ in contrast to section 4 where a_α is an object defined only in Ω.

Remark 3.1. *Observe that B_P is not symmetric, which motivates the introduction of the bilinear form B_P^*. Moreover, one sees by simple inspection that $B_P(v, w) = B_P^*(w, v)$ for all $v, w \in C_c^\infty(\mathbb{R}^n)$. This identity holds for $v, w \in H^s(\mathbb{R}^n)$ as well by density, thanks to the following boundedness lemma.*

Lemma 3.2 (Boundedness of the bilinear forms)**.** *Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ such that $2s \ge m$, and let $a_\alpha \in M(H^{s-|\alpha|} \to H^{-s})$. Then B_P and B_P^* extend as bounded bilinear forms on $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$.*

Proof of lemma 3.2. We only prove the boundedness of B_P, as for B_P^* one can proceed in the same way. The proof is a simple calculation following from inequality (2.3). Let $u, v \in C_c^\infty(\mathbb{R}^n)$. We can then estimate that

$$\begin{aligned}
|B_P(v, w)| &\le |\langle (-\Delta)^{s/2} v, (-\Delta)^{s/2} w\rangle_{\mathbb{R}^n}| + \sum_{|\alpha|\le m} |\langle a_\alpha, D^\alpha v w\rangle_{\mathbb{R}^n}| \\
&\le \|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \sum_{|\alpha|\le m} \|a_\alpha\|_{s-|\alpha|,-s}\|D^\alpha v\|_{H^{s-|\alpha|}(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)} \\
&\le \left(1 + \sum_{|\alpha|\le m} \|a_\alpha\|_{s-|\alpha|,-s}\right)\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Now the claim follows from the density of $C_c^\infty(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n)$. □

Next we shall prove existence and uniqueness of solutions for the problems (3.1) and (3.1). To this end, we will use the following form of Young's inequality, which holds for all $a, b, \eta \in \mathbb{R}^+$

and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$:

$$ab \le \frac{(q\eta)^{-p/q}}{p} a^p + \eta b^q. \tag{8}$$

The validity of (3.1) is easily proved by choosing $a_1 = a(q\eta)^{-1/q}$ and $b_1 = b(q\eta)^{1/q}$ in Young's inequality $a_1 b_1 \le a_1^p/p + b_1^q/q$.

Lemma 3.3 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$, and let $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$. There exist a real number $\mu > 0$ and a countable set $\Sigma \subset (-\mu, \infty)$ of eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \to \infty$ such that if $\lambda \in \mathbb{R} \setminus \Sigma$, for any $f \in H^s(\mathbb{R}^n)$ and $F \in (\widetilde{H}^s(\Omega))^*$ there exists unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \widetilde{H}^s(\Omega)$ and*

$$B_P(u, v) - \lambda \langle u, v \rangle_\Omega = F(v) \quad \textit{for all} \quad v \in \widetilde{H}^s(\Omega).$$

One has the estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \le C \left(\|f\|_{H^s(\mathbb{R}^n)} + \|F\|_{(\widetilde{H}^s(\Omega))^*} \right).$$

The function u is also the unique $u \in H^s(\mathbb{R}^n)$ satisfying

$$r_\Omega \left((-\Delta)^s + \sum_{|\alpha| \le m} a_\alpha D^\alpha - \lambda \right) u = F$$

in the sense of distributions in Ω and $u - f \in \widetilde{H}^s(\Omega)$. Moreover, if (3.1) holds then $0 \notin \Sigma$.

Proof. Let $\tilde{u} := u - f$. The above problem is reduced to finding a unique $\tilde{u} \in \widetilde{H}^s(\Omega)$ such that $B_P(\tilde{u}, v) - \lambda \langle \tilde{u}, v \rangle_\Omega = \tilde{F}(v)$, where $\tilde{F} := F - B_P(f, \cdot) + \lambda \langle f, \cdot \rangle_\Omega$. Observe that the modified functional $\tilde{F}$ belongs to $(\widetilde{H}^s(\Omega))^*$ as well, since by lemma 3.2 we have for all $v \in \widetilde{H}^s(\Omega)$

$$|\tilde{F}(v)| \le |F(v)| + |B_P(f, v)| + |\lambda|\, |\langle f, v \rangle_\Omega| \le (\|F\|_{(\widetilde{H}^s(\Omega))^*} + (C + |\lambda|)\|f\|_{H^s(\mathbb{R}^n)})\|v\|_{H^s(\mathbb{R}^n)}.$$

Since $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$, for any $\epsilon > 0$ we can write $a_\alpha = a_{\alpha,1} + a_{\alpha,2}$, where $a_{\alpha,1} \in C_c^\infty(\mathbb{R}^n) \cap M(H^{s-|\alpha|} \to H^{-s})$ and $\|a_{\alpha,2}\|_{s-|\alpha|,-s} < \epsilon$. Thus by formula (2.3), the continuity of the multiplication $H^r(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n)$ for large enough $r \in \mathbb{R}$ (see [3, Theorem 7.3]) and the fact that $a_{\alpha,1} \in C_c^\infty(\mathbb{R}^n) \subset H^r(\mathbb{R}^n)$ for all $r \in \mathbb{R}$ we obtain

(9)

$$\begin{aligned}
|\langle a_\alpha, D^\alpha v w \rangle| &\le |\langle a_{\alpha,1}, D^\alpha v w \rangle| + |\langle a_{\alpha,2}, D^\alpha v w \rangle| \\
&\le \|a_{\alpha,1}\|_{H^r(\mathbb{R}^n)} \|D^\alpha v\|_{H^{-s}(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)} + \|a_{\alpha,2}\|_{s-|\alpha|,-s} \|D^\alpha v\|_{H^{s-|\alpha|}(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)} \\
&\le c\|w\|_{H^s(\mathbb{R}^n)} \left(\|a_{\alpha,1}\|_{H^r(\mathbb{R}^n)} \|v\|_{H^{|\alpha|-s}(\mathbb{R}^n)} + \epsilon \|v\|_{H^s(\mathbb{R}^n)} \right)
\end{aligned}$$

where $r \in \mathbb{R}$ is large enough ($r > \max\{s, n/2\}$ is sufficient). If $|\alpha| < s$, from formulas (3.1) and (3.1) with $p = q = 2$ we get directly

$$\begin{aligned}
|\langle a_\alpha, D^\alpha v v \rangle| &\le C \left(\|v\|_{H^s(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} + \epsilon \|v\|^2_{H^s(\mathbb{R}^n)} \right) \\
&\le C(\epsilon^{-1} \|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon \|v\|^2_{H^s(\mathbb{R}^n)})
\end{aligned} \tag{10}$$

for a constant C independent of v, w, ϵ. If instead $|\alpha| > s$ (observe that we can not have $|\alpha| = s$, because s can not be an integer), we use the interpolation inequality

$$\|v\|_{H^{|\alpha|-s}(\mathbb{R}^n)} \le C \|v\|^{1-(|\alpha|-s)/s}_{L^2(\mathbb{R}^n)} \|v\|^{(|\alpha|-s)/s}_{H^s(\mathbb{R}^n)} = C \|v\|^{2-|\alpha|/s}_{L^2(\mathbb{R}^n)} \|v\|^{|\alpha|/s-1}_{H^s(\mathbb{R}^n)}$$

in order to get

$$|\langle a_\alpha, D^\alpha v w \rangle| \le C \|w\|_{H^s(\mathbb{R}^n)} \left(\|v\|^{2-|\alpha|/s}_{L^2(\mathbb{R}^n)} \|v\|^{|\alpha|/s-1}_{H^s(\mathbb{R}^n)} + \epsilon \|v\|_{H^s(\mathbb{R}^n)} \right).$$

Then by formula (3.1) with

$$a = \|v\|_{L^2(\mathbb{R}^n)}^{2-|\alpha|/s}, \quad b = \|v\|_{H^s(\mathbb{R}^n)}^{|\alpha|/s-1}, \quad p = \frac{s}{2s-|\alpha|}, \quad q = \frac{s}{|\alpha|-s}, \quad \eta = \epsilon$$

we obtain

$$|\langle a_\alpha, D^\alpha vw\rangle| \leq C\|w\|_{H^s(\mathbb{R}^n)} \left(\epsilon^{\frac{s-|\alpha|}{2s-|\alpha|}}\|v\|_{L^2(\mathbb{R}^n)} + \epsilon\|v\|_{H^s(\mathbb{R}^n)}\right)$$

for a constant C independent of v, w, ϵ. Now we use formula (3.1) again, but this time we choose

$$a = \|v\|_{L^2(\mathbb{R}^n)}, \quad b = \|v\|_{H^s(\mathbb{R}^n)}, \quad q = p = 2, \quad \eta = \epsilon^{s/(2s-|\alpha|)}.$$

This leads to

$$\begin{aligned} (11) \qquad |\langle a_\alpha, D^\alpha vv\rangle| &\leq C\left(\epsilon^{\frac{s-|\alpha|}{2s-|\alpha|}}\|v\|_{L^2(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right) \\ &\leq C\left(\epsilon^{\frac{-|\alpha|}{2s-|\alpha|}}\|v\|^2_{L^2(\mathbb{R}^n)} + 2\epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right) \\ &\leq C\left(\epsilon^{\frac{-|\alpha|}{2s-|\alpha|}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right) \\ &\leq C'\left(\epsilon^{\frac{-m}{2s-m}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right) \end{aligned}$$

where C, C' are constants changing from line to line. Observe that C' can be taken independent of α. Eventually, using (3.1) and (3.1) we get

$$\begin{aligned} (12) \qquad B_P(v,v) &\geq \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - \sum_{|\alpha|\leq m} |\langle a_\alpha, D^\alpha vv\rangle| \\ &\geq \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - C'\left((\epsilon^{\frac{-m}{2s-m}} + \epsilon^{-1})\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right). \end{aligned}$$

By the higher order Poincaré inequality (lemma 2.2) (4.1) turns into

$$\begin{aligned} B_P(v,v) &\geq c\left(\|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)}\right) - C'\left((\epsilon^{\frac{-m}{2s-m}} + \epsilon^{-1})\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right) \\ &\geq c\|v\|^2_{H^s(\mathbb{R}^n)} - C'\left((\epsilon^{\frac{-m}{2s-m}} + \epsilon^{-1})\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right) \end{aligned}$$

for some constant $c = c(\Omega, n, s)$ changing from line to line. For ϵ small enough, this eventually gives the coercivity estimate

$$(13) \qquad B_P(v,v) \geq c_0\|v\|^2_{H^s(\mathbb{R}^n)} - \mu\|v\|^2_{L^2(\mathbb{R}^n)}$$

for some constants $c_0, \mu > 0$ independent of v.

As a consequence of the coercivity estimate, $B_P(\cdot,\cdot) + \mu\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ is an equivalent inner product on $\widetilde{H}^s(\Omega)$. Therefore, by the Riesz representation theorem there exists a bounded linear operator $G_\mu : (\widetilde{H}^s(\Omega))^* \to \widetilde{H}^s(\Omega)$ associating each functional in $(\widetilde{H}^s(\Omega))^*$ to its unique representative in the inner product $B_P(\cdot,\cdot) + \mu\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n)}$ on $\widetilde{H}^s(\Omega)$. Thus $\tilde{u} := G_\mu\tilde{F}$ verifies

$$B_P(\tilde{u}, v) + \mu\langle\tilde{u}, v\rangle_{L^2(\mathbb{R}^n)} = \tilde{F}(v) \quad \text{for all} \quad v \in \widetilde{H}^s(\Omega)$$

and it is the required unique solution $\tilde{u} \in \widetilde{H}^s(\Omega)$. Moreover, G_μ induces a compact, self-adjoint and positive operator $\tilde{G}_\mu : L^2(\Omega) \to L^2(\Omega)$ by the compact Sobolev embedding theorem. The remaining claims follow from the spectral theorem for $\tilde{G}_\mu$ and from the Fredholm alternative as in [15]. □

By the above lemma 3.3, both problems (3.1) and (3.1) have a countable set of Dirichlet eigenvalues. Throughout the paper we will assume that the coefficients a_α are such that 0 is not a Dirichlet eigenvalue for either of the problems. That is, we assume that

$$(14) \qquad \begin{cases} \text{if } u \in H^s(\mathbb{R}^n) \text{ solves } (-\Delta)^s u + \sum_{|\alpha|\leq m} a_\alpha D^\alpha u = 0 \text{ in } \Omega \text{ and } u|_{\Omega_e} = 0, \\ \qquad\qquad\qquad \text{then } u \equiv 0 \end{cases}$$

and

$$(15) \quad \begin{cases} \text{if } u^* \in H^s(\mathbb{R}^n) \text{ solves } (-\Delta)^s u^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha u^*) = 0 \text{ in } \Omega \text{ and } u^*|_{\Omega_e} = 0, \\ \qquad\qquad\qquad \text{then } u^* \equiv 0. \end{cases}$$

With this in mind, we shall define the DN maps. Consider the abstract trace space $X := H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ equipped with the quotient norm

$$\|[f]\|_X := \inf_{\phi \in \widetilde{H}^s(\Omega)} \|f - \phi\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s(\mathbb{R}^n)$$

and its dual space X^*. We use these in order to define the DN maps associated to the problems (3.1) and (3.1), which we study in the following lemma.

Lemma 3.4 (DN maps)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ such that $2s > m$, and let $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$. There exist two continuous linear maps*

$$\Lambda_P : X \to X^* \quad \textit{defined by} \quad \langle \Lambda_P[f], [g] \rangle := B_P(u_f, g)$$

and

$$\Lambda_P^* : X \to X^* \quad \textit{defined by} \quad \langle \Lambda_P^*[f], [g] \rangle := B_P^*(u_f^*, g)$$

where u_f, u_f^* are the unique solutions to the equations

$$(-\Delta)^s u + \sum_{|\alpha|\leq m} a_\alpha D^\alpha u = 0 \quad \textit{in } \Omega, \quad u - f \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha u^*) = 0 \quad \textit{in } \Omega, \quad u^* - f \in \widetilde{H}^s(\Omega)$$

with $f, g \in H^s(\mathbb{R}^n)$. Moreover, the identity $\langle \Lambda_P[f], [g] \rangle = \langle [f], \Lambda_P^*[g] \rangle$ holds.

Proof. We show well-definedness and continuity only for Λ_P, the proof being similar for Λ_P^*. We note that such unique solutions exist by lemma 3.3.

If $\phi \in \widetilde{H}^s(\Omega)$, then $u_f|_{\Omega_e} = f = u_{f+\phi}|_{\Omega_e}$, and also u_f, $u_{f+\phi}$ both solve $(-\Delta)^s u + Pu = 0$ in Ω. By unicity of solutions, we must then have that u_f and $u_{f+\phi}$ coincide. On the other hand, if $\psi \in \widetilde{H}^s(\Omega)$, then $\psi|_{\Omega_e} = 0$. These two facts imply the well-definedness of Λ_P, since

$$B_P(u_{f+\phi}, g + \psi) = B_P(u_f, g) + B_P(u_f, \psi) = B_P(u_f, g).$$

The continuity of Λ_P is an easy consequence of lemma 3.2 and the estimate in lemma 3.3. If $f, g \in H^s(\mathbb{R}^n)$ and $\phi, \psi \in \widetilde{H}^s(\Omega)$, then

$$|\langle \Lambda_P[f], [g] \rangle| = |B_P(u_{f-\phi}, g - \psi)| \leq C \|u_{f-\phi}\|_{H^s} \|g - \psi\|_{H^s} \leq C \|f - \phi\|_{H^s} \|g - \psi\|_{H^s}.$$

By taking the infimum on both sides with respect to ϕ and ψ, we end up with

$$|\langle \Lambda_P[f], [g] \rangle| \leq C \inf_{\phi \in \widetilde{H}^s(\Omega)} \|f - \phi\|_{H^s} \inf_{\psi \in \widetilde{H}^s(\Omega)} \|g - \psi\|_{H^s} = C \|[f]\|_X \|[g]\|_X.$$

The well-posedness result proved above implies that for all $f, g \in H^s(\mathbb{R}^n)$ we have $\langle \Lambda_P[f], [g] \rangle = B_P(u_f, e_g)$, where e_g is a generic extension of $g|_{\Omega_e}$ from Ω_e to $\mathbb{R}^n$. In particular, $\langle \Lambda_P[f], [g] \rangle = B_P(u_f, u_g^*)$. By lemma 3.2 this leads to

$$\langle \Lambda_P[f], [g] \rangle = B_P(u_f, u_g^*) = B_P^*(u_g^*, u_f) = \langle \Lambda_P^*[g], [f] \rangle$$

which conlcudes the proof. □

Remark 3.5. *We should observe at this point that a priori Λ_P^* has no reason to be the adjoint of Λ_P, as the symbols would suggest. However, the identity we proved in lemma 3.4 shows that this is in fact true, and thus there is no abuse of notation.*

3.2. **Proof of injectivity.** The proof of injectivity is based on an Alessandrini identity and the Runge approximation property for our operator, following the scheme developed in [15].

Lemma 3.6 (Alessandrini identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ such that $2s > m$. For $j = 1, 2$, let $a_{j,\alpha} \in M_0(H^{s-|\alpha|} \to H^{-s})$. For any $f_1, f_2 \in H^s(\mathbb{R}^n)$, let $u_1, u_2^* \in H^s(\mathbb{R}^n)$ respectively solve*

$$(-\Delta)^s u_1 + \sum_{|\alpha|\le m} a_{1,\alpha} D^\alpha u_1 = 0 \quad \text{in } \Omega, \quad u_1 - f_1 \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_2^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha} u_2^*) = 0 \quad \text{in } \Omega, \quad u_2^* - f_2 \in \widetilde{H}^s(\Omega).$$

Then we have the integral identity

$$\langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2] \rangle = \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_1 u_2^* \rangle.$$

Proof. The proof is a simple computation following from lemma 3.4

$$\begin{aligned} \langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2] \rangle &= \langle \Lambda_{P_1}[f_1], [f_2] \rangle - \langle \Lambda_{P_2}[f_1], [f_2] \rangle = \langle \Lambda_{P_1}[f_1], [f_2] \rangle - \langle [f_1], \Lambda_{P_2}^*[f_2] \rangle \\ &= B_{P_1}(u_1, u_2^*) - B_{P_2}^*(u_2^*, u_1) = \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_1 u_2^* \rangle. \end{aligned}$$ □

Lemma 3.7 (Runge approximation property)**.** *Let $\Omega, W \subset \mathbb{R}^n$ respectively be a bounded open set and a non-empty open set such that $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $m \in \mathbb{N}$ be such that $2s > m$, and let $a_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$. Moreover, let $\mathcal{R} := \{ u_f - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$ where u_f solves*

$$(-\Delta)^s u_f + \sum_{|\alpha|\le m} a_\alpha D^\alpha u_f = 0 \quad \text{in } \Omega, \quad u_f - f \in \widetilde{H}^s(\Omega)$$

and $\mathcal{R}^* := \{ u_f^* - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$ where u_f^* solves

$$(-\Delta)^s u_f^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_\alpha u_f^*) = 0 \quad \text{in } \Omega, \quad u_f^* - f \in \widetilde{H}^s(\Omega).$$

Then $\mathcal{R}$ and $\mathcal{R}^*$ are dense in $\widetilde{H}^s(\Omega)$.

Proof. The proofs of the two statements are similar, so we show only the density of $\mathcal{R}$ in $\widetilde{H}^s(\Omega)$. By the Hahn-Banach theorem, it is enough to prove that any functional F acting on $\widetilde{H}^s(\Omega)$ that vanishes on $\mathcal{R}$ must be identically 0. Thus, let $F \in (\widetilde{H}^s(\Omega))^*$ and assume $F(u_f - f) = 0$ for all $f \in C_c^\infty(W)$. Let ϕ be the unique solution of

$$(-\Delta)^s \phi + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_\alpha \phi) = -F \quad \text{in } \Omega, \quad \phi \in \widetilde{H}^s(\Omega). \tag{16}$$

In other words, ϕ is the unique function in $\widetilde{H}^s(\Omega)$ such that $B_P^*(\phi, w) = -F(w)$ for all $w \in \widetilde{H}^s(\Omega)$. Then we can compute

$$\begin{aligned} 0 = F(u_f - f) &= -B_P^*(\phi, u_f - f) = B_P^*(\phi, f) \\ &= \langle (-\Delta)^{s/2} f, (-\Delta)^{s/2} \phi \rangle + \sum_{|\alpha|\le m} \langle a_\alpha, D^\alpha f \phi \rangle \\ &= \langle f, (-\Delta)^s \phi \rangle. \end{aligned} \tag{17}$$

On the first line of (3.2) we used that $\phi \in \widetilde{H}^s(\Omega)$ and u_f solves the equation in Ω, and on the last line we used the support condition for f. By the arbitrariety of $f \in C_c^\infty(W)$ we have obtained that $(-\Delta)^s \phi = 0$ in W, and on the same set we also have $\phi = 0$. Using the unique continuation result for the higher order fractional Laplacian given in lemma 2.1 we deduce $\phi \equiv 0$ on all of $\mathbb{R}^n$. The vanishing of the functional F now follows easily from the definition of ϕ. $\square$

Remark 3.8. *We remark that using the same proof one can show that $r_\Omega \mathcal{R} \subset L^2(\Omega)$ and $r_\Omega \mathcal{R}^* \subset L^2(\Omega)$ are dense in $L^2(\Omega)$, where r_Ω is the restriction to Ω. If $F \in L^2(\Omega)$, then F induces an element in $(\widetilde{H}^s(\Omega))^*$ via the inner product $F(w) := \langle F, r_\Omega w \rangle_\Omega$, where $w \in \widetilde{H}^s(\Omega)$. Hence one can choose the solution ϕ in equation (3.2) with F as a source term and complete the proof as in equation (3.2) showing that $(r_\Omega \mathcal{R})^\perp = \{0\}$ in $L^2(\Omega)$ (similarly $(r_\Omega \mathcal{R}^*)^\perp = \{0\}$).*

We are ready to prove the main result of the paper.

Proof of theorem 1.1. **Step 1.** Since one can always shrink the sets W_1 and W_2 if necessary, we can assume without loss of generality that $\overline{W_1} \cap \overline{W_2} = \emptyset$. Let $v_1, v_2 \in C_c^\infty(\Omega)$. By the Runge approximation property proved in lemma 3.7 we can find two sequences of functions $\{f_{j,k}\}_{k\in\mathbb{N}} \subset C_c^\infty(W_j)$, $j = 1, 2$, such that

$$u_{1,k} = f_{1,k} + v_1 + r_{1,k}, \quad u_{2,k}^* = f_{2,k} + v_2 + r_{2,k}$$

where $u_{1,k}$, $u_{2,k}^* \in \widetilde{H}^s(\Omega)$ respectively solve

$$(-\Delta)^s u_{1,k} + \sum_{|\alpha|\leq m} a_{1,\alpha} D^\alpha u_{1,k} = 0 \quad \text{in } \Omega, \quad u_{1,k} - f_{1,k} \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_{2,k}^* + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha} u_{2,k}^*) = 0 \quad \text{in } \Omega, \quad u_{2,k}^* - f_{2,k} \in \widetilde{H}^s(\Omega)$$

and $r_{1,k}$, $r_{2,k} \to 0$ in $\widetilde{H}^s(\Omega)$ as $k \to \infty$. By the assumption on the DN maps and the Alessandrini identity from lemma 3.6 we have

$$\begin{aligned} (18) \qquad 0 = \langle (\Lambda_{P_1} - \Lambda_{P_2})[f_{1,k}], [f_{2,k}] \rangle &= \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha u_{1,k} u_{2,k}^* \rangle \\ = \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha r_{1,k} u_{2,k}^* \rangle &+ \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 r_{2,k} \rangle \\ + \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 v_2 \rangle. \end{aligned}$$

However, for the first two terms on the right hand side of (3.2) we can deduce

$$\begin{aligned} \left| \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha r_{1,k} u_{2,k}^* \rangle \right| &\leq \sum_{|\alpha|\leq m} |\langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha r_{1,k} u_{2,k}^* \rangle| \\ &\leq C \|u_{2,k}^*\|_{H^s} \|r_{1,k}\|_{H^s} \sum_{|\alpha|\leq m} \|a_{1,\alpha} - a_{2,\alpha}\|_{s-|\alpha|,-s} \to 0 \end{aligned}$$

and

$$\begin{aligned} | \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 r_{2,k} \rangle | &\leq \sum_{|\alpha|\leq m} |\langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 r_{2,k} \rangle| \\ &\leq C \|r_{2,k}\|_{H^s} \|v_1\|_{H^s} \sum_{|\alpha|\leq m} \|a_{1,\alpha} - a_{2,\alpha}\|_{s-|\alpha|,-s} \to 0 \end{aligned}$$

as $k \to \infty$. Thus by taking the limit in formula (3.2) we obtain

$$(19) \qquad \sum_{|\alpha|\leq m} \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha v_1 v_2 \rangle = 0 \quad \text{for all} \quad v_1, v_2 \in C_c^\infty(\Omega).$$

Step 2. Assume that we have $a_{1,\alpha}|_\Omega = a_{2,\alpha}|_\Omega$ for all α such that $|\alpha| < N$ for some $N \in \mathbb{N}$. We show that the equality of the coefficients also holds for α for which $|\alpha| = N$ and this will prove the theorem by the principle of complete induction.

To this end, consider $v_2 \in C_c^\infty(\Omega)$, and then take $v_1 \in C_c^\infty(\Omega)$ such that $v_1(x) = x^\alpha$ on $\mathrm{supp}(v_2) \Subset \Omega$. Recall that since $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ is a multi-index and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, the symbol x^α is intended to mean $x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$. With this choice of v_1, v_2, equation (3.2) becomes

$$
\begin{aligned}
(20) \qquad 0 &= \sum_{|\beta|\leq m} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta v_1 v_2 \rangle = \sum_{N\leq|\beta|\leq m} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta (x^\alpha) v_2 \rangle \\
&= \sum_{N<|\beta|\leq m} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta (x^\alpha) v_2 \rangle + \sum_{|\beta|=N,\, \beta\neq\alpha} \langle (a_{1,\beta} - a_{2,\beta}), D^\beta (x^\alpha) v_2 \rangle \\
&\quad + \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha (x^\alpha) v_2 \rangle.
\end{aligned}
$$

If $|\beta| > N = |\alpha|$, then there must exist $k \in \{1, 2, ..., n\}$ such that $\beta_k > \alpha_k$. This is true also if $|\beta| = N$ with $\beta \neq \alpha$. In both cases we can compute

$$
D^\beta(x^\alpha) = (\partial_{x_1}^{\beta_1} x_1^{\alpha_1})\ (\partial_{x_2}^{\beta_2} x_2^{\alpha_2})\ ...\ (\partial_{x_n}^{\beta_n} x_n^{\alpha_n}) = 0
$$

because $\partial_{x_k}^{\beta_k} x_k^{\alpha_k} = 0$. Therefore formula (3.2) becomes

$$
0 = \langle (a_{1,\alpha} - a_{2,\alpha}), D^\alpha (x^\alpha) v_2 \rangle_{\mathbb{R}^n} = \alpha! \langle a_{1,\alpha} - a_{2,\alpha}, v_2 \rangle_{\mathbb{R}^n}
$$

which by the arbitrariety of $v_2 \in C_c^\infty(\Omega)$ implies $a_{1,\alpha}|_\Omega = a_{2,\alpha}|_\Omega$ also for α for which $|\alpha| = N$.

Step 3. We have proved that $a_{1,\alpha}|_\Omega = a_{2,\alpha}|_\Omega$ for all α of the order $|\alpha| \leq m$. Since this entails $P_1|_\Omega = P_2|_\Omega$, the proof is complete. □

4. Main theorem for bounded coefficients

We shall now study the case when the coefficients of PDOs are from the bounded spaces $H^{r_\alpha,\infty}(\Omega)$. It should be noted, however, that most of the considerations of the previous section still apply identically.

4.1. **Well-posedness of the inverse problem.** We shall define the bilinear forms for the problems (3.1) and (3.1) respectively by (3.1) and (3.1), just as in the case of singular coefficients. These will turn out to be bounded in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ as well, but the proof we give of this fact is *a fortiori* different. Since now we assume that $a_\alpha \in H^{r_\alpha,\infty}(\Omega) \subset L^\infty(\Omega)$ for $r_\alpha \geq 0$, the duality pairing $\langle a_\alpha, D^\alpha v w \rangle$ becomes an inner product over Ω and we write $\langle a_\alpha(x) D^\alpha v, w \rangle_\Omega$ to emphasize that the coefficients $a_\alpha = a_\alpha(x)$ are now functions defined in Ω.

Lemma 4.1 (Boundedness of the bilinear forms)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in* (1.2)*. Then B_P and B_P^* extend as bounded bilinear forms on $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$.*

Remark 4.2. *Since $s \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $|\alpha| \leq m < 2s$, we also have that* $\max(0, |\alpha| - s) \leq r_\alpha < s$ *for $\delta > 0$ small (see equation* (1.2)*).*

Proof of lemma 4.1. We only prove the boundedness of B_P, as for B_P^* one can proceed in the same way. If $v, w \in C_c^\infty(\mathbb{R}^n)$, then

$$
|\langle a_\alpha(x) D^\alpha v, w \rangle_\Omega| = \left| \int_\Omega a_\alpha w D^\alpha v \, dx \right| \leq \|a_\alpha w\|_{(H^{-r_\alpha}(\Omega))^*} \|D^\alpha v\|_{H^{-r_\alpha}(\Omega)}.
$$

Since Ω is a Lipschitz domain and $r_\alpha > -1/2$, $r_\alpha \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2} ...\}$, we have $(H^{-r_\alpha}(\Omega))^* = H_0^{r_\alpha}(\Omega) \subset H^{r_\alpha}(\Omega)$. Therefore

$$
\begin{aligned}
(21) \quad |\langle a_\alpha(x) D^\alpha v, w \rangle_\Omega| &\leq C \|a_\alpha w\|_{H^{r_\alpha}(\Omega)} \|D^\alpha v\|_{H^{-r_\alpha}(\Omega)} \leq C \|A_\alpha w\|_{H^{r_\alpha}(\mathbb{R}^n)} \|D^\alpha v\|_{H^{-r_\alpha}(\Omega)} \\
&\leq C \|J^{r_\alpha}(A_\alpha w)\|_{L^2(\mathbb{R}^n)} \|v\|_{H^{|\alpha|-r_\alpha}(\Omega)}
\end{aligned}
$$

where $J = (\mathrm{Id} - \Delta)^{1/2}$ is the Bessel potential and A_α is an extension of a_α from Ω to $\mathbb{R}^n$ such that $A_\alpha|_\Omega = a_\alpha$ and $\|A_\alpha\|_{H^{r_\alpha,\infty}(\mathbb{R}^n)} \leq 2\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}$. Since $r_\alpha \geq 0$, we may estimate the last term of (4.1) by the Kato-Ponce inequality given in lemma 2.3

$$\begin{aligned}\|J^{r_\alpha}(A_\alpha w)\|_{L^2(\mathbb{R}^n)} &\leq C\left(\|A_\alpha\|_{L^\infty(\mathbb{R}^n)}\|J^{r_\alpha}w\|_{L^2(\mathbb{R}^n)} + \|J^{r_\alpha}A_\alpha\|_{L^\infty(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}\right)\\ &\leq C\|A_\alpha\|_{H^{r_\alpha,\infty}(\mathbb{R}^n)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)} \leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)}.\end{aligned}$$

Substituting this into (4.1) gives

$$\begin{aligned}|\langle a_\alpha(x)D^\alpha v, w\rangle_\Omega| &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)}\|v\|_{H^{|\alpha|-r_\alpha}(\Omega)}\\ &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}\end{aligned} \tag{22}$$

given that both $r_\alpha < s$ and $|\alpha| - r_\alpha \leq s$ hold by remark 4.2. Eventually we obtain

$$\begin{aligned}|B_P(v,w)| &\leq |\langle(-\Delta)^{s/2}v, (-\Delta)^{s/2}w\rangle_{\mathbb{R}^n}| + \sum_{|\alpha|\leq m}|\langle a_\alpha D^\alpha v, w\rangle_{\mathbb{R}^n}|\\ &\leq \|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \sum_{|\alpha|\leq m} C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}\\ &\leq C\|w\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

□

Next we shall prove existence and uniqueness of solutions for the problems (3.1) and (3.1). The reasoning is similar to the one for the proof of lemma 3.3, but the details of the computations are quite different.

Lemma 4.3 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in (1.2). There exist a real number $\mu > 0$ and a countable set $\Sigma \subset (-\mu, \infty)$ of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ such that if $\lambda \in \mathbb{R} \setminus \Sigma$, for any $f \in H^s(\mathbb{R}^n)$ and $F \in (\widetilde{H}^s(\Omega))^*$ there exists a unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \widetilde{H}^s(\Omega)$ and*

$$B_P(u,v) - \lambda\langle u, v\rangle_\Omega = F(v) \quad \textit{for all} \quad v \in \widetilde{H}^s(\Omega).$$

One has the estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\left(\|f\|_{H^s(\mathbb{R}^n)} + \|F\|_{(\widetilde{H}^s(\Omega))^*}\right).$$

The function u is also the unique $u \in H^s(\mathbb{R}^n)$ satisfying

$$r_\Omega\left((-\Delta)^s + \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha - \lambda\right)u = F$$

in the sense of distributions in Ω and $u - f \in \widetilde{H}^s(\Omega)$. Moreover, if (3.1) holds then $0 \notin \Sigma$.

Proof. Again it is enough to find unique $\tilde{u} \in \widetilde{H}^s(\Omega)$ such that $B_P(\tilde{u}, v) - \lambda\langle\tilde{u}, v\rangle_\Omega = \tilde{F}(v)$, where $\tilde{F} := F - B_P(f, \cdot) + \lambda\langle f, \cdot\rangle_\Omega$. Consider $v, w \in C_c^\infty(\Omega)$ and $r_\alpha \neq 0$. Since $0 < r_\alpha < s$, the interpolation inequality

$$\|w\|_{H^{r_\alpha}(\mathbb{R}^n)} \leq C\|w\|_{L^2(\mathbb{R}^n)}^{1-r_\alpha/s}\|w\|_{H^s(\mathbb{R}^n)}^{r_\alpha/s}$$

holds. Using this and formula (4.1) we get, for a constant $C = C(\Omega, n, s, r_\alpha)$ which may change from line to line,

$$\begin{aligned}|\langle a_\alpha(x)D^\alpha v, w\rangle_\Omega| &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^{r_\alpha}(\mathbb{R}^n)}\\ &\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}^{1-r_\alpha/s}\|w\|_{H^s(\mathbb{R}^n)}^{r_\alpha/s}\\ &\leq \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\left(C\epsilon^{r_\alpha/(r_\alpha-s)}\|w\|_{L^2(\mathbb{R}^n)} + \epsilon\|w\|_{H^s(\mathbb{R}^n)}\right).\end{aligned} \tag{23}$$

In the last step of (4.1) we used formula (3.1) with

$$q = \frac{s}{r_\alpha}, \quad p = \frac{s}{s - r_\alpha}, \quad b = \|w\|_{H^s(\mathbb{R}^n)}^{r_\alpha/s}, \quad a = C\|w\|_{L^2(\mathbb{R}^n)}^{1-r_\alpha/s}, \quad \eta = \epsilon.$$

If instead $r_\alpha = 0$, just by formula (4.1) we already have

$$|\langle a_\alpha(x) D^\alpha v, w\rangle_\Omega| \leq C\|a_\alpha\|_{L^\infty(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{L^2(\mathbb{R}^n)}.$$

Moreover, the two estimates above also hold for $v, w \in \widetilde{H}^s(\Omega)$ by the density of $C_c^\infty(\Omega)$ in $\widetilde{H}^s(\Omega)$. Now we use formula (3.1) again, but this time we choose

$$q = p = 2, \quad b = \|v\|_{H^s(\mathbb{R}^n)}, \quad a = \|v\|_{L^2(\mathbb{R}^n)}, \quad \eta = \epsilon^{s/(s-r_\alpha)}.$$

This leads to

$$\begin{aligned}
|\langle a_\alpha(x) D^\alpha v, v\rangle_\Omega| &\leq \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\|v\|_{H^s(\mathbb{R}^n)}\left(C\epsilon^{r_\alpha/(r_\alpha-s)}\|v\|_{L^2(\mathbb{R}^n)} + \epsilon\|v\|_{H^s(\mathbb{R}^n)}\right)\\
&= \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\left(C\epsilon^{r_\alpha/(r_\alpha-s)}\|v\|_{L^2(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\leq \|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\left(C\epsilon^{\frac{r_\alpha+s}{r_\alpha-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon(C+1)\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\leq C\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\left(\epsilon^{\frac{r_\alpha+s}{r_\alpha-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\leq C'\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)
\end{aligned}$$

where $C = C(\Omega, n, s, r_\alpha)$ and $C' = C'(\Omega, n, s)$ are constants changing from line to line and $M \in [0, s)$ is defined by $M := \max_{|\alpha|\leq m} r_\alpha$. Eventually

$$\begin{aligned}
(24)\quad B_P(v,v) &\geq \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - \sum_{|\alpha|\leq m}|\langle a_\alpha(x)D^\alpha v, v\rangle_\Omega|\\
&\geq \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - C'\left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\sum_{|\alpha|\leq m}\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}\\
&= \|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} - C'C''\left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)
\end{aligned}$$

where $C'' := \sum_{|\alpha|\leq m}\|a_\alpha\|_{H^{r_\alpha,\infty}(\Omega)}$ is a constant independent of ϵ and v. By the higher order Poincaré inequality (lemma 2.2) (4.1) turns into

$$\begin{aligned}
B_P(v,v) &\geq c\left(\|(-\Delta)^{s/2}v\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)}\right) - C'C''\left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)\\
&\geq c\|v\|^2_{H^s(\mathbb{R}^n)} - C'C''\left(\epsilon^{\frac{M+s}{M-s}}\|v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\|v\|^2_{H^s(\mathbb{R}^n)}\right)
\end{aligned}$$

for some constant $c = c(\Omega, n, s)$ changing from line to line. For ϵ small enough (notice that $M - s < 0$), this eventually gives the coercivity estimate

$$(25) \qquad B_P(v,v) \geq c_0\|v\|^2_{H^s(\mathbb{R}^n)} - \mu\|v\|^2_{L^2(\mathbb{R}^n)}$$

for some constants $c_0, \mu > 0$ independent of v. The proof is now concluded as in lemma 3.3. □

Assuming as in Section 3 that both (3.1) and (3.1) hold, by means of the above lemma 4.3 we can define the DN-maps Λ_P, Λ_P^* just as in lemma 3.4. We also arrive at the same Alessandrini identity and Runge approximation property which we get in lemmas 3.6 and 3.7.

Lemma 4.4 (DN maps)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$. Let $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, with r_α defined as in* (1.2)*. There exist two continuous linear maps*

$$\Lambda_P : X \to X^* \quad \textit{defined by} \quad \langle\Lambda_P[f],[g]\rangle := B_P(u_f, g)$$

and

$$\Lambda_P^* : X \to X^* \quad \textit{defined by} \quad \langle\Lambda_P^*[f],[g]\rangle := B_P^*(u_f^*, g)$$

where u_f, u_f^* are the unique solutions to the equations

$$(-\Delta)^s u + \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha u = 0 \quad \textit{in } \Omega, \quad u - f \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) u^*) = 0 \quad in\ \ \Omega, \quad u^* - f \in \widetilde{H}^s(\Omega)$$

with $f, g \in H^s(\mathbb{R}^n)$. *Moreover, the identity* $\langle \Lambda_P[f], [g]\rangle = \langle [f], \Lambda_P^*[g]\rangle$ *holds.*

Lemma 4.5 (Alessandrini identity)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain and* $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ *such that* $2s > m$. *Let* $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, *with* r_α *defined as in* (1.2). *For any* $f_1, f_2 \in H^s(\mathbb{R}^n)$, *let* $u_1, u_2^* \in H^s(\mathbb{R}^n)$ *respectively solve*

$$(-\Delta)^s u_1 + \sum_{|\alpha|\le m} a_{1,\alpha}(x) D^\alpha u_1 = 0 \quad in\ \ \Omega, \quad u_1 - f_1 \in \widetilde{H}^s(\Omega)$$

and

$$(-\Delta)^s u_2^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_{2,\alpha}(x) u_2^*) = 0 \quad in\ \ \Omega, \quad u_2^* - f_2 \in \widetilde{H}^s(\Omega).$$

Then we have the integral identity

$$\langle (\Lambda_{P_1} - \Lambda_{P_2})[f_1], [f_2]\rangle = \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha u_1, u_2^*\rangle_\Omega.$$

Lemma 4.6 (Runge approximation property)**.** *Let* $\Omega, W \subset \mathbb{R}^n$ *respectively be a bounded Lipschitz domain and a non-empty open set such that* $\overline{W} \cap \overline{\Omega} = \emptyset$. *Let* $s \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ *such that* $2s > m$. *Let* $a_\alpha \in H^{r_\alpha,\infty}(\Omega)$, *with* r_α *defined as in* (1.2). *Moreover, let* $\mathcal{R} := \{ u_f - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$, *where* u_f *solves*

$$(-\Delta)^s u_f + \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u_f = 0 \quad in\ \ \Omega, \quad u_f - f \in \widetilde{H}^s(\Omega)$$

and $\mathcal{R}^* := \{ u_f^* - f : f \in C_c^\infty(W) \} \subset \widetilde{H}^s(\Omega)$, *where* u_f^* *solves*

$$(-\Delta)^s u_f^* + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x) u_f^*) = 0 \quad in\ \ \Omega, \quad u_f^* - f \in \widetilde{H}^s(\Omega).$$

Then $\mathcal{R}$ *and* $\mathcal{R}^*$ *are dense in* $\widetilde{H}^s(\Omega)$.

4.2. **Proof of injectivity.**

Proof of theorem 1.2. The proof is virtually identical to the one of theorem 1.1, the unique difference being in the way the error terms of the Runge approximation are estimated. We make use of (4.1), which relied on the Kato-Ponce inequality instead of multiplier space estimates. In this way we get

$$\begin{aligned}\left| \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha r_{1,k}, u_{2,k}^*\rangle_{\mathbb{R}^n} \right| &\le \sum_{|\alpha|\le m} |\langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha r_{1,k}, u_{2,k}^*\rangle_{\mathbb{R}^n}| \\ &\le C \|u_{2,k}^*\|_{H^s(\mathbb{R}^n)} \|r_{1,k}\|_{H^s(\mathbb{R}^n)} \sum_{|\alpha|\le m} \|a_{1,\alpha} - a_{2,\alpha}\|_{H^{r_\alpha,\infty}(\Omega)} \to 0\end{aligned}$$

and

$$\begin{aligned}| \sum_{|\alpha|\le m} \langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha v_1, r_{2,k}\rangle_{\mathbb{R}^n}| &\le \sum_{|\alpha|\le m} |\langle (a_{1,\alpha} - a_{2,\alpha}) D^\alpha v_1, r_{2,k}\rangle_{\mathbb{R}^n}| \\ &\le C \|r_{2,k}\|_{H^s(\mathbb{R}^n)} \|v_1\|_{H^s(\mathbb{R}^n)} \sum_{|\alpha|\le m} \|a_{1,\alpha} - a_{2,\alpha}\|_{H^{r_\alpha,\infty}(\Omega)} \to 0. \square\end{aligned}$$

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