JYU DISSERTATIONS 313

Atte Lohvansuu

On the Duality of Moduli

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS
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Editors
Kai Rajala
Department of Mathematics and Statistics, University of Jyväskylä
Päivi Vuorio
Open Science Centre, University of Jyväskylä

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List of included articles

This dissertation consists of an introductory part and the following publications:

[A] A. Lohvansuu and K. Rajala, *Duality of moduli in regular metric spaces*, Indiana Univ. Math. J., to appear,

[B] A. Lohvansuu, *Duality of moduli in regular toroidal metric spaces*, Ann. Acad. Sci. Fenn. Math. 46 (2021), number 1, 3-20,

[C] A. Lohvansuu, *On the duality of moduli in arbitrary codimension*, preprint.

The author of this dissertation has actively taken part in the research of the joint article [A].

INTRODUCTION

Suppose (X, d, μ) is a metric measure space with μ Borel regular. Given $1 < p < \infty$ and a family Γ of paths in X, the *p-modulus* of Γ is the number

$$\mathrm{mod}_p\Gamma := \inf_{\rho} \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ with

$$\int_\gamma \rho \, ds \geq 1$$

for every locally rectifiable $\gamma \in \Gamma$. The path modulus is a fundamental tool in geometric function theory and nonsmooth analysis [12, 23, 28]. For example, it appears in the very definition of quasiconformal maps: given an orientation preserving homeomorphism $f : U \to V$ between domains U and V of $\mathbb{R}^n$, we say that f is *quasiconformal* if it quasipreserves the conformal modulus of every path family. That is, there is a constant $K \geq 1$, such that

$$\frac{1}{K}\mathrm{mod}_n\Gamma \leq \mathrm{mod}_n f\Gamma \leq K\mathrm{mod}_n\Gamma$$

for every family Γ of paths of U. If $K = 1$, we say that f is *conformal.* This definition is often called the *geometric* definition, as opposed to the *analytic* and *metric* ones. These definitions can be extended to large classes of metric spaces. For example, if X and Y are Ahlfors Q-regular, we say that a homeomorphism $f : X \to Y$ is (geometrically) quasiconformal if it quasipreserves the Q-modulus of every path family of X. Recall that (X, d, μ) is *Ahlfors Q-regular* if there is a constant $C \geq 1$, so that

$$\frac{1}{C}r^Q \leq \mu(B(x, r)) \leq Cr^Q$$

for all $x \in X$ and $r \leq \operatorname{diam} X$.

The metric and analytic definitions can be generalized as well, and all of the definitions are quantitatively equivalent for maps between Ahlfors Q-regular spaces that support weak Q-Poincaré inequalities. Given $1 \leq p < \infty$ we say that (X, d, μ) supports a *weak p-Poincaré inequality* with constants λ and C, if

$$\fint_B |u - u_B| \, d\mu \leq Cr \left(\fint_{\lambda B} \rho^p \, d\mu \right)^{1/p}$$

for every ball B of radius r, every locally integrable function u and upper gradient ρ of u. Here

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

A positive Borel function ρ is an *upper gradient* of a function $u : X \to \mathbb{R}$ if

$$|u(a) - u(b)| \leq \int_\gamma \rho \, ds$$

for every rectifiable path $\gamma : [a, b] \to X$. For example, for locally Lipschitz functions $u : X \to \mathbb{R}$ the pointwise Lipschitz constant

$$\mathrm{Lip}(u)(x) := \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(x) - u(y)|}{r}$$

is an upper gradient of u.

In addition, quasiconformality often agrees with *quasisymmetry* [11, 27]. We say that a homeomorphism $f : X \to Y$ is (weakly) quasisymmetric if for every triple of points $x, y, z \in X$

$$d_X(x, y) \leq d_X(x, z) \text{ implies } d_Y(f(x), f(y)) \leq H d_Y(f(x), f(z)).$$

For example, all biLipschitz homeomorphisms are quasisymmetric, and so are all snowflake maps $(X, d) \hookrightarrow (X, d^\varepsilon)$, where $0 < \varepsilon < 1$. See e.g. [20, 28] for classic theory of quasiconformal maps in euclidean spaces based on the geometric definition, and [10, 12, 27, 30] and the foundational paper [11] for the metric theory.

Quasiconformal and quasisymmetric maps often arise in the study of parametrization questions. Generally these questions ask if a homeomorphism between metric spaces can be assumed to admit desirable metric or analytic properties. A classic result in this direction is the Riemann mapping theorem, or more generally the *uniformization theorem*: every simply connected Riemann surface is conformally equivalent to the plane, the open unit disk or the sphere. A notable example in the metric setting is the Bonk-Kleiner theorem [3]: an Ahlfors 2-regular metric 2-sphere is quasisymmetrically equivalent to the standard 2-sphere if and only if it is linearly locally contractible.

Recently a quasiconformal uniformization theorem was proved by Rajala [22]: a metric plane with locally finite Hausdorff 2-measure is quasiconformally equivalent to a planar domain if and only if it is *reciprocal*. This was generalized to general reciprocal surfaces by Ikonen [15].

Part of the reciprocality property requires that for every topological rectangle $D \subset X$ with boundary edges $\xi_1, \ldots, \xi_4$ in cyclic order we have

$$\kappa^{-1} \leq \mathrm{mod}_2 \Gamma(\xi_1, \xi_3; D) \cdot \mathrm{mod}_2 \Gamma(\xi_2, \xi_4; D) \leq \kappa, \tag{0.1}$$

where $\kappa \geq 1$ is a fixed constant and $\Gamma(\xi_i, \xi_j; D)$ is the collection of paths of D that connect ξ_i to ξ_j.

If $D \subset \mathbb{R}^2$ is a Jordan domain, (0.1) holds with $\kappa = 1$. In fact, it can be shown [24] that there exist functions $u, v : D \to [0, 1]$, such that $|\nabla u|$ and $|\nabla v|$ are the minimizing admissible functions of the moduli in (0.1). Moreover, for $M = \mathrm{mod}_2 \Gamma(\xi_1, \xi_3; D)$ the map $f = (u, Mv) : D \to [0, 1] \times [0, M]$ is a homeomorphism that is conformal in the interior of D and maps the edges $\xi_1, \ldots, \xi_4$ to the boundary edges of $[0, 1] \times [0, M]$.

For Jordan domains even more is true:

$$(\text{mod}_p\Gamma(\xi_1, \xi_3; D))^{1/p}(\text{mod}_q\Gamma(\xi_2, \xi_4; D))^{1/q} = 1, \tag{0.2}$$

for *every* $1 < p < \infty$ and $q = \frac{p}{p-1}$. For conformal moduli, that is $p = 2 = q$, the identity (0.2) goes back to Ahlfors and Beurling [1], and was generalized to p-moduli by Gehring [9] and Ziemer [31].

The research presented in this thesis revolves around a phenomenon called the *duality of moduli*, of which (0.2) is the most well-known instance. The main goal of the thesis is to find generalizations or analogues of (0.2) in different settings. Roughly speaking, an analogue of (0.2) is proved for metric condensers in [A] (with K. Rajala) and for metric solid tori in [B]. In [C] we study the moduli of k and $(n-k)$-dimensional slices of euclidean n-cubes for any $1 \leq k \leq n-1$.

1. Moduli of condensers

The definition of modulus can be generalized considerably. Given $1 < p < \infty$ and a family $\mathcal{M}$ of Borel measures of X, the *p-modulus* of $\mathcal{M}$ is the number

$$\text{mod}_p\mathcal{M} := \inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all positive Borel functions $\rho : X \to [0, \infty)$ with

$$\int_X \rho \, d\nu \geq 1$$

for every $\nu \in \mathcal{M}$. Such functions are called *admissible* for $\mathcal{M}$. If ρ is admissible for $\mathcal{M} - \mathcal{N}$, where $\text{mod}_p\mathcal{N} = 0$, we say that ρ is *p-weakly admissible* or simply *weakly admissible* if the choice of p is clear from the context.

Given any *condenser* (E, F, G), i.e. a domain G (in X) with disjoint continua $E, F \subset \overline{G}$, we denote by $\Gamma(E, F; G)$ the family of all paths in G that connect E to F. Dually, we denote by $\Gamma^*(E, F; G)$ the collection of compact subsets S of $\overline{G} - (E \cup F)$ that separate E from F. By separation it is meant that E and F belong to different components of $\overline{G} - S$. When G is open in $\mathbb{R}^n$, we equip each such S with the restriction of the Hausdorff measure $\mathcal{H}^{n-1}$ to S. The modulus of $\Gamma^*(E, F; G)$ is then defined to be the modulus of the corresponding collection of Hausdorff measures.

The duality (0.2) was generalized to euclidean spaces of higher dimension by Gehring and Ziemer:

Theorem 1.1. *Let $(E, F; G)$ be a condenser in $\mathbb{R}^n$. Then*

$$(\text{mod}_p\Gamma(E, F; G))^{1/p}(\text{mod}_q\Gamma^*(E, F; G))^{1/q} = 1$$

for every $1 < p < \infty$ and $q = \frac{p}{p-1}$.

Our main result in [A] shows that an analogue of this holds in sufficiently regular metric spaces.

Theorem 1.2. *Suppose X is an Ahlfors Q-regular metric space that supports a 1-Poincaré inequality. Let $(E, F; G)$ be a condenser in X, with $E, F \subset G$. Then*

$$\frac{1}{C} \le (\mathrm{mod}_p \Gamma(E, F; G))^{1/p} (\mathrm{mod}_q \Gamma^*(E, F; G))^{1/q} \le C$$

for every $1 < p < \infty$ and $q = \frac{p}{p-1}$. Here $\infty \cdot 0 = 1 = 0 \cdot \infty$, and $C \ge 1$ depends only on the constants of Ahlfors regularity and the 1-Poincaré inequality.

Here it is understood that the "surfaces" of $\Gamma^*(E, F; G)$ are equipped with the $(Q-1)$-dimensional Hausdorff measure. Theorem 1.2 leads to a characterization of (geometrically) quasiconformal maps in terms of moduli of separating sets. It is also possible to equip the separating surfaces with so-called *perimeter measures*. For results in this direction, see [17, 16].

The methods to prove the lower bounds of Theorems 1.1 and 1.2 are essentially the same. The first step is showing that the path modulus equals a capacity. For condensers that capacity is

$$\mathrm{cap}_p(E, F; G) := \inf_u \int_G \mathrm{Lip}(u)^p \, d\mu,$$

where the infimum is taken over all Lipschitz functions u, for which $u|_E \le 0$ and $u|_F \ge 1$. Now

$$\mathrm{mod}_p \Gamma(E, F; G) := \mathrm{cap}_p(E, F; G), \tag{1.1}$$

for euclidean condensers by [25] and for metric condensers (with $E, F \subset G$) in the setting of Theorem 1.2 by [18]. The lower bound is then a combination of (1.1), Hölder's inequality and a *coarea estimate*: for every Lipschitz $u : G \to \mathbb{R}$ and positive Borel function g

$$\int_{\mathbb{R}} \int_{u^{-1}(t)} g \, d\mathcal{H}^{Q-1} dt \le C \int_G \mathrm{Lip}(u) g \, d\mu, \tag{1.2}$$

where C is a quantitative constant. In $\mathbb{R}^n$ the estimate holds with an equality and $C = 1$.

The proof of the upper bound of Theorem 1.2 is more involved, and different from the proof of Theorem 1.1, which exploits the vector space structure of $\mathbb{R}^n$. However, the proofs do have a lot in common. It suffices to show that if ρ is the minimizing admissible function of $M := \mathrm{mod}_q \Gamma^*(E, F; G)$, then $CM^{-1}\rho^{q-1}$ is admissible for $\Gamma(E, F; G)$. This is achieved with the *variation inequality*:

$$M \le \int_G \phi \rho^{q-1} \, d\mu$$

for every q-integrable ϕ admissible for $\Gamma^*(E, F; G)$. Given a $\gamma \in \Gamma(E, F; G)$, the goal is to construct a sequence of admissible functions (ϕ_i), so that

$$M \le \limsup_{i \to \infty} \int_G \phi_i \rho^{q-1} \, d\mu \le C \int_\gamma \rho^{q-1} \, ds.$$

We choose $\phi_i = i\chi_{N_{1/i}(|\gamma|)}$, where N_ε denotes the ε-neighborhood. Proving that ϕ_i is indeed admissible (up to a multiplicative constant) for $\Gamma^*(E, F; G)$ requires the *relative isoperimetric inequality*, which is equivalent to the 1-Poincaré inequality in this setting.

2. Moduli in solid tori

Another classic instance of duality concerns planar annuli. Given a topological annulus $A \subset \mathbb{R}^2$ bounded by Jordan loops A_0 and A_1, we have by Theorem 1.1 that

$$(\mathrm{mod}_p \Gamma(A_0, A_1; A))^{1/p} (\mathrm{mod}_q \Gamma^*(A_0, A_1; A))^{1/q} = 1. \tag{2.1}$$

Theorem 1.1 applies equally well to annuli of higher dimension. What makes the planar case interesting is that $\mathrm{mod}_q \Gamma^*(A_0, A_1; A)$ equals the q-modulus of the family of all degree 1 loops of A for every $1 < q < \infty$. This is because every separating set of $\Gamma^*(A_0, A_1; A)$ of finite length contains the image of a degree 1 loop.

It is then natural to ask whether (2.1) admits generalizations to other settings where degree 1 loops exist. Interestingly enough, it turns out that the answer is no in general.

Freedman and He [8] showed that there exist riemannian solid tori T (spaces diffeomorphic to $\mathbb{S}^1 \times \overline{\mathbb{D}}$), for which the quantity

$$(\mathrm{mod}_p \Gamma_1(T))^{1/p} (\mathrm{mod}_q \Gamma_1^*(T))^{1/q} \tag{2.2}$$

is arbitrarily small. Here Γ_1 denotes the family of all degree 1 loops of T, and Γ_1^* denotes the family of smooth surfaces bounded by meridians of ∂T. The surfaces are equipped with the Hausdorff measure $\mathcal{H}^2$. It turns out that (2.2) fails to admit a lower bound, since (1.1) does not hold in this setting, namely for the modulus of the degree 1 and the *degree 1 capacity*. This is defined by

$$\mathrm{cap}_p T := \inf \int_T |\nabla u|^p \, d\mathcal{H}^3,$$

where the infimum is taken over all Lipschitz maps $u : T \to \mathbb{R}/\mathbb{Z}$ of degree 1. Here degree is defined in terms of singular homology, and $\mathbb{R}/\mathbb{Z}$ is equipped with the metric

$$|[x] - [y]| = \inf_{a \in \mathbb{Z}} |x + a - y|,$$

which makes it isometric to the euclidean circle of length 1.

Theorem 2.1 (Theorem 2.5 in [8]). *For every riemannian solid torus T*

$$(\mathrm{cap}_3 T)^{1/3} (\mathrm{mod}_{3/2} \Gamma_1^*(T))^{2/3} = 1$$

The proof given in [8] can be applied for general p as well. In [B] we prove an analogue of this for metric solid tori.

Theorem 2.2. *Suppose T is Ahlfors Q-regular and supports a 1-Poincaré inequality. If $\mathrm{cap}_p T$ is nonzero, then*

$$\frac{1}{C} \leq (\mathrm{cap}_p T)^{1/p} (\mathrm{mod}_q \Gamma_1^*(T))^{1/q} \leq C,$$

where C depends only on constants of Ahlfors regularity and the 1-Poincaré inequality. Moreover, $\mathrm{cap}_p T = 0$ if and only if $\mathrm{mod}_q \Gamma_1^*(T) = \infty$.

In the metric setting we define $\Gamma_1^*(T)$ to be the collection of level sets of mappings of degree 1. These level sets are equipped with the $(Q-1)$-dimensional Hausdorff measure.

The proof of the lower bound of Theorem 2.2 follows from a coarea estimate similar to (1.2). The proof of the upper bound follows the same strategy as the proof of Theorem 1.2. This time we find a minimizer for $\mathrm{cap}_p T$, prove a variation inequality and then construct suitable Lipschitz maps of degree 1 using the surfaces of $\Gamma_1^*(T)$. This approach can be seen as dual to the proof of Theorem 1.2.

3. Moduli of (co)homology classes

Since paths are of dimension 1 and separating sets are generally of codimension 1, it is natural to ask whether a duality result could hold for suitable classes of objects of higher (co)dimension as well. Moduli of such objects are not very well known, but have been used to prove highly nontrivial nonparametrization theorems [14, 21].

A refinement of the Bonk-Kleiner theorem [3] by Wildrick [29] states that an Ahlfors 2-regular, unbounded and complete metric space homeomorphic to $\mathbb{R}^2$ is quasisymmetrically equivalent to $\mathbb{R}^2$ if and only if it is linearly locally contractible. The analogous statement in higher dimensions is false, although all known counterexamples are quite complicated. One family of such examples consists of the spaces $\mathbb{R}^3/\mathrm{Wh} \times \mathbb{R}^n$, where Wh is the *Whitehead continuum.* The Whitehead continuum is a compact and connected subset of $\mathbb{R}^3$ and has the curious property, that $\mathbb{R}^3/\mathrm{Wh}$ is *not* homeomorphic to $\mathbb{R}^3$, even though its product with $\mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. Heinonen and Wu [14] showed that $\mathbb{R}^3/\mathrm{Wh}\times\mathbb{R}^n$ can be equipped with a metric that makes it Ahlfors $(n+3)$-regular and linearly locally contractible, but not quasisymmetrically equivalent to $\mathbb{R}^{n+3}$. This result was later generalized by Pankka and Wu [21].

Spheres admit similar topological behaviour. The Poincaré homology sphere is not homeomorphic to $\mathbb{S}^3$, but its double suspension – the *Edwards sphere* – is homeomorphic to $\mathbb{S}^5$. In fact, double suspensions of *all* homology spheres are spheres, due to deep results by Edwards and Cannon [4, 5, 6]. Determining how good parametrizations the Edwards sphere can admit is a major open question. In particular, it is unknown if the Edwards sphere is quasisymmetrically equivalent to $\mathbb{S}^5$, see Questions 12-14 of [13]. Indeed, one of the main motivations for studying more general moduli is providing tools to approach such problems.

Homology and cohomology classes offer the most natural way to generalize connecting paths and separating surfaces to higher (co)dimension. Moduli of general (co)homology classes were first studied by Freedman and He [8]. These moduli are closely connected to the ones we have already defined.

Given any pair of compact riemannian manifolds (M, A), with A embedded in M, and a de Rham cohomology class $[\omega] \in H^*_{dR}(M, A)$, we define the p-modulus of $[\omega]$ by

$$\mathrm{mod}_p[\omega] := \inf_{\omega' \in [\omega]} \int_Q |\omega'|^p \, d\mathcal{H}^n, \tag{3.1}$$

where n is the dimension of M. Note that the modulus of a trivial class is zero. As a dual counterpart to (3.1) Freedman and He considered

$$\text{mod}_p^*[\omega] := \inf \int_Q |\omega'|^p \, d\mathcal{H}^n,$$

where the infimum is taken over all closed differential forms ω' with

$$\int_M \omega \wedge \omega' = 1.$$

The following is proved in [8] for conformal moduli. However, the same proof goes through for general p as well.

Theorem 3.1. *Suppose M is a compact riemannian n-manifold and A is an embedded submanifold of ∂M. Let $[\omega] \in H^k_{dR}(M, A)$. Then*

$$(\text{mod}_p[\omega])^{1/p}(\text{mod}_q^*[\omega])^{1/q} = 1,$$

for every $1 < p < \infty$ and $q = \frac{p}{p-1}$.

Note that if A is $(n-1)$-dimensional, then either ∂M decomposes into two $(n-1)$-submanifolds A and B with common boundary or $A = \partial M$, in which case we set $B = \emptyset$. In either case it can be shown that the groups (vector spaces) $H^*_{dR}(M, A)$ and the dual spaces $H^{n-*}_{dR}(Q, B)^*$ are isomorphic via the integral maps

$$\int_M : H^*_{dR}(M, A) \to H^{n-*}_{dR}(M, B)^*, \qquad \int_M [\omega]([\eta]) := \int_M \omega \wedge \eta.$$

This is more or less a relative de Rham version of the Poincaré duality. Let us call a pair of classes $[\omega] \in H^k_{dR}(M, A)$, $[\eta] \in H^{n-k}_{dR}(M, B)$ *Poincaré dual* if

$$\int_M \omega \wedge \eta = 1.$$

We can now state a very interesting corollary of (the proof of) Theorem 3.1.

Corollary 3.2. *Let M, A and B be as above. If classes $[\omega] \in H^k_{dR}(M, A)$, $[\eta] \in H^{n-k}_{dR}(M, B)$ are Poincaré dual, then their moduli are dual:*

$$(\text{mod}_p[\omega])^{1/p}(\text{mod}_q[\eta])^{1/q} = (\text{mod}_p[\omega])^{1/p}(\text{mod}_q^*[\omega])^{1/q} = 1$$

for every $1 < p < \infty$ and $q = \frac{p}{p-1}$.

Theorem 2.1 and Corollary 3.2 are closely connected. In fact, it is shown in [8] that for solid tori T, as in Theorem 3.1, we have

$$\text{cap}_3 T = \text{mod}_3[\omega] \;\text{ and }\; \text{mod}_{3/2}\Gamma_1^*(T) = \text{mod}_{3/2}[\eta],$$

where $[\omega]$ and $[\eta]$ are the elements of $H^1_{dR}(T)$ and $H^2_{dR}(T, \partial T)$, for which

$$\int_\gamma \omega = 1 = \int_S \eta$$

for every smooth γ and S that represent the generators of $H_1(T)$ and $H_2(T, \partial T)$.

This leads us to define moduli for Lipschitz homology classes. Given a relative homology class $[S]$ we denote $S' \in [S]$ if S' equals the image of some singular relative cycle that generates $[S]$. We then define

$$\mathrm{mod}_p[S] := \mathrm{mod}_p\{\mathcal{H}^k \llcorner S' \mid S' \in [S]\},$$

when $[S]$ is generated by relative k-cycles. Moduli of singular classes can be defined with the same formula, but we prefer to work in the Lipschitz category.

We can use the moduli of homology classes to give yet another formulation of (0.2). Indeed, the moduli in (0.2) are equal to the moduli of generating classes of $H_1^L(D, \xi_1 \cup \xi_3)$ and $H_1^L(D, \xi_2 \cup \xi_4)$. Note that decomposing ∂D into the sets $\xi_1 \cup \xi_3$ and $\xi_2 \cup \xi_4$ corresponds to the decomposition

$$\partial I^2 = \partial(I \times I) = (\partial I \times I) \cup (I \times \partial I)$$

under some homeomorphism $D \to I^2$. More generally, suppose Q is a smooth n-submanifold of $\mathbb{R}^n$ (with corners) that is diffeomorphic to the unit cube I^n. Denote the diffeomorphism by $h : Q \to I^n$. Define

$$A := h^{-1}(\partial I^k \times I^{n-k}) \text{ and } B := h^{-1}(I^k \times \partial I^{n-k}).$$

Then A and B are $(n-1)$-dimensional submanifolds of ∂Q with $\partial Q = A \cup B$ and $\partial A = A \cap B = \partial B$. The relative de Rham cohomology groups $H^k_{dR}(Q, A)$ and $H^{n-k}_{dR}(Q, B)$ are then isomorphic to $\mathbb{R}$ and the Lipschitz homology groups $H_k^L(Q, A)$ and $H_{n-k}^L(Q, B)$ are isomorphic to $\mathbb{Z}$.

It is then natural to ask, if the homology counterpart of Corollary 3.2 holds in this setting.

Question 1. Suppose $[S]$ and $[S^*]$ are generators of $H_k^L(Q, A)$ and $H_{n-k}^L(Q, B)$, respectively. Is it then true that

$$(\mathrm{mod}_p[S])^{1/p}(\mathrm{mod}_q[S^*])^{1/q} = 1 \tag{3.2}$$

for every $1 < p < \infty$ and $q = \frac{p}{p-1}$?

Note that (3.2) holds when $n = 2, k = 1$ due to (0.2). Our main result in [C] gives the upper bound for general n and k.

Theorem 3.3. *Suppose $[S]$ and $[S^*]$ are generating classes of $H_k^L(Q, A)$ and $H_{n-k}^L(Q, B)$. Then*

$$(\mathrm{mod}_p[S])^{1/p}(\mathrm{mod}_q[S^*])^{1/q} \le 1$$

for any $1 < p < \infty$ and $q = \frac{p}{p-1}$.

The proof of Theorem 3.3 follows the same general idea as the proof of e.g. the upper bound of Theorem 1.2. We aim to show that $M^{-1}\rho^{q-1}$ is admissible for $[S^*]$, when ρ is the unique weakly admissible minimizer of $M = \mathrm{mod}_p[S]$. To achieve this, we apply the variation inequality, which in this case takes the form

$$M \le \int_Q \phi \rho^{q-1} \, d\mathcal{H}^n \tag{3.3}$$

where ϕ is any other q-integrable function admissible for $[S^*]$. Given any $S' \in [S]$, we apply (3.3) on the smooth convolutions

$$\phi_\varepsilon^{S'}(x) := \int_{S'} \phi_\varepsilon(x-y)\, d\mathcal{H}^k(y),$$

which have the desired property

$$\int_Q \phi_\varepsilon^{S'} g\, d\mathcal{H}^n \xrightarrow{\varepsilon \to 0} \int_{S'} g\, d\mathcal{H}^k$$

for every smooth g.

Here a fundamental problem is showing that each $\phi_\varepsilon^{S'}$ is admissible for $[S^*]$. Our main new innovation is showing that this can be overcome by exploiting the topological fact that the intersection of S' and the translation $S^{*\prime} + z$ of any $S^{*\prime} \in [S^*]$ is nonempty whenever $|z|$ is small enough.

4. Open questions

Let us list here some open questions. First, as mentioned above, the lower bound of Question 1 is still open. Some discussion on this question can be found in [C].

Question 2. Suppose $[S]$ and $[S^*]$ are generators of $H_k^L(Q, A)$ and $H_{n-k}^L(Q, B)$, respectively. Is it then true that

$$1 \leq (\mathrm{mod}_p[S])^{1/p} (\mathrm{mod}_q[S^*])^{1/q}$$

for every $1 < p < \infty$ and $q = \frac{p}{p-1}$?

It is interesting that none of the proofs of the duality results presented in this thesis depend on the exponent p. It is fixed in the beginning of each proof, and then left alone. One might at first expect that the conformal exponents play a special role.

Recall from the beginning of this introduction that the minimizers of the 2-moduli in (0.2) can be used to construct the components of the unique conformal map that takes D to a rectangle $[0, 1] \times [0, M]$ and the boundary edges $\xi_1, \ldots, \xi_4$ to the boundary edges of $[0, 1] \times [0, M]$. With this in mind we ask the following.

Question 3. Assuming the answer to Question 2 is positive, what is the geometric interpretation of the minimizers of the moduli $\mathrm{mod}_{\frac{n}{k}}[S]$ and $\mathrm{mod}_{\frac{n}{n-k}}[S^*]$?

Lastly, singular or Lipschitz homology classes are not the only ones for which moduli can be defined. An interesting alternative is provided by the theory of Federer-Fleming currents, see e.g. [7, 26].

Question 4. Do similar duality results hold for homology classes of currents?

Answering this question may lead to interesting results in the metric setting as well, since the theory of currents has been extended there by Ambrosio and Kirchheim [2] and Lang [19].

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Included articles

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Duality of moduli in regular metric spaces

A. Lohvansuu and K. Rajala

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DUALITY OF MODULI IN REGULAR METRIC SPACES

ATTE LOHVANSUU AND KAI RAJALA

Abstract. Gehring [3] and Ziemer [17] proved that the p-modulus of the family of paths connecting two continua is dual to the p^*-modulus of the corresponding family of separating hypersurfaces. In this paper we show that a similar result holds in complete Ahlfors-regular metric spaces that support a weak 1-Poincaré inequality. As an application we obtain a new characterization for quasiconformal mappings between such spaces.

1. Introduction

The modulus of a path family is a widely used tool in geometric function theory and its generalizations to $\mathbb{R}^n$ and furthermore to metric spaces, see [5], [10] and [12].

Given $1 \leqslant p < \infty$ and a family Γ of paths in a metric measure space (X, d, μ), the p-modulus of Γ is defined to be

$$\mathrm{mod}_p \Gamma := \inf_{\rho} \int_X \rho^p \, d\mu,$$

where the infimum is taken over all admissible functions of Γ, i.e., Borel measurable functions $\rho : X \to [0, \infty]$ that satisfy

$$\int_\gamma \rho \, ds \geqslant 1$$

for all locally rectifiable $\gamma \in \Gamma$. If no admissible functions exist, the modulus is defined to be ∞. The definition of modulus can be generalized considerably, as was done by Fuglede in his 1957 paper [2]. For example, instead of paths we can consider surfaces by defining the modulus with exactly the same formula but requiring the admissible functions to satisfy

$$\int_S \rho \, d\sigma_S \geqslant 1$$

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for all surfaces S in the family. Here σ_S denotes some Borel-measure associated to S. In our applications σ_S will be comparable to a Hausdorff measure restricted to S.

Our main result is concerned with Ahlfors Q-regular complete metric spaces that support a weak 1-Poincaré inequality. We also assume $Q > 1$. See Section 2 for all relevant definitions. Fix such a metric measure space (X, d, μ). Given a domain $G \subset\subset X$ and disjoint nondegenerate continua $E, F \subset G$ we denote by $\Gamma(E, F; G)$ the family of rectifiable paths in G that join E and F. Similarly, we denote by $\Gamma^*(E, F; G)$ the family of compact sets $S \subset \overline{G}$ that have finite $(Q-1)$-dimensional Hausdorff measure and separate E and F in G. By separation we mean that E and F belong to disjoint components of $G - S$. We equip each surface S with the restriction of the $(Q-1)$-dimensional Hausdorff measure on $S \cap G$. For $1 < p < \infty$, denote $p^* = \frac{p}{p-1}$.

The main purpose of this paper is to prove the following connection between the path modulus and the modulus of separating surfaces.

THEOREM 1.1. *Let $1 < p < \infty$. There is a constant C that depends only on the data of X such that*

$$\frac{1}{C} \leqslant \text{mod}_p \Gamma(E, F; G)^{\frac{1}{p}} \cdot \text{mod}_{p^*} \Gamma^*(E, F; G)^{\frac{1}{p^*}} \leqslant C, \tag{1}$$

for any choice of E, F and G. Here it is understood that $0 \cdot \infty = 1$.

Gehring [3] and Ziemer [17] proved that (1) holds in $\mathbb{R}^n$ with $C = 1$.

As an application of Theorem 1.1 we find a new characterization of quasiconformal maps between regular spaces. Let Y be another complete Ahlfors Q-regular space that supports a weak 1-Poincaré inequality. Recall that a homeomorphism $f : X \to Y$ is (geometrically) *K-quasiconformal* if there exists a constant $K \geqslant 1$ such that for every family Γ of paths in X

$$\frac{1}{K} \text{mod}_Q(f\Gamma) \leqslant \text{mod}_Q \Gamma \leqslant K \text{mod}_Q(f\Gamma). \tag{2}$$

Here $f\Gamma = \{f \circ \gamma \mid \gamma \in \Gamma\}$.

Corollary 1.2. *Let X and Y be as above. A homeomorphism $f : X \to Y$ is K-quasiconformal if and only if there is a constant C, such that*

$$\frac{1}{C} \text{mod}_{Q^*} \Gamma^*(E, F; G) \leqslant \text{mod}_{Q^*} \Gamma^*(fE, fF; fG) \leqslant C \text{mod}_{Q^*} \Gamma^*(E, F; G)$$

for all E, F and G as above. The constants C and K depend only on each other and the data of X and Y.

See Section 3 for the proof. We remark that the "only if" part follows also from the recent work of Jones, Lahti and Shanmugalingam [6].

This paper is organized as follows: In Section 2 we introduce the main tools for later use. In Section 3 we state our main results, Theorems

3.1 and 3.2, which are more general versions of the lower and upper bounds in (1). We also show how these results imply Corollary 1.2. Theorem 3.1 is proved in Section 4 along the lines of [3] and [17], applying coarea estimates. The proof of Theorem 3.2, which seems to be new even in the euclidean setting, is given in Section 5. Section 6 contains an example showing the necessity of the 1-Poincaré inequality in Theorem 1.1.

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2. Preliminaries

2.1. **Doubling measures.** A Borel-regular measure μ is called *doubling* with doubling constant $C_\mu > 1$ if

$$0 < \mu(2B) \leqslant C_\mu \mu(B) < \infty \tag{3}$$

for all balls $B \subset X$. Iterating (3) shows that there are constants C'_μ and $s > 0$ that depend only on C_μ such that for any $x, y \in X$ and $0 < r \leqslant R < \operatorname{diam}(X)$ with $x \in B(y, R)$,

$$\frac{\mu(B(y,R))}{\mu(B(x,r))} \leqslant C'_\mu \left(\frac{R}{r}\right)^s . \tag{4}$$

In fact, we can choose $s \geqslant \log_2 C_\mu$.

The space X is said to be *Ahlfors Q-regular*, or just Q-regular, if there are constants a and $A > 0$ such that

$$ar^Q \leqslant \mu(B(x,r)) \leqslant Ar^Q \tag{5}$$

for every $x \in X$ and $0 < r < \operatorname{diam}(X)$. It follows immediately from the definitions that Q-regular spaces are doubling.

2.2. **Moduli.** Let $\mathscr{M}$ be a collection of Borel-regular measures on X and let $1 \leqslant p < \infty$. We define the *p-modulus* of $\mathscr{M}$ to be

$$\operatorname{mod}_p \mathscr{M} = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel measurable functions $\rho : X \to [0, \infty]$ with

$$\int_X \rho \, d\nu \geqslant 1 \tag{6}$$

for all $\nu \in \mathscr{M}$. Such functions are called *admissible functions of $\mathscr{M}$*. If there are no admissible functions we define the modulus to be infinite. If ρ is an admissible function for $\mathscr{M} - \mathscr{N}$ where $\mathscr{N}$ has zero p-modulus, we say that ρ is *p-weakly admissible* for $\mathscr{M}$. As a direct consequence of the definitions we see that the p-modulus does not change if the infimum is taken over all p-weakly admissible functions. If some property holds for all $\nu \in \mathscr{M} - \mathscr{N}$ we say that it holds for *p-almost every ν* in $\mathscr{M}$.

We can also use paths instead of measures; if Γ is a family of locally rectifiable paths in X we define the *path p-modulus* of Γ as before with

$$\mathrm{mod}_p\Gamma = \inf \int_X \rho^p \, d\mu,$$

but require that

$$\int_\gamma \rho \, ds \geqslant 1$$

for every locally rectifiable $\gamma \in \Gamma$. See [16] or [1] for the definition and properties of path integrals over locally rectifiable paths. Most of the path families considered in this paper will be of the form

$$\Gamma(E, F; G) := \{\text{paths that join } E \text{ and } F \text{ in } G\},$$

where $E, F \subset G$ are disjoint continua and G is a domain in X. The modulus of $\Gamma(E, F; G)$ does not change if we consider only injective paths, see [14, Proposition 15.1]. For injective paths

$$\int_\gamma \rho \, ds = \int_{|\gamma|} \rho \, d\mathcal{H}^1,$$

as can be seen from the area formula [1, 2.10.13]. This implies that the modulus of any subfamily A of $\Gamma(E, F; G)$ is the same as the modulus of the family

$$\{\mathcal{H}^1 \llcorner |\gamma| \mid \gamma \in A\},$$

so in this sense the two definitions of the modulus are equal.

We will need the following basic lemma in multiple occasions. It is a combination of the lemmas of Fuglede and Mazur, see [5, p. 19, 131].

Lemma 2.1. *Let $\mathscr{M}$ be a set of Borel measures on X and $1 < p < \infty$. Suppose $\mathrm{mod}_p\mathscr{M} < \infty$. Then there is a sequence $(\rho_i)_{i=1}^\infty$ of admissible functions of $\mathscr{M}$ that converges in $L^p(X)$ to a p-weakly admissible function ρ of $\mathscr{M}$ such that for p-almost every $\nu \in \mathscr{M}$*

$$\int_X \rho_i \, d\nu \to \int_X \rho \, d\nu < \infty \tag{7}$$

and

$$\mathrm{mod}_p\mathscr{M} = \int_X \rho^p \, d\mu. \tag{8}$$

Remark 2.2. Lemma 2.1 holds for the path modulus of a path family Γ with the obvious modification of replacing (7) with

$$\int_\gamma \rho_i \, ds \to \int_\gamma \rho \, ds < \infty$$

for almost every $\gamma \in \Gamma$.

2.3. **Upper gradients.** A Borel function $\rho : X \to [0,\infty]$ is an *upper gradient* of a function $u : X \to \overline{\mathbb{R}}$, if

$$|u(\gamma(1)) - u(\gamma(0))| \leqslant \int_\gamma \rho \, ds \tag{9}$$

for all rectifiable paths $\gamma : [0,1] \to X$. If $|u(\gamma(0))|$ or $|u(\gamma(1))|$ equal ∞, we agree that the left side of (9) equals ∞. If (9) fails only for a family of paths of zero p-modulus, we say that ρ is a *p-weak* upper gradient. The following lemma will be useful in the sequel, and will be used without further mention. It allows the use of weak upper gradients in place of upper gradients in all the relevant results used in this paper. This is Proposition 6.2.2 of [5].

Lemma 2.3. *If $u : X \to \mathbb{R}$ has a p-weak upper gradient $\rho \in L^p(X)$ in X, then there is a decreasing sequence $(\rho_k)_{k=1}^\infty$ of upper gradients of u that converges to ρ in $L^p(X)$.*

2.4. **Maximal functions.** Suppose μ is doubling and $R > 0$. The restricted Hardy-Littlewood maximal function $\mathcal{M}_R u$ of an integrable function $u : X \to \overline{\mathbb{R}}$ is defined as

$$\mathcal{M}_R u(x) = \sup_{0<r\leqslant R} \fint_{B(x,r)} |u| \, d\mu,$$

where

$$\fint_B v \, d\mu := \frac{1}{\mu(B)} \int_B v \, d\mu.$$

The Hardy-Littlewood maximal function $\mathcal{M}u$ can then be defined as

$$\mathcal{M}u = \sup_{R>0} \mathcal{M}_R u.$$

In doubling spaces $\mathcal{M}u$ is Borel measurable whenever u is, and the assignment $u \mapsto \mathcal{M}u$ defines a bounded operator $L^p(X) \to L^p(X)$ for any $1 < p < \infty$, with bound depending only on p and the doubling constant of X, see [5, Chapter 3.5] for details.

2.5. **Codimension 1 spherical Hausdorff measure.** Given a Borel-regular measure μ, the *codimension 1 spherical Hausdorff δ-content* of a set $A \subset X$ is defined as

$$\mathcal{H}_\delta(A) := \inf \sum_i \frac{\mu(B_i)}{r_i},$$

where the infimum is taken over countable covers $\{B_i\}$ of A, and $B_i = B(x_i, r_i)$ for some $x_i \in X$ and $r_i \leqslant \delta$. The codimension 1 spherical Hausdorff measure of A is then defined to be

$$\mathcal{H}(A) := \sup_{\delta>0} \mathcal{H}_\delta(A).$$

By the Carathéodory construction $\mathcal{H}$ is also a Borel-regular measure. If X is Q-regular, $Q \geqslant 1$, and μ the Q-dimensional Hausdorff measure, then $\mathcal{H}$ is comparable to the $(Q-1)$-dimensional Hausdorff measure.

2.6. **Poincaré inequalities.** The space X is said to support a *weak p-Poincaré inequality* with constants C_P and λ_P if all balls in X have positive and finite measure, and

$$\fint_B |u - u_B| \, d\mu \leqslant C_P \mathrm{diam}\,(B) \left(\fint_{\lambda_P B} \rho^p \, d\mu \right)^{\frac{1}{p}}$$

for all functions $u \in L^1_{loc}(X)$ and all upper gradients ρ of u.

In the sequel we will encounter function-upper gradient pairs (v, ρ_v) that are defined only on some open and connected set $G \subset X$. For such pairs the Poincaré inequality can be applied on any ball B with $\lambda_P B \subset G$, or $B \subset\subset G$ if $\lambda_P = 1$. To see this, let $c > 1$ be such that $cB \subset G$ and replace v with $v' = v\chi_{cB}$ and ρ_v with $\rho' = \rho_v \chi_B + \infty \chi_{X-B}$. Then ρ' is an upper gradient of v' and v' is locally integrable on X.

2.7. **Whitney-type coverings.** We will need the following modification of Lemma 4.1.15 in [5] in multiple occasions. Here we assume that (X, d, μ) is a doubling metric measure space, $\Omega \subset X$ is open and bounded and $X - \Omega$ is nonempty.

Lemma 2.4. *Given any subset $A \subset \Omega$ and integer $n \geqslant 2$, there exists a countable collection $\mathcal{B} = \{B(x_i, r_i)\}$ of balls in Ω, such that*

(i) *$x_i \in A$ and $r_i = \frac{1}{2n} d(x_i, X - \Omega)$ for all i*

(ii) *If $B_i, B_j \in \mathcal{B}$ intersect, then*

$$\frac{1}{2} \leqslant \frac{r_i}{r_j} \leqslant 2$$

(iii) *For all $x \in \Omega$*

$$\chi_A(x) \leqslant \sum_{B \in \mathcal{B}} \chi_{2B}(x) \leqslant C,$$

where C depends only on the doubling constant of μ.

Proof. Let $A \subset \Omega$ and $2 \leqslant n \in \mathbb{Z}$. Denote $d(x) = d(x, X - \Omega)$. For any $k \in \mathbb{Z}$ let

$$A_k = \{x \in A \mid 2^{k-1} < d(x) \leqslant 2^k\}$$

and

$$\mathcal{F}_k = \{B(x, d(x)/10n) \mid x \in A_k\}.$$

Apply the $5r$-covering theorem on $\mathcal{F}_k$ to find a countable pairwise disjoint collection $\mathcal{G}_k \subset \mathcal{F}_k$ such that

$$\bigcup_{B \in \mathcal{F}_k} B \subset \bigcup_{B \in \mathcal{G}_k} 5B.$$

Denote by $\mathcal{B}$ the collection of all balls $5B$ with $B \in \mathcal{G}_k$ for some $k \in \mathbb{Z}$. Then $\mathcal{B}$ is countable and (i) is satisfied. A simple application of the triangle inequality proves (ii). The lower bound of (iii) follows from the definition of $\mathcal{B}$. Let $x \in \Omega$. By (i) and (ii) there is a $k \in \mathbb{Z}$ such that balls $B \in \mathcal{B}$ whose scaled versions $2B$ contain x must come from either $\mathcal{G}_k$ or $\mathcal{G}_{k-1}$. Now let $10B_1, \dots, 10B_N$ be balls arising from $\mathcal{G}_k$ that contain x with radii $r_1, \dots, r_N$ respectively, so that $r_1 \geqslant r_i$ for all $i = 1, \dots, N$. By the definition of $\mathcal{G}_k$ the balls B_i are disjoint, so by the doubling property and (ii)

$$\mu(100B_1) \geqslant \sum_{i=1}^{N} \mu(B_i) \geqslant CN\mu(100B_1),$$

where C depends only on the doubling constant of μ. The same argument can be applied to $\mathcal{G}_{k-1}$ and (iii) follows. □

3. Main results

Assume for the rest of the text that (X, d, μ) is a complete metric measure space that supports a weak 1-Poincaré inequality with constants C_P and λ_P. Assume also that μ is Borel-regular and doubling so that it satisfies (4) with some C'_μ and $s > 1$. Note that the doubling condition implies that X is proper and therefore also separable. By [13, Part I, II.3.11] μ is in fact a Radon-measure.

Fix a domain $G \subset\subset X$ and two disjoint nondegenerate continua $E, F \subset G$. Denote $G' = G - (E \cup F)$. Denote by Γ the set of all injective rectifiable paths $\gamma : [0,1] \to G$ with $\gamma(0) \in E$ and $\gamma(1) \in F$. For any $1 \leqslant p < \infty$ denote

$$\operatorname{mod}_p\Gamma := \operatorname{mod}_p\{\mathcal{H}^1 \llcorner |\gamma| \mid \gamma \in \Gamma\}. \tag{10}$$

Similarly, denote by Γ^* the set of all compact subsets $S \subset \overline{G}$ that separate E and F in G and have finite $\mathcal{H}$-measure in G. Abbreviate

$$\operatorname{mod}_q\Gamma^* = \operatorname{mod}_q\{\mathcal{H} \llcorner S \cap G \mid S \in \Gamma^*\}. \tag{11}$$

The requirement $\mathcal{H}(S \cap G) < \infty$ is redundant since the modulus of the family of sets with infinite $\mathcal{H}$-measure is zero. Nevertheless we prefer to work with sets of finite $\mathcal{H}$-measure.

We denote $C = C(X)$ if some constant $C > 0$ depends only on the data of X, i.e., the constants s, C_μ, C_P and λ_P. The same symbol C will be used for various different constants. Denote $p^* = \frac{p}{p-1}$ for each $1 < p < \infty$. The main results of this paper are the following:

THEOREM 3.1. *Let $1 < p < \infty$. If $\operatorname{mod}_p\Gamma \neq 0$, then*

$$C \leqslant (\operatorname{mod}_p\Gamma)^{\frac{1}{p}} (\operatorname{mod}_{p^*}\Gamma^*)^{\frac{1}{p^*}},$$

where the constant C *depends only on the data of* X*. If* $\mathrm{mod}_p\Gamma = 0$*, then* $\mathrm{mod}_{p^*}\Gamma^* = \infty$*.*

THEOREM 3.2. *Let* $1 < p < \infty$*. If* $\mathrm{mod}_{p^*}\Gamma^* < \infty$*, then*

$$(\mathrm{mod}_p\Gamma)^{\frac{1}{p}}(\mathrm{mod}_{p^*}\Gamma^*)^{\frac{1}{p^*}} \leqslant C, \tag{12}$$

where the constant C *depends only on the data of* X*. If* $\mathrm{mod}_{p^*}\Gamma^* = \infty$*, then* $\mathrm{mod}_p\Gamma = 0$*.*

Note that the conclusions in Theorems 3.1 and 3.2 are invariant under biLipschitz changes of metrics. Also recall that a complete metric space supporting a Poincaré inequality is C-quasiconvex for some $C = C(X)$. *Thus we may, and will, assume that* X *is a geodesic metric space.* Note that in geodesic spaces we can choose $\lambda_P = 1$. For these facts see Theorem 8.3.2 and Remark 9.1.19 in [5].

Theorem 1.1 follows by combining Theorems 3.1 and 3.2, and recalling that $\mathcal{H}$ is comparable to the $(Q-1)$-dimensional Hausdorff measure in Ahlfors Q-regular spaces. Theorems 3.1 and 3.2 will be proved in Sections 4 and 5, respectively. We now show how they imply Corollary 1.2.

Proof of Corollary 1.2. The "only if" part follows directly from Theorem 1.1. To prove the "if" part, notice first that Theorem 1.1 shows that (2) holds for all path families $\Gamma(E, F; G)$ joining continua E and F inside G. Injecting this estimate into the proof of Theorem 4.7 in [4] shows that f is locally quasisymmetric, with constants depending only on the given data. On the other hand, Theorem 10.9 of [15] shows that locally quasisymmetric maps satisfy (2) for all path families. The required linear local connectedness and Loewner properties of X and Y are guaranteed by [8, Theorem 3.3] and [4, Theorem 5.7]. The "if" part follows. □

4. Proof of Theorem 3.1

Let X, G, E, F, Γ and Γ^* be as in Section 3. Fix $1 < p < \infty$. Note that the constant function $1/\mathrm{dist}(E, F)$ restricted on G is admissible for Γ. Therefore $\mathrm{mod}_p\Gamma$ is finite.

We need the following result of Kallunki and Shanmugalingam [7]: The locally Lipschitz p-capacity of Γ is defined to be

$$\mathrm{cap}_p^L\Gamma = \inf_\rho \int_G \rho^p \, d\mu,$$

where the infimum is taken over every non-negative Borel-measurable function ρ that is an upper gradient to some locally Lipschitz function $u : G \to [0, 1]$ with $u|_E = 0$ and $u|_F = 1$.

Theorem 1.1 in [7] reads as follows: if $1 < p < \infty$, then

$$\mathrm{mod}_p\Gamma = \mathrm{cap}_p^L\Gamma \tag{13}$$

for any choice of E, F and G.

The proof of Theorem 3.1 is based on the following coarea estimate.

Proposition 4.1. *Let $u : G \to \mathbb{R}$ be locally Lipschitz and let ρ be a p-integrable upper gradient of u in G. Let $g : G \to [0, \infty]$ be a p^*-integrable Borel function. Then*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \; d\mathcal{H}dt \leqslant C \int_G g\rho \; d\mu \tag{14}$$

for some $C = C(X)$.

Before proving Proposition 4.1, we show how it together with (13) yields Theorem 3.1.

Proof of Theorem 3.1. First assume that $\mathrm{mod}_p\Gamma > 0$. If $\mathrm{mod}_{p^*}\Gamma^* = \infty$, there is nothing to prove. Otherwise let $g \in L^{p^*}(G)$ be admissible for Γ^*. Let $u : G \to [0, 1]$ be locally Lipschitz with $u|_E = 0$ and $u|_F = 1$. Let ρ be an upper gradient of u. We may assume that ρ is p-integrable. Note that for every $t \in (0, 1)$ the set $u^{-1}(t)$ separates E and F, and is closed in G. Moreover, by (14) $\mathcal{H}(u^{-1}(t)) < \infty$ for almost every t. Proposition 4.1 and Hölder's inequality give

$$1 \leqslant \int_{(0,1)}^{*} \int_{u^{-1}(t)} g \; d\mathcal{H}dt \leqslant C \int_G g\rho \; d\mu \leqslant C \left(\int_G g^{p^*} \, d\mu \right)^{\frac{1}{p^*}} \left(\int_G \rho^p \; d\mu \right)^{\frac{1}{p}}.$$

Now take infima over admissible functions g and ρ and apply (13) to get the lower bound. The same argument leads to a contradiction if $\mathrm{mod}_{p^*}\Gamma^*$ is finite and $\mathrm{mod}_p\Gamma = 0$. □

We start the proof of Proposition 4.1 with a classical estimate for Lipschitz functions. See [11, Theorem 7.7] for a euclidean version.

Lemma 4.2. *Let $u : G \to \mathbb{R}$ be L-Lipschitz and let A be a μ-measurable subset of G. Then*

$$\int_{\mathbb{R}}^{*} \mathcal{H}(u^{-1}(t) \cap A) \; dt \leqslant C(X)L\mu(A). \tag{15}$$

Proof. Since μ is a Radon-measure, we may assume that A is open. Let $\delta > 0$. Apply the $5r$-covering theorem to find a countable collection of disjoint balls $\{B_i\}$ with $B_i = B(x_i, r_i) \subset A$, $5r_i \leqslant \delta$ and

$$A \subset \bigcup_i 5B_i.$$

Define a Borel function $g : \mathbb{R} \to [0, \infty]$ with

$$g = \sum_i \frac{\mu(5B_i)}{5r_i} \chi_{u(5B_i)}.$$

Now for every $t \in \mathbb{R}$ we have $\mathcal{H}_\delta(u^{-1}(t) \cap A) \leqslant g(t)$, so by the doubling property of μ,

$$\begin{aligned}
\int_{\mathbb{R}}^{*} \mathcal{H}_\delta(u^{-1}(t) \cap A)\, dt &\leqslant \int_{\mathbb{R}} g(t)\, dt \\
&\leqslant \sum_i \frac{\mu(5B_i)}{5r_i} |u(5B_i)| \\
&\leqslant C(X) L \sum_i \mu(B_i) \\
&\leqslant C(X) L \mu(A).
\end{aligned}$$

Applying the monotone convergence theorem for upper integrals finishes the proof. □

The Poincaré inequality comes into play with the following lemma.

Lemma 4.3. *Let $U \subset G$ be open and connected and suppose $v : U \to \mathbb{R}$ is locally integrable and $\rho_v : X \to [0, \infty]$ is an upper gradient of v in U that vanishes outside G. Let $N \subset U$ be the set of Lebesgue points of v. Then*

$$|v(x) - v(y)| \leqslant C(X)|x - y|(\mathcal{M}_{10|x-y|}\rho_v(x) + \mathcal{M}_{10|x-y|}\rho_v(y))$$

whenever $x, y \in B \cap N$ for some ball B that satisfies $5B \subset U$.

Proof. The case $U = X$ is classical and proved in, for example, [5, Theorem 8.1.7]. We follow the same proof for the case of general U. Let $B = B(x_0, r)$ satisfy $5B \subset U$. Let $x \in B$ be a Lebesgue point of v. The first part of the proof of [5, Theorem 8.1.7] shows that

$$|v(x) - v_B| \leqslant Cr\mathcal{M}_{4r}\rho_v(x) \tag{16}$$

for some constant $C = C(X)$. Let y be another Lebesgue point of v in B. If $r \leqslant \frac{5}{2}|x - y|$, then applying (16) twice gives the desired result. Otherwise apply (16) with $B(x, 2|x - y|)$ instead. □

Proof of Proposition 4.1. By standard real analysis arguments it suffices to show that

$$\int_{[0,1]}^{*} \mathcal{H}(u^{-1}(t) \cap A)\, dt \leqslant C(X) \int_A \rho\, d\mu. \tag{17}$$

Let us first show that

$$\int_{[0,1]}^{*} \mathcal{H}(u^{-1}(t) \cap A \cap B)\, dt \leqslant C(X) \int_{A\cap B} \mathcal{M}_{10\operatorname{diam} B}\rho\, d\mu$$

for any Borel set $A \subset G$ and any ball $B \subset 5B \subset G$. Continuity of u and Lemma 4.3 give

$$|u(x) - u(y)| \leqslant C(X)|x-y|(\mathcal{M}_{10\operatorname{diam} B}\rho(x) + \mathcal{M}_{10\operatorname{diam} B}\rho(y)) \tag{18}$$

for any $x, y \in B$.

Let $B_k = \{x \in B \mid 2^k < \mathcal{M}_{10\operatorname{diam} B}\rho(x) \leqslant 2^{k+1}\}$. Abuse the notation and define the sets $B_{-\infty}$ and B_{∞} as the sets of points $x \in B$ where, respectively, $\mathcal{M}_{10\operatorname{diam} B}\rho(x) = 0$ or $\mathcal{M}_{10\operatorname{diam} B}\rho(x) = \infty$. Recall that we assume u to be locally Lipschitz. Since B is compactly contained in G, $u|_B$ is Lipschitz. Now Lemma 4.2 applied to any Lipschitz extension of $u|_B$ implies that

$$\int_{[0,1]}^{*} \mathcal{H}(u^{-1}(t) \cap A \cap B_{\infty})\, dt = 0,$$

since the integrability of $\mathcal{M}\rho$ implies that $\mu(B_\infty) = 0$. On the other hand, if $B_{-\infty} \neq \emptyset$ then $\rho = 0$ almost everywhere in $5B$. Since we may assume that X is geodesic, it moreover follows that u is constant in B. We conclude that $\mathcal{H}(u^{-1}(t) \cap A \cap B_{-\infty})$ is nonzero for at most one t.

It follows from (18) that $u|_{B_k}$ is $C(X)2^k$-Lipschitz. Let $u_k : X \to \mathbb{R}$ be any Lipschitz extension of $u|_{B_k}$ with the same Lipschitz constant. Now the previous observations together with the monotone convergence theorem, Lemma 4.2 and the definition of B_k give

$$\begin{aligned}\int_{[0,1]}^{*} \mathcal{H}(u^{-1}(t) \cap A \cap B)\, dt &= \sum_k \int_{[0,1]}^{*} \mathcal{H}(u_k^{-1}(t) \cap A \cap B_k)\, dt \\ &\leqslant C(X) \sum_k 2^k \mu(A \cap B_k) \\ &\leqslant C(X) \int_{A\cap B} \mathcal{M}_{10\operatorname{diam} B}\rho\, d\mu.\end{aligned}$$

Applying a Whitney-type covering, see Lemma 2.4, we get

$$\int_{[0,1]}^{*} \mathcal{H}(u^{-1}(t) \cap A)\, dt \leqslant C(X) \int_{A} \mathcal{M}_{10R}\rho\, d\mu,$$

where R is the supremum of the diameters of the balls used in the cover. We can make R arbitrarily small, as is implied by Lemma 2.4. The Lebesgue differentiation theorem and dominated convergence then yield (17). □

5. Proof of Theorem 3.2

Consider the sets

$$\Gamma_j^* = \{S \in \Gamma^* \mid \operatorname{dist}(S, E \cup F) > j^{-1}\}. \tag{19}$$

By applying the proof of Proposition 5.2.11 in [5] and the general Fuglede's lemma, see [2, Theorem 3], it can be shown that

$$\lim_{j\to\infty} \operatorname{mod}_{p^*}\Gamma_j^* = \operatorname{mod}_{p^*}\Gamma^*. \tag{20}$$

The following result is the key tool in connecting the two moduli.

Lemma 5.1. *(Relative isoperimetric inequality)*
Let $S \in \Gamma^*$ and let U be the component of $G - S$ that contains E. There are constants $C = C(X)$ and $\lambda = \lambda(X) > 1$ such that

$$\min\left\{\frac{\mu(B-U)}{\mu(B)}, \frac{\mu(B\cap U)}{\mu(B)}\right\} \leqslant C\frac{r}{\mu(\lambda B)}\mathcal{H}(\partial U \cap \lambda B)$$

for all balls $B \subset\subset G$.

Proof. Given a ball $B \subset\subset G$ there is a larger ball $B' \subset G$ with $B \subset B'$ and $\mathcal{H}(\partial B') < \infty$ (apply Lemma 4.2 to the distance function). Applying Theorem 6.2 of [9] shows that $B' \cap U$ is a so called *set of finite perimeter*. The relative isoperimetric inequality for sets of finite perimeter follows from the 1-Poincaré inequality by [9, Theorem 1.1]. □

Note that Lemma 5.1 requires the weak 1-Poincaré inequality. See Section 6 for examples of spaces that support a weak $(1+\varepsilon)$-Poincaré inequality for a given $\varepsilon > 0$, but no relative isoperimetric inequality.

Fix $\gamma \in \Gamma$. The idea behind the proof of Theorem 3.2 is to construct admissible functions ϕ_j^n of Γ_j^* that are supported close to $|\gamma|$, and then apply Lemma 5.2 below.

Let $n \geqslant 2$ be a natural number and let $\mathcal{B}^n$ be the collection of balls obtained by applying Lemma 2.4 with $\Omega = G'$ and $A = |\gamma| \cap G'$. Moreover, given $k \in \mathbb{Z}$ let $\mathcal{G}_k^n = \mathcal{G}_k$ be the collections of balls constructed in the proof of 2.4.

Now let $S \in \Gamma^*$. Let U be the component of $G - S$ that contains E. Let

$$T_n = \sup\left\{t \in (0,1) \;\middle|\; \frac{\mu(U\cap B)}{\mu(B)} \geqslant \frac{1}{2} \text{ for all } B \in \mathcal{B}^n \text{ such that } \gamma(t) \in B\right\}.$$

Note that there exists an $\varepsilon > 0$, so that $N_\varepsilon(E) \subset U$ and $N_\varepsilon(F) \subset G - U$. Combining this observation with Lemma 2.4 (i) and continuity of γ shows that T_n is well defined and that $T_n \in (0,1)$. It follows that there exist balls $B_i = B(x_i, r_i) \in \mathcal{B}^n$ for $i = 1, 2$ such that $B_1 \cap B_2 \neq \emptyset$ and

$$\frac{\mu(B_1 \cap U)}{\mu(B_1)} \leqslant \frac{1}{2} \leqslant \frac{\mu(B_2 \cap U)}{\mu(B_2)}.$$

Now let $x \in B_1 \cap B_2$ and let $i \in \{1, 2\}$ be the index for which $r_i = \max\{r_1, r_2\}$. Let $B = B(x, 2r_i)$. It follows from Lemma 2.4 (ii) and Lemma 5.1, that

$$C(X) \leqslant \min\left\{\frac{\mu(B - U)}{\mu(B)}, \frac{\mu(B \cap U)}{\mu(B)}\right\} \leqslant C' \frac{r_i}{\mu(\lambda B)} \mathcal{H}(\partial U \cap \lambda B)$$

for some $C' = C'(X)$ and $\lambda = \lambda(X)$. Therefore

$$\mathcal{H}(S \cap \lambda' B_i) \geqslant \mathcal{H}(\partial U \cap \lambda' B_i) \geqslant \frac{1}{C(X)} r_i^{-1} \mu(B_i),$$

where $\lambda' = 1 + 2\lambda$. We conclude that the function

$$\phi^n = C \sum_{B \in \mathcal{B}^n} r_B \mu(B)^{-1} \chi_{\lambda' B},$$

where r_B is the radius of B, is admissible for Γ^*, but it may not be p^*-integrable. This is why we consider the families Γ_j^* instead.

Note that if $5B \in \mathcal{B}^n$ satisfies $B \in \mathcal{G}_{-k}^n$ for sufficiently large k depending on j and n, then given any $S \in \Gamma_j^*$

$$\frac{\mu(U \cap B)}{\mu(B)} \in \{0, 1\}.$$

Here U is again the component of $G - S$ that contains E. Together with the construction of ϕ^n this implies that there is a $k(j, n) \in \mathbb{Z}$ such that

$$\phi_j^n = C \sum_{k \geqslant k(j,n)} \sum_{B : \frac{1}{5} B \in \mathcal{G}_k^n} r_B \mu(B)^{-1} \chi_{\lambda' B}$$

is admissible for Γ_j^*. It is p^*-integrable, since each $\mathcal{G}_k^n$ contains only finitely many balls and G is bounded.

Now let j be large enough, so that $\mathrm{mod}_{p^*} \Gamma_j^*$ is nonzero. The existence of such a j follows by combining Theorem 3.1 with (20). Apply Lemma 2.1 to find a p^*-weakly admissible function ρ_j of Γ_j^* with the property

$$\mathrm{mod}_{p^*} \Gamma_j^* = \int_G \rho_j^{p^*} \, d\mu.$$

Lemma 5.2. *Let ϕ be another p^*-integrable, p^*-weakly admissible function of Γ_j^*. Then*

$$\mathrm{mod}_{p^*} \Gamma_j^* \leqslant \int_G \phi \rho_j^{p^*-1} \, d\mu.$$

Proof. For any $t \in [0, 1]$ let $\omega_t = t\phi + (1 - t)\rho_j$. Now for any t

$$\mathrm{mod}_{p^*} \Gamma_j^* \leqslant \int_G \omega_t^{p^*} \, d\mu$$

with equality at $t = 0$. It follows that

$$0 \leqslant \frac{d}{dt}\Big|_{t=0} \int_G \omega_t^{p^*} \, d\mu = p^* \int_G (\phi - \rho_j)\rho_j^{p^*-1} \, d\mu,$$

which finishes the proof. □

Applying Lemma 5.2, the doubling property of μ, the definition of the Hardy-Littlewood maximal operator and (iii) gives

$$\begin{aligned} \operatorname{mod}_{p^*}\Gamma_j^* &\leqslant \int_G \phi_j^n \rho_j^{p^*-1} \, d\mu \\ &\leqslant C(X) \sum_{B\in\mathcal{B}^n} r_B \fint_{\lambda' B} \rho_j^{p^*-1} \, d\mu \\ &\leqslant C(X) \sum_{B\in\mathcal{B}^n} r_B \inf_{x\in B} \mathcal{M}_{C(X,G)/n}(\rho_j^{p^*-1})(x) \\ &\leqslant C(X) \int_{|\gamma|} \mathcal{M}_{C(X,G)/n}(\rho_j^{p^*-1}) \, d\mathcal{H}^1. \end{aligned}$$

Letting $n \to \infty$ and applying Fuglede's lemma [5, p. 131] we see that $C(\operatorname{mod}_{p^*}\Gamma_j^*)^{-1}\rho_j^{p^*-1}$ is weakly admissible for Γ. Therefore

$$(\operatorname{mod}_p\Gamma)^{\frac{1}{p}} \leqslant C(\operatorname{mod}_{p^*}\Gamma_j^*)^{-1}\left(\int_G \rho_j^{p^*} \, d\mu\right)^{\frac{1}{p}} = C(\operatorname{mod}_{p^*}\Gamma_j^*)^{-\frac{1}{p^*}}.$$

Applying (20) finishes the proof.

6. Counter-examples

The relative isoperimetric inequality is an instrumental part of the proof of Theorem 3.2. By [9] it is equivalent to the weak 1-Poincaré inequality. Let $\varepsilon \in (0,1)$. We now construct a space X that satisfies the hypotheses of Theorem 1.1 apart from the 1-Poincaré inequality. Instead, X will support a $(1+\varepsilon)$-Poincaré inequality.

Let $K \subset [1/4, 3/4]$ be a self-similar Cantor set with Hausdorff-dimension $1-\varepsilon$ and the following property: for all $x \in K$ and $0 < r < 1$

$$\mathcal{H}_\infty^{1-\varepsilon}(K \cap B(x,r)) \geqslant Cr^{1-\varepsilon}$$

for some $C > 0$ that does not depend on r. Let $Q = [0,1]^3 \subset \mathbb{R}^3$ and let $A = [1/4, 3/4] \times K \times \{0\} \subset Q$. Then for any $x \in A$ and $0 < r \leqslant \operatorname{diam}(Q)$

$$\mathcal{H}_\infty^{2-\varepsilon}(A \cap B(x,r)) \geqslant Cr^{2-\varepsilon} \tag{21}$$

for some (other) $C > 0$ that does not depend on r. Let Q_1 and Q_2 be two copies of the space Q. Finally, let $X = Q_1 \sqcup_A Q_2$, two cubes glued together along A. Equip X with the geodesic metric d that restricts to the metrics of the cubes in either cube, and for $x \in Q_1$ and $y \in Q_2$ set

$$d(x,y) = \inf_{a\in A}(|x-a| + |a-y|).$$

Equip X with the measure μ that restricts to the 3-dimensional Lebesgue measure on both cubes. It follows immediately from the definitions that (X, d, μ) is a complete geodesic Ahlfors 3-regular metric space. The validity of a weak $(1+\varepsilon)$-Poincaré inequality follows from (21) and [4, Theorem 6.15].

Now let $E \subset Q_1 - A$ and $F \subset Q_2 - A$ be nondegenerate continua and let $G = X$. Let Γ and Γ^* be as in Theorem 1.1. The modulus $\mathrm{mod}_3\Gamma$ is non-zero and finite, since X is Loewner, see [4]. On the other hand $\mathrm{mod}_{3^*}\Gamma^* = \infty$, since Γ^* does not admit any admissible functions. To see this, note that A separates E and F in G, but has vanishing 2-measure. We conclude that X does not satisfy the upper bound of Theorem 1.1. Note that this implies that X does not support a weak 1-Poincaré inequality. This can also be deduced from the main result of [9].

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland.

E-mail: A.L.: **atte.s.lohvansuu@jyu.fi**
K.R.: **kai.i.rajala@jyu.fi**

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Duality of moduli in regular toroidal metric spaces

A. Lohvansuu

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DUALITY OF MODULI IN REGULAR TOROIDAL METRIC SPACES

Atte Lohvansuu

University of Jyväskylä, Department of Mathematics and Statistics
P. O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland; atte.s.lohvansuu@jyu.fi

Abstract. We generalize a result of Freedman and He [4, Theorem 2.5], concerning the duality of moduli and capacities in solid tori, to sufficiently regular metric spaces. This is a continuation of the work of the author and Rajala [12] on the corresponding duality in condensers.

1. Introduction

Given a metric measure space (T, d, μ), with μ Borel-regular, and a collection Γ of paths in T, the *p-modulus* of Γ is the number

$$\mathrm{mod}_p\Gamma := \inf_\rho \int_T \rho^p \, d\mu,$$

where the infimum is taken over non-negative Borel-functions ρ that satisfy

$$\int_\gamma \rho \, ds \geqslant 1 \tag{1}$$

for all locally rectifiable $\gamma \in \Gamma$. The path modulus is a widely used tool in geometric function theory, especially in connection to quasiconformal mappings [7, 14, 15].

In the 1960s, Gehring [6] and Ziemer [16] proved that the moduli of paths connecting two compact and connected sets in $\mathbf{R}^n$ are dual to the moduli of surfaces that separate the two sets. The moduli of surface families are defined as above, but instead of condition (1) we require

$$\int_S \rho \, d\mathcal{H}^{n-1} \geqslant 1,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-Hausdorff measure. To describe these duality results in more detail, we need to introduce some notation. Given a connected bounded open subset G of any metric space, and disjoint connected compact sets $E, F \subset G$, denote by $\Gamma(E, F; G)$ the family of paths in G that intersect both E and F, and by $\Gamma^*(E, F; G)$ the family of compact subsets of G that separate E and F. We say that a set S separates E and F in G if E and F belong to different components of $G - S$. Triples (E, F, G) are called *condensers*. Let $p^* = \frac{p}{p-1}$ be the dual exponent of $1 < p < \infty$. By Gehring and Ziemer we then have

$$(\mathrm{mod}_p\Gamma(E, F; G))^{\frac{1}{p}} (\mathrm{mod}_{p^*}\Gamma^*(E, F; G))^{\frac{1}{p^*}} = 1 \tag{2}$$

in $\mathbf{R}^n$ with $n \geqslant 2$.

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It was shown by the author and Rajala that a version of (2) holds in Ahlfors q-regular metric spaces that support a 1-Poincaré inequality. In more detail, a special case of what is shown in [12] is

$$\frac{1}{C} \leqslant (\text{mod}_q \Gamma(E,F;G))^{\frac{1}{q}} (\text{mod}_{q^*} \Gamma^*(E,F;G))^{\frac{1}{q^*}} \leqslant C \tag{3}$$

for some constant C that depends only on the data of the space, i.e. the constants that appear in the definitions (see Section 2) of Ahlfors regularity and the Poincaré inequalities. Here E, F and G are as in (2), and the sets in Γ^* are equipped with the $(q-1)$-dimensional Hausdorff measure.

It should be noted that the inequalities (2) and (3) are very similar to the reciprocality condition found in [13] and [8]. One could also equip the surfaces with the so-called perimeter measures instead of the Hausdorff measure. In this direction a result similar to (3) has recently been proved by Jones and Lahti [9].

In this paper we aim to prove a different kind of duality result. Instead of condensers we consider spaces T homeomorphic to the solid torus $\mathbf{S}^1 \times \overline{\mathbf{D}}$. It is natural to ask if the duality results above remain valid for the family of paths that go around the 'hole' and the family of surfaces which are bounded by meridians on the boundary torus. It turns out that this is not the case. Freedman and He [4] studied conformal moduli on riemannian tori in connection with their research on divergence-free vector fields. They showed that the path-modulus can be arbitrarily small compared to the corresponding surface modulus, even in the smooth setting. However, they managed to prove a duality result by replacing the path modulus with a certain capacity.

Suppose now that T is equipped with a metric d and a Borel-regular measure μ, so that (T, d, μ) is Ahlfors q-regular. That is, there are constants $a, A > 0$ such that

$$ar^q \leqslant \mu(B) \leqslant Ar^q$$

for all balls B with radius $r < \text{diam}(T)$.

Following Freedman and He [4] we consider the *degree 1 capacity* instead of the path modulus. It is defined by

$$\text{cap}_p T := \inf_\phi \int_T \text{Lip}(\phi)^p \, d\mu,$$

where the infimum is taken over pointwise Lipschitz constants

$$\text{Lip}(\phi)(x) := \limsup_{r\to 0} \sup_{y \in B(x,r)} \frac{|\phi(x) - \phi(y)|}{r}$$

of Lipschitz maps $\phi\colon T \to \mathbf{S}^1$ of degree 1. Loosely speaking, a map is said to have degree 1 if it takes (oriented) loops which generate the corresponding fundamental group to (oriented) generating loops in $\mathbf{S}^1$. We assume $\mathbf{S}^1$ is equipped with a metric that makes it isometric to a euclidean circle of length 1 equipped with its geodesic metric.

The surface modulus $\text{mod}_p T$ is defined to be the p-modulus of all level sets of continuous functions of degree 1, see Section 2, equipped with the $(q-1)$-dimensional Hausdorff measure. The main results of this paper imply the following.

Theorem 1.1. *Let (T, d, μ) be a compact Ahlfors q-regular metric measure space that supports a weak 1-Poincaré inequality. Suppose T is homeomorphic to the solid* *torus* $\mathbf{S}^1 \times \overline{\mathbf{D}}$*. Let* $1 < p < \infty$*. If* $\mathrm{cap}_p T$ *is nonzero, then*

$$\frac{1}{C} \leqslant (\mathrm{cap}_p T)^{\frac{1}{p}} (\mathrm{mod}_{p^*} T)^{\frac{1}{p^*}} \leqslant C$$

where C *is a constant that depends only on the data of* T*. Moreover* $\mathrm{cap}_p T = 0$ *if and only if* $\mathrm{mod}_{p^*} T = \infty$*.*

A similar result, with $C = 1$, was proved by Freedman and He [4, Theorem 2.5] for smooth solid tori equipped with riemannian metrics.

Theorem 1.1 is obtained from slightly more general statements. These are Theorems 2.2 and 2.3, and they correspond to the lower and upper bounds of the inequality in Theorem 1.1, respectively. The proof of the lower bound is essentially the same as the proof of the lower bound of (3) found in [12]. The main difficulty of the proof of Theorem 1.1 is then the upper bound.

In [12] the proof of the upper bound boils down to showing that given any path γ that connects the two continua E and F, and a neighborhood N_γ of $|\gamma|$, there is a function admissible for the modulus of surfaces separating E and F that is supported in N_γ. This approach cannot be adopted in our current situation, since the paths have been replaced with Lipschitz maps. Instead, given any level set S of a map of degree 1 and a neighborhood N_S of S, we construct a Lipschitz map of degree 1 that is constant outside N_S. Note that this implies that the pointwise Lipschitz constant of this map can be assumed to be supported in N_S. This approach seems to be new. It can be seen as a dual to the one in [12], and as such it can in fact be used to reprove (3).

Section 2 contains some definitions and the main results. Theorems 2.2 and 2.3 are proved in Sections 3 and 4, respectively.

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2. Main results and definitions

For the rest of this text we fix a compact metric measure space (T, d, μ) that supports a weak 1-Poincaré inequality. We also assume that μ is doubling. In order to apply the theory of covering spaces later on, we also have to assume that T is semilocally simply connected (local and global path connectedness follow from the 1-Poincaré inequality [7, 8.3.2]).

We call a measure μ *doubling* if it is Borel-regular and there exists a constant $C_\mu > 1$, such that for every ball $B = B(x, r)$ with radius $r < \mathrm{diam}(T)$

$$0 < \mu(2B) < C_\mu \mu(B) < \infty.$$

Here $2B = B(x, 2r)$.

Let $\mathscr{M}$ be a set of Borel-regular measures on T and let $1 \leqslant p < \infty$. We define the *p-modulus* of $\mathscr{M}$ to be

$$\mathrm{mod}_p \mathscr{M} = \inf \int_T \rho^p \, d\mu,$$

where the infimum is taken over all Borel measurable functions $\rho \colon T \to [0, \infty]$ with

$$\int_T \rho \, d\nu \geqslant 1 \tag{4}$$

for all $\nu \in \mathscr{M}$. Such functions are called *admissible functions of* $\mathscr{M}$. If there are no admissible functions we define the modulus to be infinite. If ρ is an admissible function for $\mathscr{M} - \mathscr{N}$ where $\mathscr{N}$ has zero p-modulus, we say that ρ is *p-weakly admissible* for $\mathscr{M}$. As a direct consequence of the definitions we see that the p-modulus does not change if the infimum is taken over only p-weakly admissible functions. If some property holds for all $\nu \in \mathscr{M} - \mathscr{N}$ we say that it holds for *p-almost every ν* in $\mathscr{M}$.

Given a family Γ of paths in T, the *path p-modulus* of Γ is denoted and defined like the modulus of a family of measures, but instead of (4) it is required that

$$\int_\gamma \rho\, ds \geqslant 1$$

for every locally rectifiable path $\gamma \in \Gamma$.

A Borel function $\rho\colon T \to [0, \infty]$ is an *upper gradient* of a function $u\colon T \to Y$, where (Y, d_Y) is a metric space, if

$$d_Y(u(\gamma(a)), u(\gamma(b))) \leqslant \int_\gamma \rho\, ds \tag{5}$$

for all rectifiable paths $\gamma\colon [a, b] \to T$. The target $Y = [-\infty, \infty]$ is also allowed, but with an additional requirement that the right-hand side of (5) has to equal ∞ whenever either $|u(\gamma(a))| = \infty$ or $|u(\gamma(b))| = \infty$. If the family of paths for which (5) fails has zero p-modulus, we say that ρ is a *p-weak* upper gradient. The inequality (5) is called the *upper gradient inequality* for the pair (u, ρ) on γ.

A p-integrable p-weak upper gradient ρ of u is *minimal* if for any other p-integrable p-weak upper gradient ρ' of u we have $\rho \leqslant \rho'$ μ-almost everywhere. By [7, Theorem 6.3.20] minimal p-weak upper gradients exist whenever p-integrable upper gradients do.

The space T is said to support a *weak p-Poincaré inequality* with constants C_P and λ_P if all balls in T have positive and finite measure, and

$$\fint_B |u - u_B|\, d\mu \leqslant C_P \operatorname{diam}(B) \left(\fint_{\lambda_P B} \rho^p\, d\mu \right)^{\frac{1}{p}}$$

for all locally integrable functions u and all upper gradients ρ of u. Here

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u\, d\mu.$$

In this paper we consider toroidal spaces, meaning that we assume the fundamental group of T to be isomorphic to $\mathbf{Z}$ with respect to any basepoint. Fix a generator $[\alpha_{x_0}] \in \pi_1(T, x_0)$. We say that a loop γ with basepoint $x \in T$ is a *degree 1 loop* if it is loop-homotopic to $\alpha_x = \gamma_{xx_0} * \alpha_{x_0} * \overleftarrow{\gamma}_{xx_0}$ for some path γ_{xx_0} that starts at x and ends at x_0. It can be shown that the equivalence class $[\alpha_x] \in \pi_1(T, x)$ does not depend on the choice of γ_{xx_0}.

For every continuous map $f\colon T \to \mathbf{R}/\mathbf{Z}$ there is a unique integer $\deg f$, called the *degree* of f, so that for every $x \in T$ and every degree 1 loop γ based at x the push-forward $f_*\gamma = f \circ \gamma$ is loop-homotopic to $[f(x)] + \deg f \cdot \beta$, where $\beta\colon [0, 1] \to \mathbf{R}/\mathbf{Z}$ is the path $\beta(t) = [t]$.

Now let $1 < p < \infty$. We define the *degree 1 p-capacity* of T to be the number

$$\operatorname{cap}_p T := \inf \int_T \rho_f^p\, d\mu,$$

where the infimum is taken over all Lipschitz maps $f\colon T \to \mathbf{R}/\mathbf{Z}$ with $\deg f = 1$, and ρ_f denotes the minimal p-weak upper gradient of f. Note that for Lipschitz maps the minimal upper gradient agrees almost everywhere with the pointwise Lipschitz constant $\mathrm{Lip}(f)$, see [3] and [7, 13.5.1]. We assume here and hereafter that $\mathbf{R}/\mathbf{Z}$ is equipped with the metric

$$|[x]-[y]| = \inf_{a\in\mathbf{Z}} |x+a-y|,$$

where the equivalence classes of $\mathbf{R}/\mathbf{Z}$ are denoted by brackets. Observe that with this metric $\mathbf{R}/\mathbf{Z}$ is isometric to a 1-dimensional euclidean sphere of total length 1 equipped with its intrinsic length metric.

Denote by Γ^* the family of all level sets $\phi^{-1}[0]$ with finite codimension 1 spherical Hausdorff measure, where $\phi\colon T \to \mathbf{R}/\mathbf{Z}$ is a continuous map of degree 1. The *codimension 1 spherical Hausdorff measure* is defined by

$$\mathcal{H}(A) := \sup_{\delta>0} \mathcal{H}_\delta(A),$$

where

$$\mathcal{H}_\delta(A) := \inf \sum_i \frac{\mu(B_i)}{r_i},$$

and the infimum is taken over countable covers $\{B_i\}$ of A by balls with radii $r_i \leqslant \delta$. By the Carathéodory construction $\mathcal{H}$ is a Borel-regular measure. A simple application of a coarea estimate, see Proposition 3.1, shows that almost all level sets of Lipschitz maps have finite $\mathcal{H}$-measure. On the other hand, the relative isoperimetric inequality (Lemma 4.6) shows that level sets of Lipschitz maps of degree 1 must have nonzero $\mathcal{H}$-measure.

As a dual counterpart to $\mathrm{cap}_p T$ we consider the surface modulus of Γ^*. We abbreviate

$$\mathrm{mod}_{p^*} T = \mathrm{mod}_{p^*}\{\mathcal{H} \llcorner S \mid S \in \Gamma^*\}. \tag{6}$$

The definitions of $\mathrm{cap}_p T$ and $\mathrm{mod}_{p^*} T$ are rather trivial if Lipschitz maps of degree 1 do not exist. Although path-connected topological spaces with fundamental groups isomorphic to $\mathbf{Z}$ can fail to admit maps of nonzero degree, it seems to be unknown whether the existence of such a map is implied by the additional structure of (T, d, μ). To make life easier we simply assume that there exists at least one Lipschitz map $f\colon T \to \mathbf{R}/\mathbf{Z}$ of degree 1.

Let us gather all of the assumptions into one place for clarity and future reference.

Assumptions 2.1. The metric measure space (T, d, μ) is doubling and supports a weak 1-Poincaré inequality. The space T is compact and semilocally simply connected. The fundamental group of T with respect to any basepoint is isomorphic to $\mathbf{Z}$ and there exists at least one Lipschitz map $\phi\colon T \to \mathbf{R}/\mathbf{Z}$ of degree 1.

With these assumptions our main results are the following

Theorem 2.2. *Let $1 < p < \infty$. If $\mathrm{cap}_p T > 0$, then*

$$\frac{1}{C} \leqslant (\mathrm{cap}_p T)^{\frac{1}{p}} (\mathrm{mod}_{p^*} T)^{\frac{1}{p^*}},$$

where the constant C depends only on the data of T. If $\mathrm{cap}_p T = 0$, then $\mathrm{mod}_{p^*} T = \infty$.

Theorem 2.3. *Let $1 < p < \infty$. If $\mathrm{mod}_{p^*} \Gamma^* < \infty$, then*

$$(\mathrm{cap}_p T)^{\frac{1}{p}} (\mathrm{mod}_{p^*} T)^{\frac{1}{p^*}} \leqslant C,$$

where the constant C depends only on the data of T. If $\mathrm{mod}_{p^*}T = \infty$, then $\mathrm{cap}_p T = 0$.

We say that a constant $C > 0$ depends only on the data of T, denoted $C = C(T)$, if it depends only on the constants C_μ, C_P and λ_P appearing in the definitions of doubling measures and Poincaré inequalities. The same symbol C will be used for various different constants.

If we let the metric measure space (T, d, μ) be as in Theorem 1.1, it satisfies Assumptions 2.1. The existence of Lipschitz maps of degree 1 follows from Proposition 4.5. In Ahlfors q-regular spaces the $\mathcal{H}$-measure is comparable to the $(q-1)$-dimensional Hausdorff measure, so the surface moduli defined using either measure are comparable. Therefore Theorem 1.1 is just a combination of Theorems 2.2 and 2.3.

Note that the conclusions in Theorems 2.2 and 2.3 are invariant under biLipschitz changes of metrics. Also recall that a complete metric space supporting a Poincaré inequality is C-quasiconvex for some $C = C(T)$. This means that the change of metrics $(T, d) \to (T, d')$ is C-biLipschitz, when d' is the intrinsic length metric induced by d. It follows that we may assume without any loss of generality that d is the length metric. It is then implied by compactness that (T, d) is in fact geodesic. Note that in geodesic spaces we can choose $\lambda_P = 1$. For these facts see Theorem 8.3.2 and Remark 9.1.19 in [7].

3. Proof of Theorem 2.2

The proof of Theorem 2.2 is exactly the same as the proof of Theorem 3.1 in [12], but with a different coarea estimate.

Proposition 3.1. *Let $u\colon T \to \mathbf{R}/\mathbf{Z}$ be Lipschitz and let ρ be a p-integrable upper gradient of u in T. Let $g\colon T \to [0, \infty]$ be a p^*-integrable Borel function. Then*

$$\int^*_{\mathbf{R}/\mathbf{Z}} \int_{u^{-1}(t)} g \, d\mathcal{H} \, dt \leqslant C \int_T g\rho \, d\mu \tag{7}$$

for some $C = C(T)$.

Proposition 3.1 follows by applying [12, Proposition 4.1] in small enough balls.

Proof of Theorem 2.2. First assume that $\mathrm{cap}_p T > 0$. If $\mathrm{mod}_{p^*}T = \infty$, there is nothing to prove. Otherwise let $g \in L^{p^*}(T)$ be admissible for $\mathrm{mod}_{p^*}T$. Let $u\colon T \to \mathbf{R}/\mathbf{Z}$ be Lipschitz with degree 1 and note that u must be surjective. Let ρ be an upper gradient of u. We may assume that ρ is p-integrable. Note that by (7) $\mathcal{H}(u^{-1}(t)) < \infty$ for almost every t. Proposition 3.1 and Hölder's inequality give

$$1 \leqslant \int^*_{\mathbf{R}/\mathbf{Z}} \int_{u^{-1}(t)} g \, d\mathcal{H} \, dt \leqslant C \int_T g\rho \, d\mu \leqslant C \left(\int_T g^{p^*} \, d\mu \right)^{\frac{1}{p^*}} \left(\int_T \rho^p \, d\mu \right)^{\frac{1}{p}}.$$

The lower bound follows by taking infima over admissible functions g and ρ. The same argument would lead to a contradiction if $\mathrm{mod}_{p^*}T$ was finite when $\mathrm{cap}_p T = 0$. □

4. Proof of Theorem 2.3

Theorem 2.3 follows, once we have shown that there is a non-negative Borel function ρ_0 defined on T, such that

$$\mathrm{cap}_p T = \int_T \rho_0^p \, d\mu,$$

and that

$$\mathrm{cap}_p T \leqslant C(T) \int_S \mathcal{M}_{C(T)/n}(\rho_0^{p-1})\, d\mathcal{H} \tag{8}$$

for all $S \in \Gamma^*$ and all large enough n, depending on S. Here $\mathcal{M}_r$ for $r > 0$ denotes the restricted Hardy–Littlewood maximal operator, see [7, Chapter 3.5] for its definition and basic properties. Indeed, letting $n \to \infty$ and applying the general Fuglede's lemma [5, Theorem 3] we find that

$$\mathrm{cap}_p T \leqslant C(T) \int_S \rho_0^{p-1}\, d\mathcal{H} \tag{9}$$

for mod_{p^*}-almost every S. Now suppose $\mathrm{mod}_{p^*} T < \infty$. If $\mathrm{cap}_p T = 0$, there is nothing to prove. Otherwise it follows from (9) that the function

$$\frac{C(T)}{\mathrm{cap}_p T} \rho_0^{p-1}$$

is weakly admissible for $\mathrm{mod}_{p^*} T$. Thus

$$(\mathrm{mod}_{p^*} T)^{1/p^*} \leqslant \frac{C(T)}{\mathrm{cap}_p T} \left(\int_T \rho_0^{p^*(p-1)}\, d\mu \right)^{1/p^*} = C(T)(\mathrm{cap}_p T)^{-1/p}.$$

The same calculation shows that $\mathrm{mod}_{p^*} T$ must be finite if $\mathrm{cap}_p T$ is nonzero. This proves Theorem 2.3. The rest of this section is focused on finding ρ_0 and proving (8).

Let us begin by constructing ρ_0. We would like to apply the usual method of constructing minimizers for capacities or moduli. This method would consist of picking a minimizing sequence $(\phi_i)_i$ of Lipschitz maps of degree 1 and their upper gradients $(\rho_i)_i$, applying weak compactness properties of L^p-spaces and Mazur's lemma to find a subsequence of convex combinations of ρ_i that converges strongly to some limit ρ_0, and finally showing that ρ_0 is an upper gradient of a Lipschitz map of degree 1. The obvious flaw with this method is that it is not clear whether the proposed minimizer ρ_0 or the convex combinations of the functions ρ_i are upper gradients of Lipschitz maps of degree 1.

To fix this, we replace the collection of upper gradients of degree 1 Lipschitz maps by a slightly larger collection $\mathcal{F}$ and show in Proposition 4.3 that the capacity does not change if we take the infimum over functions of $\mathcal{F}$ instead. The collection $\mathcal{F}$ is defined using the universal cover $(\tilde{T}, \pi)$ of T, and consists of those non-negative Borel functions ρ on T for which the function $\rho \circ \pi$ is an upper gradient of a Newtonian map, which satisfies an analogue of the degree 1 -property. See Subsection 4.2 for the definition of Newtonian maps. Once we have set the proper definition of $\mathcal{F}$, it is easy to see that it is convex, and by applying the proofs of existing compactness results on Newtonian spaces we show in Proposition 4.4 that the limit ρ_0 is a member of $\mathcal{F}$ as well.

4.1. Universal cover and lifts. We denote the universal cover of T by $(\tilde{T}, \pi)$. The metric $\tilde{d}$ on $\tilde{T}$ is defined as the path metric induced by pulling back the length functional of T with π. This means that given points $\tilde{x}, \tilde{y} \in \tilde{T}$ we define

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf_\gamma \ell(\pi \circ \gamma),$$

where the infimum is taken over all paths in $\tilde{T}$ that connect $\tilde{x}$ and $\tilde{y}$, and $\ell(\pi \circ \gamma)$ is the length of the path $\pi \circ \gamma$. With this metric π becomes a local isometry.

We equip $\tilde{T}$ with the Borel-regular measure $\tilde{\mu}$ that satisfies

$$\tilde{\mu}(A) := \int_{\pi(A)} N(x, \pi, A)\, d\mu(x),$$

for all Borel sets $A \subset \tilde{T}$. Here $N(x, \pi, A)$ denotes the cardinality of $\pi^{-1}(x) \cap A$. The area formula

$$\int_{\tilde{T}} f\, d\tilde{\mu} = \int_T \sum_{y \in \pi^{-1}(x)} f(y)\, d\mu(x)$$

holds for every integrable Borel-function f.

Denote by $\tau\colon \tilde{T} \to \tilde{T}$ the unique deck transform that satisfies

$$\tau(\tilde{\gamma}(0)) = \tilde{\gamma}(1),$$

for all lifts $\tilde{\gamma}\colon [0,1] \to \tilde{T}$ of all degree 1 loops $\gamma\colon [0,1] \to T$. With the additional metric and measure theoretic structure the classic lifting theorems imply the following.

Lemma 4.1. *Suppose $f\colon T \to \mathbf{R}/\mathbf{Z}$ is a Lipschitz map of degree* 1 *and let ρ be one of its upper gradients. There exists a function $\tilde{f}\colon \tilde{T} \to \mathbf{R}$, called the* lift *of f, that satisfies the following properties.*

(1) $[\tilde{f}] = f \circ \pi$. *In particular $\tilde{f}$ is locally Lipschitz.*
(2) $\rho \circ \pi$ *is an upper gradient of* $\tilde{f}$.
(3) $\tilde{f} \circ \tau - \tilde{f} = 1$.

Moreover, if $\tilde{f}'$ is another lift that satisfies the properties above, then there is a $k \in \mathbf{Z}$ such that $\tilde{f}' = \tilde{f} \circ \tau^k = \tilde{f} + k$.

Claim (2) follows from the identity

$$\int_\gamma \rho \circ \pi\, ds = \int_{\pi \circ \gamma} \rho\, ds,$$

which holds for every rectifiable path γ in $\tilde{T}$.

Conversely, we have the following.

Lemma 4.2. *For every locally Lipschitz $g\colon \tilde{T} \to \mathbf{R}$ with $g \circ \tau - g = 1$ there is a Lipschitz map $f\colon T \to \mathbf{R}/\mathbf{Z}$ of degree 1, that satisfies $[g] = f \circ \pi$. Moreover, if ρ_f is the minimal p-weak upper gradient of f in T, then $\rho_f \circ \pi$ is the minimal p-weak upper gradient of g in $\tilde{T}$.*

Proof. We define f locally by

$$f = [g \circ \pi^{-1}].$$

Then f is well defined due to the property $g \circ \tau - g = 1$. It is certainly locally Lipschitz, has degree 1, and satisfies $[g] = f \circ \pi$.

It remains to show the relation between the upper gradients. Given any $x \in \tilde{T}$ there is a ball B' that contains x and on which π is an isometry onto $B = \pi(B')$. Clearly $\rho \circ \pi|_{B'}^{-1}$ is a p-weak upper gradient of f in B whenever ρ is a p-weak upper gradient of g in B'. Thus, if $\rho_f \circ \pi$ is a p-weak upper gradient of g in $\tilde{T}$, it must be the minimal one.

Now let $\gamma\colon [0,1] \to B'$ be a rectifiable path, so that the upper gradient inequality holds for the pair (f, ρ_f) on every subpath of $\pi \circ \gamma$. Almost every path in B' is such a path, since ρ_f is a p-weak upper gradient of f, and as an isometry $\pi|_{B'}$ preserves all path moduli. Continuity of g implies that we can decompose γ into $\gamma = \gamma_1 * \cdots * \gamma_k$, so that $\gamma_i = \gamma|_{[t_i,t_{i+1}]}$ and

$$|g(\gamma_i(t_{i+1})) - g(\gamma_i(t_i))| = |[g(\gamma_i(t_{i+1}))] - [g(\gamma_i(t_i))]| \tag{10}$$

for all $i = 1, \dots, k$. On these subpaths we have

$$\begin{aligned}\int_{\gamma_i} \rho_f \circ \pi \, ds = \int_{\pi\circ\gamma_i} \rho_f \, ds &\geqslant |f(\pi(\gamma_i(t_{i+1}))) - f(\pi(\gamma_i(t_i)))| \\ &= |[g(\gamma_i(t_{i+1}))] - [g(\gamma_i(t_i))]|.\end{aligned}$$

Combining this with triangle inequality and (10) yields

$$|g(\gamma(1)) - g(\gamma(0))| \leqslant \int_\gamma \rho_f \circ \pi \, ds.$$

Given an open set $U \subset \tilde{T}$, denote the set of all paths in U on which the upper gradient inequality fails for the pair $(g, \rho_f \circ \pi)$ by Γ_U. We need to show that $\mathrm{mod}_p \Gamma_{\tilde{T}} = 0$. Cover $\tilde{T}$ by countably many balls B_i', on which π is an isometry onto $\pi(B_i')$. Note that if the upper gradient inequality fails for the pair $(g, \rho_f \circ \pi)$ on some path η, it must fail on some subpath of η that is contained in one of the balls B_i'. In other words, for every path in the collection $\Gamma_{\tilde{T}}$ there is a subpath in one of the collections $\Gamma_{B_i'}$. Now

$$\mathrm{mod}_p \Gamma_{\tilde{T}} \leqslant \mathrm{mod}_p \left(\bigcup_i \Gamma_{B_i'} \right) \leqslant \sum_i \mathrm{mod}_p \Gamma_{B_i'} = 0,$$

since the first part of the proof shows that $\mathrm{mod}_p \Gamma_{B_i'} = 0$ for all i. □

4.2. Minimizers. Motivated by Lemmas 4.1 and 4.2 we find an alternative definition for the capacity.

We say that a function $f \colon \tilde{T} \to \mathbf{R}$ belongs to the Newtonian space $N^{1,p}(\tilde{T})$ if f is p-integrable and admits a p-weak upper gradient that is also p-integrable. See [7, Chapter 7] or [1, Chapter 5] for further properties of these spaces. We say that $f \in N^{1,p}_{\mathrm{loc}}(\tilde{T})$ if $f|_U \in N^{1,p}(U)$ for every open $U \subset\subset \tilde{T}$ (note that $\tilde{T}$ is proper). The space $N^{1,p}(U)$ is equipped with the seminorm

$$\|f\|_{N^{1,p}(U)} := \|f\|_{L^p(U)} + \inf_\rho \|\rho\|_{L^p(U)},$$

where the infimum is taken over all p-weak upper gradients ρ of f in U.

Let $\mathcal{F}$ be the collection of all positive Borel functions ρ on T, for which $\rho \circ \pi$ is a p-weak upper gradient of some $f \in N^{1,p}_{\mathrm{loc}}(\tilde{T})$ with $f \circ \tau - f = 1$ almost everywhere. Define

$$\mathrm{cap}_p^{\mathcal{F}} T := \inf_{\rho \in \mathcal{F}} \int_T \rho^p \, d\mu.$$

Note that by Lemma 4.1 every upper gradient of a map admissible for $\mathrm{cap}_p T$ belongs to $\mathcal{F}$. Therefore

$$\mathrm{cap}_p^{\mathcal{F}} T \leqslant \mathrm{cap}_p T.$$

The reverse inequality is also valid, but requires a bit more work.

Proposition 4.3.

$$\mathrm{cap}_p T = \mathrm{cap}_p^{\mathcal{F}} T.$$

Proof. We must first show that locally Lipschitz functions of degree 1 are dense in the space of degree 1 functions of $N^{1,p}_{\mathrm{loc}}(\tilde{T})$. Here having degree 1 means satisfying the property $f \circ \tau - f = 1$ almost everywhere. A result by Björn and Björn [2, Theorem 8.4.] shows that locally Lipschitz functions are dense in $N^{1,p}_{\mathrm{loc}}(\tilde{T})$. A simple modification of the proof of this result shows that the approximating locally Lipschitz maps can be chosen to be of degree 1 whenever the limit is of degree 1. We provide the main points of this modification.

Following the proof of Theorem 8.4 of [2], we start by choosing for every $x \in \tilde{T}$ a ball B_x centered at x, so that

- the 1-Poincaré inequality and the doubling property hold within B_x, in the sense of [2],
- the covering map π is an isometry on B_x.

Let $U_x := \pi^{-1}(\pi(\frac{1}{4}B_x))$. The space T is compact, so there is a finite subcollection $\{U_{x_i}\}_{i=1}^m$ that covers $\tilde{T}$. Write $B_j = \frac{1}{4}B_{x_j}$ and $U_j = U_{x_j}$. Note that U_j can be written as a disjoint union $U_j = \bigcup_{k\in\mathbf{Z}} \tau^k B_j$. We denote $cU_j = \bigcup_{k\in\mathbf{Z}} \tau^k(cB_j)$ for any $c > 0$. For each j pick a Lipschitz function $\psi'_j \colon B_j \to \mathbf{R}$ that satisfies $\chi_{B_j} \leqslant \psi'_j \leqslant \chi_{2B_j}$. Extend these to Lipschitz functions $\psi_j \colon \tilde{T} \to \mathbf{R}$ first by defining $\psi_j|_{\tau^k(2B_j)} := \psi'_j \circ \tau^{-k}$ in $2U_j$ and then extending as zero to the rest of $\tilde{T}$. Next, define Lipschitz maps $\varphi_j \colon \tilde{T} \to \mathbf{R}$ recursively with $\varphi_1 = \psi_1$ and for $j > 1$

$$\varphi_j = \psi_j \cdot \left(1 - \sum_{k=1}^{j-1} \varphi_k\right).$$

Then $\sum_{k=1}^{i} \varphi_k = 1$ in U_i and $\varphi_j = 0$ in U_i for all $j > i$. Therefore $\{(\varphi_j, U_j)\}_j$ is a partition of unity.

Now let $f \in N^{1,p}_{loc}(\tilde{T})$ be a degree 1 map, $f \circ \tau - f = 1$. Let $\varepsilon > 0$. By Lemma 8.5 of [2] there are locally Lipschitz functions $v_j \colon 2B_j \to \mathbf{R}$ with

$$\|f - v_j\|_{N^{1,p}(2B_j)} \leqslant \frac{\varepsilon}{1 + L_j},$$

where L_j is the Lipschitz constant of φ_j. Extend v_j to $2U_j$ with

$$v_j|_{\tau^k(2B_j)} = k + v_j \circ \tau^{-k}.$$

Then $v_j \circ \tau - v_j = 1$, and for all k

$$\|f - v_j\|_{N^{1,p}(\tau^k(2B_j))} \leqslant \frac{\varepsilon}{1 + L_j}.$$

As in [2] we get

$$\|\varphi_j(f - v_j)\|^p_{N^{1,p}(\tau^k(2B_j))} \leqslant 2\varepsilon^p. \tag{11}$$

The function $v := \sum_{j=1}^m \varphi_j v_j$ is locally Lipschitz, and satisfies the degree 1 property $v \circ \tau - v = 1$.

Now (11) gives

$$\|v - f\|_{N^{1,p}(U)} \leqslant C(U)\varepsilon,$$

for any domain $U \subset\subset \tilde{T}$. This proves the density of degree 1 locally Lipschitz functions in the space of $N^{1,p}_{loc}(\tilde{T})$-functions of degree 1.

Now if we let $\rho \in \mathcal{F}$, the function $\rho \circ \pi$ is a p-weak upper gradient of some $f \in N^{1,p}_{loc}(\tilde{T})$, and we find a sequence of locally Lipschitz functions (v_j) of degree 1, such that

$$\|v_j - f\|_{N^{1,p}(U)} \overset{j\to\infty}{\longrightarrow} 0.$$

for every $U \subset\subset \tilde{T}$. Let w_j be the Lipschitz projections of v_j, given by Lemma 4.2. Then the minimal upper gradients satisfy $\rho_{v_j} = \rho_{w_j} \circ \pi$. Now

$$\|\rho_{v_j} - \rho_f\|_{L^p(B_i)} \overset{j\to\infty}{\longrightarrow} 0 \tag{12}$$

for all i. Let $A_1 = \pi(B_1)$ and for $1 \leqslant j \leqslant m-1$ define $A_{j+1} := \pi(B_{j+1}) - \bigcup_{i=1}^{j} A_j$. Let $\pi_j : B_j \cap \pi^{-1}(A_j) \to A_j$ be the restriction of π and define $\rho'_f := \sum_j \chi_{A_j}\rho_f \circ \pi_j^{-1}$. The Borel sets A_j are disjoint and cover T, so a quick calculation shows that

$$\|\rho'_f\|^p_{L^p(T)} \leqslant \|\rho\|^p_{L^p(T)}, \tag{13}$$

since by definition of ρ we have $\rho_f \leqslant \rho\circ\pi$ almost everywhere. Finally, note that

$$\|\rho_{w_j} - \rho'_f\|^p_{L^p(T)} = \sum_{j=1}^{m} \|\rho_{w_j} - \rho'_f\|^p_{L^p(A_j)} \leqslant \sum_{j=1}^{m} \|\rho_{v_j} - \rho_f\|^p_{L^p(B_j)},$$

and thus (12) implies

$$\lim_{j\to\infty} \|\rho_{w_j}\|^p_{L^p(T)} = \|\rho'_f\|^p_{L^p(T)}, \tag{14}$$

since there are only finitely many sets A_j. Combining (13) and (14) yields

$$\mathrm{cap}_p T \leqslant \|\rho\|^p_{L^p(T)},$$

which finishes the proof. □

Proposition 4.4. *There is a unique minimizer $\rho_0 \in \mathcal{F}$, i.e.*

$$\mathrm{cap}_p T = \mathrm{cap}_p^{\mathcal{F}} T = \int_T \rho_0^p \, d\mu.$$

Moreover, for any other p-integrable $\rho \in \mathcal{F}$

$$\mathrm{cap}_p T \leqslant \int_T \rho_0^{p-1} \rho \, d\mu. \tag{15}$$

Proof. Note that $\mathcal{F}$ is convex. Once we know the existence of a minimizer, the proof of the variation inequality (15) is standard. See for example [12, Lemma 5.2.]. Uniqueness of the minimizer follows from the convexity of $\mathcal{F}$ and the uniform convexity of $L^p(T)$.

We now show the existence of a minimizer. First recall that we have assumed in Assumptions 2.1 that there exists at least one Lipschitz map of degree 1. It follows that $\mathrm{cap}_p T$ is finite. Let $(f_i)_i$ be a sequence of locally Lipschitz maps $f_i \colon T \to \mathbf{R}/\mathbf{Z}$ of degree 1, so that for each i the function ρ_i is an upper gradient of f_i, and

$$\mathrm{cap}_p T = \lim_{i\to\infty} \int_T \rho_i^p \, d\mu.$$

We claim that the lifts $\tilde{f}_i$ of the maps f_i can be chosen so that the sequence $(\tilde{f}_i)$ is L^p-bounded in any bounded domain of $\tilde{T}$.

To this end, note that the length of any loop-homotopically non-trivial loop γ must satisfy

$$\ell(\gamma) \geqslant c \tag{16}$$

for some $c > 0$. This is implied by the existence of Lipschitz maps of degree 1.

Let $\{x_i\}_{i=1}^N$ be a $\frac{c}{16}$-net in T, where c is the constant from (16). Note that by the net property of $\{x_i\}$ any two balls $B_i := B(x_i, \frac{c}{8})$ are connected by a chain of balls of the same form. By a chain we mean a sequence of balls, in which adjacent ones have nonempty intersection. The same chaining property holds for the balls $2B_i$, but now additionally we find that the connecting chains $(2B_{i_k})_k$ can be chosen so that for each k there is a ball $B'_k \subset 2B_{i_k} \cap 2B_{i_{k+1}}$ of radius $c/8$.

Note that by (16) the balls $2B_i$ are evenly covered. In fact, π is an isometry when restricted to any component of $\pi^{-1}(2B_i)$. Fix a component $\tilde{B}_1$ of $\pi^{-1}(B_1)$. Set $V_1 = \tilde{B}_1$. For $k \geqslant 1$ we define domains V_k recursively by adding components of $\pi^{-1}(B_i)$ for suitable B_i. At step $k+1$ we choose exactly one component of $\pi^{-1}(B_i)$, call it $\tilde{B}_i$, to be added to V_k if and only if $\pi^{-1}(B_i)$ intersects V_k and there are no components of $\pi^{-1}(B_i)$ that are contained in V_k.

After at most N steps no new balls can be added. Let $V = V_N$. It follows from the construction that V is a bounded domain on which π is surjective. It may happen that the previous construction does not define $\tilde{B}_i$ for all B_i. If so, just let $\tilde{B}_i$ be a component of $\pi^{-1}(B_i)$ that is contained in V. Thus

$$V = \bigcup_{i=1}^N \tilde{B}_i.$$

Denote by $2\tilde{B}_i$ the component of $\pi^{-1}(2B_i)$ that contains $\tilde{B}_i$.

By adding integers if necessary, we may now fix the lifts $\tilde{f}_i$ by requiring

$$0 \leqslant (\tilde{f}_i)_{2\tilde{B}_1} < 1. \tag{17}$$

If $j \neq 1$, by construction there is a chain $(2\tilde{B}_{j_k})_{k=1}^l$ with $j_1 = 1$, $j_l = j$ and $l \leqslant N$, so that for every $1 \leqslant k < l$ there is a ball $\tilde{B}'_k \subset 2\tilde{B}_{j_k} \cap 2\tilde{B}_{j_{k+1}}$ of radius $c/8$. Let $m := \min\{\mu(B_i)\} > 0$. By the Poincaré inequality and the doubling condition

$$|(\tilde{f}_i)_{2\tilde{B}_{j_k}} - (\tilde{f}_i)_{\tilde{B}'_k}| \leqslant C \fint_{2\tilde{B}_{j_k}} |\tilde{f}_i - (\tilde{f}_i)_{2\tilde{B}_{j_k}}| \, d\tilde{\mu} \leqslant C\|\rho_i\|^p_{L^p(2B_{j_k})},$$

where $C = C(T, p, m, c)$, and the same calculation shows

$$|(\tilde{f}_i)_{2\tilde{B}_{j_{k+1}}} - (\tilde{f}_i)_{\tilde{B}'_k}| \leqslant C\|\rho_i\|^p_{L^p(2B_{j_{k+1}})}$$

as well. Thus by the triangle inequality and (17)

$$|(\tilde{f}_i)_{2\tilde{B}_j}| \leqslant CN\|\rho_i\|^p_{L^p(T)} + 1.$$

Now by the Sobolev–Poincaré inequality, see [7, Thm. 9.1.2], and the local isometry of π

$$\int_{2\tilde{B}_j} |\tilde{f}_i - (\tilde{f}_i)_{2\tilde{B}_j}|^p \, d\tilde{\mu} \leqslant C(T, p, m, c)\|\rho_i\|_{L^p(T)}.$$

It follows that the sequence $(\tilde{f}_i)_i$ is bounded in $L^p(2\tilde{B}_j)$.

Since V is covered by finitely many balls $2B_j$, we find that both sequences $(\tilde{f}_i)_i$ and $(\rho_i \circ \pi)_i$ are bounded in $L^p(V)$, and also in every $L^p(W_k)$, where

$$W_k := \bigcup_{l=-k}^{k} \tau^l V.$$

Note that $W_0 = V$. Now by extracting enough subsequences we may assume that $(\tilde{f}_i)_i$ and $(\rho_i \circ \pi)_i$ converge weakly to functions $\tilde{f}^0$ and $\tilde{\rho}^0$ in $L^p(W_0)$. By Lemma 3.1 of [10] there exist sequences of convex combinations $(\tilde{f}^0_k)$ and $(\tilde{\rho}^0_k)$ of the functions $\tilde{f}_i$

and $\rho_i \circ \pi$, respectively, that converge strongly to $\tilde{f}^0$ and $\tilde{\rho}^0$. Moreover $\tilde{\rho}^0$ is a p-weak upper gradient of $\tilde{f}^0$ in W_0.

This allows us to define sequences $(\tilde{f}_i^{k+1})$ and $(\tilde{\rho}_i^{k+1})$ recursively to be the sequences in $L^p(W_{k+1})$ that are obtained by applying the argument above on W_{k+1} instead of W_0 and on sequences $(\tilde{f}_i^k)$ and $(\tilde{\rho}_i^k)_i$ instead of $(\tilde{f}_i)_i$ and $(\rho_i \circ \pi)_i$. Let $\tilde{f}^{k+1}$ and $\tilde{\rho}^{k+1}$ be the corresponding limits in $L^p(W_{k+1})$. It follows that $\tilde{f}^{k+1}|_{\Omega_k} = \tilde{f}^k$ and $\tilde{\rho}^{k+1}|_{\Omega_k} = \tilde{\rho}^k$. Define $\tilde{f}$ and $\tilde{\rho}\colon \tilde{T} \to \mathbf{R}$ by setting $\tilde{f}|_{W_k} = \tilde{f}^k$ and $\tilde{\rho}|_{W_k} = \tilde{\rho}^k$. It is immediate that $\tilde{\rho}$ is a p-weak upper gradient of $\tilde{f}$.

Consider the diagonal sequences $(\tilde{f}_j^j)_j$ and $(\tilde{\rho}_j^j)_j$. These maps are still convex combinations of the functions $\tilde{f}_i$ and $\rho_i \circ \pi$, respectively. It follows that these sequences converge to $\tilde{f}$ and $\tilde{\rho}$ in $L^p_{\text{loc}}(\tilde{T})$. Moreover $\tilde{f} \circ \tau - \tilde{f} = 1$ and $\tilde{\rho} \circ \tau - \tilde{\rho} = 0$ almost everywhere, since these hold everywhere for all maps in the respective sequences. The latter equality allows us to define ρ_0 by projecting $\tilde{\rho}$. Therefore $\rho_0 \in \mathcal{F}$ and

$$\operatorname{cap}_p^{\mathcal{F}} T = \int_T \rho_0^p \, d\mu,$$

since $(\tilde{\rho}_j^j)$ is still a minimizing sequence, due to convexity of $\mathcal{F}$. □

4.3. Competing admissible maps. Now that the minimizer ρ_0 has been found, the proof of Theorem 2.3 is only missing the proof of (8). Recall that (8) says that for all $S \in \Gamma^*$

$$\operatorname{cap}_p T \leqslant C(T) \int_S \mathcal{M}_{C(T)/n}(\rho_0^{p-1}) \, d\mathcal{H},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator. Given an $S \in \Gamma^*$ we construct suitable Lipschitz maps of degree 1 that are constant outside a small neighborhood of S. Then we can apply the variation inequality (15) of Proposition 4.4 on the upper gradients of these Lipschitz maps to conclude (8).

In this subsection we construct these Lipschitz maps. It turns out that the same construction can be used to obtain Lipschitz maps of degree 1 out of general (continuous) maps of any nonzero degree. We only need to consider maps of positive degree by composing with the antipodal map of $\mathbf{R}/\mathbf{Z}$ if necessary.

To simplify the notation, we omit some parentheses and write for example $\phi^{-1}[0]$ and $\pi\tilde{\phi}^{-1}(0)$ instead of $\phi^{-1}([0])$ and $\pi(\tilde{\phi}^{-1}(0))$ from now on.

Proposition 4.5. *Let $\phi\colon T \to \mathbf{R}/\mathbf{Z}$ be a continuous map of nonzero positive degree. There is a number $N = N(\phi)$, such that for all $n \geqslant N$ there is a finite pairwise disjoint collection of balls $\{B_i\}$ of radius $1/n$ in T, such that for all i*

$$\mathcal{H}(\phi^{-1}[0] \cap B_i) \geqslant C(T) n \mu(B_i)$$

and such that the Borel function

$$\rho = n \sum_i \chi_{5B_i}$$

is an upper gradient of a Lipschitz map $\psi\colon T \to \mathbf{R}/\mathbf{Z}$ of degree 1.

Proof of (8) *assuming Proposition 4.5.* Let $S \in \Gamma^*$. Then $S = \phi^{-1}[0]$ for some degree 1 map ϕ. Let $\{B_i\}$ be the collection of balls and let ρ be the Borel function that is obtained by applying Proposition 4.5 for some large enough n. Now

$$\mathcal{H}(S \cap B_i) \geqslant C(T) n \mu(B_i)$$

for all i. Applying this along with the variation inequality (15) of Corollary 4.4, the doubling property of μ and the definition of the Hardy-Littlewood maximal operator gives

$$\begin{aligned}\operatorname{cap}_p T &\leqslant \int_T \rho\rho_0^{p-1}\,d\mu \leqslant C(T)\sum_i n\mu(B_i)\fint_{5B_i}\rho_0^{p-1}\,d\mu\\ &\leqslant C(T)\sum_i \mathcal{H}(S\cap B_i)\inf_{x\in B_i}\mathcal{M}_{C(T)/n}(\rho_0^{p-1})(x)\\ &\leqslant C(T)\int_S \mathcal{M}_{C(T)/n}(\rho_0^{p-1})\,d\mathcal{H},\end{aligned}$$

which is exactly (8). □

The rest of the section is focused on proving Proposition 4.5. Let $\phi\colon T\to\mathbf{R}/\mathbf{Z}$ be a continuous map of nonzero positive degree. Let $x_0\in\phi^{-1}[0]$, $\tilde{x}_0\in\pi^{-1}(x_0)$ and let $\tilde{\phi}\colon\tilde{T}\to\mathbf{R}$ be the lift of ϕ that satisfies $\tilde{\phi}(\tilde{x}_0)=0$. Compactness of T implies that

$$\delta := \min\left\{d\left(\pi\tilde{\phi}^{-1}\left(\pm\frac{1}{8}\right),\pi\tilde{\phi}^{-1}(0)\right),d\left(\pi\tilde{\phi}^{-1}\left(\pm\frac{1}{4}\right),\pi\tilde{\phi}^{-1}\left(\pm\frac{1}{8}\right)\right)\right\} \tag{18}$$

is strictly positive. Denote $U^+=\pi\tilde{\phi}^{-1}(0,1/4)$ and $U^-=\pi\tilde{\phi}^{-1}(-1/4,0]$. Denote also $S=\pi\tilde{\phi}^{-1}(0)$. Observe that $S\subset\phi^{-1}[0]$, and if $\deg\phi=1$, then $S=\phi^{-1}[0]$.

For our intents and purposes the relative isoperimetric inequality takes the following form.

Lemma 4.6. (Relative isoperimetric inequality) *There are constants $C=C(T)$ and $\lambda=\lambda(T)\geqslant 1$ such that*

$$\min\left\{\frac{\mu(B\cap U^+)}{\mu(B)},\frac{\mu(B\cap U^-)}{\mu(B)}\right\}\leqslant C\frac{r}{\mu(\lambda B)}\mathcal{H}(S\cap\lambda B)$$

for all balls $B=B(x,r)$ for which $\lambda B\subset\pi\tilde{\phi}^{-1}(-1/4,1/4)$.

This formulation is essentially the same as the one used in [12, Lemma 5.1], which is just an application of Theorems 6.2 and 1.1 of [11]. The same proof is valid here as well. Note that restricting the balls to $\phi^{-1}(-1/4,1/4)$ ensures that $\partial U^+\cap\lambda B\subset S\cap\lambda B$.

Denote by Γ the set of all paths γ that connect $\pi\tilde{\phi}^{-1}(-1/8)$ to $\pi\tilde{\phi}^{-1}(1/8)$ inside $\pi\tilde{\phi}^{-1}(-1/4,1/4)$.

Corollary 4.7. *For every $n>\frac{1}{\delta}$ and $\gamma\in\Gamma$ there is a ball B_γ^n that is centered on γ, has radius $\frac{1}{n}$ and satisfies*

$$\mathcal{H}(S\cap B_\gamma^n)\geqslant Cn\mu(B_\gamma^n)$$

for some constant $C=C(T)$.

Proof. The proof is essentially contained in the discussion following Lemma 5.1 in [12]. We sketch the idea here for completeness. Given a path $\gamma\colon[0,1]\to T$ of Γ, we consider the balls $B_t:=B(\gamma(t),\frac{1}{2\lambda n})$, where λ is as in Lemma 4.6. We may assume that $|\gamma|$ is contained in $\pi\tilde{\phi}^{-1}[-1/8,1/8]$, and therefore by the definition of δ each B_t is contained in $\pi\tilde{\phi}^{-1}(-1/4,1/4)$. Now the function

$$\Phi\colon t\mapsto\frac{\mu(U^+\cap B_t)}{\mu(B_t)}$$

vanishes when t is near 0 and is equal to 1 when t is near 1. Pick

$$t_0 := \sup\{t \in (0,1) \mid \Phi(t) \leqslant 1/2\}$$

and choose $B^n_\gamma := 2\lambda B_{t_0}$. The lower bound on the measure of the boundary is then given by the relative isoperimetric inequality. □

Now let $\mathcal{F}_n$ be the collection of balls B^n_γ that arise from the paths in Γ as in Corollary 4.7 with n fixed. Apply the $5r$ covering theorem on $\mathcal{F}_n$ to find a pairwise disjoint subcollection $\mathcal{G}_n$ with the property

$$\bigcup_{B\in\mathcal{F}_n} B \subset \bigcup_{B\in\mathcal{G}_n} 5B.$$

Note that $\mathcal{G}_n$ must be finite due to the compactness of T. Write $\mathcal{G}_n = \{B_i\}_{i=1}^N$. Define a positive Borel function $\rho\colon T \to \mathbf{R}$ with

$$\rho := n \sum_{i=1}^{N} \chi_{5B_i}.$$

Let Ω be the open set that consists of the points that can be connected to $\pi\tilde{\phi}^{-1}(-1/8)$ by a rectifiable path inside $\pi\tilde{\phi}^{-1}(-1/4,1/4)$. Define a function $\tilde{\psi}\colon T \to \mathbf{R}$ inside Ω with

$$\tilde{\psi}(x) := \inf_{\gamma_x} \int_{\gamma_x} \rho\, ds,$$

where the infimum is taken over all rectifiable paths γ_x that connect $\pi\tilde{\phi}^{-1}(-1/8)$ to x inside $\pi\tilde{\phi}^{-1}(-1/4,1/4)$. Extend $\tilde{\psi}$ as zero to the rest of T. Finally, the desired competing admissible map $\psi\colon T \to \mathbf{R}/\mathbf{Z}$ is defined by

$$\psi(x) := [\min\{1, \tilde{\psi}(x)\}].$$

Lemma 4.8. *The mapping ψ is Lipschitz and ρ is one of its upper gradients.*

Proof. It is straightforward to prove that ρ is an upper gradient of both $\tilde{\psi}$ and $\min\{1,\tilde{\psi}\}$ in Ω, see e.g. [1, Lemma 5.25]. Let γ be a rectifiable path in T that connects two points $x, y \in T$. The upper gradient inequality for the pair (ψ,ρ) on γ is immediate if $x, y \in \Omega$ and $|\gamma| \subset \Omega$, or if $\psi(x) = \psi(y)$.

In order to prove the upper gradient inequality in the other possible situations we need to show that $\tilde{\psi} \geqslant 1$ on $\pi\tilde{\phi}^{-1}(1/8,1/4) \cap \Omega$. To this end, let η be a rectifiable path that connects $\pi\tilde{\phi}^{-1}(-1/8)$ to a point $x \in \pi\tilde{\phi}^{-1}(1/8,1/4)$ inside $\pi\tilde{\phi}^{-1}(-1/4,1/4)$. Then η has a subpath $\eta' \in \Gamma$. Let $B^n_{\eta'} \in \mathcal{F}_n$ be the ball obtained by applying Corollary 4.7 on η'. Now

$$\int_\eta \rho\, ds \geqslant \int_{|\eta'|} \rho\, d\mathcal{H}^1 \geqslant n \sum_{i=1}^{N} \mathcal{H}^1(|\eta'| \cap 5B_i) \geqslant n\mathcal{H}^1(|\eta'| \cap B^n_{\eta'}) \geqslant 1,$$

since $B^n_{\eta'}$ is covered by the balls $5B_i$. This holds for every connecting path η, which implies that $\tilde{\psi}(x) \geqslant 1$.

Next assume $x, y \in \Omega$ with $\tilde{\psi}(x), \tilde{\psi}(y) \in (0,1)$ and $|\gamma| \not\subset \Omega$. Note that $\min\{1,\tilde{\psi}\}$ equals 0 in $\pi\tilde{\phi}^{-1}(-1/4,-1/8) \cap \Omega$, since ρ vanishes there. This means that there exist subpaths $\gamma_1 = \gamma|_{[0,t_1]}$ and $\gamma_2 = \gamma|_{[t_2,1]}$ of γ that satisfy $|\gamma_1| \cup |\gamma_2| \subset \Omega$ and

$\psi(\gamma(t_1)) = \psi(\gamma(t_2)) = [0]$. Therefore

$$|\psi(x) - \psi(y)| \leqslant |\psi(x) - \psi(\gamma(t_1))| + |\psi(\gamma(t_2)) - \psi(y)| \leqslant \int_{\gamma_1} \rho\, ds + \int_{\gamma_2} \rho\, ds \leqslant \int_{\gamma} \rho\, ds.$$

The same argument can be applied in the case of $x \in \Omega$, $y \notin \Omega$. We omit the details.

The upper gradient inequality implies that ψ is Lipschitz, since T is geodesic and ρ is bounded. □

4.4. Degree of $\boldsymbol{\psi}$. In this subsection we prove that $\deg \psi = 1$.

Pick a rectifiable degree 1 loop γ and a point $a \in (1/8, 1/4)$. We may now assume that the endpoints of γ are on $\pi\tilde{\phi}^{-1}(a)$. Since T is geodesic and semilocally simply connected, we may assume that γ has finite length. This, and moving the starting point if necessary, allows us to decompose γ into

$$\gamma = (\gamma_1 * \eta_1) * \cdots * (\gamma_k * \eta_k), \tag{19}$$

so that each γ_i intersects $\pi\tilde{\phi}^{-1}(a)$ precisely at the endpoints, and none of the paths η_i intersect $\pi\tilde{\phi}^{-1}(-a)$.

For the next lemma we denote for brevity $\zeta := \min\{1, \tilde{\psi}\}$.

Lemma 4.9. *Let $\eta\colon [0,1] \to \Omega$ be a rectifiable path. Suppose that the endpoints of $\zeta_*\eta$ belong to $\{0, 1\}$. Then $\psi_*\eta$ is loop-homotopic to $\zeta_*\eta(1) - \zeta_*\eta(0)$ times the standard generator of $\pi_1(\mathbf{R}/\mathbf{Z}, [0])$.*

Proof. If the starting point is 0 and the end point is 1, the homotopy is given by $H\colon [0,1]^2 \to \mathbf{R}/\mathbf{Z}$,

$$H(s,t) = [s\zeta_*\eta(t) + (1-s)t].$$

It is straightforward to check all the requirements. The other cases are similar. □

Corollary 4.10. *The paths $\psi_*\eta_i$ and $\phi_*\eta_i$ are loop-contractible.*

Proof. The endpoints of the path η_i must be in the set $\pi\tilde{\phi}^{-1}(a)$. Since γ has finite length, η_i can be decomposed into

$$\eta_i = \eta_i^1 * \cdots * \eta_i^l,$$

where the endpoints of each η_i^j are in $\pi\tilde{\phi}^{-1}(a)$, and if $|\eta_i^j| \not\subset \Omega$, then there are no other intersections with $\pi\tilde{\phi}^{-1}(a)$.

Now if η_i^j is contained in Ω, Lemma 4.9 implies that it is loop-contractible. Otherwise $\psi_*\eta_i^j$ is already a constant path. Therefore $\psi_*\eta_i$ is loop-contractible as well. The path $\phi_*\eta_i$ cannot be surjective, so it is loop-contractible. □

Let $\alpha\colon \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ be the isomorphism $\alpha[x] = [x - a]$. Note that $\alpha_*\phi_*\gamma_i$ and $\psi_*\gamma_i$ are all loops with the same basepoint $[0]$.

Denote the domain of γ_i by $[a_i, b_i]$. Let $\gamma_i'\colon [a_i, b_i] \to \mathbf{R}$ be the unique lift of $\alpha_*\phi_*\gamma_i$ for which $\gamma_i'(a_i) = 0$. Further decompose each γ_i into

$$\gamma_i = \gamma_i^1 * \gamma_i^2 * \gamma_i^3,$$

where γ_i^1 and γ_i^3 intersect $\pi\tilde{\phi}^{-1}(\pm a)$ exactly at their endpoints.

Lemma 4.11. *The lifted path γ_i' intersects integer multiples of $\deg \phi$ exactly at its endpoints. In particular $\gamma_i'(b_i) = \pm \deg \phi$ or $\gamma_i'(b_i) = 0$. Moreover, γ_i^1 (respectively γ_i^3) is contained in Ω if and only if γ_i' is negative in a neighborhood of a_i ($\gamma_i' \leqslant \gamma'(b_i)$ in a neighborhood of b_i).*

Proof. Let $\tilde{\gamma}_i\colon [a_i, b_i] \to \tilde{T}$ be the lift of γ_i that satisfies $\tilde{\phi}_*\tilde{\gamma}_i(a_i) = a$. Then due to uniqueness of lifts we have $\gamma_i' = \tilde{\phi}_*\tilde{\gamma}_i - a$. Since $\tilde{\phi}$ is a lift of ϕ, we have $\tilde{\phi} \circ \tau^k = k \cdot \deg \phi + \tilde{\phi}$ for any integer k. It follows that $\gamma_i'(t) = k \cdot \deg \phi$ if and only if $\tilde{\phi}(\tau^{-k}(\tilde{\gamma}_i(t))) = a$, which can be combined with the lifting property $\pi_*\tilde{\gamma}_i = \gamma_i$ to conclude that $\gamma_i'(t)$ equals an integer multiple of $\deg \phi$ if and only if $\gamma_i(t) \in \pi\tilde{\phi}^{-1}(a)$. By construction the latter happens if and only if t equals either endpoint of $[a_i, b_i]$. This proves the first assertion of the lemma.

The definitions of γ_i^1, γ_i^3 and Ω imply that these paths are contained in Ω if and only if they are contained in $\pi\tilde{\phi}^{-1}[-a, a]$. Therefore γ_i^1 is contained in Ω if and only if the part of $\tilde{\gamma}_i$ corresponding to γ_i^1 is contained in $\tilde{\phi}^{-1}(k \cdot \deg \phi + [-a, a])$ for some fixed integer k. This k must be 0, since we chose $\tilde{\phi}_*\tilde{\gamma}_i(a_i) = a$. Thus $\gamma_i' = \tilde{\phi}_*\tilde{\gamma}_i - a$ is negative in a neighborhood of a_i if and only if γ_i^1 is contained in Ω. The path γ_i^3 can be treated similarly. □

Corollary 4.12. *The paths* $\alpha_*\phi_*\gamma_i$ *and* $\deg \phi \cdot \psi_*\gamma_i$ *are loop-homotopic.*

Proof. We need to check four different cases, corresponding to γ_i^1 and γ_i^3 being or not being contained in Ω. The proofs are essentially the same, so we write down only one of them.

Assume that γ_i^1 is not contained in Ω but γ_i^3 is. Then $\psi_*\gamma_i^1$ is a constant path, and $\psi_*\gamma_i^3$ is loop-homotopic to the standard generator by Lemma 4.9. Arguing exactly as in the proof of Corollary 4.10, we see that $\psi_*\gamma_i^2$ is loop-contractible. Therefore $\psi_*\gamma_i$ is loop-homotopic to the standard generator.

By Lemma 4.11 the lift γ_i' satisfies $\gamma_i'(a_i) = 0$ and $\gamma_i'(b_i) = \pm\deg \phi$ or $\gamma_i'(b_i) = 0$. We also find that γ_i' is positive in a neighborhood of a_i, and less than $\gamma_i'(b_i)$ in a neighborhood of b_i. Combining these gives $\gamma_i'(b_i) = \deg \phi$, which means precisely that $\alpha_*\phi_*\gamma_i$ is loop-homotopic to $\deg \phi$ times the standard generator. □

Applying Corollaries 4.10 and 4.12 to the decomposition (19) yields

$$\alpha_*\phi_*\gamma \simeq \deg \phi \cdot \psi_*\gamma.$$

Now by applying the identity $\deg(\alpha \circ \phi) = \deg \phi$, we see that $\psi_*\gamma$ is loop-homotopic to the standard generator. Therefore $\deg \psi = 1$ and the proof of Proposition 4.5 is finished.

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On the duality of moduli in arbitrary codimension

A. Lohvansuu

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ON THE DUALITY OF MODULI IN ARBITRARY CODIMENSION

ATTE LOHVANSUU

Abstract. We study the duality of moduli of k- and $(n-k)$-dimensional slices of euclidean n-cubes, and establish the optimal upper bound 1.

1. Introduction and the main result

Suppose $D \subset \mathbb{R}^2$ is a Jordan domain, whose boundary is divided into four segments $\zeta_1, \dots, \zeta_4$, in cyclic order. Let $\Gamma(\zeta_1, \zeta_3; D)$ be the family of all paths of D that connect ζ_1 and ζ_3. Then for every $1 < p < \infty$

(1) $$(\mathrm{mod}_p \Gamma(\zeta_1, \zeta_3; D))^{1/p} (\mathrm{mod}_q \Gamma(\zeta_2, \zeta_4; D))^{1/q} = 1.$$

Here $q = \frac{p}{p-1}$ and the p-modulus of a path family Γ is defined by

$$\mathrm{mod}_p \Gamma = \inf_{\rho} \int_D \rho^p \, d\mathcal{H}^2,$$

where the infimum is taken over all positive Borel-functions ρ with

$$\int_\gamma \rho \, ds \geqslant 1$$

for every locally rectifiable path $\gamma \in \Gamma$. The path modulus is a fundamental tool in geometric function theory and nonsmooth analysis [9, 17, 21].

For conformal moduli, that is $p = 2 = q$, the duality (1) was already known to Beurling and Ahlfors, see e.g. [1, Lemma 4] and [2, Ch. 14], although instead of moduli they considered their reciprocals, called *extremal lengths*. For general p the identity (1) follows from the results of [23]. It has found applications in connection with uniformization theorems [16, 11] and Sobolev extension domains [22].

The *duality of moduli* phenomenon (1) is also present in euclidean spaces [5, 7, 23] of higher dimension and sufficiently regular metric spaces [12, 13, 14]. For example, in [23] it is shown that

(2) $$(\mathrm{mod}_p \Gamma(E, F; G))^{1/p} (\mathrm{mod}_q \Gamma^*(E, F; G))^{1/q} = 1,$$

where $G \subset \mathbb{R}^n$ is open and connected, E and F are disjoint, compact and connected subsets of G and $\Gamma^*(E, F; G)$ is the set of all compact

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sets of G that separate E from F. The modulus of separating sets is a natural generalization of the definition of the path modulus. See Section 2 for definitions of moduli and other concepts appearing in the introduction.

Separating sets are generally of codimension 1, so (1) and (2) deal with objects of either dimension or codimension 1. In fact, this is a common theme in all of the results cited above. However, an observation by Freedman and He (see the discussion after Theorem 2.5 in [5]) hints that a similar duality result could be true for objects of higher (co)dimension as well. In this paper we explore this question in the setting of cubes of $\mathbb{R}^n$.

Moduli of higher (co)dimensional objects have appeared in [10, 15], where the nonexistence of quasisymmetric parametrizations of certain spaces was established. Indeed, one of the main motivations for studying more general moduli is finding tools to approach parametrization problems in higher dimensions.

Our first problem is defining suitable classes of k- and $(n-k)$-dimensional objects, since simple descriptions such as "connecting paths" or "separating surfaces" do not seem to exist. We follow [5] and define the objects as representatives of certain relative homology classes. For example, in the context of (1) we can think of the paths of $\Gamma(\zeta_1, \zeta_3; D)$ as singular relative cycles, that are representatives of either generator of $H_1(D, \zeta_1 \cup \zeta_3) \simeq \mathbb{Z}$. Since we also want to integrate over the chains, we need to assume some regularity. For this reason we will consider Lipschitz chains instead of singular chains.

Let $Q \subset \mathbb{R}^n$ be a compact set homeomorphic to the closed unit n-cube I^n. Fix a homeomorphism $h : Q \to I^n$ and an integer $0 < k < n$ and let

$$A = h^{-1}(\partial I^k \times I^{n-k}) \text{ and } B = h^{-1}(I^k \times \partial I^{n-k}).$$

Then A and B are $(n-1)$-dimensional submanifolds of ∂Q with $\partial Q = A \cup B$ and $\partial A = A \cap B = \partial B$. We assume that A, B and Q are locally Lipschitz neighborhood retracts. This includes triples (Q, A, B) that are smooth or polygonal, and cubes that are images of the standard cube under biLipschitz automorphisms of $\mathbb{R}^n$.

We denote the Lipschitz homology groups by H_*^L. We consider only groups with integer coefficients. This notation should not be confused with the Hausdorff measures, which are denoted by $\mathcal{H}^*$. Note that

$$H_k^L(Q, A) \simeq \mathbb{Z} \simeq H_{n-k}^L(Q, B),$$

since the same is true for singular homology, and the two homology theories are equivalent for pairs of locally Lipschitz retracts (see Lemma 2.1).

Let Γ_A (resp. Γ_B) be the collection of the images of relative Lipschitz k-cycles of $Q - B$ that generate $H_k^L(Q, A)$ ($(n-k)$-cycles of $Q - A$ that generate $H^L_{n-k}(Q,B)$). Define

$$\mathrm{mod}_p\Gamma_A := \inf_\rho \int_Q \rho^p \, d\mathcal{H}^n,$$

where the infimum is taken over positive Borel-functions ρ, for which

$$\int_S \rho \, d\mathcal{H}^k \geqslant 1$$

for every $S \in \Gamma_A$. The moduli $\mathrm{mod}_p\Gamma_B$ are defined analogously. In this paper we will prove the following upper bound.

THEOREM 1.1. *For every* $1 < p < \infty$

$$(\mathrm{mod}_p\Gamma_A)^{1/p}(\mathrm{mod}_q\Gamma_B)^{1/q} \leqslant 1,$$

where $q = \frac{p}{p-1}$.

It is unknown, whether Theorem 1.1 holds with an equality. We will prove Theorem 1.1 in Section 3. A similar result for de Rham cohomology classes, with an equality, is proved in the setting of Riemannian manifolds in pages 212-213 of [5].

The assumption on Q, A and B being locally Lipschitz neighborhood retracts can be relaxed. The proof of Theorem 1.1 only requires that there exists a pair of Lipschitz chains that generate $H_k(Q,A)$ and $H_{n-k}(Q,B)$. The assumption on retracts was chosen for its simplicity and its use in [4]. It is also likely that such minimal assumptions on the upper bound of Theorem 1.1 are not sufficient for the corresponding lower bound. We will discuss the lower bound in Section 4.

In light of the results of [4, Ch. 4], it would be interesting to know whether analogues of Theorem 1.1 hold for homology classes of integral currents.

2. Definitions

2.1. **Lipschitz homology.** Let us recall the definition and basic properties of the integral homology groups. See e.g. [3, 8] or other texts on basic algebraic topology for more comprehensive treatment.

For an integer $k \geqslant 0$ the *standard k-simplex* Δ_k is the convex hull of the standard unit vectors $e_0, \ldots, e_k$ of $\mathbb{R}^{k+1}$. Given a metric space (X,d), a *singular k-simplex* is a continuous map from Δ_k to X. Finite formal linear combinations

$$\sigma = \sum_i k_i \sigma_i$$

of singular k-simplices σ_i with integer coefficients k_i are called *singular k-chains*. Singular k-chains of X form a free abelian group denoted by $C_k(X)$. The *boundary* $\partial\sigma$ of a singular k-simplex σ is the singular $(k-1)$-chain

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i \sigma \circ F_k^i,$$

where $F_k^i : \Delta_{k-1} \to \Delta_k$ is the unique linear map that maps each e_j to e_j for $j < i$ and to e_{j+1} for $j \geqslant i$. For singular 0-simplices we set $\partial\sigma = 0$. The boundary defines a collection of homomorphisms $\partial : C_k(X) \to C_{k-1}(X)$, all denoted by the same symbol ∂. Then $\partial\partial = 0$.

The *image* of a singular k-simplex σ is the compact set $|\sigma| = \sigma(\Delta_k)$. The image of a k-chain $\sigma = \sum_i k_i\sigma_i$ is the compact set $|\sigma| = \bigcup_i |\sigma_i|$.

Given a subspace $Y \subset X$, we identify each singular simplex σ of Y with the singular simplex $i_Y \circ \sigma$ of X, where $i_Y : Y \hookrightarrow X$ is the inclusion map. We define the groups of *relative chains* by

$$C_k(X, Y) := \frac{C_k(X)}{C_k(Y)},$$

with the convention $C_k(X, \emptyset) = C_k(X)$. The boundary map induces homomorphisms $\partial : C_k(X, Y) \to C_{k-1}(X, Y)$, which are again denoted by the same symbol. A chain $\sigma \in C_k(X)$ is called a *cycle* relative to Y, if $\partial\sigma \in C_{k-1}(Y)$, or simply a relative cycle if the choice of Y is clear from the context. Similarly, σ is called a relative *boundary* if $\sigma = \partial\sigma' + \sigma''$, where $\sigma' \in C_{k+1}(X)$ and $\sigma'' \in C_k(Y)$.

The *singular relative homology groups* of the pair (X, Y) are the quotient groups

$$H_k(X, Y) := \frac{\ker(\partial : C_k(X, Y) \to C_{k-1}(X, Y))}{\operatorname{im}(\partial : C_{k+1}(X, Y) \to C_k(X, Y))}.$$

The homology groups of X are the groups $H_k(X) := H_k(X, \emptyset)$. The homology class of a (relative) chain σ is denoted by $[\sigma]$. The homology classes of $H_k(X, Y)$ are represented by relative k-cycles, and two relative k-cycles define the same class if and only if their difference is a relative boundary.

If X' is another metric space with a subset Y', and $f : X \to X'$ is a continuous map with $f(Y) \subset Y'$, we denote by f_* the induced homomorphisms $f_* : C_k(X, Y) \to C_k(X', Y')$, and also the homomorphisms $f_* : H_k(X, Y) \to H_k(X', Y')$. These are given by $f_*\sigma = f \circ \sigma$ for singular simplices, $f_* \sum_i k_i\sigma_i = \sum_i k_i f_*\sigma_i$ for chains and $f_*[\sigma] = [f_*\sigma]$ for homology classes.

Given a continuous homotopy $H : X \times I \to X'$ with $H(Y \times I) \subset Y'$, there exists a sequence of homomorphisms

$$P : C_k(X, Y) \to C_{k+1}(X', Y'),$$

such that

$$H_{1*} - H_{0*} = P\partial + \partial P. \tag{3}$$

Here $H_t(x) = H(x,t)$. Formula (3) is called the *homotopy formula*.

A continuous $f : X \to Y$ is called a *retraction* if $f \circ i_Y = \mathrm{id}_Y$. The set Y is then called a *retract* of X. If Y is a retract of one if its neighborhoods in X, it is called a *neighborhood retract*.

The corresponding objects in the Lipschitz category are obtained by replacing each occurrence of "singular" or "continuous" with "Lipschitz". The homotopies involved in these definitions are then required to be Lipschitz with respect to the metric $d((x,t),(x',t')) = d(x,x')+|t-t'|$. We denote the groups of Lipschitz chains by $C_*^L(X,Y)$ and the Lipschitz homology groups by $H_*^L(X,Y)$. We define *locally* Lipschitz objects similarly. However, due to compactness there is often no difference between the corresponding objects of Lipschitz and locally Lipschitz categories.

Lemma 2.1. *Let $Y \subset X \subset \mathbb{R}^n$ be locally Lipschitz neighborhood retracts. Then the inclusions*

$$i : C_*^L(X,Y) \hookrightarrow C_*(X,Y)$$

induce isomorphisms on homology.

Lemma 2.1 follows from a more general result [18, Cor. 11.1.2], which holds for pairs of *locally Lipschitz contractible* metric spaces. It is straightforward to show that the existence of locally Lipschitz neighborhood retractions implies locally Lipschitz contractibility.

2.2. **Modulus.** Given a $1 < p < \infty$ and a family $\mathcal{M}$ of Borel measures of $\mathbb{R}^n$, the *p-modulus* of $\mathcal{M}$ is the number

$$\mathrm{mod}_p\mathcal{M} := \inf_\rho \int_{\mathbb{R}^n} \rho^p \, d\mathcal{H}^n, \tag{4}$$

where the infimum is taken over all Borel functions $\rho : \mathbb{R}^n \to [0,\infty)$ with

$$\int_{\mathbb{R}^n} \rho \, d\nu \geqslant 1 \tag{5}$$

for every $\nu \in \mathcal{M}$. Such functions are called *admissible* for $\mathcal{M}$. If there exists a subfamily $\mathcal{N} \subset \mathcal{M}$ such that $\mathrm{mod}_p\mathcal{N} = 0$ and (5) holds for all $\nu \in \mathcal{M} - \mathcal{N}$, we say that ρ is *p-weakly admissible* or simply *weakly admissible* if the choice of p is clear from the context. It follows that the infimum in (4) does not change if we take it over p-weakly admissible functions instead. Let us list some useful properties of the modulus.

Lemma 2.2. *Let $\mathcal{M}$ be a collection of Borel measures of $\mathbb{R}^n$. Let $1 < p < \infty$.*

i) If ρ_i are p-integrable Borel functions that converge to a function ρ in L^p, there exists a subsequence $(\rho_{i_j})_j$ for which

$$\int_{\mathbb{R}^n} \rho_{i_j} \, d\nu \overset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} \rho \, d\nu$$

for almost every $\nu \in \mathcal{M}$*. In particular, Borel representatives of* L^p*-limits of admissible functions are weakly admissible.*

ii) If $\mathrm{mod}_p\mathcal{M} < \infty$*, then*

$$\mathrm{mod}_p\mathcal{M} = \int_{\mathbb{R}^n} \rho^p \, d\mathcal{H}^n$$

for a weakly admissible minimizer ρ*, unique up to sets of* $\mathcal{H}^n$*-measure zero. Moreover,*

$$\mathrm{mod}_p\mathcal{M} \leqslant \int_{\mathbb{R}^n} \phi\rho^{p-1} \, d\mathcal{H}^n$$

for any other p*-integrable weakly admissible* ϕ*.*

iii) If $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i$ *with* $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ *for all* i*, then*

$$\mathrm{mod}_p\mathcal{M} = \lim_{i\to\infty} \mathrm{mod}_p\mathcal{M}_i.$$

Claim $i)$ is often referred to as *Fuglede's lemma*. Proofs for $i)$ and the first part of $ii)$ can be found in [6, Thm. 3]. The second part of $ii)$ and $iii)$ are generalizations of [14, Lemma 5.2] and [24, Lemma 2.3], respectively. The same proofs apply.

In this paper we abbreviate

$$\mathrm{mod}_p\Gamma_A = \mathrm{mod}_p\{\mathcal{H}^k \llcorner S \mid S \in \Gamma_A\},$$

and

$$\mathrm{mod}_q\Gamma_B = \mathrm{mod}_q\{\mathcal{H}^{n-k} \llcorner S^* \mid S^* \in \Gamma_B\}.$$

2.3. **Rectifiable sets.** A subset of $\mathbb{R}^n$ is *k-rectifiable* if it is covered by the image of a subset of $\mathbb{R}^k$ under a Lipschitz map. A subset of $\mathbb{R}^n$ is *countably k-rectifiable* if $\mathcal{H}^k$-almost all of it is contained in a countable union of k-rectifiable sets.

See e.g. [4, 20] for basic theory on rectifiable sets. Note that the definition of countable rectifiability in [4, 3.2.14] is slightly different from ours.

Let us record some useful facts on rectifiable sets. The following Fubini-type lemma is an application of [4, 3.2.23] and [4, 2.6.2].

Lemma 2.3. *Suppose* S^* *is a countably* k*-rectifiable subset of* $\mathbb{R}^n$ *and* S *is a countable union of* l*-rectifiable subsets of* $\mathbb{R}^m$*. Then* $S^* \times S$ *is a countably* $(k+l)$*-rectifiable subset of* $\mathbb{R}^n \times \mathbb{R}^m$*, and*

$$\int_{S^*\times S} g(x,y) \, d\mathcal{H}^{k+l}(x,y) = \int_{S^*} \int_S g(x,y) \, d\mathcal{H}^l(y) \, d\mathcal{H}^k(x)$$

for any positive Borel function g *on* $\mathbb{R}^n \times \mathbb{R}^m$*.*

Lemma 2.3 is not true for general countably k-rectifiable sets S, see [4, 3.2.24]. The second tool we need is the coarea formula, see e.g. [20, 12.7].

Lemma 2.4. *Suppose $m \leqslant k$. Let S be a countably k-rectifiable subset of $\mathbb{R}^n$ and let $u : S \to \mathbb{R}^m$ be locally Lipschitz. Then*

$$\int_{\mathbb{R}^m} \int_{u^{-1}(z)} g \, d\mathcal{H}^{k-m} \, d\mathcal{H}^m(z) = \int_S g J_u^S \, d\mathcal{H}^k \tag{6}$$

for every positive Borel function g on S.

Let us define the jacobian J_u^S appearing in (6). Details can be found in [20, §12]. Suppose first, that S is an embedded C^1 k-submanifold (without boundary) of $\mathbb{R}^n$. Then u is differentiable at $\mathcal{H}^k$-almost every $x \in S$. Fix such an x, and let $\{E_1, \dots, E_k\}$ be an orthonormal basis for the tangent space of S at x. Let $Du(x)$ be the jacobian matrix of u at x with respect to standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. We set

$$J_u^S(x) := \sqrt{\det(d^S u(x) d^S u(x)^t)},$$

where $d^S u(x)$ is the matrix with columns $Du(x)E_i$. It can be shown that $J_u^S(x)$ does not depend on the choice of the basis $\{E_i\}$.

More generally, every countably k-rectifiable set S can be expressed as a disjoint union $S = \bigcup_{i=0}^{\infty} M_i$, where $\mathcal{H}^k(M_0) = 0$ and each M_i for $i \geqslant 1$ is contained in an embedded C^1 k-submanifold N_i of $\mathbb{R}^n$. Given an $x \in M_i$ with $i \geqslant 1$, we set

$$J_u^S(x) := J_u^{N_i}(x).$$

Then J_u^S is well defined $\mathcal{H}^k$-almost everywhere on S. It can be shown that J_u^S does not depend on the decomposition $S = \bigcup_{i=0}^{\infty} M_i$, up to sets of $\mathcal{H}^k$-measure zero.

3. Proof of Theorem 1.1

Given any set $S \subset \mathbb{R}^n$ and a vector $y \in \mathbb{R}^n$ we denote

$$S_y = \{x + y \mid x \in S\}$$

and

$$N_\varepsilon(S) = \{x \mid d(x, S) < \varepsilon\}.$$

Denote by Γ_A^* the collection of $(n-k)$-rectifiable subsets S^* of $Q - A$, such that the homomorphism

$$i_* : H_k^L(Q - S^*, A) \to H_k^L(Q, A)$$

induced by inclusion is trivial. Lemma 3.5 below implies that $\Gamma_B \subset \Gamma_A^*$. Every set $S^* \in \Gamma_A^*$ intersects with every $S \in \Gamma_A$ in a nonempty set. To see this, note that if $|\sigma| \cap S^*$ is empty for some Lipschitz cycle $\sigma \in C_k(Q)$ relative to A, then $[\sigma] = i_*[\sigma] = 0$ in $H_k^L(Q, A)$ by the definition of Γ_A^*.

We abbreviate

$$\mathrm{mod}_q \Gamma_A^* := \mathrm{mod}_q \{\mathcal{H}^{n-k} \llcorner S^* \mid S^* \in \Gamma_A^*\}.$$

Theorem 1.1 is then implied by the following more general result.

THEOREM 3.1. *For every* $1 < p < \infty$

$$(\mathrm{mod}_p \Gamma_A)^{1/p} (\mathrm{mod}_q \Gamma_A^*)^{1/q} \leqslant 1,$$

where $q = \frac{p}{p-1}$.

The rest of this section is focused on the proof of Theorem 3.1.

For each $\delta > 0$ let Γ_A^δ be the subcollection of Γ_A consisting of those sets whose distance to B is at least 100δ. The subcollections $\Gamma_A^{*\delta}$ are defined analogously. In light of *iii*) of Lemma 2.2, it suffices to show that

$$(\mathrm{mod}_p \Gamma_A^\delta)^{1/p} (\mathrm{mod}_q \Gamma_A^{*\delta})^{1/q} \leqslant 1 \tag{7}$$

for all δ. Fix a δ for the rest of the proof. We may assume without loss of generality that the moduli in question are nonzero and the collections Γ_A^δ and $\Gamma_A^{*\delta}$ are nonempty.

The following intersection property of the elements of Γ_A and Γ_A^* forms the topological core of Theorem 3.1.

Proposition 3.2. *The intersection* $S_z \cap S^*$ *is nonempty for every* $S \in \Gamma_A^\delta$, $S^* \in \Gamma_A^{*\delta}$ *and* $|z| < 10\delta$.

We postpone the proof to Subsection 3.1.

Let $S \in \Gamma_A^\delta$. Observe that the map

$$g \mapsto \int_S g \, d\mathcal{H}^k \tag{8}$$

is a distribution in $\mathbb{R}^n$. Thus we have by [4, 4.1.2] that

$$\int_Q \phi_\varepsilon^S g \, d\mathcal{H}^n \overset{\varepsilon \to 0}{\longrightarrow} \int_S g \, d\mathcal{H}^k \tag{9}$$

for every smooth compactly supported function g, where

$$\phi_\varepsilon^S(x) := \int_S \phi_\varepsilon(x-y) \, d\mathcal{H}^k(y)$$

is the *convolution* of the distribution (8) with respect to a smooth kernel ϕ. That is, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$ and ϕ is a positive smooth function on $\mathbb{R}^n$ that vanishes outside the unit ball $\mathbb{B}^n$ and satisfies $\int_{\mathbb{B}^n} \phi \, d\mathcal{H}^n = 1$.

Smoothness is convenient for avoiding tedious technicalities, but to see the geometry behind the arguments that follow, the reader is encouraged to repeat the proof with the nonsmooth kernel $\phi = |\mathbb{B}^n|^{-1}\chi_{\mathbb{B}^n}$.

Theorem 3.1 follows via (7) from the following proposition.

Proposition 3.3. *The convolution* $\phi_\varepsilon^{S_z}$ *is admissible for* $\Gamma_A^{*\delta}$ *for all* $\varepsilon < \delta$ *and all* $|z| < \delta$.

Proof. Fix an $\varepsilon < \delta$ and a set $S^* \in \Gamma_A^{*\delta}$. Let $z = 0$ for now. By Lemma 2.3

$$\begin{aligned}\int_{S^*} \phi_\varepsilon^S(x)\, d\mathcal{H}^{n-k}(x) &= \int_{S^*}\int_S \phi_\varepsilon(x-y)\, d\mathcal{H}^k(y) d\mathcal{H}^{n-k}(x) \\ &= \int_{S^*}\int_{S\cap N_\varepsilon(S^*)} \phi_\varepsilon(x-y)\, d\mathcal{H}^k(y) d\mathcal{H}^{n-k}(x) \\ &= \int_{(S^*\times S)\cap\{|x-y|<\varepsilon\}} \phi_\varepsilon(x-y)\, d\mathcal{H}^n(x,y).\end{aligned}$$

Now we can apply the coarea formula (Lemma 2.4) on the map $u(x,y) = x - y$ to obtain

(10)
$$\int_{S^*} \phi_\varepsilon^S(x)\, d\mathcal{H}^{n-k}(x) \geqslant \int_{\varepsilon\mathbb{B}^n}\int_{(S^*\times S)\cap\{x-y=w\}} \phi_\varepsilon(x-y)\, d\mathcal{H}^0 d\mathcal{H}^n(w)$$

since $J_u^{S^*\times S} \leqslant 1$. To see this, note for any $(n-k)$- and k-dimensional embedded C^1 submanifolds N^* and N of $\mathbb{R}^n$ the matrix $d^{N^*\times N}u$ consists of unit column vectors. Thus $J_u^{N^*\times N} \leqslant 1$. It follows that $J_u^{S^*\times S} \leqslant 1$ as well, since it can be computed via $J_u^{M_i^*\times M_j}$ with $i, j \geqslant 1$, where $S^* = \bigcup_{i=0}^\infty M_i^*$ and $S = \bigcup_{i=0}^\infty M_j$ are decompositions of S^* and S as in the discussion following Lemma 2.4. Note that the sets $M_0^* \times S$ and $S^* \times M_0$ have zero $\mathcal{H}^n$-measure by Lemma 2.3.

Finally, we apply Proposition 3.2 on (10) and obtain

$$\int_{S^*} \phi_\varepsilon^S(x)\, d\mathcal{H}^{n-k}(x) \geqslant \int_{\varepsilon\mathbb{B}^n} \phi_\varepsilon(w)\, d\mathcal{H}^n(w) = 1.$$

The proof in the case of general z reduces to the case $z = 0$ via

(11)
$$\phi_\varepsilon^{S_z}(x) = \phi_\varepsilon^S(x-z),$$

since Proposition 3.2 can still be applied. □

Proof of Theorem 3.1. The q-modulus of $\Gamma_A^{*\delta}$ is finite by Proposition 3.3. Let ρ be the unique weak minimizer of $\mathrm{mod}_q\Gamma_A^{*\delta}$ given by *ii)* of Lemma 2.2. We may assume that ρ vanishes in $N_{10\delta}(A)$ and is defined as zero outside Q. Let g_r be the smooth convolution

$$g_r(x) := \int_{r\mathbb{B}^n} \rho^{q-1}(x+y)\phi_r(y)\, d\mathcal{H}^n(y).$$

Let $S \in \Gamma_A^\delta$ and let $\varepsilon < \delta$. Proposition 3.3 and *ii)* of Lemma 2.2 imply

$$\mathrm{mod}_q\Gamma_A^{*\delta} \leqslant \int_Q \phi_\varepsilon^{S_z}\rho^{q-1}\, d\mathcal{H}^n$$

for all $|z| < \delta$ and $S \in \Gamma_A^\delta$. Note that the product $\phi_\varepsilon^{S_z}\rho^{q-1}$ vanishes in $N_{10\delta}(\partial Q)$, so by (11) and a change of variables

$$\mathrm{mod}_q\Gamma_A^{*\delta} \leqslant \int_Q \phi_\varepsilon^S(x)\rho^{q-1}(x+z)\, d\mathcal{H}^n(x)$$

for all $|z| < \delta$. Multiplying both sides by $\phi_r(z)$ and integrating over z yields

$$\mathrm{mod}_q \Gamma_A^{*\delta} \leqslant \int_Q \phi_\varepsilon^S g_r \, d\mathcal{H}^n$$

by Fubini's theorem. Letting $\varepsilon \to 0$ and then $r \to 0$ yields

$$\mathrm{mod}_q \Gamma_A^{*\delta} \leqslant \int_S \rho^{q-1} \, d\mathcal{H}^k$$

for mod_p-almost every $S \in \Gamma_A^\delta$ by (9) and i) of Lemma 2.2. Thus

$$\frac{1}{\mathrm{mod}_q \Gamma_A^{*\delta}} \rho^{q-1}$$

is weakly admissible for Γ_A^δ, so

$$\mathrm{mod}_p \Gamma_A^\delta \leqslant (\mathrm{mod}_q \Gamma_A^{*\delta})^{1-p},$$

which is a rearrangement of (7). □

3.1. **Topological lemmas.** In this subsection we complete the proof of Theorem 1.1 by proving Proposition 3.2 and showing that $\Gamma_B \subset \Gamma_A^*$. These are implied by the following two lemmas.

Lemma 3.4. *Suppose $S \in \Gamma_A^\delta$ and $|y| < 10\delta$. Then there exists a singular relative cycle σ_y, such that it generates $H_k(Q, A)$ and its image coincides with S_y outside $N_{100\delta}(A)$.*

Lemma 3.5. *Suppose σ_A and σ_B are relative singular chains that generate nontrivial elements of $H_k(Q, A)$ and $H_{n-k}(Q, B)$, respectively. Then $|\sigma_A| \cap |\sigma_B|$ is nonempty.*

Proof of Lemma 3.4. The lemma follows from the homotopy formula (3). By the definition of Γ_A there is a relative cycle σ that generates $H_k(Q, A)$ and has S as its image. By applying barycentric subdivision multiple times, if necessary, we may assume that σ splits into $\sigma = \sigma_1 + \sigma_2$, where $|\sigma_1| \subset N_{30\delta}(A)$ and $|\sigma_2| \subset Q - N_{20\delta}(\partial Q)$. Let H_t be the homotopy $H_t(x) = x + ty$. Then by (3) there exist homomorphisms $P : C_l(U) \to C_{l+1}(U_y)$ for all l and all open sets $U \subset \mathbb{R}^n$, such that

$$H_{1*} - H_{0*} = \partial P + P\partial. \tag{12}$$

Note that $P(\partial\sigma_2)$ and $H_{1*}\sigma_2$ are chains in $Q - N_{10\delta}(\partial Q)$. We let $\sigma_y = \sigma_1 - P(\partial\sigma_2) + H_{1*}\sigma_2$. Then $\sigma_y - \sigma = \partial P\sigma_2$ by (12), so σ_y belongs to the same relative homology class as σ. To prove the final part of the lemma, note that $|\partial\sigma_2| \subset N_{30\delta}(A)$, since $|\partial\sigma_2| = |\partial\sigma_1| \cap \mathrm{int}(Q)$. Thus $|P(\partial\sigma_2)| \subset N_{40\delta}(A)$ and $|\sigma_y|$, $|H_{1*}\sigma_2| = |\sigma_2|_y$ and S_y all coincide outside $N_{100\delta}(A)$. □

Proof of Lemma 3.5. The lemma follows from the theory of intersection numbers developed in [3]. We may assume that $Q = J^n$, where $J = [-1, 1]$, and respectively $A = \partial J^k \times J^{n-k}$ and $B = J^k \times \partial J^{n-k}$. Let σ_A and σ_B be representatives of some nontrivial classes of $H_k(Q, A)$

and $H_{n-k}(Q, B)$, respectively. Suppose $|\sigma_A| \cap |\sigma_B| = \emptyset$. Then we can deform σ_A and σ_B slightly, if necessary, and assume that $|\sigma_A| \cap B = \emptyset = |\sigma_B| \cap A$. This allows us to define the *intersection number* $[\sigma_A] \circ [\sigma_B] \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \simeq \mathbb{Z}$ of the classes $[\sigma_A]$ and $[\sigma_B]$, as in [3, VII.4].

The intersection number of the two classes is defined (up to sign) by pushing the outer product

$$[\sigma_A] \times [\sigma_B] \in H_n(Q \times Q, A \times Q \cup Q \times B)$$

forward with the map $u(x, y) = x - y$. Notice the analogy with the proof of Proposition 3.3. We do not describe the definition of the outer product here, as it is rather complicated and would take us too far away from the main topic.

Let us compute the intersection number by using two different pairs of representatives for $[\sigma_A]$ and $[\sigma_B]$. On one hand, since the images of the representatives σ_A and σ_B do not intersect, Propositions 4.5 and 4.6 of [3, VII] imply that $[\sigma_A] \circ [\sigma_B] = 0$. On the other hand, $[\sigma_A]$ and $[\sigma_B]$ admit representatives that are integer multiples of triangulations of the subspaces $J^k \times \{0\}$ and $\{0\} \times J^{n-k}$, so combining Proposition 4.5 and Example 4.10 of [3, VII] shows that $[\sigma_A] \circ [\sigma_B]$ is nontrivial. □

4. Lower bound and related open problems

Theorems 1.1 and 3.1 raise the question:

Question 4.1. Do the lower bounds

$$1 \leqslant (\text{mod}_p \Gamma_A)^{1/p} (\text{mod}_q \Gamma_B)^{1/q} \tag{13}$$

or

$$1 \leqslant (\text{mod}_p \Gamma_A)^{1/p} (\text{mod}_q \Gamma_A^*)^{1/q} \tag{14}$$

hold whenever Q, A and B are as in Theorem 1.1?

Since $\Gamma_B \subset \Gamma_A^*$, (13) implies (14). All existing proofs, save the one in [5], of such lower bounds rely on some variation of the coarea formula, Lemma 2.4.

In [5] a lower bound is proved for de Rham cohomology classes. Hence it may be possible to answer Question 4.1 by finding a connection between the moduli of Γ_A and Γ_B, which can be seen as moduli of homology classes, and the moduli of suitable cohomology classes. This is of course easier said than done. For instance, it is not very clear what "suitable cohomology" should mean, when Q is nonsmooth. It seems these kinds of questions are still largely unexplored.

Let us sketch a proof of (14) in the special case $k = 1$. Then A consists of two opposite faces A_0 and A_1 of Q and, recalling the notation from the introduction,

$$\text{mod}_p \Gamma_A = \text{mod}_p \Gamma(A_0, A_1; Q).$$

Moreover, by [19]

$$\text{mod}_p\Gamma(A_0, A_1; Q) = \text{cap}_p\Gamma(A_0, A_1; Q), \tag{15}$$

where the (Lipschitz) *capacity* is defined by

$$\text{cap}_p\Gamma(A_0, A_1; Q) := \inf_u \int_Q |\nabla u|^p \, d\mathcal{H}^n,$$

and the infimum is taken over Lipschitz functions $u : Q \to I$ with $u|_{A_0} = 0$ and $u|_{A_1} = 1$. Then by the coarea formula

$$1 \leqslant \int_I \int_{u^{-1}(t)} \rho \, d\mathcal{H}^{n-1} dt = \int_Q \rho |\nabla u| \, d\mathcal{H}^n$$

for any integrable ρ admissible for Γ_A^*, since by [4, 3.2.15] almost every level set $u^{-1}(t)$ is an element of Γ_A^*. Now the lower bound (14) follows from Hölder's inequality and (15).

Similar ideas can be used to prove that Theorems 1.1 and 3.1 are sharp for any n and k. Let us show that (13) holds whenever $Q = Q_1 \times Q_2$, where $Q_1 \subset \mathbb{R}^k$ and $Q_2 \subset \mathbb{R}^{n-k}$ are k- and $(n-k)$-dimensional topological cubes as in Theorem 1.1, $A = \partial Q_1 \times Q_2$ and $B = Q_1 \times \partial Q_2$. Then it suffices to show that

$$\text{mod}_p\Gamma_A = \frac{\mathcal{H}^{n-k}(Q_2)}{\mathcal{H}^k(Q_1)^{p-1}} \quad \text{and} \quad \text{mod}_q\Gamma_B = \frac{\mathcal{H}^k(Q_1)}{\mathcal{H}^{n-k}(Q_2)^{q-1}}.$$

The proofs of the two formulas are identical, so we only consider Γ_A. For every $y \in Q_2$ and ρ admissible for Γ_A

$$1 \leqslant \int_{Q_1 \times \{y\}} \rho \, d\mathcal{H}^k,$$

so by Hölder's inequality

$$1 \leqslant \left(\int_{Q_1 \times \{y\}} \rho^p \, d\mathcal{H}^k \right)^{1/p} \mathcal{H}^k(Q_1)^{1/q},$$

from which we obtain the inequality "$\geqslant$" by integrating over y and applying Fubini's theorem (or the coarea formula applied on the projection $\pi_2(x, y) = y$). The reverse inequality follows from the observation that $\mathcal{H}^k(Q_1)^{-1}\chi_Q$ is admissible for Γ_A.

It is also noteworthy that in this case $\text{mod}_q\Gamma_B = \text{mod}_q\Gamma_A^*$, and both are equal to the q-modulus of the slices $\{x\} \times Q_2$.

Observe that if we let $\lambda = \mathcal{H}^k(Q_1)^{-1/k}$ and use a scaled projection map $\lambda\pi_1(x, y) = \lambda x$ instead, we find that $\mathcal{H}^k(\lambda\pi_1(Q_1 \times Q_2)) = 1$ and $J_{\lambda\pi_1} = \mathcal{H}^k(Q_1)^{-1}\chi_Q$. That is, the minimizer of $\text{mod}_p\Gamma_A$ is the jacobian of $\lambda\pi_1$. Moreover, the level sets of $\lambda\pi_1$ are elements of Γ_B.

Inspired by this example we extend the definition of the capacity to general Q and A by

$$\text{cap}_p\Gamma_A := \inf_u \int_Q J_u^p \, d\mathcal{H}^n,$$

where the infimum is taken over all such Lipschitz maps $u : (Q, A) \to (\bar{U}, \partial U)$, that U is a domain in $\mathbb{R}^k$ normalized with $\mathcal{H}^k(U) = 1$, $(\bar{U}, \partial U)$ is homeomorphic to $(\bar{\mathbb{B}}^k, \partial \mathbb{B}^k)$, and the induced homomorphism

$$u_* : H_k(Q, A) \to H_k(\bar{U}, \partial U) \simeq \mathbb{Z} \tag{16}$$

is an isomorphism. We observe that $U \subset u(S)$ for any $S \in \Gamma_A$, so almost every level set of u is in Γ_A^*, since $H_k(\bar{U} - \{x\}, \partial U)$ is trivial for all $x \in U$. Moreover, the Cauchy-Binet formula implies that $J_u \geqslant J_u^S$, so

$$\int_S J_u \, d\mathcal{H}^k \geqslant \int_S J_u^S \, d\mathcal{H}^k \geqslant \int_U d\mathcal{H}^k = 1$$

by Lemma 2.4. Thus J_u is admissible for Γ_A and

$$\mathrm{mod}_p \Gamma_A \leqslant \mathrm{cap}_p \Gamma_A .$$

It is unknown whether the reverse inequality is true, but it would imply (14). To prove the reverse inequality one would have to be able to construct the required Lipschitz maps u. This seems to be very difficult when $k > 1$, especially with a given J_u. If $k = 1$, the situation is considerably simpler, since then $J_u = |\nabla u|$ and the unit interval I is practically the only choice of U.

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland.

E-mail: **atte.s.lohvansuu@jyu.fi**