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Admissibility versus $\mathrm{A_p}$-Conditions on Regular Trees

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Abstract: We show that the combination of doubling and $(1, p)$-Poincaré inequality is equivalent to a version of the $\mathrm{A_p}$-condition on rooted K-ary trees.

Keywords: $\mathrm{A_p}$-condition; doubling measure; Poincaré inequality; regular tree

MSC: 30L99, 31C45, 46E35

Dedicated to Professor Pekka Koskela on the occasion of his 59th birthday celebration

1 Introduction

The class of p-admissible weights for Sobolev spaces and differential equations on $\mathbb{R}^n$ was introduced in [12]. The definition was initially based on four conditions, but Theorem 2 in [10] and Theorem 5.2 in [13] reduce them to the following two conditions, see also [12, 2nd ed., Section 20].

Definition 1.1. A measure μ on $\mathbb{R}^n$ is *p-admissible*, $1 \le p < \infty$, if it is doubling and supports a $(1, p)$-Poincaré inequality. If $d\mu = w\,dx$, we also say that the weight w is p-admissible.

Here μ supports a (q, p)-Poincaré inequality, $1 \le q < \infty$, $1 \le p < \infty$, if there is a constant $C > 0$ such that

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^q \, d\mu \right)^{1/q} \le Cr \left(\fint_{B(x,r)} |\nabla u|^p \, d\mu \right)^{1/p}$$

for every $u \in C^1(\mathbb{R}^n)$, every $x \in \mathbb{R}^n$ and all $r > 0$.

In [12, Section 15], it was shown that Muckenhoupt $\mathrm{A_p}$-weights are p-admissible, but the converse is not true in $\mathbb{R}^n$, $n \ge 2$, see also [6]. Surprisingly, on the real line $\mathbb{R}$, any p-admissible measure is actually given by an $\mathrm{A_p}$-weight, see [7]. Very recently, it was also shown in [5] that a measure on $\mathbb{R}$ is locally p-admissible if and only if it is given by a local $\mathrm{A_p}$-weight. Moreover, on $\mathbb{R}^n$, p-admissible measures can be characterized by a stronger version of the Poincaré inequality, the (q, p)-Poincaré inequality with $q > p$. Under doubling, the $(1, p)$-Poincaré inequality improves to a (q, p)-Poincaré inequality with $q > p$ by [10] and any measure satisfying (q, p)-Poincaré inequality with $q > p$ is a doubling measure, see [1] and [17].

In the recent years, analysis on regular trees has been under development, see [3, 18–21]. Given a K-regular tree X (a rooted K-ary tree), $K \ge 1$, we introduce a metric structure on X by considering each edge of X to be an isometric copy of the unit interval. Then the distance between two vertices is the number of edges needed to connect them and there is a unique geodesic that minimizes this number. Let us denote the root

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by 0. If x is a vertex, we define $|x|$ to be the distance between 0 and x. Since each edge is an isometric copy of the unit interval, we may extend this distance naturally to any x belonging to an edge.

Write $d|x|$ for the length element on X and let $\mu : [0,\infty) \to (0,\infty)$ be a locally integrable function. We abuse notation and refer also to the measure generated via $d\mu(x) = \mu(|x|)d|x|$ by μ. Further, let $\lambda : [0,\infty) \to (0,\infty)$ be locally integrable and define a distance via $ds(x) = \lambda(|x|)d|x|$ by setting $d(z,y) = \int_{[z,y]} ds(x)$ whenever $z, y \in X$ and $[z,y]$ is the unique geodesic between z and y. We abuse the notation and let $\mu(x)$ and $\lambda(x)$ denote $\mu(|x|)$ and $\lambda(|x|)$, respectively, for any $x \in X$, if there is no danger of confusion. Throughout this paper, we assume additionally that the diameter of X is infinity.

Our space (X, d, μ) is a metric measure space and hence one may define a Newtonian Sobolev space $N^{1,p}(X) := N^{1,p}(X, d, \mu)$ based on upper gradients [14] and [22]. It is then natural to ask if we can characterize the p-admissibility of a given μ, see Section 2.2 for the definitions. To do so, we introduce the following $\mathrm{A_p}$-conditions on regular trees.

Before continuing, we first introduce some notations. For any $x \in X$ and $r > 0$, we denote by $\bar{x}^r$ the point in $[0,x]$ with $d(\bar{x}^r, x) = \min\{r, d(0,x)\}$ and denote by $\underline{x}_r$ a point in X such that $x \in [0, \underline{x}_r]$ with $d(\underline{x}_r, x) = r$. Hence $\bar{x}^r$ is an ancestor of x and $\underline{x}_r$ is a descendant of x, see Section 2.1 for more relations between points on regular trees. Also let

$$F(x,r) = \{y \in X : x \in [0,y], d(x,y) < r\}$$

be the downward directed “half ball". It is perhaps worth to mention that the notations $\bar{x}^r$ and $F(x,r)$ coincide with the notation “z" and $F(x,r)$ in [3, Lemma 3.2], respectively.

Given $1 < p < \infty$, we set

$$\mathrm{A_p}(x,r) = \frac{\mu(F(\bar{x}^r, 2r))}{2r} \cdot \left(\frac{1}{r} \int_{[x,\underline{x}_r]} \left(\frac{K^{j(w)-j(x)}\mu(w)}{\lambda(w)} \right)^{\frac{1}{1-p}} ds(w) \right)^{p-1} \tag{1.1}$$

and we define

$$\mathrm{A_1}(x,r) = \frac{\mu(F(\bar{x}^r, 2r))}{2r} \cdot \operatorname{ess\,sup}_{w\in[x,\underline{x}_r]} \frac{\lambda(w)}{K^{j(w)-j(x)}\mu(w)} \tag{1.2}$$

where $j(w)$ and $j(x)$ are the smallest integers such that $j(w) \geq |w|$ and $j(x) \geq |x|$, respectively. Notice that $\mathrm{A_p}(x,r)$ is independent of the choice of $\underline{x}_r$ among the points y with $x \in [0,y]$ and $d(y,x) = r$.

Definition 1.2. Let $1 \leq p < \infty$ and X be a K-regular tree with distance d and metric μ. We say that μ satisfies the *A_p-condition* if

$$\sup\left\{\mathrm{A_p}(x,r) : x \in X, r > 0\right\} < \infty. \tag{1.3}$$

We say that μ satisfies the *A_p-condition far from* 0 if

$$\sup\left\{\mathrm{A_p}(x,r) : x \in X, 0 < r \leq 8\, d(0,x)\right\} < \infty. \tag{1.4}$$

If $K = 1$ and $\lambda \equiv 1$, then the 1-regular tree (X, d, μ) is isometric to the half line $(\mathbb{R}^+, dx, \mu\, dx)$ and our $\mathrm{A_p}$-condition (1.3) is equivalent to μ being a Muckenhoupt $\mathrm{A_p}$-weight, see [5–7, 12] for more information about Muckenhoupt $\mathrm{A_p}$-weights. Above, we call (1.4) “$\mathrm{A_p}$-condition far from 0” since $0 < r \leq 8\, d(0,x)$ is equivalent to $d(0,x) \geq r/8 > 0$, which means that x has to be “far” away from the root 0 in terms of r.

The main result of this paper is the following characterization of p-admissibility on regular trees.

Theorem 1.3. *Let $1 \leq p < \infty$ and X be a K-regular tree with distance d and measure μ. Then we have:*

1. *For $K = 1$, μ is p-admissible if and only if μ satisfies the A_p-condition far from* 0.
2. *For $K \geq 2$, μ is p-admissible if and only if μ satisfies the A_p-condition.*

The characterizations for $K = 1$ and $K \geq 2$ are different. For $K \geq 2$, a K-regular tree has a kind of symmetry property with respect to the root 0, since the root has more than one branch. But for $K = 1$, the root 0 behaves like an end point.

Readers who are familiar with the results on the real line $\mathbb{R}$ may regard our K-regular tree with $K \geq 2$ as a generalized model of the real line $\mathbb{R}$. As a byproduct, a slightly modified proof of Theorem 1.3 for $K \geq 2$ gives a new proof of [7, Theorem 2]. On the other hand, for $K = 1$, one may connect the result on 1-regular trees with the result on bounded intervals (see [5, Theorem 4.6] for bounded intervals). Hence Theorem 1.3 is new and interesting even when $K = 1$ and $\lambda \equiv 1$, since it gives a full characterization of p-admissibility on the half line $\mathbb{R}^+$.

In [5, Example 4.7], one can find a weight ω on the interval $[0, 1]$ which is 1-admissible but not a Muckenhoupt A_1-weight on $(0, 1)$. By a suitable constant extension of ω on $(1, \infty)$, we obtain a weight ω' which is 1-admissible but not a Muckenhoupt A_1-weight on $\mathbb{R}^+$. As evidence towards Theorem 1.3 for $K = 1$, it is easy to check that the extended weight ω' on $\mathbb{R}^+$ satisfies the A_1-condition far from 0, i.e., condition (1.4) holds. We refer to [5] and [8] for more details.

Let us close this introduction by pointing out that the constant "8" in A_p-condition far from 0 (1.4) is not necessary. Actually replacing 8 by any constant $\infty > c > 1$, Theorem 1.3 for $K = 1$ holds. Here the requirement of $c > 1$ is sharp in the sense that there exists an example $(\mathbb{R}_+, dx, \mu\, dx)$ such that (1.4) holds for any positive constant $c' < 1$ replacing 8, but μ is not even doubling, see Remark 4.5 and Example 4.6.

The paper is organized as follows. In section 2, we introduce regular trees, p-admissibility and Newtonian spaces on our tree. We give the proof of Theorem 1.3 for $K \geq 2$ in Section 3 and the proof of Theorem 1.3 for $K = 1$ is given in Section 4.

2 Preliminaries

Throughout this paper, the letter C (sometimes with a subscript) will denote positive constants; if C depends on $a, b, \ldots$, we write $C = C(a, b, \ldots)$.

2.1 Regular trees and their boundaries

A *graph* G is a pair (V, E), where V is a set of vertices and E is a set of edges. We call a pair of vertices $x, y \in V$ neighbors if x is connected to y by an edge. The degree of a vertex is the number of its neighbors. The graph structure gives rise to a natural connectivity structure. A *tree* is a connected graph without cycles. A graph (or tree) is made into a metric graph by considering each edge as a geodesic of length one.

We call a tree X a *rooted tree* if it has a distinguished vertex called the *root*, which we will denote by 0. The neighbors of a vertex $x \in X$ are of two types: the neighbors that are closer to the root are called *parents* of x and all other neighbors are called *children* of x. Each vertex has a unique parent, except for the root itself that has none.

A *K-ary tree* is a rooted tree such that each vertex has exactly K children. Then all vertices except the root of a K-ary tree have degree $K + 1$, and the root has degree K. In this paper we say that a tree is *regular* if it is a K-ary tree for some $K \geq 1$.

For $x \in X$, let $|x|$ be the distance from the root 0 to x, that is, the length of the geodesic from 0 to x, where the length of every edge is 1 and we consider each edge to be an isometric copy of the unit interval. The geodesic connecting two points $x, y \in V$ is denoted by $[x, y]$, and its length is denoted $|x - y|$. If $|x| < |y|$ and x lies on the geodesic connecting 0 to y, we write $x < y$ and call y a descendant of the point x. More generally, we write $x \leq y$ if the geodesic from 0 to y passes through x, and in this case $|x - y| = |y| - |x|$.

On our K-regular tree X, we define the metric ds and measure $d\mu$ by setting

$$d\mu = \mu(|x|)\, d|x|, \quad ds(x) = \lambda(|x|)\, d|x|,$$

where $\lambda, \mu : [0, \infty) \to (0, \infty)$ with $\lambda, \mu \in L^1_{\mathrm{loc}}([0, \infty))$. Here $d\,|x|$ is the measure which gives each edge Lebesgue measure 1, as we consider each edge to be an isometric copy of the unit interval and the vertices are the end points of this interval. Hence for any two points $z, y \in X$, the distance between them is

$$d(z, y) = \int_{[z,y]} ds(x) = \int_{[z,y]} \lambda(|x|)\, d|x|,$$

where $[z, y]$ is the unique geodesic from z to y in X.

We abuse the notation and let $\mu(x)$ and $\lambda(x)$ denote $\mu(|x|)$ and $\lambda(|x|)$, respectively, for any $x \in X$, if there is no danger of confusion.

Throughout the paper, we let

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

denote the (open) ball in X with center x and radius r, and let $\sigma B(x, r) = B(x, \sigma r)$. Also

$$F(x, r) = \{y \in X : x \in [0, y], d(x, y) < r\}$$

is the downward directed half ball. For any $x \in X$ and $r > 0$, we denote by $\bar{x}^r$ the point in $[0, x]$ with $d(\bar{x}^r, x) = \min\{r, d(0, x)\}$ and denote by $\underline{x}_r$ a point in X such that $x \in [0, \underline{x}_r]$ with $d(\underline{x}_r, x) = r$. Hence $\bar{x}^r$ is the ancestor of any point $y \in B(x, r)$. Usually, the choice of $\underline{x}_r$ is not unique, but we will not specify it since the results and proofs in this paper are independent of the choice of $\underline{x}_r$.

2.2 Admissibility

Let $u \in L^1_{\mathrm{loc}}(X)$. We say that a Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if

$$|u(z) - u(y)| \leq \int_\gamma g\, ds \tag{2.1}$$

whenever $z, y \in X$ and γ is the geodesic from z to y. In the setting of a tree any rectifiable curve with end points z and y contains the geodesic connecting z and y, and therefore the upper gradient defined above is equivalent to the definition which requires that inequality (2.1) holds for all rectifiable curves with end points z and y. In [9, 15], the notion of a p-weak upper gradient is given. A Borel function $g : X \to [0, \infty]$ is called a p-weak upper gradient of u if (2.1) holds on p-a.e. curve. Here we say that a property holds for *p-a.e. curve* if it fails only for a rectifiable curve family Γ with *zero p-modulus*, i.e., there is Borel function $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho\, ds = \infty$ for every curve $\gamma \in \Gamma$. We refer to [9, 15] for more information about p-weak upper gradients.

The notion of upper gradients is due to Heinonen and Koskela [14]; we refer interested readers to [2, 9, 15, 22] for a more detailed discussion on upper gradients.

The Newtonian space $N^{1,p}(X)$, for $1 \leq p < \infty$, is defined as the collection of the functions for which the given norm

$$\|u\|_{N^{1,p}(X)} := \left(\int_X |u|^p\, d\mu + \inf_g \int_X |g|^p\, d\mu \right)^{1/p}$$

is finite, where the infimum is taken over all p-weak upper gradients g of u.

A measure μ is doubling if there exists a positive constant C_d such that for all balls $B(x, r)$ with $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)), \tag{2.2}$$

where the constant C_d is called the *doubling constant*.

(X, d, μ) supports a $(1, p)$-Poincaré inequality if there exist positive constants $C_P > 0$ and $\sigma \geq 1$ such that for all balls $B(x, r)$ with $x \in X$ and $r > 0$, every integrable function u on $\sigma B(x, r)$ and all upper gradients g,

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C_P r \left(\fint_{\sigma B(x,r)} g^p\, d\mu \right)^{1/p} \tag{2.3}$$

where $u_B := \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu$. We say that μ is *p-admissible* if μ is a doubling measure and (X, d, μ) supports a $(1, p)$-Poincaré inequality.

The doubling property (2.2) and $(1, p)$-Poincaré inequality (2.3) can be defined on general metric measure spaces. In particular, on $\mathbb{R}^n$, in view of [16, Theorem 2] or [15, Theorem 8.4.2], the $(1, p)$-Poincaré inequality (2.3) is equivalent to the $(1, p)$-Poincaré inequality given in the Introduction. It perhaps worth to point out that, since our K-regular trees are geodesic spaces, if μ is p-admissible, the dilation constant σ in (2.3) can be taken to 1, see [10] and [11].

3 Proof of Theorem 1.3 for $K \geq 2$

In this section, we give the proof of Theorem 1.3 for $K \geq 2$. To do so, we establish the following lemmas.

Lemma 3.1. *Let $1 \leq p < \infty$ and X be a K-regular tree with distance d and measure μ where $K \geq 1$. Assume that μ satisfies the A_p-condition. Then μ is p-admissible.*

Proof. For $1 \leq p < \infty$, let

$$C_A := \sup \left\{ \mathrm{A_p}(x, r) : x \in X, r > 0 \right\}.$$

Since μ satisfies the $\mathrm{A_p}$-condition, $0 < C_A < \infty$.

Case $p = 1$**:** We first show that μ is a doubling measure. Let $x \in X$ and $r > 0$ be arbitrary. Notice that $\mathrm{A_1}(x, 2r) \leq C_A$. Then it follows from (1.2) that

$$\operatorname{ess\,sup}_{w \in [x, \underline{x}_{2r}]} \frac{\lambda(w)}{K^{j(w)-j(x)} \mu(w)} \leq \frac{4rC_A}{\mu(F(\bar{x}^{2r}, 4r))}.$$

Hence

$$\begin{aligned} r = \int_{[x, \underline{x}_r]} ds &= \int_{[x, \underline{x}_r]} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right) \left(\frac{\lambda(w)}{K^{j(w)-j(x)} \mu(w)} \right) ds(w) \\ &\leq \left(\int_{[x, \underline{x}_r]} \frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} ds(w) \right) \left(\frac{4rC_A}{\mu(F(\bar{x}^{2r}, 4r))} \right). \end{aligned} \tag{3.1}$$

Notice that

$$\int_{[x, \underline{x}_r]} \frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} ds(w) = \mu(F(x, r)) \leq \mu(B(x, r))$$

and that

$$\mu(F(\bar{x}^{2r}, 4r)) \geq \mu(B(x, 2r)).$$

It follows from estimate (3.1) that

$$r \leq 4C_A r \frac{\mu(B(x, r))}{\mu(B(x, 2r))},$$

which proves that μ is a doubling measure with doubling constant $4C_A$ since $r > 0$ and the pair (x, r) is arbitrary.

Next we prove that (X, d, μ) supports a $(1, 1)$-Poincaré inequality. Consider an arbitrary ball $B(x, r)$ with $x \in X$ and $r > 0$. By the triangle inequality, we obtain that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq 2 \fint_{B(x,r)} |u(y) - u(\bar{x}^r)| d\mu(y) \tag{3.2}$$

for the left-hand side of our Poincaré inequality. By the definition of upper gradients and the Fubini theorem, for any upper gradient g_u of u, the right-hand side of (3.2) rewrites as

$$
\begin{aligned}
2 \fint_{B(x,r)} |u(y) - u(\bar{x}^r)| d\mu(y) &\le 2 \fint_{B(x,r)} \int_{[\bar{x}^r, y]} g_u(w) ds(w) d\mu(y) \\
&= 2 \fint_{B(x,r)} g_u(w) \frac{\lambda(w)}{\mu(w)} \left(\int_{B(x,r)} \chi_{[\bar{x}^r, y]}(w)\, d\mu(y) \right) d\mu(w) \\
&= 2 \fint_{B(x,r)} g_u(w) \frac{\lambda(w)}{\mu(w)} \mu(\{y \in B(x,r) : w \in [0,y]\}) d\mu(w). \qquad (3.3)
\end{aligned}
$$

Here the last equality holds since $\chi_{[\bar{x}^r, y]}(w)$ is not zero only if $w \in [0, y]$.

Since the measure μ satisfies the A_1-condition, $A_1(\bar{x}^r, 2r) < C_A$. It follows from (1.2) that

$$
\frac{\mu(F(\bar{x}^{3r}, 4r))}{4r} \cdot \operatorname{ess\,sup}_{w \in [\bar{x}^r, \underline{x}_r]} \frac{\lambda(w)}{K^{j(w)-j(\bar{x}^r)} \mu(w)} \le C_A.
$$

Combining with the fact that $K^{j(\bar{x}^{3r})} \le K^{j(\bar{x}^r)}$, we obtain that

$$
\begin{aligned}
\frac{\lambda(w)}{\mu(w)} \mu(\{y \in B(x,r) : w \in [0,y]\}) &= \frac{\lambda(w)}{\mu(w)} \int_{\{y \in [w, \underline{w}_r] \cap B(x,r)\}} \frac{K^{j(y)-j(w)} \mu(y)}{\lambda(y)} ds(y) \\
&\le \frac{\lambda(w) K^{j(\bar{x}^{3r})}}{\mu(w) K^{j(w)}} \int_{[\bar{x}^{3r}, \underline{x}_r]} \frac{K^{j(y)-j(\bar{x}^{3r})} \mu(y)}{\lambda(y)} ds(y) \\
&\le \frac{\lambda(w)}{\mu(w) K^{j(w)-j(\bar{x}^r)}} \mu(F(\bar{x}^{3r}, 4r)) \le 4 C_A r \qquad (3.4)
\end{aligned}
$$

for any $w \in B(x, r)$. Combining (3.2)-(3.4), yields

$$
\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \le 8 C_A r \fint_{B(x,r)} g_u \, d\mu
$$

for all balls $B(x, r)$.

Case $p > 1$**:** Let us first prove that μ is a doubling measure. Let $B(x, r)$ be an arbitrary ball in X. Since μ satisfies the A_p-condition, we have $A_p(x, 2r) \le C_A$, and hence

$$
\frac{\mu(F(\bar{x}^{2r}, 4r))}{4r} \cdot \left[\frac{1}{2r} \int_{[x, \underline{x}_{2r}]} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right)^{\frac{1}{1-p}} ds(w) \right]^{p-1} \le C_A. \qquad (3.5)
$$

A simple calculation using the Hölder inequality shows that

$$
\begin{aligned}
r &= \int_{[x, \underline{x}_r]} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right)^{1/p} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right)^{-1/p} ds(w) \\
&\le \left(\int_{[x, \underline{x}_r]} \frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} ds(w) \right)^{1/p} \left[\int_{[x, \underline{x}_r]} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right)^{\frac{1}{1-p}} ds(w) \right]^{\frac{p-1}{p}} \\
&\le \mu(F(x,r))^{1/p} (2r)^{\frac{p-1}{p}} \left[\frac{1}{2r} \int_{[x, \underline{x}_{2r}]} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right)^{\frac{1}{1-p}} ds(w) \right]^{\frac{p-1}{p}}.
\end{aligned}
$$

Inserting (3.5) into the above estimate yields

$$r \leq (2r)^{\frac{p-1}{p}} \mu(F(x,r))^{1/p} \left[\frac{\mu(F(\bar{x}^{2r},4r))}{4rC_A}\right]^{\frac{-1}{p}} = C_A{}^{1/p} 2^{\frac{p+1}{p}} r \left(\frac{\mu(F(x,r))}{\mu(F(\bar{x}^{2r},4r))}\right)^{1/p}. \tag{3.6}$$

Note that $\mu(F(x,r)) \leq \mu(B(x,r))$ and $\mu(F(\bar{x}^{2r},4r)) \geq \mu(B(x,2r))$. Then the estimate (3.6) implies that

$$r \leq C_A{}^{1/p} 2^{\frac{p+1}{p}} r \left(\frac{\mu(B(x,r))}{\mu(B(x,2r))}\right)^{1/p},$$

which gives that μ is a doubling measure with doubling constant $C_A 2^{p+1}$, since $r > 0$ and $B(x,r)$ is arbitrary.

Next we show that (X,d,μ) supports a $(1,p)$-Poincaré inequality. Suppose $B(x,r)$ is an arbitrary ball with center $x \in X$ and radius $r > 0$. Since the measure μ satisfies the $\mathrm{A_p}$-condition, then $\mathrm{A_p}(\bar{x}^r, 2r) < C_A$. It follows from (1.1) that

$$\frac{\mu(F(\bar{x}^{3r},4r))}{4r} \cdot \left[\frac{1}{2r}\int_{[\bar{x}^r,\underline{x}_r]} \left(\frac{K^{j(w)-j(\bar{x}^r)}\mu(w)}{\lambda(w)}\right)^{\frac{1}{1-p}} ds(w)\right]^{p-1} \leq C_A. \tag{3.7}$$

Recall that the left-hand side of our Poincaré inequality can be estimated by (3.3). A simple calculation shows that

$$\begin{aligned}\frac{\lambda(w)}{\mu(w)}\mu(\{y \in B(x,r) : w \in [0,y]\}) &= \frac{\lambda(w)}{\mu(w)} \int_{\{y \in [w,\underline{w}_r] \cap B(x,r)\}} \frac{K^{j(y)-j(w)}\mu(y)}{\lambda(y)} ds(y)\\ &\leq \frac{\lambda(w)}{\mu(w)K^{j(w)-j(\bar{x}^r)}} \int_{[\bar{x}^r,\underline{x}_r]} \frac{K^{j(y)-j(\bar{x}^r)}\mu(y)}{\lambda(y)} ds(y)\\ &= \frac{\lambda(w)}{\mu(w)K^{j(w)-j(\bar{x}^r)}} \mu(F(\bar{x}^r,2r))\end{aligned} \tag{3.8}$$

for any point $w \in B(x,r)$. Inserting the estimate (3.8) into (3.3) yields that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq 2 \left(\fint_{B(x,r)} g_u(w) \frac{\lambda(w)}{\mu(w)K^{j(w)-j(\bar{x}^r)}} d\mu(w)\right) \mu(F(\bar{x}^r,2r)).$$

Applying the Hölder inequality for the right-hand side of the above inequality, it follows that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq 2 \left(\fint_{B(x,r)} g_u{}^p d\mu\right)^{1/p} \left[\fint_{B(x,r)} \left(\frac{\lambda(w)}{K^{j(w)-j(\bar{x}^r)}\mu(w)}\right)^{\frac{p}{p-1}} d\mu(w)\right]^{\frac{p-1}{p}} \mu(F(\bar{x}^r,2r)). \tag{3.9}$$

By using the estimate (3.7), we obtain that

$$\begin{aligned}\left[\fint_{B(x,r)} \left(\frac{\lambda(w)}{K^{j(w)-j(\bar{x}^r)}\mu(w)}\right)^{\frac{p}{p-1}} d\mu(w)\right]^{\frac{p-1}{p}} \mu(F(\bar{x}^r,2r)) &\leq \frac{\mu(F(\bar{x}^r,2r))}{\mu(B(x,r))^{\frac{p-1}{p}}} \left[\int_{F(\bar{x}^r,2r)} \left(\frac{\lambda(w)}{K^{j(w)-j(\bar{x}^r)}\mu(w)}\right)^{\frac{p}{p-1}} d\mu(w)\right]^{\frac{p-1}{p}}\\ &\leq \frac{\mu(F(\bar{x}^r,2r))}{\mu(B(x,r))^{\frac{p-1}{p}}} (2r)^{\frac{p-1}{p}} \left[\frac{1}{2r}\int_{[\bar{x}^r,\underline{x}_r]} \left(\frac{K^{j(w)-j(\bar{x}^r)}\mu(w)}{\lambda(w)}\right)^{\frac{1}{1-p}} ds(w)\right]^{\frac{p-1}{p}}\\ &\leq \frac{\mu(F(\bar{x}^r,2r))}{\mu(B(x,r))^{\frac{p-1}{p}}} (2r)^{\frac{p-1}{p}} \left[\frac{\mu(F(\bar{x}^{3r},4r))}{4rC_A}\right]^{\frac{-1}{p}}\\ &= C_A{}^{1/p} 2^{\frac{p+1}{p}} r \frac{\mu(F(\bar{x}^r,2r))}{\mu(B(x,r))^{\frac{p-1}{p}} \mu(F(\bar{x}^{3r},4r))^{1/p}}.\end{aligned} \tag{3.10}$$

Note that $F(\bar{x}^r, 2r)) \subset B(x, 4r)$ and that $B(x, r) \subset F(\bar{x}^{3r}, 4r)$. Since μ is a doubling measure with doubling constant $C_A 2^{p+1}$, we have that

$$\frac{\mu(F(\bar{x}^r, 2r))}{\mu(B(x, r))^{\frac{p-1}{p}} \mu(F(\bar{x}^{3r}, 4r))^{1/p}} \le \frac{\mu(B(x, 4r))}{\mu(B(x, r))} \le (C_A 2^{p+1})^2.$$

Inserting the above estimate into the estimate (3.10), we have

$$\left[\fint_{B(x,r)} \left(\frac{\lambda(w)}{K^{j(w)-j(\bar{x}^r)} \mu(w)} \right)^{\frac{p}{p-1}} d\mu(w) \right]^{\frac{p-1}{p}} \mu(F(\bar{x}^r, 2r)) \le C_A^{2+\frac{1}{p}} 2^{\frac{p+1}{p}+2(p+1)} r. \tag{3.11}$$

Thanks to the estimates (3.9) and (3.11), we obtain

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \le C_A^{2+\frac{1}{p}} 2^{\frac{1}{p}+2p+4} r \left(\fint_{B(x,r)} g_u{}^p \, d\mu \right)^{\frac{1}{p}}$$

for all balls $B(x, r)$. □

Lemma 3.2. *Let $1 \le p < \infty$ and X be a K-regular tree with distance d and measure μ where $K \ge 2$. Suppose that μ is p-admissible. Then μ satisfies the A_p-condition.*

Proof. Let $x \in X$ and $r > 0$ be arbitrary. Let ε be an arbitrary positive number. Let $x_1 \in X$ be a closest vertex of x with $|x_1| > |x|$. Then we define

$$T_{x_1} := \{y \in X : x_1 \in [0, y]\} \quad \text{and} \quad T_1 := [x, x_1] \cup T_{x_1}$$

Since μ is p-admissible, we may assume that μ satisfies the doubling condition (2.2) and the $(1, p)$-Poincaré inequality (2.3).

Case $p = 1$: Let

$$m = \operatorname{ess\,inf}_{w \in [x, x_{\frac{r}{2}}]} \frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)}.$$

In order to test the $(1, 1)$-Poincaré inequality (2.3), we define

$$u(y) = \begin{cases} 0 & \text{if } y \in X \setminus T_1, \\ \int_{[x,y]} \chi_{E_\varepsilon}(w) ds(w) & \text{if } y \in F(x, r/2) \cap T_1, \\ a & \text{otherwise} \end{cases}$$

where $E_\varepsilon := \left\{ w \in F(x, \frac{r}{2}) : \frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} < m + \varepsilon \right\}$ and $a = \int_{[x, x_{\frac{r}{2}}]} \chi_{E_\varepsilon}(w) \, ds(w)$. Note that E_ε is a non-empty set by the definition of m and that

$$r > a = \int_{[x, x_{\frac{r}{2}}]} \chi_{E_\varepsilon}(w) ds(w) > 0.$$

By the definition of u, we obtain that $g_u := \chi_{E_\varepsilon}$ is an upper gradient of u. Hence the right-hand side of the $(1, 1)$-Poincaré inequality (2.3) is

$$\begin{aligned} C_P r \fint_{\sigma B(x,r)} g_u \, d\mu &= C_P r \fint_{\sigma B(x,r)} \chi_{E_\varepsilon}(w) d\mu(w) \\ &= \frac{C_P r}{\mu(\sigma B(x, r))} \int_{F(x,r/2)} \chi_{E_\varepsilon}(w) d\mu(w) \\ &= \frac{C_P r}{\mu(\sigma B(x, r))} \int_{[x, x_{\frac{r}{2}}]} \chi_{E_\varepsilon}(w) \frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} ds(w). \end{aligned}$$

Here the second equality holds since $\chi_{E_\varepsilon}(w)$ is non-zero only if $w \in F(x, r/2)$. Note that $\mu(\sigma B(x, r)) \geq \mu(B(x, r))$. Then it follows from the definition of E_ε that

$$C_P r \fint_{\sigma B(x,r)} g_u \, d\mu \leq \frac{C_P r}{\mu(B(x, r))}(m + \varepsilon)a. \tag{3.12}$$

Let

$$E_1 := B(x, r) \setminus T_1 \text{ and } E_2 := T_1 \cap F(x, r) \setminus F(x, r/2). \tag{3.13}$$

Note that $u \equiv 0$ on E_1 and $u \equiv a$ on E_2. Hence, at least one of the following holds:

$$|u - u_{B(x,r)}| \geq \frac{a}{2} \text{ on } E_1 \quad \text{or} \quad |u - u_{B(x,r)}| \geq \frac{a}{2} \text{ on } E_2. \tag{3.14}$$

Since $K \geq 2$, then E_1 and E_2 are not empty. Notice that $K\mu(E_2) \geq \mu(F(x, r) \setminus F(x, r/2))$. Furthermore, the doubling property of μ gives

$$K\mu(E_2) \geq \mu(F(x, r) \setminus F(x, r/2)) \geq \mu(B(x_{\frac{3r}{4}}, r/4)) \geq C_d{}^{-4}\mu(B(x_{\frac{3r}{4}}, 4r)) \geq C_d{}^{-4}\mu(B(x, r))$$

and

$$\mu(E_1) \geq \mu(B(z, r/2)) \geq C_d{}^{-3}\mu(B(z, 4r)) \geq C_d{}^{-3}\mu(B(x, r)),$$

for some $z \notin T_1$ with $d(x, z) = r/2$. Consequently,

$$\min\{\mu(E_1), \mu(E_2)\} \geq C_d{}^{-4}K^{-1}\mu(B(x, r)). \tag{3.15}$$

Then it follows from (3.14) and (3.15) that the left-hand side of the $(1, 1)$-Poincaré inequality (2.3) is

$$\begin{aligned} \fint_{B(x,r)} |u - u_{B(x,r)}| d\mu &\geq \frac{1}{\mu(B(x, r))} \max\left\{ \int_{E_1} |u - u_{B(x,r)}| d\mu, \int_{E_2} |u - u_{B(x,r)}| d\mu \right\} \\ &\geq \frac{a}{2C_d{}^4 K}. \end{aligned} \tag{3.16}$$

Combining the estimates (3.12) and (3.16), we obtain that

$$\frac{a}{2C_d{}^4 K} \leq \frac{C_P r}{\mu(B(x, r))}(m + \varepsilon)a.$$

Since $a > 0$ and $\mu(F(\bar{x}^{\frac{r}{2}}, r)) \leq \mu(B(x, 2r)) \leq C_d\mu(B(x, r))$, it follows that

$$0 < \frac{\mu(F(\bar{x}^{\frac{r}{2}}, r))}{r} \leq 2C_d{}^5 C_P K \cdot (m + \varepsilon).$$

Since ε and the pair (x, r) are arbitrary, letting $\varepsilon \to 0$, the A_1-condition holds.

Case $p > 1$**:** We define

$$u(y) = \begin{cases} 0 & \text{if } y \in X \setminus T_1, \\ \int_{[x,y]} \left(\frac{K^{j(w)-j(x)}\mu(w)}{\lambda(w)}\right)^{\frac{1}{1-p}} ds(w) & \text{if } y \in F(x, r/2) \cap T_1, \\ b & \text{ortherwise} \end{cases}$$

where

$$b = \int_{[x, x_{\frac{r}{2}}]} \left(\frac{K^{j(w)-j(x)}\mu(w)}{\lambda(w)}\right)^{\frac{1}{1-p}} ds(w).$$

By the definition of u, we obtain that

$$g_u(y) := \left(\frac{K^{j(y)-j(x)}\mu(y)}{\lambda(y)}\right)^{\frac{1}{1-p}} \chi_{F(x,r/2)}(y) \tag{3.17}$$

is an upper gradient of u. Note that $u \equiv 0$ on E_1 and $u \equiv b$ on E_2 where E_1 and E_2 are defined as for $p = 1$. Therefore, by an argument similar to the one in $p = 1$ case, the left-hand side of the $(1, p)$-Poincaré inequality (2.3) can be estimated as

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \geq \frac{b}{2{C_d}^4 K}. \tag{3.18}$$

For the right-hand side, we have that

$$\begin{aligned} C_P r \left(\fint_{\sigma B(x,r)} {g_u}^p \, d\mu \right)^{1/p} &= \frac{C_P r}{\mu(\sigma B(x,r))^{1/p}} \left[\int_{F(x,r/2)} \left(\frac{K^{j(y)-j(x)} \mu(y)}{\lambda(y)} \right)^{\frac{p}{1-p}} d\mu(y) \right]^{1/p} \\ &= \frac{C_P r}{\mu(\sigma B(x,r))^{1/p}} \left[\int_{[x, x_{\frac{r}{2}}]} \left(\frac{K^{j(y)-j(x)} \mu(y)}{\lambda(y)} \right)^{\frac{1}{1-p}} ds(y) \right]^{1/p} \\ &= \frac{C_P r}{\mu(\sigma B(x,r))^{1/p}} b^{1/p}. \end{aligned}$$

Since $\mu(\sigma B(x,r)) \geq \mu(B(x,r))$, it follows that

$$C_P r \left(\fint_{\sigma B(x,r)} {g_u}^p \, d\mu \right)^{1/p} \leq \frac{C_P r}{\mu(B(x,r))^{1/p}} b^{1/p}. \tag{3.19}$$

Combining (3.18) and (3.19), we obtain that

$$\frac{b}{2{C_d}^4 K} \leq \frac{C_P r}{\mu(B(x,r))^{1/p}} b^{1/p}.$$

Notice that $\mu(F(\bar{x}^{\frac{r}{2}}, r)) \leq \mu(B(x, 2r)) \leq C_d \mu(B(x,r))$. Hence we have

$$0 < \frac{\mu(F(\bar{x}^{\frac{r}{2}}, r))^{1/p}}{r} \leq 2{C_d}^{4+\frac{1}{p}} C_P K b^{\frac{1-p}{p}}.$$

Recalling the definition of b, the above estimate can be rewritten as

$$0 < \frac{\mu(F(\bar{x}^{\frac{r}{2}}, r))}{r} \leq 2^p {C_d}^{4p+1} {C_P}^p K^p \left(\frac{1}{r} \int_{[x, x_{\frac{r}{2}}]} \left(\frac{K^{j(w)-j(x)} \mu(w)}{\lambda(w)} \right)^{\frac{1}{1-p}} ds(w) \right)^{1-p}.$$

Since the pair (x, r) is arbitrary, the above estimate implies that μ satisfies the $\mathrm{A_p}$-condition. □

Proof of Theorem 1.3 for $K \geq 2$. The proof follows from Lemma 3.1 and Lemma 3.2. □

4 Proof of Theorem 1.3 for $K = 1$

Lemma 4.1. *Let $1 \leq p < \infty$ and X be a 1-regular tree with distance d and measure μ. Suppose that μ is p-admissible. Then μ satisfies the A_p-condition far from 0, i.e.,*

$$\sup \left\{ A_p(x,r) : x \in X, 0 < r \leq 8d(0,x) \right\} < \infty.$$

Proof. Let (x, r) be an arbitrary pair with $d(0, x) \geq r/16 > 0$. Since $K = 1$, we may let $T_1 := F(x, \infty) = \{y \in X : |y| \geq |x|\}$ and repeat the proof of Lemma 3.2. The only danger is whether (3.15) holds, since, for $K = 1$, E_1 could be empty. But here we required that $d(0, x) \geq r/16 > 0$, which gives a version of (3.15). Then the proof of Lemma 3.2 gives that $\mathrm{A_p}(x, \frac{r}{2}) \le C(p, K, C_d, C_P)$, where $C(p, K, C_d, C_P)$ is a constant only depending on p, K, C_d and C_P. Since the pair (x, r) is arbitrary with $d(0, x) \ge r/16 > 0$, we obtain that

$$\sup\left\{\mathrm{A_p}\left(x, \frac{r}{2}\right) : x \in X, 0 < \frac{r}{2} \le 8d(0, x)\right\} < \infty,$$

which gives the result. □

Lemma 4.2. *Let $1 \le p < \infty$ and X be a 1-regular tree with distance d and measure μ. Assume that μ satisfies the A_p-condition far from 0. Then we have:*

1. *The measure μ is doubling.*
2. *There exists a positive constant $C_p > 0$ such that for all balls $B(x, r)$ with $x \in X$ and $0 < r \le \frac{4}{5}d(0, x)$, every integrable function u on $B(x, r)$ and all upper gradients g of u,*

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r \left(\fint_{B(x,r)} g^p\, d\mu \right)^{1/p}. \tag{4.1}$$

Proof. **Claim 1:** Recall the proof of Lemma 3.1. It actually shows that for any pair (x, r) with $\mathrm{A_p}(x, 2r) \le C_A$, we have

$$\mu(B(x, 2r)) \le C(C_A)\mu(B(x, r)),$$

where $C(C_A)$ is a constant only depending on C_A. In this lemma, since μ only satisfies the $\mathrm{A_p}$-condition far from 0, i.e.,

$$M_A := \sup\left\{\mathrm{A_p}(x, r) : x \in X, 0 < r \le 8\, d(0, x)\right\} < \infty,$$

we obtain that there is a positive constant $C := C(M_A)$ only depending on M_A such that

$$\mu(B(x, r)) \le C\mu(B(x, r/2)) \tag{4.2}$$

for all balls $B(x, r)$ with $d(0, x) \ge r/8 > 0$.

To get that μ is a doubling measure, it is sufficient to show that (4.2) holds for all balls $B(x, r)$ with $d(0, x) < r/8$. Note that $d(0, \underline{0}_{\frac{r}{2}}) = \frac{r}{2} \ge \max\{4r/8, 2r/8, r/8\}$. Applying (4.2) for $B(\underline{0}_{\frac{r}{2}}, 4r)$, $B(\underline{0}_{\frac{r}{2}}, 2r)$ and $B(\underline{0}_{\frac{r}{2}}, r)$ in turns, we obtain that

$$\mu(B(\underline{0}_{\frac{r}{2}}, 4r)) \le C\mu(B(\underline{0}_{\frac{r}{2}}, 2r)) \le C^2\mu(B(\underline{0}_{\frac{r}{2}}, r)) \le C^3\mu(B(\underline{0}_{\frac{r}{2}}, r/2)).$$

Hence

$$\mu(B(\underline{0}_{\frac{r}{2}}, 4r)) \le C^3\mu(B(\underline{0}_{\frac{r}{2}}, r/2)). \tag{4.3}$$

From $B(\underline{0}_{\frac{r}{2}}, r/2) \subset B(0, r)$ and $B(0, 2r) \subset B(\underline{0}_{\frac{r}{2}}, 4r)$, we have

$$\mu(B(0, 2r)) \le \mu(B(\underline{0}_{\frac{r}{2}}, 4r)), \quad \mu(B(\underline{0}_{\frac{r}{2}}, r/2)) \le \mu(B(0, r))$$

for all $r > 0$. Combining with (4.3), we get that

$$\mu(B(0, 2r)) \le C^3\mu(B(0, r))$$

for all $r > 0$. In particular,

$$\mu(B(0, 2r)) \le C^9\mu(B(0, r/4)) \tag{4.4}$$

for all $r > 0$. Let $B(x, r)$ be an arbitrary ball with $d(0, x) < r/8$. By $B(x, r) \subset B(0, 2r)$ and $B(0, r/4) \subset B(x, r/2)$, it follows from (4.4) that

$$\mu(B(x, r)) \le \mu(B(0, 2r)) \le C^9\mu(B(0, r/4)) \le C^9\mu(B(x, r/2))$$

for all balls $B(x, r)$ with $d(0, x) < r/8$. Combining with (4.2), we conclude that μ is a doubling measure.

Claim 2: Recall the proof of Lemma 3.1. It actually shows that for any pair (x, r) with $\mathrm{A_p}(\bar{x}^r, 2r) \le C_A$, there exists a constant $C_p(C_A)$ such that for every integrable function u on $B(x, r)$ and all upper gradients g of u, the $(1, p)$-Poincaré inequality (4.1) holds for $B(x, r)$, where $C_p(C_A)$ is a constant only depending on C_A. In this lemma, μ only satisfies the $\mathrm{A_p}$-condition far from 0, i.e.,

$$M_A := \sup\left\{\mathrm{A_p}(x, r) : x \in X, 0 < r \le 8\, d(0, x)\right\} < \infty.$$

Since

$$0 < 2r \le 8\, d(0, \bar{x}^r) \iff d(0, \bar{x}^r) \ge r/4 > 0 \iff d(0, x) \ge 5r/4 > 0,$$

we obtain that there is a positive constant $C_p := C(M_A)$ only depending on M_A such that the Claim 2 holds. □

We say (X, d, μ) supports a $(1, p)$-*Poincaré inequality at* 0, $1 \le p < \infty$, if there are positive constants $C_0, \sigma_0 \ge 1$ such that for any $r > 0$, every integrable function u on $\sigma_0 B(0, r)$ and all upper gradients g of u,

$$\fint_{B(0,r)} |u - u_{B(0,r)}|\, d\mu \le C_0 r \left(\fint_{\sigma_0 B(0,r)} g^p\, d\mu \right)^{1/p}. \tag{4.5}$$

Proposition 4.3. *Let* $1 \le p < \infty$ *and* (X, d, μ) *be as in Lemma 4.2. Assume additionally that* (X, d, μ) *supports a* $(1, p)$*-Poincaré inequality at* 0. *Then* μ *is* p*-admissible.*

Proof. It follows from Claim 2 of Lemma 4.2 that it suffices to check the $(1, p)$-Poincaré inequality (2.3) for balls $B(x, r)$ with $d(0, x) < 5r/4$.

Fix an arbitrary ball $B(x, r)$ with $d(0, x) < 5r/4$. By the triangle inequality, the left-hand side of a $(1, p)$-Poincaré inequality (2.3) can be estimated as

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \le 2 \fint_{B(x,r)} |u - u_{B(0,4r)}| d\mu. \tag{4.6}$$

It follows from Claim 1 of Lemma 4.2 that μ is a doubling measure. Without loss of generality, we may assume that the doubling constant is C_d. Since $d(0, x) < 5r/4$, then $B(0, 4r) \subset B(x, 8r)$. Hence by doubling property,

$$\mu(B(x, r)) \ge {C_d}^{-3} \mu(B(x, 8r)) \ge {C_d}^{-3} \mu(B(0, 4r)).$$

Combining with (4.5), the estimate (4.6) can be rewritten as

$$\begin{aligned} \fint_{B(x,r)} |u - u_{B(x,r)}| d\mu &\le 2{C_d}^3 \fint_{B(0,4r)} |u - u_{B(0,4r)}| d\mu \\ &\le 8{C_d}^3 C_0 r \left(\fint_{\sigma_0 B(0,4r)} g^p\, d\mu \right)^{1/p}. \end{aligned} \tag{4.7}$$

An easy verification shows that

$$\fint_{\sigma_0 B(0,4r)} g^p\, d\mu \le {C_d}^2 \fint_{\sigma_0 B(x,8r)} g^p\, d\mu, \tag{4.8}$$

since $\sigma_0 B(0, 4r) \subset \sigma_0 B(x, 8r)$ and $\mu(\sigma_0 B(x, 8r)) \le {C_d}^2 \mu(\sigma_0 B(x, 2r)) \le {C_d}^2 \mu(\sigma_0 B(0, 4r))$ by doubling. Combining (4.7) and (4.8), we deduce that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \le 8{C_d}^{3+2/p} C_0 r \left(\fint_{8\sigma_0 B(x,r)} g^p\, d\mu \right)^{1/p}. \tag{4.9}$$

Since $B(x, r)$ is an arbitrary ball with $d(0, x) < 5r/4$, combining (4.9) with Claim 1 and 2 of Lemma 4.2, it shows that μ is p-admissible. □

The following lemma shows that the assumption in Lemma 4.2 is sufficient to obtain a $(1, p)$-Poincaré inequality at 0, which means that the additional assumption in Proposition 4.3 is redundant. The core idea of the proof comes from the proof of [10, Theorem 1].

Lemma 4.4. *Let $1 \le p < \infty$ and (X, d, μ) be as in Lemma 4.2. Then (X, d, μ) supports a $(1, p)$-Poincaré inequality at* 0.

Proof. It follows from Lemma 4.2 that μ is doubling and (X, d, μ) supports the $(1, p)$-Poincaré inequality (4.1). For any $R > 0$, since X is a 1-regular tree, we have $B(0, R) = [0, x_R)$, where $x_R \in X$ with $|x_R| = R$. By using the geometry of the 1-regular tree, we are able to modify the proof of [10, Theorem 1] by using a better chain condition $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$ which requires additionally that $r_i < \frac{4}{5} d(x_i, 0)$ (since (4.1) only works for balls $B(x, r)$ with $r < \frac{4}{5} d(x, 0)$). Hence it follows from the proof of [10, Theorem 1] that there is a constant C independent of R such that

$$\fint_{B(0,R)} |u - u_{B(0,R)}|\, d\mu \le CR \left(\fint_{B(0,R)} g^p\, d\mu \right)$$

for all integrable functions u and all upper gradients g of u. □

Proof of Theorem 1.3 for $K = 1$. The claim follows from Lemma 4.2, Proposition 4.3 and Lemma 4.4. □

Remark 4.5. Fix any $\infty > c > 1$, if we change the $\mathrm{A_p}$-condition far from 0, i.e., the condition (1.4) to

$$\sup \left\{ \mathrm{A_p}(x, r) : x \in X, 0 < r \le c\, d(0, x) \right\} < \infty, \tag{4.10}$$

repeating the proof Theorem 1.3 and related lemmas, it follows that the condition (4.10) is also equivalent to μ being p-admissible.

Example 4.6. The following example from [1, Example 4] or [4, Example 6.2] gives a 1-regular tree with a non-doubling measure which satisfies (4.10) for any $0 < c < 1$. Let $X = (\mathbb{R}_+, dx, \mu\, d\mu)$ with $\mu(x) = \min\{1, x^{-1}\}$. Then it follows from [4] and [1] that μ is not a doubling measure, hence μ is not p-admissible for any $1 \le p < \infty$. It remains to show that (4.10) holds for any $0 < c < 1$ and $1 \le p < \infty$.

Fix $0 < c < 1$. Let $R = \frac{1}{1-c}$. To show (4.10) holds, it suffices to show that

$$\sup \left\{ \mathrm{A_p}(t, \beta t) : 0 < \beta \le c, t \in (R, \infty) \right\} < \infty, \tag{4.11}$$

since

$$\sup \left\{ \mathrm{A_p}(t, \beta t) : 0 < \beta \le c, t \in [0, R] \right\} < \infty$$

is given by the fact that $(R + cR)^{-1} \le \mu(x) \le 1$ for any $x \in F(\bar{t}^{\beta t}, 2\beta t)$ with $t \le R$ and $0 < \beta \le c$. For any $0 < \beta \le c$, since $F(\bar{t}^{\beta t}, 2\beta t) = [t - \beta t, t + \beta t]$ and $t - \beta t > 1$ for any $t > R$, we have that

$$\mu(F(\bar{t}^{\beta t}, 2\beta t)) \le \int_{(1-\beta)t}^{(1+\beta)t} x^{-1}\, dx = \log\left(\frac{1+\beta}{1-\beta}\right) \le \log\left(\frac{1+c}{1-c}\right).$$

On the other hand, we have that for $p > 1$,

$$\left(\frac{1}{\beta t} \int_t^{t+\beta t} x^{\frac{1}{p-1}}\, dx \right)^{p-1} = t \left(\frac{(1+\beta)^{p/(p-1)} - 1}{\beta} \right)^{p-1} \le C(c, p) t,$$

where $C(c, p)$ is a constant only depending on c and p, and that

$$\operatorname{ess\,sup}_{x\in[t,t+\beta t]} x = (1 + \beta)t \le (1 + c)t.$$

Hence condition (4.11) holds.

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