JYU DISSERTATIONS 260

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UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Editors
Mikko Parviainen
Department of Mathematics and Statistics, University of Jyväskylä
Päivi Vuorio
Open Science Centre, University of Jyväskylä

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Jarkko Siltakoski

List of included articles

This dissertation consists of an introductory part and the following articles:

[A] J. Siltakoski. Equivalence of viscosity and weak solutions for the normalized $p(x)$-Laplacian. *Calc. Var. Partial Differential Equations*, 57(4):Art. 95, 20, 2018.

[B] J. Siltakoski. Equivalence of viscosity and weak solutions for a p-parabolic equation. Submitted.

[C] J. Siltakoski. Equivalence between radial solutions of different non-homogeneous p-Laplacian type equations. Submitted.

In the introduction these articles will be referred to as [A], [B], and [C], whereas other references will be referred as [AHP17], [APR17], ...

Introduction

A *classical solution* to a partial differential equation is a suitably smooth function that satisfies an equation pointwise in a domain. However, many equations that appear in applications admit no such solutions and therefore the notion of solution needs to be extended. One such extension is achieved by integration by parts in the theory of *distributional weak solutions*. Another class of extended solutions is the *viscosity solutions* defined by generalized pointwise derivatives. If both viscosity and weak solutions can be meaningfully defined, it is natural to ask whether they coincide. This dissertation studies the equivalence of solutions to different equations related to the p-Laplacian and stochastic tug-of-war games.

1. Backgrounds

1.1. **Viscosity solutions.** Crandall and Lions [CL83] introduced viscosity solutions as a uniqueness criterion for first order equations, though related ideas were also published by Evans [Eva78, Eva80]. Viscosity solutions to second order equations remained of limited interest for several years as the uniqueness of solutions was known only in some special cases. A major breakthrough took place when Jensen [Jen88] proved the uniqueness of viscosity solutions to equations of the form $F(u, Du, D^2u) = 0$ under certain assumptions. His results were further extended by Ishii [Ish89] to include equations that depend on x.

The name of viscosity solutions originates from the so called *vanishing viscosity method* in which one adds a vanishing viscosity term to an equation and passes to the limit to obtain the existence of solutions. However, this method is no longer central. To illustrate the basic idea and definition of viscosity solutions, consider a partial differential equation

$$F(x, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $F : \Omega \times \mathbb{R}^N \times S^N \to \mathbb{R}$ is continuous, D^2u is the Hessian matrix of u and $\Omega \subset \mathbb{R}^N$ is a bounded domain. Here S^N denotes the set of symmetric real valued $N \times N$ matrices and is equipped with the usual partial ordering where $X \leq Y$ if $\eta' X\eta \leq \eta' Y\eta$ for all $\eta \in \mathbb{R}^N$. Suppose moreover that F is *degenerate elliptic*, meaning that

$$F(x, \eta, X) \geq F(x, \eta, Y) \quad \text{whenever } X \leq Y.$$

To give an example, the Laplace equation would now correspond to

$$F(x, \eta, X) := -\operatorname{tr} X := -\sum_{i=1}^{N} X_{ii}.$$

Definition. A lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ is a *viscosity supersolution* to (1.1) in Ω if $u \not\equiv \infty$ and whenever $\varphi \in C^2(\Omega)$ is such that $u - \varphi$ has a local minimum at $x \in \Omega$, we have

$$F(x, D\varphi(x), D^2\varphi(x)) \geq 0. \tag{1.2}$$

Similarly, an upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is a *viscosity subsolution* to (1.1) in Ω if $u \not\equiv -\infty$ and whenever $\varphi \in C^2(\Omega)$ is such that $u - \varphi$ has a local maximum at $x \in \Omega$, we have

$$F(x, D\varphi(x), D^2\varphi(x)) \leq 0.$$

A function is a *viscosity solution* if it is both viscosity sub- and supersolution.

Too see that the concept of viscosity solutions extends classical solutions, recall from calculus that if $\psi \in C^2$ has a local minimum at x, then we have

$$D\psi(x) = 0 \quad \text{and} \quad D^2\psi(x) \geq 0.$$

Therefore, if $u, \varphi \in C^2$ are such that $u - \varphi$ has a local minimum at x, we have

$$Du(x) = D\varphi(x) \quad \text{and} \quad D^2u(x) \geq D^2\varphi(x).$$

Consequently, if u is a classical supersolution to (1.1), it follows from degenerate ellipticity that

$$F(x, D\varphi(x), D^2\varphi(x)) \geq F(x, Du(x), D^2u(x)) \geq 0,$$

which means that u is a viscosity supersolution. Similarly we see that it is a viscosity subsolution and thus classical solutions are viscosity solutions.

A useful equivalent definition of viscosity supersolutions requires that the inequality (1.2) holds whenever $\varphi \in C^2$ *touches* u *from below at* x, i.e. whenever $\varphi(x) = u(x)$ and $\varphi(y) < u(y)$ for $y \neq x$. Analogously, definition of subsolutions uses test functions that touch from above. So far we assumed that F is continuous. For singular equations the definition of viscosity solutions needs to be adjusted, see the sections discussing the articles [B] and [C].

Viscosity solutions have turned out to be the natural class of solutions for many applications. They appear for example in optimal control (Hamilton–Jacobi–Bellman equation) [CL83, BCD97], stochastic games [PS08, PSSW09, MPR10, MPR12, BR19] and geometric flows [CGG91, ES91, AD00, FLM14]. In particular viscosity solutions can be formulated for equations in a non-divergence form or even for fully nonlinear equations. Viscosity solutions also have good stability properties with respect to uniform convergence. The existence of viscosity solutions can be often achieved via Perron's method. The standard reference to viscosity solutions is the paper by Crandall, Ishii and Lions [CIL92], see also the books by Caffarelli and Cabre [CC95], Koike [Koi12] and Katzourakis [Kat15].

1.1.1. *Viscosity solutions and tug-of-war.* Tug-of-war games provide an important motivation for the equations which we study in articles [A] and [C]. The connection of tug-of-war game to viscosity solutions was first discovered by Peres, Schramm, Sheffield and Wilson [PSSW09]. They introduced a two-player zero-sum stochastic game called *tug-of-war* and showed that the value function of the game is related to the solutions of the so called ∞-*Laplace equation*

$$-\Delta_\infty u := -(Du)' D^2u Du = -\sum_{i,j=1}^{N} D_{ij}u D_i u D_j u = 0, \tag{1.3}$$

where $(Du)'$ denotes the transpose of the column vector Du. The ∞-Laplacian was first studied by Aronsson in the 60s. It is related to optimal Lipschitz extensions [Aro67, Jen93]. The equation $-\Delta_\infty u = 0$ needs to be understood in the viscosity sense as it is in a non-divergence form and classical solutions turn out to be too restrictive. Indeed, any C^2 solution to $-\Delta_\infty u = 0$ must be a constant if it has any critical points [Aro68, Yu06]. This implies non-existence of classical solutions to the Dirichlet problem with suitable C^2 boundary data.

In 2008 Peres and Sheffield [PS08] introduced *tug-of-war with noise*, this time in connection to the *normalized (*or *game-theoretic) p-Laplacian* which for smooth u with a non-vanishing gradient can be written as

$$\Delta_p^N u := |Du|^{2-p} \operatorname{div}(|Du|^{p-2} Du) = \Delta u + (p-2) |Du|^{-2} \Delta_\infty u, \tag{1.4}$$

where $1 < p < \infty$ and the N stands for "normalized". In the time dependent case this leads to a *normalized p-parabolic equation*, see [MPR10, BG15].

The normalized p-Laplacian was also studied in the context of image processing [Doe11, ETT15]. To describe its connection to game theory, consider the following version of tug-of-war with noise by Manfredi, Parviainen and Rossi [MPR12]. Suppose that $p > 2$ for simplicity and that $\partial\Omega$ is suitably regular. A step size $\varepsilon > 0$ is fixed and a token is placed at x_0 in a domain Ω. A biased coin is tossed so that it lands heads with probability $\alpha = (p-2)/(p+N)$ and tails with probability $1-\alpha$. If it lands tails, the token moves to a random position $x_1 \in B_\varepsilon(x_0)$ according to a uniform probability distribution. Otherwise, a tug-of-war step is played: a fair coin is tossed and the winning player is allowed to select a new position $x_1 \in B_\varepsilon(x_0)$ for the token. Once the token exits the domain, the game ends and Player II pays Player I the amount $g(x_\tau)$, where x_τ is the final location of the token and $g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ is called a payoff function. There is a well defined concept of a value for this game. At each point $x \in \Omega$, the value of the game $u_\varepsilon(x)$ equals the amount of money that Player I is expected to win from Player II if the game starts from x. The value function satisfies the *dynamic programming principle*

$$u_\varepsilon(x) = \frac{\alpha}{2}\left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon\right) + (1-\alpha)\fint_{B_\varepsilon(x)} u_\varepsilon(y)\,dy$$

from which one can essentially read the rules of the game.

To heuristically link the dynamic programming principle to the normalized p-Laplacian, suppose for the moment that u is a smooth solution to $-\Delta_p^N u = 0$ whose gradient does not vanish. By Taylor's theorem we have

$$u(y) = u(x) + (y-x)\cdot Du(x) + \frac{1}{2}(y-x)'D^2u(x)(y-x) + o(|x-y|^2).$$

Taking an average over $B_\varepsilon(x)$ we obtain

$$\fint_{B_\varepsilon(x)} u(y)\,dy = u(x) + \frac{\varepsilon^2}{2(N+2)}\Delta u(x) + o(\varepsilon^2).$$

On the other hand, since the maximum and minimum of the Taylor expansion in $B_\varepsilon(x)$ are roughly at the points $y = x \pm \varepsilon Du(x)/\left|Du(x)\right|$, we heuristically have

$$\frac{1}{2}(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u) \approx u(x) + \frac{\varepsilon^2}{2}\left|Du(x)\right|^{-2}(Du(x))'D^2u(x)Du(x) + o(\varepsilon^2).$$

Combining the last two displays, recalling that $\alpha = (p-2)/(p+N)$ and using that u is a solution to $-\Delta_p^N u = 0$, we would see that u satisfies the dynamic programming principle with a small error.

When the step size approaches zero, the value function converges uniformly up to a subsequence to a viscosity solution of the Dirichlet problem

$$\begin{cases} -\Delta_p^N u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

This result can be extended to the case of space dependent probabilities [AHP17], which provides motivation for the study of the *normalized $p(x)$-Laplacian*

$$\Delta_{p(x)}^N u := \Delta u + (p(x)-2)\left|Du\right|^{-2}\Delta_\infty u$$

in article [A]. It is also possible to include a running payoff by requiring that Player II pays an amount equal to $\varepsilon^2 f(x_k)$ whenever the token is moved from x_k. In this case the value function converges to a solution of the non-homogeneous equation $-\Delta_p^N u = f$ [Ruo16] which is a special case of the equation studied in article [C].

1.2. **Distributional weak solutions.** The origins of distributional weak solutions (weak solutions for short) go back about a century to the works of Levi, Morrey, Sobolev, Tonelli and others on Hilbert's 20th problem. This problem asks if every regular variational problem has a solution with given boundary values, provided that the notion of solution is extended if needed. It turns out that the correct space in which to look for a solution is a Sobolev space. Furthermore, solutions to a regular variational problem coincide with weak solutions to the corresponding Euler-Lagrange equation. While Hilbert's 20th problem is concerned with existence, his 19th problem asks if a solution to a regular variational problem is always analytic. This was resolved independently by De Giorgi [Gio57] and Nash [Nas58] who proved that weak solutions to the corresponding Euler-Lagrange equation are Hölder continuous. By previous results this yielded analyticity of the solutions.

Today weak solutions are a central part of the analysis of partial differential equations with a large number of applications and vast literature. For an introduction to the topic, see for example the textbooks by Gilbarg and Trudinger [GT01], Wu, Yin and Wang [WYW06] or Evans [Eva10].

Let us recall the idea of weak solutions by considering the *p-Laplace equation*, which for a smooth function u with a non-vanishing gradient can be written as

$$\begin{aligned} -\Delta_p u &:= -\operatorname{div}(|Du|^{p-2} Du) \\ &= -|Du|^{p-2}\left(\Delta u + (p-2)|Du|^{-2}\Delta_\infty u\right) = 0, \end{aligned} \tag{1.5}$$

where $1 < p < \infty$. The p-Laplace equation is a model for quasilinear equations in a divergence form. It is in a sense singular when $p < 2$ and degenerate when $p > 2$. The cases $p = 1$ and $p \to \infty$ are related to the mean-curvature operator and the ∞-Laplacian, respectively. The book by Heinonen, Kilpeläinen and Martio [HKM06] and the notes by Lindqvist [Lin17] are good introductions to the topic. The p-Laplace equation is the Euler-Lagrange equation corresponding to the problem of minimizing the Dirichlet energy

$$\int_\Omega |Du|^p \, dx$$

among all functions in Ω with the same boundary values. By applying the direct method in calculus of variations, one finds that the minimizer must satisfy

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = 0 \tag{1.6}$$

for all $\varphi \in C_0^\infty(\Omega)$. The equation (1.6) is called the *weak form* of the equation $-\Delta_p u = 0$. Observe that it only contains first derivatives even though the original equation is of second order. For weak solutions we merely require these derivatives to exist in the distributional sense.

Definition. A function $u \in W^{1,p}_{loc}(\Omega)$ is a *weak solution* to

$$-\Delta_p u = 0 \quad \text{in } \Omega$$

if (1.6) holds for all $\varphi \in C_0^\infty(\Omega)$, where Du is understood in the distributional sense. For *weak supersolutions* we require that the integral in (1.6) is non-negative for all non-negative $\varphi \in C_0^\infty(\Omega)$. Analogously, for *weak subsolutions* we require that the integral is non-positive.

To see that classical solutions are weak solutions, let $\varphi \in C_0^\infty(\Omega)$ be arbitrary and suppose that u is a smooth solution to $-\Delta_p u = 0$ whose gradient does not vanish. Multiplying the equation by φ, integrating over Ω and applying the

Gauss-Green theorem, we obtain

$$\begin{aligned} 0 &= -\int_\Omega \varphi \operatorname{div}(|Du|^{p-2} Du)\, dx \\ &= \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx - \int_{\partial\Omega} \varphi\, |Du|^{p-2} Du \cdot \boldsymbol{n}\, dS. \end{aligned}$$

The normal vector $\boldsymbol{n}$ makes sense only for suitably regular $\partial\Omega$ but this is not a problem since any domain can be exhausted with smooth domains. Since φ has a compact support, the surface integral vanishes and we obtain the equation (1.6). Conversely, a C^2 weak solution is a classical solution since by the fundamental lemma in the calculus of variations the integral $\int_\Omega \varphi \operatorname{div}(|Du|^{p-2} Du)\, dx$ vanishes for all $\varphi \in C_0^\infty(\Omega)$ only if $\operatorname{div}(|Du|^{p-2} Du) \equiv 0$. Weak solutions are therefore a generalization of classical solutions.

1.3. **Equivalence of solutions.** The relationship between viscosity and weak solutions has been extensively studied starting from the work of Ishii [Ish95] on linear equations. The equivalence of viscosity and weak solutions to the p-Laplace equation and its parabolic version was first proved by Juutinen, Lindqvist and Manfredi [JLM01] for $1 < p < \infty$. In fact, they showed that $u : \Omega \to (-\infty, \infty]$ is a viscosity supersolution to the equation $-\Delta_p u = 0$ in Ω if and only if u is *p-superharmonic* in Ω. Recall that $u : \Omega \to (-\infty, \infty]$ is p-superharmonic if it is lower semicontinuous, $u \not\equiv \infty$ and it satisfies the comparison principle on each subdomain $D \Subset \Omega$: if $v \in C(\overline{D})$ is a weak solution to $-\Delta_p v = 0$ in D, then

$$u \geq v \text{ on } \partial D \quad \text{implies} \quad u \geq v \text{ in } D.$$

For example the so called *fundamental solution*

$$V(x) = \begin{cases} |x|^{\frac{p-N}{p-1}}, & p \neq N, \\ \log(|x|), & p = N, \end{cases}$$

is p-superharmonic in $\mathbb{R}^N$, but it is not a weak supersolution to the p-Laplace equation when $p \leq N$ because it fails to be in the correct Sobolev space. However, a locally bounded p-superharmonic function is a weak supersolution and a lower semicontinuous weak supersolution is p-superharmonic [Lin86]. The proof in [JLM01] relies on the comparison principle of viscosity and weak solutions. Later Julin and Juutinen [JJ12] proved the equivalence of viscosity and weak solutions to the p-Laplace equation without relying on the comparison principle of viscosity solutions. Equivalence of solutions to the *$p(x)$-Laplace equation*

$$-\Delta_{p(x)} u := -\operatorname{div}(|Du|^{p(x)-2} Du) = 0$$

was showed by Juutinen, Lukkari and Parviainen [JLP10] for $p \in C^1(\Omega)$ and $1 < \inf_\Omega p < \sup_\Omega p < \infty$. More recently, Attouchi, Parviainen and Ruosteenoja [APR17] proved and used the equivalence of solutions to obtain $C^{1,\alpha}$ regularity of solutions to the *normalized p-Poisson problem*

$$-\Delta_p^N u = f,$$

where $p \geq 2$ and $f \in C(\Omega)$ is in a suitable Lebesgue space. Ochoa and Medina [MO19] proved the equivalence of solutions to the non-homogeneous p-Laplace equation

$$-\Delta_p u = f(x, u, Du)$$

under suitable assumptions on f. Parviainen and Vazquez [PV] showed that radial viscosity solutions to the parabolic equation

$$\partial_t u = |Du|^{q-2} \Delta_p^N u,$$

where $q, p > 1$ coincide with weak solutions to a one-dimensional equation related to the radial q-Laplacian. Fractional equations were considered for example by Servadei and Valdinoci [SV14] and Korvenpää, Kuusi and Lindgren [KKL19]. Equivalence questions have been studied also in non-Euclidean settings. For example, Bieske [Bie06] proved the equivalence of viscosity and weak solutions to the p-Laplace equation in the Heisenberg group and recently Bieske and Freeman [BF] considered the $p(x)$-Laplace equation in Carnot groups.

In this dissertation we show equivalence of viscosity and weak solutions in three different cases. In article [A] we study the normalized or game-theoretic $p(x)$-Laplacian which appears in stochastic tug-of-war games. In article [B] we consider a parabolic p-Laplace equation with a gradient term. Finally, in [C] we study radial solutions to a non-homogeneous equation that includes both the normalized and standard p-Laplace equations. Though the main results are equivalence theorems, in [A] and [C] we also derive some applications from the equivalence of solutions.

2. The normalized $p(x)$-Laplacian and article [A]

In [A] we study the *normalized $p(x)$-Laplace equation* which for smooth u with a non-vanishing gradient can be written as

$$-\Delta_{p(x)}^N u := -\Delta u - \frac{p(x)-2}{|Du|^2}\Delta_\infty u = 0 \quad \text{in } \Omega, \tag{2.1}$$

where $\Delta_\infty u$ is the ∞-Laplacian defined in (1.3), $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p : \Omega \to \mathbb{R}$ is Lipschitz continuous with $p_{min} := \inf_\Omega p > 1$. As mentioned, the study of (2.1) is partially motivated by its connection to stochastic tug-of-war games with space dependent probabilities [AHP17].

Our main result is that viscosity solutions to (2.1) coincide with weak solutions once the equation is written in an appropriate divergence formulation. To find the divergence formulation, suppose for the moment that u is a smooth function whose gradient does not vanish. Then a direct calculation yields

$$|Du|^{p(x)-2}\,\Delta_{p(x)}^N u = \operatorname{div}(|Du|^{p(x)-2}\,Du) - |Du|^{p(x)-2}\log(|Du|)Du\cdot Dp,$$

where the logarithm appears because of the variable exponent inside the divergence. The right-hand side is the so called *strong $p(x)$-Laplacian* $\Delta_{p(x)}^S u$ which was introduced by Adamowicz and Hästö [AH10, AH11] in connection with mappings of finite distortion. We show that viscosity solutions to (2.1) are equivalent to weak solutions of the strong $p(x)$-Laplace equation

$$-\Delta_{p(x)}^S u = 0. \tag{2.2}$$

Weak solutions to (2.2) are defined using appropriate variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$. Under our assumptions they are Banach spaces and have similar properties as the usual Sobolev spaces. For details we refer the reader to the monograph by Diening, Harjulehto, Hästö and Růžička [DHHR11]. The precise definitions of solutions to the the strong and normalized $p(x)$-Laplace equations are below.

Definition 2.1. [A, Definition 3.1] A function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a *weak supersolution* to $-\Delta_{p(x)}^S u = 0$ in Ω if

$$\int_\Omega |Du|^{p(x)-2}\,Du\cdot D\varphi + |Du|^{p(x)-2}\log\left(|Du|\right)Du\cdot Dp\,\varphi\,dx \geq 0$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega)$ with a compact support. We say that u is a *weak subsolution* if $-u$ is a weak supersolution and that u is a *weak solution* if it is both weak super- and subsolution.

At the beginning of this introduction we mentioned that viscosity solutions are based on generalized pointwise derivatives. This refers to the so called *second order semi-jets*. For example, the *subjet* of a function u at x is defined by setting $(\eta, X) \in J^{2,-}u(x)$ if

$$u(y) \geq u(x) + (y-x)\cdot\eta + \frac{1}{2}(y-x)'X(y-x) + o(|y-x|^2) \quad \text{as } y \to x.$$

Using Taylor's theorem one can show that

$$J^{2,-}u(x) = \left\{(D\varphi(x), D^2\varphi(x)) : \varphi \in C^2 \text{ and } u-\varphi \text{ has a min at } x\right\}.$$

Therefore it is clear that viscosity solutions can be also defined using semi-jets.

Definition 2.2. [A, Definition 3.3] A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity supersolution* to $-\Delta^N_{p(x)}u = 0$ in Ω if, whenever $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$ and $\eta \neq 0$, then

$$-\operatorname{tr}(X) - \frac{(p(x)-2)}{|\eta|^2}\eta' X\eta \geq 0.$$

A function u is a *viscosity subsolution* if $-u$ is a viscosity supersolution, and a *viscosity solution* if it is both viscosity super- and subsolution.

Observe that in Definition 2.2 we ignore test functions whose gradient vanishes at the point of touching. This can be done because the equation is homogeneous; it would not lead to a reasonable definition if the right-hand side was non-zero.

To show that viscosity solutions are weak solutions, we adapt the method introduced by Julin and Juutinen [JJ12]. This way we can avoid relying on the uniqueness of solutions which to the best of our knowledge is still open for both the normalized and strong $p(x)$-Laplace equations. The idea is to fix a bounded viscosity supersolution u to $-\Delta^N_{p(x)}u \geq 0$ and approximate it by inf-convolution

$$u_\varepsilon(x) := \inf_{y\in\Omega}\left\{u(y) + \frac{1}{\hat{q}\varepsilon^{\hat{q}-1}}|x-y|^{\hat{q}}\right\},$$

where $\varepsilon > 0$ and $\hat{q} > 2$ is so large that $p_{min} - 2 + (\hat{q}-2)/(\hat{q}-1) \geq 0$. If there was no x-dependence in $-\Delta^N_{p(x)}u = 0$, it would be straightforward to show that u_ε is still a viscosity supersolution in the smaller set

$$\Omega_{r(\varepsilon)} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r(\varepsilon)\},$$

where $r(\varepsilon) \to 0$. To deal with the x-dependence, we modify an argument from [Ish95] to prove the following lemma. Roughly speaking it says that u_ε is a viscosity supersolution to $-\Delta^N_{p(x)}u \geq 0$ in $\Omega_{r(\varepsilon)}$ up to some small error. The proof is based on the Theorem of sums.

Lemma 2.3. *[A, Lemma 5.3] Assume that u is a uniformly continuous viscosity supersolution to $-\Delta^N_{p(x)}u = 0$ in Ω. Then, whenever $(\eta, X) \in J^{2,-}u_\varepsilon(x)$, $\eta \neq 0$ and $x \in \Omega_{r(\varepsilon)}$, it holds*

$$-|\eta|^{\min(p(x)-2,0)}\left(\operatorname{tr} X + \frac{(p(x)-2)}{|\eta|^2}\eta' X\eta\right) \geq E(\varepsilon),$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$. The error function E depends only on p, q and the modulus of continuity of u.

The inf-convolution u_ε is semi-concave in $\Omega_{r(\varepsilon)}$ and therefore twice differentiable almost everywhere by Alexandrov's theorem. This combined with the previous lemma essentially means that u_ε satisfies the equation $-\Delta^N_{p(x)}u_\varepsilon \geq 0$

pointwise almost everywhere in $\Omega_{r(\varepsilon)}$ up to some error. Moreover, the proof of Alexandrov's theorem in [EG15] establishes that we can approximate the semi-concave function u_ε with smooth functions $u_{\varepsilon,j}$ so that

$$u_{\varepsilon,j} \to u_\varepsilon, \quad Du_{\varepsilon,j} \to Du_\varepsilon \quad \text{and} \quad D^2 u_{\varepsilon,j} \to D^2 u_\varepsilon$$

almost everywhere in $\Omega_{r(\varepsilon)}$. We also denote by p_j the standard mollification of p. Very roughly speaking, we can now compute that

$$|Du_{\varepsilon,j}|^{p_j(x)-2}\Delta^N_{p_j(x)} u_{\varepsilon,j} = \Delta^S_{p_j(x)} u_{\varepsilon,j}$$

and let $j \to \infty$ at both sides to obtain from Lemma 2.3 that u_ε is a weak supersolution to $-\Delta^S_{p(x)} u_\varepsilon \geq 0$ with some error. However, there are additional technicalities on the way since $\Delta^N_{p_j(x)}$ is singular when $p_j(x) < 2$. To overcome this, we first regularize the equation by considering the identity

$$\begin{aligned}
&(|Du_{\varepsilon,j}|^2 + \delta)^{\frac{p_j(x)-2}{2}} (\Delta u_{\varepsilon,j} + \frac{p_j(x) - 2}{\delta + |Du_{\varepsilon,j}|^2} \Delta_\infty u_{\varepsilon,j}) \\
&= \operatorname{div}((\delta + |Du_{\varepsilon,j}|^2)^{\frac{p_j(x)-2}{2}} Du_{\varepsilon,j}) \\
&\quad - \frac{1}{2}(\delta + |Du_{\varepsilon,j}|^2)^{\frac{p_j(x)-2}{2}} \log(\delta + |Du_{\varepsilon,j}|^2) Du_{\varepsilon,j} \cdot Dp_j.
\end{aligned}$$

Then we let $j \to \infty$ and $\delta \to 0$, in that order. This is the part where the choice of large enough $\hat{q}$ in the definition of inf-convolution gets into play. Heuristically speaking, it makes the inf-convolution so flat that the singularity of $\Delta^S_{p(x)}$ gets canceled and we can pass to the limit. In the end we obtain the following lemma, which says that u_ε is a weak supersolution to $-\Delta^S_{p(x)} u_\varepsilon \geq 0$ with some error.

Lemma 2.4. *[A, Lemma 5.5] Assume that u is a uniformly continuous viscosity supersolution to $-\Delta^S_{p(x)} u = 0$ in Ω. Let u_ε be the inf-convolution of u. Then*

$$\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot (D\varphi + \log|Du_\varepsilon|\, Dp\, \varphi)\, dx \geq E(\varepsilon) \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{s(x)} \varphi\, dx$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega_{r(\varepsilon)})$ with compact support, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $s(x) = \max(p(x) - 2, 0)$.

With this lemma at hand, we use a Caccioppoli type estimate to conclude that Du_ε is bounded in $L^{p(\cdot)}(\Omega')$ for any $\Omega' \Subset \Omega$ with respect to ε. To do this, we test the inequality of Lemma 2.4 with $\varphi := (L - u_\varepsilon)\xi^{p_{\max}}$, where $L := \sup_{\varepsilon, x\in\Omega'} u$ and $\xi \in C_0^\infty(\Omega)$ is a cut-off function such that $\xi \equiv 1$ in Ω'. This yields

$$\begin{aligned}
\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)} \xi^{p_{max}}\, dx \leq &\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-1} \xi^{p_{max}-1} (L - u_\varepsilon) p_{\max} |D\xi|\, dx \\
&+ \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-1} |\log|Du_\varepsilon|| |Dp| (L - u_\varepsilon)\xi^{p_{\max}}\, dx \\
&+ |E(\varepsilon)| \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{\max(p(x)-2,0)} (L - u_\varepsilon)\xi^{p_{\max}}\, dx.
\end{aligned}$$

The terms containing $|Du_\varepsilon|$ can be absorbed to the left-hand side by using Young's inequality, and we obtain that

$$\int_{\Omega'} |Du_\varepsilon|^{p(x)}\, dx \leq C(p, L, Dp, D\xi).$$

Since Du_ε is bounded in the variable exponent Lebesgue space, it has a weakly converging subsequence. Using the inequality of Lemma 2.4 again and some algebraic inequalities, we obtain a further subsequence for which Du_ε converges strongly in the variable exponent Lebesgue space. It then remains to pass to the limit in the inequality of Lemma 2.4 to see that u is a weak supersolution.

Theorem 2.5. *[A, Theorem 5.8] If $u \in C(\Omega)$ is a viscosity supersolution to $-\Delta^N_{p(x)}u \geq 0$ in Ω, then u is a weak supersolution to $-\Delta^S_{p(x)}u \geq 0$ in Ω.*

Corollary 2.6. *[A, Corollary 5.10] Since weak solutions to $-\Delta^S_{p(x)}u = 0$ are $C^{1,\alpha}$ regular* [ZZ12]*, Theorem 2.5 implies the $C^{1,\alpha}$ regularity of viscosity solutions to $-\Delta^N_{p(x)}u = 0$.*

To show the other direction of the equivalence, we adapt a standard argument used for example in [JLM01]. We suppose on the contrary that a weak solution u to $-\Delta^S_{p(x)}u = 0$ is not a viscosity supersolution to $-\Delta^N_{p(x)}u \geq 0$. This means that there is a function $\varphi \in C^2$ that touches u from below at $x \in \Omega$ so that

$$-\Delta^N_{p(x)}\varphi(x) < 0$$

and $D\varphi \neq 0$ near x. By continuity the above inequality holds in some neighborhood of x where the gradient of φ does not vanish. Therefore a direct computation yields

$$-\Delta^S_{p(y)}\varphi(y) = -\left|D\varphi(y)\right|^{p(y)-2}\Delta^N_{p(y)}\varphi(y) < 0$$

for all y in some ball $B_r(x)$. In other words, φ is a classical subsolution to the strong $p(x)$-Laplace equation. On the other hand, since φ touches the C^1 function u from below, we have $Du(x) = D\varphi(x) \neq 0$. Therefore, by taking smaller $r > 0$ if necessary, we can ensure that the gradient of u does not vanish in $B_r(x)$. Next we lift φ slightly by setting

$$\tilde{\varphi} := \varphi + l,$$

where $l := \sup_{y \in \partial B_r(x)}(u - \varphi) > 0$. Then $\tilde{\varphi}$ is still a subsolution and we have $\tilde{\varphi} \leq u$ on $\partial B_r(x)$. Using that u and $\tilde{\varphi}$ have non-vanishing gradients in $B_r(x)$, we can prove a comparison principle to show that $u \leq \tilde{\varphi}$ in $B_r(x)$. This yields a contradiction since $\tilde{\varphi}(x) = \varphi(x) + l = u(x) + l$ and $l > 0$.

Theorem 2.7. *[A, Theorem 4.1] Let $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ be a weak solution to $-\Delta^S_{p(x)}u = 0$ in Ω. Then it is a viscosity solution to $-\Delta^N_{p(x)}u = 0$ in Ω.*

3. A parabolic p-Laplace equation and article [B]

In [B] we study the relationship of viscosity and weak supersolutions to the parabolic equation

$$\partial_t u - \Delta_p u = f(Du) \quad \text{in } \Xi, \tag{3.1}$$

where Δ_p is the p-Laplace operator defined in (1.5), $p > 1$, $f \in C(\mathbb{R})$ satisfies suitable assumptions and $\Xi \subset \mathbb{R}^{N+1}$ is a bounded domain. Our main result is that bounded viscosity supersolutions to (3.1) coincide with lower semicontinuous weak supersolutions. Our proof is different than in [JLM01] even for $f \equiv 0$. The lower semicontinuity of weak supersolutions is needed since by definition they are only in a parabolic Sobolev space. However, under slightly stronger assumptions on f and in the range $p \geq 2$, we show that weak supersolutions are in fact lower semicontinuous.

For a domain $\Omega \subset \mathbb{R}^N$, we denote the space-time cylinder $\Omega_{t_1,t_2} := \Omega \times (t_1, t_2)$, where $t_1 < t_2$. A Lebesgue measurable function $u : \Omega_{t_1,t_2} \to \mathbb{R}$ belongs to the *parabolic Sobolev space* $L^p(t_1, t_2; W^{1,p}(\Omega))$ if $u(\cdot, t) \in W^{1,p}(\Omega)$ for almost all $t \in (t_1, t_2)$ and the norm

$$\left(\int_{\Omega_{t_1,t_2}} |u|^p + |Du|^p \, dx \, dt\right)^{\frac{1}{p}}$$

is finite.

In the first part of the article we suppose the following *growth condition* on the gradient term

$$|f(\xi)| \le C_f(1+|\xi|^\beta) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{3.2}$$

where $C_f > 0$ and $1 \le \beta < p$. This ensures in particular that $f(Du)$ is summable when $Du \in L^p$. The precise definitions of weak and viscosity solutions are below.

Definition 3.1. [B, Definition 2.1] A function $u : \Xi \to \mathbb{R}$ is a *weak supersolution* to (3.1) in Ξ if $u \in L^p(t_1, t_2; W^{1,p}(\Omega))$ whenever $\Omega_{t_1,t_2} \Subset \Xi$ and

$$\int_\Xi -u\partial_t\varphi + |Du|^{p-2}\, Du \cdot D\varphi - \varphi f(Du)\, dx\, dt \ge 0$$

for all non-negative *test functions* $\varphi \in C_0^\infty(\Omega_{t_1,t_2})$. For *weak subsolutions* the inequality is reversed and a function is a *weak solution* if it is both super- and subsolution.

Definition 3.2. [B, Definition 2.2] A lower semicontinuous and bounded function $u : \Xi \to \mathbb{R}$ is a *viscosity supersolution* to (3.1) in Ξ if whenever $\varphi \in C^2(\Xi)$ and $(x_0, t_0) \in \Xi$ are such that

$$\begin{cases} \varphi(x_0,t_0) = u(x_0,t_0), & \\ \varphi(x,t) < u(x,t) & \text{when } (x,t) \ne (x_0,t_0), \\ D\varphi(x,t) \ne 0 & \text{when } x \ne x_0, \end{cases}$$

then

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\ x\ne x_0}} \left(\partial_t\varphi(x,t) - \Delta_p\varphi(x,t) - f(D\varphi(x,t))\right) \ge 0.$$

An upper semicontinuous and bounded function $u : \Xi \to \mathbb{R}$ is a *viscosity subsolution* to (3.1) in Ξ if whenever $\varphi \in C^2(\Xi)$ and $(x_0, t_0) \in \Xi$ are such that

$$\begin{cases} \varphi(x_0,t_0) = u(x_0,t_0), & \\ \varphi(x,t) > u(x,t) & \text{when } (x,t) \ne (x_0,t_0), \\ D\varphi(x,t) \ne 0 & \text{when } x \ne x_0, \end{cases}$$

then

$$\liminf_{\substack{(x,t)\to(x_0,t_0)\\ x\ne x_0}} \left(\partial_t\varphi(x,t) - \Delta_p\varphi(x,t) - f(D\varphi(x,t))\right) \le 0.$$

A function that is both viscosity sub- and supersolution is a *viscosity solution.*

The limiting process in the definition of viscosity solutions is in the spirit of [JLM01]. It is used to deal with the singularity of Δ_p when $1 < p < 2$. When $p \ge 2$, the operator is degenerate and the limiting process disappears.

To show that viscosity supersolutions are weak solutions, we adapt the method of Julin and Juutinen [JJ12] to the parabolic case. This was previously done in [PV] for radial solutions. The inf-convolution needs to be adapted to the parabolic setting and it takes the form

$$u_\varepsilon(x,t) := \inf_{(y,s)\in\Xi} \left\{ u(y,s) + \frac{|x-y|^{\hat q}}{\hat q \varepsilon^{q-1}} + \frac{|t-s|^2}{2\varepsilon} \right\},$$

where $\varepsilon > 0$ and $\hat q \ge 2$ is a constant so large that $p - 2 + (\hat q - 2)/(\hat q - 1) > 0$. If u is a weak supersolution to (3.1) in Ξ, then u_ε is still a weak supersolution to (3.1) in a smaller set Ξ_ε. Moreover, if u_ε is differentiable in time and twice differentiable in space at $(x,t) \in \Xi_\varepsilon$ and $Du_\varepsilon(x,t) = 0$, then $\partial_t u_\varepsilon(x,t) - f(0) \ge 0$. Using these observations we show that u_ε is a weak supersolution to (3.1) in Ξ_ε.

Lemma 3.3. *[B, Lemmas 4.1 and 4.2] Let u be a bounded viscosity supersolution to (3.1) in Ξ. Then u_ε is a weak supersolution to (3.1) in Ξ_ε.*

In [B, Lemma 4.3] we show that Lipschitz continuous weak supersolutions to (3.1) satisfy the Caccioppoli's inequality

$$\int_\Xi \xi^p \left|Du\right|^p \, dx\, dt \le C \int_\Xi M^2 \partial_t \xi^p + M^p \left|D\xi\right|^p + (M^{\frac{p}{p-\beta}} + M)\xi^p \, dx\, dt, \qquad (3.3)$$

where $\xi \in C_0^\infty(\Xi)$ and $M = \|u\|_{L^\infty(\operatorname{spt}\xi)}$. This implies that the sequence Du_ε converges weakly in $L^p_{loc}(\Xi)$ up to a subsequence. However, this is not enough to pass to the limit under the integral sign of

$$\int_\Xi -u_\varepsilon \partial_t \varphi + \left|Du_\varepsilon\right|^{p-2} Du_\varepsilon \cdot D\varphi - \varphi f(Du_\varepsilon) \, dx\, dt \ge 0. \qquad (3.4)$$

To this end, we prove the next lemma.

Lemma 3.4. *[B, Lemma 4.4] Suppose that (u_j) is a sequence of locally Lipschitz continuous weak supersolutions to (3.1) such that $u_j \to u$ in $L^p_{loc}(\Xi)$. Then (Du_j) is a Cauchy sequence in $L^r_{loc}(\Xi)$ for any $1 < r < p$.*

The proof is more involved than in the elliptic setting and it is based on the proof of Lemma 5 in [LM07], see also Theorem 5.3 in [KKP10]. The idea is to fix $1 < r < p$ and use the test functions $(\delta - w_{jk})\theta$ and $(\delta + w_{jk})\theta$, where $\theta \in C_0^\infty(\Xi)$ is a cut-off function with $\theta \equiv 1$ in $U \Subset \Xi$ and

$$w_{jk} := \begin{cases} \delta, & u_j - u_k > \delta, \\ u_j - u_k, & |u_j - u_k| \le \delta, \\ -\delta, & u_j - u_k < -\delta. \end{cases}$$

This gives us information about the behavior of $|Du_j - Du_k|$ in the set where $|u_j - u_k| < \delta$. More precisely, we obtain after estimations that

$$\int_{U \cap \{|u_j - u_k| < \delta\}} \left|Du_j - Du_k\right|^r dz \le C \delta^{\frac{r}{\max(2,p)}},$$

where C is independent of j, k and δ. To handle the set where $|u_j - u_k| \ge \delta$, we apply Hölder's and Chebysheff's inequalities as well as the Caccioppoli's inequality (3.3) to obtain

$$\int_{U \cap \{|u_j - u_k| \ge \delta\}} \left|Du_j - Du_k\right|^r dz \le C \delta^{r-p} ||u_j - u_k||^{p-r}_{L^p(U)}.$$

By taking first small $\delta > 0$ and then large j, k, we see that $||Du_j - Du_k||_{L^r(U)}$ can be made arbitrarily small.

With Lemma 3.4 at hand, we can pass to the limit in (3.4) and conclude that u is a weak supersolution.

Theorem 3.5. *[B, Theorem 4.5] Let $1 < p < \infty$ and suppose that (3.2) holds. Let u be a bounded viscosity supersolution to (3.1) in a domain Ξ. Then u is a weak supersolution to (3.1) in Ξ.*

To prove the other part of the equivalence, we apply a parabolic version of the argument described at the end of the section discussing article [A]. Most of the work is therefore in proving suitable comparison principles for the equation (3.1) at the weak side. To state this part of the equivalence, we define the *lower semicontinuous regularization* of a function $u : \Xi \to \mathbb{R}$ by

$$u_*(x,t) := \lim_{R \to 0} \operatorname*{ess\,inf}_{B_R(x) \times (t - R^p, t + R^p)} u.$$

Theorem 3.6. *[B, Theorem 3.5] Let $1 < p < \infty$ and suppose that (3.2) holds. Let u be a bounded lower semicontinuous weak supersolution to (3.1) in Ξ for which $u = u_*$ almost everywhere in Ξ. Then u_* is a viscosity supersolution to (3.1) in Ξ.*

The lower semicontinuous regularization is needed because weak supersolutions are not semicontinuous by definition. In other words, it is not clear if all weak supersolutions satisfy the assumption $u = u_*$. By adapting the work of Kuusi [Kuu09], we show that this is the case at least when $p \geq 2$, provided that $f(0) = 0$ and the following stronger growth condition holds

$$|f(\xi)| \leq C_f \left(1 + |\xi|^{p-1}\right).$$

The proof first applies Moser's iteration technique to obtain essential supremum estimates for weak subsolutions. These estimates are then used to show that a weak supersolution coincides with its lower semicontinuous regularization at its Lebesgue points.

4. Radial solutions to $-|Du|^{q-2}\Delta_p^N u = f$ and article [C]

In [C] we study radial solutions to the equation

$$-|Du|^{q-2}\Delta_p^N u = f(|x|) \quad \text{in } B_R \subset \mathbb{R}^N, \tag{4.1}$$

where $f \in C[0, \mathbb{R})$, $p, q \in (1, \infty)$, $N \geq 2$ and Δ_p^N denotes the normalized p-Laplacian defined in (1.4). The use of viscosity solutions is appropriate as the equation (4.1) may be in a non-divergence form: the left-hand side is the normalized p-Laplacian when $q = 2$ and the usual p-Laplacian when $q = p$. Since we are interested in radial solutions, it is natural to restrict to a ball at the origin and assume that the source term is radial.

Our main result is that bounded radial viscosity supersolutions to (4.1) coincide with bounded weak solutions of a one-dimensional equation related to the p-Laplacian. This kind of equivalence was recently obtained by Parviainen and Vázquez [PV] for solutions of the parabolic equation

$$\partial_t u = |Du|^{q-2}\Delta_p^N u.$$

Stated slightly more precisely, we show that $u(x) = v(|x|)$ is a bounded viscosity supersolution to (4.1) if and only if v is a bounded weak supersolution to the one-dimensional equation

$$-\kappa\Delta_q^d v = f \quad \text{in } (0, R), \tag{4.2}$$

where

$$\Delta_q^d v = |v'|^{q-2}\left((q-1)v'' + \frac{d-1}{r}v'\right)$$

and the positive constants κ and d are given in (4.4). Heuristically speaking, the operator Δ_q^d is the radial q-Laplacian in a fictitious dimension d which is not necessarily an integer. However, we show in [C, Theorem 5.3] that if d happens to be an integer, then weak supersolutions to (4.2) correspond to radial weak supersolutions of the equation

$$-\Delta_q u = f(|x|) \quad \text{in } B_R \subset \mathbb{R}^d.$$

In order to derive the one dimensional equation (4.2), suppose for the moment that $u : B_R \to \mathbb{R}$ is a smooth radial function. This means that there exists a smooth function $v : [0, R) \to \mathbb{R}$ such that $u(x) = v(|x|)$. Then by a direct computation we have for $r > 0$

$$Du(re_1) = e_1 v'(r) \quad \text{and} \quad D^2u(re_1) = e_1 \otimes e_1 v''(r) + \frac{1}{r}(I - e_1 \otimes e_1)v'(r).$$

In particular, $|Du(re_1)| = |v'(r)|$. Assuming that the gradient does not vanish, we have by the definition of the normalized p-Laplacian

$$\begin{aligned}\Delta_p^N u(re_1) =&\Delta u(re_1) + \frac{p-2}{|Du(re_1)|^2}\sum_{i,j=1}^N D_{ij}u(re_1)D_iu(re_1)D_ju(re_1)\\ =&v''(r) + \frac{N-1}{r}v'(r) + \frac{p-2}{|v'(r)|^2}v''(r)\,|v'(r)|^2\\ =&(p-1)v''(r) + \frac{N-1}{r}v'(r). \end{aligned}\tag{4.3}$$

Denoting

$$\kappa := \frac{p-1}{q-1}, \quad d := \frac{(N-1)(q-1)}{p-1} + 1 \tag{4.4}$$

and multiplying (4.3) by $|Du(re_1)|^{q-2}$, we obtain

$$|Du(re_1)|^{q-2}\,\Delta_p^N u(re_1) = \kappa\,|v'(r)|^{q-2}\,((q-1)v''(r) + \frac{d-1}{r}v'(r)) = \kappa\Delta_q^d v(r).$$

This suggests that $u(x) = v(|x|)$ solves (4.1) whenever v solves (4.2). However, to make this rigorous, we must carefully exploit the precise definitions of viscosity and weak solutions.

Weak solutions to (4.2) are defined using appropriate weighted Sobolev spaces. A Lebesgue measurable function $v : (0,R) \to \mathbb{R}$ is in $W^{1,q}(r^{d-1},(0,R))$ if the norm

$$\|v\|_{W^{1,q}(r^{d-1},(0,R))} := \Big(\int_0^R |v|^q\,r^{d-1}\,dr + \int_0^R |v'|^q r^{d-1}\,dr\Big)^{1/q}$$

is finite, where v' denotes the distributional derivative of v. For details on these spaces, see [Kuf85]. To derive the weak formulation of (4.2), we multiply the equation by r^{d-1} to obtain

$$\begin{aligned}fr^{d-1} =& -\kappa r^{d-1}\,|v'|^{q-2}\,((q-1)v'' + \frac{d-1}{r}v')\\ =& -\kappa(|v'|^{q-2}\,v'r^{d-1})'.\end{aligned}$$

This is in a divergence form and the precise definition of weak solutions to (4.2) is below. Observe that we require the test function space to be $C_0^\infty(-R,R)$ instead of $C_0^\infty(0,R)$. This is necessary as otherwise there may exist weak solutions that do not correspond to radial viscosity solutions of (4.1) [C, Example 2.3].

Definition 4.1. [C, Definition 2.2] We say that v is a *weak supersolution* to (4.2) in $(0,R)$ if $v \in W^{1,q}(r^{d-1},(0,R'))$ for all $R' \in (0,R)$ and we have

$$\int_0^R \kappa\,|v'|^{q-2}\,v'\varphi' r^{d-1} - \varphi f r^{d-1}\,dr \geq 0$$

for all $\varphi \in C_0^\infty(-R,R)$. For weak subsolutions the inequality is reversed. Furthermore, $v \in C[0,R)$ is a weak solution if it is both weak sub- and supersolution.

Viscosity solutions to (4.1) are defined as follows.

Definition 4.2. [C, Definition 2.1] A bounded lower semicontinuous function $u : B_R \to \mathbb{R}$ is a *viscosity supersolution* to (4.1) in B_R if whenever $\varphi \in C^2$ touches u from below at x_0 and $D\varphi(x) \neq 0$ when $x \neq x_0$, then we have

$$\limsup_{x_0 \neq y \to x_0}\Big(-\,|D\varphi(y)|^{q-2}\,\Delta_p^N\varphi(y)\Big) - f(|x_0|) \geq 0.$$

A bounded upper semicontinuous function $u : B_R \to \mathbb{R}$ is a *viscosity subsolution* to (4.1) in B_R if whenever $\varphi \in C^2$ touches u from above at x_0 and $D\varphi(x) \neq 0$ when $x \neq x_0$, then we have

$$\liminf_{x_0 \neq y \to x_0} \Big(- |D\varphi(y)|^{q-2}\, \Delta_p^N \varphi(y) \Big) - f(|x_0|) \leq 0.$$

A function is a *viscosity solution* if it is both viscosity sub- and supersolution.

The precise equivalence result is now contained in the following theorems.

Theorem 4.3. *[C, Theorem 3.1] Let v be a bounded weak supersolution to (4.2) in $(0, R)$. Let $u(x) := v_*(|x|)$, where*

$$v_*(r) := \lim_{S \to 0} \operatorname*{ess\,inf}_{s \in (r-S, r+S) \cap (0,R)} v(s) \quad \textit{for all } r \in [0, R).$$

Then u is a viscosity supersolution to (4.1) in $B_R \subset \mathbb{R}^N$.

Theorem 4.4. *[C, Theorem 4.1] Let u be a bounded radial viscosity supersolution to (4.1) in $B_R \subset \mathbb{R}^N$. Then $v(r) := u(re_1)$ is a weak supersolution to (4.2) in $(0, R)$.*

Since the equation (4.2) satisfies a comparison principle [C, Theorem 3.4], we obtain the uniqueness of radial viscosity solutions to (4.1) as a consequence of the equivalence. To the best of our knowledge this was previously known only for $f \equiv 0$ or f with a constant sign [KMP12]. However, the full uniqueness and comparison principle remain open.

Corollary 4.5. *[C, Corollary 4.3] Let $u, h \in C(\overline{B_R})$ be radial viscosity solutions to (4.1) in B_R such that $u = h$ on ∂B_R. Then $u = h$.*

To show Theorem 4.3, we adapt the basic argument and suppose on the contrary that $u(x) := v_*(|x|)$ is not a viscosity solution. Roughly speaking, this implies that there exists a smooth function φ touching u from below at $x_0 \in B_R$ so that φ is a subsolution to (4.1) near x_0. We use φ to construct a new function ϕ that is a weak subsolution to (4.2) and touches v_* from below. Since v_* is a weak supersolution, this violates a comparison principle and we arrive at the desired contradiction. A special argument is needed if the point of touching is the origin. We also exploit a different but equivalent definition of viscosity solutions by Birindelli and Demengel [BD04] to avoid technicalities that might arise should the gradient of φ vanish at the point of touching.

To prove Theorem 4.4, we fix a bounded radial viscosity supersolution u to (4.1) in B_R. We begin by approximating u by its inf-convolution u_ε. Then $u_\varepsilon \to u$ pointwise and it is standard to show that u_ε is a viscosity supersolution to

$$-|Du_\varepsilon|^{q-2} \Delta_p^N u_\varepsilon \geq f_\varepsilon(|x|) \quad \text{in } B_{R_\varepsilon}, \tag{4.5}$$

where

$$f_\varepsilon(r) := \inf_{|r-s| \leq \rho(\varepsilon)} f(s),$$

$R_\varepsilon := R - \rho(\varepsilon)$ and $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since u_ε is semi-concave, it is twice differentiable almost everywhere by Alexandrov's theorem and therefore satisfies (4.5) almost everywhere in B_{R_ε}. Since $u(x) = v(|x|)$ is a radial function, so is its inf-convolution and we have $u_\varepsilon(x) = v_\varepsilon(|x|)$ for some $v_\varepsilon : (0, R) \to \mathbb{R}$. Therefore we can perform a radial transformation on (4.5) to roughly obtain that v_ε satisfies $-\kappa \Delta_q^d v_\varepsilon \geq f_\varepsilon$ for almost every $r \in (0, R_\varepsilon)$. More precisely, we have the following lemma.

Lemma 4.6. *[C, Lemma 4.6] Assume that u is a bounded radial viscosity supersolution to (4.1) in B_R. Set $v_\varepsilon(r) := u_\varepsilon(re_1)$ and assume that v_ε is twice differentiable at $r \in (0, R_\varepsilon)$. Then, if $q > 2$ or $v_\varepsilon'(r) \neq 0$, we have*

$$-\kappa \left|v_\varepsilon'(r)\right|^{q-2} \Big((q-1)v_\varepsilon''(r) + \frac{d-1}{r} v_\varepsilon'(r)\Big) - f_\varepsilon(r) \geq 0.$$

Moreover, if $1 < q \leq 2$ with $v_\varepsilon'(r) = 0$, then we have $f_\varepsilon(r) \leq 0$.

Combining the above lemma with mollification and regularization arguments, we obtain the next lemma which states that v_ε is a weak supersolution to $-\kappa\Delta_q^d v_\varepsilon \geq f_\varepsilon$ in $(0, R_\varepsilon)$.

Lemma 4.7. *[C, Lemmas 4.7 and 4.8] Let $1 < q < \infty$. Assume that u is a bounded radial viscosity supersolution to (4.1) in B_R. Then the function $v_\varepsilon(r) := u_\varepsilon(re_1)$ is a weak supersolution to $-\kappa\Delta_q^d v_\varepsilon \geq f_\varepsilon$ in $(0, R_\varepsilon)$.*

Using that v_ε is a weak supersolution, we show that it satisfies a Caccioppoli's inequality

$$\int_0^R \left|v_\varepsilon'\right|^q \xi^q r^{d-1}\, dr \leq C \int_0^R \Big(\left|\xi'\right|^q + \xi^q \left|f\right|\Big) r^{d-1}\, dr$$

for all $\xi \in C_0^\infty(-R_\varepsilon, R_\varepsilon)$, where C depends only on κ, q and $\|v\|_{L^\infty(0,R)}$. It follows that v_ε is a bounded sequence in the weighted Sobolev space $W^{1,q}(r^{d-1}, (0, R'))$ for any $R' \in (0, R)$ and therefore we can extract a weakly converging subsequence. Using the supersolution property of v_ε again, we find a further subsequence that converges strongly. It then remains to fix a test function $\varphi \in C_0^\infty(-R, R)$ and pass to the limit in the inequality

$$\int_0^R \kappa \left|v_\varepsilon'\right|^{q-2} v_\varepsilon' \varphi' r^{d-1} - \varphi f_\varepsilon r^{d-1}\, dr \geq 0$$

to conclude that v is a weak supersolution to $-\Delta_q^d v \geq f$ in $(0, R)$.

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Equivalence of viscosity and weak solutions for the normalized $p(x)$-Laplacian

Jarkko Siltakoski

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EQUIVALENCE OF VISCOSITY AND WEAK SOLUTIONS FOR THE NORMALIZED $p(x)$-LAPLACIAN

JARKKO SILTAKOSKI

Abstract. We show that viscosity solutions to the normalized $p(x)$-Laplace equation coincide with distributional weak solutions to the strong $p(x)$-Laplace equation when p is Lipschitz and $\inf p > 1$. This yields $C^{1,\alpha}$ regularity for the viscosity solutions of the normalized $p(x)$-Laplace equation. As an additional application, we prove a Radó-type removability theorem.

1. Introduction

In this paper, we study viscosity solutions to the *normalized $p(x)$-Laplace equation* which is defined by

$$-\Delta_{p(x)}^N u := -\Delta u - \frac{p(x)-2}{|Du|^2}\Delta_\infty u = 0, \tag{1.1}$$

where

$$\Delta_\infty u := \left\langle D^2 u Du, Du \right\rangle .$$

There has been recent interest in normalized equations, see e.g. [JS17, IJS, BG15]. We are partly motivated by the connection to stochastic tug-of-war games [PS08, PSSW09] as the case of space dependent probabilities leads to (1.1) [AHP17].

The objective of this work is to show that viscosity solutions to (1.1) coincide with solutions in the distributional weak sense, when the equation is rewritten in an appropriate divergence formulation. One approach to this kind of equivalence results [JLM01, Ish95] is based on the uniqueness of solutions. However, it seems difficult to use uniqueness in our case because the uniqueness of solutions is an open problem for the equation (1.1) as pointed out in [JLP10]. The equation (1.1) is in non-divergence form. In order to find the appropriate weak formulation, we note that for $u \in C^2(\Omega)$ with non-vanishing gradient it holds that

$$-|Du|^{p(x)-2}\Delta_{p(x)}^N u = -\operatorname{div}\left(|Du|^{p(x)-2} Du\right) + |Du|^{p(x)-2}\log\left(|Du|\right) Du\cdot Dp.$$

Thus the weak formulation of (1.1) should be the *strong $p(x)$-Laplace equation*

$$-\Delta_{p(x)}^S u := -\operatorname{div}(|Du|^{p(x)-2} Du) + |Du|^{p(x)-2}\log|Du|\, Du\cdot Dp = 0. \tag{1.2}$$

Our main result, Theorem 5.9, is that viscosity solutions to (1.1) coincide with weak solutions to (1.2) when the function p is Lipschitz with $\inf p > 1$. With these assumptions weak solutions to (1.2) in a domain are locally $C^{1,\alpha}$ continuous [ZZ12]. Thus our equivalence result yields local $C^{1,\alpha}$ regularity also for viscosity solutions to (1.1). As an application, we prove a Radó-type removability theorem for the strong $p(x)$-Laplacian. The theorem follows from the equivalence result since in the definition of a viscosity solution we may ignore the test functions whose gradient vanishes. The equivalence result also implies that the equation (1.2) is homogeneous: if u is a solution, so is λu. This is not completely obvious and was established in [AH10].

That viscosity solutions to (1.1) are weak solutions to (1.2) is proven by applying the method of [JJ12]. The idea is to approximate a viscosity solution through a sequence of inf-convolutions, show that the inf-convolutions are essentially weak supersolutions, and then pass to the limit.

First, in Lemma 5.3 we show that the inf-convolution u_ε of a viscosity supersolution u to (1.1) is still, in essence, a viscosity supersolution up to some error. This fact is a key part of our proof. If there was no x-dependence in (1.1), it would be straightforward to see that the inf-convolution of a viscosity supersolution is still a viscosity supersolution. This is because a test function that touches the inf-convolution from below also touches the original function from below at a nearby point once we add some constant to it. From this it would follow that the inf-convolution is a supersolution to the original equation. However, the equation (1.1) has x-dependence caused by $p(x)$. Thus the inf-convolution no longer satisfies the original equation.

In Lemma 5.5 we use the standard mollification on u_ε and p to deduce from Lemma 5.3 that u_ε is "almost" a weak solution to $-\Delta^S_{p(x)} u_\varepsilon \geq 0$. Applying Caccioppoli type estimates and vector inequalities we are then able to deduce that the sequence of inf-convolutions converges to the viscosity supersolution in $W^{1,p(\cdot)}_{loc}(\Omega)$ as $\varepsilon \to 0$. This allows us to pass to the limit and conclude that the function u satisfies $-\Delta^S_{p(x)} u \geq 0$ in the weak sense.

Due to the variable exponent, the operator $\Delta^S_{p(x)}$ can be singular in some subsets and degenerate in others. Therefore we apply different arguments in the cases $p(x) < 2$ and $p(x) \geq 2$, and finally need to be able to combine them.

The equivalence of weak and viscosity solutions to the usual p-Laplace equation was first proven by Juutinen, Lindqvist and Manfredi [JLM01]. Later Julin and Juutinen [JJ12] presented a more direct way to show that viscosity solutions to $-\Delta_p u = f$ are also weak solutions. This proof was adapted in [APR17] to show that viscosity solutions to $-\Delta^N_p u = f$ coincide with weak solutions to $-\Delta_p u = |Du|^{p-2} f$ when $p \geq 2$. Similar arguments were also used in [MO] to study the equivalence of solutions to $-\Delta_p u = f(x, u, Du)$. The variable exponent case was explored in [JLP10] where the equivalence of weak and viscosity solutions was proven for the $p(x)$-Laplace equation using techniques of [JLM01].

As mentioned, the equation (1.1) appears in stochastic tug-of-war games. Let us illustrate this in the case where $p > 2$ is a constant by considering the following two-player, zero-sum game from [MPR12]. A step size $\varepsilon > 0$ is fixed and a token is placed at x_0 in a domain Ω. The players toss a biased coin that is heads with the probability $\alpha = \frac{p-2}{p+N}$ and tails with the probability $\beta = 1 - \alpha$. If the outcome is heads, the following tug-of-war step is played: a fair coin is tossed and the winning player is allowed to move the token to any position $x_1 \in B_\varepsilon(x_0)$. If the outcome is tails, the token moves to a random position in $x_1 \in B_\varepsilon(x_0)$. Once the token exits the domain, the game ends and player I pays player II according to the final location of the token. When the players optimize over their strategies, we obtain a value of the game. Then, as the step-size approaches zero, the value function converges uniformly to a viscosity solution of $-\Delta^N_p u = 0$ in Ω. This result can be extended to the general case $1 < p(x) < \infty$, see [PS08, AHP17].

The equation (1.2) was introduced by Adamowicz and Hästö [AH10] in connection with mappings of finite distortion. Unlike the standard $p(x)$-Laplace equation, the equation (1.2) is homogeneous and its solutions satisfy a classical Harnack inequality [AH11]. The equation (1.2) has been further studied for example in [ZZ12, PL13, ZZZ17].

The paper is organized as follows: in Section 2 we recall the variable exponent Lebesgue and Sobolev spaces. Section 3 contains the rigorous definitions of solutions to equations (1.1) and (1.2). In Section 4 we show that weak solutions of (1.2) are viscosity solutions to (1.1) and the converse statement is proven in

Section 5. Finally, in Section 6 we formulate and prove a Radó-type removability theorem for weak solutions of (1.2).

2. Variable exponent lebesgue and sobolev spaces

We briefly recall basic facts about these spaces. For general reference see e.g. [DHHR11]. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $p : \Omega \to (1, \infty)$ be a measurable function. We denote

$$p_{\max} := \operatorname*{ess\,sup}_{x \in \Omega} p(x) \quad \text{and} \quad p_{\min} := \operatorname*{ess\,inf}_{x \in \Omega} p(x).$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ for which the $p(\cdot)$-modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} \, dx$$

is finite. It is a Banach space equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Given that $p_{\max} < \infty$ or $\varrho_{p(\cdot)}(u) > 0$, the norm and the modular satisfy the inequality (see [DHHR11, p75])

$$\min \left\{ \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\min}}}, \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\max}}} \right\} \leq \|u\|_{L^{p(\cdot)}(\Omega)} \leq \max \left\{ \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\min}}}, \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\max}}} \right\}. \tag{2.1}$$

A version of Hölder's inequality holds [DHHR11, p81] : if $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for a.e. $x \in \Omega$, then

$$\int_\Omega |u| \, |v| \, dx \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

As a consequence of the Hölder's inequality we have that

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leq 2 \left(1 + |\Omega|\right) \|u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ if $q(x) \leq p(x)$ for a.e. $x \in \Omega$.

If $1 < p_{\min} \leq p_{\max} < \infty$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the set of functions in $u \in L^{p(\cdot)}(\Omega)$ for which the weak gradient Du belongs in $L^{p(\cdot)}(\Omega)$. It is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)}.$$

The space $W_0^{1,p}(\Omega)$ is the closure of compactly supported Sobolev functions in the space $W^{1,p(\cdot)}(\Omega)$. A function belongs to the the local Lebesgue space $L^{p(\cdot)}_{loc}(\Omega)$ if it belongs to $L^{p(\cdot)}(\Omega')$ for all $\Omega' \Subset \Omega$. The space $W^{1,p(\cdot)}_{loc}(\Omega)$ is defined analogically.

3. The strong and normalized $p(x)$-Laplace equations

In this section, we define weak solutions to the strong $p(x)$-Laplace equation and viscosity solutions to the normalized $p(x)$-Laplace equation.

From now on we assume that p is Lipschitz continuous and $p_{\min} > 1$.

Definition 3.1. A function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a *weak supersolution* to $-\Delta_{p(x)}^S u \geq 0$ in Ω if

$$\int_\Omega |Du|^{p(x)-2} Du \cdot D\varphi + |Du|^{p(x)-2} \log(|Du|) Du \cdot Dp\, \varphi \, dx \geq 0$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega)$ with compact support. We say that u is a *weak subsolution* to $-\Delta_{p(x)}^S u \leq 0$ if $-u$ is a supersolution and that u is a *weak solution* to $-\Delta_{p(x)}^S u = 0$ if u is both supersolution and subsolution.

Lemma 3.2. *It is enough to consider $C_0^\infty(\Omega)$ test functions in the previous definition.*

Proof. Assume that $\varphi \in W^{1,p(\cdot)}(\Omega)$ has a compact support in an open set $\Omega' \Subset \Omega$. Since p is log-Hölder continuous and bounded as a Lipschitz function, there is a sequence of functions $\varphi_j \in C_0^\infty(\Omega')$ such that $\varphi_j \to \varphi$ in $W^{1,p(\cdot)}(\Omega')$ (see [DHHR11, p347]). We set $\psi_j := \varphi - \varphi_j$. Then it is enough to show that

$$\int_{\Omega'} |Du|^{p(x)-2} Du \cdot D\psi_j \, dx + \int_{\Omega'} |Du|^{p(x)-2} \log(|Du|) Du \cdot Dp\, \psi_j \, dx \to 0$$

as $j \to \infty$. The first integral convergences to zero by Hölder's inequality so we focus on the second integral. We may assume that $N > 1$. We set $q(x) := \frac{p(x)}{p(x)-1+\frac{1}{N}}$. Using the inequality $a^s \log a \leq N a^{s+\frac{1}{N}} + \frac{1}{s}$ for $a, s > 0$ we get

$$\begin{aligned} &\int_{\Omega'} |Du|^{p(x)-1} |\log |Du|| \, |Dp| \, |\psi_j| \, dx \\ &\leq \|Dp\|_{L^\infty(\Omega')} \left(\int_{\Omega'} \frac{|\psi_j|}{p(x)-1} \, dx + N \int_{\Omega'} |Du|^{p(x)-1+\frac{1}{N}} |\psi_j| \, dx \right) \\ &\leq C(p,\Omega) \left(\|\psi_j\|_{L^{p(\cdot)}(\Omega')} + \left\| |Du|^{p(x)-1+\frac{1}{N}} \right\|_{L^{q(\cdot)}(\Omega')} \|\psi_j\|_{L^{q'(\cdot)}(\Omega')} \right). \end{aligned}$$

We take $r \in (1, N)$ such that $q'^{+} \leq r^* := \frac{Nr}{N-r}$. Then we have $q'(x) = \frac{Np(x)}{N-1} \leq \min(p^*(x), r^*)$, where $p^*(x) := \frac{Np(x)}{N-p(x)}$. Therefore

$$\|\psi_j\|_{L^{q'(\cdot)}(\Omega')} \leq 2\,(1 + |\Omega|) \, \|\psi_j\|_{L^{\min(p^*(\cdot), r^*)}(\Omega')}.$$

Since $\psi_j \in W_0^{1,\min(p(\cdot),r)}(\Omega')$, we have by a variable exponent version of the Sobolev inequality (see e.g. [DHHR11, p265])

$$\|\psi_j\|_{L^{\min(p^*(\cdot),r^*)}(\Omega')} \leq C \, \|D\psi_j\|_{L^{\min(p(\cdot),r)}(\Omega')} \leq 2C(1+|\Omega|) \, \|D\psi_j\|_{L^{p(\cdot)}(\Omega')}.$$

These estimates imply the claim since $\|\psi_j\|_{W^{1,p}(\Omega')} \to 0$ as $j \to \infty$. □

In order to define viscosity solutions to $-\Delta_{p(x)}^N u = 0$, we set

$$F(x,\eta,X) := -\left(\mathrm{tr} X + \frac{p(x)-2}{|\eta|^2} \langle X\eta, \eta \rangle \right)$$

for all $(x,\eta,X) \in \Omega \times \left(\mathbb{R}^N \setminus \{0\}\right) \times S^N$ where S^N is the set of symmetric $N \times N$ matrices. We also recall the concept of semi-jets. The *subjet of a function* $u : \Omega \to \mathbb{R}$ *at* x is defined by setting $(\eta, X) \in J^{2,-}u(x)$ if

$$u(y) \geq u(x) + \eta \cdot (y-x) + \frac{1}{2} \langle X(y-x), (y-x) \rangle + o(|y-x|^2) \text{ as } y \to x. \quad (3.1)$$

The *closure of a subjet* is defined by setting $(\eta, X) \in \overline{J}^{2,-}u(x)$ if there is a sequence $(\eta_i, X_i) \in J^{2,-}u(x_i)$ such that $(x_i, \eta_i, X_i) \to (x, \eta, X)$. The *superjet* $J^{2,+}u(x)$ and its closure $\overline{J}^{2,+}u(x)$ are defined in the same manner except that the inequality (3.1) is reversed.

Definition 3.3. A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity supersolution* to $-\Delta^N_{p(x)} u \geq 0$ in Ω if, whenever $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$ and $\eta \neq 0$, then

$$F(x, \eta, X) \geq 0.$$

A function u is a *viscosity subsolution* to $-\Delta^N_{p(x)} u \leq 0$ if $-u$ is a viscosity supersolution, and a *viscosity solution* to $-\Delta^N_{p(x)} u = 0$ if it is both viscosity super- and subsolution.

Remark. Observe that in the previous definition we require nothing in the case $(0, X) \in J^{2,-}u(x)$.

Viscosity solutions may be equivalently defined using the jet-closures or test functions. The next proposition follows easily from the proof of Proposition 2.6 in [Koi12].

Proposition 3.4. *Let $u : \Omega \to \mathbb{R}$ be lower semicontinuous. Then the following conditions are equivalent.*

(i) *The function u is a viscosity supersolution to $-\Delta^N_{p(x)} u \geq 0$ in Ω.*
(ii) *Whenever $(\eta, X) \in \overline{J}^{2,-}u(x)$ with $x \in \Omega$, $\eta \neq 0$, we have $F(x, \eta, X) \geq 0$.*
(iii) *Whenever $\varphi \in C^2(\Omega)$ is such that $\varphi(x) = u(x)$, $D\varphi(x) \neq 0$ and $\varphi(y) < u(y)$ for all $y \neq x$, it holds $F(x, D\varphi(x), D^2\varphi(x)) \geq 0$.*

When φ is as in the third condition above, we say that φ *touches u from below at x.*

4. Weak solutions are Viscosity solutions

We show that if u is a weak solution to $-\Delta^S_{p(x)} u = 0$, then it is a viscosity solution to $-\Delta^N_{p(x)} u = 0$.

Juutinen, Lukkari and Parviainen [JLP10] showed that weak solutions to the standard $p(x)$-Laplace equation are also viscosity solutions. This was accomplished with the help of the comparison principle. For if u is a weak supersolution to $-\Delta_{p(x)} u \geq 0$ that is not a viscosity supersolution, then there is a test function $\varphi \in C^2$ touching u from below at x so that $-\Delta_{p(x)}\varphi < 0$ in some ball $B(x)$. Lifting φ slightly produces a new function $\tilde{\varphi}$ still satisfying $-\Delta_{p(x)}\tilde{\varphi} < 0$ in $B(x)$ and $\tilde{\varphi} \leq u$ in $\partial B(x)$. Comparison principle now implies that $\tilde{\varphi} \leq u$ in $B(x)$ which is a contradiction since $\tilde{\varphi}(x) > \varphi(x) = u(x)$.

Our difficulty is that, to the best of our knowledge, the comparison principle is an open problem for the strong $p(x)$-Laplacian. Our strategy is therefore to consider a ball so small that the gradient of the test function does not vanish. Then the comparison principle holds and we arrive at a contradiction.

Theorem 4.1. *If $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a weak solution to $-\Delta^S_{p(x)} u = 0$, then it is a viscosity solution to $-\Delta^N_{p(x)} u = 0$ in Ω.*

Proof. Zhang and Zhou [ZZ12] showed that weak solutions of $-\Delta^S_{p(x)} u = 0$ are in $C^1(\Omega)$. Therefore it suffices to show that if $u \in C^1(\Omega)$ is a weak supersolution to $-\Delta^S_{p(x)} u \geq 0$, then it is also a viscosity supersolution to $-\Delta^N_{p(x)} u \geq 0$. Assume on the contrary that there is $\varphi \in C^2(\Omega)$ touching u from below at $x_0 \in \Omega$, $D\varphi(x_0) \neq 0$ and

$$0 > -h > F(x_0, D\varphi(x_0), D^2\varphi(x_0)).$$

Then by continuity there is $r > 0$ such that in $B_r(x_0)$ it holds

$$-h\,|D\varphi|^{p(x)-2} \geq -\,|D\varphi|^{p(x)-2}\left(\Delta\varphi + \frac{p(x)-2}{|D\varphi|^2}\Delta_\infty\varphi\right). \tag{4.1}$$

Since $Du(x_0) = D\varphi(x_0) \neq 0$, we may also assume that there is $m > 0$ such that

$$\inf_{x\in B_r(x_0)} |D\varphi|^{p(x)-2} \geq m \tag{4.2}$$

and

$$\operatorname*{ess\,sup}_{x\in B_r(x_0)} |Dp| \left| |D\varphi|^{p(x)-2} \log(|D\varphi|)\, D\varphi - |Du|^{p(x)-2} \log(|Du|)\, Du \right| \leq \frac{hm}{2}. \tag{4.3}$$

Let $l := \min_{x\in\partial B_r(x_0)} (u-\varphi) > 0$ and set $\psi(x) := \max(\varphi(x) + l - u(x), 0)$. Then $\psi \in W_0^{1,2}(B_r(x_0))$ so there are $\psi_j \in C_0^\infty(B_r(x_0))$ such that $\psi_j \to \psi$ in $W^{1,2}(B_r(x_0))$. Let p_j be the standard mollification of p. Multiplying (4.1) by ψ and integrating over $B_r(x_0)$ yields

$$\begin{aligned}
-h&\int_{B_r(x_0)} |D\varphi|^{p(x)-2}\, \psi\, dx \\
&\geq \int_{B_r(x_0)} -|D\varphi|^{p(x)-2} \left(\Delta\varphi + \frac{p(x)-2}{|D\varphi|^2} \Delta_\infty \varphi\right) \psi\, dx \\
&= \lim_{j\to\infty} \int_{B_r(x_0)} -|D\varphi|^{p_j(x)-2} \left(\Delta\varphi + \frac{p_j(x)-2}{|D\varphi|^2} \Delta_\infty \varphi\right) \psi_j\, dx,
\end{aligned} \tag{4.4}$$

where the last equality holds because $\psi_j \to \psi$ in $W^{1,2}(B_r(x_0))$ and $p_j \to p$ uniformly in $B_r(x_0)$. Calculating the divergence of $|D\varphi|^{p_j(x)-2} D\varphi$ and integrating by parts we get

$$\begin{aligned}
&\int_{B_r(x_0)} -|D\varphi|^{p_j(x)-2} \left(\Delta\varphi + \frac{p_j(x)-2}{|D\varphi|^2} \Delta_\infty \varphi\right) \psi_j\, dx \\
&\quad= \int_{B_r(x_0)} -\operatorname{div}\left(|D\varphi|^{p_j(x)-2} D\varphi\right) \psi_j + |D\varphi|^{p_j(x)-2} \log(|D\varphi|)\, D\varphi \cdot Dp_j\, \psi_j\, dx \\
&\quad= \int_{B_r(x_0)} |D\varphi|^{p_j(x)-2} D\varphi \cdot \left(D\psi_j + \log(|D\varphi|)\, Dp_j\, \psi_j\right) dx.
\end{aligned} \tag{4.5}$$

By the convergence of ψ_j and p_j, it follows from (4.4) and (4.5) that

$$-h \int_{B_r(x_0)} |D\varphi|^{p(x)-2}\, \psi\, dx \geq \int_{B_r(x_0)} |D\varphi|^{p(x)-2} D\varphi \cdot \left(D\psi + \log(|D\varphi|)\, Dp\, \psi\right) dx. \tag{4.6}$$

Since u is a weak supersolution to $\Delta^S_{p(x)} u = 0$ and $\psi \in W^{1,p(\cdot)}(\Omega)$ has a compact support in Ω, we have

$$\int_{B_r(x_0)} |Du|^{p(x)-2} Du \cdot \left(D\psi + \log|Du|\, Dp\, \psi\right) dx \geq 0. \tag{4.7}$$

Denoting $A := \{x \in B_r(x_0) : \psi(x) > 0\}$ and combining (4.6) and (4.7) we arrive at

$$\begin{aligned}
&\int_A \left(|D\varphi|^{p(x)-2} D\varphi - |Du|^{p(x)-2} Du\right) \cdot (D\varphi - Du)\, dx \\
&\leq \int_A \left| |Du|^{p(x)-2} \log(|Du|)\, Du - |D\varphi|^{p(x)-2} \log(|D\varphi|)\, D\varphi \right| |Dp|\, \psi\, dx \\
&\quad - h \int_A |D\varphi|^{p(x)-2}\, \psi\, dx \\
&\leq -\frac{hm}{2} \int_A \psi\, dx,
\end{aligned} \tag{4.8}$$

where the last inequality follows from (4.2) and (4.3). Since

$$\left(|a|^{p(x)-2}\,a-|b|^{p(x)-2}\,b\right)\cdot(a-b)\geq 0$$

for any two vectors $a,b\in\mathbb{R}^N$ when $p(x)>1$, it follows from (4.8) that $|A|=0$. But this is impossible since $\varphi(x_0)=u(x_0)$ and $l>0$. □

5. Viscosity solutions are Weak solutions

We show that if u is a viscosity supersolution to $-\Delta^N_{p(x)}u\geq 0$, then it is a weak supersolution to $-\Delta^S_{p(x)}u\geq 0$. The same statement for subsolutions then follows by analogy.

We recall the usual partial ordering for symmetric $N\times N$ matrices by setting $X\leq Y$ if $\langle X\xi,\xi\rangle\leq\langle Y\xi,\xi\rangle$ for all $\xi\in\mathbb{R}^N$. For a matrix X we also set $\|X\|:=\max\{|\lambda|:\lambda\text{ is an eigenvalue of }X\}$ and for vectors $\xi,\eta\in\mathbb{R}^N$ we use the notation $\xi\otimes\eta:=\xi\eta'$, i.e. $\xi\otimes\eta$ is an $N\times N$ matrix whose (i,j) entry is $\xi_i\eta_j$.

Definition 5.1 (Inf-convolution)**.** Let $q\geq 2$ and $\varepsilon>0$. The inf-convolution of a bounded function $u\in C(\Omega)$ is defined by

$$u_\varepsilon(x):=\inf_{y\in\Omega}\left\{u(y)+\frac{1}{q\varepsilon^{q-1}}\,|x-y|^q\right\}. \tag{5.1}$$

The inf-convolution is well known to provide good approximations of viscosity supersolutions and often one only needs to consider it for $q=2$ (see e.g. [CIL92]). However, as the authors in [JJ12] observed, considering large enough q essentially cancels the singularity in the usual p-Laplace operator when $1<p<2$. In similar fashion it also cancels the singularity of the operator $\Delta^S_{p(x)}$. This is due to the property (v) in the next lemma. We also list some other properties of the inf-convolution.

Lemma 5.2. *Let $u\in C(\Omega)$ be a bounded function. Then the inf-convolution u_ε as defined in (5.1) has the following properties.*

(i) *We have $u_\varepsilon\leq u$ in Ω and $u_\varepsilon\to u$ locally uniformly in Ω as $\varepsilon\to 0$.*

(ii) *There exists $r(\varepsilon)>0$ such that*

$$u_\varepsilon(x)=\inf_{y\in B_{r(\varepsilon)}(x)\cap\Omega}\left\{u(y)+\frac{1}{q\varepsilon^{q-1}}\,|x-y|^q\right\}$$

and $r(\varepsilon)\to 0$ as $\varepsilon\to 0$. In fact we can choose $r(\varepsilon)=\left(q\varepsilon^{q-1}\operatorname{osc}_\Omega u\right)^{\frac{1}{q}}$.

(iii) *The function u_ε is semi-concave in $\Omega_{r(\varepsilon)}$, that is, the function $x\mapsto u_\varepsilon(x)-\frac{q-1}{2\varepsilon^{q-1}}r(\varepsilon)^{q-2}\,|x|^2$ is concave.*

(iv) *If $x\in\Omega_{r(\varepsilon)}:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>r(\varepsilon)\}$, then there exists a point $x_\varepsilon\in B_{r(\varepsilon)}(x)$ such that $u_\varepsilon(x)=u(x_\varepsilon)+\frac{1}{q\varepsilon^{q-1}}\,|x-x_\varepsilon|^q$.*

(v) *If $(\eta,X)\in J^{2,-}u_\varepsilon(x)$ with $x\in\Omega_{r(\varepsilon)}$, then $\eta=\frac{(x-x_\varepsilon)}{\varepsilon^{q-1}}\,|x_\varepsilon-x|^{q-2}$ and $X\leq\frac{q-1}{\varepsilon}\,|\eta|^{\frac{q-2}{q-1}}I$, where x_ε is as in* (iv)*.*

These properties are well known, see appendix of [JJ12] and also [Kat15b] where more general "flat inf-convolution" is considered. Regardless, we give a proof of (v) based on [Kat15a, p53] due to its critical role in the proof of Lemma 5.5.

Proof of property (v) in Lemma 5.2. Let $(\eta,X)\in J^{2,-}u_\varepsilon(x)$. Then there is a function $\varphi\in C^2(\mathbb{R}^N)$ such that it touches u_ε from below at x and $D\varphi(x)=\eta$, $D^2\varphi(x)=X$. Therefore for all $y,z\in\Omega$ we have

$$u(y)+\frac{|y-z|^q}{q\varepsilon^{q-1}}-\varphi(z)\geq u_\varepsilon(z)-\varphi(z)\geq 0.$$

Choosing $y = x_\varepsilon$, we obtain

$$\varphi(z) - \frac{|x_\varepsilon - z|^q}{q\varepsilon^{q-1}} \leq u(x_\varepsilon) \text{ for all } z \in \Omega.$$

Since $\varphi(x) = u_\varepsilon(x) = u(x_\varepsilon) + \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}}$, the above inequality means that the function

$$z \mapsto \varphi(z) - \frac{|x_\varepsilon - z|^q}{q\varepsilon^{q-1}} =: \varphi(z) - \psi(z)$$

has a maximum at x. Thus $\eta = D\psi(x) = \frac{(x - x_\varepsilon)}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}$ and

$$\begin{aligned} X \leq D^2\psi(x) =& \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \Big((q-2)(x_\varepsilon - x) \otimes (x_\varepsilon - x) + |x_\varepsilon - x|^2 I \Big) \\ \leq& \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \Big((q-2) \left\| (x_\varepsilon - x) \otimes (x_\varepsilon - x) \right\| I + |x_\varepsilon - x|^2 I \Big) \\ =& \frac{q-1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2} I \\ =& \frac{q-1}{\varepsilon^{q-1}} \Big(\varepsilon |\eta|^{\frac{1}{q-1}} \Big)^{q-2} I \\ =& \frac{q-1}{\varepsilon} |\eta|^{\frac{q-2}{q-1}} I. \end{aligned}$$

□

We will show that the inf-convolution provides approximations of viscosity supersolutions to $-\Delta_{p(x)}^N u \geq 0$. If there was no x-dependence in the equation, it would be straightforward to show that the inf-convolution of a supersolution is still a supersolution. However, the equation $-\Delta_{p(x)}^N u \geq 0$ has x-dependence caused by $p(x)$. Regardless, in [Ish95, Thm 3] it is shown that with some assumptions on G, the inf-convolution u_ε of a viscosity supersolution to $G(x, u, Du, D^2u) \geq 0$ is still a viscosity supersolution to $G(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) \geq E(\varepsilon)$, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We prove a modified version of this theorem for the solutions of $-\Delta_{p(x)}^N u \geq 0$. The important modification is the term $|\eta|^{\min(p(x)-2,0)}$ in (5.2) as it cancels a singular gradient term that appears due to the error term in the proof of Lemma 5.5, see (5.14). Another difference is that we consider inf-convolution with the exponent $q \geq 2$.

Lemma 5.3. *Assume that u is a uniformly continuous viscosity supersolution to $-\Delta_{p(x)}^N u \geq 0$ in Ω. Then, whenever $(\eta, X) \in J^{2,-} u_\varepsilon(x)$, $\eta \neq 0$ and $x \in \Omega_{r(\varepsilon)}$, it holds*

$$|\eta|^{\min(p(x)-2,0)} F(x, \eta, X) \geq E(\varepsilon), \tag{5.2}$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$. The error function E depends only on p, q and the modulus of continuity of u.

Proof. Fix $x \in \Omega_{r(\varepsilon)}$ and $(\eta, X) \in J^{2,-} u_\varepsilon(x)$, $\eta \neq 0$. Then by Lemma 5.2 there is $x_\varepsilon \in B_{r(\varepsilon)}(x)$ such that

$$u_\varepsilon(x) = u(x_\varepsilon) + \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}} \tag{5.3}$$

and $\eta = \frac{(x - x_\varepsilon)}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}$. There exists a function $\varphi \in C^2(\mathbb{R}^N)$ such that it touches u_ε from below at x and $D\varphi(x) = \eta$, $D^2\varphi(x) = X$. By the definition of inf-convolution

$$u(y) - \varphi(z) + \frac{|y - z|^q}{q\varepsilon^{q-1}} \geq u_\varepsilon(z) - \varphi(z) \geq 0 \text{ for all } y, z \in \Omega_{r(\varepsilon)}. \tag{5.4}$$

Since by (5.3) we have $u(x_\varepsilon) = \varphi(x) - \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}}$, it follows from (5.4) that the expression $u(y) - \varphi(z) + \frac{|y-z|^q}{q\varepsilon^{q-1}}$ reaches its minimum at $(y,z) = (x_\varepsilon, x)$. Thus

$$\max_{(y,z)\in\Omega_{r(\varepsilon)}\times\Omega_{r(\varepsilon)}} -u(y) + \varphi(z) - \frac{|y-z|^q}{q\varepsilon^{q-1}} = -u(x_\varepsilon) + \varphi(x) - \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}}.$$

We denote $\Phi(y,z) := \frac{1}{q\varepsilon^{q-1}} |y-z|^q$ and invoke the Theorem of sums (see [CIL92]). There exist $Y, Z \in S^N$ such that

$$(\eta, -Y) \in \overline{J}^{2,-} u(x_\varepsilon), \quad (\eta, -Z) \in \overline{J}^{2,+} \varphi(x)$$

and

$$\begin{pmatrix} Y & 0 \\ 0 & -Z \end{pmatrix} \leq D^2\Phi(x_\varepsilon, x) + \varepsilon^{q-1} \left(D^2\Phi(x_\varepsilon, x)\right)^2 \tag{5.5}$$

where

$$D^2\Phi(x_\varepsilon, x) = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix}$$

with $M = \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \Big((q-2)(x_\varepsilon - x)\otimes(x_\varepsilon - x) + |x_\varepsilon - x|^2 I\Big)$ and

$$\left(D^2\Phi(x_\varepsilon, x)\right)^2 = 2\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix}.$$

The above implies $Y \leq Z \leq -D^2\varphi(x) = -X$. Multiplying (5.5) by the $\mathbb{R}^{2N}$ vector $(\frac{\eta}{|\eta|}\sqrt{p(x_\varepsilon)-1}, \frac{\eta}{|\eta|}\sqrt{p(x)-1})$ from both sides yields

$$\frac{(p(x_\varepsilon)-1)}{|\eta|^2}\langle Y\eta, \eta\rangle - \frac{(p(x)-1)}{|\eta|^2}\langle Z\eta, \eta\rangle \leq \varLambda^2 \left\langle \left(M + 2\varepsilon^{q-1}M^2\right)\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle, \tag{5.6}$$

where $\varLambda = \sqrt{p(x)-1} - \sqrt{p(x_\varepsilon)-1}$. We have

$$\begin{aligned}
0 \leq & F(x_\varepsilon, \eta, -Y) \\
= & F(x, \eta, Z) - F(x_\varepsilon, \eta, Y) - F(x, \eta, Z) \\
= & (p(x_\varepsilon)-1)\left\langle Y\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle - (p(x)-1)\left\langle Z\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle \\
& + \operatorname{tr}(Y) - \left\langle Y\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle - \operatorname{tr}(Z) + \left\langle Z\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle + F(x, \eta, -Z) \\
\leq & \varLambda^2 \left\langle \left(M + 2\varepsilon^{q-1}M^2\right)\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle + F(x, \eta, X),
\end{aligned} \tag{5.7}$$

where we used (5.6) and the fact that $Y \leq Z$ implies

$$\operatorname{tr}(Y - Z) - \left\langle (Y-Z)\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle \leq 0.$$

We have the estimate

$$\begin{aligned}
\|M\| \leq & \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \Big((q-2)\|(x_\varepsilon - x)\otimes(x_\varepsilon - x)\| + |x_\varepsilon - x|^2 \|I\|\Big) \\
= & \frac{q-1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}.
\end{aligned}$$

Since p is Lipschitz continuous and $p_{\min} > 1$, we have also

$$\varLambda^2 = \frac{|p(x) - p(x_\varepsilon)|^2}{\left|\sqrt{p(x)-1} + \sqrt{p(x_\varepsilon)-1}\right|^2} \leq C(p)\,|x - x_\varepsilon|^2.$$

Combining these with (5.7) we get (we may assume that $r(\varepsilon) < 1$)

$$\begin{aligned}
-F(x,\eta,X) \leq& \Lambda^2 \left(\|M\| + 2\varepsilon^{q-1} \|M\|^2 \right) \\
\leq& \Lambda^2 \left(\frac{q-1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2} + 2\varepsilon^{q-1} \left(\frac{q-1}{\varepsilon^{q-1}} \right)^2 |x_\varepsilon - x|^{2(q-2)} \right) \\
\leq& \frac{3(q-1)^2}{\varepsilon^{q-1}} \Lambda^2 |x_\varepsilon - x|^{q-2} \\
\leq& C(p,q) \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^q . \qquad (5.8)
\end{aligned}$$

Moreover, by uniform continuity of u there is a modulus of continuity ω such that $\omega(t) \to 0$ as $t \to 0$ and $|u(y) - u(z)| \leq \omega(|y-z|)$ for all $y, z \in \Omega$. Hence by (5.3)

$$|x_\varepsilon - x| \leq \left(q\varepsilon^{q-1} \left(u(x) - u(x_\varepsilon) \right) \right)^{\frac{1}{q}} \leq q^{\frac{1}{q}} \varepsilon^{\frac{q-1}{q}} \omega(r(\varepsilon))^{\frac{1}{q}}. \qquad (5.9)$$

We now consider the situations $p(x) \leq 2$ and $p(x) > 2$ separately.
If $p(x) \leq 2$, we multiply (5.8) by $|\eta|^{p(x)-2}$ and estimate using (5.9). We get

$$\begin{aligned}
-|\eta|^{p(x)-2} F(x,\eta,X) \leq& C(p,q) \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^q |\eta|^{p(x)-2} \\
=& C(p,q) \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^q \left| \frac{1}{\varepsilon^{q-1}} (x - x_\varepsilon) |x_\varepsilon - x|^{q-2} \right|^{p(x)-2} \\
=& C(p,q) \left(\frac{1}{\varepsilon} \right)^{(q-1)(p(x)-1)} |x_\varepsilon - x|^{q+(q-1)(p(x)-2)} \\
\leq& C(p,q) \left(\frac{1}{\varepsilon} \right)^{(q-1)(p(x)-1)} \left(q^{\frac{1}{q}} \varepsilon^{\frac{q-1}{q}} \omega(r(\varepsilon))^{\frac{1}{q}} \right)^{q+(q-1)(p(x)-2)} \\
=& C(p,q) \left(\frac{1}{\varepsilon} \right)^{\left(\frac{q-1}{q}\right)(p(x)-2)} \omega(r(\varepsilon))^{\frac{q+(q-1)(p(x)-2)}{q}} \\
\leq& C(p,q) \omega(r(\varepsilon))^{\frac{q+(q-1)(p_{\min}-2)}{q}},
\end{aligned}$$

where the last inequality is true when $\varepsilon < 1$ is so small that $\omega(r(\varepsilon)) < 1$. This proves (5.2) when $p(x) \leq 2$.
If $p(x) > 2$, we estimate (5.8) directly using (5.9). We get

$$-F(x,\eta,X) \leq C(p,q) \frac{1}{\varepsilon^{q-1}} \left(q^{\frac{1}{q}} \varepsilon^{\frac{q-1}{q}} \omega(r(\varepsilon))^{\frac{1}{q}} \right)^q = C(p,q)\omega(r(\varepsilon))),$$

which proves (5.2) when $p(x) > 2$. □

Next we will use the previous lemma to show that inf-convolution of a viscosity supersolution to $-\Delta^N_{p(x)} u \geq 0$ in Ω is a weak supersolution to $-\Delta^S_{p(x)} u \geq 0$ in $\Omega_{r(\varepsilon)}$ up to some error term. Before proceeding we make some remarks about the point-wise differentiability of inf-convolution.

Remark 5.4. It follows from semi-concavity that the inf-convolution u_ε is locally Lipschitz in $\Omega_{r(\varepsilon)}$ (see [EG15, p267]). Therefore it belongs in $W^{1,\infty}_{loc}(\Omega_{r(\varepsilon)})$, is differentiable almost everywhere in $\Omega_{r(\varepsilon)}$, and its derivative agrees with its Sobolev derivative almost everywhere in $\Omega_{r(\varepsilon)}$ (see [EG15, p155 and p265]).

By Lemma 5.2 the function $\phi(x) := u_\varepsilon(x) - C(q,\varepsilon,u)|x|^2$ is concave in $\Omega_{r(\varepsilon)}$. Thus Alexandrov's theorem implies that u_ε is twice differentiable almost everywhere in $\Omega_{r(\varepsilon)}$. Furthermore, the proof of Alexandrov's theorem in [EG15, p273] establishes that if ϕ_j is the standard mollification of ϕ, then $D^2\phi_j \to D^2\phi$ almost everywhere in $\Omega_{r(\varepsilon)}$.

Lemma 5.5. *Assume that u is a uniformly continuous viscosity supersolution to $-\Delta^N_{p(x)}u \geq 0$ in Ω. Let $q > 2$ be so large that $p_{\min} - 2 + \frac{q-2}{q-1} \geq 0$ and let u_ε be the inf-convolution of u as defined in (5.1). Then*

$$\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot (D\varphi + \log|Du_\varepsilon|\, Dp\, \varphi)\, dx \geq E(\varepsilon) \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{s(x)} \varphi\, dx$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega_{r(\varepsilon)})$ with compact support, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $s(x) = \max(p(x) - 2, 0)$.

Proof. It is enough to consider $\varphi \in C_0^\infty(\Omega_{r(\varepsilon)})$. This can be proven as Lemma 3.2, but since $u_\varepsilon \in W^{1,\infty}_{loc}(\Omega_{r(\varepsilon)})$, the proof is even simpler.

(Step 1) We show that u_ε satisfies the auxiliary inequality (5.11) for all $0 < \delta < 1$. As mentioned in Remark 5.4, the function $\phi(x) := u_\varepsilon(x) - C(q, \varepsilon, u)\,|x|^2$ is concave in $\Omega_{r(\varepsilon)}$ and therefore we can approximate it by smooth concave functions ϕ_j so that $(\phi_j, D\phi_j, D^2\phi_j) \to (\phi, D\phi, D^2\phi)$ almost everywhere in $\Omega_{r(\varepsilon)}$. We define

$$u_{\varepsilon,j}(x) := \phi_j(x) + C(q, \varepsilon, u)\,|x|^2$$

and denote by p_j the standard mollification of p. Since $u_{\varepsilon,j}$ and p_j are smooth, we calculate

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} -\left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} \left(\Delta u_{\varepsilon,j} + \frac{p_j(x)-2}{\delta + |Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)\varphi\, dx \\
&\quad = \int_{\Omega_{r(\varepsilon)}} -\operatorname{div}\left(\left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} Du_{\varepsilon,j}\right)\varphi \\
&\qquad + \frac{1}{2}\left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} \log\left(\delta + |Du_{\varepsilon,j}|^2\right) Du_{\varepsilon,j} \cdot Dp_j\, \varphi\, dx \\
&\quad = \int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}} Du_{\varepsilon,j} \cdot \left(D\varphi + \frac{1}{2}\log\left(\delta + |Du_{\varepsilon,j}|^2\right) Dp_j\, \varphi\right) dx.
\end{aligned} \tag{5.10}$$

We let $j \to \infty$ in (5.10) and intend to use Fatou's lemma at the LHS and the Dominated convergence theorem at the RHS. This results in the auxiliary inequality

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} -\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{\delta + |Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right)\varphi\, dx \\
&\quad \leq \int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} Du_\varepsilon \cdot \left(D\varphi + \frac{1}{2}\log\left(\delta + |Du_\varepsilon|^2\right) Dp\, \varphi\right) dx,
\end{aligned} \tag{5.11}$$

where $D^2 u_\varepsilon$ is the Hessian of u_ε in the Alexandrov's sense. We still need to check that the assumptions of the Dominated convergence theorem and Fatou's lemma hold. By Lipschitz continuity of u_ε and p there is $M \geq 1$ such that

$$\sup_j \|Du_{\varepsilon,j}\|_{L^\infty(\operatorname{supp}\varphi)}, \sup_j \|Dp_j\|_{L^\infty(\operatorname{supp}\varphi)} \leq M.$$

This justifies our use of the Dominated convergence theorem. In order to justify our use of Fatou's lemma, we notice first that by concavity of ϕ_j we have $D^2 u_{\varepsilon,j} \leq C(q, \varepsilon, u)I$. Thus the integrand at the LHS of (5.10) is clearly bounded from below by a constant independent of j if $Du_{\varepsilon,j}=0$. If $Du_{\varepsilon,j}\neq 0$, we have

$$
\begin{aligned}
&\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}}\left(\Delta u_{\varepsilon,j}+\frac{p_j(x)-2}{\delta+|Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)\\
&=\frac{\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p_j(x)-2}{2}}}{\delta+|Du_{\varepsilon,j}|^2}\left(|Du_{\varepsilon,j}|^2\left(\Delta u_{\varepsilon,j}+\frac{p_j(x)-2}{|Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)+\delta\Delta u_{\varepsilon,j}\right)\\
&\leq\frac{\delta^{\frac{p_j(x)-2}{2}}+\left(\delta+M^2\right)^{\frac{p_j(x)-2}{2}}}{\delta+|Du_{\varepsilon,j}|^2}C(q,\varepsilon,u)\left(|Du_{\varepsilon,j}|^2\left(N+p_j(x)-2\right)+\delta N\right)\\
&\leq C(q,\varepsilon,u)\left(\delta^{\frac{p_{\min}-2}{2}}+\left(\delta+M^2\right)^{\frac{p_{\max}-2}{2}}\right)\left(2N+p_{\max}-2\right),
\end{aligned}
$$

where the first inequality follows like estimate (5.7) since $p_j\geq p_{\min}>1$.

(Step 2) We let $\delta\to 0$ in the auxiliary inequality (5.11). The RHS becomes

$$
\int_{\Omega_{r(\varepsilon)}\setminus\{Du_\varepsilon=0\}}|Du_\varepsilon|^{p(x)-2}Du_\varepsilon\cdot\left(D\varphi+\log|Du_\varepsilon|\,Dp\,\varphi\right)\,dx
$$

by the Lebesgue's dominated convergence theorem. We intend to apply Fatou's lemma on the LHS. We have $\left(Du_\varepsilon(x),D^2u_\varepsilon(x)\right)\in J^{2,-}u_\varepsilon(x)$ for almost every $x\in\Omega_{r(\varepsilon)}$. Therefore by Lemma 5.3 it holds that

$$
|Du_\varepsilon|^{\min(p(x)-2,0)}F(x,Du_\varepsilon,D^2u_\varepsilon)\geq E(\varepsilon)\text{ in }\left\{x\in\Omega_{r(\varepsilon)}:Du_\varepsilon\neq 0\right\}\tag{5.12}
$$

and by the property (v) in Lemma 5.2 we have

$$
D^2u_\varepsilon\leq\frac{q-1}{\varepsilon}|Du_\varepsilon|^{\frac{q-2}{q-1}}I.\tag{5.13}
$$

Observe that since $q>2$, the condition (5.13) implies that the Hessian D^2u_ε is negative semi-definite in the set where the gradient Du_ε vanishes. Using this fact, Fatou's lemma and (5.12) we get

$$
\begin{aligned}
&\liminf_{\delta\to 0}\int_{\Omega_{r(\varepsilon)}}-\left(|Du_\varepsilon|^2+\delta\right)^{\frac{p(x)-2}{2}}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2+\delta}\Delta_\infty u_\varepsilon\right)\varphi\,dx\\
&\quad\geq\liminf_{\delta\to 0}\int_{\{Du_\varepsilon\neq 0\}}-\left(|Du_\varepsilon|^2+\delta\right)^{\frac{p(x)-2}{2}}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2+\delta}\Delta_\infty u_\varepsilon\right)\varphi\,dx\\
&\qquad+\liminf_{\delta\to 0}\int_{\{Du_\varepsilon=0\}}-\delta^{\frac{p(x)-2}{2}}\Delta u_\varepsilon\varphi\,dx\\
&\quad\geq\int_{\{Du_\varepsilon\neq 0\}}-|Du_\varepsilon|^{p(x)-2}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right)\varphi\,dx\\
&\quad\geq E(\varepsilon)\int_{\{Du_\varepsilon\neq 0\}}|Du_\varepsilon|^{\max(p(x)-2,0)}\varphi\,dx,
\end{aligned}\tag{5.14}
$$

and thus we arrive at the desired inequality. Our use of Fatou's lemma is justified since if $Du_\varepsilon \neq 0$ and $p(x) \leq 2$, we have by (5.13)

$$\begin{aligned}
&\Big(|Du_\varepsilon|^2+\delta\Big)^{\frac{p(x)-2}{2}}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2+\delta}\Delta_\infty u_\varepsilon\right)\\
&\quad=\frac{\Big(|Du_\varepsilon|^2+\delta\Big)^{\frac{p(x)-2}{2}}}{|Du_\varepsilon|^2+\delta}\left(|Du_\varepsilon|^2\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right)+\delta\Delta u_\varepsilon\right)\\
&\quad\leq\frac{\Big(|Du_\varepsilon|^2+\delta\Big)^{\frac{p(x)-2}{2}}}{|Du_\varepsilon|^2+\delta}\frac{q-1}{\varepsilon}\left(|Du_\varepsilon|^{\frac{q-2}{q-1}+2}(N+p(x)-2)+|Du_\varepsilon|^{\frac{q-2}{q-1}}\delta N\right)\\
&\quad\leq|Du_\varepsilon|^{p(x)-2+\frac{q-2}{q-1}}\left(\frac{q-1}{\varepsilon}\right)(2N+p(x)-2)\\
&\quad\leq\Big(\|Du_\varepsilon\|_{L^\infty(\operatorname{supp}\varphi)}+1\Big)^{p_{\max}-2+\frac{q-2}{q-1}}\left(\frac{q-1}{\varepsilon}\right)(2N+p_{\max}-2),
\end{aligned}$$

where the last inequality follows from $p_{\min}-2+\frac{q-2}{q-1}\geq 0$. If $Du_\varepsilon \neq 0$ and $p(x) > 2$, we have

$$\begin{aligned}
&\Big(|Du_\varepsilon|^2+\delta\Big)^{\frac{p(x)-2}{2}}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2+\delta}\Delta_\infty u_\varepsilon\right)\\
&\qquad\leq\Big(\|Du_\varepsilon\|^2_{L^\infty(\operatorname{supp}\varphi)}+1\Big)^{\frac{p_{\max}-2}{2}+\frac{q-2}{q-1}}(\frac{q-1}{\varepsilon})(N+p_{\max}-2). \qquad \square
\end{aligned}$$

In the next two lemmas we use Caccioppoli type estimates and algebraic inequalities to show that the sequence of inf-convolutions converges to the viscosity supersolution in $W^{1,p(\cdot)}_{loc}(\Omega)$.

Lemma 5.6. *Under the assumptions of Lemma 5.5, the function u belongs in $W^{1,p(\cdot)}_{loc}(\Omega)$ and for any $\Omega' \Subset \Omega$ we have $Du_\varepsilon \to Du$ weakly in $L^{p(\cdot)}(\Omega')$ for some subsequence.*

Proof. Take a cut-off function $\xi \in C_0^\infty(\Omega')$ such that $0 \leq \xi \leq 1$ in Ω and $\xi \equiv 1$ in Ω'. Then assume that ε is so small that $\operatorname{supp}\xi =: K \subset \Omega_{r(\varepsilon)}$. We define a test function $\varphi := (L-u_\varepsilon)\xi^{p_{\max}}$ where $L := \sup_{\varepsilon, x\in\Omega'}|u_\varepsilon(x)|$ is finite since $u_\varepsilon \to u$ locally uniformly. We have

$$D\varphi = -Du_\varepsilon\,\xi^{p_{\max}} + (L-u_\varepsilon)p^+\xi^{p_{\max}-1}D\xi$$

and therefore by Lemma 5.5

$$\begin{aligned}
\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p(x)}\,\xi^{p_{\max}}\,dx \leq &\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p(x)-1}\,\xi^{p_{\max}-1}\,(L-u_\varepsilon)\,p_{\max}\,|D\xi|\,dx\\
&+\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p(x)-1}\,|\log|Du_\varepsilon||\,|Dp|\,(L-u_\varepsilon)\,\xi^{p_{\max}}\,dx\\
&+|E(\varepsilon)|\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{\max(p(x)-2,0)}\,(L-u_\varepsilon)\,\xi^{p_{\max}}\,dx\\
=:&I_1+I_2+I_3.
\end{aligned}$$

We estimate these integrals using Young's inequality. The first integral is estimated by the facts $\frac{p(x)(p_{\max}-1)}{p(x)-1} \geq p_{\max}$ and $\xi \leq 1$ as follows

$$\begin{aligned}
I_1 \leq & \int_{\Omega_{r(\varepsilon)}} \delta \left| Du_\varepsilon \right|^{p(x)} \xi^{\frac{p(x)(p_{\max}-1)}{p(x)-1}} + \left(\frac{2}{\delta} L p_{\max} \left| D\xi \right| \right)^{p(x)} dx \\
\leq & \delta \int_{\Omega_{r(\varepsilon)}} \left| Du_\varepsilon \right|^{p(x)} \xi^{p_{\max}} \, dx + C(\delta, p, L, D\xi).
\end{aligned}$$

To estimate I_2, we also use the inequality $a^s \left| \log a \right| \leq a^{s+\frac{1}{2}} + \frac{1}{s}$ for $a > 0$ and $s > 0$,

$$\begin{aligned}
I_2 \leq & \int_{\Omega_{r(\varepsilon)}} \left(\left| Du_\varepsilon \right|^{p(x)-\frac{1}{2}} + \frac{1}{p(x)-1} \right) \xi^{p_{\max}} \left| Dp \right| 2L \, dx \\
\leq & \int_{\Omega_{r(\varepsilon)}} \delta \left| Du_\varepsilon \right|^{p(x)} \xi^{\frac{p_{\max} p(x)}{p(x)-\frac{1}{2}}} + \left(\frac{2}{\delta} \left| Dp \right| L \right)^{2p(x)} + \frac{2L \left| Dp \right| \xi^{p_{\max}}}{p_{\min} - 1} \, dx \\
\leq & \delta \int_{\Omega_{r(\varepsilon)}} \left| Du_\varepsilon \right|^{p(x)} \xi^{p_{\max}} \, dx + C(\delta, p, Dp, L).
\end{aligned}$$

The last integral is estimated by the two alternatives in $\max(p(x)-2, 0)$ as follows (we may assume that $\left| E(\varepsilon) \right| \leq 1$)

$$\begin{aligned}
I_3 \leq & \int_{\Omega_{r(\varepsilon)} \cap \{p(x) > 2\}} \left| Du_\varepsilon \right|^{p(x)-2} \xi^{p_{\max}} 2L \, dx + \int_{\Omega_{r(\varepsilon)} \cap \{p(x) \leq 2\}} 2L \xi^{p_{\max}} \, dx \\
\leq & \int_{\Omega_{r(\varepsilon)} \cap \{p(x) > 2\}} \delta \left| Du_\varepsilon \right|^{p(x)} \xi^{\frac{p_{\max} p(x)}{p(x)-2}} + \left(\frac{2}{\delta} L \right)^{\frac{p(x)}{2}} dx + C(p, L) \\
\leq & \delta \int_{\Omega_{r(\varepsilon)}} \left| Du_\varepsilon \right|^{p(x)} \xi^{p_{\max}} \, dx + C(\delta, p, L).
\end{aligned}$$

Taking small δ we conclude that Du_ε is bounded in $L^{p(\cdot)}(\Omega')$ with respect to ε. Since $L^{p(\cdot)}(\Omega')$ is a reflexive Banach space [DHHR11, p76 and p89], it follows that there is a function $Du \in L^{p(\cdot)}(\Omega')$ such that $Du_\varepsilon \to Du$ weakly in $L^{p(\cdot)}(\Omega')$ for some subsequence. Consequently $u \in W^{1,p(\cdot)}(\Omega')$ with Du as its weak derivative. □

Lemma 5.7. *Under the assumptions of Lemma 5.5, for any $\Omega' \Subset \Omega$ we have $Du_\varepsilon \to Du$ in $L^{p(\cdot)}(\Omega')$ for some subsequence.*

Proof. Take a cut-off function $\xi \in C_0^\infty(\Omega)$ such that $\xi \equiv 1$ in Ω' and define a test function $\varphi := (u - u_\varepsilon)\xi$. Then assume that ε is so small that $\operatorname{supp} \xi =: K \subset \Omega_{r(\varepsilon)}$.

Since $\varphi \in W^{1,p(\cdot)}(\Omega_{r(\varepsilon)})$ with compact support it follows from Lemma 5.5 that

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} \left(|Du|^{p(x)-2} Du - |Du_\varepsilon|^{p(x)-2} Du_\varepsilon\right) \cdot (Du - Du_\varepsilon)\, \xi \, dx \\
&\quad \le \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot D\xi \,(u-u_\varepsilon)\, dx \\
&\quad\quad + \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} \log(|Du_\varepsilon|)\, Du_\varepsilon \cdot Dp\,(u-u_\varepsilon)\xi \, dx \\
&\quad\quad + |E(\varepsilon)| \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{\max(p(x)-2,0)} (u-u_\varepsilon)\xi\, dx \\
&\quad\quad + \int_{\Omega_{r(\varepsilon)}} |Du|^{p(x)-2} Du \cdot (Du - Du_\varepsilon)\, \xi\, dx \\
&\quad \le \|u-u_\varepsilon\|_{L^\infty(K)} \int_K \left(C(p_{\min}) + |Du_\varepsilon|^{p(x)}\right)(D\xi + |Dp| + |E(\varepsilon)|)\ dx \\
&\quad\quad + \int_K |Du|^{p(x)-2} Du \cdot (Du - Du_\varepsilon)\, \xi\, dx.
\end{aligned} \tag{5.15}$$

According to Lemma 5.6 we have $u_\varepsilon \to u$ locally uniformly and $Du_\varepsilon \to Du$ weakly in $L^{p(\cdot)}(K)$ for a subsequence. Thus by passing to a subsequence we may assume that the right hand side of (5.15) converges to zero. The claim now follows from the inequalities (see e.g. [Lin17, Chapter 12])

$$\begin{aligned}
&\left(|a|^{p(x)-2} a - |b|^{p(x)-2} b\right) \cdot (a-b) \\
&\qquad \ge \begin{cases} (p(x)-1)\,|a-b|^2 \left(1+|a|^2+|b|^2\right)^{\frac{p(x)-2}{2}} & p(x) < 2 \\ 2^{2-p(x)}\,|a-b|^{p(x)} & p(x) \ge 2 \end{cases}
\end{aligned}$$

for $a, b \in \mathbb{R}^N$. Indeed, we immediately get that $\int_{\Omega' \cap \{p(x) \ge 2\}} |Du - Du_\varepsilon|^{p(x)}\ dx \to 0$. To deal with the set $\{p(x) < 2\}$, we first apply the above algebraic inequality and then estimate using Hölder's inequality, the modular inequality (2.1) and the definition of the $\|\cdot\|_{L^{p(\cdot)}}$-norm. We get

$$\begin{aligned}
&\int_{\Omega' \cap \{p(x)<2\}} |Du - Du_\varepsilon|^{p(x)}\ dx \\
&\le \int_{\Omega' \cap \{p(x)<2\}} \left(\left(|Du|^{p(x)-2} Du - |Du_\varepsilon|^{p(x)-2} Du_\varepsilon\right) \cdot (Du - Du_\varepsilon)\right)^{\frac{p(x)}{2}} \\
&\quad \cdot \left(\frac{1}{p(x)-1}\right)^{\frac{p(x)}{2}} \left(1 + |Du|^2 + |Du_\varepsilon|^2\right)^{\frac{p(x)(2-p(x))}{4}}\ dx \\
&\le \left\| \left(\left(|Du|^{p(x)-2} Du - |Du_\varepsilon|^{p(x)-2} Du_\varepsilon\right) \cdot (Du - Du_\varepsilon)\right)^{\frac{p(x)}{2}} \right\|_{L^{\frac{2}{p(\cdot)}}(\Omega' \cap \{p(x)<2\})} \\
&\quad \cdot \frac{2}{p_{\min}-1} \left\| \left(1 + |Du|^2 + |Du_\varepsilon|^2\right)^{\frac{p(x)(2-p(x))}{4}} \right\|_{L^{\frac{2}{2-p(\cdot)}}(\Omega' \cap \{p(x)<2\})} \\
&\le \left(\int_{\Omega_{r(\varepsilon)}} \left(|Du|^{p(x)-2} Du - |Du_\varepsilon|^{p(x)-2} Du_\varepsilon\right) \cdot (Du - Du_\varepsilon)\, \xi\, dx \right)^s \\
&\quad \cdot \frac{2}{p_{\min}-1} \left(1 + \int_{\Omega' \cap \{p(x)<2\}} \left(1 + |Du|^2 + |Du_\varepsilon|^2\right)^{\frac{p(x)}{2}}\ dx \right),
\end{aligned}$$

where $s \in \left\{\frac{p_{\max}}{2}, \frac{p_{\min}}{2}\right\}$. The last integral is bounded since the sequence Du_ε is bounded in $L^{p(\cdot)}(\Omega')$ by its weak convergence. The RHS therefore converges to zero by (5.15). □

Next, we use the previous convergence result to pass to the limit in the inequality of Lemma 5.5 and conclude that viscosity supersolutions to $-\Delta^N_{p(x)} u \geq 0$ are weak supersolutions to $-\Delta^S_{p(x)} u \geq 0$.

Theorem 5.8. *If $u \in C(\Omega)$ is a viscosity supersolution to $-\Delta^N_{p(x)} u \geq 0$ in Ω, then u is a weak supersolution to $-\Delta^S_{p(x)} u \geq 0$ in Ω.*

Proof. It is clear from the definition of weak supersolutions to $-\Delta^S_{p(x)} u \geq 0$ that we can without loss of generality assume that u is uniformly continuous in Ω by restricting to a smaller domain. Fix a non-negative test function $\varphi \in C_0^\infty(\Omega)$ and take an open $\Omega' \Subset \Omega$ such that $\operatorname{supp}\varphi \subset \Omega'$. Let q and u_ε be as in Lemma 5.5 and assume that ε is so small that $\Omega' \subset \Omega_{r(\varepsilon)}$. Then the claim follows from Lemma 5.5 if we show that

$$\lim_{\varepsilon \to 0} \int_{\Omega'} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot D\varphi \, dx = \int_{\Omega'} |Du|^{p(x)-2} Du \cdot D\varphi \, dx \tag{5.16}$$

and

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\Omega'} |Du_\varepsilon|^{p(x)-2} \log\left(|Du_\varepsilon|\right) Du_\varepsilon \cdot Dp \, \varphi \, dx \\ &= \int_{\Omega'} |Du|^{p(x)-2} \log\left(|Du|\right) Du \cdot Dp \, \varphi \, dx \end{aligned} \tag{5.17}$$

as well as

$$\lim_{\varepsilon \to 0} E(\varepsilon) \int_{\Omega'} |Du_\varepsilon|^{\max(p(x)-2,0)} \varphi \, dx = 0. \tag{5.18}$$

By Lemma 5.7 we have that $u_\varepsilon \to u$ in $W^{1,p(\cdot)}(\Omega')$.
Claim (5.16) follows from the inequalities (see e.g. [Lin17, Chapter 12])

$$\left| |a|^{p(x)-2} a - |b|^{p(x)-2} b \right| \leq \begin{cases} 2^{2-p(x)} |a-b|^{p(x)-1} & p(x) < 2 \\ 2^{-1} \left(|a|^{p(x)-2} + |b|^{p(x)-2} \right) |a-b| & p(x) \geq 2 \end{cases} \tag{5.19}$$

for $a, b \in \mathbb{R}^N$. Indeed, when ε is so small that $\int_{\Omega'} |Du_\varepsilon - Du|^{p(x)} \, dx < 1$ we have by Hölder's inequality and the modular inequality

$$\begin{aligned} &\int_{\Omega'} \left| |Du_\varepsilon|^{p(x)-2} Du_\varepsilon - |Du|^{p(x)-2} Du \right| \, dx \\ \leq &2 \int_{\Omega' \cap \{p(x)<2\}} |Du_\varepsilon - Du|^{p(x)-1} \, dx \\ &+ 2^{-1} \int_{\Omega' \cap \{p(x) \geq 2\}} \left(|Du_\varepsilon|^{p(x)-2} + |Du|^{p(x)-2} \right) |Du_\varepsilon - Du| \, dx \\ \leq &C(p,\Omega) \left(\int_{\Omega'} |Du_\varepsilon - Du|^{p(x)} \, dx \right)^{\frac{1}{p_{\max}}} \\ &+ C(p,\Omega) \left(1 + \int_{\Omega'} |Du_\varepsilon|^{p(x)} + |Du|^{p(x)} \, dx \right) \|Du_\varepsilon - Du\|_{L^{p(\cdot)}(\Omega')}. \end{aligned}$$

Claim (5.18) holds since $\int_{\Omega'} |Du_\varepsilon|^{p(x)} \, dx$ is bounded and $E(\varepsilon) \to 0$.
Claim (5.17) follows if we show that

$$\lim_{\varepsilon \to 0} \int_{\Omega'} \left| |Du_\varepsilon|^{p(x)-2} \log\left(|Du_\varepsilon|\right) Du_\varepsilon - |Du|^{p(x)-2} \log\left(|Du|\right) Du \right| \, dx = 0. \tag{5.20}$$

To this end, fix $0<\epsilon<1$. The mapping $(a,x)\mapsto |a|^{p(x)-2}\log(|a|)\,a$ is uniformly continuous in bounded sets of $\mathbb{R}^N\times\Omega'$. Hence there exists $\delta=\delta(\epsilon)<\epsilon$ such that whenever $x\in\Omega'$ and $a,b\in\overline{B}(0,3)$ satisfy $|a-b|<\delta$, it holds

$$\left||a|^{p(x)-2}\log(|a|)\,a-|b|^{p(x)-2}\log(|b|)\,b\right|\leq\epsilon. \tag{5.21}$$

If $|a|,|b|\geq 1$ and $|a-b|<\delta$, then we use (5.19) to get the estimate

$$\begin{aligned}
&\left|\,|a|^{p(x)-2}\log(|a|)\,a-|b|^{p(x)-2}\log(|b|)\,b\right|\\
&\leq |b|^{p(x)-1}\left|\log|a|-\log|b|\right|+\left|\log|a|\right|\left||a|^{p(x)-2}\,a-|b|^{p(x)-2}\,b\right|\\
&\leq |b|^{p(x)}\,|a-b|+|a|\cdot\begin{cases}2^{2-p(x)}\,|a-b|^{p(x)-1}, & p(x)<2\\ 2^{-1}\left(|a|^{p(x)-2}+|b|^{p(x)-2}\right)|a-b|, & p(x)\geq 2\end{cases}\\
&\leq(1+2^{-1})\left(|a|^{p(x)}+|b|^{p(x)}\right)|a-b|+2\,|a|\,|a-b|^{p(x)-1}\\
&\leq C\left(|a|^{p(x)}+|b|^{p(x)}\right)\epsilon^{\min(p_{\min}-1,1)}.
\end{aligned} \tag{5.22}$$

We denote

$$F_\varepsilon=\left\{x\in\Omega':|Du_\varepsilon(x)-Du(x)|\geq\delta\right\}.$$

The strong convergence of Du_ε to Du in $L^{p(\cdot)}(\Omega')$ implies that $Du_\varepsilon\to Du$ in measure in Ω' (see [DHHR11, Lemma 3.2.10]). Thus there is $\varepsilon_0=\varepsilon_0(\delta)$ such that for all $\varepsilon<\varepsilon_0$ it holds $|F_\varepsilon|\leq\delta$. Using the inequality $a^s\,|\log a|\leq a^{s+\frac{1}{2}}+\frac{1}{s}$ for $a,s>0$, we get for all $\varepsilon<\varepsilon_0$

$$\begin{aligned}
&\int_{F_\varepsilon}\left||Du_\varepsilon|^{p(x)-2}\log(|Du_\varepsilon|)\,Du_\varepsilon-|Du|^{p(x)-2}\log(|Du|)\,Du\right|\,dx\\
&\leq\int_{F_\varepsilon}\frac{2}{p(x)-1}+|Du_\varepsilon|^{p(x)-\frac{1}{2}}+|Du|^{p(x)-\frac{1}{2}}\,dx\\
&\leq C(p_{\min})\,|F_\varepsilon|+\|1\|_{L^{2p(\cdot)}(F_\varepsilon)}\left(\|Du_\varepsilon\|_{L^{\frac{p(\cdot)}{p(\cdot)-\frac{1}{2}}}(F_\varepsilon)}+\|Du\|_{L^{\frac{p(\cdot)}{p(\cdot)-\frac{1}{2}}}(F_\varepsilon)}\right)\\
&\leq C(p_{\min})\,|F_\varepsilon|+|F_\varepsilon|^{\frac{1}{2p_{\max}}}\left(2+\int_{F_\varepsilon}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}\,dx\right)\\
&\leq C(p_{\min})\left(1+\int_{\Omega'}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}\,dx\right)\epsilon^{\frac{1}{2p_{\max}}}.
\end{aligned} \tag{5.23}$$

If $x\in\Omega'\setminus F_\varepsilon$, then either $|Du_\varepsilon|,|Du|\leq 3$ or $|Du_\varepsilon|,|Du|\geq 1$. Hence by (5.21) and (5.22) we have

$$\begin{aligned}
&\int_{\Omega'\setminus F_\varepsilon}\left||Du_\varepsilon|^{p(x)-2}\log(|Du_\varepsilon|)\,Du_\varepsilon-|Du|^{p(x)-2}\log(|Du|)\,Du\right|\,dx\\
&\qquad\leq C\left(\int_{\Omega'}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}+1\,dx\right)\epsilon^{\min(p_{\min}-1,1)}.
\end{aligned} \tag{5.24}$$

Combining (5.24) and (5.23) proves (5.20) since ϵ was arbitrary. □

Merging Theorems 4.1 and 5.8 yields the following equivalence result.

Theorem 5.9. *A function u is a viscosity solution to $-\Delta^N_{p(x)}u=0$ in Ω if and only if it is a weak solution to $-\Delta^S_{p(x)}u=0$ in Ω.*

Since the weak solutions to the strong $p(x)$-Laplace equation are locally $C^{1,\alpha}$ continuous [ZZ12], our equivalence result yields local $C^{1,\alpha}$ regularity also for viscosity solutions of the normalized $p(x)$-Laplace equation.

Corollary 5.10. *If u is a viscosity solution to $-\Delta^N_{p(x)}u = 0$ in a bounded domain Ω, then $u \in C^{1,\alpha}(\Omega)$ with $\alpha \in (0,1)$.*

6. An Application: A Radó-type removability theorem

The classical theorem of Radó says that if a continuous complex-valued function f defined on a domain $\Omega \subset \mathbb{C}$ is holomorphic in $\Omega \setminus \{f = 0\}$, then it is holomorphic in the whole Ω. Similar results have been proven for solutions of partial differential equations. We prove a Radó-type removability theorem for the strong $p(x)$-Laplace equation. It is worth pointing out that it could be difficult to show this kind of result without appealing to viscosity solutions whereas it is straightforward to do so with the help of the equivalence result. The theorem follows by observing that weak solutions to $\Delta^S_{p(x)}u = 0$ coincide with viscosity solutions of an equation that satisfies the assumptions of a Radó-type removability theorem in [JL05].

Recall that we ignore the test functions whose gradient vanishes at the point of touching in the Definition 3.3 of viscosity solutions to $-\Delta^N_{p(x)}u = 0$. Sometimes this kind of solutions are called *feeble viscosity solutions* (e.g. [JL05, Kat15b]). We will observe that these feeble viscosity solutions to $-\Delta^N_{p(x)}u = 0$ are exactly the usual viscosity solutions to

$$-\mathrm{tr}(A(x, Du)D^2u) = 0, \tag{6.1}$$

where $A(x, Du) := |Du|^2 I + (p(x) - 2)\, Du \otimes Du$. To be precise, we define the viscosity solutions to (6.1).

Definition 6.1. A lower semicontinuous function u is a *viscosity supersolution* to (6.1) in Ω if, whenever $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$, then

$$-\mathrm{tr}(A(x, \eta)X) \geq 0.$$

A function u is a *viscosity subsolution* to (6.1) if $-u$ is a supersolution, and a *viscosity solution* if it is both viscosity super- and subsolution.

Lemma 6.2. *A function u is a viscosity solution to $-\Delta^N_{p(x)}u = 0$ if and only if it is a viscosity solution to (6.1).*

Proof. It is enough to consider supersolutions. Take $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$. If $\eta = 0$, then the conditions for both definitions are satisfied, so we may assume that $\eta \neq 0$. Then we have

$$F(x, \eta, X) \geq 0$$

if and only if

$$-\left(|\eta|^2 \mathrm{tr}(X) + (p(x) - 2) \langle X\eta, \eta\rangle \right) \geq 0,$$

where

$$\begin{aligned} |\eta|^2 \mathrm{tr}(X) + (p(x) - 2) \langle X\eta, \eta\rangle &= |\eta|^2 \mathrm{tr}(X) + (p(x) - 2)\, \mathrm{tr}(\eta \otimes \eta X) \\ &= \mathrm{tr}\big(\big(|\eta|^2 I + (p(x) - 2)\, \eta \otimes \eta\big)X\big). \end{aligned}$$

Hence the definitions are equivalent. □

Theorem 6.3 (A Radó-type removability theorem)**.** *Let $u \in C^1(\Omega)$ be a weak solution to $-\Delta^S_{p(x)}u = 0$ in $\Omega \setminus \{u = 0\}$. Then u is a weak solution to $-\Delta^S_{p(x)}u = 0$ in the whole Ω.*

Proof. By Lemma 6.2 and our equivalence result weak solutions to $-\Delta^S_{p(x)}u = 0$ coincide with viscosity solutions to (6.1). Therefore it suffices to show that if u is a viscosity solution to (6.1) in $\Omega \setminus \{u = 0\}$, it is a viscosity solution to (6.1) in the whole Ω. This on the other hand follows from [JL05, Theorem 2.2]. The matrix A satisfies the assumptions of the theorem as it is symmetric, has continuous entries and $A(x,0,0)=0$ for all $x\in\Omega$. It is also positive semi-definite since for all $\xi\in\mathbb{R}^N$ we have

$$\begin{aligned}\xi'\left(|\eta|^2 I+(p(x)-2)\,\eta\otimes\eta\right)\xi &\geq \xi'\left(|\eta|^2 I-\eta\otimes\eta\right)\xi\\ &\geq |\xi|^2\left(|\eta|^2-\|\eta\otimes\eta\|\right)=0.\end{aligned}$$

□

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Jarkko Siltakoski, Department of Mathematics and Statistics, P.O.Box 35, FIN-40014, University of Jyväskylä, Finland

E-mail address: jarkko.j.m.siltakoski@student.jyu.fi

[B]

Equivalence of viscosity and weak solutions for a p-parabolic equation

Jarkko Siltakoski

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EQUIVALENCE OF VISCOSITY AND WEAK SOLUTIONS FOR A p-PARABOLIC EQUATION

JARKKO SILTAKOSKI

Abstract. We study the relationship of viscosity and weak solutions to the equation

$$\partial_t u - \Delta_p u = f(Du)$$

where $p > 1$ and $f \in C(\mathbb{R}^N)$ satisfies suitable assumptions. Our main result is that bounded viscosity supersolutions coincide with bounded lower semicontinuous weak supersolutions. Moreover, we prove the lower semicontinuity of weak supersolutions when $p \geq 2$.

1. Introduction

A classical solution to a partial differential equation is a smooth function that satisfies the equation pointwise. Since many equations that appear in applications admit no such solutions, a more general class of solutions is needed. One such class is the extensively studied distributional weak solutions defined by integration by parts. Another is the celebrated viscosity solutions based on generalized pointwise derivatives. When both classes of solutions can be meaningfully defined, it is naturally crucial that they coincide. This has been profusely studied starting from [Ish95]. In [JLM01] the equivalence of solutions was proved for the parabolic p-Laplacian. The objective of the present work is to prove this equivalence in a different way while also allowing the equation to depend on a first-order term. To the best of our knowledge, the proof is new even in the homogeneous case, at least when $1 < p < 2$.

More precisely, we study the parabolic equation

$$\partial_t u - \Delta_p u = f(Du) \tag{1.1}$$

where $1 < p < \infty$ and $f \in C(\mathbb{R}^N)$ satisfies a certain growth condition, for details see Section 2. We show that bounded viscosity supersolutions to (1.1) coincide with bounded lower semicontinuous weak supersolutions. Moreover, we prove the lower semicontinuity of weak supersolutions in the range $p \geq 2$ under slightly stronger assumptions on f.

To show that viscosity supersolutions are weak supersolutions, we apply the technique introduced by Julin and Juutinen [JJ12]. In contrast to [JLM01], we do not employ the uniqueness machinery of viscosity solutions. Instead, our strategy is to approximate a viscosity supersolution u by its inf-convolution u_ε. It is straightforward to show that u_ε is still a viscosity supersolution in a smaller set. This and the pointwise properties of the inf-convolution imply that u_ε is also a weak supersolution in the smaller set.

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Furthermore, it follows from Caccioppoli's estimates that u_ε converges to u in a suitable Sobolev space. It then remains to pass to the limit to see that u is a weak supersolution.

To show that weak supersolutions are viscosity supersolutions, we apply the argument from [JLM01] that is based on the comparison principle of weak solutions. However, we could not find a reference for comparison principle for the equation (1.1). Therefore we give a detailed proof of such a result.

To prove the lower semicontinuity of weak supersolutions, we adapt the strategy of [Kuu09]. First we prove estimates for the essential supremum of a subsolution using the Moser's iteration technique. Then we use those estimates to deduce that a supersolution is lower semicontinuous at its Lebesgue points.

The equivalence of viscosity and weak solutions for the p-Laplace equation and its parabolic version was first proven in [JLM01]. A different proof in the elliptic case was found in [JJ12]. Recently the equivalence of solutions has been studied for various equations. These include the normalized p-Poisson equation [APR17], a non-homogeneous p-Laplace equation [MO19] and the normalized $p(x)$-Laplace equation [Sil18]. Moreover, in [PV] the equivalence is shown for the radial solutions of a parabolic equation. We also mention that an unpublished version of [Lin12] applies [JJ12] to sketch the equivalence of solutions to (1.1) in the homogeneous case when $p \geq 2$.

Comparison principles for quasilinear parabolic equations have been studied by several authors. In [Jun93] comparison is proven for $\partial_t u - \Delta_p u + f(u,x,t) = 0$ when $p > 2$ and f is a continuous function such that $|f(u,x,t)| \leq g(u)$ for some $g \in C^1$. The homogeneous case for the p-parabolic equation is considered also in [KL96] and the general equation $\partial_t u - \operatorname{div} \mathcal{A}(x,t,Du) = 0$ in [KKP10]. Equations with gradient terms are studied for example in [Att12], where comparison principle is shown for the equation $\partial_t u - \Delta_p u - |Du|^\beta = 0$ when $p > 2$ and $\beta > p-1$. In the recent papers [BT14, BT], both positive results and counter examples are provided for the comparison, strong comparison and maximum principles for the equation $\partial_t u - \Delta_p u - \lambda |u|^{p-2} u - f(x,t) = 0$. Furthermore, according to [BGKT16], the equation $\partial_t u - \Delta_p u = q(x) |u|^\alpha$ can admit multiple solutions with zero boundary values when $0 < \alpha < 1$.

The paper is organized as follows. Section 2 contains the precise definitions of weak and viscosity solutions. In Section 3 we show that weak supersolutions are viscosity supersolutions, and the converse is shown in Section 4. Finally, the lower semicontinuity of weak supersolutions is considered in Section 5.

2. Preliminaries

The symbols Ξ and Ω are reserved for bounded domains in $\mathbb{R}^N \times \mathbb{R}$ and $\mathbb{R}^N$, respectively. For $t_1 < t_2$, we define the cylinder $\Omega_{t_1,t_2} := \Omega \times (t_1,t_2)$ and its *parabolic boundary* $\partial_p \Omega_{t_1,t_2} := (\overline{\Omega} \times \{t_1\}) \cup (\partial\Omega \times (t_1,t_2])$. Moreover, for $T > 0$ we set $\Omega_T := \Omega_{0,T}$.

The *Sobolev space* $W^{1,p}(\Omega)$ contains the functions $u \in L^p(\Omega)$ for which the distributional gradient Du exists and belongs in $L^p(\Omega)$. It is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}.$$

A Lebesgue measurable function $u : \Omega_{t_1,t_2} \to \mathbb{R}$ belongs to the *parabolic Sobolev space* $L^p(t_1,t_2; W^{1,p}(\Omega))$ if $u(\cdot,t) \in W^{1,p}(\Omega)$ for almost every $t \in (t_1,t_2)$ and the norm

$$\left(\int_{\Omega_{t_1,t_2}} |u|^p + |Du|^p \, dz \right)^{\frac{1}{p}}$$

is finite. By dz we mean integration with respect to space and time variables, i.e. $dz = dx\,dt$. Integral average is denoted by

$$\fint_{\Omega_T} u\,dz := \frac{1}{|\Omega_T|}\int_{\Omega_T} u\,dz.$$

Growth condition. Unless otherwise stated, the function $f \in C(\mathbb{R}^N)$ is assumed to satisfy the growth condition

$$|f(\xi)| \leq C_f(1+|\xi|^\beta) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{G1}$$

where $C_f > 0$ and $1 \leq \beta < p$.

Definition 2.1 (Weak solution)**.** A function $u : \Xi \to \mathbb{R}$ is a *weak supersolution* to (1.1) in Ξ if $u \in L^p(t_1, t_2; W^{1,p}(\Omega))$ whenever $\Omega_{t_1,t_2} \Subset \Xi$, and

$$\int_\Xi -u\partial_t\varphi + |Du|^{p-2}\,Du\cdot D\varphi - \varphi f(Du)\,dz \geq 0$$

for all non-negative *test functions* $\varphi \in C_0^\infty(\Omega_{t_1,t_2})$. For *weak subsolutions* the inequality is reversed and a function is a *weak solution* if it is both super- and subsolution.

To define viscosity solutions to (1.1), we set for all $\varphi \in C^2$ with $D\varphi \neq 0$

$$\Delta_p\varphi := |D\varphi|^{p-2}\left(\Delta\varphi + \frac{p-2}{|D\varphi|^2}\left\langle D^2\varphi D\varphi, D\varphi\right\rangle\right).$$

Definition 2.2 (Viscosity solution)**.** A lower semicontinuous and bounded function $u : \Xi \to \mathbb{R}$ is a *viscosity supersolution* to (1.1) in Ξ if whenever $\varphi \in C^2(\Xi)$ and $(x_0, t_0) \in \Xi$ are such that

$$\begin{cases} \varphi(x_0,t_0) = u(x_0,t_0), \\ \varphi(x,t) < u(x,t) & \text{when } (x,t) \neq (x_0,t_0), \\ D\varphi(x,t) \neq 0 & \text{when } x \neq x_0, \end{cases}$$

then

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\ x\neq x_0}} \left(\partial_t\varphi(x,t) - \Delta_p\varphi(x,t) - f(D\varphi(x,t))\right) \geq 0.$$

An upper semicontinuous and bounded function $u : \Xi \to \mathbb{R}$ is a *viscosity subsolution* to (1.1) in Ξ if whenever $\varphi \in C^2(\Xi)$ and $(x_0, t_0) \in \Xi$ are such that

$$\begin{cases} \varphi(x_0,t_0) = u(x_0,t_0), \\ \varphi(x,t) > u(x,t) & \text{when } (x,t) \neq (x_0,t_0), \\ D\varphi(x,t) \neq 0 & \text{when } x \neq x_0, \end{cases}$$

then

$$\liminf_{\substack{(x,t)\to(x_0,t_0)\\ x\neq x_0}} \left(\partial_t\varphi(x,t) - \Delta_p\varphi(x,t) - f(D\varphi(x,t))\right) \leq 0.$$

A function that is both viscosity sub- and supersolution is a *viscosity solution.*

If a function φ is like in the definition of viscosity supersolution, we say that φ *touches u from below at* (x_0, t_0). The limit supremum in the definition is needed because the operator Δ_p is singular when $1 < p < 2$. When $p \geq 2$, the operator is degenerate and the limit supremum disappears.

3. Weak solutions are viscosity solutions

We show that bounded, lower semicontinuous weak supersolutions to (1.1) are viscosity supersolutions when $1 < p < \infty$ and $f \in C(\mathbb{R}^N)$ satisfies the growth condition (G1). One way to prove this kind of results is by applying the comparison principle [JLM01]. However, we could not find the comparison principle for the equation (1.1) in the literature and therefore we prove it first. To this end, we first prove comparison Lemmas 3.2 and 3.3 for locally Lipschitz continuous f. The local Lipschitz continuity allows us to absorb the first-order terms into the terms that appear due to the p-Laplacian, see Step 2 in proof of Lemma 3.2. To deal with general f, we take a locally Lipschitz continuous approximant f_δ such that $\|f - f_\delta\|_{L^\infty(\mathbb{R}^N)} < \delta/4T$. Then for sub- and supersolutions u and v, we consider the functions

$$u_\delta := u - \frac{\delta}{T - t/2} \quad \text{and} \quad v_\delta := v + \frac{\delta}{T - t/2}.$$

These functions will be sub- and supersolutions to (1.1) where f is replaced by f_δ. Since f_δ is locally Lipschitz continuous, it follows from the Lemmas 3.2 and 3.3 that $u_\delta \le v_\delta$. Letting $\delta \to 0$ then yields that $u \le v$.

For similar comparison results, see [Att12, Proposition 2.1] and [Jun93]. See also Chapters 3.5 and 3.6 in [PS07] for the elliptic case. A minor difference in our results is that instead of requiring that both the subsolution and the supersolution have uniformly bounded gradients, we only require this for the subsolution.

To prove the comparison principle, we need to use a test function that depends on the supersolution itself. However, supersolutions do not necessarily have a time derivative. One way to deal with this is to use mollifications in the time direction. For a compactly supported $\varphi \in L^p(\Omega_T)$ we define its *time-mollification* by

$$\varphi^\epsilon(x,t) = \int_{\mathbb{R}} \phi(x, t-s)\rho_\epsilon(s)\, ds,$$

where ρ_ϵ is a standard mollifier whose support is contained in $(-\epsilon, \epsilon)$. Then φ^ϵ has time derivative and $\varphi^\epsilon \to \varphi$ in $L^p(\Omega_T)$. Furthermore, the time-mollification of a supersolution satisfies a reguralized equation in the sense of the following lemma.

Lemma 3.1. *Let $v \in L^\infty(\Omega_T)$ be a weak supersolution (subsolution) to (1.1) in Ω_T. Then we have*

$$\int_{\Omega_T} -v^\epsilon \partial_t \varphi + \left(|Dv|^{p-2} Dv\right)^\epsilon \cdot D\varphi - \varphi\,(f(Dv))^\epsilon \, dz \ge (\le)\, 0 \tag{3.1}$$

for all $\varphi \in W^{1,p}(\Omega_T) \cap L^\infty(\Omega_T)$ with compact support in Ω_T. Moreover, if the stronger growth condition (G2) holds, then the assumption $\varphi \in L^\infty(\Omega_T)$ is not needed.

If φ is smooth, then testing the weak formulation of (1.1) with φ^ϵ, changing variables and using Fubini's theorem yields (3.1). The general case follows by approximating φ in $W^{1,p}(\Omega_T)$ with the standard mollification. We omit the details.

Lemma 3.2. *Let $1 < p < 2$ and let f be locally Lipschitz. Let u, $v \in L^\infty(\Omega_T)$ respectively be weak sub- and supersolutions to (1.1) in Ω_T. Assume that for all $(x_0, t_0) \in \partial_p \Omega_T$*

$$\operatorname*{ess\,lim\,sup}_{(x,t)\to(x_0,t_0)} u(x,t) \le \operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} v(x,t).$$

Suppose also that $Du \in L^\infty(\Omega_T)$. Then $u \le v$ a.e. in Ω_T.

Proof. **(Step 1)** Let $l > 0$ and set $w := (u - v - l)_+$. Let also $s \in (0, T)$. We want to use $w \cdot \chi_{[0,s]}$ as a test function, but since it is not smooth, we must perform mollifications. Let $h > 0$ and define

$$\varphi := \eta\left((u - v - l)^{\,\epsilon}\right)_+,$$

where

$$\eta(t) = \begin{cases} 1, & t \in (0, s-h], \\ (-t + s + h)/2h, & t \in (s-h, s+h), \\ 0, & t \in [s+h, T). \end{cases}$$

The function φ is compactly supported and belongs in $W^{1,p}(\Omega_T)$. Therefore by Lemma 3.1 we have

$$\begin{aligned} &\int_{\Omega_T} -(u-v)^\epsilon \partial_t \varphi \, dz \\ &\quad \leq \int_{\Omega_T} \left(\left(|Dv|^{p-2} Dv\right)^\epsilon - \left(|Du|^{p-2} Du\right)^\epsilon\right) \cdot D\varphi + \varphi\left(f(Du)^\epsilon - f(Dv)^\epsilon\right) dz. \end{aligned} \tag{3.2}$$

We use the linearity of convolution and integration by parts to eliminate the time derivative. We obtain

$$\begin{aligned} &\int_{\Omega_T} -(u-v)^\epsilon \partial_t \varphi \, dz \\ &\quad = -\int_{\Omega_T} (u-v)^\epsilon \left((u-v-l)^\epsilon\right)_+ \partial_t \eta + \eta (u-v)^\epsilon \partial_t \left((u-v-l)^\epsilon\right)_+ dz \\ &\quad = -\int_{\Omega_T} (u-v-l)^\epsilon ((u-v-l)^\epsilon)_+ \partial_t \eta + l\left((u-v-l)^\epsilon\right)_+ \partial_t \eta \\ &\qquad\qquad + \eta (u-v-l)^\epsilon \partial_t \left((u-v-l)^\epsilon\right)_+ + l \eta \partial_t \left((u-v-l)^\epsilon\right)_+ dz \\ &\quad = -\int_{\Omega_T} ((u-v-l)^\epsilon)_+^2 \partial_t \eta + \frac{1}{2} \eta \partial_t ((u-v-l)^\epsilon)_+^2 \, dz \\ &\quad = -\frac{1}{2} \int_{\Omega_T} ((u-v-l)^\epsilon)_+^2 \partial_t \eta \, dz \\ &\quad \underset{\epsilon \to 0}{\longrightarrow} -\frac{1}{2} \int_{\Omega_T} (u-v-l)_+^2 \partial_t \eta \, dz. \end{aligned}$$

Moreover, by the Lebesgue differentiation theorem for a.e. $s \in (0, T)$ it holds

$$-\frac{1}{2} \int_{\Omega_T} (u-v-l)_+^2 \partial_t \eta \, dz = \frac{1}{4h} \int_{s-h}^{s+h} \int_\Omega w^2(x,t) \, dx \, dt \underset{h \to 0}{\longrightarrow} \frac{1}{2} \int_\Omega w^2(x,s) \, dx.$$

The terms at the right-hand side of (3.2) converge similarly. Hence for a.e. $s \in (0, T)$ we have

$$\begin{aligned} &\frac{1}{2} \int_\Omega w^2(x,s) \, dx \\ &\quad \leq \int_{\Omega_s} |f(Du) - f(Dv)| \, w \, dz - \int_{\Omega_s} \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv\right) \cdot Dw \, dz \\ &\quad =: I_1 - I_2. \end{aligned} \tag{3.3}$$

(Step 2) We seek to absorb some of I_1 into I_2 so that we can conclude from Grönwall's inequality that $w \equiv 0$ almost everywhere. Since f is locally Lipschitz continuous, there are constants $M \geq \max(2\|Du\|_{L^\infty(\Omega_T)}, 1)$ and $L = L(M)$ such that

$$|f(\xi) - f(\eta)| \leq L\,|\xi - \eta| \text{ when } |\xi|, |\eta| < M. \tag{3.4}$$

We denote $\Omega_s^+ := \{x \in \Omega_s : w \geq 0\}$,

$$A := \Omega_s^+ \cap \{|Dv| < M\} \text{ and } B := \Omega_s^+ \cap \{|Dv| \geq M\}.$$

Observe that in B we have by the growth condition (G1), choice of M and the assumption that $\beta \geq 1$

$$|f(Du)| \leq C_f(1 + |Du|^\beta) \leq C_f(M + M^\beta) \leq 2C_f M^\beta \leq 2C_f\,|Dv|^\beta \tag{3.5}$$

and

$$|f(Dv)| \leq C_f(1 + |Dv|^\beta) \leq 2C_f\,|Dv|^\beta. \tag{3.6}$$

It follows from (3.4), (3.5), (3.6) and Young's inequality that

$$\begin{aligned} I_1 &\leq \int_A L\,|Du - Dv|\,w\,dz + \int_B (|f(Du)| + |f(Dv)|)\,w\,dz \\ &\leq \int_A L\,|Du - Dv|\,w\,dz + \int_B 4C_f\,|Dv|^\beta\,w\,dz \\ &\leq \int_A \epsilon\,|Du - Dv|^2 + C(\epsilon, L)w^2\,dz + \int_B \epsilon\,|Dv|^{\frac{\beta p}{\beta}} + C(\epsilon, p, \beta, L, C_f)w^{\frac{p}{p-\beta}}\,dz \\ &\leq \epsilon \int_A |Du - Dv|^2\,dz + \epsilon \int_B |Dv|^p\,dz + C(\epsilon, p, \beta, L, C_f, \|w\|_{L^\infty}) \int_{\Omega_s} w^2\,dz, \end{aligned} \tag{3.7}$$

where in the last step we used that $\frac{p}{p-\beta} > 2$ to estimate

$$\int_{\Omega_s} w^{p/(p-\beta)}\,dz = \int_{\Omega_s} w^{p/(p-\beta)-2} w^2\,dz \leq \|w\|_{L^\infty(\Omega_T)}^{p/(p-\beta)-2} \int_{\Omega_s} w^2\,dz.$$

Using the vector inequality

$$\left(|a|^{p-2} a - |b|^{p-2} b\right) \cdot (a - b) \geq (p-1)\,|a - b|^2 \left(1 + |a|^2 + |b|^2\right)^{\frac{p-2}{2}}, \tag{3.8}$$

which holds when $1 < p < 2$ [Lin17, p98], we get

$$\begin{aligned} I_2 &= \int_{\Omega_s} \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv\right) \cdot Dw\,dz \\ &\geq (p-1) \int_{\Omega_s^+} \frac{|Du - Dv|^2}{\left(1 + |Du|^2 + |Dv|^2\right)^{\frac{2-p}{2}}}\,dz \\ &\geq (p-1) \int_A \frac{|Du - Dv|^2}{(1 + M^2 + M^2)^{\frac{2-p}{2}}}\,dz + (p-1) \int_B \frac{(|Dv| - |Du|)^2}{\left(3\,|Dv|^2\right)^{\frac{2-p}{2}}}\,dz \\ &\geq C(p, M) \int_A |Du - Dv|^2\,dz + (p-1) \int_B \frac{\left(|Dv| - \frac{1}{2}M\right)^2}{\left(3\,|Dv|^2\right)^{\frac{2-p}{2}}}\,dz \\ &\geq C(p, M) \int_A |Du - Dv|^2\,dz + (p-1) \int_B \frac{\left(\frac{1}{2}\,|Dv|\right)^2}{\left(3\,|Dv|^2\right)^{\frac{2-p}{2}}}\,dz \\ &= C(p, M) \int_A |Du - Dv|^2\,dz + C(p) \int_B |Dv|^p\,dz, \end{aligned} \tag{3.9}$$

where $C(p,M), C(p) > 0$. Combining the estimates (3.7) and (3.9) we arrive at

$$I_1 - I_2 \leq (\epsilon - C(p,M)) \int_A |Du - Dv|^2 \, dz + (\epsilon - C(p)) \int_B |Dv|^p \, dz + C_0 \int_{\Omega_s} w^2 \, dz,$$

where $C_0 = C(\epsilon, p, \beta, L, C_f, \|w\|_{L^\infty})$. Recalling (3.3) and taking small enough ϵ yields

$$\int_\Omega w^2(x,s) \, dx \leq 2C_0 \int_{\Omega_s} w^2 \, dz.$$

Since this holds for a.e. $s \in (0,T)$, Grönwall's inequality implies that $w \equiv 0$ a.e. in Ω_T. Finally, letting $l \to 0$ yields that $u - v \leq 0$ a.e. in Ω_T. □

Lemma 3.3. *Let $p \geq 2$ and let f be locally Lipschitz. Let $v \in L^\infty(\Omega_T)$ be a weak supersolution to (1.1) and let $u \in L^\infty(\Omega_T)$ be a weak subsolution to*

$$\partial_t u - \Delta_p u - f(Du) \leq -\delta \quad \text{in } \Omega_T$$

for some $\delta > 0$. Assume that for all $(x_0,t_0) \in \partial_p \Omega_T$

$$\operatorname*{ess\,lim\,sup}_{(x,t)\to(x_0,t_0)} u(x,t) \leq \operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} v(x,t).$$

Suppose also that $Du \in L^\infty(\Omega_T)$. Then $u \leq v$ a.e. in Ω_T.

Proof. Let $l > 0$ and set $w := (u - v - l)_+$. Let also $s \in (0,T)$. Repeating the first step of the proof of Lemma 3.2, we arrive at the inequality

$$\begin{aligned}
&\frac{1}{2} \int_\Omega w^2(x,s) \, dx \\
&\quad \leq \int_{\Omega_s} |f(Du) - f(Dv)| \, w \, dz - \int_{\Omega_s} \left(|Du|^{p-2} Du - |Dv|^{p-2} Dv\right) \cdot Dw \, dz - \int_{\Omega_s} \delta w \, dz \\
&\quad =: I_1 - I_2 - \int_{\Omega_s} \delta w \, dz.
\end{aligned} \tag{3.10}$$

Moreover, we define the constants M and L, and the sets A and B, exactly in the same way as in the proof of Lemma 3.2. Then by (3.4), (3.5), (3.6) and Young's inequality

$$\begin{aligned}
I_1 &\leq \int_A L |Du - Dv| \, w \, dz + \int_B (|f(Du)| + |f(Dv)|) \, w \, dz \\
&\leq \int_A \epsilon |Du - Dv|^p + C(\epsilon, L) w^{\frac{p}{p-1}} \, dz + \int_B 4C_f |Dv|^\beta \, w \, dz \\
&\leq \epsilon \int_A |Du - Dv|^p \, dz + \epsilon \int_B |Dv|^p \, dz + C(\epsilon, p, \beta, L, C_f) \int_{\Omega_s} w^{\frac{p}{p-1}} + w^{\frac{p}{p-\beta}} \, dz.
\end{aligned} \tag{3.11}$$

Using the vector inequality

$$\left(|a|^{p-2} a - |b|^{p-2} b\right) \cdot (a - b) \geq 2^{2-p} |a - b|^p, \tag{3.12}$$

which holds when $p \geq 2$ [Lin17, p95], we get

$$I_2 \geq C(p) \int_A |Du - Dv|^p \, dz + C(p) \int_B |Du - Dv|^p \, dz.$$

Furthermore, since in B it holds

$$|Du - Dv|^p \geq (|Dv| - |Du|)^p \geq \left(|Dv| - \frac{1}{2}M\right)^p \geq C(p) |Dv|^p,$$

we arrive at

$$I_2 \geq C(p) \int_A |Du - Dv|^p \, dz + C(p) \int_B |Dv|^p \, dz. \tag{3.13}$$

Combining (3.11) and (3.13) with (3.10) we get

$$\frac{1}{2}\int_\Omega w^2\,dx \le (\epsilon - C(p))\left(\int_A |Du-Dv|^p\,dz + \int_B |Dv|^p\,dz\right) + \int_{\Omega_s} C(\epsilon,p,\beta,L,C_f)\left(w^{\frac{p}{p-1}} + w^{\frac{p}{p-\beta}}\right) - \delta w\,dz.$$

By taking small enough $\epsilon = \epsilon(p)$, the above becomes

$$\int_\Omega w^2(x,s)\,dx \le \int_{\Omega_s} C(p,\beta,L,C_f)\left(w^{\frac{p}{p-1}} + w^{\frac{p}{p-\beta}}\right) - \delta w\,dz. \tag{3.14}$$

Observe that since w is bounded and $\frac{p}{p-1}, \frac{p}{p-\beta} > 1$, the integrand at the right-hand side is bounded by some constant times w^2. To argue this rigorously, we write down the following algebraic fact.

If $a_0, \delta, \gamma > 0$ and $\alpha > 1$, then there exists $C(\alpha,\gamma,\delta,a_0) > 0$ such that

$$\gamma a^\alpha \le \delta a + C(\alpha,\gamma,\delta,a_0)a^2 \text{ for all } a \in [0,a_0).$$

To see this, let first $\alpha < 2$. Then by Young's inequality

$$\begin{aligned}\gamma a^\alpha = \gamma a \cdot a^{\alpha-1} \le& \frac{\delta}{1+a_0^{\frac{2}{3-\alpha}}} a^{\frac{2}{3-\alpha}} + C(\alpha,\gamma,\delta,a_0)a^{(\alpha-1)\cdot\frac{2}{\alpha-1}} \\ \le& \delta a + C(\alpha,\gamma,\delta,a_0)a^2.\end{aligned}$$

If $\alpha \ge 2$, then

$$\gamma a^\alpha = \gamma a^{\alpha-2}\cdot a^2 \le \gamma a_0^{\alpha-2}a^2.$$

Applying the algebraic fact on (3.14) we get

$$\int_\Omega w^2(x,s)\,dx \le C(p,\beta,L,C_f,\delta,\|w\|_{L^\infty})\int_{\Omega_s} w^2\,dz.$$

The conclusion now follows from Grönwall's inequality and letting $l \to 0$. □

Next we use the previous comparison results to prove the comparison principle for general continuous f.

Theorem 3.4. *Let $1 < p < \infty$. Let $u, v \in L^\infty(\Omega_T)$ respectively be weak sub- and supersolutions to (1.1) in Ω_T. Assume that for all $(x_0,t_0) \in \partial_p\Omega_T$*

$$\operatorname*{ess\,lim\,sup}_{(x,t)\to(x_0,t_0)} u(x,t) \le \operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} v(x,t).$$

Assume also that $Du \in L^\infty(\Omega_T)$. Then $u \le v$ a.e. in Ω_T.

Proof. For $\delta > 0$, define

$$u_\delta := u - \frac{\delta}{T-t/2}.$$

Then for any non-negative test function $\varphi \in C_0^\infty(\Omega_T)$ we have by integration by parts

$$\begin{aligned}\int_{\Omega_T} -u_\delta \partial_t\varphi\,dz &= \int_{\Omega_T} -u\partial_t\varphi + \frac{\delta}{T-t/2}\partial_t\varphi\,dz \\ &= \int_{\Omega_T} -u\partial_t\varphi - \varphi\frac{\delta}{2\,(T-t/2)^2}\,dz \\ &\le \int_{\Omega_T} -u\partial_t\varphi - \varphi\frac{\delta}{2T^2}\,dz.\end{aligned}$$

Since f is continuous, there is a locally Lipschitz continuous function f_δ such that $\|f-f_\delta\|_{L^\infty(\mathbb{R}^N)} \le \frac{\delta}{4T}$ (see e.g. [Mic00]). Then, since u is a weak subsolution, we have

$$\begin{aligned}
&\int_{\Omega_T} -u_\delta\partial_t\varphi + |Du_\delta|^{p-2} Du_\delta\cdot D\varphi - \varphi f_\delta(Du_\delta)\,dz\\
&\le \int_{\Omega_T} -u\partial_t\varphi + |Du|^{p-2} Du\cdot D\varphi - \varphi f(Du) + \varphi\,\|f-f_\delta\|_{L^\infty(\mathbb{R}^N)} - \varphi\frac{\delta}{2T^2}\,dz\\
&\le \int_{\Omega_T} -\frac{\delta}{4T^2}\varphi\,dz.
\end{aligned}$$

Hence u_δ is a weak subsolution to

$$\partial_t u_\delta - \Delta_p u_\delta - f_\delta(Du_\delta) \le -\frac{\delta}{4T^2} \quad \text{in } \Omega_T.$$

Similarly, since v is a weak supersolution, we define

$$v_\delta := v + \frac{\delta}{T-t/2}$$

and deduce that v_δ is a weak supersolution to

$$\partial_t v_\delta - \Delta_p v_\delta - f_\delta(Dv_\delta) \ge 0 \quad \text{in } \Omega_T.$$

Now it follows from the comparison Lemmas 3.2 and 3.3 that $u_\delta \le v_\delta$ a.e. in Ω_T. Thus

$$u \le v + \frac{2\delta}{T-t/2} \quad \text{a.e. in } \Omega_T.$$

Letting $\delta \to 0$ finishes the proof. □

Now that the comparison principle is proven, we are ready to show that weak solutions are viscosity solutions. To state this part of the equivalence, we define the *lower semicontinuous regularization* of a function $u : \Xi \to \mathbb{R}$ by

$$u_*(x,t) := \operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)} u(y,s) := \lim_{R\to 0} \operatorname*{ess\,inf}_{B_R(x)\times(t-R^p,t+R^p)} u.$$

The time scaling R^p is technically convenient in Section 5. We have included it here for notational consistency.

Theorem 3.5. *Let $1<p<\infty$. Let $u \in L^\infty_{loc}(\Xi)$ be a weak supersolution to (1.1) in Ξ for which $u = u_*$ almost everywhere in Ξ. Then u_* is a viscosity supersolution to (1.1) in Ξ.*

Proof. Assume on the contrary that there is $\phi \in C^2(\Xi)$ touching u_* from below at $(x_0,t_0) \in \Xi$, $D\phi(x,t) \ne 0$ for $x \ne x_0$ and

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\ x\ne x_0}} \left(\partial_t\phi(x,t) - \Delta_p\phi(x,t) - f(D\phi(x,t))\right) < 0. \tag{3.15}$$

Denote $Q_r := B_r(x_0)\times(t_0-r,t_0+r)$. It follows from above that there are $r>0$ and $\delta>0$ such that

$$\partial_t\phi - \Delta_p\phi - f(D\phi) < -\delta \quad \text{in } Q_r \setminus \{x = x_0\}\,. \tag{3.16}$$

Indeed, otherwise there would be a sequence $(x_n,t_n) \to (x_0,t_0)$ such that $x_n \ne x_0$ and

$$\partial_t\phi(x_n,t_n) - \Delta_p\phi(x_n,t_n) - f(D\phi(x_n,t_n)) > -\frac{1}{n},$$

but this contradicts (3.15). Using Gauss's theorem and (3.16) we obtain for any non-negative test function $\varphi \in C_0^\infty(Q_r)$ that

$$\begin{aligned}
&\int_{Q_r} - \phi \partial_t \varphi + |D\phi|^{p-2} D\phi \cdot D\varphi - \varphi f(D\phi)\, dz \\
&= \lim_{\rho \to 0} \int_{Q_r \setminus \{|x-x_0| \le \rho\}} -\phi \partial_t \varphi + |D\phi|^{p-2} D\phi \cdot D\varphi - \varphi f(D\phi)\, dz \\
&= \lim_{\rho \to 0} \Big(\int_{Q_r \setminus \{|x-x_0| \le \rho\}} \varphi \partial_t \phi - \varphi \operatorname{div}(|D\phi|^{p-2} D\phi) - \varphi f(D\phi)\, dz \\
&\qquad + \int_{t_0-r}^{t_0+r} \int_{\{|x-x_0| = \rho\}} \varphi \, |D\phi|^{p-2} D\phi \cdot \frac{(x-x_0)}{\rho}\, dS\, dt \Big) \\
&= \lim_{\rho \to 0} \int_{Q_r \setminus \{|x-x_0| \le \rho\}} \varphi \left(\partial_t \phi - \Delta_p \phi - f(D\phi) \right) dz \\
&\le \int_{Q_r} -\delta \varphi \, dz.
\end{aligned}$$

Let $l := \min_{\partial_p Q_r} (u_* - \phi) > 0$ and set $\widetilde{\phi} := \phi + l$. Then by the above inequality, $\widetilde{\phi}$ is a weak subsolution to

$$\partial_t \widetilde{\phi} - \Delta_p \widetilde{\phi} - f(D\widetilde{\phi}) \le -\delta \quad \text{in } Q_r$$

and on $\partial_p Q_r$ it holds $\widetilde{\phi} = \phi + l \le \phi + u_* - \phi = u_*$. Hence Theorem 3.4 implies that $\widetilde{\phi} \le u$ almost everywhere in Q_r. By the definition of u_*, it follows that

$$\widetilde{\phi} \le u_* \quad \text{everywhere in } Q_r, \tag{3.17}$$

which is a contradiction since in particular $\widetilde{\phi}(x_0, t_0) = \phi(x_0, t_0) + l > u_*(x_0, t_0)$.

To see (3.17), fix $(y_0, s_0) \in Q_r$ and let $\varepsilon > 0$. By continuity of $\widetilde{\phi}$ and the definition of u_*, there is $R > 0$ such that

$$\left| \widetilde{\phi}(y, s) - \widetilde{\phi}(y_0, s_0) \right| \le \varepsilon \quad \text{for all } (y, s) \in Q'_R$$

and

$$\left| \operatorname*{ess\,inf}_{Q'_R} u - u_*(y_0, s_0) \right| < \varepsilon,$$

where we denoted $Q'_R := B_R(y_0) \times (s_0 - R^p, s_0 + R^p)$. In particular

$$u_*(y_0, s_0) \ge \operatorname*{ess\,inf}_{Q'_R} u - \varepsilon.$$

By the definition of $\operatorname{ess\,inf}_{Q'_R} u$, there is $A \subset Q'_R$ with $|A| > 0$ such that

$$\operatorname*{ess\,inf}_{Q'_R} u + \varepsilon > u(y, s) \quad \text{for all } (y, s) \in A.$$

Moreover, since $\widetilde{\phi} \le u$ almost everywhere in Q_r, we can take $(y, s) \in A$ such that

$$\widetilde{\phi}(y, s) \le u(y, s).$$

Now we have by the last three displays

$$u_*(y_0, s_0) \ge \operatorname*{ess\,inf}_{Q'_R} u - \varepsilon > u(y, s) - 2\varepsilon \ge \widetilde{\phi}(y, s) - 2\varepsilon \ge \widetilde{\phi}(y_0, s_0) - 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that $u_*(y_0, s_0) \ge \widetilde{\phi}(y_0, s_0)$. $\square$

4. Viscosity solutions are weak solutions

We show that bounded viscosity supersolutions to (1.1) are weak supersolutions when $1 < p < \infty$ and $f \in C(\mathbb{R}^N)$ satisfies the growth condition (G1). We use the method developed in [JJ12]. The method of [JJ12] was previously applied to parabolic equations in [PV], but for radially symmetric solutions.

The idea is to approximate a viscosity supersolution u to (1.1) by the *inf-convolution*

$$u_\varepsilon(x,t) := \inf_{(y,s)\in\Xi} \left\{ u(y,s) + \frac{|x-y|^q}{q\varepsilon^{q-1}} + \frac{|t-s|^2}{2\varepsilon} \right\},$$

where $\varepsilon > 0$ and $q \geq 2$ is a fixed constant so large that $p - 2 + \frac{q-2}{q-1} > 0$. It is straightforward to show that the inf-convolution u_ε is a viscosity supersolution in the smaller set

$$\Xi_\varepsilon = \left\{ (x,t) \in \Xi : B_{r(\varepsilon)}(x) \times (t - t(\varepsilon), t + t(\varepsilon)) \Subset \Xi \right\},$$

where $r(\varepsilon)$, $t(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, u_ε is semi-concave by definition and therefore it has a second derivative almost everywhere. It follows from these pointwise properties that u_ε is a weak supersolution to (1.1) in Ξ_ε. Caccioppoli type estimates then imply that u_ε converges to u in a parabolic Sobolev space and consequently u is a weak supersolution.

The standard properties of the inf-convolution are postponed to the end of this section. Instead, we begin by proving the key observation: that the inf-convolution of a viscosity supersolution is a weak supersolution in the smaller set Ξ_ε. When $p \geq 2$, the idea is the following. Since u_ε is a viscosity supersolution to (1.1) that is twice differentiable almost everywhere, it satisfies the equation pointwise almost everywhere. Hence we may multiply the equation by a non-negative test function φ and integrate over Ξ_ε so that the integral will be non-negative. Then we approximate this expression through smooth functions $u_{\varepsilon,j}$ defined via the standard mollification. Since $u_{\varepsilon,j}$ is smooth, we may integrate by parts to reach the weak formulation of the equation, see (4.1). It then remains to let $j \to \infty$ to conclude that u_ε is a weak supersolution. The range $1 < p < 2$ is more delicate because of the singularity of the p-Laplace operator

$$\Delta_p u := |Du|^{p-2} \left(\Delta u + \frac{(p-2)}{|Du|^2} \left\langle D^2 u Du, Du \right\rangle \right),$$

and therefore we consider the case $p \geq 2$ first.

Lemma 4.1. *Let $p \geq 2$. Let u be a bounded viscosity supersolution to (1.1) in Ξ. Then u_ε is a weak supersolution to (1.1) in Ξ_ε.*

Proof. Fix a non-negative test function $\varphi \in C_0^\infty(\Xi_\varepsilon)$. By Remark 4.8, the function

$$\phi(x,t) := u_\varepsilon(x,t) - C(q,\varepsilon,u) \left(|x|^2 + t^2 \right)$$

is concave in Ξ_ε and we can approximate it by smooth concave functions ϕ_j so that $(\phi_j, \partial_t \phi_j, D\phi_j, D^2\phi_j) \to (\phi, \partial_t \phi, D\phi, D^2\phi)$ a.e. in Ξ_ε. We define

$$u_{\varepsilon,j}(x,t) := \phi_j(x,t) + C(q,\varepsilon,u) \left(|x|^2 + t^2 \right).$$

Since $u_{\varepsilon,j}$ is smooth and φ is compactly supported in Ξ_ε, we integrate by parts to get

$$\begin{aligned}
&\int_{\Xi_\varepsilon} \varphi\Big(\partial_t u_{\varepsilon,j} - |Du_{\varepsilon,j}|^{p-2}\left(\Delta u_{\varepsilon,j} + \frac{(p-2)}{|Du_{\varepsilon,j}|^2}\left\langle D^2 u_{\varepsilon,j} Du_{\varepsilon,j}, Du_{\varepsilon,j}\right\rangle\right) - f(Du_{\varepsilon,j})\Big)\,dz \\
&= \int_{\Xi_\varepsilon} \varphi\partial_t u_{\varepsilon,j} - \varphi \operatorname{div}\left(|Du_{\varepsilon,j}|^{p-2} Du_{\varepsilon,j}\right) - \varphi f(Du_{\varepsilon,j})\,dz \\
&= \int_{\Xi_\varepsilon} -u_{\varepsilon,j}\partial_t\varphi + |Du_{\varepsilon,j}|^{p-2} Du_{\varepsilon,j}\cdot D\varphi - \varphi f(Du_{\varepsilon,j})\,dz. \qquad (4.1)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\liminf_{j\to\infty}\int_{\Xi_\varepsilon} \varphi\Big(\partial_t u_{\varepsilon,j} - |Du_{\varepsilon,j}|^{p-2}\left(\Delta u_{\varepsilon,j} + \frac{(p-2)}{|Du_{\varepsilon,j}|^2}\left\langle D^2 u_{\varepsilon,j} Du_{\varepsilon,j}, Du_{\varepsilon,j}\right\rangle\right) - f(Du_{\varepsilon,j})\Big)\,dz \\
&\qquad \le \lim_{j\to\infty}\int_{\Xi_\varepsilon} -u_{\varepsilon,j}\partial_t\varphi + |Du_{\varepsilon,j}|^{p-2} Du_{\varepsilon,j}\cdot D\varphi - \varphi f(Du_{\varepsilon,j})\,dz.
\end{aligned}$$

We intend to use Fatou's lemma at the left-hand side and dominated convergence at the right-hand side. Once we verify their assumptions, we arrive at the inequality

$$\int_{\Xi_\varepsilon} \varphi\left(\partial_t u_\varepsilon - \Delta_p u_\varepsilon - f(Du_\varepsilon)\right)\,dz \le \int_{\Xi_\varepsilon} -u_\varepsilon\partial_t\varphi + |Du_\varepsilon|^{p-2} Du_\varepsilon\cdot D\varphi - \varphi f(Du_\varepsilon)\,dz.$$

The left-hand side is non-negative since by Lemma 4.7 the inf-convolution u_ε is still a viscosity supersolution in Ξ_ε. Consequently u_ε is a weak supersolution in Ξ_ε as desired. It remains to justify our use of Fatou's lemma and the dominated convergence theorem. It follows from Remark 4.8 that $|u_{\varepsilon,j}|$, $|\partial_t u_{\varepsilon,j}|$ and $|Du_{\varepsilon,j}|$ are uniformly bounded by some constant $M > 0$ in the support of φ with respect to j. This justifies our use of the dominated convergence theorem. Observe then that since ϕ_j is concave, we have $D^2 u_{\varepsilon,j} \le C(q,\varepsilon,u)I$. Hence

$$\begin{aligned}
&\partial_t u_{\varepsilon,j} - |Du_{\varepsilon,j}|^{p-2}\left(\Delta u_{\varepsilon,j} + \frac{(p-2)}{|Du_{\varepsilon,j}|^2}\left\langle D^2 u_{\varepsilon,j} Du_{\varepsilon,j}, Du_{\varepsilon,j}\right\rangle\right) - f(Du_{\varepsilon,j}) \\
&\qquad \ge -M - C(q,\varepsilon,u)M^{p-2}(N+p-2) - \sup_{|\xi|\le M}|f(\xi)|.
\end{aligned}$$

The integrand at the left-hand side of (4.1) is therefore bounded from below with respect to j, justifying our use of Fatou's lemma. □

Next we consider the singular case $1 < p < 2$. We cannot directly repeat the previous proof because $\Delta_p u_\varepsilon$ no longer has a clear meaning at the points where $Du_\varepsilon = 0$. To deal with this, we consider the regularized terms

$$\Delta_{p,\delta} u := \left(\delta + |Du|^2\right)^{\frac{p-2}{2}}\left(\Delta u + \frac{p-2}{\delta + |Du|^2}\Delta_\infty u\right), \qquad (4.2)$$

where $\Delta_\infty u = \langle D^2 u Du, Du\rangle$.

Lemma 4.2. *Let $1 < p < 2$. Let u be a bounded viscosity supersolution to (1.1) in Ξ. Then u_ε is a weak supersolution to (1.1) in Ξ_ε.*

Proof. **(Step 1)** Let $\varphi \in C_0^\infty(\Xi_\varepsilon)$ be a non-negative test function. We set

$$\phi(x,t) := u_\varepsilon(x,t) - C(q,\varepsilon,u)\left(|x|^2 + t^2\right),$$

where $C(q,\varepsilon,u)$ is the semi-concavity constant of u_ε in Ξ_ε. Then by Remark 4.8 we can approximate ϕ by smooth concave functions ϕ_j so that $(\phi_j,\partial_t\phi_j,D\phi_j,D^2\phi_j)\to(\phi,\partial_t\phi,D\phi,D^2\phi)$ a.e. in Ξ_ε. We define

$$u_{\varepsilon,j}(x,t):=\phi_j(x,t)+C(q,\varepsilon,u)\left(|x|^2+t^2\right).$$

Let $\delta\in(0,1)$. Since $u_{\varepsilon,j}$ is smooth and φ is compactly supported in Ξ_ε, we calculate via integration by parts

$$\begin{aligned}&\int_{\Xi_\varepsilon}\varphi\Bigg(\partial_t u_{\varepsilon,j}-\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}\left(\Delta u_{\varepsilon,j}+\frac{p-2}{\delta+|Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)-f(Du_{\varepsilon,j})\Bigg)\,dz\\&=\int_{\Xi_\varepsilon}\varphi\partial_t u_{\varepsilon,j}-\varphi\operatorname{div}\left(\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}Du_{\varepsilon,j}\right)-\varphi f(Du_{\varepsilon,j})\,dz\\&=\int_{\Xi_\varepsilon}-u_{\varepsilon,j}\partial_t\varphi+\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}Du_{\varepsilon,j}\cdot D\varphi-\varphi f(Du_{\varepsilon,j})\,dz.\end{aligned}$$

Recalling the shorthand $\Delta_{p,\delta}$ defined in (4.2), we deduce from the above that

$$\begin{aligned}&\liminf_{j\to\infty}\int_{\Xi_\varepsilon}\varphi\left(\partial_t u_{\varepsilon,j}-\Delta_{p,\delta}u_{\varepsilon,j}-f(Du_{\varepsilon,j})\right)\,dz\\&\quad\le\lim_{j\to\infty}\int_{\Xi_\varepsilon}-u_{\varepsilon,j}\partial_t\varphi+\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}Du_{\varepsilon,j}\cdot D\varphi-\varphi f(Du_{\varepsilon,j})\,dz.\end{aligned}\tag{4.3}$$

We use Fatou's lemma at the left-hand side and the dominated convergence at the right-hand side. Once we verify their assumptions, we arrive at the auxiliary inequality

$$\begin{aligned}&\int_{\Xi_\varepsilon}\varphi\left(\partial_t u_\varepsilon-\Delta_{p,\delta}u_\varepsilon-f(Du_\varepsilon)\right)\,dz\\&\quad\le\int_{\Xi_\varepsilon}-u_\varepsilon\partial_t\varphi+\left(\delta+|Du_\varepsilon|^2\right)^{\frac{p-2}{2}}Du_\varepsilon\cdot D\varphi-\varphi f(Du_\varepsilon)\,dz.\end{aligned}\tag{4.4}$$

Next we verify the assumptions of Fatou's lemma and the dominated convergence theorem. By Remark 4.8, the functions $|u_{\varepsilon,j}|$, $|\partial_t u_{\varepsilon,j}|$ and $|Du_{\varepsilon,j}|$ are uniformly bounded by some constant $M>1$ in the support of φ with respect to j. Hence the assumptions of the dominated convergence theorem are satisfied. Observe then that the concavity of ϕ_j implies that $D^2u_{\varepsilon,j}\le C(q,\varepsilon,u)I$. Thus the integrand at the left-hand side of (4.3) has a lower bound independent of j when $Du_{\varepsilon,j}=0$. When $Du_{\varepsilon,j}\neq0$, we have

$$\begin{aligned}&\partial_t u_{\varepsilon,j}-\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}\left(\Delta u_{\varepsilon,j}+\frac{p-2}{\delta+|Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)-f(Du_{\varepsilon,j})\\&=\partial_t u_{\varepsilon,j}-\frac{\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}}{\delta+|Du_{\varepsilon,j}|^2}\left(|Du_{\varepsilon,j}|^2\left(\Delta u_{\varepsilon,j}+\frac{p-2}{|Du_{\varepsilon,j}|^2}\Delta_\infty u_{\varepsilon,j}\right)+\delta\Delta u_{\varepsilon,j}\right)-f(Du_{\varepsilon,j})\\&\ge-\partial_t u_{\varepsilon,j}-\frac{\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}}{\delta+|Du_{\varepsilon,j}|^2}C(q,\varepsilon,u)\left(|Du_{\varepsilon,j}|^2(N+p-2)+\delta N\right)-f(Du_{\varepsilon,j})\\&\ge-\partial_t u_{\varepsilon,j}-C(q,\varepsilon,u)\left(\delta+|Du_{\varepsilon,j}|^2\right)^{\frac{p-2}{2}}(2N+p-2)-f(Du_{\varepsilon,j})\\&\ge-M-C(q,\varepsilon,u)\delta^{\frac{p-2}{2}}(2N+p-2)-\sup_{|\xi|\le M}|f(\xi)|,\end{aligned}$$

so that our use of Fatou's lemma is justified.

(Step 2) We let $\delta \to 0$ in the auxiliary inequality (4.4). Since u_ε is Lipschitz continuous, the dominated convergence theorem implies

$$\begin{aligned}\liminf_{\delta\to 0}\int_{\Xi_\varepsilon} &\varphi\left(\partial_t u_\varepsilon - \Delta_{p,\delta} u_\varepsilon - f(Du_\varepsilon)\right)\,dz \\ &\leq \int_{\Xi_\varepsilon} -u_\varepsilon \partial_t \varphi + |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot D\varphi - \varphi f(Du_\varepsilon)\,dz. \end{aligned} \tag{4.5}$$

Applying Fatou's lemma (we verify assumptions at the end), we get

$$\begin{aligned}\liminf_{\delta\to 0}\int_{\Xi_\varepsilon} &\varphi\left(\partial_t u_\varepsilon - \Delta_{p,\delta} u_\varepsilon - f(Du_\varepsilon)\right)\,dz \\ &\geq \int_{\Xi_\varepsilon} \liminf_{\delta\to 0} \varphi\left(\partial_t u_\varepsilon - \Delta_{p,\delta} u_\varepsilon - f(Du_\varepsilon)\right)\,dz \\ &= \int_{\Xi_\varepsilon\cap\{Du_\varepsilon\neq 0\}} \liminf_{\delta\to 0} \varphi\left(\partial_t u_\varepsilon - \Delta_{p,\delta} u_\varepsilon - f(Du_\varepsilon)\right)\,dz \\ &\quad + \int_{\Xi_\varepsilon\cap\{Du_\varepsilon= 0\}} \liminf_{\delta\to 0} \varphi(\partial_t u_\varepsilon - \delta^{\frac{p-2}{2}}\Delta u_\varepsilon - f(0))\,dz \\ &= \int_{\Xi_\varepsilon\cap\{Du_\varepsilon\neq 0\}} \varphi\left(\partial_t u_\varepsilon - \Delta_{p} u_\varepsilon - f(Du_\varepsilon)\right)\,dz \\ &\quad + \int_{\Xi_\varepsilon\cap\{Du_\varepsilon= 0\}} \varphi\left(\partial_t u_\varepsilon - f(0)\right)\,dz \geq 0, \end{aligned} \tag{4.6}$$

where the last inequality follows from Lemma 4.7 since u_ε is twice differentiable almost everywhere. Combining (4.5) and (4.6), we find that u_ε is a weak supersolution in Ξ_ε. It remains to verify the assumptions of Fatou's lemma, i.e. that the integrand at the left-hand side of (4.5) has a lower bound independent of δ. When $Du_\varepsilon = 0$, this follows directly from the inequality

$$D^2 u_\varepsilon \leq \frac{q-1}{\varepsilon}\left|Du_\varepsilon\right|^{\frac{q-2}{q-1}} I,$$

which holds by Lemma 4.6. When $Du_\varepsilon \neq 0$, we recall that by Lipschitz continuity $\partial_t u_\varepsilon$ and Du_ε are uniformly bounded in Ξ_ε, and estimate

$$\begin{aligned} &-\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}}\left(\Delta u_\varepsilon + \frac{p-2}{\delta+|Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right) \\ &= -\frac{\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}}}{\delta+|Du_\varepsilon|^2}\left(|Du_\varepsilon|^2\left(\Delta u_\varepsilon + \frac{p-2}{|Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right) + \delta\Delta u_\varepsilon\right) \\ &\geq -\frac{\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}}}{\delta+|Du_\varepsilon|^2}\frac{(q-1)}{\varepsilon}\left(|Du_\varepsilon|^{\frac{q-2}{q-1}+2}(N+p-2) + |Du_\varepsilon|^{\frac{q-2}{q-1}}\delta N\right) \\ &\geq -\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}}\frac{(q-1)}{\varepsilon}|Du_\varepsilon|^{\frac{q-2}{q-1}}(2N+p-2) \\ &\geq -|Du_\varepsilon|^{p-2+\frac{q-2}{q-1}}\frac{(q-1)}{\varepsilon}(2N+p-2) \\ &\geq -\|Du_\varepsilon\|_{L^\infty(\Xi_\varepsilon)}^{p-2+\frac{q-2}{q-1}}\frac{(q-1)}{\varepsilon}(2N+p-2), \end{aligned}$$

where we used that $p-2+\frac{q-2}{q-1} > 0$. Hence the assumptions of Fatou's lemma hold. □

If u_ε is the sequence of inf-convolutions of a viscosity supersolution to (1.1), then by next Caccioppoli's inequality the sequence Du_ε converges weakly in $L^p_{loc}(\Xi)$ up to a subsequence. However, we need stronger convergence to pass to the limit under the integral sign of

$$\int_\Xi -u_\varepsilon \partial_t \varphi + |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot D\varphi - \varphi f(Du_\varepsilon)\, dz \geq 0.$$

For this end, we show in Lemma 4.4 that Du_ε converges in $L^r_{loc}(\Xi)$ for all $1 < r < p$.

Lemma 4.3 (Caccioppoli's inequality)**.** *Let $1 < p < \infty$. Assume that u is a locally Lipschitz continuous weak supersolution to (1.1) in Ξ. Then there is a constant $C = C(p, \beta, C_f)$ such that for any test function $\xi \in C_0^\infty(\Xi)$ we have*

$$\int_\Xi \xi^p |Du|^p \, dz \leq C \int_\Xi M^2 \partial_t \xi^p + M^p |D\xi|^p + (M^{\frac{p}{p-\beta}} + M)\xi^p \, dz,$$

where $M = \|u\|_{L^\infty(\operatorname{spt} \xi)}$.

Proof. Since u is locally Lipschitz continuous, the function $\varphi := (M - u)\, \xi^p$ is an admissible test function. Testing the weak formulation of (1.1) with φ yields

$$\int_\Xi \xi^p |Du|^p \, dz \leq \int_\Xi u \partial_t \varphi + p\xi^{p-1}(M - u) |Du|^{p-1} |D\xi| + \varphi f(Du)\, dz. \tag{4.7}$$

We have by integration by parts

$$\begin{aligned}\int_\Xi u \partial_t \varphi \, dz &= \int_\Xi -\xi^p u \partial_t u + u(M-u)\partial_t \xi^p \, dz \\ &= \int_\Xi -\frac{1}{2}\xi^p \partial_t u^2 + u(M-u)\partial_t \xi^p \, dz \\ &= \int_\Xi \frac{1}{2} u^2 \partial_t \xi^p + u(M-u)\partial_t \xi^p \, dz \leq \int_\Xi C M^2 \partial_t \xi^p \, dz.\end{aligned}$$

By Young's inequality

$$\int_\Xi p\xi^{p-1}(M-u)|Du|^{p-1}|D\xi| \, dz \leq \int_\Xi \frac{1}{4}\xi^p |Du|^p \, dz + C(p) \int_\Xi M^p |D\xi|^p \, dz.$$

Using the growth condition (G1) and Young's inequality we get

$$\begin{aligned}\int_\Xi \varphi f(Du)\, dz &\leq \int_\Xi (M-u)\, \xi^p C_f \left(1 + |Du|^\beta\right) dz \\ &= \int_\Xi C_f\, (M-u)\, \xi^{p-\beta} \xi^\beta |Du|^\beta + C_f (M-u)\xi^p \, dz \\ &\leq \int_\Xi \frac{1}{4}\xi^p |Du|^p + C(p,\beta,C_f)\,(M-u)^{\frac{p}{p-\beta}}\, \xi^p + C_f\,(M-u)\,\xi^p \, dz \\ &\leq \int_\Xi \frac{1}{4}\xi^p |Du|^p + C(p,\beta,C_f)\left(M^{\frac{p}{p-\beta}} + M\right)\xi^p \, dz.\end{aligned}$$

Combining these estimates with (4.7) and absorbing the terms with Du to the left-hand side yields the desired inequality. □

The proof of Lemma 4.4 is based on that of Lemma 5 in [LM07], see also Theorem 5.3 in [KKP10]. For the convenience of the reader, we give the full details.

Lemma 4.4. *Let $1 < p < \infty$. Suppose that (u_j) is a sequence of locally Lipschitz continuous weak supersolutions to (1.1) such that $u_j \to u$ in $L^p_{loc}(\Xi)$. Then (Du_j) is a Cauchy sequence in $L^r_{loc}(\Xi)$ for any $1 < r < p$.*

Proof. Let $U \Subset \Xi$ and take a cut-off function $\theta \in C_0^\infty(\Xi)$ such that $0 \le \theta \le 1$ and $\theta \equiv 1$ in U. For $\delta > 0$, we set

$$w_{jk} = \begin{cases} \delta, & u_j - u_k > \delta, \\ u_j - u_k, & |u_j - u_k| \le \delta, \\ -\delta, & u_j - u_k < -\delta. \end{cases}$$

Then the function $(\delta - w_{jk})\theta$ is an admissible test function with a time derivative since it is Lipschitz continuous. Since u_j is a weak supersolution, testing the weak formulation of (1.1) with $(\delta - w_{jk})\theta$ yields

$$\begin{aligned} 0 &\le \int_\Xi -u_j \partial_t((\delta - w_{jk})\theta) + |Du_j|^{p-2} Du_j \cdot D((\delta - w_{jk})\theta) - (\delta - w_{jk})\theta f(Du_j)\, dz \\ &= \int_\Xi -\theta\, |Du_j|^{p-2} Du_j \cdot Dw_{jk} + (\delta - w_{jk})\, |Du_j|^{p-2} Du_j \cdot D\theta - (\delta - w_{jk})\theta f(Du_j) \\ &\qquad + u_j \partial_t(w_{jk}\theta) - (\delta - w_{jk}) u_j \partial_t \theta\, dz. \end{aligned}$$

Since $|w_{jk}| \le \delta$ and $Dw_{jk} = \chi_{\{|u_j - u_k| < \delta\}} (Du_j - Du_k)$, the above becomes

$$\begin{aligned} &\int_{\{|u_j-u_k|<\delta\}} \theta\, |Du_j|^{p-2} Du_j \cdot (Du_j - Du_k)\, dz \\ &\qquad \le \int_\Xi 2\delta\, |Du_j|^{p-1} |D\theta| + 2\delta\theta\, |f(Du_j)| + u_j \partial_t(w_{jk}\theta) + 2\delta\, |u_j|\, |\partial_t \theta|\, dz. \end{aligned}$$

Since u_k is a weak supersolution, the same arguments as above but testing this time with $(\delta + w_{jk})\theta$ yield the analogous estimate

$$\begin{aligned} &\int_{\{|u_j-u_k|<\delta\}} -\theta\, |Du_k|^{p-2} Du_k \cdot (Du_j - Du_k)\, dz \\ &\qquad \le \int_\Xi 2\delta\, |Du_k|^{p-1} |D\theta| + 2\delta\theta\, |f(Du_k)| - u_k \partial_t (w_{jk}\theta) + 2\delta\, |u_k|\, |\partial_t \theta|\, dz. \end{aligned}$$

Summing up these two inequalities we arrive at

$$\begin{aligned} &\int_{\{|u_j-u_k|<\delta\}} \theta \left(|Du_j|^{p-2} Du_j - |Du_k|^{p-2} Du_k \right) \cdot (Du_j - Du_k)\, dz \\ &\qquad \le 2\delta \int_\Xi |D\theta| \left(|Du_j|^{p-1} + |Du_k|^{p-1} \right) dz + 2\delta \int_\Xi \theta\, (|f(Du_j)| + |f(Du_k)|)\, dz \\ &\qquad\quad + \int_\Xi (u_j - u_k) \partial_t (w_{jk}\theta)\, dz + 2\delta \int_\Xi (|u_j| + |u_k|)\, |\partial_t \theta|\, dz \\ &\qquad =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.8}$$

We proceed to estimate these integrals. Denoting $M := \sup_j \|u_j\|_{L^\infty(\operatorname{spt}\theta)} < \infty$, we have by the Caccioppoli's inequality Lemma 4.3

$$\sup_j \int_{\operatorname{spt}\theta} |Du_j|^p\, dz \le C(p, \beta, C_f, \theta, M). \tag{4.9}$$

The estimate (4.9) and Hölder's inequality imply that

$$I_1 \le \delta C(p, \beta, C_f, \theta, M).$$

To estimate I_2, we also use the growth condition (G1) and the assumption $\beta < p$. We get

$$I_2 \le 2\delta \int_\Xi \theta C_f (2 + |Du_j|^\beta + |Du_k|^\beta)\, dz \le \delta C(p, \beta, C_f, \theta, M).$$

The integral I_3 is estimated using integration by parts and that $|w_{jk}| \le \delta$

$$\begin{aligned} I_3 &= \int_\Xi \theta(u_j - u_k)\partial_t\,(w_{jk}) + (u_j - u_k)\, w_{jk}\partial_t \theta\, dz = \int_\Xi \frac{1}{2}\theta\partial_t w_{jk}^2 + (u_j - u_k)\, w_{jk}\partial_t\theta\, dz \\ &= \int_\Xi -\frac{1}{2} w_{jk}^2 \partial_t \theta + (u_j - u_k) w_{jk} \partial_t \theta\, dz \le \delta C(\theta, M). \end{aligned}$$

For the last integral we have directly $I_4 \le \delta C(\theta, M)$. Combining these estimates with (4.8) we arrive at

$$\int_{\{|u_j - u_k| < \delta\}} \theta \left(|Du_j|^{p-2} Du_j - |Du_k|^{p-2} Du_k \right) \cdot (Du_j - Du_k)\; dz \le \delta C_0, \tag{4.10}$$

where $C_0 = C(p, \beta, C_f, \theta, M)$. If $1 < p < 2$, Hölder's inequality and the algebraic inequality (3.8) give the estimate (recall that $1 < r < p$ and $\theta \equiv 1$ in U)

$$\begin{aligned} &\int_{U\cap\{|u_j - u_k| < \delta\}} |Du_j - Du_k|^r\; dz \\ &\le \left(\int_{U\cap\{|u_j - u_k| < \delta\}} \left(1 + |Du_j|^2 + |Du_k|^2\right)^{\frac{r(2-p)}{2(2-r)}}\; dz \right)^{\frac{2-r}{2}} \\ &\quad \cdot \left(\int_{U\cap\{|u_j - u_k| < \delta\}} \frac{|Du_j - Du_k|^2}{\left(1 + |Du_j|^2 + |Du_k|^2\right)^{\frac{2-p}{2}}}\, dz \right)^{\frac{r}{2}} \\ &\le C(p, \beta, r, C_f, \theta, M) \\ &\quad \cdot \left(\int_{\{|u_j - u_k| < \delta\}} \theta \left(|Du_j|^{p-2} Du_j - |Du_k|^{p-2} Du_k \right) \cdot (Du_j - Du_k)\; dz \right)^{\frac{r}{2}}, \end{aligned}$$

where in the last inequality we also used (4.9) with the knowledge $\frac{r(2-p)}{(2-r)} \le \frac{p(2-p)}{2-p} = p$. If $p \ge 2$, Hölder's inequality and the algebraic inequality (3.12) imply

$$\begin{aligned} &\int_{U\cap\{|u_j - u_k| < \delta\}} |Du_j - Du_k|^r\; dz \\ &\le \left(\int_\Xi 1\, dz \right)^{\frac{p-r}{p}} \left(\int_{U\cap\{|u_j - u_k| < \delta\}} |Du_j - Du_k|^p\; dz \right)^{\frac{r}{p}} \\ &\le C(p, r) \left(\int_{\{|u_j - u_k| < \delta\}} \theta \left(|Du_j|^{p-2} Du_j - |Du_k|^{p-2} Du_k \right) \cdot (Du_j - Du_k)\; dz \right)^{\frac{r}{p}}. \end{aligned}$$

Hence (4.10) leads to

$$\int_{U\cap\{|u_j - u_k| < \delta\}} |Du_j - Du_k|^r\; dz \le \delta^{\frac{r}{\max(2,p)}} C(p, \beta, r, C_f, \theta, M).$$

On the other hand, Hölder's and Tchebysheff's inequalities with (4.9) imply

$$\begin{aligned}&\int_{U\cap\{|u_j-u_k|\geq\delta\}}|Du_j-Du_k|^r\,dz\\&\leq|U\cap\{|u_j-u_k|\geq\delta\}|^{\frac{p-r}{p}}\left(\int_{U\cap\{|u_j-u_k|\geq\delta\}}|Du_j-Du_k|^p\,dz\right)^{\frac{r}{p}}\\&\leq\delta^{r-p}\,\|u_j-u_k\|_{L^p(U)}^{p-r}\,C(p,\beta,r,C_f,\theta,M).\end{aligned}$$

So we arrive at

$$\int_U|Du_j-Du_k|^r\,dz\leq(\delta^{\frac{r}{\max(2,p)}}+\delta^{r-p}\,\|u_j-u_k\|_{L^p(U)}^{p-r})C(p,\beta,r,C_f,\theta,M).$$

Taking first small $\delta>0$ and then large j,k, we can make the right-hand side arbitrarily small. □

Now we are ready to prove the main result of this section which states that bounded viscosity supersolutions are weak supersolutions.

Theorem 4.5. *Let $1<p<\infty$. Let u be a bounded viscosity supersolution to (1.1) in Ξ. Then u is a weak supersolution to (1.1) in Ξ.*

Proof. Fix a non-negative test function $\varphi\in C_0^\infty(\Xi)$ and take an open cylinder $\Omega_{t_1,t_2}\Subset\Xi$ such that $\operatorname{spt}\varphi\Subset\Omega_{t_1,t_2}$. Let $\varepsilon>0$ be so small that $\Omega_{t_1,t_2}\Subset\Xi_\varepsilon$. Then Lemma 4.2 implies that u_ε is a weak supersolution to (1.1) in Ξ_ε. Therefore by the Caccioppoli's inequality Lemma 4.3, Du_ε is bounded in $L^p(\Omega_{t_1,t_2})$. Hence Du_ε converges weakly in $L^p(\Omega_{t_1,t_2})$ up to a subsequence. Since also $u_\varepsilon\to u$ in $L^p(\Omega_{t_1,t_2})$ by dominated convergence and the fact that $u_\varepsilon\to u$ pointwise in Ω_{t_1,t_2}, it follows that $u\in L^p(t_1,t_2;W^{1,p}(\Omega))$.

Since u_ε is a weak supersolution, it remains to show that up to a subsequence

$$\lim_{\varepsilon\to0}\int_{\Omega_{t_1,t_2}}u_\varepsilon\partial_t\varphi+|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot D\varphi\,dz=\int_{\Omega_{t_1,t_2}}u\partial_t\varphi+|Du|^{p-2}Du\cdot D\varphi\,dz\tag{4.11}$$

and

$$\lim_{\varepsilon\to0}\int_{\Omega_{t_1,t_2}}\varphi f(Du_\varepsilon)\,dz=\int_{\Omega_{t_1,t_2}}\varphi f(Du)\,dz.\tag{4.12}$$

Since $u_\varepsilon\to u$ in $L^p(\Omega_{t_1,t_2})$ and $Du_\varepsilon\to Du$ in $L^r(\Omega_{t_1,t_2})$ for any $1<r<p$ by Lemma 4.4, the claim (4.11) follows by applying the vector inequality (see [Lin17, p95-96])

$$\left||a|^{p-2}a-|b|^{p-2}b\right|\leq\begin{cases}2^{2-p}|a-b|^{p-1} & \text{when } p<2,\\ 2^{-1}\left(|a|^{p-2}+|b|^{p-2}\right)|a-b| & \text{when } p\geq2.\end{cases}$$

To show (4.12), let $M\geq1$ and write using the growth condition (G1)

$$\begin{aligned}&\int_{\Omega_{t_1,t_2}}|f(Du_\varepsilon)-f(Du)|\,dz\\&\leq\int_{\{|Du_\varepsilon|<M\}}|f(Du_\varepsilon)-f(Du)|\,dz+\int_{\{|Du_\varepsilon|\geq M\}}C_f(2+|Du_\varepsilon|^\beta+|Du|^\beta)\,dz\\&=:I_1+I_2.\end{aligned}$$

Then by Hölder's inequality

$$\begin{aligned} I_2 &= C_f \int_{\{|Du_\varepsilon|\geq M\}} \frac{2\,|Du_\varepsilon|^p}{|Du_\varepsilon|^p} + \frac{|Du_\varepsilon|^p}{|Du_\varepsilon|^{p-\beta}} + \frac{|Du|^\beta\,|Du_\varepsilon|^{p-\beta}}{|Du_\varepsilon|^{p-\beta}}\,dz \\ &\leq C_f\left(\frac{2}{M^p} + \frac{1}{M^{p-\beta}}\right)\|Du_\varepsilon\|^p_{L^p(\Omega_{t_1,t_2})} + C_f\,\frac{1}{M^{p-\beta}}\,\|Du\|^\beta_{L^p(\Omega_{t_1,t_2})}\,\|Du_\varepsilon\|^{p-\beta}_{L^p(\Omega_{t_1,t_2})} \\ &\leq \frac{1}{M^{p-\beta}} C(p,\beta,C_f,\|Du\|_{L^p(\Omega_{t_1,t_2})}, \sup_\varepsilon \|Du_\varepsilon\|_{L^p(\Omega_{t_1,t_2})}). \end{aligned}$$

On the other hand, we have $|f(Du_\varepsilon) - f(Du)| \to 0$ a.e. in Ω_{t_1,t_2} up to a subsequence and the integrand in I_1 is dominated by an integrable function since the growth condition (G1) implies

$$|f(Du_\varepsilon) - f(Du)| \leq C_f(2 + |M|^\beta + |Du|^\beta) \quad \text{when} \quad |Du_\varepsilon| < M.$$

Hence, for any $M \geq 1$, we have $I_1 \to 0$ as $\varepsilon \to 0$ by the dominated convergence theorem. By taking first large $M \geq 1$ and then small $\varepsilon > 0$, we can make $I_1 + I_2$ arbitrarily small. □

The rest of this section is devoted to the properties of the inf-convolution. The facts in the following lemma are well known, see e.g. [CIL92], [JJ12], [Kat15] or [PV].

Lemma 4.6. *Assume that $u : \Xi \to \mathbb{R}$ is lower semicontinuous and bounded. Then u_ε has the following properties.*

(i) *We have $u_\varepsilon \leq u$ in Ξ and $u_\varepsilon \to u$ pointwise as $\varepsilon \to 0$.*

(ii) *Denote $r(\varepsilon) := (q\varepsilon^{q-1} \operatorname{osc}_\Xi u)^{\frac{1}{q}}$, $t(\varepsilon) := (2\varepsilon \operatorname{osc}_\Xi u)^{\frac{1}{2}}$. For $(x,t) \in \mathbb{R}^{N+1}$, set*

$$\Xi_\varepsilon := \left\{(x,t) \in \Xi : B_{r(\varepsilon)}(x) \times (t - t(\varepsilon), t + t(\varepsilon)) \Subset \Xi\right\}.$$

Then for any $(x,t) \in \Xi_\varepsilon$ there exists $(x_\varepsilon, t_\varepsilon) \in \overline{B}_{r(\varepsilon)}(x) \times [t - t(\varepsilon), t + t(\varepsilon)]$ such that

$$u_\varepsilon(x,t) = u(x_\varepsilon, t_\varepsilon) + \frac{|x - x_\varepsilon|^q}{q\varepsilon^{q-1}} + \frac{|t - t_\varepsilon|^2}{2\varepsilon}.$$

(iii) *The function u_ε is semi-concave in Ξ_ε with a semi-concavity constant depending only on u, q and ε.*

(iv) *Assume that u_ε is differentiable in time and twice differentiable in space at $(x,t) \in \Xi_\varepsilon$. Then*

$$\begin{aligned} \partial_t u_\varepsilon(x,t) &= \frac{t - t_\varepsilon}{\varepsilon}, \\ Du_\varepsilon(x,t) &= (x - x_\varepsilon)\,\frac{|x - x_\varepsilon|^{q-2}}{\varepsilon^{q-1}}, \\ D^2 u_\varepsilon(x,t) &\leq \frac{q-1}{\varepsilon}\,|Du_\varepsilon|^{\frac{q-2}{q-1}}\, I. \end{aligned}$$

Next we show that the inf-convolution of a viscosity supersolution to (1.1) is still a supersolution in the smaller set Ξ_ε. Since the inf-convolution is "flat enough", that is, since $q > p/(p-1)$, the inf-convolution essentially cancels the singularity of the p-Laplace operator. This allows us to extract information on the time derivative at those points of differentiability where Du_ε vanishes.

Lemma 4.7. *Let $1 < p < \infty$. Let u be a viscosity supersolution to (1.1) in Ξ. Then the inf-convolution u_ε is also a viscosity supersolution to (1.1) in Ξ_ε.*

Moreover, if u_ε is differentiable in time and twice differentiable in space at $(x,t) \in \Xi_\varepsilon$ and $Du_\varepsilon(x,t) = 0$, then $\partial_t u_\varepsilon(x,t) - f(0) \geq 0$.

Proof. Assume that φ touches u_ε from below at $(x,t) \in \Xi_\varepsilon$. Let $(x_\varepsilon, t_\varepsilon)$ be like in the property (ii) of Lemma 4.6. Then

$$\varphi(x,t) = u_\varepsilon(x,t) = u(x_\varepsilon, t_\varepsilon) + \frac{|x - x_\varepsilon|^q}{q\varepsilon^{q-1}} + \frac{|t - t_\varepsilon|^2}{2\varepsilon}, \tag{4.13}$$

$$\varphi(y,\tau) \leq u_\varepsilon(y,\tau) \leq u(z,s) + \frac{|y - z|^q}{q\varepsilon^{q-1}} + \frac{|\tau - s|^2}{2\varepsilon} \text{ for all } (y,\tau),(z,s) \in \Xi. \tag{4.14}$$

Set

$$\psi(z,s) := \varphi(z + x - x_\varepsilon, s + t - t_\varepsilon) - \frac{|x - x_\varepsilon|^q}{q\varepsilon^{q-1}} - \frac{|t - t_\varepsilon|^2}{2\varepsilon}.$$

Then ψ touches u from below at $(x_\varepsilon, t_\varepsilon)$ since by (4.13)

$$\psi(x_\varepsilon, t_\varepsilon) = \varphi(x,t) - \frac{|x - x_\varepsilon|^q}{q\varepsilon^{q-1}} - \frac{|t - t_\varepsilon|^2}{2\varepsilon} = u(x_\varepsilon, t_\varepsilon)$$

and selecting $(y,\tau) = (z + x - x_\varepsilon, s + t - t_\varepsilon)$ in (4.14) gives

$$\psi(z,s) = \varphi(z + x - x_\varepsilon, s + t - t_\varepsilon) - \frac{|x - x_\varepsilon|^q}{q\varepsilon^{q-1}} - \frac{|t - t_\varepsilon|^2}{2\varepsilon} \leq u(z,s).$$

Since u is a viscosity supersolution, it follows that

$$\begin{aligned} 0 \leq & \limsup_{\substack{(z,s)\to(x_\varepsilon,t_\varepsilon) \\ z \neq x_\varepsilon}} \left(\partial_s \psi(z,s) - \Delta_p \psi(z,s) - f(D\psi(z,s))\right) \\ = & \limsup_{\substack{(z,s)\to(x,t) \\ z \neq x}} \left(\partial_s \varphi(z,s) - \Delta_p \varphi(z,s) - f(D\varphi(z,s))\right), \end{aligned}$$

and the first claim is proven. To prove the second claim, assume that u_ε is differentiable in time and twice differentiable in space at $(x,t) \in \Xi_\varepsilon$ and $Du_\varepsilon(x,t) = 0$. By the property (iv) in Lemma 4.6, we have $x = x_\varepsilon$, so that

$$u_\varepsilon(x,t) = u(x,t_\varepsilon) + \frac{|t - t_\varepsilon|^2}{2\varepsilon}.$$

Hence by the definition of inf-convolution

$$u(y,s) + \frac{|x - y|^q}{q\varepsilon^{q-1}} + \frac{|t - s|^2}{2\varepsilon} \geq u_\varepsilon(x,t) = u(x,t_\varepsilon) + \frac{|t - t_\varepsilon|^2}{2\varepsilon} \text{ for all } (y,s) \in \Xi.$$

Arranging the terms as

$$u(y,s) \geq u(x,t_\varepsilon) - \frac{|x - y|^q}{q\varepsilon^{q-1}} - \frac{|t - s|^2}{2\varepsilon} + \frac{|t - t_\varepsilon|^2}{2\varepsilon} =: \phi(y,s),$$

we see that the function ϕ touches u from below at (x, t_ε). Since u is a viscosity supersolution and $D\phi(y,s) \neq 0$ when $y \neq x$, we have

$$\limsup_{\substack{(y,s)\to(x,t_\varepsilon) \\ y \neq x}} \left(\partial_s \phi(y,s) - \Delta_p \phi(y,s) - f(D\phi(y,s))\right) \geq 0.$$

On the other hand, since $q > p/(p-1)$, we have $\Delta_p \phi(y,s) \to 0$ as $y \to x$. Hence we get

$$0 \leq \partial_s \phi(x, t_\varepsilon) - f(0) = \frac{t - t_\varepsilon}{\varepsilon} - f(0) = \partial_t u_\varepsilon(x,t) - f(0),$$

where the last equality follows from the property (iv) in Lemma 4.6. □

Remark 4.8. Semi-concavity implies that the inf-convolution u_ε is locally Lipschitz in Ξ_ε (see [EG15, p267]). Therefore u_ε is differentiable almost everywhere in Ξ_ε, $\partial_t u_\varepsilon \in L^\infty_{loc}(\Xi_\varepsilon)$ and $u_\varepsilon \in L^\infty(t_1, t_2; W^{1,\infty}(\Omega))$ for any $\Omega_{t_1,t_2} \Subset \Xi_\varepsilon$ (see [EG15, p266]).

Moreover, since the function $\phi(x,t) := u_\varepsilon(x,t) - C(q,\varepsilon,u)(|x|^2 + |t|^2)$ is concave, Alexandrov's theorem implies that u_ε is twice differentiable almost everywhere in Ξ_ε. Furthermore, the proof of Alexandrov's theorem in [EG15, p273] establishes that if ϕ_j is the standard mollification of ϕ, then $D^2\phi_j \to D^2\phi$ almost everywhere in Ξ_ε.

5. Lower semicontinuity of supersolutions

We show the lower semicontinuity of weak supersolutions when $p \geq 2$ and the function $f \in C(\mathbb{R}^N)$ satisfies that $f(0) = 0$ as well as the stronger growth condition

$$|f(\xi)| \leq C_f \left(1 + |\xi|^{p-1}\right). \tag{G2}$$

Our proof follows the method of Kuusi [Kuu09], but the first-order term causes some modifications. In particular, our essential supremum estimate is slightly different, see Theorem 5.3 and the brief discussion before it. The assumption $f(0) = 0$ is used to ensure that the positive part u_+ of a subsolution is still a subsolution.

We begin by proving estimates for the essential supremum of a subsolution using the Moser's iteration technique. We first need the following Caccioppoli's inequalities.

Lemma 5.1 (Caccioppoli's inequalities)**.** *Assume that $p \geq 2$ and that (G2) holds. Suppose that u is a non-negative weak subsolution to (1.1) in Ω_{t_1,t_2} and $u \in L^{p-1+\lambda}(\Omega_{t_1,t_2})$ for some $\lambda \geq 1$. Then there exists a constant $C = C(p, C_f)$ that satisfies the estimates*

$$\begin{aligned}\operatorname*{ess\,sup}_{t_1<\tau<t_2} &\int_\Omega u^{1+\lambda}(x,\tau)\zeta^p(x,\tau)\,dx \\ &\leq C \int_{\Omega_{t_1,t_2}} \lambda u^{p-1+\lambda}|D\zeta|^p + u^{1+\lambda}|\partial_t\zeta|\zeta^{p-1} + \lambda\left(u^\lambda + u^{p-1+\lambda}\right)\zeta^p\,dz\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_{t_1,t_2}} &\left|D(u^{\frac{p-1+\lambda}{p}}\zeta)\right|^p dz \\ &\leq C \int_{\Omega_{t_1,t_2}} \lambda^p u^{p-1+\lambda}|D\zeta|^p + \lambda^{p-1}u^{1+\lambda}|\partial_t\zeta|\zeta^{p-1} + \lambda^p\left(u^\lambda + u^{p-1+\lambda}\right)\zeta^p\,dz\end{aligned}$$

for all non-negative $\zeta \in C^\infty(\Omega \times [t_1,t_2])$ such that $\operatorname{spt}\zeta(\cdot,t) \Subset \Omega$ *and* $\zeta(x,t_1) = 0$.

Proof. We test the regularized equation in Lemma (3.1) with $\varphi := \min(u^\epsilon, k)^{\lambda-1}u^\epsilon\zeta^p\eta$, where η is the following cut-off function

$$\eta(t) = \begin{cases} 0, & t \in (t_1, s-h), \\ (t-s+h)/2h, & t \in [s-h, s+h], \\ 1, & t \in (s+h, \tau-h), \\ (-t+\tau+h)/2h, & t \in [\tau-h, \tau+h], \\ 0, & t \in (\tau+h, t_2), \end{cases}$$

and $t_1 < s < \tau < t_2$, $h > 0$. We denote $g(l) := \int_0^l \min(r,k)^{\lambda-1} r\,dr$. Then integration by parts and Lebesgue's differentiation theorem yield for a.e. $s, \tau \in (t_1, t_2)$

$$\begin{aligned} &\int_{\Omega_{t_1,t_2}} \partial_t(u^\epsilon)\min(u^\epsilon,k)^{\lambda-1}u^\epsilon\zeta^p\eta\,dz \\ &\quad= \int_{\Omega_{t_1,t_2}} \partial_t g(u^\epsilon)\zeta^p\eta\,dz \\ &\quad= \int_{\Omega_{t_1,t_2}} -\eta g(u^\epsilon)\partial_t(\zeta^p) - \zeta^p g(u^\epsilon)\partial_t\eta\,dz \\ &\underset{\epsilon\to 0, h\to 0}{\longrightarrow} \int_{\Omega_{s,\tau}} -g(u)\partial_t(\zeta^p)\,dz - \int_\Omega \zeta^p(x,s)g(u(x,s))\,dx + \int_\Omega \zeta^p(x,\tau)g(u(x,\tau))\,dx. \end{aligned}$$

Letting $s \to t_1$ and observing that the other terms of (3.1) converge as well, we obtain for a.e. $\tau \in (t_1, t_2)$ that

$$\begin{aligned} &\int_\Omega g(u(x,\tau))\zeta^p(x,\tau)\,dx \\ &\quad\le \int_{\Omega_{t_1,\tau}} g(u)\partial_t(\zeta^p) - |Du|^{p-2}Du\cdot D(u_k^{\lambda-1}u\zeta^p) + u_k^{\lambda-1}u\zeta^p f(Du)\,dz, \end{aligned}$$

where we have denoted $u_k := \min(u,k)$. Since

$$Du_k^{\lambda-1} = \chi_{\{u<k\}}(\lambda-1)u^{\lambda-2}Du,$$

we have by Young's inequality

$$\begin{aligned} -|Du|^{p-2}Du\cdot D(u_k^{\lambda-1}u\zeta^p) \le& -\zeta^p\left((\lambda-1)\chi_{\{u<k\}}u^{\lambda-1} + u_k^{\lambda-1}\right)|Du|^p \\ &+ p\zeta^{p-1}u_k^{\lambda-1}u\,|Du|^{p-1}|D\zeta| \\ \le& -\frac{1}{2}\zeta^p u_k^{\lambda-1}|Du|^p + C(p)u^{p-1+\lambda}|D\zeta|^p. \end{aligned}$$

Moreover, by the growth condition (G2) and Young's inequality

$$\begin{aligned} u_k^{\lambda-1}u\zeta^p f(Du) \le& C_f\zeta^p u_k^{\lambda-1}u + C_f\zeta^p u_k^{\lambda-1}u\,|Du|^{p-1} \\ \le& C_f\zeta^p u^{\lambda-1} + C(p,C_f)\zeta^p u^{p-1+\lambda} + \frac{1}{4}\zeta^p u_k^{\lambda-1}|Du|^p. \end{aligned}$$

Collecting the estimates, moving the terms with Du to the left-hand side and letting $k \to \infty$, we arrive at

$$\begin{aligned} &\lambda^{-1}\int_\Omega u^{\lambda+1}\zeta^p(x,\tau)\,dx + \int_{\Omega_{t_1,\tau}} \frac{1}{4}\zeta^p u^{\lambda-1}|Du|^p\,dz \\ &\le C(p,C_f)\int_{\Omega_{t_1,\tau}} \lambda^{-1}u^{\lambda+1}|\partial_t\zeta^p| + u^{p-1+\lambda}|D\zeta|^p + \zeta^p(u^{\lambda-1}+u^{p-1+\lambda})\,dz. \qquad (5.1) \end{aligned}$$

Since the integrals are positive, this yields the first inequality of the lemma by taking essential supremum over τ. The second inequality follows from (5.1) by using that

$$\int_{\Omega_{t_1,t_2}} \left|D(u^{\frac{p-1+\lambda}{p}}\zeta)\right|^p dz \le C(p)\int_{\Omega_{t_1,t_2}} u^{p-1+\lambda}\left|D\zeta\right|^p + \lambda^p\zeta^p u^{\lambda-1}\left|Du\right|^p dz. \qquad \square$$

We first prove the following essential supremum estimate where we assume that the subsolution is bounded away from zero.

Lemma 5.2. *Assume that* $p \ge 2$ *and that (G2) holds. Suppose that* u *is a weak subsolution to (1.1) in* Ξ *and* $B_R(x_0)\times(t_0-T,t_0) \Subset \Xi$ *where* $R, T < 1$ *are such that*

$$\frac{R^p}{T} \le 1 \quad and \quad u \ge \left(\frac{R^p}{T}\right)^{\frac{1}{p-1}}. \tag{5.2}$$

Then there exists a constant $C(N,p,C_f)$ *such that*

$$\operatorname*{ess\,sup}_{B_{\sigma R(x_0)}\times(t_0-\sigma^pT,t_0)} u \le C\left(\frac{T}{R^p}(1-\sigma)^{-N-p}\fint_{B_R(x_0)\times(t_0-T,t_0)} u^{p-2+\delta}\,dz\right)^{1/\delta}$$

for every $1/2 \le \sigma < 1$ *and* $1<\delta<2$.

Proof. Let $\sigma R \le s < S < R$. For $j \in 0,1,2,\dots$, we set

$$R_j := S-(S-s)\left(1-2^{-j}\right)$$

and

$$U_j := B_j\times\Gamma_j := B_{R_j}(x_0)\times(t_0-(R_j/S)^pT,t_0).$$

We choose test functions $\varphi_j \in C^\infty(\overline{U_j})$ such that $\operatorname{spt}\varphi_j(\cdot,t)\Subset B_{R_j}(x_0)$,

$$0\le\varphi_j\le 1,\quad \varphi_j\equiv 0 \text{ on } \partial_pU_j,\quad \varphi_j\equiv 1 \text{ in } U_{j+1}$$

and

$$|D\varphi_j| \le \frac{C}{S-s}2^j,\quad |\partial_t\varphi_j| \le \frac{R^p}{T}\frac{C}{(S-s)^p}2^{jp}.$$

We set $\gamma := 1+p/N$ and

$$\lambda_j := 2\gamma^j-1, j=0,1,2,\dots.$$

Assuming that we already know that $u\in L^{p-1+\lambda_j}(U_j)$, then we have by a parabolic Sobolev's inequality (see [DiB93, p7])

$$\begin{aligned}\int_{U_{j+1}} u^{\kappa\alpha}\,dz \le& \int_{U_j}\left(u^{\alpha/p}\varphi_j^{\beta/p}\right)^{\kappa p} dz\\ \le& C(N,p)\int_{U_j}\left|D(u^{\alpha/p}\varphi_j^{\beta/p})\right|^p dz\left(\operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j}\left(u^{\alpha/p}\varphi_j^{\beta/p}\right)^{(\kappa-1)N} dx\right)^{p/N},\end{aligned} \tag{5.3}$$

where

$$\alpha = p-1+\lambda_j,\quad \kappa = 1+\frac{p(1+\lambda_j)}{N(p-1+\lambda_j)},\quad \beta=\frac{p(p-1+\lambda_j)}{1+\lambda_j}.$$

The first estimate in Lemma 5.1 gives

$$\begin{aligned}\operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j}\left(u^{\alpha/p}\varphi_j^{\beta/p}\right)^{(\kappa-1)N} dx =& \operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j} u^{1+\lambda_j}\varphi_j^p\,dx\\ \le& C\lambda_j\int_{U_j} u^{p-1+\lambda_j}\left|D\varphi_j\right|^p + u^{1+\lambda_j}\left|\partial_t\varphi_j\right|\varphi_j^{p-1} + \left(u^{\lambda_j}+u^{p-1+\lambda_j}\right)\varphi_j^p\,dz.\end{aligned} \tag{5.4}$$

Using the second estimate with $\zeta = \varphi_j^{\beta/p}$ we obtain

$$\begin{aligned}&\int_{U_j} \left| D(u^{\alpha/p}\varphi_j^{\beta/p})\right|^p \, dz \\ \leq & C\lambda_j^p \int_{U_j} u^{p-1+\lambda_j} \left| D\varphi_j\right|^p + u^{1+\lambda_j} \left| \partial_t \varphi_j \right| \varphi_j^{p-1} + \left(u^{\lambda_j} + u^{p-1+\lambda_j}\right) \varphi_j^p \, dz. \end{aligned} \tag{5.5}$$

Combining (5.3) with (5.4) and (5.5) we arrive at

$$\left(\int_{U_{j+1}} u^{\kappa\alpha} \, dz \right)^{\frac{1}{\gamma}} \leq C\lambda_j^p \int_{U_j} \frac{2^{jp}}{(S-s)^p} u^{p-1+\lambda_j} + \frac{R^p 2^{jp}}{T(S-s)^p} u^{1+\lambda_j} + u^{\lambda_j} \, dz, \tag{5.6}$$

where $\gamma = 1 + p/N$. We wish to iterate this inequality, but having multiple terms at the right-hand side is a problem. This is where the assumption (5.2) comes into play. Since $u \geq (R^p/T)^{1/(p-1)}$, we have

$$u^{\lambda_j} = \left(\frac{1}{u} \right)^{p-1} u^{p-1+\lambda_j} \leq \left(\frac{T}{R^p} \right)^{\frac{p-1}{p-1}} u^{p-1+\lambda_j} \leq \frac{1}{(S-s)^p} u^{p-1+\lambda_j}$$

and since $T/R^p \geq 1$, we have also

$$u^{1+\lambda_j} = \left(\frac{1}{u} \right)^{p-2} u^{p-1+\lambda_j} \leq \left(\frac{T}{R^p} \right)^{\frac{p-2}{p-1}} u^{p-1+\lambda_j} \leq \frac{T}{R^p} u^{p-1+\lambda_j}.$$

Using these estimates it follows from (5.6) that

$$\left(\int_{U_{j+1}} u^{\kappa\alpha} \, dz \right)^{\frac{1}{\gamma}} \leq \frac{C\lambda_j^p 2^{jp}}{(S-s)^p} \int_{U_j} u^{p-1+\lambda_j} \, dz. \tag{5.7}$$

Observe that

$$\kappa\alpha = p - 1 + \lambda_j(1 + p/N) + p/N = p - 1 + \lambda_{j+1}.$$

Hence by denoting $Y := C(S-s)^{-p}$, the inequality (5.7) becomes

$$\left(\int_{U_{j+1}} u^{p-1+\lambda_{j+1}} \, dz \right)^{\frac{1}{\gamma}} \leq Y (2\gamma)^{jp} \int_{U_j} u^{p-1+\lambda_j} \, dz.$$

We iterate this inequality. When $j = 0$, it reads as

$$\left(\int_{U_1} u^{p-1+\lambda_1} \, dz \right)^{\frac{1}{\gamma}} \leq Y \int_{U_0} u^p \, dz.$$

Then, when $j = 1$, we have

$$\left(\int_{U_2} u^{p-1+\lambda_2} \, dz \right)^{\frac{1}{\gamma^2}} \leq Y^{\frac{1}{\gamma}} (2\gamma)^{p\frac{1}{\gamma}} \left(\int_{U_1} u^{p-1+\lambda_1} \, dz \right)^{\frac{1}{\gamma}} \leq Y^{1+\frac{1}{\gamma}} (2\gamma)^{p\frac{1}{\gamma}} \int_{U_0} u^p \, dz.$$

Continuing this way we arrive at

$$\begin{aligned}\left(\int_{U_{j+1}} u^{p-1+\lambda_{j+1}} \, dz \right)^{\frac{1}{\gamma^{j+1}}} &\leq Y^{1+\frac{1}{\gamma}+\ldots+\frac{1}{\gamma^j}} (2\gamma)^{p(\frac{1}{\gamma}+\frac{2}{\gamma^2}+\ldots+\frac{j}{\gamma^j})} \int_{U_0} u^p \, dz \\ &\leq C Y^{\frac{N}{p}+1} \int_{U_0} u^p \, dz,\end{aligned}$$

so that

$$\left(\int_{U_{j+1}} u^{p-1+\lambda_{j+1}} \, dz \right)^{\frac{1}{p-1+\lambda_{j+1}}} \leq \left(C Y^{\frac{N}{p}+1} \int_{U_0} u^p \, dz \right)^{\frac{\gamma^{j+1}}{p-1+\lambda_{j+1}}}.$$

Since $\gamma^{j+1}/(p-1+\lambda_{j+1}) \to 1/2$ and $p-1+\lambda_{j+1} \to \infty$ as $j \to \infty$, we obtain that

$$\operatorname*{ess\,sup}_{Q(s)} u \le C\left((S-s)^{-N-p}\int_{Q(S)} u^p\,dz\right)^{1/2},$$

where $Q(s) = B(x_0,s)\times(t_0-(s/S)^pT,t_0)$. By Young's inequality we have for every $1<\delta<2$ that

$$\begin{aligned}\operatorname*{ess\,sup}_{Q(s)} u \le& \left(\operatorname*{ess\,sup}_{Q(S)} u^{2-\delta}(S-s)^{-N-p}\int_{Q(S)} u^{p-2+\delta}\,dz\right)^{1/2}\\ \le&\frac{1}{2}\operatorname*{ess\,sup}_{Q(S)} u + \left((S-s)^{-N-p}\int_{B_R(x_0)\times(t_0-T,t_0)} u^{p-2+\delta}\,dz\right)^{1/\delta}. \end{aligned} \tag{5.8}$$

A standard iteration argument such as [GG82, Lemma 1.1] now finishes the proof. Indeed, if $f:[T_0,T_1]\to\mathbb{R}$ is a non-negative bounded function such that all $T_0\le t\le \tau\le T_1$ satisfy

$$f(t)\le \theta f(\tau)+(\tau-t)^{-\eta}A, \tag{5.9}$$

where $A,\theta,\eta\ge 0$ with $\theta<1$, then

$$f(T_0)\le C(\eta,\theta)(T_1-T_0)^{-\eta}A.$$

Selecting $T_0 := \sigma R$, $T_1 := (\sigma R+R)/2$ and the other variables so that (5.8) implies (5.9), we get the desired estimate. □

Next we consider the case where the non-negative subsolution is not necessarily bounded away from zero. Observe that the estimate differs from the usual estimate for the p-Laplacian because of the power $1/(p-1)$ in the first term (cf. [DiB93, Theorem 4.1] or [Kuu09, Theorem 3.4]). However, we have the additional assumption (5.10).

Theorem 5.3. *Assume that $p\ge 2$ and that (G2) holds. Suppose that u is a non-negative weak subsolution to (1.1) in Ξ and $B_R(x_0)\times(t_0-T,t_0)\Subset\Xi$ with $R,T<1$ such that*

$$\frac{R^p}{T}\le 1. \tag{5.10}$$

Then there exists a constant $C=C(N,p,C_f,\delta)$ such that we have the estimate

$$\operatorname*{ess\,sup}_{B(x_0,R/2)\times(t_0-T/2^p,t_0)} u \le C\left(\frac{R^p}{T}\right)^{\frac{1}{p-1}\cdot\frac{\delta-1}{\delta}} + C\left(\frac{T}{R^p}\fint_{t_0-T}^{t_0}\fint_{B_R(x_0)} u^{p-2+\delta}\,dx\,dt\right)^{\frac{1}{\delta}}$$

for all $1<\delta<2$.

Proof. We denote

$$\Lambda := (1-\sigma)^{-N-p},\ \ \theta:=\left(\frac{R^p}{T}\right)^{\frac{1}{p-1}}.$$

Using Lemma 5.2 on the subsolution $v:=\theta+u$ we get the estimate

$$\begin{aligned}\operatorname*{ess\,sup}_{B_{\sigma R}(x_0)\times(t_0-\sigma^pT,t_0)} u \le& C\left(\Lambda\frac{T}{R^p}\fint_{B_R(x_0)\times(t_0-T,t_0)}(\theta+u)^{p-2+\delta}\,dz\right)^{\frac{1}{\delta}}\\ \le& C\Lambda^{\frac{1}{\delta}}\left(\frac{T}{R^p}\theta^{p-2+\delta}\right)^{\frac{1}{\delta}} + C\Lambda^{\frac{1}{\delta}}\left(\frac{T}{R^p}\fint_{B_R(x_0)\times(t_0-T,t_0)} u^{p-2+\delta}\,dz\right)^{\frac{1}{\delta}},\end{aligned}$$

where

$$\frac{T}{R^p}\theta^{p-2+\delta}=T^{1-\frac{p-2+\delta}{p-1}}R^{-p+\frac{p(p-2+\delta)}{p-1}}=\left(T^{1-\delta}R^{p(\delta-1)}\right)^{\frac{1}{p-1}}=\left(\frac{R^p}{T}\right)^{\frac{\delta-1}{p-1}}.$$

Taking $\sigma=1/2$ now yields the desired inequality. □

Lemma 5.4. *Assume that $p\geq 2$ and that $f(0)=0$. Let u be a weak subsolution to (1.1) in Ω_{t_1,t_2}. Then $u_+=\max(u,0)$ is also a weak subsolution.*

Proof. Fix a non-negative test function $\zeta\in C_0^\infty(\Omega_{t_1,t_2})$. We test the regularized equation in Lemma 3.1 with $\min\{k(u^\epsilon)_+,1\}\zeta$. Then by similar arguments as in the proof of Lemma 5.1 we get the estimate

$$\begin{aligned}\int_{\Omega_{t_1,t_2}}&\min\{ku_+,1\}\left(-u\partial_t\zeta+|Du|^{p-2}Du\cdot D\zeta-\zeta f(Du)\right)dz\\&\leq-\frac{1}{2k}\int_{\Omega_{t_1,t_2}}\left(\min\{ku_+,1\}\right)^2\partial_t\zeta\,dz-k\int_{\{0<ku<1\}}\zeta\,|Du|^p\,dz.\end{aligned}$$

Letting $k\to\infty$ this implies

$$\int_{\{u>0\}}-u\partial_t\zeta+|Du|^{p-2}Du\cdot D\zeta-\zeta f(Du)\,dz\leq 0.$$

Since $f(0)=0$ and $u_+\partial_t\zeta=0=Du_+$ a.e. in $\{u\leq 0\}$, we get that

$$\int_{\Omega_{t_1,t_2}}-u_+\partial_t\zeta+|Du_+|^{p-2}Du_+\cdot D\zeta-\zeta f(Du_+)\,dz\leq 0.$$ □

Theorem 5.5. *Assume that $p\geq 2$, (G2) holds and that $f(0)=0$. Suppose that u is a weak supersolution to (1.1) in Ξ. Let u_* denote the lower semicontinuous regularization of u, that is,*

$$u_*(x,t):=\operatorname*{ess\,lim\,inf}_{(y,s)\to(x,t)}u(y,s):=\lim_{R\to 0}\operatorname*{ess\,inf}_{B_R(x)\times(t-R^p,t+R^p)}u.$$

Then $u=u_*$ almost everywhere.

Proof. For all $M\in\mathbb{N}$, we define the cylinders

$$Q_R^M(x,t):=B_R(x)\times(t-MR^p,t+MR^p).$$

We denote by E_M the set of Lebesgue points with respect to the basis $\{Q_R^M\}$, that is,

$$E_M:=\left\{(x,t)\in\Xi:\lim_{R\to 0}⨍_{Q_R^M(x,t)}|u(x,t)-u(y,s)|^{p-\frac{1}{2}}\,dy\,ds=0\right\}.$$

Then $E_M\subset E_{M+1}$ so that

$$E:=\bigcap_{M\in\mathbb{N}}E_M=E_1.$$

Moreover, we have $|E|=|\Xi|$, which follows from [Ste93, p13] by a simple argument, see for example [EG15, p54].

We now claim that if $(x_0,t_0)\in E$, then

$$u(x_0,t_0)\leq\operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)}u(x,t).\tag{5.11}$$

We make the counter assumption

$$u(x_0,t_0)-\operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)}u(x,t)=\varepsilon>0.$$

Let R_0 be a radius such that

$$\left|\operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} u(x,t) - \operatorname*{ess\,inf}_{Q_R^1(x_0,t_0)} u\right| \le \varepsilon/2$$

for all $0 < R \le R_0$. For such R we have

$$u(x_0,t_0) - \operatorname*{ess\,inf}_{Q_R^1(x_0,t_0)} u \ge \varepsilon/2. \tag{5.12}$$

We set $v := (u(x_0,t_0) - u)_+$. Since $(x_0,t_0) \in E$, we find for any $M \in \mathbb{N}$ a radius $R_1 = R_1(M)$ such that

$$\fint_{Q_{R_1}^M(x_0,t_0)} v^{p-\frac{1}{2}}\,dx\,dt \le \fint_{Q_{R_1}^M(x_0,t_0)} |u(x_0,t_0) - u|^{p-\frac{1}{2}}\,dx\,dt \le \left(\frac{1}{M}\right)^2. \tag{5.13}$$

On the other hand, by Lemma 5.4 the function v is a weak subsolution to

$$\partial_t v + \Delta_p v - g(Dv) \le 0,$$

where $g(\xi) = -f(-\xi)$. Observe also that the cylinder $Q_{R_1}^M(x_0,t_0)$ satisfies the condition (5.10) since $R_1^p/(MR_1^p) \le 1$. Hence we may apply Theorem 5.3 with $\delta = 3/2$ and then use (5.13) to get

$$\begin{aligned}
\operatorname*{ess\,sup}_{Q_{(R_1)/2}^M(x_0,t_0)} v \le& C\left(\frac{R_1^p}{R_1^p M}\right)^{\frac{1}{3(p-1)}} + C\left(\frac{R_1^p M}{R_1^p}\fint_{Q_{R_1}^M(x_0,t_0)} v^{p-\frac{1}{2}}\,dx\,dt\right)^{\frac{2}{3}} \\
\le& \frac{C}{M^{3(p-1)}} + C\left(M\cdot\frac{1}{M^2}\right)^{\frac{2}{3}} \\
\le& C\left(\frac{1}{M}\right)^{\frac{1}{3}}.
\end{aligned}$$

Now we first fix M so large that $C/M^{\frac{1}{3}} \le \varepsilon/4$ and this will also fix R_1. Then we take $R \in (0,R_0]$ so small that $Q_R^1(x_0,t_0) \subset Q_{(R_1)/2}^M(x_0,t_0)$. Then (5.12) leads to a contradiction since

$$\varepsilon/4 \ge \operatorname*{ess\,sup}_{Q_{(R_1)/2}^M(x_0,t_0)} v \ge \operatorname*{ess\,sup}_{Q_R^1(x_0,t_0)} v \ge u(x_0,t_0) - \operatorname*{ess\,inf}_{Q_R^1(x_0,t_0)} u \ge \varepsilon/2.$$

Hence (5.11) holds and we have

$$u(x_0,t_0) \le \operatorname*{ess\,lim\,inf}_{(x,t)\to(x_0,t_0)} u(x,t) \le \lim_{R\to 0}\fint_{Q_R^1} u(x,t)\,dx\,dt = u(x_0,t_0).$$

Thus $u_* = u$ almost everywhere and it is easy to show that u_* is lower semicontinuous. □

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Jarkko Siltakoski, Department of Mathematics and Statistics, P.O.Box 35, FIN-40014, University of Jyväskylä, Finland

E-mail address: jarkko.j.m.siltakoski@student.jyu.fi

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Equivalence between radial solutions of different non-homogeneous p-Laplacian type equations

Jarkko Siltakoski

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EQUIVALENCE BETWEEN RADIAL SOLUTIONS OF DIFFERENT NON-HOMOGENEOUS p-LAPLACIAN TYPE EQUATIONS

JARKKO SILTAKOSKI

Abstract. We study radial viscosity solutions to the equation

$$-|Du|^{q-2}\Delta_p^N u = f(|x|) \quad \text{in } B_R \subset \mathbb{R}^N,$$

where $f \in C[0,R)$, $p,q \in (1,\infty)$ and $N \geq 2$. Our main result is that $u(x) = v(|x|)$ is a bounded viscosity supersolution if and only if v is a bounded weak supersolution to $-\kappa\Delta_q^d v = f$ in $(0,R)$, where $\kappa > 0$ and Δ_q^d is heuristically speaking the radial q-Laplacian in a fictitious dimension d. As a corollary we obtain the uniqueness of radial viscosity solutions. However, the full uniqueness of solutions remains an open problem.

1. Introduction

In this paper, we study radial viscosity solutions to the equation

$$-|Du|^{q-2}\Delta_p^N u = f(|x|) \quad \text{in } B_R, \tag{1.1}$$

where

$$\Delta_p^N u := \Delta u + \frac{(p-2)}{|Du|^2}\sum_{i,j=1}^{N} D_{ij}uD_iuD_ju$$

is the normalized p-Laplacian, $f \in C[0,R)$, $B_R \subset \mathbb{R}^N$, $N \geq 2$ and $p,q \in (1,\infty)$. The left-hand side of the equation (1.1) is the usual p-Laplacian when $q = p$ and the normalized p-Laplacian when $q = 2$. In particular, the equation (1.1) may be in a non-divergence form and therefore the use of viscosity solutions is appropriate. Since we are interested in radial solutions, it is natural to restrict to a ball at the origin and assume that the source term is radial.

Recently Parviainen and Vázquez [PV] proved that radial viscosity solutions to the parabolic equation $\partial_t u = |Du|^{q-2}\Delta_p^N u$ coincide with weak solutions of a one dimensional equation related to the usual radial q-Laplacian. The objective of the present work is to obtain a similar equivalence result for the equation (1.1) while also considering supersolutions. Since the one dimensional equation satisfies a comparison principle, we obtain the uniqueness of radial solutions to (1.1) as a corollary. To the best of our knowledge, this was previously known only for $f = 0$ or f with a constant sign [KMP12] and the full uniqueness remains an open problem.

Stated more precisely, our main result is that $u(x) := v(|x|)$ is a bounded viscosity supersolution to (1.1) if and only if v is a bounded weak supersolution to the

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one-dimensional equation

$$-\kappa\Delta_q^d v \geq f \quad \text{in } (0,R) \subset \mathbb{R}, \tag{1.2}$$

where

$$\Delta_q^d v := |v'|^{q-2}\left((q-1)v'' + \frac{d-1}{r}v'\right)$$

and κ and d are given in (1.4). Heuristically speaking, the operator Δ_q^d is the usual radial q-Laplacian in a fictitious dimension d. Indeed, we show that if d is an integer, then supersolutions to (1.2) coincide with radial supersolutions to the equation $-\Delta_q u \geq f(|x|)$ in $B_R \subset \mathbb{R}^d$. The precise definition of weak supersolutions to (1.2) uses certain weighted Sobolev spaces and is given in Section 2.

Let us illustrate the relationship between equations (1.1) and (1.2) by a few formal computations. Assume that $u : \mathbb{R}^N \to \mathbb{R}$ is a smooth function such that $u(x) = v(|x|)$ for some $v : [0,\infty) \to \mathbb{R}$. Then by a simple calculation, we have $Du(re_1) = e_1 v'(r)$ and $D^2u(re_1) = e_1 \otimes e_1 v''(r) + r^{-1}(I - e_1 \otimes e_1)v'(r)$ for $r > 0$. In particular we have $|Du(re_1)| = |v'(r)|$. Assuming that the gradient does not vanish, we obtain

$$\begin{aligned}\Delta_p^N u(re_1) &= \Delta u + \frac{p-2}{|Du(re_1)|^2}\sum_{i,j=1}^{N} D_{ij}uD_iuD_ju \\ &= (p-1)v''(r) + \frac{N-1}{r}v'(r).\end{aligned} \tag{1.3}$$

Denoting

$$\kappa := \frac{p-1}{q-1}, \quad d := \frac{(N-1)(q-1)}{p-1} + 1 \tag{1.4}$$

and multiplying (1.3) by $|Du(re_1)|^{q-2}$, it follows that

$$|Du(re_1)|^{q-2}\,\Delta_p^N u(re_1) = \kappa\,|v'(r)|^{q-2}\left((q-1)v''(r) + \frac{d-1}{r}v'(r)\right),$$

where the right-hand side equals $\kappa\Delta_q^d v(r)$. Thus at least formally there is an equivalence between the equations (1.1) and (1.2). However, to make this rigorous, we need to carefully exploit the exact definitions of viscosity and weak supersolutions.

To show that v is a weak supersolution to (1.2) whenever u is a viscosity supersolution to (1.1), we apply the method developed by Julin and Juutinen [JJ12]. The idea is to approximate u using its inf-convolution u_ε. Since u_ε is still radial, there is v_ε such that $u_\varepsilon(x) = v_\varepsilon(|x|)$. Using the pointwise properties of inf-convolution, we show that v_ε is a weak supersolution to (1.2). It then suffices to pass to the limit to see that v is also a weak supersolution.

To show that u is a viscosity supersolution to (1.1) whenever v is a weak supersolution to (1.2), we adapt a standard argument used for example in [JLM01]. Thriving for a contradiction, we assume that u is not a viscosity supersolution. Roughly speaking, this means that there exists a smooth function φ that touches u from below and (1.1) fails at the point of touching. We use φ to construct a new function ϕ that is a weak subsolution to (1.2) and touches v from below. This violates a comparison principle and produces the desired contradiction. To avoid technicalities that might occur should the gradient of φ vanish, we use an equivalent definition of viscosity supersolutions proposed by Birindelli and Demengel [BD04]. Extra care is also needed if the point of touching is the origin.

The equation (1.1) has received an increasing amount of attention in the last several years. For example, the $C^{1,\alpha}$ regularity of radial solutions to (1.1) was shown by Birindelli and Demengel [BD12]. Using a different technique Imbert and Silvestre [IS12] proved the $C^{1,\alpha}$ regularity of solutions to $|Du|^{q-2} F(D^2u) = f$ when $q > 2$. More recently Attouchi and Ruosteenoja [AR18] obtained $C^{1,\alpha}$ regularity results for any solution of (1.1) and also proved some $W^{2,2}$ estimates.

The equivalence of viscosity and weak solutions was first studied by Ishii [Ish95] in the case of linear equations. The equivalence of solutions for p-Laplace equation was first obtained by Manfredi, Lindqvist and Juutinen [JLM01], later in a different way by Julin and Juutinen [JJ12] and for the $p(x)$-Laplace equation by Juutinen, Lukkari and Parviainen [JLP10]. Recent papers on this matter include the works of Attouchi, Parviainen and Ruosteenoja [APR17] on the normalized p-Poisson problem where the equivalence was used to obtain $C^{1,\alpha}$ regularity of solutions, Medina and Ochoa [MO19] on a non-homogeneous p-Laplace equation, Siltakoski [Sil18] on a normalized $p(x)$-Laplace equation and Bieske and Freeman [BF] on the $p(x)$-Laplace equation in Carnot groups.

The paper is organized as follows. Section 2 contains the precise definitions of viscosity solutions and weak solutions in our context. In Section 3 we show that weak supersolutions to (1.2) are viscosity supersolutions to (1.1) and the converse is proved in Section 4. In Section 5 we consider the special case where d is an integer and finally the Appendix contains some properties of the weighted Sobolev spaces.

2. Preliminaries

2.1. **Viscosity solutions.** Let $\varphi, u : B_R \to \mathbb{R}$. We say that φ *touches* u *from below at* $x_0 \in B_R$ if $\varphi(x_0) = u(x_0)$ and $\varphi(x) < u(x)$ when $x \neq x_0$.

Definition 2.1. A bounded lower semicontinuous function $u : B_R \to \mathbb{R}$ is a *viscosity supersolution* to (1.1) in B_R if whenever $\varphi \in C^2$ touches u from below at x_0 and $D\varphi(x) \neq 0$ when $x \neq x_0$, then we have

$$\limsup_{x_0 \neq y \to x_0} \Big(- |D\varphi(y)|^{q-2} \Delta_p^N \varphi(y) \Big) - f(|x_0|) \geq 0.$$

A bounded upper semicontinuous function $u : B_R \to \mathbb{R}$ is *a viscosity subsolution* to (1.1) in B_R if whenever $\varphi \in C^2$ touches u from above at x_0 and $D\varphi(x) \neq 0$ when $x \neq x_0$, then we have

$$\liminf_{x_0 \neq y \to x_0} \Big(- |D\varphi(y)|^{q-2} \Delta_p^N \varphi(y) \Big) - f(|x_0|) \leq 0.$$

A function is a *viscosity solution* if it is both viscosity sub- and supersolution.

The limit procedure in Definition 2.1 is needed because of the discontinuity in the equation when $q \leq 2$. When $q > 2$ the equation is continuous and the limit procedure is unnecessary.

2.2. **Weak solutions.** In order to define weak solutions, we must first define the appropriate Sobolev spaces. The *weighted Lebesgue space* $L^q(r^{d-1}, (0, R))$ is defined as the set of all measurable functions $v : (0, R) \to \mathbb{R}$ such that the norm

$$\|v\|_{L^q(r^{d-1},(0,R))} := \Big(\int_0^R |v|^q \, r^{d-1} \, dr \Big)^{1/q}$$

is finite. We define the *weighted Sobolev space* $W^{1,q}(r^{d-1},(0,R))$ as the set of all functions $v \in L^q(r^{d-1},(0,R))$ whose distributional derivative v' is in $L^q(r^{d-1},(0,R))$. As usual, by distributional derivative we mean that v' satisfies

$$\int_0^R v'\varphi\,dr = -\int_0^R v\varphi'\,dr$$

for all $\varphi \in C_0^\infty(0,R)$. We equip $W^{1,q}(r^{d-1},(0,R))$ with the norm

$$\|v\|_{W^{1,q}(r^{d-1},(0,R))} := \Big(\int_0^R |v|^q\, r^{d-1}\,dr + \int_0^R |v'|^q r^{d-1}\,dr\Big)^{1/q}.$$

Then $W^{1,q}(r^{d-1},(0,R))$ is a separable Banach space, see e.g. [KO84] or [Kuf85]. Since $d>1$, it follows from Theorem 7.4 in [Kuf85] that the set

$$C^\infty[0,R] := \Big\{u_{|(0,R)} : u \in C^\infty(\mathbb{R})\Big\}$$

is dense in $W^{1,q}(r^{d-1},(0,R))$. For the benefit of the reader we have also included a proof in the appendix, see Theorem A.1. We point out that any $v \in W^{1,q}(r^{d-1},(0,R))$ has a representative that is continuous in $(0,R]$. Indeed, for any $\delta>0$ we have $\delta^{d-1} < r^{d-1} < R^{d-1}$ when $r \in (\delta,R)$ and consequently the restriction $v_{|(\delta,R)}$ is in the usual Sobolev space $W^{1,q}(\delta,R)$.

In addition to [Kuf85], weighted Sobolev spaces have been studied for example in [HKM06]. However, the weight $w:\mathbb{R}\to\mathbb{R}$, $w(x)=|x|^{d-1}$ is not necessarily q-admissible in the sense of [HKM06]. Indeed, in the one dimensional setting q-admissible weights coincide with Muckenhoupt's A_q-weights [JBK06]. Thus w is q-admissible if and only if $d-1<p-1$ which by (1.4) is equivalent to $p>N$.

With the weighted Sobolev spaces at hand, we can define weak solutions. Recall that formally the equation (1.2) reads as

$$-\kappa\,|v'|^{q-2}\Big((q-1)v'' + \frac{d-1}{r}v'\Big) = f \quad \text{in } (0,R),$$

where κ and d are the constants given in (1.4). If v is smooth and the gradient does not vanish, this can be equivalently written as

$$-\kappa\Big(|v'|^{q-2}\, v' r^{d-1}\Big)' - f r^{d-1} = 0 \quad \text{in } (0,R).$$

Definition 2.2. We say that v is a *weak supersolution* to (1.2) in $(0,R)$ if $v \in W^{1,q}(r^{d-1},(0,R'))$ for all $R' \in (0,R)$ and we have

$$\int_0^R \kappa\,|v'|^{q-2}\, v'\varphi' r^{d-1} - \varphi f r^{d-1}\,dr \geq 0 \tag{2.1}$$

for all non-negative $\varphi \in C_0^\infty(-R,R)$. For *weak subsolutions* the inequality (2.1) is reversed. Furthermore, $v \in C[0,R)$ is a *weak solution* if it is both weak sub- and supersolution.

Recall that our goal is to establish an equivalence between radial viscosity supersolutions of (1.1) and weak supersolutions of (1.2). For this reason the class of test functions in Definition 2.2 needs to be $C_0^\infty(-R,R)$ instead of $C_0^\infty(0,R)$, see the example below. We also point out that if d is an integer, then weak supersolutions in the sense of Definition 2.2 coincide with radial weak supersolutions to $\Delta_q u \geq f(|x|)$, where Δ_q is the usual q-Laplacian in d-dimensions, see Theorem 5.3.

Example 2.3. Let $p>N$, $f\equiv 0$, and define $v:(0,R)\to\mathbb{R}$ by

$$v(r) := \frac{1}{1-\alpha} r^{1-\alpha}, \quad \text{where } \alpha := \frac{N-1}{p-1}.$$

Then $v \in W^{1,q}(r^{d-1},(0,R))$ and it satisfies (2.1) for all non-negative $\varphi \in C_0^\infty(0,R)$, but $u(x) := v(|x|)$ is not a viscosity supersolution to (1.1). To verify this, observe first that v is in the correct Sobolev space. Indeed, the distributional derivative of v is $v'(r) = r^{-\alpha}$ and thus $v' \in L^q(r^{d-1},(0,R))$ since

$$-\alpha q + d - 1 = -\frac{N-1}{p-1}q + \frac{(N-1)(q-1)}{p-1} = -\frac{N-1}{p-1} > -1.$$

Moreover, for any non-negative $\varphi \in C_0^\infty(0,R)$, we have

$$\int_0^R \kappa \left|v'\right|^{q-2} v'\varphi' r^{d-1}\,dr = \int_0^R \kappa r^{-(q-1)\alpha}\varphi' r^{\frac{(N-1)(q-1)}{p-1}}\,dr = \int_0^R \kappa\varphi'\,dr = 0.$$

To see that the function $u(x) = v(|x|)$ is not a viscosity supersolution to (1.1), set $\phi(x) := (x_1-1)^2$. Then $u-\phi$ has a local minimum at 0 and $D\phi(0) \neq 0$, but

$$\begin{aligned}
&-\left|D\varphi(0)\right|^{q-2}\Delta_p^N\varphi(0)\\
&= -\left|2\right|^{q-2}\Big(\operatorname{tr}(2e_1\otimes e_1) + \frac{(p-2)}{2^2}(-2e_1)'(2e_1\otimes e_1)(-2e_1)\Big) < 0,
\end{aligned}$$

which means that u is not a supersolution.

Lemma 2.4. *We may extend the class of test functions in Definition 2.2 to $\varphi \in W^{1,q}(r^{d-1},(0,R))$ such that $\operatorname{spt}\varphi \subset [0,R')$ for some $R' \in (0,R)$.*

Proof. Take a cut-off function $\xi \in C_0^\infty(-R,R)$ such that $\xi \equiv 1$ in $[0,R']$. Take $\varphi_j \in C^\infty[0,R]$ such that $\varphi_j \to \varphi$ in $W^{1,q}(r^{d-1},(0,R))$. Set $\phi_j := \varphi_j\xi$. Then $\phi_j \in C_0^\infty(-R,R)$ and hence

$$\begin{aligned}
0 &\le \int_0^R \left|v'\right|^{q-2} v'\phi_j' r^{d-1} - f\phi_j r^{d-1}\,dr\\
&= \int_0^R \left|v'\right|^{q-2} v'\varphi_j'\xi r^{d-1} - f\varphi_j\xi r^{d-1}\,dr + \int_0^R \left|v'\right|^{q-2} v'\varphi_j\xi' r^{d-1}\,dr. \qquad (2.2)
\end{aligned}$$

Since $\xi \equiv 1$ in $\operatorname{spt}\varphi$, we have $\varphi'\xi = \varphi'$ and so

$$\begin{aligned}
&\int_0^R \left|v'\right|^{q-2} v'\varphi_j'\xi r^{d-1} - f\varphi_j\xi r^{d-1}\,dr\\
&= \int_0^R \left|v'\right|^{q-2} v'(\varphi_j'-\varphi')\xi r^{d-1} - f(\varphi_j-\varphi)\xi r^{d-1}\,dr\\
&\quad + \int_0^R \left|v'\right|^{q-2} v'\varphi' r^{d-1} - f\varphi r^{d-1}\,dr.
\end{aligned}$$

Combining this with (2.2), we get

$$\begin{aligned}
\int_0^R \left|v'\right|^{q-2} v'\varphi' r^{d-1} - f\varphi r^{d-1}\,dr &\ge -\int_0^R \left|v'\right|^{q-1}\left|\varphi_j'-\varphi'\right|\xi r^{d-1} + |f|\left|\varphi_j-\varphi\right|\xi r^{d-1}dr\\
&\quad -\int_0^R \left|v'\right|^{q-1}\left|\varphi_j\right|\left|\xi'\right| r^{d-1}\,dr.
\end{aligned}$$

The first integral at the right-hand side converges to zero by Hölder's inequality. Moreover, since $\varphi\xi' \equiv 0$ in $(0,R)$, we have

$$\int_0^R \left|v'\right|^{q-1}\left|\varphi_j\right|\left|\xi'\right| r^{d-1}\,dr = \int_0^R \left|v'\right|^{q-1}\left|\varphi_j-\varphi\right|\left|\xi'\right| r^{d-1}\,dr \to 0. \qquad \square$$

3. Weak solutions are viscosity solutions

We show that bounded weak supersolutions to (1.2) are radial viscosity supersolutions to (1.1). In order to formulate the precise statement, we recall that the lower semicontinuous reguralization of a function $v : (0,R) \to \mathbb{R}$ is defined by

$$v_*(r) := \operatorname*{ess\,lim\,inf}_{s\to r} v(s) := \lim_{S\to 0} \operatorname*{ess\,inf}_{s\in(r-S,r+S)\cap(0,R)} v(s)$$

for all $r \in [0,R]$. Observe that since any function $v \in W^{1,q}(r^{d-1},(0,R))$ admits a continuous representative, we have $v = v_*$ almost everywhere in $(0,R)$ for such v.

Theorem 3.1. *Assume that v is a bounded weak supersolution to (1.2) in $(0,R)$. Then $u(x) := v_*(|x|)$ is a viscosity supersolution to (1.1) in B_R.*

To prove Theorem 3.1, we use the following definition of viscosity supersolutions introduced by Birindelli and Demengel [BD04]. Its advantage is that we may restrict to test functions whose gradient does not vanish. It is shown in [AR18] that Definitions 2.1 and 3.2 are equivalent.

Definition 3.2. A bounded and lower semicontinuous function $u : B_R \to \mathbb{R}$ is a viscosity supersolution to (1.1) if for any $x_0 \in B_R$ one of the following conditions holds.

(i) The function u is not a constant in $B_\delta(x_0)$ for any $\delta > 0$, and whenever $\varphi \in C^2$ touches u from below at x_0 with $D\varphi(x_0) \neq 0$, we have

$$-\left|D\varphi(x_0)\right|^{q-2} \Delta_p^N \varphi(x_0) \geq f(|x_0|). \tag{3.1}$$

(ii) The function u is a constant in $B_\delta(x_0)$ for some $\delta > 0$, and we have

$$f(|x|) \leq 0 \quad \text{for all } x \in B_\delta(x_0).$$

Proof of Theorem 3.1. Let $x_0 \in B_R$. We first consider the case where u is a constant in $B_\delta(x_0)$ for some $\delta > 0$. In this case also v is a constant a.e. in $I := (0,R) \cap (|x_0| - \delta, |x_0| + \delta)$. This implies that $v' \equiv 0$ a.e. in I and thus, since v is a weak supersolution to (1.2), we have

$$\int_I \varphi f r^{d-1}\, dr \leq 0$$

for all non-negative $\varphi \in C_0^\infty(I)$. Since f is continuous, it follows that $f \leq 0$ in I and consequently $f(|x|) \leq 0$ in $B_\delta(x_0)$, as desired.

Assume then that u is not a constant near x_0. Suppose on the contrary that the condition (i) of Definition 3.2 fails at x_0, that is, there exists $\varphi \in C^2$ touching u from below at x_0 with $D\varphi(x_0) \neq 0$ and

$$f(|x_0|) > -\left|D\varphi(x_0)\right|^{q-2} \Delta_p^N \varphi(x_0). \tag{3.2}$$

We consider the case $x_0 \neq 0$ first and argue like in the proof of Proposition A.3 in [PV]. Let Q be an orthogonal matrix such that $x_0 = r_0 Q e_1$, where $r_0 := |x_0|$ and define $\psi(x) := \varphi(Qx)$. Then ψ touches u from below at $r_0 e_1$ and we have $D\psi(x) = Q' D\varphi(Qx)$ and $D^2\psi(x) = Q' D^2\varphi(Qx) Q$. From these and (3.2) it follows that

$$f(r_0) > -\left|D\psi(r_0 e_1)\right|^{q-2} \Delta_p^N \psi(r_0 e_1). \tag{3.3}$$

Since ψ touches the radial function u from below at $r_0 e_1 \neq 0$, we have $D_i\psi(r_0 e_1) = 0$ and $D_{ii}\psi(r_0 e_1) \leq \frac{1}{r_0} D_1\psi(r_0 e_1)$ for $1 < i \leq N$ (see Lemma 3.3 below). Thus by setting $\phi(r) := \psi(re_1)$, we obtain from (3.3)

$$\begin{aligned} f(r_0) &> -\left|D_1\psi(r_0e_1)\right|^{q-2}\Big(D_{11}\psi(r_0e_1) + \sum_{i=2}^{N} D_{ii}\psi(r_0e_1) + (p-2)D_{11}\psi(r_0e_1)\Big) \\ &\geq -\left|D_1\psi(r_0e_1)\right|^{q-2}\Big((p-1)D_{11}\psi(r_0e_1) + \frac{N-1}{r_0}D_1\psi(r_0e_1)\Big) \\ &= -\kappa\left|\phi'(r_0)\right|^{q-2}\Big((q-1)\phi''(r_0) + \frac{d-1}{r_0}\phi'(r_0)\Big), \end{aligned}$$

where we used that $\kappa = \frac{p-1}{q-1}$ and $d = \frac{(N-1)(q-1)}{p-1} + 1$. Since the above inequality is strict, by continuity it remains true in some interval $I \Subset (0, R)$ containing r_0. In other words, for any $r \in I$ it holds that

$$\begin{aligned} f(r)r^{d-1} &> -\kappa\left|\phi'(r)\right|^{q-2}\Big((q-1)\phi''(r) + \frac{d-1}{r}\phi'(r)\Big)r^{d-1} \\ &= -\kappa\Big(\left|\phi'(r)\right|^{q-2}\phi'(r)r^{d-1}\Big)'. \end{aligned}$$

Multiplying this by a non-negative function $\eta \in C_0^\infty(I)$ and integrating by parts we find that

$$\int_I \kappa\left|\phi'\right|^{q-2}\phi'\eta' r^{d-1} - \eta f r^{d-1}\,dr \leq 0. \tag{3.4}$$

We set

$$\overline{\phi}(r) := \phi(r) + l,$$

where $l := \min_{r\in\partial I}(v_*(r) - \phi(r)) > 0$. Then $\overline{\phi}$ still satisfies (3.4). Since $\overline{\phi} \leq v_*$ on ∂I, it follows from a comparison principle that $\overline{\phi} \leq v_*$ in I (see Lemma 3.5 below). But this is a contradiction since $\overline{\phi}(r_0) = v_*(r_0)$ and $l > 0$.

Consider then the case $x_0 = 0$. Denote $\xi := D\varphi(0)/\left|D\varphi(0)\right|$ and define a function $\phi : [0, R) \to \mathbb{R}$ by

$$\phi(r) := \varphi(r\xi).$$

Then for $r > 0$ we have

$$\phi'(r) = \xi \cdot D\varphi(r\xi) \quad \text{and} \quad \phi''(r) = \xi' D^2\varphi(r\xi)\xi.$$

Since $\xi \cdot D\varphi(0) = \left|D\varphi(0)\right| > 0$, it follows by continuity that there are constants $M, \delta > 0$ such that

$$\phi'(r) \geq M \quad \text{when } r \in (0, \delta). \tag{3.5}$$

Hence the quantity $(d-1)r^{-1}\phi'(r)$ is large when $r > 0$ is small. Therefore, since f and ϕ'' are bounded in $(0, R')$ for any $R' < R$, there exists $\delta > 0$ such that for all $r \in (0, \delta)$ we have

$$\begin{aligned} f(r) &\geq -\kappa\left|\phi'(r)\right|^{q-2}\Big((q-1)\phi''(r) + \frac{d-1}{r}\phi'(r)\Big) \\ &= -\kappa\Big(\left|\phi'(r)\right|^{q-2}\phi'(r)r^{d-1}\Big)' r^{1-d}. \end{aligned}$$

In other words, for all $r \in (0, \delta)$ it holds that

$$-\kappa\Big(\left|\phi'(r)\right|^{q-2}\phi'(r)r^{d-1}\Big)' - f(r)r^{d-1} \leq 0. \tag{3.6}$$

On the other hand, since ϕ' is bounded in $(0,\delta)$ we have $\phi \in W^{1,q}(r^{d-1},(0,\delta))$. Moreover, for any non-negative $\zeta \in C_0^\infty(-\delta,\delta)$ we obtain using integration by parts

$$\begin{aligned}&\int_0^\delta \kappa\left|\phi'\right|^{q-2}\phi'\zeta' r^{d-1} - \zeta f r^{d-1}\,dr\\&= \lim_{h\to 0}\int_h^\delta \kappa\left|\phi'\right|^{q-2}\phi'\zeta' r^{d-1} - \zeta f r^{d-1}\,dr\\&= \lim_{h\to 0}\left(\int_h^\delta -\kappa\Big(\left|\phi'\right|^{q-2}\phi' r^{d-1}\Big)'\zeta - \zeta f r^{d-1}\,dr - \kappa\left|\phi'(h)\right|^{q-2}\phi'(h)h^{d-1}\zeta(h)\right) \leq 0,\end{aligned}$$

where we used (3.6) and noticed that the last term converges to zero because $d-1>0$ and $\phi' \geq M > 0$ in $(0,\delta)$. Thus ϕ is a weak subsolution to (1.2) in $(0,\delta)$. We set

$$\overline{\phi}(r) := \phi(r) + l,$$

where $l := \phi(\delta) - v_*(\delta) > 0$. Then $\overline{\phi} \leq v_*$ in $(0,\delta)$ by Theorem 3.4. Hence it follows from continuity of $\overline{\phi}$ and definition of v_* that $\overline{\phi}(0) \leq v_*(0)$. But this is a contradiction since $\overline{\phi}(0) = \varphi(0) + l = v_*(0) + l$ and $l > 0$. ☐

We still need to prove the lemmas used in the previous proof: the comparison theorems and the following fact about the derivatives of test functions.

Lemma 3.3. *Let $u : B_R \to \mathbb{R}$ be radial. Assume that $\varphi \in C^2$ touches u from below at $re_1 \neq 0$. Then for $1 < i \leq N$ we have*

$$D_i\varphi(re_1) = 0 \quad \text{and} \quad D_{ii}\varphi(re_1) \leq \frac{1}{r}D_1\varphi(re_1).$$

Proof. Since $\varphi \in C^2$, we have

$$\varphi(y) = \varphi(re_1) + (y - re_1)\cdot D\varphi(re_1) + \frac{1}{2}(y-re_1)'D^2\varphi(re_1)(y-re_1) + o(|y-re_1|^2)$$

as $y \to re_1$. Letting $y = re_1 + he_i$, where $h > 0$ and $1 < i \leq N$, the above implies that

$$hD_i\varphi(re_1) + \frac{1}{2}h^2 D_{ii}\varphi(re_1) = \varphi(re_1 + he_i) - \varphi(re_1) + o(|h|^2) \text{ as } h \to 0. \tag{3.7}$$

Let now

$$S(h) := r - \sqrt{r^2 - h^2}$$

so that the vector $re_1 + he_i - S(h)e_1$ lies on the boundary of the ball $B_r(0)$. Since u is constant on $\partial B_r(0)$, the assumption that φ touches u from below at re_1 implies

$$\varphi(re_1) = u(re_1) = u(re_1 + he_i - S(h)e_1) \geq \varphi(re_1 + he_i - S(h)e_1).$$

Combining this with (3.7) we obtain

$$hD_i\varphi(re_1) + \frac{1}{2}h^2 D_{ii}\varphi(re_1) \leq \varphi(re_1 + he_i) - \varphi(re_1 + he_i - S(h)e_1) + o(|h|^2). \tag{3.8}$$

Since $\varphi \in C^2$, there is $M > 0$ such that for all $a, z \in B_1(re_1)$ we have the estimate

$$\varphi(a) - \varphi(z) \leq -(z-a)\cdot D\varphi(a) + M\left|z-a\right|^2.$$

Setting $a = re_1 + he_i$ and $z = re_1 + he_i - S(h)e_1$, the above and (3.8) lead to

$$\frac{1}{h}D_i\varphi(re_1) + \frac{1}{2}D_{ii}\varphi(re_1) \leq \frac{S(h)}{h^2}D_1\varphi(re_1 + he_i) + M\frac{\left|S(h)\right|^2}{h^2} + \frac{o(|h|^2)}{h^2}.$$

Observe that $\frac{S(h)}{h^2} \to \frac{1}{2r}$ as $h \to 0$, so the left hand side of the above inequality tends to $\frac{1}{2r} D_1\varphi(re_1)$. Thus we must have $D_i\varphi(re_1) \le 0$. On the other hand, repeating the previous arguments, but instead selecting $y = re_1 - he_i$ at the beginning, we can deduce the estimate

$$-\frac{1}{h} D_i\varphi(re_1) + \frac{1}{2} D_{ii}\varphi(re_1) \le \frac{S(h)}{h^2} D_1\varphi(re_1 - he_i) + M\frac{|S(h)|^2}{h^2} + \frac{o(|h|^2)}{h^2},$$

from which it follows that $D_i\varphi(re_1) \ge 0$. Thus $D_i\varphi(re_1) = 0$ and we may let $h \to 0$ to obtain that $D_{ii}(re_1) \le \frac{1}{r} D_1\varphi(re_1)$. ☐

Theorem 3.4 (Comparison principle)**.** *Let w and v respectively be bounded weak sub- and supersolutions to (1.2) in $(0, R)$. Assume that we have*

$$\limsup_{r\to R} w(r) \le \liminf_{r\to R} v(r)$$

Then $w \le v$ a.e. in $(0, R)$.

Proof. Let $\varepsilon > 0$. Then there is $0 < R' < R$ such that $w - v - \varepsilon < 0$ in (R', R). We set

$$\varphi := \max(w - v - \varepsilon, 0).$$

By Lemma A.4 we have $\varphi \in W^{1,q}(r^{d-1}, (0, R))$ with

$$\varphi' = \begin{cases} w' - v', & \text{a.e. in } \{w > v + \varepsilon\}, \\ 0, & \text{a.e. in } (0, R) \setminus \{w > v + \varepsilon\}. \end{cases}$$

By Lemma 2.4 we may use φ as a test function in (2.1) for w and v. This yields the inequalities

$$\int_{\{w>v+\varepsilon\}} \kappa\, |w'|^{q-2}\, w'(w' - v')r^{d-1}\, dr \le \int_0^R \varphi f r^{d-1}\, dr,$$
$$\int_{\{w>v+\varepsilon\}} \kappa\, |v'|^{q-2}\, v'(w' - v')r^{d-1}\, dr \ge \int_0^R \varphi f r^{d-1}\, dr.$$

Subtracting the second inequality from the first we get

$$\int_{\{w>v+\varepsilon\}} \kappa(|w'|^{q-2}\, w' - |v'|^{q-2}\, v')(w' - v')r^{d-1}\, dr \le 0.$$

Since $(|a|^{q-2}\, a - |b|^{q-2}\, b)\,(a - b) \ge 0$ for all $a, b \in \mathbb{R}$, it follows that $w' - v' \equiv 0$ in $\{w > v + \varepsilon\}$. Hence $\varphi' \equiv 0$ a.e. in $(0, R)$. This implies that also $\varphi \equiv 0$ a.e. in $(0, R)$ since we have $\varphi \in W^{1,q}_{loc}(0, R)$ and $\varphi \equiv 0$ in (R', R). Consequently $w \le v - \varepsilon$ a.e. in $(0, R)$ and letting $\varepsilon \to 0$ finishes the proof. ☐

Lemma 3.5. *Let v be a bounded weak supersolution to (1.2) in $(0, R)$. Let $I \Subset (0, R)$ be an interval and suppose that $\phi \in C^2(\overline{I})$ satisfies*

$$\int_I |\phi'|^{q-2}\, \phi'\varphi' r^{d-1} - \varphi f r^{d-1}\, dr \le 0 \tag{3.9}$$

for all $\varphi \in C_0^\infty(I)$. Assume also that for all $r_0 \in \partial I$ we have

$$\limsup_{r\to r_0} \phi(r) \le \liminf_{r\to r_0} v(r).$$

Then $\phi \le v$ a.e. in I.

Proof. Since $v \in W^{1,q}(r^{d-1},(0,R))$ we have $v_{|I} \in W^{1,q}(I)$ with $(v_{|I})' = v'$ in I. Moreover, we have

$$\int_0^R |(v_{|I})'|^{q-2}(v_{|I})'\varphi' r^{d-1} - \varphi f r^{d-1}\, dr \geq 0 \tag{3.10}$$

for all $\varphi \in C_0^\infty(I)$. For $\varepsilon > 0$, we set

$$\varphi := (\phi - v_{|I} - \varepsilon)_+.$$

Then $\varphi \in W^{1,q}(I)$ and $\operatorname{spt}\varphi \Subset I$. Thus we may after approximation use φ as a test function in (3.9) and (3.10). It then follows similarly as in the proof of Theorem 3.4 that $\varphi \equiv 0$ a.e. in I and letting $\varepsilon \to 0$ finishes the proof. □

4. Viscosity solutions are weak solutions

We show that bounded radial viscosity supersolutions to (1.1) are weak supersolutions to (1.2). More precisely, we prove the following theorem.

Theorem 4.1. *Let u be a bounded radial viscosity supersolution to (1.1) in B_R. Then $v(r) := u(re_1)$ is a weak supersolution to (1.2) in $(0,R)$.*

As a corollary of Theorem 4.1, we obtain the uniqueness of radial viscosity solutions to (1.1). We also have the following comparison result for radial super- and subsolutions. However, the full uniqueness and comparison principle still remain open as far as we know.

Lemma 4.2. *Let $h, u \in C(B_R)$ be bounded radial viscosity sub- and supersolutions to (1.1) in B_R, respectively. Assume that for all $x_0 \in \partial B_R$ it holds*

$$\limsup_{x\to x_0} h(x) \leq \liminf_{x\to x_0} u(x).$$

Then $h \leq u$ in B_R.

Proof. By Theorem 4.1, the functions $w(r) := h(re_1)$ and $v(r) := u(re_1)$ are weak sub- and supersolutions to (1.2) in $(0,R)$, respectively. Hence by Theorem 3.4 we have $w \leq v$ a.e. in $(0,R)$. It follows from continuity that $h \leq u$ in B_R. □

Corollary 4.3. *Let $u, h \in C(\overline{B_R})$ be radial viscosity solutions to (1.1) in B_R such that $u = h$ on ∂B_R. Then $u = h$.*

One way to prove that viscosity solutions are weak solutions is by using a comparison principle [JLM01]. As mentioned however, full comparison principle for the equation (1.1) is open and Lemma 4.2 is not *a priori* available. Therefore we use the method developed by Julin and Juutinen [JJ12]. The idea is to approximate a viscosity supersolution u by its inf-convolution

$$u_\varepsilon(x) := \inf_{y\in B_R}\left\{u(y) + \frac{|x-y|^{\hat{q}}}{\hat{q}\varepsilon^{\hat{q}-1}}\right\},$$

where $\varepsilon > 0$ and $\hat{q} > 2$ is a fixed constant so large that $q - 2 + (\hat{q}-2)/(\hat{q}-1) > 0$. Then $u_\varepsilon \to u$ pointwise in B_R and it is standard to show that u_ε is a viscosity supersolution to

$$-\left|Du_\varepsilon\right|^{q-2}\Delta_p^N u_\varepsilon \geq f_\varepsilon(|x|) \quad \text{in } B_{R_\varepsilon}, \tag{4.1}$$

where $f_\varepsilon(r) := \inf_{|r-s|\leq\rho(\varepsilon)} f(s)$, $R_\varepsilon := R - \rho(\varepsilon)$ and $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, u_ε is semi-concave by definition and thus twice differentiable almost everywhere by Alexandrov's theorem (see e.g. [EG15, p273]). Hence u_ε satisfies the equation (4.1) pointwise almost everywhere. Since u_ε is still radial, we can perform a radial transformation on (4.1) to obtain after mollification arguments that $v_\varepsilon(r) := u_\varepsilon(re_1)$ is a weak supersolution to $-\kappa\Delta_q^d v_\varepsilon = f_\varepsilon$ in $(0, R_\varepsilon)$. Caccioppoli's estimate then implies that v_ε converges to v in the weighted Sobolev space up to a subsequence and we obtain that v_ε is a weak supersolution.

Before beginning the proof of Theorem 4.1, we collect some well known properties of inf-convolution in the following lemma (see e.g. [CIL92, Kat15, JJ12]).

Lemma 4.4. *Assume that $u : B_R \to \mathbb{R}$ is bounded and lower semicontinuous. Then the inf-convolution u_ε has the following properties.*

(i) *We have $u_\varepsilon \le u$ and $u_\varepsilon \to u$ pointwise in B_R as $\varepsilon \to 0$.*
(ii) *There exists $\rho(\varepsilon) > 0$ such that*

$$u_\varepsilon(x) = \inf_{y \in B_{\rho(\varepsilon)}(x) \cap B_R} \left\{ u(y) + \frac{1}{q\varepsilon^{\hat{q}-1}} |x-y|^{\hat{q}} \right\}$$

and $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. In fact we can choose $\rho(\varepsilon) = \left(q\varepsilon^{\hat{q}-1} \operatorname{osc}_{B_R} u\right)^{\frac{1}{\hat{q}}}$.
(iii) *Denote $R_\varepsilon := R - \rho(\varepsilon)$. Then u_ε is semi-concave in B_{R_ε}. Moreover, for any $x \in B_{R_\varepsilon}$ there is $x_\varepsilon \in \overline{B_{\rho(\varepsilon)}(x)}$ such that $u_\varepsilon(x) = u(x_\varepsilon) + \frac{1}{\hat{q}\varepsilon^{\hat{q}-1}} |x - x_\varepsilon|^{\hat{q}}$.*
(iv) *If u_ε is twice differentiable at $x \in B_{R_\varepsilon}$, then*

$$\begin{aligned} Du_\varepsilon(x) =& (x - x_\varepsilon) \frac{|x - x_\varepsilon|^{\hat{q}-2}}{\varepsilon^{\hat{q}-1}}, \\ D^2 u_\varepsilon(x) \le& (\hat{q}-1) \frac{|x - x_\varepsilon|^{\hat{q}-2}}{\varepsilon^{\hat{q}-1}} I. \end{aligned} \tag{4.2}$$

Remark. Observe that if u is radial, then so is u_ε. Moreover, if we set $v(r) := u_\varepsilon(re_1)$ and assume that v is twice differentiable at $r \in (0, R_\varepsilon)$, then by (iv) of Lemma 4.4 we have

$$v_\varepsilon'(r) = (r - r_\varepsilon) \frac{|r - r_\varepsilon|^{\hat{q}-2}}{\varepsilon^{\hat{q}-1}}, \tag{4.3}$$

$$v_\varepsilon''(r) \le \frac{\hat{q}-1}{\varepsilon} |v_\varepsilon'(r)|^{\frac{\hat{q}-2}{\hat{q}-1}}, \tag{4.4}$$

where $r_\varepsilon \in (r - \rho(\varepsilon), r + \rho(\varepsilon))$.

Lemma 4.5. *Assume that u is a bounded viscosity supersolution to (1.1) in B_R. Then u_ε is a viscosity supersolution to (4.1) in B_{R_ε}.*

Proof. Suppose that $\varphi \in C^2$ touches u_ε from below at $x \in B_{R_\varepsilon}$ and that $D\varphi(y) \ne 0$ when $y \ne x$. Let x_ε be as in (iii) of Lemma 4.4. Then

$$\varphi(x) = u_\varepsilon(x) = u(x_\varepsilon) + \frac{1}{\hat{q}\varepsilon^{\hat{q}-1}} |x - x_\varepsilon|^{\hat{q}}, \tag{4.5}$$

$$\varphi(y) \le u_\varepsilon(y) \le u(z) + \frac{1}{\hat{q}\varepsilon^{\hat{q}-1}} |y - z|^{\hat{q}} \quad \text{for all } y, z \in B_R. \tag{4.6}$$

Set

$$\psi(z) = \varphi(z + x - x_\varepsilon) - \frac{|x - x_\varepsilon|^{\hat{q}}}{\hat{q}\varepsilon^{\hat{q}-1}}.$$

It follows from (4.5) and (4.6) that ψ touches u from below at x_ε. Therefore, since u is a viscosity supersolution to (1.1), we have

$$\begin{aligned} 0 &\le \limsup_{x\neq z\to x_\varepsilon} \Big(-|D\psi(z)|^{q-2}\,\Delta_p^N \psi(z)\Big) - f(|x_\varepsilon|) \\ &\le \limsup_{x\neq y\to x} \Big(-|D\varphi(y)|^{q-2}\,\Delta_p^N \varphi(y)\Big) - f_\varepsilon(|x|), \end{aligned}$$

where we used that $|x-x_\varepsilon|\le \rho(\varepsilon)$ and $f_\varepsilon(r)=\inf_{|r-s|\le\rho(\varepsilon)} f(s)$. Consequently u_ε is a viscosity supersolution to (4.1). □

Next we combine the previous lemma with the radial transformation of (4.1).

Lemma 4.6. *Assume that u is a bounded radial viscosity supersolution to (1.1) in B_R. Set $v_\varepsilon(r):=u_\varepsilon(re_1)$ and assume that v_ε is twice differentiable at $r\in(0,R_\varepsilon)$. Then, if $q>2$ or $v_\varepsilon'(r)\neq 0$, we have*

$$-\kappa\,|v_\varepsilon'(r)|^{q-2}\Big((q-1)v_\varepsilon''(r)+\frac{d-1}{r}v_\varepsilon'(r)\Big) - f_\varepsilon(r)\ge 0. \tag{4.7}$$

Moreover, if $1<q\le 2$ with $v_\varepsilon'(r)=0$, then we have $f_\varepsilon(r)\le 0$.

Proof. Consider first the case $q>2$ or $v_\varepsilon'(r)\neq 0$. Since u_ε is twice differentiable at re_1, it follows from the definition of viscosity supersolutions that

$$-|Du_\varepsilon(re_1)|^{q-2}\left(\Delta u_\varepsilon(re_1)+(p-2)\Delta_\infty^N u_\varepsilon(re_1)\right) - f_\varepsilon(r)\ge 0, \tag{4.8}$$

where

$$\Delta_\infty^N u_\varepsilon = |Du_\varepsilon|^{-2}\sum_{i,j=1}^N D_{ij}u_\varepsilon D_i u_\varepsilon D_j u_\varepsilon. \tag{4.9}$$

Moreover, we have

$$Du_\varepsilon(re_1)=e_1 v_\varepsilon'(r) \quad\text{and}\quad D^2u_\varepsilon(re_1)=e_1\otimes e_1 v_\varepsilon''(r)+\frac{1}{r}\Big(I-e_1\otimes e_1\Big)v_\varepsilon'(r).$$

It is now straightforward to compute that

$$\Delta u_\varepsilon(re_1)=\operatorname{tr}D^2u_\varepsilon(re_1)=v_\varepsilon''(r)+\frac{N-1}{r}v_\varepsilon'(r)$$

and using (4.9)

$$\Delta_\infty^N u_\varepsilon(re_1)=v_\varepsilon''(r).$$

Combining these with (4.8) and recalling that $d-1=(N-1)(q-1)/(p-1)$, $\kappa=(p-1)/(q-1)$, we obtain (4.7).

Consider then the case $1<q\le 2$ and $v_\varepsilon'(r)=0$. Denote $x:=re_1$. Then $Du(x)=0$ and so by (4.2) we have $x_\varepsilon=x$. Therefore by the definition of inf-convolution

$$u(y)+\frac{|x-y|^{\hat q}}{\hat q\varepsilon^{\hat q-1}}\ge u_\varepsilon(x)=u(x)\quad\text{for all } y\in B_R.$$

Rearranging the terms, we find that

$$\phi(y):=u(x)-\frac{|x-y|^{\hat q}}{\hat q\varepsilon^{\hat q-1}}\le u(y)\quad\text{for all } y\in B_R.$$

In other words, the function ϕ touches u from below at x. Since u is a viscosity supersolution and $D\phi(y)\neq 0$ when $y\neq x$, it follows that

$$\limsup_{y\to x,y\neq x}\Big(|D\phi(y)|^{q-2}\,\Delta_p^N\phi(y)-f(|y|)\Big)\ge 0.$$

This implies that $-f(|x|) = -f(r) \geq 0$ since $|D\phi(y)|^{q-2} \Delta_p^N \phi(y) \to 0$ as $y \to x$. Indeed, we have

$$\begin{aligned}|D\phi(y)|^{q-2} \left|\Delta_p^N \phi(y)\right| \leq & C(q, \hat{q}, \varepsilon) |y - x|^{(q-2)(\hat{q}-1)} (N + |p-2|) ||D^2\phi(y)|| \\ \leq & C(q, \hat{q}, p, N, \varepsilon) |y - x|^{(q-2)(\hat{q}-1)+\hat{q}-2},\end{aligned}$$

where $(q-2)(\hat{q}-1) + \hat{q} - 2 > 0$ by definition of $\hat{q}$. □

Next we show that the inf-convolution is a weak supersolution to $-\kappa\Delta_q^d u_\varepsilon = f_\varepsilon$ in $(0, R_\varepsilon)$. We consider the case $q > 2$ first.

Lemma 4.7. *Let $q > 2$. Assume that u is a bounded radial viscosity supersolution to (1.1) in B_R. Then the function $v_\varepsilon(r) := u_\varepsilon(re_1)$ is a weak supersolution to $-\kappa\Delta_q^d u = f_\varepsilon$ in $(0, R_\varepsilon)$.*

Proof. Since u_ε is semi-concave in B_{R_ε}, it is also locally Lipschitz continuous there [EG15, p267]. Consequently we have $v_\varepsilon \in W^{1,q}(r^{d-1}, (0, R'))$ for all $R' \in (0, R_\varepsilon)$ by Lemma A.2. Observe then that since $\phi(x) := u_\varepsilon(x) - C(\hat{q}, \varepsilon, u) |x|^2$ is concave in B_{R_ε}, it is twice differentiable almost everywhere by Alexandrov's theorem. Moreover, the proof of Alexandrov's theorem in [EG15, p273] establishes that we can approximate ϕ by smooth concave radial functions ϕ_j with the standard mollification. Therefore, by setting $u_{\varepsilon,j}(x) := \phi_j(x) + C(\hat{q}, \varepsilon, u) |x|^2$, the following pointwise limits hold almost everywhere in B_{R_ε}

$$u_{\varepsilon,j} \to u_\varepsilon, \quad Du_{\varepsilon,j} \to Du_\varepsilon \quad \text{and} \quad D^2u_{\varepsilon,j} \to D^2u_\varepsilon.$$

Thus, since u_ε is radial, setting $v_{\varepsilon,j}(r) := u_{\varepsilon,j}(re_1)$ we have

$$v_{\varepsilon,j} \to v_\varepsilon, \quad v'_{\varepsilon,j} \to v'_\varepsilon \quad \text{and} \quad v''_{\varepsilon,j} \to v''_\varepsilon$$

almost everywhere in $(0, R_\varepsilon)$. Since $v_{\varepsilon,j}$ is smooth and $q > 2$, a direct calculation yields for $r \in (0, R_\varepsilon)$

$$-\kappa|v'_{\varepsilon,j}|^{q-2}\Big((q-1)v''_{\varepsilon,j} + \frac{d-1}{r}v'_{\varepsilon,j}\Big)r^{d-1} = -\kappa(|v'_{\varepsilon,j}|^{q-2}v'_{\varepsilon,j}r^{d-1})'. \tag{4.10}$$

Fix a non-negative $\varphi \in C_0^\infty(-R_\varepsilon, R_\varepsilon)$. Then, integrating by parts we find for $h > 0$

$$\begin{aligned}&\int_h^R -\varphi\kappa(|v'_{\varepsilon,j}|^{q-2}v'_{\varepsilon,j}r^{d-1})'\,dr \\ &= \int_h^R \kappa|v'_{\varepsilon,j}|^{q-2}v'_{\varepsilon,j}r^{d-1}\varphi'\,dr + \varphi(h)\kappa|v'_{\varepsilon,j}(h)|^{q-2}v'_{\varepsilon,j}(h)h^{d-1}.\end{aligned}$$

Combining this with (4.10), letting $h \to 0$ and subtracting $\int_0^R \varphi f_\varepsilon r^{d-1}\,dr$ from both sides, we obtain

$$\begin{aligned}\int_0^R & -\kappa\varphi|v'_{\varepsilon,j}|^{q-2}\Big((q-1)v''_{\varepsilon,j} + \frac{d-1}{r}v'_{\varepsilon,j}\Big)r^{d-1} - \varphi f_\varepsilon r^{d-1}\,dr \\ &= \int_0^R \kappa|v'_{\varepsilon,j}|^{q-2}v'_{\varepsilon,j}\varphi' r^{d-1} - \varphi f_\varepsilon r^{d-1}\,dr.\end{aligned}$$

Since $v_{\varepsilon,j}$ is Lipschitz continuous, we have $M := \sup_j ||v'_{\varepsilon,j}||_{L^\infty(\operatorname{spt}\varphi)} < \infty$. Thus we may let $j \to \infty$ in the above inequality and apply the dominated convergence theorem at the right hand side to obtain

$$\liminf_{j\to\infty}\int_0^R -\kappa\varphi|v'_{\varepsilon,j}|^{q-2}\Big((q-1)v''_{\varepsilon,j}+\frac{d-1}{r}v'_{\varepsilon,j}\Big)r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr$$
$$\le\int_0^R \kappa|v'_\varepsilon|^{q-2}v'_\varepsilon\varphi' r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr. \tag{4.11}$$

It now suffices to show that the left-hand side is non-negative to finish the proof. Observe that $v''_{\varepsilon,j}\le C(\hat q,\varepsilon,u)$ since ϕ_j is concave. Thus

$$-|v'_{\varepsilon,j}|^{q-2}\Big((q-1)v''_{\varepsilon,j}+\frac{d-1}{r}v'_{\varepsilon,j}\Big)\ge -M^{q-2}\Big(C(q,\hat q,\varepsilon,u)+\frac{d-1}{r}M\Big).$$

Since $d-2>-1$, it follows from the above inequality that the integral at the left hand side of (4.11) has an integrable lower bound. Hence by Fatou's lemma

$$\liminf_{j\to\infty}\int_0^R -\kappa\varphi|v'_{\varepsilon,j}|^{q-2}\Big((q-1)v''_{\varepsilon,j}+\frac{d-1}{r}v'_{\varepsilon,j}\Big)r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr$$
$$\ge\int_0^R -\kappa\varphi|v'_\varepsilon|^{q-2}\Big((q-1)v''_\varepsilon+\frac{d-1}{r}v'_\varepsilon\Big)r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr\ge 0,$$

where the last inequality follows from Lemma 4.6. □

Next we consider the case $1<q\le 2$. We need an additional regularization step because of the singularity of $|Du|^{q-2}\,\Delta_p^N u$ at the points where the gradient vanishes.

Lemma 4.8. *Let $1<q\le 2$. Assume that u is a bounded radial viscosity supersolution to (1.1) in B_R. Then the function $v_\varepsilon(r):=u_\varepsilon(re_1)$ is a weak supersolution to $-\kappa\Delta_q^d u\ge f_\varepsilon$ in $(0,R_\varepsilon)$.*

Proof. **(Step 1)** We define the smooth semi-concave functions $v_{\varepsilon,j}$ exactly as in the proof of Lemma 4.7. Then again

$$v_{\varepsilon,j}\to v_\varepsilon,\quad v'_{\varepsilon,j}\to v'_\varepsilon\quad\text{and}\quad v''_{\varepsilon,j}\to v''_\varepsilon$$

almost everywhere in $(0,R_\varepsilon)$. Let $\delta>0$. We regularize the radial transformation of equation (1.1) by considering the following term

$$G_\delta(v):=-\kappa(|v'|^2+\delta)^{\frac{q-2}{2}}\bigg(\Big(1+(q-2)\frac{|v'|^2}{|v'|^2+\delta}\Big)v''+\frac{d-1}{r}v'\bigg).$$

Since $v_{\varepsilon,j}$ is smooth, a direct calculation yields for $r\in(0,R_\varepsilon)$

$$G_\delta(v_{\varepsilon,j})r^{d-1}=-\kappa\Big((|v'_{\varepsilon,j}|^2+\delta)^{\frac{q-2}{2}}v'_{\varepsilon,j}r^{d-1}\Big)'. \tag{4.12}$$

Fix a non-negative $\varphi\in C_0^\infty(-R_\varepsilon,R_\varepsilon)$. Then, integrating by parts we have for $h>0$

$$\int_h^R -\kappa\varphi\Big((|v'_{\varepsilon,j}|^2+\delta)^{\frac{q-2}{2}}v'_{\varepsilon,j}r^{d-1}\Big)'\,dr$$
$$=\int_h^R \kappa(|v'_{\varepsilon,j}|^2+\delta)^{\frac{q-2}{2}}v'_{\varepsilon,j}r^{d-1}\varphi'\,dr+\varphi(h)\kappa(|v'_{\varepsilon,j}(h)|^2+\delta)^{\frac{q-2}{2}}v'_{\varepsilon,j}(h)h^{d-1}.$$

Combining this with (4.12), letting $h\to 0$ and subtracting $\int_0^R\varphi f_\varepsilon r^{d-1}\,dr$ from both sides we obtain

$$\int_0^R\varphi G_\delta(v_{\varepsilon,j})r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr=\int_0^R\kappa(|v'_{\varepsilon,j}|^2+\delta)^{\frac{q-2}{2}}v'_{\varepsilon,j}\varphi' r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr.$$

This implies that

$$\begin{aligned}\liminf_{j\to\infty}&\int_0^R \varphi G_\delta(v_{\varepsilon,j})r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr\\ &\le \lim_{j\to\infty}\int_0^R \kappa(|v_{\varepsilon,j}'|^2+\delta)^{\frac{q-2}{2}}v_{\varepsilon,j}'\varphi' r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr. \qquad (4.13)\end{aligned}$$

We intend to apply Fatou's lemma at the left-hand side and the dominated convergence theorem at the right-hand side. Since v_ε is Lipschitz continuous, we have $M := \sup_j ||v_{\varepsilon,j}'||_{L^\infty(\operatorname{spt}\varphi)} < \infty$, which justifies the use of the dominated convergence theorem. Observe then that $v_{\varepsilon,j}'' \le C(\hat{q},\varepsilon,u)$ by semi-concavity. Hence

$$\begin{aligned}G_\delta(v_{\varepsilon,j}) =& -\kappa(|v_{\varepsilon,j}'|^2+\delta)^{\frac{q-2}{2}}\Big(\Big(1+(q-2)\frac{|v_{\varepsilon,j}|^2}{|v_{\varepsilon,j}|^2+\delta}\Big)v_{\varepsilon,j}''+\frac{d-1}{r}v_{\varepsilon,j}'\Big)\\ \ge& -\kappa\delta^{\frac{q-2}{2}}(C(q,\hat{q},\varepsilon,u)+\frac{d-1}{r}M).\end{aligned}$$

Since $d-2>-1$, it follows from the above estimate that the integrand at the left-hand side of (4.13) has an integrable lower bound independent of j. Thus

$$\int_0^R \varphi\,(G_\delta(v_\varepsilon)-f_\varepsilon)\,r^{d-1}\,dr \le \int_0^R \kappa(|v_\varepsilon'|^2+\delta)^{\frac{q-2}{2}}v_\varepsilon'\varphi' r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr. \qquad (4.14)$$

(Step 2) We let $\delta\to 0$ in the auxiliary inequality (4.14) and obtain

$$\begin{aligned}\liminf_{\delta\to 0}&\int_0^R \varphi(G_\delta(v_\varepsilon)-f_\varepsilon)r^{d-1}dr\\ &\le \lim_{\delta\to 0}\int_0^R \kappa(|v_\varepsilon'|^2+\delta)^{\frac{q-2}{2}}v_\varepsilon'\varphi' r^{d-1}-\varphi f_\varepsilon r^{d-1}dr\\ &=\int_0^R \kappa\,|v_\varepsilon'|^{q-2}\,v_\varepsilon'\varphi' r^{d-1}-\varphi f_\varepsilon r^{d-1}\,dr, \qquad (4.15)\end{aligned}$$

where the use of the dominated convergence theorem was justified since v_ε is Lipschitz continuous. It now suffices to show that the left-hand side of (4.15) is non-negative to finish the proof. By (4.4) we have

$$v_\varepsilon'' \le \frac{\hat{q}-1}{\varepsilon}\,|v_\varepsilon'|^{\frac{\hat{q}-2}{\hat{q}-1}} \qquad (4.16)$$

almost everywhere in $(0,R_\varepsilon)$. Hence, when $v_\varepsilon' \ne 0$, it holds that

$$\begin{aligned}G_\delta(v_\varepsilon) &= -\kappa(|v_\varepsilon'|^2+\delta)^{\frac{q-2}{2}}\Big(\Big(1+(q-2)\frac{|v_\varepsilon'|^2}{|v_\varepsilon'|^2+\delta}\Big)v_\varepsilon''+\frac{d-1}{r}v_\varepsilon'\Big)\\ &\ge -\kappa\,|v_\varepsilon'|^{q-2}\,(C(q,\hat{q},\varepsilon)\,|v_\varepsilon'|^{\frac{\hat{q}-2}{\hat{q}-1}}+\frac{d-1}{r}\,|v_\varepsilon'|)\\ &= -\kappa(C(q,\hat{q},\varepsilon)\,|v_\varepsilon'|^{q-2+\frac{\hat{q}-2}{\hat{q}-1}}+\frac{d-1}{r}\,|v_\varepsilon'|^{q-1}),\end{aligned}$$

where $q-2+\frac{\hat{q}-2}{\hat{q}-1}\ge 0$ by definition of $\hat{q}$. Moreover, when $v_\varepsilon'=0$, we have $G_\delta(v_\varepsilon)\ge 0$ directly by (4.16). Since v_ε is Lipschitz continuous in the support of φ, these estimates imply that the integrand at the left-hand side of (4.15) has an integrable lower bound independent of δ. Thus by Fatou's lemma

$$\begin{aligned}
&\liminf_{\delta\to 0}\int_0^R \varphi\left(G_\delta(v_\varepsilon)-f_\varepsilon\right)r^{d-1}\,dr\\
&\geq \int_0^R \liminf_{\delta\to 0}\varphi\left(G_\delta(v_\varepsilon)-f_\varepsilon\right)r^{d-1}\,dr\\
&=\int_{\{v_\varepsilon'\neq 0\}}\varphi\Big(-\kappa\left|v_\varepsilon'\right|^{q-2}\Big((q-1)v_\varepsilon''+\frac{d-1}{r}v_\varepsilon'\Big)-f_\varepsilon\Big)r^{d-1}\,dr\\
&\quad+\int_{\{v_\varepsilon'=0\}}\liminf_{\delta\to 0}\varphi(-\kappa\delta^{\frac{q-2}{2}}v_\varepsilon''-f_\varepsilon)r^{d-1}\,dr\\
&=:A_1+A_2.
\end{aligned}\tag{4.17}$$

It follows directly from Lemma 4.6 that $A_1\geq 0$. Moreover, if $r\in\{v_\varepsilon'=0\}$, then Lemma 4.6 implies that $f_\varepsilon(r)\leq 0$ and inequality (4.16) reads as $v_\varepsilon''(r)\leq 0$. Hence also $A_2\geq 0$. Combining (4.15) and (4.17) we have thus established the desired inequality. □

We use the following Caccioppoli's estimate to show that the sequence v_ε is bounded in the weighted Sobolev space.

Lemma 4.9 (Caccioppoli's estimate)**.** *Let v be a bounded weak supersolution to (1.2) in $(0,R)$. Suppose moreover that v is Lipschitz continuous in $(0,R')$ for any $R'\in(0,R)$. Then for any non-negative $\xi\in C_0^\infty(-R,R)$ we have*

$$\int_0^R \left|v'\right|^q\xi^q r^{d-1}\,dr\leq C\int_0^R\Big(\left|\xi'\right|^q+\xi^q\left|f\right|\Big)r^{d-1}\,dr,$$

where $C=C(\kappa,q,M)$ and $M=\left\|v\right\|_{L^\infty((0,R)\cap\operatorname{spt}\xi)}$.

Proof. Since $\xi\in C_0^\infty(-R,R)$, we can use $\varphi:=(M-v)\xi^q$ as a test function by Lemma 2.4. This yields

$$0\leq\int_0^R\kappa\left|v'\right|^{q-2}v'(-v'\xi^q+(M-v)q\xi'\xi^{q-1})r^{d-1}-(M-v)\xi^q f r^{d-1}\,dr.$$

Rearranging the terms and using that $(M-v)\leq 2M$, we obtain

$$\int_0^R\kappa\left|v'\right|^q\xi^q r^{d-1}\,dr\leq 2M\int_0^R q\left|v'\right|^{q-1}\left|\xi'\right|\xi^{q-1}r^{d-1}+\xi^q\left|f\right|r^{d-1}\,dr.\tag{4.18}$$

By Young's inequality, we have for any $\epsilon>0$

$$q\left|v'\right|^{q-1}\left|\xi'\right|\xi^{q-1}r^{d-1}\leq\epsilon\left|v'\right|^q\xi^q r^{d-1}+C(q,\epsilon)\left|\xi'\right|^q r^{d-1}.$$

Applying this to (4.18), taking small enough $\epsilon>0$ and absorbing the term with v' to the left-hand side, we obtain the desired estimate. Absorbing the term is justified as it is finite by the Lipschitz continuity of v. □

It now remains to use the Caccioppoli's estimate to obtain a subsequence of v_ε that converges to v in the weighted Sobolev space. Then we can pass to the limit to see that v is a weak supersolution to (1.2) in $(0,R)$.

Proof of Theorem 4.1. Set $v_\varepsilon(r):=u_\varepsilon(re_1)$ and let $0<R''<R$. We start by showing that $v_\varepsilon\to v$ in $W^{1,q}(r^{d-1},(0,R''))$. By assuming that ε is small enough, we find R' such that

$$R''<R'<R_\varepsilon<R.$$

Since by Lemmas 4.7 and 4.8 the function v_ε is a weak supersolution to $-\kappa\Delta_q^d v_\varepsilon\geq f_\varepsilon$ in $(0,R_\varepsilon)$, Lemma 4.9 implies that v_ε' is bounded in $L^q(r^{d-1},(0,R'))$. Thus by

Lemma A.3 we have $v \in W^{1,q}(r^{d-1},(0,R'))$ and $v'_\varepsilon \to v'$ weakly in $L^q(r^{d-1},(0,R'))$ up to a subsequence. We set

$$\varphi := (v - v_\varepsilon)\xi^q,$$

where $\xi \in C_0^\infty(-R',R')$ is a non-negative cut-off function such that $\xi \equiv 1$ in $(0,R'')$. Using φ as a test function in the weak formulation of $-\kappa\Delta_q^d v_\varepsilon \geq f_\varepsilon$ we obtain

$$0 \leq \int_0^{R'} \kappa|v'_\varepsilon|^{q-2}v'_\varepsilon\Big((v' - v'_\varepsilon)\xi^q + q\xi'\xi^{q-1}(v - v_\varepsilon)\Big)r^{d-1} - (v - v_\varepsilon)\xi^q f_\varepsilon r^{d-1}\,dr.$$

Rearranging the terms and adding $\int_0^{R'} \kappa|v'|^{q-2}v'(v' - v'_\varepsilon)\xi^q r^{d-1}\,dr$ to both sides of the inequality, we get

$$\begin{aligned}
&\int_0^{R'} \kappa(|v'|^{q-2}v' - |v'_\varepsilon|^{q-2}v'_\varepsilon)(v' - v'_\varepsilon)\xi^q r^{d-1}\,dr \\
&\leq \int_0^{R'} \kappa q|v'_\varepsilon|^{q-1}|v - v_\varepsilon||\xi'|\xi^{q-1}r^{d-1}\,dr \\
&\quad + \int_0^{R'} |v - v_\varepsilon|\,\xi^q\,|f_\varepsilon|\,r^{d-1}\,dr \\
&\quad + \int_0^{R'} \kappa|v'|^{q-2}v'(v' - v'_\varepsilon)\xi^q r^{d-1}\,dr \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Since v'_ε is bounded in $L^q(r^{d-1},(0,R'))$, $v_\varepsilon \to v$ in $L^q(r^{d-1},(0,R'))$ and $f \in L^\infty$, it follows from Hölder's inequality that $A_1, A_2 \to 0$ as $\varepsilon \to 0$. Moreover, since $v'_\varepsilon \to v'$ weakly in $L^q(r^{d-1},(0,R'))$, also A_3 converges to zero. We conclude that $v'_\varepsilon \to v'$ strongly in $L^q(r^{d-1},(0,R''))$ by applying Hölder's inequality and the following inequality (see [Lin17, p95-96])

$$(|a|^{q-2}\,a - |b|^{q-2}\,b)\,(a - b) \geq \begin{cases}(q-1)\,|a-b|^2\,(1 + |a|^2 + |b|^2)^{\frac{q-2}{2}}, & 1 < q < 2,\\ 2^{2-q}\,|a-b|^q, & q \geq 2.\end{cases}$$

Recall then that since v_ε is a weak supersolution to $-\kappa\Delta_q^d v_\varepsilon \geq f_\varepsilon$ in $(0,R_\varepsilon)$, any $\varphi \in C_0^\infty(-R'',R'')$ satisfies

$$\int_0^{R''} \kappa|v'_\varepsilon|^{q-2}v'_\varepsilon\varphi' r^{d-1} - \varphi f_\varepsilon r^{d-1}\,dr \geq 0.$$

Since $v'_\varepsilon \to v'$ strongly in $L^q(r^{d-1},(0,R''))$, we may let $\varepsilon \to 0$ in the above inequality. Since $R'' < R$ was arbitrary, the proof is finished. □

5. The case of integer d

We show that if d is an integer, then weak supersolutions to (1.2) coincide with radial weak supersolutions to $-\Delta_q u \geq f(|x|)$, where Δ_q is the usual q-Laplacian in d-dimensions. We begin by recalling the definition of weak supersolutions to the latter equation.

Definition 5.1. Let d be an integer and let $B_R \subset \mathbb{R}^d$ be a ball centered at the origin. A function $u \in W^{1,q}_{loc}(B_R)$ is a weak supersolution to $-\Delta_q u \geq f(|x|)$ in B_R if

$$\int_{B_R} |Du|^{q-2}\,Du \cdot D\varphi - \varphi f(|x|)\,dx \geq 0$$

for all non-negative $\varphi \in C_0^\infty(B_R)$.

We will use the following lemma which states that the weighted Sobolev space $W^{1,q}(r^{d-1},(0,R))$ can be identified with the space of radial Sobolev functions in d-dimensions. Similar results hold also for higher-order Sobolev spaces, see [dFdSM11].

Lemma 5.2. *Let d be an integer. Assume that $u : B_R \to \mathbb{R}$ is radial, i.e. $u(x) = v(|x|)$ for all $x \in B_R$. Then $u \in W^{1,q}(B_R)$ if and only if $v \in W^{1,q}(r^{d-1},(0,R))$. Moreover, we have*

$$Du(x) = \frac{x}{|x|} v'(|x|) \quad \textit{for a.e. } x \in B_R. \tag{5.1}$$

Proof. Suppose first that $v \in W^{1,q}(r^{d-1},(0,R))$. By Lemma A.1 there is a sequence $v_n \in C^\infty[0,R]$ such that $v_n \to v$ in $W^{1,q}(r^{d-1},(0,R))$. Setting $u_n(x) := v_n(|x|)$ we have $u_n \in W^{1,q}(B_R)$ by Lipschitz continuity and

$$Du_n(x) = \frac{x}{|x|} v_n'(|x|) \quad \text{for all } x \in B_R \setminus \{0\}\,. \tag{5.2}$$

We obtain using the formula (9) in [SS05, p280]

$$\begin{aligned}\int_{B_R} |u_n - u|^q \, dx &= \int_{\partial B_1} \int_0^R |u_n(rz) - u(rz)|^q \, r^{d-1} \, dr \, d\sigma(z) \\ &= \int_{\partial B_1} \int_0^R |v_n(|rz|) - v(|rz|)|^q \, r^{d-1} \, dr \, d\sigma(z) \\ &= \sigma(\partial B_1) \int_0^R |v_n(r) - v(r)|^q \, r^{d-1} \, dr,\end{aligned}$$

where σ is the spherical measure. Similarly, but now also using (5.2), we compute

$$\begin{aligned}\int_{B_R} \left| Du_n - \frac{x}{|x|} v'(|x|) \right|^q dx &= \int_{B_R} |v_n'(|x|) - v'(|x|)|^q \, dx \\ &= \sigma(\partial B_1) \int_0^R |v_n'(r) - v'(r)|^q \, r^{d-1} \, dr.\end{aligned}$$

Since $v_n \to v$ in $W^{1,q}(r^{d-1},(0,R))$, it follows from the last two displays that $u \in W^{1,q}(B_R)$ and that (5.1) holds.

Suppose then that $u \in W^{1,q}(B_R)$. Since u is radial, there exists a sequence of radial functions $u_n(x) = v_n(|x|)$ such that $u_n \in C^\infty(B_R)$ and $u_n \to u$ in $W^{1,q}(B_R)$. Now we have

$$\begin{aligned}\sigma(\partial B_1) \int_0^R |v_n(r) - v(r)|^q \, r^{d-1} \, dr &= \int_{\partial B_1} \int_0^R |v_n(|rz|) - v(|rz|)|^q \, r^{d-1} \, dr \, d\sigma(z) \\ &= \int_{B_R} |u_n(x) - u(x)|^q \, dx,\end{aligned}$$

which means that $v_n \to v$ in $L^q(r^{d-1},(0,R))$. Observe then that for all $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\sigma(\partial B_1) \int_0^R |v_n'(r) - v_m'(r)| \, r^{d-1} \, dr &= \int_{B_R} |v_n'(|x|) - v_m'(|x|)|^q \, dx \\ &= \int_{B_R} \left| \frac{x}{|x|} \cdot Du_n(x) - \frac{x}{|x|} \cdot Du_m(x) \right|^q dx \\ &\le \int_{B_R} |Du_n(x) - Du_m(x)|^q \, dx.\end{aligned}$$

In other words, v_n is Cauchy in $W^{1,q}(r^{d-1},(0,R))$ and thus converges to some function. This function has to be v since $v_n \to v$ in $L^q(r^{d-1},(0,R))$. Hence we have established that $v \in W^{1,q}(r^{d-1},(0,R))$. The formula (5.1) now follows from the first part of the proof. □

Theorem 5.3. *Let d be an integer. Then v is a radial weak supersolution to (1.2) in $(0,R)$ if and only if the function $u(x) := v(|x|)$ is a weak supersolution to $-\Delta_q u = f(|x|)$ in $B_R \subset \mathbb{R}^d$.*

Proof. Suppose first that v is a weak supersolution to (1.2) in $(0,R)$. By Lemma 5.2 we have at least $u \in W^{1,q}_{loc}(B_R)$. Let $\varphi \in C_0^\infty(B_R)$ be a non-negative test function. Then by [SS05, p280] and (5.1) we have

$$\begin{aligned}&\int_{B_R} |Du|^{q-2}\, Du \cdot D\varphi - \varphi f(|x|)\, dx\\ &= \int_{\partial B_1}\int_0^R |Du(rz)|^{q-2}\, Du(rz)\cdot D\varphi(rz) r^{d-1} - \varphi(rz) f(|rz|) r^{d-1}\, dr\, d\sigma(z)\\ &= \int_{\partial B_1}\int_0^R |v'(r)|^{q-2}\, v'(r) z \cdot D\varphi(rz) r^{d-1} - \varphi(rz) f(r) r^{d-1}\, dr\, d\sigma(z) \geq 0,\end{aligned}$$

where the last inequality follows from the assumption that v is a weak supersolution to (1.2) and that $\phi(r) := \varphi(rz)$, $z \in \partial B_1$, is an admissible test function in Definition 2.2. Thus u is a weak supersolution to $-\Delta_q u \geq f(|x|)$ in B_R.

Suppose then that u is a radial weak supersolution to $-\Delta_q u \geq f(|x|)$ in B_R. By Lemma 5.2 we have $v \in W^{1,q}(r^{d-1},(0,R'))$ for all $R' \in (0,R)$. Let $\phi \in C_0^\infty(-R,R)$ be a non-negative test function and set $\varphi(x) := \phi(|x|)$. Then φ is a Lipschitz continuous function that is compactly supported in B_R and therefore an admissible test function in Definition 5.1. Using formula (5.1) we obtain

$$\begin{aligned}0 &\geq \int_{B_R} |Du|^{q-2}\, Du \cdot D\varphi - \varphi f(|x|)\, dx\\ &= \int_{B_R} \left|\frac{x}{|x|} v'(|x|)\right|^{q-2} v'(|x|)\frac{x}{|x|}\cdot\frac{x}{|x|}\phi'(|x|) - \phi(|x|) f(|x|)\, dx\\ &= \int_{B_R} |v'(|x|)|^{q-2}\, v'(|x|)\phi'(|x|) - \phi(|x|) f(|x|)\, dx\\ &= \sigma(\partial B_1)\int_0^R |v'(r)|^{q-2}\, v'(r)\phi'(r) r^{d-1} - \phi(r) f(r) r^{d-1}\, dr,\end{aligned}$$

which means that v is a weak supersolution to (1.2) in $(0,R)$. □

Combining Theorems 3.1, 4.1 and 5.3 we get the following corollary.

Corollary 5.4. *Let d be an integer. Then $u(x) := v_*(|x|)$ is a bounded viscosity supersolution to (1.1) in $B_R \subset \mathbb{R}^N$ if and only if $w(x) := v(|x|)$ is a bounded weak supersolution to $-\Delta_q w = f(|x|)$ in $B_R \subset \mathbb{R}^d$.*

Remark 5.5. Let us conclude this section with a brief remark on the special case where (1.1) is simply the homogeneous p-Laplace equation ($q = p$ and $f \equiv 0$). Recall that p-superharmonic functions are defined as lower semicontinuous functions that satisfy a comparison principle with respect to the solutions of the p-Laplace equation [Lin86]. In particular, the so called fundamental solution

$$V(x) = \begin{cases} |x|^{\frac{p-N}{p-1}}, & p \neq N,\\ \log(|x|), & p = N,\end{cases}$$

is p-superharmonic. It is possible to show that if $u(x) := v(|x|)$ is a radial p-superharmonic function, then v satisfies a comparison principle with respect to weak solutions of (1.2). The converse is also true. If $v : [0, R) \to (-\infty, \infty]$, $v \not\equiv \infty$, is a lower semicontinuous function that satisfies a comparison principle with respect to weak solutions of (1.2), then u is p-superharmonic. However, for expository reasons we have decided to not discuss this further here.

Appendix A. Some properties of the weighted Sobolev space

In this section we collect some basic facts about the weighted Sobolev space $W^{1,q}(r^{d-1}, (0, R))$, where $d > 1$. In particular, we have the following theorem from [Kuf85] about the density of smooth functions.

Theorem A.1. *The set*

$$C^\infty[0, R] := \left\{ v_{|(0,R)} : v \in C^\infty(\mathbb{R}) \right\}$$

is dense in $W^{1,q}(r^{d-1}, (0, R))$.

Proof. Let $v \in W^{1,q}(r^{d-1}, (0, R))$. Take θ_1,$\theta_2 \in C^\infty(\mathbb{R})$ such that $0 \le \theta_i \le 1$, $\theta_1 + \theta_2 = 1$ in $[0, R]$ and $\operatorname{spt} \theta_1 \subset (-\infty, R')$, $\operatorname{spt} \theta_2 \subset (R'', \infty)$ for some $0 < R'' < R' < R$. Then we have

$$v = \theta_1 v + \theta_2 v.$$

Since $\theta_2 v$ vanishes near zero, we have $\theta_2 v \in W^{1,q}(0, R)$. Hence by [Bre11, Theorem 8.2] there exists a sequence of functions in $C^\infty[0, R]$ that converges to $\theta_2 v$ in $W^{1,q}(0, R)$ and thus also in $W^{1,q}(r^{d-1}, (0, R))$. Consequently it remains to approximate the function

$$w := \theta_1 v.$$

For $\lambda > 0$, we define the function $w_\lambda : (-\lambda, R) \to \mathbb{R}$ by setting

$$w_\lambda(r) := w(r + \lambda).$$

We show that $w_\lambda \to w$ in $W^{1,q}(r^{d-1}, (0, R))$ as $\lambda \to 0$. We start with the estimate

$$\begin{aligned}
&\int_0^R |w_\lambda' - w'|^q r^{d-1} \, dr \\
&= \int_0^R |w_\lambda' r^{\frac{d-1}{q}} - w_\lambda' \cdot (r + \lambda)^{\frac{d-1}{q}} + w_\lambda' \cdot (r + \lambda)^{\frac{d-1}{q}} - w' r^{\frac{d-1}{q}}|^q \, dr \\
&\le 2^{q-1} \Big(\int_0^R |w_\lambda'|^q |r^{\frac{d-1}{q}} - (r + \lambda)^{\frac{d-1}{q}}|^q \, dr + \int_0^R |w_\lambda' \cdot (r + \lambda)^{\frac{d-1}{q}} - w' r^{\frac{d-1}{q}}|^q \, dr \Big) \\
&=: 2^{q-1}(I_1 + I_2). \qquad \text{(A.1)}
\end{aligned}$$

To see that $I_1 \to 0$ as $\lambda \to 0$, fix $\varepsilon > 0$. Since $w' \in L^q(r^{d-1}, (0, R))$, we can take positive $\delta = \delta(\varepsilon) < 1$ such that

$$\int_0^{2\delta} |w'|^q \, r^{d-1} \, dr < \varepsilon. \qquad \text{(A.2)}$$

Then for all $0 < \lambda < \delta$ we have

$$\begin{aligned}
I_1 =& \int_0^R |w'(r+\lambda)|^q\,(r+\lambda)^{d-1}\left|1-\frac{r^{\frac{d-1}{q}}}{(r+\lambda)^{\frac{d-1}{q}}}\right|^q dr\\
\leq& \int_0^\delta |w'(r+\lambda)|^q\,(r+\lambda)^{d-1}\,dr + \int_\delta^R |w'(r+\lambda)|^q\,(r+\lambda)^{d-1}\left|1-\frac{r^{\frac{d-1}{q}}}{(r+\lambda)^{\frac{d-1}{q}}}\right|^q dr\\
=& \int_\lambda^{\delta+\lambda} |w'(r)|^q\,r^{d-1}\,dr + \int_{\delta+\lambda}^{R+\lambda} |w'(r)|^q\,r^{d-1}\left|1-\frac{(r-\lambda)^{\frac{d-1}{q}}}{r^{\frac{d-1}{q}}}\right|^q dr\\
\leq& \varepsilon + \int_\delta^{R+1} |w'(r)|^q\,r^{d-1}\left|1-\frac{(r-\lambda)^{\frac{d-1}{q}}}{r^{\frac{d-1}{q}}}\right|^q dr, \qquad \text{(A.3)}
\end{aligned}$$

where in the last estimate we used (A.2). Since the term

$$\left|1-\frac{(r-\lambda)^{\frac{d-1}{q}}}{r^{\frac{d-1}{q}}}\right|^q$$

is bounded by 1 and converges to zero as $\lambda \to 0$ for all $r > \delta$, it follows from Lebesgue's dominated convergence theorem that for small enough $\lambda = \lambda(\varepsilon) < \delta$ we have

$$\int_\delta^{R+1} |w'(r)|^q\,r^{d-1}\left|1-\frac{(r-\lambda)^{\frac{d-1}{q}}}{r^{\frac{d-1}{q}}}\right|^q dr < \varepsilon. \qquad \text{(A.4)}$$

It follows from (A.3) and (A.4) that $I_1 \to 0$ as $\lambda \to 0$. Observe then that

$$I_2 = \int_0^R |w'(r+\lambda)(r+\lambda)^{\frac{d-1}{q}} - w'(r)r^{\frac{d-1}{q}}|^q\,dr = \int_0^R |g(r+\lambda)-g(r)|^q\,dr, \qquad \text{(A.5)}$$

where $g(r) = w'(r)r^{\frac{d-1}{q}}$. Since $w' \in L^q(r^{d-1},(0,R))$, we have $g \in L^q(0,R)$. Thus g is q-mean continuous by [PKJF12, Theorem 3.3.3]. This means that the integral at the right-hand side of (A.5) converges to zero as $\lambda \to 0$ and so also $I_2 \to 0$. It now follows from (A.1) that $w'_\lambda \to w'$ in $L^q(r^{d-1},(0,R))$ and the convergence $w_\lambda \to w$ is seen in the same way. Consequently, for any $\varepsilon > 0$ we may take $\lambda_\varepsilon > 0$ such that

$$\|w_{\lambda_\varepsilon} - w\|_{W^{1,q}(r^{d-1},(0,R))} < \varepsilon. \qquad \text{(A.6)}$$

Observe now that $w_{\lambda_\varepsilon} \in W^{1,q}(-\mu,R)$ for some $\mu \in (0,\lambda_\varepsilon)$. Hence there is a function $\psi \in C^\infty[-\mu,R]$ such that

$$\|w_{\lambda_\varepsilon} - \psi\|_{W^{1,q}(-\mu,R)} < \varepsilon. \qquad \text{(A.7)}$$

Using (A.6) and (A.7) we obtain

$$\begin{aligned}
\|w-\psi\|_{W^{1,q}(r^{d-1},(0,R))} \leq& \|w_{\lambda_\varepsilon}-\psi\|_{W^{1,q}(r^{d-1},(0,R))} + \|w_{\lambda_\varepsilon}-w\|_{W^{1,q}(r^{d-1},(0,R))}\\
\leq& \Big(\int_0^R |w_{\lambda_\varepsilon}-\psi|^q r^{d-1} + |w'_{\lambda_\varepsilon}-\psi'|^q r^{d-1}\,dr\Big)^{1/q} + \varepsilon\\
\leq& R^{\frac{d-1}{q}}\Big(\int_0^R |w_{\lambda_\varepsilon}-\psi|^q + |w'_{\lambda_\varepsilon}-\psi'|^q\,dr\Big)^{1/q} + \varepsilon\\
\leq& R^{\frac{d-1}{q}}\,\|w_{\lambda_\varepsilon}-\psi\|_{W^{1,q}(-\mu,R)} + \varepsilon\\
\leq& \varepsilon(R^{\frac{d-1}{q}}+1).
\end{aligned}$$

Thus w can be approximated by functions in $C^\infty[0,R]$ and the proof is finished. □

Lemma A.2. *The usual Sobolev space* $W^{1,q}(0,R)$ *is contained in* $W^{1,q}(r^{d-1},(0,R))$.

Proof. If $v \in W^{1,q}(0,R)$, then v has a distributional derivative v'. The claim then follows from the inclusion $L^q(0,R) \subset L^q(r^{d-1},(0,R))$ which holds since

$$\int_0^R |v|^q \, r^{d-1} \, dr \leq \int_0^R |v|^q \, R^{d-1} \, dr. \qquad \square$$

Lemma A.3. *Let $v_n \in W^{1,q}(r^{d-1},(0,R))$ be a sequence such that*

$$v_n \to v \text{ weakly in } L^q(r^{d-1},(0,R))$$

and v_n' is bounded in $L^q(r^{d-1},(0,R))$. Then $v \in W^{1,q}(r^{d-1},(0,R))$ and

$$v_n' \to v' \text{ weakly in } L^q(r^{d-1},(0,R))$$

up to a subsequence.

Proof. Since v_n' is bounded in $L^q(r^{d-1},(0,R))$, there is $g \in L^q(r^{d-1},(0,R))$ such that $v_n' \to g$ in $L^q(r^{d-1},(0,R))$ weakly up to a subsequence (see e.g. [Yos80, p126]). Let $\varphi \in C_0^\infty(0,R)$. Then

$$\begin{aligned}\int_0^R g\varphi \, dr = \int_0^R g \frac{\varphi}{r^{d-1}} r^{d-1} \, dr &= \lim_{n\to\infty} \int_0^R v_n' \frac{\varphi}{r^{d-1}} r^{d-1} \, dr \\ &= \lim_{n\to\infty} \int_0^R v_n' \varphi \, dr \\ &= \lim_{n\to\infty} -\int_0^R v_n \varphi' \, dr \\ &= -\int_0^R v \varphi' \, dr.\end{aligned}$$

Hence $g \in L^q(r^{d-1},(0,R))$ is the distributional derivative of v, as desired. $\square$

Lemma A.4. *If $v \in W^{1,q}(r^{d-1},(0,R))$, then $v_+ := \max(v,0) \in W^{1,q}(r^{d-1},(0,R))$ with*

$$(v_+)' = \begin{cases} v' & \text{a.e. in } \{r \in (0,R) : v > 0\}, \\ 0 & \text{a.e. in } \{r \in (0,R) : v \leq 0\}. \end{cases}$$

Proof. Let $\varphi \in C_0^\infty(0,R)$ with $\operatorname{spt}\varphi \subset I$, where $I \Subset (0,R)$ is an interval. Since the restriction $v_{|I}$ is in the standard Sobolev space $W^{1,q}(I)$, we have also $(v_{|I})_+ \in W^{1,q}(I)$ and

$$((v_{|I})_+)' = \begin{cases} v' & \text{a.e. in } \{r \in I : v > 0\}, \\ 0 & \text{a.e. in } \{r \in I : v \leq 0\}. \end{cases}$$

Therefore

$$\int_0^R v_+ \varphi' \, dr = \int_I (v_{|I})_+ \varphi' \, dr = -\int_I ((v_{|I})_+)' \varphi \, dr = \int_0^R (v_+)' \varphi \, dr.$$

Since clearly $(v_+)' \in L^q(r^{d-1},(0,R))$, it follows that $v \in W^{1,q}(r^{d-1},(0,R))$. $\square$

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Jarkko Siltakoski, Department of Mathematics and Statistics, P.O.Box 35, FIN-40014, University of Jyväskylä, Finland

E-mail address: jarkko.j.m.siltakoski@student.jyu.fi