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Malliavin smoothness on the Lévy space with Hölder continuous or BV functionals

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Abstract

We consider Malliavin smoothness of random variables $f(X_1)$, where X is a pure jump Lévy process and the function f is either bounded and Hölder continuous or of bounded variation. We show that Malliavin differentiability and fractional differentiability of $f(X_1)$ depend both on the regularity of f and the Blumenthal-Getoor index of the Lévy measure.

Keywords: Lévy process, Malliavin calculus, interpolation
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1. Introduction

Consider a Lévy process Y and the according Malliavin Sobolev space $\mathbb{D}_{1,2}$ based on the Itô chaos decomposition on the Lévy space of square integrable random variables. We recall the space $\mathbb{D}_{1,2}$ in Section 2.1. We are interested in the ways that Malliavin differentiability of $f(Y_1)$ depends on the properties of f and the properties of Y.

The process Y consists of three components

$$Y_t = \gamma t + \sigma B_t + X_t,$$

where $\gamma, \sigma \in \mathbb{R}$, B is a standard Brownian motion and X is a pure jump process. For the Brownian motion we have that $f(B_1) \in \mathbb{D}_{1,2}$ if and only if $f \in W^{1,2}(\mathbb{R}; \mathbb{P}_{B_1})$ (see, for instance, Nualart [23, Exercise 1.2.8]). We also examine fractional differentiability which is determined by the real interpolation spaces $(L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ between $L_2(\mathbb{P})$ and $\mathbb{D}_{1,2}$ (see Section 2.2). The fractional smoothness of $f(B_1)$ means that f is in a weighted Besov space (see S. Geiss and Hujo [15], for example). In this paper we focus on the pure jump Lévy process with $\gamma = 0$ and $\sigma = 0$. We search for properties of the function f and the Lévy measure ν of X, which are related to the smoothness of $f(X_1)$. It

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turns out that Malliavin smoothness is in connection to the Blumenthal-Getoor index

$$\beta = \inf\{\xi \geq 0 : m_\xi < \infty\}, \quad \text{where} \quad m_\xi := \int_{\mathbb{R}} \left(|x|^\xi \wedge 1\right) \nu(\mathrm{d}x).$$

We show that the smaller the index β is, the higher smoothness of $f(X_1)$ we have for a given f which is Hölder continuous or of bounded variation.

So far little is known about the question for which f and for which ν one has $f(X_1) \in \mathbb{D}_{1,2}$ or $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$. The note [22] enlightens the case where $\nu(\mathbb{R}) < \infty$: Then

$$f(X_1) \in \mathbb{D}_{1,2} \quad \text{if and only if} \quad \mathbb{E}\left[f^2(X_1)\left(N((0,1] \times \mathbb{R}) + 1\right)\right] < \infty$$

and

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,2} \quad \text{if and only if} \quad \mathbb{E}\left[f^2(X_1)\left(N((0,1] \times \mathbb{R})^\theta + 1\right)\right] < \infty,$$

where N is the Poisson random measure associated with X (see Section 2).

A Lévy measure ν always satisfies the property $m_2 < \infty$, and from Solé, Utzet and Vives [26] we know that

$$\|f(X_1)\|^2_{\mathbb{D}_{1,2}} = \|f(X_1)\|^2_{L_2(\mathbb{P})} + \int_{\mathbb{R}} \mathbb{E}\left[(f(X_1 + x) - f(X_1))^2\right] \nu(\mathrm{d}x).$$

Since $m_2 < \infty$, it follows that $f(X_1) \in \mathbb{D}_{1,2}$ for any f that is Lipschitz continuous and bounded. On the other hand, if the Lévy measure ν is finite, then it is sufficient that f is bounded to have $f(X_1) \in \mathbb{D}_{1,2}$. In Section 3 we shall examine intermediate cases, namely that f is bounded and Hölder continuous, that is, in C_b^α. In Theorem 3 we prove that

$$f(X_1) \in \mathbb{D}_{1,2} \text{ for all } f \in C_b^\alpha \quad \text{if and only if} \quad m_{2\alpha} < \infty,$$

where the necessity of the condition $m_{2\alpha} < \infty$ holds under assumption (**A2**) given in Section 2.3. For fractional smoothness we obtain in Theorem 5 for $0 < \alpha \leq \theta < 1$, that

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \text{ for all } f \in C_b^\alpha \quad \text{if} \quad m_{2\alpha/\theta} < \infty,$$

and under assumption (**A3**), that

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \text{ for all } f \in C_b^\alpha \quad \text{only if} \quad m_{2\alpha/\theta+\varepsilon} < \infty$$

for all $\varepsilon > 0$. In Section 5.1 we see that if the process X is strictly stable and symmetric and $2\alpha/\theta$ is equal to the Blumenthal-Getoor index β, then $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$ for all $f \in C_b^\alpha$ eventhough $m_{2\alpha/\theta} = m_\beta = \infty$.

We also consider normalized functions of bounded variation (NBV, see Section 4). In Theorem 6 we prove that under assumptions (**A1**) and (**A2**) it holds that

$$f(X_1) \in \mathbb{D}_{1,2} \text{ for all } f \in NBV \quad \text{if and only if} \quad m_1 < \infty.$$

In [11, Section 4.2] it was shown that $\mathbb{1}_{(K,\infty)}(Y_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{1/2,\infty}$, when Y_1 has a bounded density. We obtain a sharper smoothness index for the pure jump process: Theorem 7 states that under assumption (**A1**) it holds that

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \text{ for all } f \in NBV \quad \text{if} \quad m_{1/\theta} < \infty,$$

and under assumption (**A3**) it holds that

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \text{ for all } f \in NBV \quad \text{only if} \quad m_{1/\theta+\varepsilon} < \infty$$

for all $\varepsilon > 0$. In Section 5.1 we see that if the process X is strictly stable and symmetric and $1/\theta = \beta$, then $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$ for all $f \in NBV$ eventhough $m_{1/\theta} = m_\beta = \infty$.

The method in Section 5 is based on a characterization of fractional smoothness which was introduced for the Brownian motion by S. Geiss and Hujo [15], and which we translate for jump processes in Lemma 9.

1.1. Motivation

Malliavin smoothness and fractional smoothness play a role for example in discrete approximation of stochastic integrals and in the investigation of properties of backward stochastic differential equations (BSDEs): Consider the orthogonal Galtchouk-Kunita-Watanabe decomposition of $f(Y_1)$, that is,

$$f(Y_1) = c + \int_0^1 \varphi_t \, \mathrm{d}Y_t + \mathcal{E}.$$

Then the convergence rate of the equidistant Riemann-approximation of the integral depends on the smoothness parameter of $f(Y_1)$. On the other hand, if $f(Y_1)$ admits fractional smoothness, then it is possible to adjust the discretization points to obtain the best possible convergence rate. (See Geiss et al. [11].) The L_p-variation of the solution of certain BSDEs depends on the Malliavin fractional smoothness of the terminal condition $f(Y_1)$. This was shown with more general terminal conditions for the Brownian motion by C. Geiss, S. Geiss and Gobet [10] and S. Geiss and Ylinen [17] and for $p = 2$ for general L_2-Lévy processes by C. Geiss and Steinicke [13].

2. Preliminaries

Consider a pure jump Lévy process $X = (X_t)_{t\geq 0}$ with càdlàg paths on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the completion of the sigma-algebra generated by X. The Lévy-Itô decomposition of a pure jump Lévy process is

$$X_t = \iint_{(0,t]\times\{|x|>1\}} xN(\mathrm{d}s, \mathrm{d}x) + \iint_{(0,t]\times\{0<|x|\leq 1\}} x\tilde{N}(\mathrm{d}s, \mathrm{d}x),$$

where N is a Poisson random measure on $\mathcal{B}([0,\infty) \times \mathbb{R})$ and $\tilde{N}(\mathrm{d}s, \mathrm{d}x) = N(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\nu(\mathrm{d}x)$ is the compensated Poisson random measure. The measure $\nu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ is the Lévy measure of X satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(\mathrm{d}x) < \infty$ and $\nu(B) = \mathbb{E}[N((0,1] \times B)]$.

2.1. Itô chaos decomposition and the Malliavin Sobolev space

Denote $\mathbb{R}_+ := [0,\infty)$. We consider the following measure $\mathbbm{m}\colon \mathcal{B}(\mathbb{R}_+\times\mathbb{R}) \to [0,\infty]$ defined as

$$\mathbbm{m}(A) := \int_A x^2 \mathrm{d}t\nu(\mathrm{d}x) = \mathbb{E}\left[\left(\int_A x\tilde{N}(\mathrm{d}t,\mathrm{d}x)\right)^2\right].$$

For $n = 1,2,\dots$ we write $L_2(\mathbbm{m}^{\otimes n}) := L_2\left((\mathbb{R}_+\times\mathbb{R})^n, \mathcal{B}(\mathbb{R}_+\times\mathbb{R})^{\otimes n}, \mathbbm{m}^{\otimes n}\right)$ and set $L_2(\mathbbm{m}^{\otimes 0}) := \mathbb{R}$. A function $f_n : (\mathbb{R}_+\times\mathbb{R})^n \to \mathbb{R}$ is said to be symmetric, if it coincides with its symmetrization $\tilde{f}_n$,

$$\tilde{f}_n((s_1,x_1),\dots,(s_n,x_n)) = \frac{1}{n!}\sum_\pi f_n\left((s_{\pi(1)},x_{\pi(1)}),\dots,(s_{\pi(n)},x_{\pi(n)})\right),$$

where the sum is taken over all permutations $\pi : \{1,\dots,n\} \to \{1,\dots,n\}$.

We consider Itô's multiple stochastic integral $I_n : L_2(\mathbbm{m}^{\otimes n}) \to L_2(\mathbb{P})$ of order n with respect to the measure $x\tilde{N}(\mathrm{d}t,\mathrm{d}x)$. According to [19, Theorem 2] it holds that

$$L_2(\mathbb{P}) = \mathbb{R} \oplus \bigoplus_{n=1}^\infty \{I_n(f_n) : f_n \in L_2(\mathbbm{m}^{\otimes n})\}.$$

The functions f_n in the representation $F = \sum_{n=0}^\infty I_n(f_n)$ in $L_2(\mathbb{P})$ are unique when they are chosen to be symmetric, which is always possible since $I_n(f_n) = I_n(\tilde{f}_n)$. Moreover, we have

$$\mathbb{E}\left[I_n(f_n)I_k(g_k)\right] = \begin{cases} 0, & \text{if } n \neq k \\ n!(\tilde{f}_n,\tilde{g}_n)_{L_2(\mathbbm{m}^{\otimes n})} & \text{if } n = k \end{cases}$$

and

$$\|F\|^2_{L_2(\mathbb{P})} = \sum_{n=0}^\infty n! \left\|\tilde{f}_n\right\|^2_{L_2(\mathbbm{m}^{\otimes n})}.$$

In this paper we focus on random variables of the form $f(X_1)$, where $f : \mathbb{R} \to \mathbb{R}$ is a Borel function. We will take advantage of the following lemma in Sections 3 and 5.

Lemma 1. *Let $f(X_1) = \sum_{n=0}^\infty I_n(f_n) \in L_2(\mathbb{P})$ and let $(\mathcal{F}_t)_{t\geq 0}$ be the augmented natural filtration of X. Then*

(a) there are functions $g_n \in L_2\left((x^2\nu(\mathrm{d}x))^{\otimes n}\right)$ such that

$$\tilde{f}_n((t_1,x_1),\dots,(t_n,x_n)) = g_n(x_1,\dots,x_n)\mathbb{1}_{[0,1]^{\times n}}(t_1,\dots,t_n)$$

for $\mathbbm{m}^{\otimes n}$-a.e. $((t_1,x_1),\dots,(t_n,x_n)) \in (\mathbb{R}_+\times\mathbb{R})^{\times n}$ and

(b) $\mathbb{E}\left[\mathbb{E}\left[f(X_1)|\mathcal{F}_t\right]^2\right] = \sum_{n=0}^\infty t^n n! \|\tilde{f}_n\|^2_{L_2(\mathbbm{m}^{\otimes n})}$.

Proof. (a) Follows from [3, Remark 6.7]. (b) By analogous argumentation to [23, Lemma 1.2.4] we see that $\mathbb{E}[f(X_1)|\mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(g_n \mathbb{1}_{[0,t]^{\times n}})$. The claim follows from $\|\tilde{f}_n\|_{L_2(\mathfrak{m}^{\otimes n})} = \|g_n\|_{L_2((x^2\nu(\mathrm{d}x))^{\otimes n})}$. □

We define the Malliavin Sobolev space using Itô's chaos decomposition (as [24, 8, 26, 27, 1, 12] and many others). We denote by $\mathbb{D}_{1,2}$ the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L_2(\mathbb{P})$ such that

$$\|F\|_{\mathbb{D}_{1,2}}^2 := \|F\|_{L_2(\mathbb{P})}^2 + \sum_{n=1}^{\infty} n n! \left\|\tilde{f}_n\right\|_{L_2(\mathfrak{m}^{\otimes n})}^2 = \sum_{n=0}^{\infty} (n+1)! \left\|\tilde{f}_n\right\|_{L_2(\mathfrak{m}^{\otimes n})}^2 < \infty.$$

Let us write $L_2(\mathfrak{m} \otimes \mathbb{P}) := L_2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}, \mathfrak{m} \otimes \mathbb{P})$. The Malliavin derivative $D : \mathbb{D}_{1,2} \to L_2(\mathfrak{m} \otimes \mathbb{P})$ is defined for $F \in \mathbb{D}_{1,2}$ by

$$D_{t,x}F = \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, (t,x))) \quad \text{in } L_2(\mathfrak{m} \otimes \mathbb{P}).$$

From [26, Proposition 5.4] we have in the canonical probability space that

$$\begin{aligned} &\|f(X_1)\|_{\mathbb{D}_{1,2}}^2 \\ &= \|f(X_1)\|_{L_2(\mathbb{P})}^2 + \int_{[0,1]\times\mathbb{R}\setminus\{0\}} \mathbb{E}\left[\left(\frac{f(X_1+x) - f(X_1)}{x}\right)^2\right] \mathfrak{m}(\mathrm{d}t, \mathrm{d}x) \\ &= \|f(X_1)\|_{L_2(\mathbb{P})}^2 + \int_{\mathbb{R}} \mathbb{E}\left[\left(f(X_1+x) - f(X_1)\right)^2\right] \nu(\mathrm{d}x), \end{aligned} \tag{1}$$

and when $f(X_1) \in \mathbb{D}_{1,2}$, then

$$D_{t,x} f(X_1) = \frac{f(X_1+x) - f(X_1)}{x} \mathbb{1}_{[0,1]\times\mathbb{R}\setminus\{0\}}(t,x) \quad \mathfrak{m} \otimes \mathbb{P}\text{-a.e.} \tag{2}$$

The result was converted to the general probability space in [14, Lemma 3.2].

For the Brownian motion B, the space $\mathbb{D}_{1,2}$ is defined in an analogous way by a chaos decomposition, but the property (1) can not be formulated (see [23]).

2.2. Interpolation and Malliavin fractional smoothness

The interpolation space $(A_0, A_1)_{\theta,q}$ is a Banach space, intermediate between two Banach spaces A_0 and A_1 which are a compatible couple, that is, they are continuously embedded into a Hausdorff topological vector space.

When (A_0, A_1) is a compatible couple, the *K-functional* of $a \in A_0 + A_1$ is the mapping $K(a, \cdot; A_0, A_1) : (0, \infty) \to [0, \infty)$ defined by

$$K(a,t;A_0,A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : \ a = a_0 + a_1, \ a_0 \in A_0, \ a_1 \in A_1\}.$$

Let $\theta \in (0,1)$ and $q \in [1,\infty]$. The *real interpolation space* $(A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1 := \{a_0 + a_1 : a_0 \in A_0, \, a_1 \in A_1\}$ such that the norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \begin{cases} \left[\int_0^{\infty} \left(t^{-\theta} K(a,t;A_0,A_1)\right)^q \frac{\mathrm{d}t}{t}\right]^{\frac{1}{q}}, & q \in [1,\infty) \\ \sup_{t>0} t^{-\theta} K(a,t;A_0,A_1), & q = \infty \end{cases}$$

is finite. If $A_1 \subseteq A_0$ with continuous embedding, then

$$A_1 \subseteq (A_0, A_1)_{\theta,q} \subseteq (A_0, A_1)_{\eta,p} \subseteq (A_0, A_1)_{\eta,q} \subseteq A_0 \tag{3}$$

for $0 < \eta < \theta < 1$ and $1 \le p \le q \le \infty$.

From the Reiteration Theorem we know that for $\eta, \theta \in (0,1)$ and $q \in [1,\infty]$ one has

$$(A_0, (A_0, A_1)_{\eta,\infty})_{\theta,q} = (A_0, A_1)_{\eta\theta,q} \tag{4}$$

with

$$\|a\|_{(A_0,A_1)_{\eta\theta,\infty}} \le \|a\|_{(A_0,(A_0,A_1)_{\eta,\infty})_{\theta,\infty}} \le 3\|a\|_{(A_0,A_1)_{\eta\theta,\infty}} \tag{5}$$

for all $a \in (A_0, A_1)_{\eta\theta,\infty} = (A_0, (A_0, A_1)_{\eta,\infty})_{\theta,\infty}$. In the literature the Reiteration Theorem is usually given in a more general context and the constants 1 and 3 in the norm equivalence (5) are not computed explicitely. Therefore we verify (5) in Lemma 16. For further properties of interpolation spaces, see for instance [4], [5] or [30].

We say that a random variable admits fractional smoothness of order (θ, q) if it belongs to the interpolation space

$$(L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q},$$

where $\theta \in (0,1)$ and $q \in [1,\infty]$.

2.3. Assumptions about a density

Some of the assertions in this paper rest on the following assumptions:

(**A1**) X_1 has a bounded density p_1.

(**A2**) X_1 has a density p_1 and there exist $a, b, c \in \mathbb{R}$ with $c > 0$ and $b - a > 0$ such that $p_1(x) \ge c$ for all $x \in [a,b]$.

(**A3**) There exist $t_0 \in (0,1)$ and $a, b, c \in \mathbb{R}$ with $c > 0$ and $b - a > 0$ such that for all $t \in [t_0, 1]$, the random variable X_t has a density p_t such that $p_t(x) \ge c$ for all $x \in [a,b]$.

Note that the conditions (**A1**), (**A2**) and (**A3**) are satisfied, for example, when the condition

$$\ell := \liminf_{|u|\to\infty} \frac{\int_{\mathbb{R}} \sin^2(ux)\nu(\mathrm{d}x)}{\log|u|} > \frac{1}{2}$$

of Hartman and Wintner [18] holds. We formulate the argumentation in a lemma as it will be used later.

Lemma 2. *Assume that $\ell > 1/2$. Then (**A1**), (**A2**) and (**A3**) are satisfied.*

Proof. By [18, Section 13, statement II], X_t has a bounded and continuous density for all $t > \frac{1}{2\ell}$. The conditions (**A1**) and (**A2**) follow immediately. Let us prove (**A3**). Let $r>0$. Due to stochastic continuity of Lévy processes, there is $t_0 \in \left(\frac{1}{2\ell},1\right)$ such that

$$\mathbb{P}(|X_{t-t_0}| \le r) \ge 1/2 \quad \text{for all } t \in [t_0,1].$$

Since $\ell > 1/2$, [25, Theorem 24.10] implies that either the support of X_{t_0} is a half line $[\kappa,\infty)$ (or $(-\infty,\kappa]$) for some $\kappa \in \mathbb{R}$, or the support of X_s is $\mathbb{R}$ for all $s>0$. The continuous density p_{t_0}, if supported on a half line, is strictly positive on the open half line (κ,∞) (or $(-\infty,\kappa)$) by [28, Chapter IV, Theorem 8.6]. If X_s has a bounded and continuous density supported on the whole real line for $\frac{1}{2\ell} < s < t_0$, then [28, Chapter IV, Theorem 8.6] implies that p_{t_0} is strictly positive. In any case p_{t_0} is continuous and strictly positive on at least a half line, so that we find $K \in \mathbb{R}$ and $c>0$ such that $p_{t_0}(x) \ge c$ for all $x \in [K-2r,K+2r]$. For any $x \in [K-r,K+r]$ and $t \in [t_0,1]$ it holds that

$$\begin{aligned} p_t(x) &= \int_{\mathbb{R}} p_{t_0}(x-y)\mathbb{P}_{X_{t-t_0}}(\mathrm{d}y) \ge \int_{[-r,r]} p_{t_0}(x-y)\mathbb{P}_{X_{t-t_0}}(\mathrm{d}y) \\ &\ge c\mathbb{P}(|X_{t-t_0}| \le r) \ge c/2. \end{aligned}$$

□

3. Hölder continuous functions and Malliavin smoothness

For $\alpha \in (0,1]$, the spaces $B(\mathbb{R})$, C^α and C_b^α are spaces of Borel measurable functions f such that

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|, \quad \|f\|_{C^\alpha} = \sup_{x\ne y} \tfrac{|f(x)-f(y)|}{|x-y|^\alpha} \quad \text{or} \quad \|f\|_{C_b^\alpha} = \|f\|_\infty + \|f\|_{C^\alpha},$$

respectively, is finite. We frequently use the notation $Lip := C_b^1$. Note that $(B(\mathbb{R}),\|\cdot\|_\infty)$ and $(C_b^\alpha,\|\cdot\|_{C_b^\alpha})$ are Banach spaces and $\|\cdot\|_{C^\alpha}$ is a seminorm. Recall the notation

$$m_{2\alpha} = \int_{\mathbb{R}} \left(|x|^{2\alpha} \wedge 1\right) \nu(\mathrm{d}x).$$

3.1. Smoothness of first order

Theorem 3. *Let $\alpha \in (0,1)$ and $A := [0,1] \times \{x : |x| > 1\}$ and assume that $f(X_1) \in L_2(\mathbb{P})$.*

(a) If $f \in C^\alpha$ and $\int_{\mathbb{R}} |x|^{2\alpha}\nu(\mathrm{d}x) < \infty$, then $f(X_1) \in \mathbb{D}_{1,2}$ and

$$\|f(X_1)\|_{\mathbb{D}_{1,2}}^2 \le \|f(X_1)\|_{L_2(\mathbb{P})}^2 + \|f\|_{C^\alpha}^2 \int_{\mathbb{R}} |x|^{2\alpha}\nu(\mathrm{d}x).$$

(b) If $f \in C^\alpha$, $m_{2\alpha} < \infty$ and $\mathbb{E}\left[f^2(X_1)N(A)\right] < \infty$, then $f(X_1) \in \mathbb{D}_{1,2}$ and

$$\begin{aligned} &\|f(X_1)\|_{\mathbb{D}_{1,2}}^2 \\ &\le \|f\|_{C^\alpha}^2 m_{2\alpha} + \mathbb{E}\left[f^2(X_1)N(A)\right] + \|f(X_1)\|_{L_2(\mathbb{P})}^2(1+\nu(\{|x|>1\})). \end{aligned}$$

(c) If $f \in C_b^\alpha$ and $m_{2\alpha} < \infty$, then $f(X_1) \in \mathbb{D}_{1,2}$ and

$$\|f(X_1)\|_{\mathbb{D}_{1,2}}^2 \le (1 + 4m_{2\alpha}) \|f\|_{C_b^\alpha}^2. \tag{6}$$

*(d) Assume that (**A2**) holds and choose $\ell \in \{0, 1, 2, \dots\}$ such that there exist $k \in \mathbb{Z}$ and $c > 0$ with $p_1(x) \ge c$ for all $x \in \left[k2^{-\ell}, (k+1)2^{-\ell}\right]$. Then for the function $g^{\alpha,\ell}(x) = \sum_{n=\ell}^{\infty} 2^{-\alpha n} d(2^n x, \mathbb{Z})$ from Lemma 4 it holds that $g^{\alpha,\ell} \in C_b^\alpha$, and*

$$g^{\alpha,\ell}(X_1) \in \mathbb{D}_{1,2} \quad \text{only if} \quad m_{2\alpha} < \infty.$$

Proof. (a) The claim follows from [26, Proposition 5.4] (see (1)) and the α-Hölder continuity.

(c) The claim follows from $\|f(X_1)\|_{L_2(\mathbb{P})}^2 \le \|f\|_{C_b^\alpha}^2$ and (1), since

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}\left[|f(X_1 + x) - f(X_1)|^2\right] \nu(\mathrm{d}x) \\
&\le \int_{\{|x|\le 1\}} \|f\|_{C^\alpha}^2 |x|^{2\alpha} \nu(\mathrm{d}x) + \int_{\{|x|>1\}} 4\|f\|_\infty^2 \nu(\mathrm{d}x) \\
&\le \|f\|_{C_b^\alpha}^2 \cdot 4 \int_{\mathbb{R}} \left(|x|^{2\alpha} \wedge 1\right) \nu(\mathrm{d}x).
\end{aligned}$$

(b) Consider the chaos expansion $f(X_1) = \sum_{n=0}^\infty I_n(f_n)$ and recall that

$$\|f(X_1)\|_{\mathbb{D}_{1,2}}^2 = \|f(X_1)\|_{L_2(\mathbb{P})}^2 + \sum_{n=1}^\infty n n! \left\|\tilde{f}_n\right\|_{L_2(\mathbb{m}^{\otimes n})}^2.$$

We show first that

$$\begin{aligned}
\sum_{n=1}^\infty n n! \left\|\tilde{f}_n\right\|_{L_2(\mathbb{m}^{\otimes n})}^2 = &\int_{[-1,1]} \mathbb{E}\left[|f(X_1 + x) - f(X_1)|^2\right] \nu(\mathrm{d}x) \\
&+ \sum_{n=1}^\infty n n! \left\|\tilde{f}_n \mathbb{1}_{(\mathbb{R}_+ \times \mathbb{R})^{\times(n-1)} \times A}\right\|_{L_2(\mathbb{m}^{\otimes n})}^2.
\end{aligned} \tag{7}$$

In fact, it holds that

$$\begin{aligned}
&\int_{\mathbb{R}_+ \times \mathbb{R}\setminus\{0\}} \mathbb{E}\left[\left|\frac{f(X_1 + x) - f(X_1)}{x} \mathbb{1}_{[0,1]\times\{0<|x|\le 1\}}(t, x)\right|^2\right] \mathbb{m}(\mathrm{d}t, \mathrm{d}x) \\
&= \int_{[-1,1]} \mathbb{E}\left[|f(X_1 + x) - f(X_1)|^2\right] \nu(\mathrm{d}x) \le \|f\|_{C^\alpha}^2 \int_{[-1,1]} |x|^{2\alpha} \nu(\mathrm{d}x) < \infty,
\end{aligned} \tag{8}$$

so that there is a chaos representation

$$\frac{f(X_1+x)-f(X_1)}{x}\mathbb{1}_{[0,1]\times\{0<|x|\leq 1\}}(t,x)=\sum_{n=0}^{\infty} I_n(h_{n+1}(\cdot,(t,x))) \quad \text{in } L_2(\mathbb{m}\otimes\mathbb{P})$$

where $h_{n+1}\in L_2(\mathbb{m}^{\otimes(n+1)})$ is symmetric in the first n pairs of variables (see [23, Lemma 1.3.1] or [24, Section 4]). Let $\varphi_k=-k\vee(f\wedge k)$ so that $\varphi_k\in C_b^{\alpha}$ and $\varphi_k(X_1)\in\mathbb{D}_{1,2}$ by (c). Consider the chaos expansion $\varphi_k(X_1)=\sum_{n=0}^{\infty} I_n(f_n^{(k)})$. Then $\tilde{f}_n^{(k)}\to\tilde{f}_n$ in $L_2(\mathbb{m}^{\otimes n})$, since $\varphi_k(X_1)\to f(X_1)$ in $L_2(\mathbb{P})$. It also holds that

$$\int_{[0,1]\times\{0<|x|\leq 1\}}\mathbb{E}\left[\left|\frac{\varphi_k(X_1+x)-\varphi_k(X_1)}{x}-\frac{f(X_1+x)-f(X_1)}{x}\right|^2\right]\mathbb{m}(\mathrm{d}t,\mathrm{d}x)$$

converges to 0 as $k\to\infty$ by dominated convergence, since $|\varphi_k(X_1+x)-\varphi_k(X_1)|\leq|f(X_1+x)-f(X_1)|$. From (2) we have that

$$\frac{\varphi_k(X_1+x)-\varphi_k(X_1)}{x}\mathbb{1}_{[0,1]\times\mathbb{R}\setminus\{0\}}(t,x)=D_{t,x}\varphi_k(X_1)=\sum_{n=1}^{\infty} nI_{n-1}(\tilde{f}_n^{(k)}(\cdot,(t,x)),$$

in $L_2(\mathbb{m}\otimes\mathbb{P})$, which gives

$$\begin{aligned}h_n&=\lim_{k\to\infty} n\tilde{f}_n^{(k)}\mathbb{1}_{(\mathbb{R}_+\times\mathbb{R})^{\times(n-1)}\times([0,1]\times\{0<|x|\leq 1\})}\\&=n\tilde{f}_n\mathbb{1}_{(\mathbb{R}_+\times\mathbb{R})^{\times(n-1)}\times([0,1]\times\{0<|x|\leq 1\})}\end{aligned}$$

in $L_2(\mathbb{m}^{\otimes n})$ for $n=1,2,\ldots$ Therefore

$$\begin{aligned}&\frac{f(X_1+x)-f(X_1)}{x}\mathbb{1}_{[0,1]\times\{0<|x|\leq 1\}}(t,x)\\&=\sum_{n=1}^{\infty} nI_{n-1}(\tilde{f}_n(\cdot,(t,x))\mathbb{1}_{[0,1]\times\{0<|x|\leq 1\}}(t,x))\end{aligned}$$

in $L_2(\mathbb{m}\otimes\mathbb{P})$. This together with Lemma 1(a) proves equation (7). For the second term on the right hand side of (7) we have by [22, Proposition 3.4] that

$$\sum_{n=1}^{\infty} nn!\left\|\tilde{f}_n\mathbb{1}_{(\mathbb{R}_+\times\mathbb{R})^{\times(n-1)}\times A}\right\|^2_{L_2(\mathbb{m}^{\otimes n})}\leq\mathbb{E}\left[f^2(X_1)N(A)\right]+\mathbb{E}[f^2(X_1)]\mathbb{E}[N(A)].$$

Thus, from (7), (8) and the above inequality we get that

$$\sum_{n=1}^{\infty} nn!\left\|\tilde{f}_n\right\|^2_{L_2(\mathbb{m}^{\otimes n})}\leq\|f\|^2_{C^{\alpha}}m_{2\alpha}+\mathbb{E}\left[f^2(X_1)N(A)\right]+\mathbb{E}[f^2(X_1)]\mathbb{E}[N(A)].$$

Noting that $\mathbb{E}[N(A)]=\nu(\{|x|>1\})$, we obtain the claim.

(d) We have $g^{\alpha,\ell} \in C_b^\alpha$ by Lemma 4 below. If $g^{\alpha,\ell}(X_1) \in \mathbb{D}_{1,2}$, then by (1) and Lemma 4 it holds that

$$\begin{aligned}\infty &> \int_{\mathbb{R}} \mathbb{E}\left[\left(g^{\alpha,\ell}(X_1+x) - g^{\alpha,\ell}(X_1)\right)^2\right]\nu(\mathrm{d}x) \\ &\geq \int_{|x|\leq 2^{-\ell-3}} \left[c\int_{k2^{-\ell}}^{(k+1)2^{-\ell}} (g(y+x)-g(y))^2\,\mathrm{d}y\right]\nu(\mathrm{d}x) \\ &\geq c2^{-\ell}2^{8\alpha-10}\int_{|x|\leq 2^{-\ell-3}} |x|^{2\alpha}\nu(\mathrm{d}x).\end{aligned}$$

Hence it must be $m_{2\alpha} < \infty$. □

The idea for the construction of the function $g^{\alpha,\ell}$ below is based on the decomposition of Ciesielski [7].

Lemma 4. *Let $\ell \in \{0,1,2\ldots\}$ and $g^{\alpha,\ell}(x) = \sum_{n=\ell}^{\infty} 2^{-\alpha n} g_n(x)$, where*

$$g_n(x) = d(2^n x, \mathbb{Z}) = \inf\{|2^n x - z| : z \in \mathbb{Z}\}.$$

Then $g^{\alpha,\ell} \in C_b^\alpha$, and for all $k \in \mathbb{Z}$ and $|x| \leq 2^{-\ell-3}$ it holds that

$$\int_{k2^{-\ell}}^{(k+1)2^{-\ell}} \left[g^{\alpha,\ell}(y+x) - g^{\alpha,\ell}(y)\right]^2 \mathrm{d}y \geq 2^{-\ell}2^{8\alpha-10}|x|^{2\alpha}.$$

Proof. Since $|g_n(x)| \leq 1/2$ for all $x \in \mathbb{R}$, it is clear that $\|g^{\alpha,\ell}\|_\infty < \infty$. Since we also have that $|g_n(x) - g_n(y)| \leq 2^n|x-y|$ for all $x, y \in \mathbb{R}$, we get for any $m \geq \ell$ and $2^{-m-1} \leq |x-y| \leq 2^{-m}$, that

$$\begin{aligned}|g^{\alpha,\ell}(x) - g^{\alpha,\ell}(y)| &\leq \sum_{n=\ell}^{\infty} 2^{-\alpha n}|g_n(x) - g_n(y)| \\ &\leq \sum_{n=0}^{m} 2^{-\alpha n}2^n 2^{-m} + \sum_{n=m+1}^{\infty} 2^{-\alpha n} \\ &\leq \frac{2(2^{-m-1})^\alpha}{2^{1-\alpha}-1} + \frac{(2^{-m-1})^\alpha}{1-2^{-\alpha}} \\ &\leq \left(\frac{1}{(2^{1-\alpha}-1)(1-2^{-\alpha})}\right)|x-y|^\alpha.\end{aligned}$$

Thus $g^{\alpha,\ell} \in C_b^\alpha$.

The function g_m is periodic with period length 2^{-n} for all $m \geq n$, so that via dominated convergence we get that

$$\int_{k2^{-\ell}}^{(k+1)2^{-\ell}} \left[g^{\alpha,\ell}(y+x) - g^{\alpha,\ell}(y)\right]^2 \mathrm{d}y$$
$$= \sum_{n=\ell}^{\infty} 2^{n-\ell-2\alpha n} \int_0^{2^{-n}} \left[g_n(y+x) - g_n(y)\right]^2 \mathrm{d}y$$
$$+ 2 \sum_{m>n\geq\ell} 2^{n-\ell-\alpha(n+m)} \int_0^{2^{-n}} \left[g_n(y+x) - g_n(y)\right]\left[g_m(y+x) - g_m(y)\right] \mathrm{d}y.$$

Let $m > n \geq \ell$. Since g_m is periodic with period length 2^{-n-1} and

$$g_n(y+x) - g_n(y) = -\left(g_n(y + 2^{-n-1} + x) - g_n(y + 2^{-n-1})\right)$$

for all $x, y \in \mathbb{R}$, we have that

$$\int_0^{2^{-n}} \left[g_n(y+x) - g_n(y)\right]\left[g_m(y+x) - g_m(y)\right] \mathrm{d}y$$
$$= \int_0^{2^{-n-1}} \left[g_n(y+x) - g_n(y)\right]\left[g_m(y+x) - g_m(y)\right] \mathrm{d}y$$
$$+ \int_{2^{-n-1}}^{2^{-n}} \left[g_n(y+x) - g_n(y)\right]\left[g_m(y+x) - g_m(y)\right] \mathrm{d}y$$
$$= 0.$$

Let $0 < |x| \leq 2^{-\ell-3}$ and $m \geq \ell$ such that $2^{-m-4} < |x| \leq 2^{-m-3}$. Since $|g_m(y+x) - g_m(y)| = 2^m|x|$ when both $y + x \in \left(0, 2^{-m-1}\right)$ and $y \in \left(0, 2^{-m-1}\right)$, we obtain that

$$\int_0^{2^{-m}} \left[g_m(y+x) - g_m(y)\right]^2 \mathrm{d}y \geq \int_{2^{-m-3}}^{3\cdot 2^{-m-3}} \left[2^m|x|\right]^2 \mathrm{d}y = 2^{m-2}x^2.$$

Since $2^{m-2}x^2 \geq 2^{m-2}(2^{-m-4})^{2-2\alpha}|x|^{2\alpha} = 2^{-m+2\alpha m+8\alpha-10}|x|^{2\alpha}$, we get

$$\sum_{n=\ell}^{\infty} 2^{n-\ell-2\alpha n} \int_0^{2^{-n}} \left[g_n(y+x) - g_n(y)\right]^2 \mathrm{d}y \geq 2^{m-\ell-2\alpha m} 2^{-m+2\alpha m+8\alpha-10}$$
$$\geq 2^{-\ell} 2^{8\alpha-10}|x|^{2\alpha}.$$

□

Remark 1. The function $g^{\alpha,\ell}$ in Theorem 3(d) and Lemma 4 is irregular on the whole real line. If a C_b^α-function is "more smooth", then Theorem 3(d) does not necessarily give the best condition: Take for example $f(x) = |x|^\alpha \wedge 1$, which is C_b^α but not $C_b^{\alpha'}$ for any $\alpha' > \alpha$, and assume that (**A1**) holds. Then for

$0 < |x| \le 1$ we have that

$$
\begin{aligned}
&\mathbb{E}\left[\left(|X_1+x|^\alpha \wedge 1 - |X_1|^\alpha \wedge 1\right)^2\right] \\
&\le \|p_1\|_\infty \int_{-2}^{2} \left(|y+x|^\alpha - |y|^\alpha\right)^2 \mathrm{d}y \\
&= \|p_1\|_\infty |x|^{2\alpha+1} \int_{-\frac{2}{|x|}}^{\frac{2}{|x|}} \left(\left|z+\tfrac{x}{|x|}\right|^\alpha - |z|^\alpha\right)^2 \mathrm{d}z \\
&\le \|p_1\|_\infty |x|^{2\alpha+1} \left[\int_{|z|<2} 1\mathrm{d}z + \alpha^2 \int_{2\le|z|\le\frac{2}{|x|}} (|z|-1)^{2\alpha-2}\mathrm{d}z\right] \\
&\le \begin{cases} \|p_1\|_\infty |x|^{2\alpha+1}\left[4+\frac{2\alpha^2}{1-2\alpha}\right], & \text{for } \alpha<\frac{1}{2} \\ \|p_1\|_\infty |x|^{2}\left[4+2\log\frac{2}{|x|}\right], & \text{for } \alpha=\frac{1}{2} \\ \|p_1\|_\infty |x|^{2\alpha+1}\left[4+\frac{2^{2\alpha}\alpha^2}{2\alpha-1}|x|^{1-2\alpha}\right], & \text{for } \alpha>\frac{1}{2} \end{cases}.
\end{aligned}
$$

Since $\mathbb{E}\left[\left(|X_1+x|^\alpha \wedge 1 - |X_1|^\alpha \wedge 1\right)^2\right] \le 1$, we get from (1) that $|X_1|^\alpha \wedge 1 \in \mathbb{D}_{1,2}$, if one of the following three conditions holds: 1. $0 < \alpha < 1/2$ and $m_{2\alpha+1} < \infty$, 2. $\alpha = 1/2$ and $\int_{\{0<|x|\le 1\}} x^2 \log(1/|x|)\nu(\mathrm{d}x) < \infty$ or 3. $\alpha > 1/2$. Note that for the Brownian motion B we have $|B_1|^\alpha \wedge 1 \in \mathbb{D}_{1,2}$ if and only if $\alpha > 1/2$. This can be easily seen using [23, Example 1.2.8].

3.2. Fractional smoothness

To find fractional smoothness for $f(X_1)$ with $f \in C_b^\alpha$ in Corollary 5 below, we take advantage of the fact that $C_b^\alpha = (B(\mathbb{R}), Lip)_{\alpha,\infty}$ with

$$\|\cdot\|_{C_b^\alpha} \le 3\|\cdot\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}} \le 6\|\cdot\|_{C_b^\alpha} \tag{9}$$

(see Lemma 17 and also [30, Theorem 2.7.2/1] in a slightly different setting).

Theorem 5. *Let $0 < \alpha \le \theta < 1$.*

(a) If $f \in C_b^\alpha$ and $m_{2\alpha/\theta} < \infty$, then

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$$

and

$$\|f(X_1)\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,\infty}} \le 18\sqrt{1+4m_{2\alpha/\theta}}\|f\|_{C_b^\alpha}.$$

*(b) Assume that (**A3**) holds and choose $t_0 \in (0,1)$ and $\ell \in \{0,1,2,\ldots\}$ such that there exist $k \in \mathbb{Z}$ and $c > 0$ with $p_t(x) \ge c$ for all $t \in [t_0,1]$ and all $x \in [(k-1)2^{-\ell},(k+2)2^{-\ell}]$. For the function $g^{\alpha,\ell} \in C_b^\alpha$ of Lemma 4 it holds that*

$$g^{\alpha,\ell}(X_1) \in (L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,\infty} \quad \textit{only if} \quad m_{2\alpha/\theta+\varepsilon} < \infty \textit{ for all } \varepsilon > 0.$$

Proof. (a) One finds for every $t > 0$ and $\varepsilon > 0$ a function $f_t \in C_b^{\alpha/\theta}$ such that

$$\left(\|f - f_t\|_\infty + t\|f_t\|_{C_b^{\alpha/\theta}}\right) \le K(f, t; B(\mathbb{R}), C_b^{\alpha/\theta}) + \varepsilon.$$

Using inequality (6) for $f_t(X_1)$ we get

$$\begin{aligned} K(f(X_1), t; L_2(\mathbb{P}), \mathbb{D}_{1,2}) &\le \|(f - f_t)(X_1)\|_{L_2(\mathbb{P})} + t\|f_t(X_1)\|_{\mathbb{D}_{1,2}} \\ &\le \|f - f_t\|_\infty + t\|f_t\|_{C_b^{\alpha/\theta}} \sqrt{1 + 4m_{2\alpha/\theta}} \\ &\le \sqrt{1 + 4m_{2\alpha/\theta}} \left(K(f, t; B(\mathbb{R}), C_b^{\alpha/\theta}) + \varepsilon\right) \end{aligned}$$

so that

$$\|f(X_1)\|_{(L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}} \le \sqrt{1 + 4m_{2\alpha/\theta}} \|f\|_{(B(\mathbb{R}), C_b^{\alpha/\theta})_{\theta,\infty}}.$$

Using the first inequality of (9), (5), and the second inequality of (9), we obtain that

$$\begin{aligned} \|f\|_{(B(\mathbb{R}), C_b^{\alpha/\theta})_{\theta,\infty}} &\le 3\|f\|_{(B(\mathbb{R}), (B(\mathbb{R}), Lip)_{\alpha/\theta,\infty})_{\theta,\infty}} \\ &\le 9\|f\|_{(B(\mathbb{R}), Lip)_{\alpha,\infty}} \\ &\le 18\|f\|_{C_b^\alpha} \end{aligned}$$

and this finishes the proof of (a). The proof of assertion (b) is given in Section 5. □

Remark 2. Assertion (a) of Theorem 5 implies that $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\alpha,\infty}$ for all $f \in C_b^\alpha$ for any pure jump Lévy process X. Also for the Brownian motion B we obtain the smoothness of level (α, ∞) for $f(B_1)$ for any $f \in C_b^\alpha$: choose $f_t \in C_b^1 = Lip$ like in the proof of Theorem 5 and use the fact that

$$\|f_t(B_1)\|_{\mathbb{D}_{1,2}} \le c\|f_t\|_{Lip}$$

from [29, Lemma A.5], where $c > 0$ is a constant not depending on f_t.

4. Functions of bounded variation and smoothness

Let us first recall the space of *normalized functions of bounded variation*, the space NBV. The variation function of f is given by

$$T_f(x) = \sup\left\{\sum_{i=1}^n |f(x_i) - f(x_{i-1})| : -\infty < x_0 < x_1 < \cdots < x_n = x,\ n \ge 1\right\}$$

and the total variation of f is $V(f) = \lim_{x\to\infty} T_f(x)$. The space of functions of bounded variation is

$$BV = \left\{f : \mathbb{R} \to \mathbb{R} : \|f\|_{BV} = \limsup_{x\to-\infty} |f(x)| + V(f) < \infty\right\}.$$

Note that when $V(f) < \infty$, then the limit $f(-\infty) := \lim_{x\to-\infty} f(x)$ exists ([9, Theorem 3.27(c)]) and for $f \in BV$ we may write $\|f\|_{BV} = |f(-\infty)| + V(f)$. Furthermore,

$$\|f\|_\infty \le \|f\|_{BV}.$$

We denote by NBV the space of normalized functions of bounded variation, that is, the space of all $f \in BV$ such that f is right continuous and $f(-\infty) = 0$. When $f \in NBV$, then by [9, Theorem 3.29] there exists a finite signed measure μ_f such that

$$f(x) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty,x]}(u)\mu_f(\mathrm{d}u) = \int_{\mathbb{R}} \mathbb{1}_{[u,\infty)}(x)\mu_f(\mathrm{d}u) = \int_{\mathbb{R}} \mathbb{1}_{[0,\infty)}(x-u)\mu_f(\mathrm{d}u) \tag{10}$$

for all $x \in \mathbb{R}$. Furthermore, μ_f admits the Jordan decomposition $\mu_f = \mu_f^+ - \mu_f^-$, where μ_f^+ and μ_f^- are nonnegative finite measures. We write $|\mu_f| = \mu_f^+ + \mu_f^-$ so that $|\mu_f|(\mathbb{R}) = \|f\|_{BV}$.

4.1. Smoothness of first order

Theorem 6 ([21, Example 3.1])**.** *For normalized functions of bounded variation we have the following.*

*(a) Assume that (**A1**) holds. If $f \in NBV$ and $m_1 < \infty$, then $f(X_1) \in \mathbb{D}_{1,2}$ and*

$$\|f(X_1)\|_{\mathbb{D}_{1,2}} \le \sqrt{1 + (1 \vee \|p_1\|_\infty)m_1}\,\|f\|_{BV}.$$

*(b) Suppose that X_1 satisfies (**A2**) and let $K \in \mathbb{R}$ be such that there is $r > 0$ and $c > 0$ such that the density p_1 of X_1 satisfies $p_1(x) \ge c$ for all $x \in [K-r, K+r]$. Then $\mathbb{1}_{[K,\infty)}(X_1) \in \mathbb{D}_{1,2}$ only if $m_1 < \infty$.*

Proof. (a) Let $f \in NBV$ and μ_f be the according signed measure from (10). We use Hölder's inequality to get

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}\left[(f(X_1+x) - f(X_1))^2\right]\nu(\mathrm{d}x)\\
&= \int_{\mathbb{R}} \mathbb{E}\left[\left(\int_{\mathbb{R}} \left(\mathbb{1}_{[u,\infty)}(X_1+x) - \mathbb{1}_{[u,\infty)}(X_1)\right)\mu_f(\mathrm{d}u)\right)^2\right]\nu(\mathrm{d}x)\\
&\le |\mu_f|(\mathbb{R}) \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}\left[\left(\mathbb{1}_{[u,\infty)}(X_1+x) - \mathbb{1}_{[u,\infty)}(X_1)\right)^2\right]|\mu_f|(\mathrm{d}u)\nu(\mathrm{d}x)\\
&\le |\mu_f|(\mathbb{R}) \int_{\mathbb{R}}\int_{\mathbb{R}} (\|p_1\|_\infty |x| \wedge 1)\,|\mu_f|(\mathrm{d}u)\nu(\mathrm{d}x)\\
&\le \|f\|_{BV}^2 (1 \vee \|p_1\|_\infty) \int_{\mathbb{R}} (|x| \wedge 1)\nu(\mathrm{d}x).
\end{aligned}$$

Hence from (1) we obtain that

$$\begin{aligned}
\|f(X_1)\|_{\mathbb{D}_{1,2}}^2 &= \|f(X_1)\|_{L_2(\mathbb{P})}^2 + \int_{\mathbb{R}} \mathbb{E}\left[(f(X_1+x) - f(X_1))^2\right]\nu(\mathrm{d}x)\\
&\le \|f\|_{BV}^2 + \|f\|_{BV}^2 (1 \vee \|p_1\|_\infty) m_1.
\end{aligned}$$

(b) Let $r > 0$ and $c > 0$ be such that $p_1(x) \geq c$ for all $x \in [K - r, K + r]$. Let $f = \mathbb{1}_{[K,\infty)}$. Then $f \in NBV$ and

$$\begin{aligned}&\int_{\mathbb{R}} \mathbb{E}\left[|f(X_1 + x) - f(X_1)|^2\right] \nu(\mathrm{d}x) \\ &= \int_{(-\infty,0)} \mathbb{E}\left[\mathbb{1}_{[K,K-x)}(X_1)\right] \nu(\mathrm{d}x) + \int_{(0,\infty)} \mathbb{E}\left[\mathbb{1}_{[K-x,K)}(X_1)\right] \nu(\mathrm{d}x) \\ &\geq c \int_{0<|x|\leq r} |x|\nu(\mathrm{d}x).\end{aligned}$$

By (1) it holds that $m_1 < \infty$, if $f(X_1) \in \mathbb{D}_{1,2}$. □

4.2. Fractional smoothness

If $m_1 < \infty$ does not hold, it is still possible to attain fractional smoothness with functions in NBV. In [11, Example 4.2(a)] it is verified that $\mathbb{1}_{(K,\infty)}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{1/2,\infty}$. Note that in [11, Example 4.2(a)] it is assumed a small ball estimate for the distribution and this assumption is equivalent with (**A1**) (one can easily see this by using the steps of the proof of [2, Theorem 2.4(iii)]). In the following theorem we show that the smoothness level increases as the Blumenthal-Getoor index decreases.

Theorem 7. *Let $1/2 \leq \theta < 1$.*

*(a) Assume that (**A1**) holds. If $f \in NBV$ and $m_{1/\theta} < \infty$, then*

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$$

and

$$\|f(X_1)\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,\infty}} \leq \left(\sqrt{\|p\|_\infty} + \sqrt{1 + 2\left(\|p\|_\infty \vee 1\right) m_{1/\theta}}\right) \|f\|_{BV}.$$

Especially, $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\frac{1}{2},\infty}$ for any Lévy measure ν.

*(b) Assume that (**A3**) holds and let $t_0 \in (0, 1)$ and $K \in \mathbb{R}$ be such that there exist $r > 0$ and $c > 0$ with $p_t(x) \geq c$ for all $x \in [K - 2r, K + 2r]$ and all $t \in [t_0, 1]$. Then*

$$\mathbb{1}_{[K,\infty)}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \quad \text{only if} \quad m_{1/\theta+\varepsilon} < \infty \text{ for all } \varepsilon > 0.$$

Proof. (a) Let $f \in NBV$ and μ_f be the according signed measure from (10). For $t \in (0, 1)$ we define

$$g_t(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{t} x^{\frac{1}{2\theta}}, & 0 < x < t^{2\theta} \\ 1, & x \geq t^{2\theta} \end{cases} \qquad \text{and} \qquad f_t(x) = \int_{\mathbb{R}} g_t(x - u)\mu_f(\mathrm{d}u).$$

Then

$$
\begin{aligned}
&\mathbb{E}\left[(f_t(X_1+x)-f_t(X_1))^2\right] \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} [g_t(y+x-u)-g_t(y-u)]\, \mu_f(\mathrm{d}u) \right)^2 p(y)\mathrm{d}y \\
&\le |\mu_f|(\mathbb{R}) \|p\|_\infty \int_{\mathbb{R}} \int_{\mathbb{R}} (g_t(y+x-u)-g_t(y-u))^2 \, |\mu_f|(\mathrm{d}u)\mathrm{d}y \\
&= |\mu_f|(\mathbb{R})^2 \|p\|_\infty \int_{\mathbb{R}} (g_t(z+x)-g_t(z))^2 \, \mathrm{d}z.
\end{aligned}
$$

Note that $g_t(\cdot+x)-g_t$ is nonzero only on an interval of length $t^{2\theta}+|x|$ and

$$
\begin{aligned}
|g_t(z+x)-g_t(z)| &= \left| \int_z^{z+x} \frac{1}{2\theta t} u^{\frac{1}{2\theta}-1} \mathbb{1}_{(0,t^{2\theta})}(u)\mathrm{d}u \right| \\
&\le \int_0^{|x|} \frac{1}{2\theta t} u^{\frac{1}{2\theta}-1} \mathbb{1}_{(0,t^{2\theta})}(u)\mathrm{d}u \\
&= g_t(|x|) \le 1
\end{aligned}
$$

for all $x, z \in \mathbb{R}$, since $\frac{1}{2\theta} - 1 \le 0$. When $|x| \ge t^{2\theta}$, then

$$
\int_{\mathbb{R}} (g_t(z+x)-g_t(z))^2 \, \mathrm{d}z \le 2|x| = 2t^{2(\theta-1)}|x|t^{2(1-\theta)} \le 2t^{2(\theta-1)}|x|^{1/\theta}.
$$

When $|x| < t^{2\theta}$, then

$$
\int_{\mathbb{R}} (g_t(z+x)-g_t(z))^2 \, \mathrm{d}z \le 2t^{2\theta} g_t^2(|x|) = 2t^{2(\theta-1)}|x|^{1/\theta}.
$$

On the other hand,

$$
\begin{aligned}
&\mathbb{E}\left[(f_t(X_1+x)-f_t(X_1))^2\right] \\
&= \mathbb{E}\left[\left(\int_{\mathbb{R}} (g_t(X_1+x-u)-g_t(X_1-u))\, \mu_f(\mathrm{d}u) \right)^2 \right] \\
&\le |\mu_f|^2(\mathbb{R}),
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbb{E}\left[(f_t(X_1+x)-f_t(X_1))^2\right] \nu(\mathrm{d}x) \\
&\le \int_{\mathbb{R}} |\mu_f|(\mathbb{R})^2 \left(\|p\|_\infty \vee 1\right) \left(2t^{2(\theta-1)}|x|^{1/\theta} \wedge 1\right) \nu(\mathrm{d}x) \\
&\le |\mu_f|(\mathbb{R})^2 \left(\|p\|_\infty \vee 1\right) 2t^{2(\theta-1)} m_{1/\theta}
\end{aligned}
$$

since $0 < t < 1$, and therefore $f_t(X_1) \in \mathbb{D}_{1,2}$. It also holds, by (10), that

$$\begin{aligned}\|(f-f_t)(X_1)\|^2_{L_2(\mathbb{P})} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left[\mathbb{1}_{[0,\infty)}(y-u) - g_t(y-u) \right] \mu_f(\mathrm{d}u) \right)^2 \mathbb{P}_{X_1}(\mathrm{d}y) \\ &\le |\mu_f|(\mathbb{R})^2 \|p\|_\infty \int_{\mathbb{R}} \left(\mathbb{1}_{[0,\infty)}(y) - g_t(y) \right)^2 \mathrm{d}y \\ &\le |\mu_f|(\mathbb{R})^2 \|p\|_\infty t^{2\theta}\end{aligned}$$

and

$$\|f_t(X_1)\|_{L_2(\mathbb{P})} \le |\mu_f|(\mathbb{R}).$$

We obtain for $t \in (0,1)$ that

$$\begin{aligned}&t^{-\theta} \left(\|(f-f_t)(X_1)\|_{L_2(\mathbb{P})} + t\sqrt{\|f_t(X_1)\|^2_{L_2(\mathbb{P})} + \|Df_t(X_1)\|^2_{L_2(\mathrm{m}\otimes\mathbb{P})}} \right) \\ &\le t^{-\theta} \left(\sqrt{\|p\|_\infty} |\mu_f|(\mathbb{R}) t^\theta + t\sqrt{|\mu_f|(\mathbb{R})^2 + |\mu_f|(\mathbb{R})^2 \left(\|p\|_\infty \vee 1 \right) 2t^{2(\theta-1)} m_{1/\theta}} \right) \\ &\le \left(\sqrt{\|p\|_\infty} + \sqrt{1 + 2\left(\|p\|_\infty \vee 1 \right) m_{1/\theta}} \right) |\mu_f|(\mathbb{R}).\end{aligned}$$

Thus

$$\begin{aligned}&\|f(X_1)\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,\infty}} \\ &= \sup_{t>0} t^{-\theta} \inf\{\|Y_0\|_{L_2(\mathbb{P})} + t\|Y_1\|_{\mathbb{D}_{1,2}} : Y_0 + Y_1 = f(X_1)\} \\ &\le \sup_{t\in(0,1)} t^{-\theta} \left(\|(f-f_t)(X_1)\|_{L_2(\mathbb{P})} + t\sqrt{\|f_t(X_1)\|^2_{L_2(\mathbb{P})} + \|Df_t(X_1)\|^2_{L_2(\mathrm{m}\otimes\mathbb{P})}} \right) \\ &\qquad \vee \|f(X_1)\|_{L_2(\mathbb{P})} \\ &\le \left(\sqrt{\|p\|_\infty} + \sqrt{1 + 2\left(\|p\|_\infty \vee 1 \right) m_{1/\theta}} \right) \|f\|_{BV}.\end{aligned}$$

The proof of assertion (b) is given in Section 5. □

5. Sharpness of the connection between the smoothness index and the Blumenthal-Getoor index

In Lemma 9 below, we adapt the characterisation for fractional smoothness from [15, Corollary 2.3], where it is written for the Brownian motion.

Definition 1. For a sequence of Banach spaces $E = (E_n)_{n=0}^\infty$ with $E_n \neq \{0\}$ we let $\ell_2(E)$ and $d_{1,2}(E)$ be the Banach spaces of all $a = (a_n)_{n=0}^\infty \in E$ such that

$$\|a\|_{\ell_2(E)} := \left(\sum_{n=0}^\infty \|a_n\|^2_{E_n} \right)^{\frac{1}{2}} \quad \text{and} \quad \|a\|_{d_{1,2}(E)} := \left(\sum_{n=0}^\infty (n+1)\|a_n\|^2_{E_n} \right)^{\frac{1}{2}}$$

respectively, are finite. For $a \in E$ we let $Ta : [0,1] \to \mathbb{R}$ be defined by

$$(Ta)(t) := \sum_{n=0}^\infty \|a_n\|^2_{E_n} t^n.$$

We use the notation $A \sim_c B$ for $\frac{1}{c}B \le A \le cB$, where $A, B \in [0,\infty]$ and $c \ge 1$.

Lemma 8 ([15, Theorem 2.2])**.** *For $\theta \in (0,1)$, $q \in [1,\infty]$ and $a \in \ell_2(E)$ one has*

$$\begin{aligned} &\|a\|_{(\ell_2(E), d_{1,2}(E))_{\theta,q}} \\ &\sim_c \|a\|_{\ell_2(E)} + \left\| (1-t)^{\frac{1-\theta}{2}} \sqrt{(Ta)'(t)} \right\|_{L_q\left((0,1),\mathcal{B}(0,1),\frac{\mathrm{d}t}{1-t}\right)} \\ &\sim_c \|a\|_{\ell_2(E)} + \left\| (1-t)^{-\frac{\theta}{2}} \sqrt{(Ta)(1)-(Ta)(t)} \right\|_{L_q\left((0,1),\mathcal{B}(0,1),\frac{\mathrm{d}t}{1-t}\right)}, \end{aligned}$$

where $c \ge 1$ depends only on (θ, q), and the expressions may be infinite.

We will apply this theorem to the Itô chaos decomposition. Let $(\mathcal{F}_t)_{t\ge 0}$ be the augmented natural filtration of X. Throughout this section we let $\bar{X}$ be an independent copy of X on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. We will use the notation $\bar{\mathbb{E}}$ for the expectation with respect to the measure $\bar{\mathbb{P}}$.

Lemma 9. *For $\theta \in (0,1)$, $q \in [1,\infty]$ and $f(X_1) \in L_2(\mathbb{P})$ one has*

$$\begin{aligned} &\|f(X_1)\|_{(L_2(\mathbb{P}),\mathbb{D}_{1,2})_{\theta,q}} \\ &\sim_c \|f(X_1)\|_{L_2(\mathbb{P})} + \left\| (1-t)^{-\frac{\theta}{2}} \|f(X_1) - \mathbb{E}\left[f(X_1)|\mathcal{F}_t\right]\|_{L_2(\mathbb{P})} \right\|_{L_q\left((0,1),\mathcal{B}(0,1),\frac{\mathrm{d}t}{1-t}\right)} \\ &= \|f(X_1)\|_{L_2(\mathbb{P})} + \frac{1}{\sqrt{2}} \left\| (1-t)^{-\frac{\theta}{2}} \left\| \|f(X_1) - f(X_t + \bar{X}_{1-t})\|_{L_2(\mathbb{P})} \right\|_{L_2(\bar{\mathbb{P}})} \right\|_{L_q\left(\frac{\mathrm{d}t}{1-t}\right)}, \end{aligned}$$

where $c \ge 1$ depends only on (θ, q) and the expressions may be infinite.

Proof. Let $f(X_1) = \sum_{n=0}^\infty I_n(f_n) \in L_2(\mathbb{P})$, $E = (L_2(\mathbb{m}^{\otimes n}))_{n=0}^\infty$ and $a = \left(\sqrt{n!}\tilde{f}_n\right)_{n=0}^\infty$. By orthogonality the equality

$$\sum_{n=0}^\infty I_n(f_n) = \sum_{n=0}^\infty I_n(g_n) + \sum_{n=0}^\infty I_n(h_n)$$

holds in $L_2(\mathbb{P})$ if and only if $\tilde{f}_n = \tilde{g_n} + \tilde{h_n}$ holds $\mathbb{m}^{\otimes n}$-a.e. Therefore

$$\begin{aligned} &K(f(X_1), t; L_2(\mathbb{P}), \mathbb{D}_{1,2}) \\ &= \inf_{\tilde{f}_n = \tilde{g_n} + \tilde{h_n}} \left(\sqrt{\sum_{n=0}^\infty n! \|\tilde{g_n}\|^2_{L_2(\mathbb{m}^{\otimes n})}} + t \sqrt{\sum_{n=0}^\infty (n+1)! \|\tilde{h_n}\|^2_{L_2(\mathbb{m}^{\otimes n})}} \right) \\ &= K(a, t; \ell_2(E), d_{1,2}(E)) \end{aligned}$$

and Lemma 1(b) gives

$$\begin{aligned} \|f(X_1) - \mathbb{E}\left[f(X_1)|\mathcal{F}_t\right]\|^2_{L_2(\mathbb{P})} &= \mathbb{E}\left[f(X_1)^2\right] - \mathbb{E}\left[\mathbb{E}\left[f(X_1)|\mathcal{F}_t\right]^2\right] \\ &= (Ta)(1) - (Ta)(t). \end{aligned}$$

The equivalence follows now from Lemma 8. To conclude with the equality below, we use the facts that $\mathbb{E}\left[f(X_1)|\mathcal{F}_t\right] = \bar{\mathbb{E}}\left[f(X_t + \bar{X}_{1-t})\right]$ a.s. and $X_t + \bar{X}_{1-t} \stackrel{d}{=} X_1$ to get that

$$\begin{aligned}
&\|f(X_1) - \mathbb{E}\left[f(X_1)|\mathcal{F}_t\right]\|^2_{L_2(\mathbb{P})} \\
&= \mathbb{E}\left[f(X_1)\big(f(X_1) - \mathbb{E}\left[f(X_1)|\mathcal{F}_t\right]\big)\right] \\
&= \bar{\mathbb{E}}\mathbb{E}\left[f(X_1)\big(f(X_1) - f(X_t + \bar{X}_{1-t})\big)\right] \\
&= -\bar{\mathbb{E}}\mathbb{E}\left[f(X_t + \bar{X}_{1-t})\big(f(X_1) - f(X_t + \bar{X}_{1-t})\big)\right] \\
&= \frac{1}{2}\bar{\mathbb{E}}\mathbb{E}\left[\big(f(X_1) - f(X_t + \bar{X}_{1-t})\big)^2\right],
\end{aligned}$$

where the last line is obtained as the average of the two previous lines. □

Lemma 10. *Let $\tilde{X}$ be a pure jump Lévy process with càdlàg paths on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Let $\tilde{\nu}$ be its Lévy measure and β be its Blumenthal-Getoor index. Let $t_0 > 0$ and define a constant κ by letting*

$$\kappa = \begin{cases} \int_{\{|x|\le 1\}} x\tilde{\nu}(\mathrm{d}x), & \text{if } \int_{\{|x|\le 1\}} |x|\tilde{\nu}(\mathrm{d}x) < \infty \\ 0, & \text{if } \int_{\{|x|\le 1\}} |x|\tilde{\nu}(\mathrm{d}x) = \infty. \end{cases}$$

(a) For all $\beta' > \beta$ it holds that

$$\int_0^{t_0} \tilde{\mathbb{P}}\left(\frac{|\tilde{X}_t + \kappa t|}{t^{1/\beta'}} > c\right)\frac{\mathrm{d}t}{t} < \infty \qquad \text{for all } c > 0.$$

(b) For any $\beta'' < \beta$ there exists $c' > 0$ such that

$$\int_0^{t_0} \tilde{\mathbb{P}}\left(\frac{|\tilde{X}_t + \kappa t|}{t^{1/\beta''}} > c'\right)\frac{\mathrm{d}t}{t} = \infty.$$

(c) It holds that

$$\int_0^{t_0} \tilde{\mathbb{P}}\left(|\tilde{X}_t| > c\right)\frac{\mathrm{d}t}{t} < \infty \qquad \text{for all } c > 0.$$

Proof. By [6, Theorems 3.1 and 3.3] it holds for all $\beta'' < \beta < \beta'$ that

$$\lim_{t\to 0} \frac{\tilde{X}_t + \kappa t}{t^{1/\beta'}} = 0 \text{ a.s.} \quad \text{and} \quad \limsup_{t\to 0} \frac{|\tilde{X}_t + \kappa t|}{t^{1/\beta''}} = \infty \text{ a.s.}$$

By a result of Khintchine [20, Section 2], we have that if $u : (0, t_0) \to (0, \infty)$ is non-decreasing and $\lim_{t\to 0} u(t) = 0$, then for any Lévy process Y it holds that

$$\lim_{t\to 0} \frac{Y_t}{u(t)} = 0 \text{ a.s.} \quad \text{if and only if} \quad \int_0^{t_0} \mathbb{P}\left(\frac{|Y_t|}{u(t)} > c\right)\frac{\mathrm{d}t}{t} < \infty \text{ for all } c > 0.$$

The claims (a) and (b) follow by choosing $Y_t = \tilde{X}_t + \kappa t$ and $u(t) = t^{1/\beta'}$ in (a) and $u(t) = t^{1/\beta''}$ in (b).
(c) Let $u(t) = t^{1/3} \wedge 1$. Then

$$\lim_{t\to 0} \frac{\tilde{X}_t}{u(t)} \le \lim_{t\to 0} \frac{|\tilde{X}_t + \kappa t|}{t^{1/3}} + \frac{|\kappa t|}{t^{1/3}} = 0,$$

by (a) so that [20, Section 2] implies that

$$\int_0^{t_0} \tilde{\mathbb{P}}\left(|\tilde{X}_t| > c\right) \frac{\mathrm{d}t}{t} \le \int_0^{t_0} \tilde{\mathbb{P}}\left(\frac{|\tilde{X}_t|}{u(t)} > c\right) \frac{\mathrm{d}t}{t} < \infty \quad \text{for all } c > 0.$$

□

Lemma 11. *Assume that (**A3**) holds and let $R > 0$, $a < b$, $t_0 \in (0,1)$ and $c_1 > 0$ be such that $p_t(x) \ge c_1$ for all $x \in [a-R, b+R]$ and $t \in [t_0, 1]$. If $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable and there exist $r > 0$, $c_2 > 0$ and $\eta > 0$ such that*

$$\int_a^b [f(y+x) - f(y)]^2 \,\mathrm{d}y \ge c_2 |x|^\eta \quad \textit{for all } |x| \le r, \tag{11}$$

then

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \quad \textit{only if} \quad m_{\eta/\theta+\varepsilon} < \infty$$

for all $\varepsilon > 0$.

Proof. The assumptions (**A3**) and (11) yield for $t \in [t_0, 1]$ that

$$\begin{aligned}
&\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|_{L_2(\bar{\mathbb{P}})}^2 \\
&= \bar{\mathbb{E}}\mathbb{E}\left[\int_{\mathbb{R}} \left(f(y + X_1 - X_t) - f(y + \bar{X}_{1-t}) \right)^2 p_t(y)\mathrm{d}y \right] \\
&= \bar{\mathbb{E}}\mathbb{E}\left[\int_{\mathbb{R}} \left(f(y + X_{1-t} - \bar{X}_{1-t}) - f(y) \right)^2 p_t(y - \bar{X}_{1-t})\mathrm{d}y \right] \\
&\ge \bar{\mathbb{E}}\mathbb{E}\left[\int_a^b \left(f(y + X_{1-t} - \bar{X}_{1-t}) - f(y) \right)^2 c_1 \mathrm{d}y \mathbb{1}_{\{|\bar{X}_{1-t}| \le R\}} \right] \\
&\ge c_1 c_2 \bar{\mathbb{E}}\mathbb{E}\left[|X_{1-t} - \bar{X}_{1-t}|^\eta \mathbb{1}_{\{|X_{1-t} - \bar{X}_{1-t}| \le r, |\bar{X}_{1-t}| \le R\}} \right].
\end{aligned} \tag{12}$$

Since X and $\bar{X}$ are independent, the process $\tilde{X} = X - \bar{X}$ with $\tilde{X}_t(\omega, \bar{\omega}) = X_t(\omega) - \bar{X}_t(\bar{\omega})$ is a Lévy process on $(\Omega \times \bar{\Omega}, \mathcal{F} \otimes \bar{\mathcal{F}}, \mathbb{P} \otimes \bar{\mathbb{P}})$ with Lévy measure $\tilde{\nu}(B) = \nu(B) + \nu(-B)$, and its Blumenthal-Getoor index is the same β as for X. Let $0 < \theta' < \theta$ and $c > 0$ and set $c_3 = c_1 c_2 c^\eta$. Then (12) has the lower bound

$$\begin{aligned}
&c_3 (1-t)^{\theta'} (\mathbb{P} \otimes \bar{\mathbb{P}}) \left(\frac{|X_{1-t} - \bar{X}_{1-t}|^\eta}{(1-t)^{\theta'} c^\eta} > 1, |X_{1-t} - \bar{X}_{1-t}| \le r, |\bar{X}_{1-t}| \le R \right) \\
&\ge c_3 (1-t)^{\theta'} \left[\tilde{\mathbb{P}}\left(\frac{|\tilde{X}_{1-t}|}{(1-t)^{\theta'/\eta}} > c \right) - \tilde{\mathbb{P}}\left(|\tilde{X}_{1-t}| > r \right) - \bar{\mathbb{P}}(|\bar{X}_{1-t}| > R) \right].
\end{aligned}$$

If $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$, then $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta',2}$ by (3). Using Lemma 8 we get that

$$\begin{aligned}
\infty &> \int_0^1 (1-t)^{-\theta'} \left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|_{L_2(\bar{\mathbb{P}})}^2 \frac{\mathrm{d}t}{1-t} \\
&\geq c_3 \int_{1-t_0}^1 \left[\tilde{\mathbb{P}} \left(\frac{|\tilde{X}_{1-t}|}{(1-t)^{\theta'/\eta}} > c \right) - \tilde{\mathbb{P}} \left(|\tilde{X}_{1-t}| > r \right) - \bar{\mathbb{P}}(|\bar{X}_{1-t}| > R) \right] \frac{\mathrm{d}t}{1-t} \\
&= c_3 \left[\int_0^{t_0} \tilde{\mathbb{P}} \left(\frac{|\tilde{X}_t|}{t^{\theta'/\eta}} > c \right) \frac{\mathrm{d}t}{t} - \int_0^{t_0} \tilde{\mathbb{P}}(|\tilde{X}_t| > r) \frac{\mathrm{d}t}{t} - \int_0^{t_0} \bar{\mathbb{P}}(|\bar{X}_t| > R) \frac{\mathrm{d}t}{t} \right],
\end{aligned}$$

where

$$\int_0^{t_0} \tilde{\mathbb{P}}(|\tilde{X}_t| > r) \frac{\mathrm{d}t}{t} + \int_0^{t_0} \bar{\mathbb{P}}(|\bar{X}_t| > R) \frac{\mathrm{d}t}{t} < \infty$$

by Lemma 10(c). Hence

$$\int_0^{t_0} \tilde{\mathbb{P}} \left(\frac{|\tilde{X}_t|}{t^{\theta'/\eta}} > c \right) \frac{\mathrm{d}t}{t} < \infty \quad \text{for all } c > 0 \text{ and for all } 0 < \theta' < \theta.$$

Since $\tilde{\nu}$ is symmetric, the constant κ of Lemma 10 is zero and Lemma 10(b) implies $\beta \leq \eta/\theta'$ for all $0 < \theta' < \theta$, so that $\beta \leq \eta/\theta$. □

Proof of Theorem 5(b). By Lemma 4, the function $g^{\alpha,\ell}$ satisfies (11) with $[a,b] = [k2^{-\ell}, (k+1)2^{-\ell}]$, $r = 2^{-\ell-3}$, $c_2 = 2^{-\ell}2^{8\alpha-10}$ and $\eta = 2\alpha$. If $g^{\alpha,\ell}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$, then by Lemma 11 it holds that $\beta \leq 2\alpha/\theta$. □

Proof of Theorem 7(b). We have that

$$\begin{aligned}
&\int_{K-r}^{K+r} \left(\mathbb{1}_{[K,\infty)}(y+x) - \mathbb{1}_{[K,\infty)}(y) \right)^2 \mathrm{d}y \\
&= \int_{K-r}^{K+r} \left(\mathbb{1}_{[K-x,K)}(y) \mathbb{1}_{(0,\infty)}(x) + \mathbb{1}_{[K,K-x)}(y) \mathbb{1}_{(-\infty,0)}(x) \right) \mathrm{d}y \\
&= |x| \qquad (13)
\end{aligned}$$

for all $|x| \leq r$, so that $\mathbb{1}_{[K,\infty)}$ satisfies (11) with $[a,b] = [K-r, K+r]$. Choosing $R = r$ it now follows from Lemma 11, that if $\mathbb{1}_{[K,\infty)}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$ then $\beta \leq 1/\theta$. □

Remark 3. (a) If $m_\beta < \infty$ and (**A3**) holds, then we get for $0 < \alpha \leq \theta < 1$ from Theorem 5 the "if and only if"-condition

$$m_{2\alpha/\theta} < \infty \iff f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \, \forall f \in C_b^\alpha$$

and if also (**A1**) holds, then Theorem 7 implies for $1/2 \leq \theta < 1$ that

$$m_{1/\theta} < \infty \iff f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \, \forall f \in NBV.$$

Note that $m_\beta < \infty$ is indeed possible: choose for example

$$\nu(\mathrm{d}x) = \frac{b}{|x|^{1+\beta}(\log^2 x + 1)}\mathrm{d}x \quad \text{for some } b > 0$$

for $\beta \in (0, 2]$. Using Lemma 2 we see that this process satisfies (**A1**)-(**A3**).

(b) If $m_\beta = \infty$, then Theorems 5 and 7 do not give an "if and only if"-result in general: In Theorems 13-15 in Section 5.1 we consider the symmetric strictly stable process with

$$\nu(\mathrm{d}x) = \frac{b}{|x|^{1+\beta}}\mathrm{d}x \quad \text{for some } b > 0 \text{ and } \beta \in (0, 1),$$

and the process satisfies (**A1**)-(**A3**) by Lemma 2. Theorems 13 and 14 show that when $0 < \alpha < \theta < 1$, then

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}\, \forall f \in C_b^\alpha \quad \text{for } 2\alpha/\theta = \beta,$$

and that for $\frac{1}{2} \le \theta < 1$ it holds that

$$f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}\, \forall f \in NBV \quad \text{for } 1/\theta = \beta,$$

eventhough $m_\beta = \infty$. However, we obtain for $0 < \alpha < \theta < 1$ from Theorem 13, that

$$\begin{aligned} m_{2\alpha/\theta} < \infty &\iff f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}\, \forall f \in C_b^\alpha \text{ for some } q \in [1, \infty) \\ &\iff f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}\, \forall f \in C_b^\alpha \text{ for all } q \in [1, \infty). \end{aligned}$$

Theorems 14 and 15 imply for $0 < \theta < 1$ that

$$\begin{aligned} m_{1/\theta} < \infty &\iff f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}\, \forall f \in NBV \text{ for some } q \in [1, \infty) \\ &\iff f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}\, \forall f \in NBV \text{ for all } q \in [1, \infty). \end{aligned}$$

5.1. Symmetric strictly stable process

We consider the symmetric strictly stable process which has the characteristic function $\varphi(u) = e^{-c|u|^\beta}$ for some $c > 0$ and $\beta \in (0, 2]$ ([25, Theorem 14.14]). If $\beta = 2$, then the process is the Brownian motion $\sqrt{2c}B$, and otherwise it is a pure jump Lévy process X with Lévy measure

$$\nu(\mathrm{d}x) = b|x|^{-\beta-1}\mathrm{d}x \quad \text{for some } b > 0,$$

where β is the Blumenthal-Getoor index of the process. We will later take advantage of the property that $X_t \overset{d}{=} t^{1/\beta} X_1$, which follows from

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{-tc|u|^\beta} = e^{-c\left|ut^{1/\beta}\right|^\beta} = \mathbb{E}\left[e^{iut^{1/\beta}X_1}\right].$$

Using Lemma 2 one can easily check that assumptions (**A1**), (**A2**) and (**A3**) are satisfied. For the rest of this section we assume that X is the symmetric and strictly stable process of index $\beta \in (0, 2)$.

Lemma 12. *Let $a < b$ and $t_0 \in (0,1)$. If $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable and there exist $r > 0$, $c_2 > 0$ and $\eta > 0$ such that (11) holds, then there exists $c > 0$ such that*

$$\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|_{L_2(\bar{\mathbb{P}})}^2 \geq c(1-t)^{\eta/\beta} \quad \text{for all } t \in [t_0, 1].$$

Proof. Let $R > 0$. Since X_1 has the support $\mathbb{R}$ by [25, Theorem 24.10(ii)], then p_1 is strictly positive and continuous on $\mathbb{R}$ by the proof of Lemma 2. Hence we find $c_1 > 0$ such that $p_1(x) \geq c_1$ for all $-|a-R|t_0^{-1/\beta} \leq x \leq |b+R|t_0^{-1\beta}$. Using the fact that $X_t \stackrel{d}{=} t^{1/\beta} X_1$, we obtain for any $x \in [a-R, b+R]$ that

$$p_t(x) = t^{-1/\beta} p_1(t^{-1/\beta} x) \geq p_1(t^{-1/\beta} x) \geq c_1$$

for all $t \in [t_0, 1]$. Using (12) we get that

$$\begin{aligned}
&\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|_{L_2(\bar{\mathbb{P}})}^2 \\
&\geq c_1 c_2 \bar{\mathbb{E}}\mathbb{E} \left[|X_{1-t} - \bar{X}_{1-t}|^\eta \mathbb{1}_{\{|X_{1-t} - \bar{X}_{1-t}| \leq r, |\bar{X}_{1-t}| \leq R\}} \right] \\
&= c_1 c_2 (1-t)^{\eta/\beta} \bar{\mathbb{E}}\mathbb{E} \left[|X_1 - \bar{X}_1|^\eta \mathbb{1}_{\{|X_1 - \bar{X}_1| \leq r(1-t)^{-1/\beta}, |\bar{X}_1| \leq R(1-t)^{-1/\beta}\}} \right] \\
&\geq c_1 c_2 (1-t)^{\eta/\beta} \bar{\mathbb{E}}\mathbb{E} \left[|X_1 - \bar{X}_1|^\eta \mathbb{1}_{\{|X_1 - \bar{X}_1| \leq r, |\bar{X}_1| \leq R\}} \right] \\
&\geq c(1-t)^{\eta/\beta}
\end{aligned}$$

for some $c > 0$, where we used the fact that since $X_1 - \bar{X}_1$ is strictly stable with Lévy measure 2ν, it must be that $\bar{\mathbb{E}}\mathbb{E}\left[|X_1 - \bar{X}_1| \mathbb{1}_{\{|X_1 - \bar{X}_1| \leq r, |\bar{X}_1| \leq R\}}\right]$ is strictly positive. □

Theorem 13. *Let $0 < \alpha < \theta < 1$ and assume that $f \in C_b^\alpha$.*

(a) It holds that $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$, if $\beta \leq 2\alpha/\theta$.

(b) Let $q \in [1, \infty)$ and $\ell \in \{0, 1, 2, \ldots\}$. For the function $g^{\alpha,\ell} \in C_b^\alpha$ from Lemma 4 we have that

(i)

$$g^{\alpha,\ell}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q} \quad \text{if and only if} \quad \beta < 2\alpha/\theta$$

and

(ii)

$$g^{\alpha,\ell}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty} \quad \text{if and only if} \quad \beta \leq 2\alpha/\theta.$$

Proof. (a) If $\beta \leq 2\alpha$, then $m_{2\alpha/\theta} < \infty$ and the claim follows from Theorem 5(a). Assume now that $\beta > 2\alpha$. We have

$$\begin{aligned}
\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|_{L_2(\bar{\mathbb{P}})}^2 &\leq 2\bar{\mathbb{E}}\mathbb{E}\left[|X_{1-t} - \bar{X}_{1-t}|^{2\alpha} \|f\|_{C_b^\alpha}^2 \right] \\
&\leq 2\bar{\mathbb{E}}\mathbb{E}\left[|(1-t)^{1/\beta}(X_1 - \bar{X}_1)|^{2\alpha} \|f\|_{C_b^\alpha}^2 \right] \\
&\leq 2(1-t)^{2\alpha/\beta} \|f\|_{C_b^\alpha}^2 \bar{\mathbb{E}}\mathbb{E}\left[|X_1 - \bar{X}_1|^{2\alpha} \right].
\end{aligned}$$

Since the process $X - \bar{X}$ on $\Omega \times \bar{\Omega}$ has the Lévy measure 2ν and $\beta > 2\alpha$, we get that

$$\int_{\{|x|>1\}} |x|^{2\alpha} 2\nu(\mathrm{d}x) = 2\int_{\{|x|>1\}} |x|^{2\alpha-\beta-1}\mathrm{d}x < \infty,$$

which implies $\bar{\mathbb{E}}\mathbb{E}\left[|X_1 - \bar{X}_1|^{2\alpha}\right] < \infty$ by [25, Theorem 25.3]. Thus

$$\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|^2_{L_2(\bar{\mathbb{P}})} \le C(1-t)^{2\alpha/\beta}$$

for all $t \in (0,1)$ for some $C \in (0,\infty)$ and the claim (a) follows from Lemma 9. The "if"-parts of (b) follow from (a) and (3). By Lemma 4, the function $g^{\alpha,\ell}$ satisfies (11) with $[a,b] = [k2^{-\ell}, (k+1)2^{-\ell}]$, $r = 2^{-\ell-3}$, $c_2 = 2^{-\ell}2^{8\alpha-10}$ and $\eta = 2\alpha$. Thus, Lemma 12 implies that

$$\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|^2_{L_2(\bar{\mathbb{P}})} \ge c(1-t)^{2\alpha/\beta}$$

for some $c > 0$, and with the use of Lemma 9 this proves the "only if"-parts of (b). □

Theorem 14. *Let $f \in NBV$.*

(a) If $\beta < 1$, then $f(X_1) \in \mathbb{D}_{1,2}$.

(b) If $\beta = 1$, then $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ for all $\theta \in (0,1)$ and $q \in [1,\infty]$.

(c) Let $\theta \in (0,1)$. If $\beta \le 1/\theta$, then $f(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$.

Proof. (a) The claim follows from Theorem 6(a).
(b) The claim follows from Theorem 7 and (3).
(c) If $\beta \le 1$, then the claim follows from (b). Assume that $\beta > 1$. We have

$$\begin{aligned}
&\left\| \left\| f(X_1) - f(X_t + \bar{X}_{1-t}) \right\|_{L_2(\mathbb{P})} \right\|^2_{L_2(\bar{\mathbb{P}})} \\
&= \bar{\mathbb{E}}\mathbb{E}\left[\left(\int_{\mathbb{R}} \mathbb{1}_{[u,\infty)}(X_1) - \mathbb{1}_{[u,\infty)}(X_t + \bar{X}_{1-t}) \mu_f(\mathrm{d}u) \right)^2 \right] \\
&\le |\mu_f|(\mathbb{R}) \int_{\mathbb{R}} \bar{\mathbb{E}}\mathbb{E}\left[\mathbb{1}_{[u-X_{1-t}, u-\bar{X}_{t-1}) \cup [u-\bar{X}_{1-t}, u-X_{t-1})}(X_t) \right] \mu_f(\mathrm{d}u) \\
&\le |\mu_f|^2(\mathbb{R}) \|p_t\|_\infty \bar{\mathbb{E}}\mathbb{E}\left[|\bar{X}_{1-t} - X_{1-t}| \right] \\
&= |\mu_f|^2(\mathbb{R}) t^{-1/\beta} \|p_1\|_\infty \bar{\mathbb{E}}\mathbb{E}\left[(1-t)^{1/\beta} |\bar{X}_1 - X_1| \right] \\
&\le (1-t)^{1/\beta} |\mu_f|^2(\mathbb{R}) t^{-1/\beta} \|p_1\|_\infty \bar{\mathbb{E}}\mathbb{E}\left[|\bar{X}_1 - X_1| \right].
\end{aligned}$$

Since the process $X - \bar{X}$ has the Lévy measure 2ν and

$$\int_{\{|x|>1\}} |x| 2\nu(\mathrm{d}x) = 2\int_{\{|x|>1\}} |x|^{-\beta}\mathrm{d}x < \infty,$$

for $\beta > 1$, we get $\bar{\mathbb{E}}\mathbb{E}\left[|\bar{X}_1 - \hat{X}_1|\right] < \infty$ from [25, Theorem 25.3]. Thus

$$\left\| \| f(X_1) - f(X_t + \bar{X}_{1-t}) \|_{L_2(\mathbb{P})} \right\|^2_{L_2(\bar{\mathbb{P}})} \le C(1-t)^{1/\beta}$$

for all $t \in (1/2, 1)$ for some $C \in (0, \infty)$. When $t \in (0, 1/2]$, then

$$\left\| \| f(X_1) - f(X_t + \bar{X}_{1-t}) \|_{L_2(\mathbb{P})} \right\|^2_{L_2(\bar{\mathbb{P}})} \le \|f\|^2_{BV} \le \|f\|^2_{BV} 2^{1/\beta}(1-t)^{1/\beta}$$

and the claim follows from Lemma 9. □

Theorem 15. *Let $K \in \mathbb{R}$.*

(a) It holds that $\mathbb{1}_{[K,\infty)}(X_1) \in \mathbb{D}_{1,2}$ if and only if $\beta < 1$.

(b) It holds that $\mathbb{1}_{[K,\infty)}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ for all $\theta \in (0,1)$ and $q \in [1,\infty]$ if and only if $\beta \le 1$.

(c) Let $\theta \in (0,1)$ and $q \in [1,\infty)$. Then

(i) $\mathbb{1}_{[K,\infty)}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ if and only if $\beta < 1/\theta$ and

(ii) $\mathbb{1}_{[K,\infty)}(X_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$ if and only if $\beta \le 1/\theta$.

(d) Let $\theta \in (0,1)$ and $q \in [1,\infty)$. For the Brownian motion B we have that

(i) $\mathbb{1}_{[K,\infty)}(B_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,q}$ if and only if $2 < 1/\theta$ and

(ii) $\mathbb{1}_{[K,\infty)}(B_1) \in (L_2(\mathbb{P}), \mathbb{D}_{1,2})_{\theta,\infty}$ if and only if $2 \le 1/\theta$.

Proof. (a) The claim follows from Theorem 6(a) and the proof of Theorem 6(b), since by [25, Theorem 24.10(ii)] the continuous density of X_1 is strictly positive on the whole real line.
(b) The "if" follows from Theorem 14 and the "only if" follows from (c), since $\beta < 1/\theta$ for all $\theta \in (0,1)$.
(c) The "if"-parts of (i) and (ii) follow from Theorem 14(c) and (3).
Fix $r > 0$ and $t_0 \in (0,1)$. By (13), the function $\mathbb{1}_{[K,\infty)}$ satisties (11) with $[a,b] = [K-r, K+r]$, $c_2 = 1$ and $\eta = 1$. Thus, Lemma 12 implies that

$$\left\| \| f(X_1) - f(X_t + \bar{X}_{1-t}) \|_{L_2(\mathbb{P})} \right\|^2_{L_2(\bar{\mathbb{P}})} \ge c(1-t)^{1/\beta}$$

for some $c > 0$. The "only if"-parts of (c) follow now from Lemma 9.
(d) We choose E and a like in the proof of Lemma 9 on the corresponding Wiener chaos. The claim follows from Lemma 8 and the proof in [16, Example 4.7], where it is shown that $(Ta)'(t) \sim (1-t)^{-1/2}$. □

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Appendix A.

The reiteration theorem states that $(A_0, A_1)_{\eta\theta,q} = (A_0, (A_0, A_1)_{\eta,\infty})_{\theta,q}$ for all $\eta, \theta \in (0,1)$ and $q \in [1,\infty]$ with equivalent norms. In the following lemma we compute the explicit constants for the equivalence of the norms for $q = \infty$.

Lemma 16. *Let (A_0, A_1) be a compatible couple and $\eta, \theta \in (0,1)$. Then*

$$\|f\|_{(A_0,A_1)_{\eta\theta,\infty}} \leq \|f\|_{(A_0,(A_0,A_1)_{\eta,\infty})_{\theta,\infty}} \leq 3\|f\|_{(A_0,A_1)_{\eta\theta,\infty}}$$

for all $f \in (A_0, A_1)_{\eta\theta,\infty} = (A_0, (A_0, A_1)_{\eta,\infty})_{\theta,\infty}$.

Proof. First inequality: Let $t > 0$ and $\varepsilon > 0$. There exist $f_0, g_0 \in A_0$, $g \in (A_0, A_1)_{\eta,\infty}$ and $g_1 \in A_1$ such that $f = f_0 + g = f_0 + g_0 + g_1$ and

$$\begin{aligned}
K(f, t^\eta; A_0, (A_0, A_1)_{\eta,\infty}) &\geq \|f_0\|_{A_0} + t^\eta \|g\|_{(A_0,A_1)_{\eta,\infty}} - \frac{\varepsilon}{2} \\
&\geq \|f_0\|_{A_0} + t^\eta t^{-\eta} \left(\|g_0\|_{A_0} + t\|g_1\|_{A_1} - \frac{\varepsilon}{2} \right) - \frac{\varepsilon}{2} \\
&\geq \|f_0 + g_0\|_{A_0} + t\|g_1\|_{A_1} - \varepsilon \\
&\geq K(f, t; A_0, A_1) - \varepsilon.
\end{aligned}$$

Thus

$$\begin{aligned}
\|f\|_{(A_0,(A_0,A_1)_{\eta,\infty})_{\theta,\infty}} &= \sup_{t>0} (t^\eta)^{-\theta} K(f, t^\eta; A_0, (A_0, A_1)_{\eta,\infty}) \\
&\geq \sup_{t>0} t^{-\eta\theta} K(f, t; A_0, A_1) \\
&= \|f\|_{(A_0,A_1)_{\eta\theta,\infty}}.
\end{aligned}$$

Second inequality: Let $f \in (A_0, A_1)_{\eta\theta,\infty}$ and $\varepsilon > 0$. For all $t > 0$ we find $g_t \in A_0$ and $h_t \in A_1$ such that $f = g_t + h_t$ and

$$\|g_t\|_{A_0} + t\|h_t\|_{A_1} \leq K(f, t; A_0, A_1) + \frac{\varepsilon}{2} t^{\eta\theta}.$$

Then

$$\begin{aligned}
&K(g_t, s; A_0, A_1) \leq \|g_t\|_{A_0} \leq K(f, t; A_0, A_1) + \frac{\varepsilon}{2} t^{\eta\theta} \text{ and} \\
&K(h_t, s; A_0, A_1) \leq s\|h_t\|_{A_1} \leq \frac{s}{t} \left[K(f, t; A_0, A_1) + \frac{\varepsilon}{2} t^{\eta\theta} \right]
\end{aligned}$$

for all $s \in (0,\infty)$. These inequalities give, keeping in mind that $h_t = f - g_t$, that

$$
\begin{aligned}
t^{\eta}\|h_t\|_{(A_0,A_1)_{\eta,\infty}} &= t^{\eta}\sup_{s>0} s^{-\eta} K(h_t,s;A_0,A_1) \\
&\leq \left(\sup_{0<s\leq t}\left(\frac{s}{t}\right)^{-\eta}\frac{s}{t}\left[K(f,t;A_0,A_1)+\frac{\varepsilon}{2}t^{\eta\theta}\right]\right)\vee \\
&\quad \left(\sup_{s\geq t}\left(\frac{s}{t}\right)^{-\eta}\left[K(f,s;A_0,A_1)+K(g_t,s;A_0,A_1)\right]\right) \\
&\leq \left(K(f,t;A_0,A_1)+\frac{\varepsilon}{2}t^{\eta\theta}\right)\vee \\
&\quad \left(\sup_{s\geq t}\left(\frac{s}{t}\right)^{-\eta}\left[K(f,s;A_0,A_1)+K(f,t;A_0,A_1)+\frac{\varepsilon}{2}t^{\eta\theta}\right]\right) \\
&\leq K(f,t;A_0,A_1)+\frac{\varepsilon}{2}t^{\eta\theta}+\sup_{s\geq t}\left(\frac{s}{t}\right)^{-\eta\theta} K(f,s;A_0,A_1).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\|f\|_{(A_0,(A_0,A_1)_{\eta,\infty})_{\theta,\infty}} \\
&= \sup_{t>0}\left(t^{\eta}\right)^{-\theta} K\left(f,t^{\eta};A_0,(A_0,A_1)_{\eta,\infty}\right) \\
&\leq \sup_{t>0} t^{-\eta\theta}\left(\|g_t\|_{A_0}+t^{\eta}\|h_t\|_{(A_0,A_1)_{\eta,\infty}}\right) \\
&\leq \sup_{t>0} t^{-\eta\theta}\left(2K(f,t;A_0,A_1)+\varepsilon t^{\eta\theta}+\sup_{s\geq t}\left(\frac{s}{t}\right)^{-\eta\theta} K(f,s;A_0,A_1)\right) \\
&\leq 3\sup_{s>0} s^{-\eta\theta}K(f,s;A_0,A_1)+\varepsilon \\
&= 3\|f\|_{(A_0,A_1)_{\eta\theta,\infty}}+\varepsilon.
\end{aligned}
$$

□

Lemma 17. *Let $\alpha \in (0,1)$. Then $C_b^{\alpha} = (B(\mathbb{R}), Lip)_{\alpha,\infty}$ with*

$$\|\cdot\|_{C_b^{\alpha}} \leq 3\|\cdot\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}} \leq 6\|\cdot\|_{C_b^{\alpha}}.$$

Proof. First inequality: Let $f \in (B(\mathbb{R}), Lip)_{\alpha,\infty}$ and $\varepsilon > 0$. For all $t > 0$ we find $f_t \in Lip$ such that

$$t^{-\alpha}\left(\|f-f_t\|_{\infty}+t\|f_t\|_{Lip}\right) \leq \|f\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}}+\varepsilon.$$

Let $x \neq y \in \mathbb{R}$ and $t = |x-y| > 0$. By the triangle inequality we have

$$
\begin{aligned}
\frac{|f(x)-f(y)|}{|x-y|^{\alpha}} &\leq |x-y|^{-\alpha}\left(|f(x)-f_t(x)|+|f(y)-f_t(y)|+|f_t(x)-f_t(y)|\right) \\
&\leq t^{-\alpha}\left(2\|f-f_t\|_{\infty}+t\|f_t\|_{Lip}\right) \\
&\leq 2\left(\|f\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}}+\varepsilon\right).
\end{aligned}
$$

It also holds that

$$\|f\|_\infty \le \|f - f_1\|_\infty + \|f_1\|_\infty \le \|f\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}} + \varepsilon,$$

so that

$$\|f\|_{C_b^\alpha} = \|f\|_\infty + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \le 3\|f\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}}.$$

Second inequality: Let $f \in C_b^\alpha$ and $t > 0$ and define f_t so that $f_t(kt) = f(kt)$ for $k \in \mathbb{Z}$ and f_t is linear on each interval $[kt,(k+1)t]$, $k \in \mathbb{Z}$. Then for $x \in [kt,(k+1)t]$ there is $s \in [0,1]$ such that $f_t(x) = sf(kt) + (1-s)f((k+1)t)$ and we get that

$$\begin{aligned}
\|f - f_t\|_\infty &= \sup_{k\in\mathbb{Z}} \sup_{x\in[kt,(k+1)t]} |f(x) - f_t(x)| \\
&\le \sup_{k\in\mathbb{Z}} \sup_{x\in[kt,(k+1)t]} \big(s|f(x) - f(kt)| + (1-s)|f(x) - f((k+1)t)|\big) \\
&\le \sup_{|x-y|\le t} |f(x) - f(y)| \\
&\le t^\alpha \|f\|_{C_b^\alpha}.
\end{aligned}$$

For the function f_t it holds for $0 < t \le 1$ that

$$\begin{aligned}
\|f_t\|_{Lip} &= \|f_t\|_\infty + \sup_{x\neq y} \frac{|f_t(x) - f_t(y)|}{|x-y|} \\
&\le \|f\|_\infty + \sup_{k\in\mathbb{Z}} \frac{|f(kt) - f((k+1)t)|}{t} \\
&\le \|f\|_\infty + t^{\alpha-1} \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \\
&\le t^{\alpha-1}\|f\|_{C_b^\alpha}.
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
\|f\|_{(B(\mathbb{R}),Lip)_{\alpha,\infty}} &\le \left[\sup_{0<t\le 1} t^{-\alpha}\left(\|f - f_t\|_\infty + t\|f_t\|_{Lip}\right)\right] \vee \sup_{t\ge 1} t^{-\alpha}\|f\|_\infty \\
&\le 2\|f\|_{C_b^\alpha}.
\end{aligned}$$

□

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