JYU DISSERTATIONS 232

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Existence of Optimal Transport Maps with Applications in Metric Geometry

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS
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List of included articles

This dissertation consists of an introductory part and the following four publications:

[A] T. Schultz, *Existence of optimal transport maps in very strict $CD(K,\infty)$ -spaces*, Calc. Var. Partial Differential Equations. **57**, (2018), no. 5, Art. 139.

[B] T. Rajala and T. Schultz, *Optimal transport maps on Alexandrov spaces revisited*, preprint, arXiv:1803.10023.

[C] T. Schultz, *Equivalent definitions of very strict $CD(K,N)$ -spaces*, preprint, arXiv:1906.07693.

[D] T. Schultz, *On one-dimensionality of metric measure spaces*, Proc. Amer. Math. Soc., to appear.

The author of this dissertation has actively taken part in the research of the joint article [B].

INTRODUCTION

The main structure present in the dissertation is the structure of metric measure space. Throughout the thesis, by a metric measure space $(X, d, \mathfrak{m})$, we mean a complete and separable metric space (X, d) equipped with a locally finite Borel measure $\mathfrak{m}$. The space (X, d) is most of the time assumed (explicitly or implicitly) to be a length space, that is, the distance between any two points in the space X can be realised as an infimum of lengths of paths connecting them. If the infimum is a minimum for any pair of points, we say that the space is a geodesic space.

Any constant speed curve parametrised by $[0, 1]$ whose length is equal to the distance between the endpoints is called a geodesic, and the set of all geodesics in X is denoted by $\mathrm{Geo}(X)$. The space of geodesics $\mathrm{Geo}(X)$ is equipped with the supremum distance. It is complete and separable as a closed subset of $C([0, 1], X)$. We will denote by $e_t \colon \mathrm{Geo}(X) \to X$ the evaluation map $\gamma \mapsto \gamma_t$, and by $\mathrm{restr}_t^s \colon \mathrm{Geo}(X) \to \mathrm{Geo}(X)$ the map that sends a geodesic γ to a geodesic $\gamma|_{[t,s]}$ reparametrised by $[0, 1]$.

We say that two geodesics γ^1 and γ^2 branch, if $\gamma^1|_{[0,t]} = \gamma^2|_{[0,t]}$ for some $t \in (0, 1)$, but $\gamma^1 \neq \gamma^2$. Moreover, a set $\Gamma \subset \mathrm{Geo}(X)$ is said to be non-branching, if $\mathrm{restr}_0^t|_\Gamma$ is injective for all $t \in (0, 1)$. A space (X, d) is called non-branching, if $\mathrm{Geo}(X)$ is non-branching, and a metric measure space $(X, d, \mathfrak{m})$ is called essentially non-branching if any optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ between probability measures $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$ is concentrated on a set of non-branching geodesics, that is, there exists a non-branching set Γ so that $\pi(\Gamma) = 1$, see Section 1.1 for the necessary definitions.

In the dissertation, the existence of so-called optimal transport maps is proven for metric (measure) spaces satisfying a certain generalised sectional or Ricci curvature lower bound. In the case of the sectional curvature bound, in Alexandrov spaces, the advantage of the Riemannian-like structure is taken to give a geometric proof for the existence of optimal transport maps under the assumption of the starting measure being purely $(n-1)$-unrectifiable.

Regarding the Ricci curvature bound, a modified curvature dimension condition, the *very strict* $CD(K, N)$ *-condition*, is introduced in order to obtain the existence of optimal transport maps on spaces that are not essentially non-branching. The existence of optimal maps is further used to obtain stronger convexity inequalities, both pointwise and integral ones, for the densities along optimal transport plans, leading to the equivalence of two variants of very strict $CD(K, N)$ -conditions, one given in the spirit of $CD(K, N)$-condition à la Sturm [52, 53], and the other in the spirit of $CD(K, N)$-condition à la Lott and Villani [37].

Motivated by the existence of optimal maps, one-dimensionality of metric measure spaces is studied. More precisely, the optimal maps together with the existence of a one-dimensional part, at local or infinitesimal level, of a metric measure space is used to guarantee that the space in question is a one-dimensional manifold. Thus, for very strict $CD(K,N)$ -spaces, and for essentially non-branching $MCP(K,N)$ -spaces, having a one-dimensional part in the space immediately implies that the space is globally one-dimensional.

1. Optimal mass transportation

The role of the optimal mass transportation in this dissertation is threefold. First of all, the question of the existence of optimal transport maps is the basis for the thesis. Secondly, the optimal mass transportation is built into the definition of Ricci curvature lower bounds on non-smooth spaces. Thirdly, the existence of optimal transport maps is used as a tool in the study of the spaces in question.

The theory of optimal mass transportation boils down to the study of the so-called *Monge–Kantorovich* minimisation problem which reads as follows. Let $c\colon X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function, and let μ and ν be probability measures on the spaces X and Y, respectively. Consider the minimisation

$$\inf_{\sigma} \int c(x,y)\,\mathrm{d}\sigma(x,y), \tag{1.1}$$

where the infimum is taken over all probability measures σ that have μ as the first marginal and ν as the second marginal (i.e. $\mathrm{P}^1_{\#}\sigma = \mu$ and $\mathrm{P}^2_{\#}\sigma = \nu$). Such an admissible measure σ is called a *transport plan*. The function c is called a *cost function*.

The infimum (1.1) is realised under reasonably mild assumptions including the framework of the dissertation (see e.g. [54]); from here on we will assume that (X,d) is a complete and separable metric space, $X = Y$, μ and ν are Borel probability measures on X (and any transport plan σ is a Borel probability measure on $X \times X$). Furthermore, the cost used is the quadratic cost $c = d^2$. A plan σ that realises the infimum (1.1) is called an *optimal transport plan* and the set of all optimal plans between μ and ν is denoted by $\mathrm{Opt}(\mu,\nu)$. The set of transport plans is denoted by $\mathcal{A}(\mu,\nu)$

With the assumptions above the interpretation of the Monge–Kantorovich problem as a mass transportation problem is quite intuitive: given an initial distribution μ of the mass (soil for example) and a final distribution ν, a transport plan σ tells that mass from position x to position y is to be transported if $(x,y) \in \operatorname{spt}\sigma$. The cost function $c = d^2$ evaluated at (x,y) tells how much it costs to transport a unit mass from x to y, and the total cost for the transport plan σ is given by the integral

$$\int d^2(x,y)\,\mathrm{d}\sigma(x,y).$$

Even though the quadratic cost is probably the most studied one among all cost functions, it is worth pointing out that there are many other cost functions used in different fields of mathematics. As an example of a cost function leading to a quite different interpretation compared to the one described here, we mention the so-called Coulomb cost used in the (multi-marginal) optimal transport formulation of the strictly correlated electron functional in the density functional theory [14].

To study a minimiser of the Monge–Kantorovich problem more closely, it is convenient to have different characterisations for the optimality of transport plans. There are two highly useful characterisations dual to each other originating from the Kantorovich duality (that is, a dual formulation of the optimal transport problem) via convex analytic approach. The first one states that a transport plan is optimal if and only if it is concentrated on a c-subdifferential of a c-convex function. The second one, which is used multiple times in this dissertation, arises from the connection of subdifferentials and cyclically monotone sets: a transport plan is optimal if and only if it is concentrated on a c-cyclically monotone set.

Definition 1.1 (c-cyclically monotone set). *A set $\Gamma \subset X \times X$ is said to be* c-cyclically monotone *if for any $N \in \mathbb{N}$ and any $\{(x_i, y_i)\}_{i=1}^N \subset \Gamma$ it holds that*

$$\sum_i c(x_i, y_i) \leq \sum_i c(x_i, y_{\tau(i)}),$$

for any permutation τ.

Theorem (See, e.g. [54]). *Let μ and ν be Borel probability measures with*

$$\inf_{\sigma \in \mathcal{A}(\mu,\nu)} \int c(x, y) \, \mathrm{d}\sigma(x, y) < \infty.$$

Then a transport plan $\sigma \in \mathcal{A}(\mu, \nu)$ is optimal if and only if there exists a c-cyclically monotone set $\Gamma \subset X \times X$ such that

$$\sigma(\Gamma) = 1.$$

1.1. **The Wasserstein space.** The choice $c = d^2$ of the cost function leads to the following geometric interpretation of the transport problem. Denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X, and by $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ the subset of probability measures with finite second moment, i.e. the set of measures $\mu \in \mathcal{P}(X)$ for which

$$\int d^2(\,\cdot\,, x_0) \, \mathrm{d}\mu < \infty$$

for some $x_0 \in X$. Defining $W_2 \colon \mathcal{P}_2(X) \times \mathcal{P}_2(X) \to [0, \infty)$ by the formula

$$W_2^2(\mu_0, \mu_1) := \inf_{\sigma \in \mathcal{A}(\mu_0,\mu_1)} \int d^2(x, y) \, \mathrm{d}\sigma(x, y),$$

that is, as a square root of the optimal mass transportation cost in the quadratic optimal mass transportation problem, one obtains the so-called *Wasserstein distance* (or 2-Wasserstein distance, to be more precise) on the set $\mathcal{P}_2(X)$. It is straightforward (with the help of a suitable "gluing" lemma guaranteed by the disintegration theorem) to prove that the Wasserstein distance W_2 is an actual distance function on the set $\mathcal{P}_2(X)$, see for example [54]. The metric space $(\mathcal{P}_2(X), W_2)$ is called the Wasserstein space. It can be shown that since the space (X, d) is complete and separable, so is the Wasserstein space $(\mathcal{P}_2(X), W_2)$. The Wasserstein space inherits also other properties from the original space, namely the space $(\mathcal{P}_2(X), W_2)$ is a length space if and only if (X, d) is, and it is a geodesic space if and only if the space (X, d) is. The geodesics of the space $(\mathcal{P}_2(X), W_2)$ are called Wasserstein geodesics.

In the case of (X, d) being a geodesic space, it is quite intuitive that the optimal way of transporting the mass should be along the geodesics and according to the optimal plan. This intuition is made rigorous in the following lifting property of Wasserstein geodesics.

Theorem ([36]). *Let (X, d) be a complete and separable metric space. Then (X, d) is a length space if and only if $(\mathcal{P}_2(X), W_2)$ is. Moreover, a curve $t \mapsto \mu_t \in \mathcal{P}_2(X)$ is a geodesic if and only if there exists a measure $\pi \in \mathcal{P}(\mathrm{Geo}(X))$ so that $\mu_t = (e_t)_\# \pi$ for all $t \in [0, 1]$, and $(e_0, e_1)_\# \pi \in \mathrm{Opt}(\mu_0, \mu_1)$.*

Any such π is called an *optimal dynamical plan*, or just an optimal plan for short, and the set of all optimal dynamical plans from μ_0 to μ_1 is denoted by $\mathrm{OptGeo}(\mu_0, \mu_1)$.

1.2. **Optimal transport maps.** The Monge–Kantorovich problem (1.1) is a relaxation of Monge's original formulation[1] (put into the modern mathematical language) in which only measures of the form $\sigma = (\mathrm{id}, T)_\# \mu$ were considered. While the existence of optimal transport plans is fairly easy to prove via direct method in the calculus of variations, the existence of a minimiser of the form $\sigma = (\mathrm{id}, T)_\# \mu$ is highly non-trivial and actually requires more assumptions on the marginals μ and ν, and on the space X.

When the optimal transport plan is of the form $\sigma = (\mathrm{id}, T)_\# \mu$, it is said to be induced by the map T (from μ). Such a map is called an *optimal transport map*. The first positive result about the existence of optimal transport maps was given by Brenier [9, 10] (see also [50, 49]). Brenier's approach was via the Kantorovich duality: any optimal plan is concentrated on the c-subdifferential of a c-convex function, the so-called Kantorovich potential, and so by the differentiability results from convex analysis the existence of an optimal transport map is obtained whenever the first marginal measure μ is absolutely continuous with respect to the Lebesgue measure. This analytic approach has been generalised and refined both in the smooth setting – in the Riemannian framework by McCann [40] and Gigli [25], and in the sub-Riemannian setting by Ambrosio and Rigot [5], Agrachev and Lee [1], and by Figalli and Rifford [23] – and in the non-smooth setting – in Alexandrov spaces by Bertrand [7, 8].

In this thesis the viewpoint is more geometric rather than analytic. Instead of relying on differentiability properties of the Kantorovich potential, we use the c-cyclical monotonicity of the optimal plan together with geometric properties of the underlying space. The advantage of such a direct approach is that it does not require any smooth structure (even in the weak sense of Alexandrov spaces) and thus opens up possibilities in the metric space setting. Indeed, many results about the existence of optimal transport maps have been proven following this philosophy.

In [26], Gigli proved the existence and uniqueness of the optimal transport map for absolutely continuous measures in the setting of non-branching $CD(K, N)$ spaces. In [48] (see also [30]), Rajala and Sturm generalised the result of Gigli (still following the same proof idea) to the context of strong $CD(K, N)$ spaces, in which the non-branching

[1] Actually, Gaspard Monge used the cost $c(x, y) = d(x, y)$ in his original work [42].

assumption was not needed. Rajala and Sturm introduced a notion of essentially non-branching metric spaces and proved that the strong $CD(K,N)$ condition is strong enough to imply such a property. The essential non-branching together with the Ricci curvature lower bound was then enough to push through the strategy used in [26] for the existence of optimal transport maps.

The connection of the essential non-branching property with the existence of optimal transport maps was further developed in [17] by Cavalletti and Mondino. There it was proven that in an essentially non-branching metric measure space satisfying the so-called measure contraction property, the existence of optimal transport maps holds under the assumption of absolute continuity of the first marginal. In the paper [15] of Cavalletti and Huesmann, the emphasis was put on more general cost functions. They proved in a non-branching setting that under a qualitative non-degeneracy condition (which is implied for example by the $MCP(K,N)$ condition) on the reference measure, the existence of optimal transport maps holds for any increasing, strictly convex cost function of the distance. Continuing from, and putting together [17] and [15], Kell proved in [32] for qualitatively non-degenerate spaces that the property of having the existence and uniqueness of optimal transport maps is characterised by the essential non-branching property.

In papers [A], [C] and [D], the focus is mostly on spaces which might fail to satisfy the essentially non-branching assumption. Due to this fact, the uniqueness of optimal transport plans is out of reach. However, still following the ideas from [26] and [48], the existence of optimal transport maps is proven in very strict $CD(K,N)$ -spaces.

Theorem 1.2 ([A] Theorem 1, [C] Theorem 3.1)**.** *Let $(X,d,\mathfrak{m})$ be a metric measure space satisfying the very strict $CD(K,N)$ -condition for $N \in (1,\infty]$, and let $\mu_0,\mu_1 \in \mathcal{P}_2^{ac}(X)$. Then there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ that is induced by a Borel map $T\colon X \to \mathrm{Geo}(X)$, i.e. $\pi = T_\#\mu_0$ and $e_0 \circ T = \mathrm{id}$.*

The set $\mathcal{P}_2^{ac}(X)$ is the subset of probability measures that are absolutely continuous with respect to the reference measure $\mathfrak{m}$. If $N < \infty$, the assumption of absolute continuity of the final measure μ_1 can be dropped, see Theorem 3.3 in [C].

Here the optimal transport maps are thought of as maps to the space of geodesics: instead of telling only where to send a point x, the map also tells via which route to transport it. In geodesic spaces, the question about the existence of optimal transport maps on the level of $\mathrm{Opt}(\mu_0,\mu_1)$ and on the level of $\mathrm{OptGeo}(\mu_0,\mu_1)$ are the same. Indeed, if $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ is given by a map T, then the plan $\sigma := (e_0,e_1)_\#\pi$ is induced by the map $e_1 \circ T$. On the other hand, if $\sigma \in \mathrm{Opt}(\mu_0,\mu_1)$ is given by a map T, then by a measurable selection argument, there exists a measurable map $S\colon X \times X \to \mathrm{Geo}(X)$ selecting a geodesic from x to y for any pair of points $(x,y) \in X \times X$. Then the pushforward of μ_0 under the composed map $S \circ (\mathrm{id},T)$ is an optimal dynamical plan induced by the map $S \circ (\mathrm{id},T)$.

Even though the existence result might not change depending on the viewpoint,"static" or "dynamic", there is a crucial difference when studying the finer properties of the optimal plan. For example, let μ_0 be a uniform measure on a square in $\mathbb{R}^2$ equipped with the supremum norm, and let μ_1 be the measure obtained by pushing μ_0 forward by a horizontal translation T. Then the plan $\sigma = (\mathrm{id}, T)_\# \mu_0$ is an optimal plan, and the obvious choice for the dynamical plan would be via the map S that for any pair of points $(x, T(x))$ assigns the Euclidean geodesic (that is, the line segment). However, since there is an enormous amount of geodesics connecting x to $T(x)$, one can as well make the selection of geodesics in a way that the measure $\mu_t := (e_t)_\# \pi$ would be a purely singular measure with respect to the Lebesgue measure for all $t \in (0, 1)$. Therefore, knowing the static plan σ does not say too much about the dynamical plan.

In fact, in [A] and [C] the formulation of the statement of Theorem 1.2 was not only claiming the existence of a generic optimal map, but also that the plan induced by a map has other nice properties, namely the plan is the one given by the definition of very strict $CD(K, N)$ -spaces and thus satisfies a suitable entropic convexity property, see Section 2.

In Theorem 1.2 the focus is on the assumptions on the underlying metric measure space, while the conditions on the marginals μ_0 and μ_1 are relatively strong (mainly because of the lack of structure on the space). In the Euclidean setting, McCann proved that it suffices to assume that the measure μ_0 gives zero measure to sets of co-dimension at least one to deduce the existence and uniqueness of optimal transport maps [38]. In [24] it was realised that it is enough to assume that μ_0 vanishes on so-called $c - c$ hypersurfaces. In [25] Gigli further proved that this assumption was also sharp in the sense that for any μ_0 that gives positive mass for some $c - c$ hypersurface, there exists a measure μ_1 and optimal plan $\sigma \in \mathrm{Opt}(\mu_0, \mu_1,)$ which is not induced by a map. Gigli also proved this sharp version of the theorem in the Riemannian setting. The result using $c - c$ hypersurfaces was further generalised to Alexandrov spaces by Bertrand in [8] where DC structure of the regular part of the space and the size of the singular set was used (here the sharpness of the result was not recovered).

In [B] we give a geometric proof for a slightly weaker form of Bertrand's result in Alexandrov spaces, namely instead of measures vanishing on $c - c$ hypersurfaces, we considered so-called purely unrectifiable measures.

Definition 1.3. *A measure μ is* purely k-unrectifiable *if $\mu(G) = 0$ for all sets G of the form $G = f(E)$, where $E \subset \mathbb{R}^k$ is a Borel set, and $f \colon E \to X$ is Lipschitz.*

Theorem 1.4 ([B] Theorem 1.1)**.** *Let (X, d) be an n-dimensional Alexandrov space, and let $\mu_0 \in \mathcal{P}_2(X)$ be purely $(n-1)$-unrectifiable. Then for every $\mu_1 \in \mathcal{P}_2(X)$ there exists a unique optimal transport plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ which is induced by a map from μ_0.*

2. Curvature lower bounds

In this dissertation two classes of curvature bounds are considered, namely generalised sectional and Ricci curvature lower bounds. Motivated by the study of intrinsic geometry of non-smooth surfaces, convex surfaces in particular, A. D. Alexandrov introduced in the 50's the notion of curvature lower bounds for metric spaces. The definition is given in terms of a comparison geometry: one of the several equivalent definitions requires that to be a space with curvature bounded below by a constant κ, the "thickness" of geodesic triangles in a simply connected surface with constant curvature κ should serve as a lower bound for a thickness of geodesic triangles in the metric space itself (cf. Toponogov's triangle comparison theorem). For a Riemannian manifold to have the curvature bounded from below by κ in the sense of Alexandrov is equivalent to having the sectional curvature bounded from below by κ.

A natural question is whether it is possible to have generalisations of other curvature (bounds) from the smooth setting to singular metric spaces. In regard to Ricci curvature lower bounds, this question was answered independently by Sturm in [52, 53] and by Lott and Villani in [37]. The definition is based on a connection of Ricci curvature lower bounds and (displacement) convexity properties of suitable entropy functionals on the Wasserstein space of Riemannian manifold. These connections were present at a formal level in [45] in the case of non-negative curvature, rigorously treated in [21], and finally generalised and used to obtain a synthetic formulation of Ricci curvature lower bounds for general bound K in [55] (see also [51]).

While the notion of Alexandrov spaces is defined in purely metric terms, the definition of Ricci curvature lower bounds requires additional structure, namely the structure of metric measure spaces.

2.1. **Synthetic Ricci curvature lower bounds.** There are a few variants of the curvature dimension condition $CD(K, N)$ ("curvature bounded from below by K, and dimension bounded from above by N") present in this dissertation with subtle differences which will be unraveled. The case $N = \infty$ will be presented first to give the correct picture without the unavoidable technicalities coming from the finiteness of the dimension upper bound N. After that, also the more involved definitions for the general case will be covered.

Let $(X, d, \mathfrak{m})$ be a complete and separable length metric space (X, d) equipped with a locally finite Borel measure $\mathfrak{m}$. Define an entropy functional $\mathrm{Ent}_\infty\colon \mathcal{P}_2(X) \to \mathbb{R} \cup \{\pm\infty\}$ as

$$\mathrm{Ent}_\infty(\mu) := \begin{cases} \int \rho \log \rho \, \mathrm{d}\mathfrak{m}, & \text{if } \mu \ll \mathfrak{m} \text{ and } (\rho \log \rho)_+ \in \mathrm{L}^1(\mathfrak{m}) \\ \infty & \text{otherwise,} \end{cases}$$

where ρ is the density (i.e. the Radon-Nikodym derivative) of μ with respect to the reference measure $\mathfrak{m}$, and $(\rho \log \rho)_+ = \max\{\rho \log \rho, 0\}$.

Definition 2.1 ((Weak) $CD(K, \infty)$-condition)**.** *The space $(X, d, \mathfrak{m})$ is called a $CD(K, \infty)$-space if for every $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$, there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ for which*

$$\mathrm{Ent}_\infty(\mu_t) \leq (1-t)\mathrm{Ent}_\infty(\mu_0) + t\mathrm{Ent}_\infty(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0, \mu_1), \tag{2.1}$$

for any $t \in [0, 1]$, where $\mu_t = (e_t)_\# \pi$.

We say that the entropy Ent_∞ is K-convex along the plan π if (2.1) is satisfied. If the convexity is required to hold along *every* optimal plan π, instead of just one, the space is called a *strong* $CD(K, \infty)$-space.

Notice that in Definition 2.1 the convexity condition is actually on the level of Wasserstein geodesics and not on the geodesic plans. Such an unnecessary complication is motivated by the following definition of the so-called *very strict* $CD(K, \infty)$ -spaces which is more relevant for the purposes of this dissertation.

Definition 2.2 (Very strict $CD(K,\infty)$ -condition)**.** *The space* $(X,d,\mathfrak{m})$ *is called a* very strict $CD(K,\infty)$ *-space if for every* $\mu_0,\mu_1 \in \mathcal{P}_2^{ac}(X)$*, there exists an optimal plan* $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ *so that the entropy* Ent_∞ *is* K*-convex along* $(\mathrm{restr}_{t_0}^{t_1})_\#(f\pi)$ *for every* $t_0,t_1 \in [0,1]$ *with* $t_0 < t_1$*, and for every non-negative and bounded Borel function* $f\colon \mathrm{Geo}(X) \to \mathbb{R}$ *with* $\int f \,\mathrm{d}\pi = 1$*.*

The definition of very strict $CD(K,\infty)$ -spaces is a modification of (possibly) less restrictive *strict* $CD(K,\infty)$ -condition (see [4]), hence the name. The difference is that in the very strict $CD(K,\infty)$ setting the convexity is required between any three points t_0, t_1, and t_2, and not only between 0, t, and 1.

On a Riemannian manifold one has a well-defined dimension, and so in principle one could work with a fixed curvature lower bound and fixed dimension. This is however not the case in the non-smooth setting: already when considering the limits of Riemannian manifolds (with fixed Ricci curvature lower bound and of fixed dimension), one might end up with a space of lower dimension. Moreover, when proving geometric and analytic properties under the assumption on curvature bounds, it is convenient to work directly with formulation that captures the finite dimensionality.

To give the corresponding definition of very strict $CD(K,N)$ -spaces for $K \in \mathbb{R}$ and for finite N, we need to introduce the following volume distortion coefficients. First, define for $K \in \mathbb{R}$ and $N \in (0,\infty]$, the coefficients $[0,1]\times\mathbb{R}_+ \to \mathbb{R}\cup\{\infty\}$, $(t,\theta)\mapsto \sigma_{K,N}^{(t)}(\theta)$ as

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} t, & \text{if } N=\infty \\ \infty, & \text{if } K\theta^2 \geq N\pi^2 \\ \frac{\sin(t\theta\sqrt{\frac{K}{N}})}{\sin(\theta\sqrt{\frac{K}{N}})}, & \text{if } 0 < K\theta^2 < N\pi^2 \\ t, & \text{if } K=0 \\ \frac{\sinh(t\theta\sqrt{\frac{-K}{N}})}{\sinh(\theta\sqrt{\frac{-K}{N}})}, & \text{if } K<0. \end{cases}$$

Then for $N \in (1,\infty]$, define the coefficients $\tau_{K,N}^{(t)}(\theta)$ as

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}}(\sigma_{K,N-1}^{(t)}(\theta))^{\frac{N-1}{N}}.$$

We also need the finite dimensional counterpart for the entropy: for $N > 1$, define $\mathrm{Ent}_N\colon \mathcal{P}_2(X) \to \mathbb{R}\cup\{\pm\infty\}$ as

$$\mathrm{Ent}_N(\mu) := -\int \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}$$

for $\mu = \rho\mathfrak{m} + \mu^\perp$, where $\mu^\perp$ is the singular part of μ with respect to $\mathfrak{m}$.

Definition 2.3 (Very strict $CD(K,N)$ -condition)**.** *We say that a metric measure space* $(X,d,\mathfrak{m})$ *is a very strict* $CD(K,N)$ *-space for* $K\in\mathbb{R}, N>1$*, if for any* $\mu_0,\mu_1 \in \mathcal{P}_2^{ac}(X)$*, there exists* $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ *such that for all non-negative and bounded Borel functions* $F\colon \mathrm{Geo}(X)\to\mathbb{R}$ *with* $\int F\,\mathrm{d}\pi = 1$*, and for all* $t_1,t_2\in[0,1]$*,* $t_1<t_2$ *it holds that*

$$\mathrm{Ent}_N(\tilde{\mu}_t) \leq -\int \tau_{K,N}^{(1-t)}(d(\gamma_0,\gamma_1))\tilde{\rho}_0^{-\frac{1}{N}}(\gamma_0) + \tau_{K,N}^{(t)}(d(\gamma_0,\gamma_1))\tilde{\rho}_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\tilde{\pi}, \tag{2.2}$$

for $\tilde{\mu}_t := (e_t)_\#\tilde{\pi} := (e_t)_\#(\mathrm{restr}_{t_1}^{t_2})_\# F\pi$, *and for all* $t \in [0,1]$.

Notice that the inequality (2.2) is a distorted convexity inequality, and in the case of $K = 0$ it is just the convexity of the entropy Ent_N along the plan $\tilde{\pi}$ i.e.

$$\mathrm{Ent}_N(\tilde{\mu}_t) \leq (1-t)\mathrm{Ent}_N(\tilde{\mu}_0) + t\mathrm{Ent}_N(\tilde{\mu}_1).$$

It is worth pointing out that in some cases when the problem in question is possible to localise, (see for example [C] Theorem 3.1) the exact form of the distortion coefficients is not important, but instead what matters is the fact that $\tau_{K,N}^{(t)}(\theta) \sim t$ when θ is close to zero. In such a localisable situation one reduces the problem, at least morally, to the case of $K = 0$.

We will also need the definition of very strict $CD^*(K,N)$ -spaces. The definition is the same as the definition of very strict $CD(K,N)$ -spaces with the difference that the distortion coefficients $\tau_{K,N}$ are replaced by the slightly smaller coefficients $\sigma_{K,N}$. It is readily proven that very strict $CD(K,N)$ implies very strict $CD^*(K,N)$ for all K and N (see Proposition 2.5 [6]). For an essentially non-branching metric measure space $(X,d,\mathfrak{m})$ with finite reference measure $\mathfrak{m}$, it has recently been proven by Cavalletti and Milman that also the other implication is true [16].

The curvature dimension condition should be thought of as the integrated version of the convexity inequality

$$\rho_t^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(d(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0) + \tau_{K,N}^{(t)}(d(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1).$$

In fact, this pointwise inequality is equivalent to the $CD(K,N)$-condition in the Riemannian setting, and more generally in the context of essentially non-branching spaces (see [53, 16]). A natural question is then whether the pointwise convexity inequality could be taken as the definition for the generalisation of Ricci curvature lower bounds in metric setting. While such a pointwise condition gives stronger information about the space, the drawback is that it is less robust in the sense that it is not necessarily preserved under limiting procedures (see Section 4 for an example).

To make this claim more precise, we mention the following stability properties of the different curvature dimension conditions. First of all, the $CD(K,N)$-condition (both à la Sturm and à la Lott–Villani) without any further assumptions is stable under pointed measured Gromov–Hausdorff convergence, that is, if $(X_i,d_i,\mathfrak{m}_i)$ is a sequence of $CD(K,N)$-spaces converging to $(X,d,\mathfrak{m})$ in pointed measured Gromov–Hausdorff[2] convergence, then $(X,d,\mathfrak{m})$ is a $CD(K,N)$-space. On the other hand, if one requires each metric measure space $(X_i,d_i,\mathfrak{m}_i)$ in addition to be essentially non-branching (and hence to satisfy the pointwise convexity condition), the limit space $(X,d,\mathfrak{m})$ might fail to be essentially non-branching and to satisfy the pointwise convexity condition (at least in the strong form of essentially non-branching spaces). Therefore also the strong $CD(K,N)$ condition fails

[2]One could also consider other notions of convergence such as the so-called **D**-convergence introduced by Sturm in [52]. However, the discussion of specific choice of convergence does not lie in the core of the dissertation, and we do not elaborate further on that. We refer to [29] for more detailed analysis on notions of convergence applicable (especially) in the framework of lower Ricci curvature bounds.

to pass to the limit in pointed measured Gromov–Hausdorff convergence. Thus, it is not satisfactory to study only spaces that are essentially non-branching – at least when allowing purely Finsler-like structures.

One goal of this dissertation is to better understand the stable situation and hence to allow also branching behaviour for the metric measure space. Towards this goal, the very strict $CD(K,N)$ -condition was introduced in [A] to obtain the existence (but not uniqueness) of optimal transport maps also in branching spaces. It should be emphasised however, that it is not known whether very strict (or strict) $CD(K,N)$ -condition is stable under suitable convergence, see Section 4 for further discussion. As a last remark, we point out that the $CD(K,N)$-condition coupled with the so-called infinitesimal Hilbertianity (the so-called $RCD(K,N)$-condition [4, 3, 27], R standing for *Riemannian*), and the so-called measure contraction property ($MCP(K,N)$ [53, 43]) are also stable under convergence of metric measure spaces. The MCP is suitable (also) for studying sub-Riemannian spaces, while RCD is "selecting" the Riemannian-like spaces among all $CD(K,N)$-spaces. One of the many nice features of $RCD(K,N)$-condition is that it is at the same time stable under convergence and implying essentially non-branching structure for the space in question [4, 3, 22, 48].

As mentioned before, the pointwise convexity condition is giving more precise information about the optimal transportation than the integrated one. Thus, it is natural to ask whether one could obtain pointwise information from the convexity of the entropy along optimal geodesic plans. The following theorem proven in [C] gives a positive answer to that in very strict $CD(K,N)$ -spaces.

Theorem 2.4 ([C] Proposition 4.2)**.** *A metric measure space $(X,d,\mathfrak{m})$ is a very strict $CD(K,N)$ -space for $N\in(1,\infty]$ if and only if for every $\mu_0,\mu_1\in\mathcal{P}_2^{ac}(X)$, there exists an optimal plan $\pi\in\mathrm{OptGeo}(\mu_0,\mu_1)$ so that $\mu_t:=(e_t)_\#\pi\ll\mathfrak{m}$ for every $t\in[0,1]$, and that the following two conditions hold:*

(i) For all $t\in(0,1)$, there exists a Borel map $T_t\colon X\to\mathrm{Geo}(X)$ for which $\pi=(T_t)_\#\mu_t$ and $e_t\circ T_t=\mathrm{id}$.

(ii) For every $t_1,t_2,t_3\in[0,1]$, $t_1<t_2<t_3$, the inequality

$$\begin{aligned}\log\rho_{t_2}(\gamma_{t_2})\leq&\frac{(t_3-t_2)}{(t_3-t_1)}\log\rho_{t_1}(\gamma_{t_1})+\frac{(t_2-t_1)}{(t_3-t_1)}\log\rho_{t_3}(\gamma_{t_3})\\&-\frac{K}{2}\frac{(t_3-t_2)}{(t_3-t_1)}\frac{(t_2-t_1)}{(t_3-t_1)}d^2(\gamma_{t_1},\gamma_{t_3})\end{aligned}\qquad(N=\infty)$$

$$-\rho_{t_2}^{-\frac{1}{N}}(\gamma_{t_2})\leq-\tau_{K,N}^{\frac{(t_3-t_2)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_1}^{-\frac{1}{N}}(\gamma_{t_1})-\tau_{K,N}^{\frac{(t_2-t_1)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_3}^{-\frac{1}{N}}(\gamma_{t_3})\qquad(N<\infty)$$

holds for π-almost every γ, where ρ_t is the density of μ_t with respect to the reference measure $\mathfrak{m}$.

As indicated before, the definitions of $CD(K,N)$-spaces by Sturm and by Lott and Villani differ from each other slightly. While Sturm required the convexity inequality to hold for specific entropy functionals $\mathrm{Ent}_{N'}$ (for *all* $N'\geq N$), Lott and Villani required it to hold for a wider class of entropy functionals, namely for functionals in the so-called *displacement convexity class* $\mathcal{DC}_N$ introduced by McCann in [39]. In the case of essentially non-branching spaces these two definitions agree (see [54, Theorem 30.32] for the proof in non-branching case). The key step to obtain the equivalence is to prove the pointwise convexity inequality for $CD(K,N)$ spaces. In [C], we generalise this to the very strict $CD(K,N)$ -spaces, or more precisely we give a definition for a Lott–Villani type analogue of the very strict $CD(K,N)$ -condition which is then proven to be equivalent to the very strict $CD(K,N)$ -condition.

To give the precise definition of the Lott–Villani analogue, we need some auxiliary definitions. First, we need yet another set of distortion coefficients. For $t > 0$ and $N > 1$, define $\beta^{(t)}_{K,N}(\theta)$ as

$$\beta^{(t)}_{K,N}(\theta) := \begin{cases} e^{\frac{K}{6}(1-t^2)\theta^2}, & \text{if } N = \infty \\ \infty, & \text{if } N < \infty,\ K\theta^2 > (N-1)\pi^2 \\ \left(\frac{\sin(t\theta\sqrt{\frac{K}{N-1}})}{t\sin(\theta\sqrt{\frac{K}{N-1}})}\right)^{N-1}, & \text{if } 0 < K\theta^2 \le (N-1)\pi^2 \\ 1, & \text{if } N < \infty,\ K = 0 \\ \left(\frac{\sinh(t\theta\sqrt{\frac{-K}{N-1}})}{t\sinh(\theta\sqrt{\frac{-K}{N-1}})}\right)^{N-1}, & \text{if } N < \infty,\ K < 0, \end{cases}$$

and $\beta^{(0)}_{K,N} \equiv 1$. In other words, $\beta^{(t)}_{K,N}(\theta) = t^{1-N}(\sigma^{(t)}_{K,N-1}(\theta))^{N-1}$. Secondly, we need the notion of the displacement convexity class $\mathcal{DC}_N$. We say that a convex and continuous function $U\colon \mathbb{R}_+ \to \mathbb{R}$ is in the class $\mathcal{DC}_N$ of dimension $N \in (1,\infty]$ if $U(0) = 0$, and if the function $u\colon (0,\infty) \to \mathbb{R}$,

$$s \mapsto u(s) := \begin{cases} s^N U(s^{-N}) & \text{if } N < \infty \\ e^s U(e^{-s}) & \text{if } N = \infty \end{cases}$$

is convex. We recall that the displacement convexity classes $\mathcal{DC}_N$ are nested, that is, $\mathcal{DC}_{N'} \subset \mathcal{DC}_N$ whenever $N < N'$. Finally, for every $U \in \mathcal{DC}_N$, we define the entropy functional $U_{\mathfrak{m}}\colon \mathcal{P}_2(X) \to \mathbb{R}\cup\{\infty\}$ as

$$U_{\mathfrak{m}}(\mu) := \int U \circ \rho \,\mathrm{d}\mathfrak{m} + \int U'(\infty)\,\mathrm{d}\mu^{\perp},$$

where $U'(\infty) := \lim_{s\to\infty} \frac{U(s)}{s}$, and given $\pi \in \mathcal{P}(\mathrm{Geo}(X))$, we define the distorted entropy functional $U^{(t)}_{\pi,\mathfrak{m}}\colon \mathcal{P}_2(X) \to \mathbb{R}\cup\{\infty\}$ as

$$U^{(t)}_{\pi,\mathfrak{m}}(\mu) := \int_X \int_{\mathrm{Geo}(X)} U\left(\frac{\rho(\gamma_0)}{\beta^{(t)}_{K,N}(\gamma_0,\gamma_1)}\right) \beta^{(t)}_{K,N}(\gamma_0,\gamma_1)\,\mathrm{d}\pi_x(\gamma)\,\mathrm{d}\mathfrak{m}(x) + \int_X U'(\infty)\,\mathrm{d}\mu^{\perp},$$

where $\{\pi_x\}$ is a disintegration of π with respect to the evaluation map e_0. The definition above will be only used in the case where $\pi \in \mathrm{OptGeo}(\mu,\nu)$ with μ and ν having bounded support, in which case the definition makes perfect sense, see [54, Theorem 17.28]. Therefore, we do not elaborate on the possible ill-definedness issue of the functional in general.

Now we have all the ingredients for a Lott–Villani type definition of very strict $CD(K,N)$ -spaces.

Definition 2.5. *We say that a metric measure space $(X,d,\mathfrak{m})$ is a very strict $CD(K,N)$ -space* in the spirit of Lott–Villani *if for every $\mu_0,\mu_1\in\mathcal{P}_2^{ac}(X)$ with bounded supports, there exists an optimal plan $\pi\in\mathrm{OptGeo}(\mu_0,\mu_1)$ such that for all non-negative and bounded Borel functions $F\colon\mathrm{Geo}(X)\to\mathbb{R}$ with $\int F\,\mathrm{d}\pi=1$, and for all $t_1,t_2\in[0,1]$, $t_1<t_2$, we have that*

$$U_{\mathfrak{m}}(\tilde\mu_t)\le(1-t)U^{(1-t)}_{\tilde\pi,\mathfrak{m}}(\tilde\mu_0)+tU^{(t)}_{\tilde\pi^{-1},\mathfrak{m}}(\tilde\mu_1)$$

for all $t\in[0,1]$ and for all $U\in\mathcal{DC}_N$, where $\tilde\mu_t\coloneqq(e_t)_\#\tilde\pi\coloneqq(e_t)_\#(\mathrm{restr}_{t_1}^{t_2})_\#F\pi$.

Remark 2.6.

(i) By choosing $U_N(s)\coloneqq-s^{1-\frac{1}{N}}$, and $U_\infty(s)\coloneqq s\log s$, one deduces that a metric measure space satisfying the very strict $CD(K,N)$ -condition in the spirit of Lott–Villani also satisfies the very strict $CD(K,N)$ -condition up to the fact that here the convexity is required to hold only along optimal plans between measures with bounded supports.

(ii) Due to the fact that the displacement convexity classes $\mathcal{DC}_N$ are nested, it follows immediately from the definition that if $(X,d,\mathfrak{m})$ is a very strict $CD(K,N)$ -space in the spirit of Lott–Villani, then it is a very strict $CD(K,N')$ -space in the spirit of Lott–Villani for all $N'>N$.

Theorem 2.7 ([C] Theorem 4.4)**.** *Let $(X,d,\mathfrak{m})$ be a metric measure space. Then the following are equivalent:*

(i) The space $(X,d,\mathfrak{m})$ is a very strict $C(K,N)$ -space (with the slightly modified definition, where $\mu_0,\mu_1\in\mathcal{P}_2^{ac}(X)$ are assumed to have bounded support).
(ii) The space $(X,d,\mathfrak{m})$ is a very strict $CD(K,N)$ -space in the spirit of Lott–Villani.

We do not a priori have that very strict $CD(K,N)$ -condition implies very strict $CD(K,N')$ -condition for $N'\ge N$. Hence, the following corollary is non-trivial.

Corollary 2.8. *Let $(X,d,\mathfrak{m})$ be a metric measure space satisfying very strict $CD(K,N)$ -condition. Then $(X,d,\mathfrak{m})$ is a very strict $CD(K,N')$ -space for every $N'>N$.*

2.2. **Synthetic sectional curvature lower bounds.** In this section we will briefly introduce the notion of Alexandrov spaces, that is, a notion of sectional curvature lower bounds for metric spaces. We refer to [12, 44] for a comprehensive survey. As mentioned before, the definition is based on a comparison of (for example) geodesic triangles of a metric space (X,d) with the ones in the constant curvature model space.

Let $\kappa\in\mathbb{R}$, and let M_κ be a simply connected surface with constant sectional curvature κ, and denote by $|x-y|$ the distance between points $x,y\in M_\kappa$. Then M_κ is the scaled hyperbolic plane if $\kappa<0$, the Euclidean plane if $\kappa=0$, and the scaled sphere if $\kappa>0$. For

$x, y, z \in X$, by a geodesic triangle $\Delta(x, y, z)$ we mean a collection of points x, y, and z, and (a choice of) geodesics $[x, y]$, $[y, z]$, and $[x, z]$ connecting them. Here we use the notation $[x, y]$ for a geodesic to make explicit that it is a geodesic connecting x to y. Furthermore, we denote by $\Delta(\tilde{x}, \tilde{y}, \tilde{z}) \subset M_\kappa$ a *comparison triangle*, that is, $\Delta(\tilde{x}, \tilde{y}, \tilde{z})$ is a triangle in the model space M_κ with vertices $\tilde{x}, \tilde{y}$, and $\tilde{z}$ such that $d(x, y) = |\tilde{x} - \tilde{y}|$, $d(z, y) = |\tilde{z} - \tilde{y}|$, and $d(x, z) = |\tilde{x} - \tilde{z}|$.

Definition 2.9 (Alexandrov space)**.** *We say that a complete, locally compact length space (X, d) is an Alexandrov space with curvature bounded from below by $\kappa \in \mathbb{R}$, if for every point $p \in X$, there exists a neighbourhood U of p for which the following holds. For every triangle $\Delta(x, y, z) \subset U$ and its comparison triangle $\Delta(\tilde{x}, \tilde{y}, \tilde{z}) \subset M_\kappa$, and for every pair of points $w \in [x, y]$ and $\tilde{w} \in [\tilde{x}, \tilde{y}]$ with $d(x, w) = |\tilde{x} - \tilde{w}|$, we have $d(w, z) \geq |\tilde{w} - \tilde{z}|$.*

We recall that the dimension of an Alexandrov space is well defined in the sense that the local Hausdorff dimension of an Alexandrov space (X, d) is constant. Moreover, it is either integer or infinity. We remark the following (natural) connection of Alexandrov curvature and synthetic Ricci curvature.

Theorem ([35, 47])**.** *Let (X, d) be a finite dimensional Alexandrov space with curvature bounded from below by κ and with dimension* $\dim X = n$. *Then $(X, d, \mathcal{H}^n)$ is an $RCD((n-1)\kappa, n)$ space.*

3. One-dimensionality of metric measure spaces

Motivated by the existence results of optimal transport maps on very strict $CD(K, N)$ -spaces, and on essentially non-branching $MCP(K, N)$ -spaces, we prove in [D], that a metric measure space, in which the existence of optimal maps holds true whenever the starting measure is absolutely continuous with respect to the reference measure, is a one-dimensional manifold, possibly with boundary, if it has at least one open set isometric to an open interval. Moreover, if the transport maps are also unique, then the structure of one-dimensional manifold is guaranteed once it has a point at which the Gromov–Hausdorff tangent is unique and isometric to the real line.

The result can be viewed as a special case of one of the following: from regularity point of view, it states that a space, which a priori does not have any structure preventing singularities, in fact has a nice smooth structure. Another way of looking at it is from dimensional point of view. It states that once the space has a one-dimensional part, then it is one-dimensional everywhere. It should be pointed out that one-dimensionality plays a crucial role in the proofs, and it is not clear whether the approach could be generalised to the study of higher dimensional spaces.

To put the result into a context, we remark on the following results. As mentioned in Section 2, the Hausdorff dimension of an Alexandrov space is a constant, either an integer or infinity [13]. Moreover, up to neglecting a singular set of co-dimension at least two, (the interior of) a finite dimensional Alexandrov space admits biLipschitz charts with Lipschitz constant arbitrarily close to one [13]. The charts may be chosen to be even better, namely the transition maps may be required to be the differences of convex functions [46, 2].

Furthermore, a finite dimensional Alexandrov space admits weak Riemannian structure compatible with the charts [44, 2].

In the case of Ricci curvature lower bounds, similar results hold true. For a non-collapsed Ricci-limit space, the dimensional bound for the singular set is still $n-2$, while the biLipschitz charts are replaced by biHölder ones [18]. For general Ricci-limits, the constancy of the dimension was proven in [20]. In the RCD setting, existence of Euclidean tangents was proven in [28], uniqueness of the tangents almost everywhere, and measurable (as opposed to open ones) biLipschitz charts were obtained in [41], and finally in [11] the constancy of the dimension was shown.

Specifically, the results in [D] generalise the ones obtained for Ricci limit spaces in [31] (see also [19]), and for RCD^*-spaces in [34]. More precisely, versions of the following theorem are obtained.

Theorem ([34, Theorem 3.7])**.** *Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume that there exists a point $x_0 \in X$ such that there exists a unique measured Gromov-Hausdorff tangent of $(X, d, \mathfrak{m})$ at x_0 isomorphic to $(\mathbb{R}, |\cdot|, c\mathcal{L}^1, 0)$. Then for any $x \in X$, there exists a positive number $\varepsilon > 0$ such that $B(x, \varepsilon)$ is isometric to $(-\varepsilon, \varepsilon)$ or to $[0, \varepsilon)$.*

The approach taken in [D] is making explicit the role of the optimal transport maps, leading to statements where the existence, and in some cases uniqueness, of the optimal transport maps is taken as an assumption. As a consequence, one-dimensionality results for very strict $CD(K, N)$ -spaces, $MCP(K, N)$-spaces, and qualitatively non-degenerate spaces are obtained.

Theorem 3.1 ([D] Theorem 3.1 and Theorem 3.10)**.** *Let $(X, d, \mathfrak{m})$ be a metric measure space satisfying the following:*

(i) For every $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ that is induced by a map from μ_0.

(ii) There exists a point $x \in X$, and a neighbourhood U of x isometric to an open interval in $\mathbb{R}$.

Then X is a one-dimensional manifold, possibly with boundary.

In the case where the optimal plans are also unique, and the space is locally metrically doubling, the assumption (ii) may be substituted by the following:

(ii') There exists a point $x \in X$ so that $\mathrm{Tan}(X, x) = \{(\mathbb{R}, 0)\}$.

Here by $\mathrm{Tan}(X, x)$ we denote the set of (equivalence classes of) Gromov-Hausdorff tangents of X at x. Let us refer to condition (ii) by saying that the space X is one-dimensional near the point x, and to condition (ii') by saying that the space X is one-dimensional at the point x. Then an immediate corollary of Theorem 3.1 may be rephrased as follows.

Corollary 3.2. *A metric measure space $(X, d, \mathfrak{m})$ is a one-dimensional manifold if one of the following conditions holds:*

(i) $(X, d, \mathfrak{m})$ is a very strict $CD(K, N)$ -space, $N < \infty$, and X is one-dimensional near a point x.

(ii) $(X, d, \mathfrak{m})$ *is an essentially non-branching* $MCP(K, N)$*-space, and* X *is one-dimensional at a point* x*.*
(iii) $(X, d, \mathfrak{m})$ *is an essentially non-branching, qualitatively non-degenerate space, and* X *is one-dimensional at a point* x*.*

4. Open questions

We present here a list of open questions arising naturally from the articles of the dissertation. We will discuss more in details two of them, which we believe that are the most fundamental ones. Let us begin with these two.

As mentioned before, on one hand, it is known that the class of essentially non-branching $CD(K, N)$-spaces is not stable under a convergence of metric measure spaces while the $CD(K, N)$-condition is, and on the other hand, it is not known what is the case for very strict $CD(K, N)$ -spaces. Hence, the question:

Question 1. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of very strict $CD(K, N)$ -spaces converging to a limit space $(X, d, \mathfrak{m})$. Is the limit $(X, d, \mathfrak{m})$ a very strict $CD(K, N)$ -space?

It is known that the class of very strict $CD(K, N)$ -spaces is strictly larger than the class of essentially non-branching $CD(K, N)$-spaces, but it is not known, whether a general $CD(K, N)$-space is a very strict $CD(K, N)$ -space. If that would be the case, then the answer to Question 1 would be affirmative.

Question 2. Let $(X, d, \mathfrak{m})$ be a $CD(K, N)$-space. Is $(X, d, \mathfrak{m})$ a very strict $CD(K, N)$ -space?

The example in Section 4.1 shows that the strategy for the proof of stability of $CD(K, N)$-condition does not generalise in an obvious way to the very strict $CD(K, N)$ -setting. It also tells that one has to be more clever when choosing the sequence of optimal plans from which the limit optimal plan with the desired properties will be obtained. Therefore, the following question might be easier to answer due to the uniqueness of optimal plans in essentially non-branching $CD(K, N)$-spaces.

Question 3. Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of essentially non-branching $CD(K, N)$-spaces converging to a limit space $(X, d, \mathfrak{m})$. Is $(X, d, \mathfrak{m})$ a very strict $CD(K, N)$ -space?

In articles [A], [B], and [C], the existence of optimal transport maps was proven under suitable assumptions. In [A] and in [C], the focus was on the assumption on the underlying space, leaving the following natural question unanswered.

Question 4. Does the existence of optimal transport maps hold in general $CD(K, N)$-space?

The article [B] was more about the conditions on the marginals. Due to the geometric nature of the proof one may hope that similar results could be proven also in a setting where strong analytic tools are not available. For instance, one could ask the question in infinite dimensional spaces.

Question 5. Let (X,d) be an infinite dimensional Alexandrov space, and $\mu_0 \in \mathcal{P}_2(X)$ be a measure giving zero measure to sets of co-dimension at least one (in a suitable sense), and let $\mu_1 \in \mathcal{P}_2(X)$. Does there exist a (unique) optimal plan that is induced by a map from μ_0?

Viewing the results in [D] from the dimensional point of view, they are of the form "if the space has a one-dimensional part, then it is entirely one-dimensional". Hence, it is natural to ask whether the results generalise to higher dimensions. A vague question could be

Question 6. In which setting, one can deduce the dimensional homogeneity from the existence (and uniqueness) of optimal transport maps?

while a more precise one could read as

Question 7. Is the dimension of a (very strict) $CD(K,N)$ -space a constant? Is the dimension of an essentially non-branching $MCP(K,N)$ -space a constant?

We remark that the answer to the Question 7 is negative in the case of general $MCP(K,N)$-spaces by [33], and is positive in the case of $RCD(K,N)$-spaces by [11].

4.1. **Counterexample for the naive approach to the stability problem.** The idea for the counterexample is to construct metric measure spaces $(X_n, d_n, \mathfrak{m}_n)$ satisfying very strict $CD(K,\infty)$ -condition such that they converge to some limit space $(X,d,\mathfrak{m})$, then construct converging sequences of marginals $\mu_n \rightharpoonup \mu$, $\nu_n \rightharpoonup \nu$ and optimal plans $\mathrm{OptGeo}(\mu_n,\nu_n) \ni \pi_n \rightharpoonup \pi \in \mathrm{OptGeo}(\mu,\nu)$ so that every π_n satisfies the convexity condition in the definition of very strict $CD(K,\infty)$ -space, but the limit measure π does not. Moreover, the entropies of the marginals will converge to the entropies of the limit marginals.

In this example the situation is actually fairly simple: $(X_n, d_n, \mathfrak{m}_n) := (\mathbb{R}^m, d_{\sup}, \mathcal{L}^m) =: (X,d,\mathfrak{m})$, $\mu_n = \mu$, and $\nu_n = \nu$ for all $n \in \mathbb{N}$. Also the measures μ and ν are just restrictions of the Lebesgue measure to some cuboids.

Let $I = [0,1]$. Define $\mu := \mathfrak{m}|_{I^3}$ and $\nu := \mathfrak{m}|_{(2,1,1)+I\times\frac{1}{2}I\times I} + \mathfrak{m}|_{(2,1,-1)+I\times\frac{1}{2}I\times I}$. Then for all $n \in \mathbb{N}$, define π_n as the pushforward of the measure μ under the map $T_n \colon X \to \mathrm{Geo}(X)$, where the maps T_n are constructed as follows, see Figure 1. For every $n \in \mathbb{N}$, write $I = \cup_{j=1}^{2^n} I_j$, where $I_j = [(j-1)2^{-n}, j2^{-n}]$ are the dyadic intervals of side length 2^{-n}. Then for $j \in 2\mathbb{N}$, and for $x \in I \times I_j \times I$, define

$$T_n(x)(t) := \begin{cases} (2t, 0, 2t) + x, & \text{if } t \in [0, \frac{1}{2}] \\ (2t, \frac{1}{2}(\frac{1}{2} - 2^{-n}j)(2t-1), 0) + (1,0,1) + x, & \text{if } t \in [\frac{1}{2}, 1], \end{cases}$$

and similarly for $j \in 2\mathbb{N} - 1$ and for $x \in I \times I_j \times I$ define

$$T_n(x)(t) := \begin{cases} (2t, 0, -2t) + x, & \text{if } t \in [0, \frac{1}{2}] \\ (2t, \frac{1}{2}(\frac{1}{2} - 2^{-n}(j-1))(2t-1), 0) + (1,0,-1) + x, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

One can easily check that $T_n(x) \in \mathrm{Geo}(X)$ for all x, and that for each $t \in [0,1]$, the map $x \mapsto T_n(x)(t)$ is just a collection of translations of the strips $I \times I_j \times I$ so that

$$T_n(I \times I_i \times I)(t) \cap T_n(I \times I_j \times I)(t) = \varnothing,$$

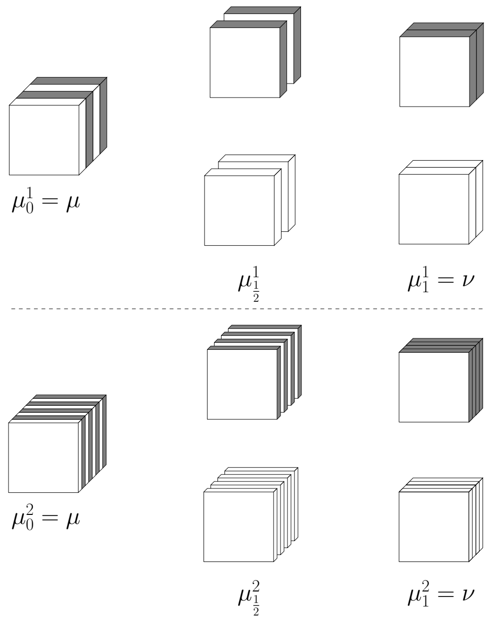

FIGURE 1. Optimal plans π_1 and π_2 at time 0, $\frac{1}{2}$ and 1.

for all $i \neq j$. Moreover, the plan $\pi_n \coloneqq (T_n)_\# \mu \in \mathrm{OptGeo}(\mu, \nu)$ and satisfies the convexity condition required in the definition of strict $CD(K, N)$ -spaces, since the entropy is actually constant along all plan $f\pi$ with Borel weight $f \colon \mathrm{Geo}(X) \to \mathbb{R}_+$.

Let us next prove that the $\pi_n \rightharpoonup \pi \coloneqq \frac{1}{2}(T^1_\# \mu + T^2_\# \mu)$, where

$$T^1(x)(t) = \begin{cases} (2t, 0, 2t) + x, & \text{if } t \in [0, \frac{1}{2}] \\ (2t, \frac{1}{2}(\frac{1}{2} - x_2)(2t-1), 0) + (1, 0, 1) + x, & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and

$$T^2(x)(t) = \begin{cases} (2t, 0, -2t) + x, & \text{if } t \in [0, \frac{1}{2}] \\ (2t, \frac{1}{2}(\frac{1}{2} - x_2)(2t-1), 0) + (1, 0, -1) + x, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

We will do this by showing that the Wasserstein distance between π_n and π tends to zero when n tends to infinity. Let us define couplings between π_n and π in the following way. First define maps $f_n \colon I^3 \to I^3$ as

$$f_n(x) = \begin{cases} (x_1, 2(x_2 - 2^{-n}(j-1)) + 2^{-n}(j-1), x_3), & \text{if } x \in I \times I_j \times I,\ j \in 2\mathbb{N} \\ (x_1, 2(x_2 - 2^{-n}j) + 2^{-n}j, x_3), & \text{if } x \in I \times I_j \times I,\ j \in 2\mathbb{N} - 1. \end{cases}$$

Then we have that $f_n(I^3_+) = I^3$ and $f_n(I^3_-) = I^3$, where $I^3_+ = \cup_{j \in 2\mathbb{N}} I \times I_j \times I$ and $I^3_- = \cup_{j \in 2\mathbb{N}-1} I \times I_j \times I$. Furthermore, we have that $(f_n|_{I^3_+})_\# \mu|_{I^3_+} = \frac{1}{2}\mu$ and similarly $(f_n|_{I^3_-})_\# \mu|_{I^3_-} = \frac{1}{2}\mu$. Hence, $\pi = (F_n)_\# \pi_n$, where

$$F_n(T_n(x)) \coloneqq \begin{cases} T^1 \circ f_n(x), & \text{if } x \in I^3_+ \\ T^2 \circ f_n(x), & \text{if } x \in I^3_-. \end{cases}$$

Now one can easily check that

$$\int d^2(\gamma, F_n(\gamma))\,\mathrm{d}\pi_n \to 0,$$

when $n \to \infty$, and hence

$$W_2(\pi_n, \pi) \to 0.$$

It remains to show that the limit plan π is not good in the sense that it does not satisfy the entropy condition required by the definition of very strict $CD(K, \infty)$ -space. This follows immediately by Theorem 1.2 from the fact that it is not induced by a map. Notice however, that the entropy is convex along π and thus $CD(K, \infty)$-condition is satisfied by π.

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Included articles

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Existence of optimal transport maps in very strict $CD(K,\infty)$ -spaces

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EXISTENCE OF OPTIMAL TRANSPORT MAPS IN VERY STRICT $CD(K,\infty)$ -SPACES

TIMO SCHULTZ

Abstract. We introduce a more restrictive version of the strict $CD(K,\infty)$ -condition, the so-called very strict $CD(K,\infty)$ -condition, and show the existence of optimal maps in very strict $CD(K,\infty)$ -spaces despite the possible lack of uniqueness of optimal plans.

1. Introduction

Consider a (complete and separable) metric space (X,d). In the theory of optimal mass transportation probably the most studied problem is the one with the quadratic cost i.e. the minimisation problem

$$\inf \int_{X\times X} d^2(x,y)\,\mathrm{d}\varpi(x,y), \tag{1}$$

where the infimum is taken over all transport plans ϖ between a given starting measure μ_0 and a final measure μ_1, in other words over all Borel probability measures ϖ on $X\times X$ with marginals μ_0 and μ_1. This problem is interpreted as an optimal mass transportation problem in the following way. The quantity $d^2(x,y)$ tells how much it costs to transport a unit mass from x to y, and for a given transport plan ϖ, mass from x is to be transported to y, if and only if (x,y) belongs to the support of ϖ. The total cost of sending all the mass μ_0 to μ_1 according to the plan ϖ is given by the integral

$$\int_{X\times X} d^2(x,y)\,\mathrm{d}\varpi(x,y).$$

The above formulation is the so-called Kantorovich formulation of the optimal mass transportation problem, which is a relaxed version of the original Monge formulation of the problem, where instead of the infimum (1) one considers the infimum

$$\inf \int_{X} d^2(x,T(x))\,\mathrm{d}\mu_0(x), \tag{2}$$

where the infimum is taken over all Borel mappings $T\colon X\to X$ for which $T_{\#}\mu_0=\mu_1$ i.e. all the Borel maps sending the mass μ_0 to μ_1. One natural and interesting question is whether these two infima agree and when is the optimal plan in (1) given by an optimal

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map in (2). More precisely, when is the optimal plan ϖ of the form $\varpi = (\mathrm{id}, T)_{\#}\mu_0$ for some Borel mapping $T\colon X \to X$.

The existence of optimal map was proven in the Euclidean setting for absolutely continuous measures by Brenier [6]. Later this result was generalized to the Riemannian framework by McCann [18], and further to some cases of the sub-Riemannian setting by Ambrosio and Rigot [2], Agrachev and Lee [1], and by Figalli and Rifford [10]. In metric space setting Bertrand proved the existence of optimal map in the so-called Alexandrov spaces [5]. Under the assumption of non-branching of geodesics, the existence of optimal map was proven for metric measure spaces with Ricci curvature bounded from below i.e. for the spaces satisfying the so-called curvature dimension condition ($CD(K,N)$-condition for short) by Gigli [11] (see Section 2.2 for the definition of $CD(K,\infty)$-space), and with milder assumptions by Cavalletti and Huesmann [8].

The non-branching assumption plays a crucial role in both of those proofs leaving the existence of optimal maps in general $CD(K,\infty)$-spaces open. On the other hand, if one considers spaces satisfying only the so-called measure contraction property, which is weaker type of Ricci curvature lower bound condition, the existence of optimal maps may fail as what follows from the example of Ketterer and Rajala in [15].

In this paper we go towards understanding the question in general $CD(K,\infty)$-spaces by considering a possibly more restrictive version of $CD(K,\infty)$-property without the non-branching assumption, namely we require the entropy to be convex not only along one optimal geodesic plan but instead along all plans that we get by restricting and weighting a given particular plan (see 2.1 for the definition of very strict $CD(K,\infty)$ -space). We prove the existence of optimal maps in very strict $CD(K,\infty)$ -spaces between measures μ_0 and μ_1 that are absolutely continuous with respect to the reference measure $\mathfrak{m}$. We actually prove the following stronger statement saying that there exists an optimal dynamical transport plan that is induced by a map.

Theorem 1.1. *Let $(X,d,\mathfrak{m})$ be a very strict $CD(K,\infty)$ -space and let $\mu_0,\mu_1 \in \mathcal{P}_2(X)$ be absolutely continuous with respect to $\mathfrak{m}$. Then there exists a measure $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ that is induced by a map i.e. there exists a Borel mapping $T\colon X \to \mathrm{Geo(X)}$ so that $\pi = T_{\#}\mu_0$.*

In [19] Rajala and Sturm (see also [13]) were able to remove the a priori non-branching assumption by considering a more restrictive version of Ricci curvature lower bounds, namely the strong $CD(K,\infty)$ -property. They introduced the definition of essential non-branching spaces and proved that strong $CD(K,\infty)$ -spaces are essentially non-branching, from which they deduced by using the idea of the proof of Gigli [11] that every optimal plan is given by a map. The result of Rajala and Sturm applies also in the measured Gromov Hausdorff -stable setting of metric measure spaces with Riemannian Ricci curvature bounded from below (the so-called $RCD(K,\infty)$ -spaces, see [4, 3]). While the $CD(K,\infty)$-condition is stable [20, 12], and the strong $CD(K,\infty)$ is not, it remains open whether very strict $CD(K,\infty)$ -, or strict $CD(K,\infty)$ -condition is stable.

Question 1. *Let $(X_i, \mathrm{d}_i, \mathfrak{m}_i)_{i\in\mathbb{N}}$ be a sequence of (very) strict $CD(K\infty)$ -spaces that converge to a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ in suitable sense (for example in pointed measured Gromov sense). Is $(X, \mathrm{d}, \mathfrak{m})$ a (very) strict $CD(K,\infty)$ -space?*

Another related open question is the relation of $CD(K,\infty)$-, strict $CD(K,\infty)$ -, and very strict $CD(K,\infty)$ -conditions:

Question 2. *Does the strict $CD(K,\infty)$ -condition imply very strict $CD(K,\infty)$ -condition? Does $CD(K,\infty)$ imply (very) strict $CD(K,\infty)$?*

With the new notion of essential non-branching introduced by Rajala and Sturm, Cavalletti and Mondino further proved the existence of optimal transport maps for essentially non-branching spaces with the measure contraction property [9]. Continuing from the work of Cavalletti and Huesmann [8] and the work of Cavalletti and Mondino [9], Kell proved that in a metric space endowed with qualitatively non-degenerate measure, and therefore especially in spaces satisfying the measure contraction property, the condition of being essentially non-branching is equivalent with having the existence and uniqueness of the optimal transport maps [14].

In the previous results the existence of optimal map is shown by proving that every optimal plan is given by a map and hence also the uniqueness of the plan is guaranteed. In very strict $CD(K,\infty)$ -spaces optimal plans may fail to be unique – which can be observed by looking for example at the space $\mathbb{R}^n$ equipped with supremum norm – and therefore this strategy cannot work in our setting. Instead, we should consider one special plan that is given by the defintion of very strict $CD(K,\infty)$ -space. Notice that even though very strict $CD(K,\infty)$ -spaces may fail to be non-branching (even in the sense of essential non-branchingness), still this specific optimal plan does not see any branching geodesics.

Our proof follows the ideas of Rajala and Sturm in [19] and of Gigli in [11]. Instead of proving the existence via the non-branchingness of the optimal plan, we do the proof directly, since it is not clear how to implement the idea of the mixing procedure of [19] in the very strict $CD(K,\infty)$ -setting.

2. Preliminaries

Throughout this paper $(X, d, \mathfrak{m})$ is assumed to be a complete and separable metric space endowed with a locally finite Borel measure $\mathfrak{m}$. Since we are considering only transportations between absolutely continuous measures, all the Wasserstein geodesics of our concern live in the set of absolutely continuous measures due to the K-convexity of the entropy. Thus we may restrict to the case where $X = \operatorname{spt}\mathfrak{m}$.

By a geodesic we mean a constant speed curve $\gamma\colon [0,1] \to X$ that is length minimizing i.e. $l(\gamma) = d(\gamma_0, \gamma_1)$, where l denotes the length of γ. We denote by Geo(X) the set of all geodesics of the space X endowed with the supremum metric.

2.1. Optimal mass transportation and Wasserstein geodesics.

We denote by $\mathcal{P}(X)$ the space of all Borel probability measures on X, and by $\mathcal{P}_2(X)$ the set of all $\mu \in \mathcal{P}(X)$

with finite second moment i.e. those μ for which we have

$$\int d^2(x, x_0)\,\mathrm{d}\mu < \infty$$

for some – and thus for all – $x_0 \in X$.

We define the Wasserstein 2-distance W_2 in $\mathcal{P}_2(X)$ as

$$W_2(\mu, \nu) := \left(\inf_{\sigma \in \mathcal{A}(\mu,\nu)} \int_{X \times X} d^2(x, y)\,\mathrm{d}\sigma \right)^{\frac{1}{2}},$$

where $\mathcal{A}(\mu, \nu) := \left\{\sigma \in \mathcal{P}(X \times X) : \mathrm{P}^1_{\#}\sigma = \mu,\ \mathrm{P}^2_{\#}\sigma = \nu\right\}$ is the set of all admissible plans. The square of the Wasserstein distance is nothing else but the total cost in the mass transportation problem with quadratic cost between the masses μ and ν. We denote the set of admissible plans realising the above infimum as $\mathrm{Opt}(\mu, \nu)$.

Even though we do not assume the space (X, d) to be geodesic, at the end of Section 2 we point out that from the definition of very strict $CD(K, \infty)$ -space we actually get that the space X is a length space – keeping in mind that $X = \operatorname{spt} \mathfrak{m}$. Since X is a complete and separable metric space with length structure, we also have that the Wasserstein space $(\mathcal{P}_2(X), W_2)$ is a complete and separable length space (see [21] and [16]). Furthermore, a curve $t \mapsto \mu_t$ in $\mathcal{P}_2(X)$ is a geodesic if and only if there exists $\pi \in \mathcal{P}(\mathrm{Geo(X)})$ such that $(e_t)_{\#}\pi = \mu_t$ and $(e_0, e_1)_{\#}\pi \in \mathrm{Opt}(\mu_0, \mu_1)$. Here $e_t \colon \mathrm{Geo(X)} \to X$ is the evaluation map $\gamma \mapsto \gamma_t := \gamma(t)$. The set of all $\pi \in \mathcal{P}(\mathrm{Geo(X)})$ for which $(e_0, e_1)_{\#}\pi \in \mathrm{Opt}(\mu_0, \mu_1)$ is denoted by $\mathrm{OptGeo}(\mu_0, \mu_1)$.

2.2. **Ricci curvature bounded from below.** The notion of synthetic Ricci curvature lower bounds for metric measure spaces were first introduced by Sturm [20] and independently by Lott and Villani [17]. The definitions are based on the connection of Ricci curvature and optimal mass transportation; namely, the convexity properties of suitable entropy functionals along Wasserstein geodesic. For the definition of Ricci curvature lower bounds let us first introduce the entropy functional $\mathrm{Ent}_\infty \colon \mathcal{P}(X) \to [-\infty, \infty]$ that is defined as

$$\mathrm{Ent}_\infty(\mu) = \begin{cases} \int \rho \log \rho \,\mathrm{d}\mathfrak{m} & \text{, if } \mu \ll \mathfrak{m} \text{ and } (\rho \log \rho)_+ \in \mathrm{L}^1(\mathfrak{m}), \\ \infty & \text{otherwise,} \end{cases}$$

where ρ is the density of μ with respect to $\mathfrak{m}$ i.e. $\mu = \rho\mathfrak{m}$, and $(\rho \log \rho)_+ = \max\{\rho \log \rho, 0\}$.

A metric measure space $(X, d, \mathfrak{m})$ is said to have Ricci curvature bounded from below by $K \in \mathbb{R}$, if for every $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ absolutely continuous with respect to $\mathfrak{m}$, there exists $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that the entropy Ent_∞ is K-convex along π, that is the inequality

$$\mathrm{Ent}_\infty(\mu_t) \leq (1-t)\mathrm{Ent}_\infty(\mu_0) + t\mathrm{Ent}_\infty(\mu_1) - \frac{K}{2} t(1-t)\mathrm{W}_2^2(\mu_0, \mu_1) \tag{3}$$

holds for all $t \in [0, 1]$, where $\mu_t := (e_t)_{\#}\pi$. Such a space is called a $CD(K, \infty)$-space. If the K-convexity holds along every $f\pi$, where $f \colon \mathrm{Geo(X)} \to \mathbb{R}$ is any non-negative Borel function for which $\int f\,\mathrm{d}\pi = 1$, then the space is called a strict $CD(K, \infty)$ -space (see [4]). In this paper a more restrictive version of strict $CD(K, \infty)$ -condition is used, namely the convexity of the entropy is not only required between points 0, t and 1, but also between any points $t_1 < t_2 < t_3$. To emphasise the difference, we call these spaces *very strict* $CD(K,\infty)$ -spaces.

Definition 2.1. *A metric measure space* $(X, d, \mathfrak{m})$ *is called a very strict* $CD(K,\infty)$ *-space, if for every* $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ *absolutely continuous with respect to the reference measure* $\mathfrak{m}$ *there exists* $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *so that the entropy* Ent_∞ *is* K*-convex along* $(\mathrm{restr}_{t_0}^{t_1})_\#(f\pi)$ *for all* $t_0, t_1 \in [0,1]$, $t_0 < t_1$, *and for all non-negative Borel functions* $f\colon \mathrm{Geo(X)} \to \mathbb{R}$ *with* $\int f \,\mathrm{d}\pi = 1$.

Here $\mathrm{restr}_{t_0}^{t_1}\colon \mathrm{Geo(X)} \to \mathrm{Geo(X)}$ is the restriction map defined as

$$\mathrm{restr}_{t_0}^{t_1}(\gamma)(t) := \gamma(t(t_1 - t_0) + t_0).$$

It is worth of noticing that due to the Radon-Nikodym theorem, in the above definition one could equivalently require the convexity to hold along $(\mathrm{restr}_{t_0}^{t_1})_\#\tilde{\pi}$ for all $\tilde{\pi} \in \mathcal{P}(X)$ that are absolutely continuos with respect to π.

As mentioned before, the very strict $CD(K,\infty)$ -condition implies that the space X is a length space: let $x, y \in X$ and define for $\varepsilon > 0$ the measures $\mu_0 := \mathfrak{m}|_{B(x,\varepsilon)}$ and $\mathfrak{m}|_{B(y,\varepsilon)}$. Let $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ which exists by the definition. Now any $\gamma \in \mathrm{spt}\pi$ is a geodesic from $B(x,\varepsilon)$ to $B(y,\varepsilon)$ and thus the point $\gamma_{\frac{1}{2}}$ is an ε-midpoint of x and y. Thus by completeness we have that X is a length space [7].

3. The Main Theorem

In the paper [19] Rajala and Sturm prove the existence of the optimal map by first proving that every optimal plan $\pi \in \mathrm{OptGeo}(X)$ is essentially non-branching and then as a corollary of this they prove that every such π is actually given by a map. While the proof for the essential non-branching of the $\pi \in \mathrm{OptGeo}(X)$ given by the definition of very strict $CD(K,\infty)$ -space can be carried through with relatively small changes, the proof of the corollary is more problematic.

In their proof of the existence of optimal map they first divide the original measure π into two measures π^1 and π^2 that intersect at time t, and then construct a new measure π^{mix} by mixing these measures π^1 and π^2 essentially in the way that at time t you may change from a geodesic in the support of one of the measures to a geodesic in the support of the other. By doing this they end up with a new plan that is still optimal, but due to this mixing the plan is not essentially non-branching anymore. The problem in applying this strategy to our case is that the mixing procedure should be done in such a way that in the end the constructed measure π^{mix} is absolutely continuous with respect to the original measure π, which is something that one should not expect from this kind of mixing. To overcome this obstacle, we prove directly the existence of a map by still using the idea from the proof of the essential non-branching in [19].

In the proof of the existence of optimal map we will use the following two lemmas.

Lemma 3.1 (cf. [11, Lemma 3.2]). *Let* $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *be the plan given by the definition of very strict* $CD(K,\infty)$. *Then for all* $\tilde{\pi} \ll \pi$ *with* $\mathfrak{m}(\{\tilde{\rho}_0 > 0\}) < \infty$ *and*

$\mathrm{Ent}_\infty(\tilde{\mu}_0), \mathrm{Ent}_\infty(\tilde{\mu}_1) \in \mathbb{R}$ *it holds that*

$$\mathfrak{m}(\{\tilde{\rho}_0 > 0\}) \leq \liminf_{t \to 0} \mathfrak{m}(\{\tilde{\rho}_t > 0\}),$$

where $\tilde{\rho}_t$ *is the density of* $(e_t)_\# \tilde{\pi}$ *with respect to* $\mathfrak{m}$.

Lemma 3.2. *Let* (X, d) *be a separable metric space. Then for any* $\sigma \in \mathcal{P}(X \times X)$ *for which* $\sigma(\{(x, x) : x \in X\}) = 0$ *there exists* $E \subset X$ *so that* $\sigma(E \times (X \setminus E)) > 1/5$.

The proof of Lemma 3.1 is the same as the proof of [11, Lemma 3.2] and the proof of Lemma 3.2 can be found in [19]. We will also use the following simple lemma.

Lemma 3.3. *Let* $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ *be absolutely continuous with respect to* $\mathfrak{m}$ *with densities* ρ_0 *and* ρ_1 *such that* $\mathrm{Ent}_\infty(\mu_0), \mathrm{Ent}_\infty(\mu_1) \in \mathbb{R}$ *and* $\rho_0, \rho_1 < C$, *and let* $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *be an optimal plan concentrated on a set of geodesics with length bounded by some constant* L *such that the convexity inequality* (3) *holds for all restrictions of* π. *Then there exists a constant* $M < \infty$ *such that for all* $t \in [0,1]$ *we have* $\rho_t \leq M$ μ_t*-almost everywhere, where* $\mu_t := (e_t)_\# \pi$ *with density* ρ_t.

Proof. We argue by contradiction. Assume that for all $M > 0$ there exists $t_M \in (0,1)$ so that $N_M := \mu_{t_M}(\{\rho_{t_M} > M\}) > 0$. Define

$$\hat{\pi} := \pi|_{e_{t_M}^{-1}(\{\rho_{t_M} > M\})}$$

and denote $\hat{\mu}_t := (e_t)_\# \hat{\pi}$ and the density of $\hat{\mu}_t$ by $\hat{\rho}_t$. Then we have that

$$\begin{aligned}&\int \frac{\hat{\rho}_{t_M}}{N_m} \log \frac{\hat{\rho}_{t_M}}{N_m} \, \mathrm{d}\mathfrak{m} - (1 - t_M) \int \frac{\hat{\rho}_0}{N_m} \log \frac{\hat{\rho}_0}{N_m} \, \mathrm{d}\mathfrak{m} - t_M \int \frac{\hat{\rho}_1}{N_m} \log \frac{\hat{\rho}_1}{N_m} \, \mathrm{d}\mathfrak{m} \\ &\geq \log \frac{M}{N_M} - (1 - t_M) \log \frac{C}{N_M} - t_M \log \frac{C}{N_M} = \log M - \log C \to \infty,\end{aligned}$$

when

$$M \to \infty.$$

On the other hand by the K-convexity we have for all M that

$$\begin{aligned}&\int \frac{\hat{\rho}_{t_M}}{N_m} \log \frac{\hat{\rho}_{t_M}}{N_m} \, \mathrm{d}\mathfrak{m} - (1 - t_M) \int \frac{\hat{\rho}_0}{N_m} \log \frac{\hat{\rho}_0}{N_m} \, \mathrm{d}\mathfrak{m} - t_M \int \frac{\hat{\rho}_1}{N_m} \log \frac{\hat{\rho}_1}{N_m} \, \mathrm{d}\mathfrak{m} \\ &\leq -\frac{K}{2} t_M (1 - t_M) \mathrm{W}_2^2\Big(\frac{\hat{\mu}_0}{\mathrm{N_M}}, \frac{\hat{\mu}_1}{\mathrm{N_M}}\Big) \leq \frac{|K|}{2} L^2 < \infty.\end{aligned}$$

which is a contradiction. Hence there exists M so that $\rho_t \leq M$ for all $t \in [0,1]$. $\square$

Proof of Theorem 1.1. Let $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ be measures that are absolutely continuous with respect to $\mathfrak{m}$, and let ρ_0 and ρ_1 be densities of μ_0 and μ_1 with respect to $\mathfrak{m}$. We will prove that the measure $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ given by the definition of very strict $CD(K, \infty)$ -space is induced by a map. We will argue by contradiction. Assume that π is not induced by a map. As in [19], we may assume that $\rho_0, \rho_1 < C < \infty$ and that the space X is bounded.

We may also assume that $\mathfrak{m}(X)$ is finite. The argument goes as follows. Since X is complete and separable, so is Geo(X) (as a closed subset of a complete and separable space of continuos curves). Therefore π is a Radon measure. In particular there exists an increasing sequence of compact subsets $K_i \subset \mathrm{Geo(X)}$ for which $\pi(K_i) \geq 1 - \frac{1}{i}$. Since π is not induced by a map, there exists $n \in \mathbb{N}$ so that $\pi|_{K_n}$ is not induced by a map. Furthermore $\mathrm{spt}(e_t)_\#\pi|_{K_n} \subset e_t(K_n) \subset \bigcup_{t\in[0,1]} e_t(K_t)$. Thus the relevant part of the space X lives inside the compact set $\bigcup_{t\in[0,1]} e_t(K_t)$, which by local finiteness of $\mathfrak{m}$ has finite measure. In particular, by using Jensen's inequality we may also assume that $\mathrm{Ent}_\infty(\mu_0), \mathrm{Ent}_\infty(\mu_1) \in \mathbb{R}$.

As in the proof of the essential non-branching of strong $CD(K,\infty)$ -spaces in [19], we want to make the square of the Wasserstein distance $\mathrm{W}_2^2(\mu_0,\mu_1)$ arbitrary small, so that we can basically consider only the convexity of the entropy and forget the K dependent error term in (3). This is done by the following lemma

Lemma 3.4. *If $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ is not induced by a map, then for every $k \in \mathbb{N}$ there exists an interval $[i/k,(i+1)/k]$ so that $(\mathrm{restr}_{i/k}^{(i+1)/k})_\#\pi$ is not induced by a map i.e. $(\mathrm{restr}_{i/k}^{(i+1)/k})_\#\pi \neq T_\#((e_{i/k})_\#\pi)$.*

We will postpone the proof of Lemma 3.4 to the end of the paper. Using the above lemma we may restrict the plan π so that we have the inequality

$$L^2 \leq \frac{\log 2}{6|K|+1},$$

where

$$L := \operatorname*{ess\,sup}_{\gamma\in\mathrm{Geo(X)}} l(\gamma)$$

and the essential supremum is taken with respect to the (restricted) measure π.

Step 1: As in the proof of essential non-branching of strong $CD(K,\infty)$ -spaces, we lift the measure π to a measure in $\mathcal{P}(\mathrm{Geo(X)}^2)$. Let $\{\pi_x\}$ be a disintegration of the measure π with respect to the evaluation map $e_0\colon \mathrm{Geo(X)} \to X$. We define $\sigma \in \mathcal{P}(\mathrm{Geo(X)}^2)$ by defining the integral of any Borel function $f\colon \mathrm{Geo(X)}^2 \to [0,\infty]$ with respect to σ as

$$\int_{\mathrm{Geo(X)}^2} f \,\mathrm{d}\sigma := \int_X \int_{e_0^{-1}(x)\times e_0^{-1}(x)} f \,\mathrm{d}(\pi_x \times \pi_x)\,\mathrm{d}\mu_0.$$

Note that σ is well defined since the map $x \mapsto \pi_x \times \pi_x(A)$ is Souslin measurable for every Borel set $A \subset \mathrm{Geo(X)}^2$. This can be seen by applying Dynkin's $\pi - \lambda$ theorem to a π-system $\mathcal{B}(\mathrm{Geo(X)}) \times \mathcal{B}(\mathrm{Geo(X)})$ and a λ-system $\{A \in \mathcal{B}(\mathrm{Geo(X)}^2) : x \mapsto \pi_x \times \pi_x(A)$ is Souslin measurable$\}$.

Since π is not induced by a map, there exists $H \subset X$ with positive μ_0-measure so that for any $x \in H$ the measure π_x is not a dirac mass. Therefore, there exists $F \subset \mathrm{Geo(X)}^2$ for which $\sigma(F) > 0$, and it holds that for any $(\gamma^1,\gamma^2) \in F$ we have that $\gamma_0^1 = \gamma_0^2$ and $\gamma^1 \neq \gamma^2$. By Lemma 3.2 there exists $E \subset \mathrm{Geo(X)}$ so that $\sigma((E \times (X \setminus E)) \cap F) > \sigma(F)/5$. Let $\eta > 0$ be such that

$$\sigma(A) > \frac{1}{10}\sigma(F)$$

for the set $A := \{(\gamma^1,\gamma^2) \in (E \times (X \setminus E)) \cap F : d(\gamma^1,\gamma^2) > \eta\}$. Let $m \in \mathbb{N}$ be large enough so that $\frac{1}{m} < \frac{\eta}{4L}$ and divide the interval $[0,1]$ into m intervals $\{I_j\}_{j=1}^m$ of equal length. Then for every $(\gamma^1,\gamma^2) \in A$ there exist $j \in \{1,\dots,m\}$ and $t \in I_j$ so that $\mathrm{d}(\gamma^1_t,\gamma^2_t) > \eta$ and thus for any $s \in I_j$, since $|t-s| \le \frac{1}{m}$, we have that

$$\mathrm{d}(\gamma^1_s,\gamma^2_s) \ge \mathrm{d}(\gamma^1_t,\gamma^2_t) - \mathrm{d}(\gamma^1_t,\gamma^1_s) - \mathrm{d}(\gamma^2_t,\gamma^2_s) \ge \eta - 2\frac{\eta}{4L}L \ge \frac{\eta}{2}.$$

Hence there exists i so that $\sigma(A_i) > 0$ for $A_i := \{(\gamma^1,\gamma^2) \in A : \mathrm{d}(\gamma^1_t,\gamma^2_t) > \frac{\eta}{2}\ \forall t \in I_i\}$. Take a countable partition $P := \{Q_j\}$ of X with $\mathrm{d}(Q_j) < \frac{\eta}{4}$. Then for some $Q \in P$ we have that $\sigma((e_S^{-1}(Q) \times \mathrm{Geo(X)}) \cap A_i) > 0$, where $S \in I_i$ is the mid-point of the interval I_i. Now for any $(\gamma^1,\gamma^2) \in (e_S^{-1}(Q) \times \mathrm{Geo(X)}) \cap A_i$ it holds that $\gamma^1_S \in Q$ and $\gamma^2_S \notin Q$. Define $\tilde\sigma$ as the restriction $\tilde\sigma := \sigma|_{(e_S^{-1}(Q)\times \mathrm{Geo(X)})\cap A_i}$. Furthermore define $\pi^1 := \mathrm{P}^1_{\#}\sigma$ and $\pi^2 := \mathrm{P}^2_{\#}\sigma$. Then $\pi^1,\pi^2 \ll \pi$, $\mu^1_0 := (e_0)_{\#}\pi^1 = (e_0)_{\#}\pi^2 =: \mu^2_0$ and $\mu^1_S \perp \mu^2_S$. By restricting the measures π^1 and π^2 we may assume that $\rho^i_j < C < \infty$ for some $C > 0$ and for all $i \in \{1,2\}$ and $j \in \{0,1\}$, where ρ^i_j is the density of $(e_j)_{\#}\mu^i$ with respect to $\mathfrak{m}$.

Step 2: As in the proof of [11, Theorem 3.3] we will find a time $T \in (0,S)$ so that the intersection of the sets $\{\rho^1_T > 0\}$ and $\{\rho^2_T > 0\}$ has positive measure. We repeat the argument here. Since $\mathfrak{m}$ is locally finite and $\operatorname{spt}\mu^1_0$ is compact, there exists a neighbourhood of $\operatorname{spt}\mu^1_0$ with finite $\mathfrak{m}$-measure. Furthermore there exists an ε-neighbourhood D of $\operatorname{spt}\mu^1_0$ for which $\mathfrak{m}(D) \le \frac{3}{2}\mathfrak{m}(\operatorname{spt}\mu^1_0)$. By Lemma 3.1 and by the fact that $L < \infty$ there exists $T \in (0,S)$ so that $\mathfrak{m}(\{\rho^i_T > 0\}) > \frac{3}{4}\mathfrak{m}(\{\rho^i_0 > 0\})$ and $\{\rho^i_T > 0\} \subset D$. Hence $\mathfrak{m}(\{\rho^1_T > 0\} \cap \{\rho^2_T > 0\}) > 0$. Define now

$$\hat\pi^i := \frac{1}{\int \min\{\rho^1_T,\rho^2_T\}\mathrm{d}\mathfrak{m}} \left(\frac{\min\{\rho^1_T,\rho^2_T\}}{\rho^i_T} \circ e_T\right)\pi^i$$

for $i \in \{1,2\}$. Then we have that $\hat\pi^1,\hat\pi^2 \ll \pi$, $\hat\mu^1_T = \hat\mu^2_T$, $\hat\mu^1_S \perp \hat\mu^2_S$ and $\hat\rho^1_j,\hat\rho^2_j < C < \infty$ for some $C > 0$ and for $j \in \{0,1\}$. By Lemma 3.3 there exists $C > 0$ so that $\hat\rho^1_t,\rho^2_t < C < \infty$ for every $t \in [0,1]$.

Step 3: Let $T = t_0 < t_1 < \cdots t_k = S$ be a partition of the interval $[T,S]$ into subintervals of equal length and define $f(i) := \int \min\{\hat\rho^1_{t_i},\hat\rho^2_{t_i}\}\mathrm{d}\mathfrak{m}$. Then $f(0) = 1$ and $f(k) = 0$. Therefore there exists $i \in \{0,\dots,k\}$ so that $f(i) - f(i+1) \ge \frac{1}{k}$. Define now $\tilde\pi^1 := \hat\pi^1|_{e^{-1}_{t_{i+1}}(\{\hat\rho^1_{t_{i+1}} > \hat\rho^2_{t_{i+1}}\})}$ and $\tilde\pi^2 := \hat\pi^2|_{e^{-1}_{t_i+1}(\{\hat\rho^1_{t_{i+1}} \le \hat\rho^2_{t_{i+1}}\})}$, and further for $j \in \{1,2\}$

$$\bar\pi^j := \left(\frac{\min\{\tilde\rho^1_{t_i},\tilde\rho^2_{t_i}\}}{\tilde\rho^j_{t_i}} \circ e_{t_i}\right)\tilde\pi^j.$$

Then $\bar\mu^1_{t_{i+1}}(\{\hat\rho^1_{t_{i+1}} > \hat\rho^2_{t_{i+1}}\}) = \bar\mu^1_{t_{i+1}}(X)$ and $\bar\mu^2_{t_{i+1}}(\{\hat\rho^1_{t_{i+1}} \le \hat\rho^2_{t_{i+1}}\}) = \bar\mu^2_{t_{i+1}}(X)$. Thus we have that $\bar\mu^1_{t_{i+1}} \perp \bar\mu^2_{t_{i+1}}$. By definition we also have that $\bar\mu^1_{t_i} = \bar\mu^2_{t_i}$. Let us prove that

$$\bar\mu^1_{t_i}(X) \ge \frac{1}{k}. \tag{4}$$

By definition we have that $\bar{\rho}^j_{t_i} \leq \tilde{\rho}^j_{t_i} \leq \hat{\rho}^j_{t_i}$ for $j \in \{1,2\}$. Also we have that

$$\begin{aligned}\tilde{\mu}^1_{t_i}(X) + \tilde{\mu}^2_{t_i}(X) &= \tilde{\mu}^1_{t_{i+1}}(X) + \tilde{\mu}^2_{t_{i+1}}(X)\\ &= \int \hat{\rho}^1_{t_{i+1}} + \hat{\rho}^2_{t_{i+1}} - \min\{\hat{\rho}^1_{t_{i+1}}, \hat{\rho}^2_{t_{i+1}}\}\mathrm{d}\mathfrak{m}\\ &= 2 - f(i+1).\end{aligned}$$

Therefore we get that

$$\begin{aligned}\bar{\mu}^1_{t_i}(X) &= \int \min\{\tilde{\rho}^1_{t_i}, \tilde{\rho}^2_{t_i}\}\mathrm{d}\mathfrak{m} = \tilde{\mu}^1_{t_i}(X) + \tilde{\mu}^2_{t_i}(X) - \int \max\{\tilde{\rho}^1_{t_i}, \tilde{\rho}^2_{t_i}\}\mathrm{d}\mathfrak{m}\\ &\geq 2 - f(i+1) - \int \max\{\hat{\rho}^1_{t_i}, \hat{\rho}^2_{t_i}\} = f(i) - f(i+1) \geq \frac{1}{k}.\end{aligned}$$

Final step: Now we are ready to arrive to a contradiction with the convexity of the entropy. As in [19] we consider three measures $\bar{\pi}^1/M$, $\bar{\pi}^2/M$ and $(\bar{\pi}^1 + \bar{\pi}^2)/(2M)$ along which the entropy is K-convex. Here $M := \bar{\pi}^i(\mathrm{Geo(X)})$. For these measures we have that $\bar{\rho}^i_t < C$ for all $t \in [0,1]$ and $i \in \{1,2\}$, $M > 1/k$, $\bar{\mu}^1_{t_i} = \bar{\mu}^2_{t_i}$ and $\bar{\mu}^1_{t_{i+1}} \perp \bar{\mu}^2_{t_{i+1}}$. From these facts we get that

$$\begin{aligned}&\int \frac{\bar{\rho}^1_{t_i}}{M}\log\frac{\bar{\rho}^1_{t_i}}{M}\mathrm{d}\mathfrak{m}\\ &\leq \frac{1}{kt_{i+1}}\int \frac{\bar{\rho}^1_0 + \bar{\rho}^2_0}{2M}\log\frac{\bar{\rho}^1_0 + \bar{\rho}^2_0}{2M}\mathrm{d}\mathfrak{m} + \frac{t_i}{t_{i+1}}\int \frac{\bar{\rho}^1_{t_{i+1}} + \bar{\rho}^2_{t_{i+1}}}{2M}\log\frac{\bar{\rho}^1_{t_{i+1}} + \bar{\rho}^2_{t_{i+1}}}{2M}\mathrm{d}\mathfrak{m}\\ &\quad + \frac{|K|}{2}\frac{t_i}{t_{i+1}}\mathrm{W}_2^2(\bar{\mu}_0, \bar{\mu}_1)\\ &\leq \frac{\log\frac{C}{M}}{kt_{i+1}} - \frac{t_i}{t_{i+1}}\log 2 + \frac{t_i}{2t_{i+1}}\left(\int \frac{\bar{\rho}^1_{t_{i+1}}}{M}\log\frac{\bar{\rho}^1_{t_{i+1}}}{M}\mathrm{d}\mathfrak{m} + \int \frac{\bar{\rho}^2_{t_{i+1}}}{M}\log\frac{\bar{\rho}^2_{t_{i+1}}}{M}\mathrm{d}\mathfrak{m}\right)\\ &\quad + \frac{|K|}{2}\frac{t_i}{t_{i+1}}L^2\\ &\leq \frac{\log\frac{C}{M}}{kt_{i+1}} - \frac{t_i}{t_{i+1}}\log 2 + \frac{t_i}{2t_{i+1}}\left[\frac{2(1-t_{i+1})}{1-t_i}\int \frac{\bar{\rho}^1_{t_i}}{M}\log\frac{\bar{\rho}^1_{t_i}}{M}\mathrm{d}\mathfrak{m}\right.\\ &\quad \left. + \frac{1}{k(1-t_i)}\left(\int \frac{\bar{\rho}^1_1}{M}\log\frac{\bar{\rho}^1_1}{M}\mathrm{d}\mathfrak{m} + \int \frac{\bar{\rho}^2_1}{M}\log\frac{\bar{\rho}^2_1}{M}\mathrm{d}\mathfrak{m}\right)\right] + |K|\frac{t_i}{t_{i+1}}L^2\\ &\leq \left(\frac{1}{kt_{i+1}} + \frac{t_i}{t_{i+1}k(1-t_{i+1})}\right)\log\frac{C}{M} - \frac{5}{6}\frac{t_i}{t_{i+1}}\log 2 + \frac{t_i(1-t_{i+1})}{t_{i+1}(1-t_i)}\int \frac{\bar{\rho}^1_{t_i}}{M}\log\frac{\bar{\rho}^1_{t_i}}{M}\mathrm{d}\mathfrak{m}\end{aligned}$$

from which we obtain

$$\frac{1}{kt_{i+1}(1-t_i)}\int \frac{\bar{\rho}^1_{t_i}}{M}\log\frac{\bar{\rho}^1_{t_i}}{M}\mathrm{d}\mathfrak{m} \leq \frac{1}{kt_{i+1}(1-t_i)}\log\frac{C}{M} - \frac{5}{6}\frac{t_i}{t_{i+1}}\log 2$$

and further

$$\int \frac{\bar\rho^1_{t_i}}{M}\log\frac{\bar\rho^1_{t_i}}{M}\mathrm{d}\mathfrak{m} \le \log\frac{C}{M} - \frac{5k}{6}(1-t_i)t_i\log 2.$$

Choosing k large enough so that $T, (1-S)^2 \ge \frac{1}{k}$, and using the above inequality together with the convexity of the entropy along $\bar\pi^1$ we get that

$$\begin{aligned}\int \frac{\bar\rho^1_{t_{i+1}}}{M}\log\frac{\bar\rho^1_{t_{i+1}}}{M}\mathrm{d}\mathfrak{m} &\le \frac{t_{i+1}-t_i}{1-t_i}\log\frac{C}{M} + \frac{1-t_{i+1}}{1-t_i}\int\frac{\rho^1_{t_i}}{M}\log\frac{\rho^1_{t_i}}{M} \\ &+ \frac{|K|}{2}\frac{(1-t_{i+1})}{k(1-t_i)^2}L^2 \\ &\le \left(\frac{t_{i+1}-t_i}{1-t_i}+\frac{1-t_{i+1}}{1-t_i}\right)\log\frac{C}{M} - \frac{5k}{6}t_i(1-t_{i+1})\log 2 \\ &+ \frac{|K|}{2}kt_i(1-t_{i+1})L^2 \\ &\le \log\frac{C}{M} - \frac{k}{3}t_i(1-t_{i+1})\log 2\end{aligned}$$

from which by (4) and Jensen's inequality we obtain

$$\begin{aligned}\log\frac{1}{k\mathfrak{m}(X)} - \log M &\le \int\frac{\bar\rho^1_{t_{i+1}}}{M}\log\bar\rho^1_{t_{i+1}}\mathrm{d}\mathfrak{m} - \log M = \int\frac{\bar\rho^1_{t_{i+1}}}{M}\log\frac{\bar\rho^1_{t_{i+1}}}{M}\mathrm{d}\mathfrak{m} \\ &\le \log\frac{C}{M} - \frac{k}{3}t_i(1-t_{i+1})\log 2.\end{aligned}$$

Hence we arrive to a contradiction

$$\frac{3}{k}(\log\frac{1}{k\mathfrak{m}(X)} - \log C) \le -t_i(1-t_{i+1})\log 2 \le -T(1-S)\log 2$$

since the left hand side goes to zero when k goes to infinity while the right hand side is negative and bounded away from zero. □

Proof of Lemma 3.4. By an induction argument it suffices to prove that if $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ is not induced by a map, then for each $t \in (0,1)$ we have that $(\mathrm{restr}_0^t)_\#\pi$ or $(\mathrm{restr}_t^1)_\#\pi$ is not induced by a map.

Suppose that there is $t \in (0,1)$ so that both $\pi^1 := (\mathrm{restr}_0^t)_\#\pi$ and $\pi^2 := (\mathrm{restr}_t^1)_\#\pi$ are induced by maps T^1 and T^2 respectively. Denote $\{\pi_x\}$, $\{\pi^1_x\}$, $\{\pi^2_x\}$ the disintegrations of π, π^1 and π^2 with respect to e_0. It suffices to prove that $\mathrm{spt}\,\pi_x$ is a singleton for μ_0-a.e. $x \in X$. We will do this by proving that actually

$$\mathrm{spt}\,\pi_x = \mathrm{spt}\,\pi^1_x * \mathrm{spt}\,\pi^2_{\hat T(x)},$$

where $\hat T := e_1 \circ T^1$ is the optimal map from μ_0 to μ_t, and $*$ is the concatenation of paths.

We begin by observing that for μ_0-a.e. x we have, by the definition of disintegration, that $\pi^1_x = (\mathrm{restr}_0^t)_\#\pi_x$. In particular we have that $\mathrm{spt}\,\pi^1_x = \mathrm{restr}_0^t\,\mathrm{spt}\,\pi_x$. Thus, since both $\operatorname{spt}\pi_x^1$ and $\operatorname{spt}\pi^2_{\hat{T}(x)}$ are singletons for μ_0-a.e. $x \in X$, it suffices to prove that $\pi_x((\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(x)}) = 1$ for μ_0-a.e. $x \in X$. Suppose this is not the case. Then the set

$$E \coloneqq \{x : \pi_x((\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(x)}) < 1\}$$

has positive μ_0-measure. Consider the set $F \coloneqq \cup_{x\in E}\operatorname{spt}\pi^2_{\hat{T}(x)}$. Then

$$\begin{aligned}
\pi^2(F) &= \pi\left((\mathrm{restr}_t^1)^{-1}F\right) = \int \pi_x\left(\bigcup_{y\in E}(\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x)\\
&= \int_E \pi_x\left(\bigcup_{y\in E}(\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x) + \int_{X\setminus E} \pi_x\left(\bigcup_{y\in E}(\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x)\\
&= \int_E \pi_x\left((\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(x)}\right)\,\mathrm{d}\mu_0(x) + \int_{X\setminus E} \pi_x\left(\bigcup_{y\in E}(\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x)\\
&< \mu_0(E) + \int_{X\setminus E} \pi_x\left(\bigcup_{y\in E}(\mathrm{restr}_t^1)^{-1}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x)\\
&= \mu_0(E) + \mu_0(\{x \in X\setminus E : \exists y \in E \text{ for which } \hat{T}(x) = \hat{T}(y)\})\\
&= \int_E \pi^2_{\hat{T}(x)}\left(\bigcup_{y\in E}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x) + \int_{X\setminus E} \pi^2_{\hat{T}(x)}\left(\bigcup_{y\in E}\operatorname{spt}\pi^2_{\hat{T}(y)}\right)\,\mathrm{d}\mu_0(x)\\
&= \pi^2(F)
\end{aligned}$$

which is a contradiction. Thus we have proven the lemma. □

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Optimal transport maps on Alexandrov spaces revisited

T. Rajala and T. Schultz

Preprint

OPTIMAL TRANSPORT MAPS ON ALEXANDROV SPACES REVISITED

TAPIO RAJALA AND TIMO SCHULTZ

Abstract. We give an alternative proof for the fact that in n-dimensional Alexandrov spaces with curvature bounded below there exists a unique optimal transport plan from any purely $(n-1)$-unrectifiable starting measure, and that this plan is induced by an optimal map.

1. Introduction

The problem of optimal mass transportation has a long history, starting from the work of Monge [27] in the late 18th century. In the original formulation of the problem, nowadays called the Monge-formulation, the problem is to find the transport map T minimizing the transportation cost

$$\int_{\mathbb{R}^n} c(x, T(x))\,\mathrm{d}\mu_0(x), \tag{1.1}$$

among all Borel maps $T\colon \mathbb{R}^n \to \mathbb{R}^n$ transporting a given probability measure μ_0 to another given probability measure μ_1, that is, $T_\sharp\mu_0 = \mu_1$. In the original problem of Monge, the cost function $c(x,y)$ was the Euclidean distance. Later, other cost functions have been considered, in particular much of the study has involved the distance squared cost, $c(x,y) = |x-y|^2$, which is the cost studied also in this paper.

In the Monge-formulation (1.1) of the optimal mass transportation problem the class of admissible maps T that send μ_0 to μ_1 is in most cases not closed in any suitable topology. To overcome this problem, Kantorovich [20, 19] considered a larger class of optimal transports, namely, measures π on $\mathbb{R}^n \times \mathbb{R}^n$ such that the first marginal of π is μ_0 and the second is μ_1. Such measures π are called transport plans. Kantorovich's relaxation leads to the so-called Kantorovich-formulation of the problem,

$$\inf_\pi \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)\,\mathrm{d}\pi(x,y). \tag{1.2}$$

Due to the closedness of the admissible transport plans and the lower semi-continuity of the cost, minimizers exist in the Kantorovich-formulation under very mild assumptions on the underlying space and the cost c.

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For the quadratic cost in the Euclidean space, it was shown independently by Brenier [8] and Smith and Knott [35] that having μ_0 absolutely continuous with respect to the Lebesgue measure guarantees that the optimal transport plans (minimizer of (1.2)) are unique and given by a transport map. Moreover, the optimal transport map is given by a gradient of a convex function.

The results of Brenier and of Smith and Knott have been generalized in many ways. The most important directions of generalization have been: going from the underlying space $\mathbb{R}^n$ to other metric spaces, considering other cost functions, and relaxing the assumption of the starting measure being absolutely continuous with respect to the reference measure (here the Lebesgue measure). In this paper, we study the direction of relaxing the absolute continuity in a more general metric space setting, the Alexandrov spaces. We note that one should be able to generalize our proof for more general costs, such as the distance to a power $p \in (1, \infty)$. In order to keep the presentation simpler, we concentrate here on the distance squared cost.

The existence of optimal transportation maps in Alexandrov spaces with curvature bounded below for starting measures that are absolutely continuous with respect to the reference Hausdorff measure was proven by Bertrand [6]. Later Bertrand improved this result [7] by relaxing the assumption on the starting measure to give zero measure to $c-c$-hypersurfaces. Here we provide an alternative proof for the result of Bertrand under the slightly stronger assumption on the starting measure of pure $(n-1)$-unrectifiability (see Definition 2.1 for the definition of pure $(n-1)$-unrectifiability).

Theorem 1.1. *Let (X, d) be an n-dimensional Alexandrov space with curvature bounded below. Then for any pair of measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ such that μ_0 is purely $(n-1)$-unrectifiable, there exists a unique optimal transport plan from μ_0 to μ_1 and this transport plan is induced by a map.*

The contribution of this paper is to provide a different approach to showing the existence and uniqueness of optimal transport maps than what was used by Bertrand in [6, 7]. In [6], Bertrand used the local $(1+\varepsilon)$-biLipschitz maps to $\mathbb{R}^n$ on the regular set of X, and the general existence of Kantorovich potentials and their Lipschitzness. Since the singular set of X is at most $(n-1)$-dimensional, and the Rademacher's theorem on $\mathbb{R}^n$ can be restated in X via the biLipschitz maps, Bertrand concluded that the optimal transport is concentrated on a graph that is given by applying the exponential map to the gradient of the Kantorovich potential. In [7], Bertrand considered the problem in boundaryless Alexandrov spaces. He used Perelman's DC calculus to translate the problem to differentiability of convex functions on Euclidean spaces. Then the result follows from the characterization of nondifferentiability points of convex functions due to Zajíček [39].

In this paper, we translate a contradiction argument (Lemma 2.11) from the Euclidean space (which uses just cyclical monotonicity in certain geometric configurations) to the space X via the $(1+\varepsilon)$-biLipschitz charts. In order to use the contradiction argument, we need to get all the used distances to be comparable. For this we use the fact that the directions of geodesics are well-defined in the biLipschitz charts (Theorem 2.7) and thus we can contract along the geodesics without changing the geometric configuration too much. Finally, the geometric configurations that result in the contradiction via cyclical monotonicity are given by the pure $(n-1)$-unrectifiability (Lemma 2.2).

Let us comment on the history of the sufficient assumptions on μ_0. The assumption of pure $(n-1)$-unrectifiability was shown by McCann [26] to be sufficient for the existence of optimal maps in the case of Riemannian manifolds. A sharper condition based on the characterization by Zajíček [39] of the set of nondifferentiability points of convex functions was first used in the Euclidean context by Gangbo and McCann [15] when they showed that having an initial measure that gives zero mass to $c-c$ -hypersurfaces is sufficient to give the existence of optimal maps. It was then shown by Gigli [16] that even in the Riemannian manifold context the sharp requirement for the starting measure to have optimal maps for any target measure is indeed that it gives zero measure to $c-c$ -hypersurfaces. It still remains open whether zero measure of $c-c$ -hypersurfaces also gives a full characterization in the case of Alexandrov spaces. One of the directions, the sufficiency, was obtained by Bertrand [7].

The existence of optimal maps has been studied in wider classes of metric measure spaces that satisfy some form of Ricci curvature lower bounds or weak versions of measure contraction property. These classes include $CD(K,N)$-spaces that were introduced by Lott and Villani [25], and by Sturm [37, 36], $MCP(K,N)$-spaces (see Ohta [28]), and $RCD(K,N)$ spaces that were first introduced by Ambrosio, Gigli and Savaré [3] (see also the improvements and later work by Ambrosio, Gigli, Mondino and Rajala [1], Erbar, Kuwada and Sturm [13] and Ambrosio, Mondino and Savaré [4]). All of these classes contain Alexandrov spaces with curvature lower bounds, see Petrunin [30].

It was first shown by Gigli [17], that in nonbranching $CD(K,N)$-spaces you do have the existence of optimal maps provided that the starting measure is absolutely continuous with respect to the reference measure. In all the subsequent work, the assumption has been the same for the starting measure, and it would be interesting to see if it can be relaxed also in the more general context of metric measure spaces with Ricci curvature lower bounds.

Also a metric version of Brenier's theorem was studied by Ambrosio, Gigli and Savaré [2]. They did not obtain the existence of optimal maps, but showed that at least the transportation distance is given by the Kantorovich potential. Later, Ambrosio and Rajala [5] showed that under sufficiently strong nonbranching assumptions one can conclude the existence of optimal maps.

Rajala and Sturm [32] noticed that strong $CD(K,\infty)$ spaces, and hence $RCD(K,\infty)$ spaces are at least essentially nonbranching, and that this weaker form of nonbranching is sufficient for carrying out Gigli's proof. This result was later improved by Gigli, Rajala and Sturm [18]. Essential nonbranching was then studied together with the measure contraction property $MCP(K,N)$ by Cavalletti and Mondino [12] (see also Cavalletti and Huesmann [11] where the case of nonbranching and a weaker version of $MCP(K,N)$ was considered), and finally it was shown by Kell [22] that under a weak type measure contraction property, the essential nonbranching characterizes the uniqueness of optimal transports and that the unique optimal transport is given by a map for absolutely continuous starting measures.

The existence of optimal transport maps in $CD(K,N)$ spaces without any extra assumption on nonbranching is still an open problem. An intermediate definition between

$CD(K,N)$ and essentially nonbranching $CD(K,N)$, called very strict $CD(K,N)$, was studied by Schultz [33]. He showed that in these spaces one still has optimal transport maps even if the space could be highly branching and the optimal plans non-unique. It is also worth noting that if one drops the assumption of essential nonbranching for $MCP(K,N)$, then optimal transport maps need not exist. This is seen from the examples by Ketterer and Rajala [23].

The paper is organized as follows. In Section 2 we recall basic things about rectifiability, Alexandrov spaces and optimal mass transportation. While doing this, we also present a few facts that easily follow from well-known results: purely $n-1$-unrectifiable measures have mass in all directions (Lemma 2.2), the singular set in an Alexandrov space is $(n-1)$-rectifiable (Theorem 2.5), gradients of geodesics exist in charts in Alexandrov spaces (Theorem 2.7) and the failure of cyclical monotonicity persists after small perturbations (Lemma 2.11). In Section 3 we then put these things together and prove Theorem 1.1.

2. Preliminaries

In this paper (X,d) always refers to a complete and locally compact length space. By a length space we mean a metric space where the distance between any two points x and y is equal to the infimum of lengths of curves connecting x and y. By the Hopf-Rinow-Cohn-Vossen Theorem, our spaces (X,d) are then geodesic, proper and, in particular, separable. A space is called geodesic, if any two points in the space can be connected by a geodesic. By a geodesic we mean a constant speed length minimizing curve $\gamma\colon [0,1] \to X$. Notice that we parametrize all the geodesics by the unit interval. We denote the space of geodesics of X by $\mathrm{Geo}(X)$ and equip it with the supremum-distance. By a (geodesic) triangle $\Delta(x,y,z)$ we mean points $x,y,z \in X$ and any choice of geodesics $[x,y]$, $[y,z]$ and $[x,z]$ pairwise connecting them.

2.1. Rectifiability.

For our Theorem 1.1 the starting measure μ_0 is diffused enough if it is purely $n-1$-unrectifiable. Let us recall this notion.

Definition 2.1. A set $A \subset X$ is called *(countably) k-rectifiable* if there exist Lipschitz maps $f_i\colon E_i \to X$ from Borel sets $E_i \subset \mathbb{R}^k$ for $i \in \mathbb{N}$, such that $A \subset \bigcup_{i\in\mathbb{N}} f_i(E_i)$.

A measure μ is called *purely k-unrectifiable*, if $\mu(A) = 0$ for every k-rectifiable set A.

The property of purely unrectifiable measures that we use is that they have mass in all directions. This is made precise using (one-sided) cones that are defined as follows. Given $x \in \mathbb{R}^n$, $\theta \in \mathbb{S}^{n-1}$, $\alpha > 0$ and $r > 0$, we denote the open cone at x in direction θ with opening angle α, by

$$C(x,\theta,\alpha) := \{y \in \mathbb{R}^n \,:\, \langle y-x, \theta\rangle > \cos(\alpha)|y-x|\}\,.$$

Lemma 2.2. *Let μ be a purely $(n-1)$-unrectifiable measure on $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ with $\mu(E) > 0$. Then at μ-almost every $x \in E$ we have $C(x,\theta,\alpha) \cap B(x,r) \cap E \neq \emptyset$ for all $\theta \in \mathbb{S}^{n-1}$, $\alpha > 0$ and $r > 0$.*

Proof. Suppose that there is a subset $E_0 \subset E$ with $\mu(E_0) > 0$ such that the conclusion fails, i.e. for every $x \in E_0$ there exist $\theta_x \in \mathbb{S}^{n-1}$, $\alpha_x > 0$ and $r_x > 0$ such that $C(x, \theta_x, \alpha_x) \cap B(x, r_x) \cap E = \emptyset$. Since

$$C(x, \theta, \alpha) \cap B(x, r) \subset C(x, \theta, \alpha') \cap B(x, r')$$

if $\alpha' \geq \alpha$ and $r' \geq r$, there exist $r > 0$ and $\alpha > 0$ such that the subset

$$\{x \in E_0 \,:\, C(x, \theta_x, \alpha) \cap B(x, r) \cap E = \emptyset\}$$

has positive μ-measure. By considering a countable dense set of directions $\{\theta_i\}_{i \in \mathbb{N}}$, we have that there exists one fixed direction θ_i such that the set

$$E_1 := \{x \in E_0 \,:\, C(x, \theta_i, \alpha/2) \cap B(x, r) \cap E = \emptyset\}$$

has positive μ-measure. But now, for evey $x \in \mathbb{R}^n$, the set $E_1 \cap B(x, r/2)$ is contained in a Lipschitz graph and hence E_1 is an $(n-1)$-rectifiable set, giving a contradiction with the pure $(n-1)$-unrectifiability of μ. □

2.2. **Alexandrov spaces.** Let us recall some basics about Alexandrov spaces. Unless we provide another source, all the following definitions and results can be found in [9].

Alexandrov spaces generalize sectional curvature bounds by means of comparison to constant curvature model spaces. Alexandrov spaces can be defined for instance by comparing geodesic triangles of a metric space to the corresponding ones in a model space. Let us next give precise definitions.

For each $k \in \mathbb{R}$, let M_k be a simply connected surface with constant sectional curvature equal to k, that is, for negative k, M_k is a scaled hyperbolic plane, for $k = 0$, M_k is the Euclidean plane, and for positive k, M_k is a (round) sphere. Let us denote the distance between two points $x, y \in M_k$ by $|x - y|$.

Let $k \in \mathbb{R}$. For a triplet $x, y, z \in X$, let $\tilde{x}, \tilde{y}, \tilde{z} \in M_k$ be points so that the triangles $\Delta(x, y, z)$ and $\Delta(\tilde{x}, \tilde{y}, \tilde{z})$ have the same side lengths, that is, $d(x, y) = |\tilde{x} - \tilde{y}|, d(y, z) = |\tilde{y} - \tilde{z}|, d(x, z) = |\tilde{x} - \tilde{z}|$. We call the triangle $\Delta(\tilde{x}, \tilde{y}, \tilde{z})$ a comparison triangle for $\Delta(x, y, z)$. For a triangle $\Delta(x, y, z)$ in X we denote by $\tilde{\measuredangle}_k(y, x, z)$ the comparison angle at $\tilde{x}$ in the comparison triangle $\Delta(\tilde{x}, \tilde{y}, \tilde{z})$ in M_k.

Definition 2.3 (Alexandrov space)**.** We say that (X, d) is an Alexandrov space (with curvature bounded below by k) if there exists $k \in \mathbb{R}$ so that for each point $p \in X$ there exists a neighbourhood U of p for which the following holds. If $\Delta(x, y, z) \subset U$, $\Delta(\tilde{x}, \tilde{y}, \tilde{z})$ its comparison triangle in M_k, and $w \in [x, y]$, $\tilde{w} \in [\tilde{x}, \tilde{y}]$ with $d(x, w) = |\tilde{x} - \tilde{w}|$, then $d(w, z) \geq |\tilde{w} - \tilde{z}|$.

An Alexandrov space might have infinite (Hausdorff) dimension. In this paper we study only finite dimensional Alexandrov spaces. Recall that in an Alexandrov space every open nonempty set has the same dimension, so the dimension of an Alexandrov space is always well defined. Moreover, the dimension is either an integer or infinity. From now on, the space (X, d) is assumed to be an n-dimensional Alexandrov space with curvature bounded below by $k \in \mathbb{R}$ with $n \in \mathbb{N}$.

We will use the fact that our purely $(n-1)$-unrectifiable starting measures μ_0 live on the regular set of the space, that has nice charts. Let us recall the notion of regular and singular points.

Definition 2.4. A point $p \in X$ is called *regular*, if the space of directions Σ_p at p is isometric to the standard sphere $\mathbb{S}^{n-1}$, or equivalently, if the Gromov-Hausdorff tangent at p is the Euclidean $\mathbb{R}^n$. A point $p \in X$ that is not regular is called *singular*. The set of regular points of X is denoted by $\mathrm{Reg}(X)$ and the set of singular points by $\mathrm{Sing}(X)$.

The following result is from [29] (see also [10]). It implies that our starting measures μ_0 give zero measure to the singular set.

Theorem 2.5. *The set* $\mathrm{Sing}(X)$ *is* $(n-1)$*-rectifiable.*

Proof. Notice that [29, Theorem A] states that $\mathrm{Sing}(X)$ has Hausdorff dimension at most $n-1$. However, the proof easily gives the stronger conclusion of $(n-1)$-rectifiability. Namely, observe that in the proof of [29, Theorem A] Otsu and Shioya show that $\mathrm{Sing}(X)$ is contained in Lipschitz images from subsets of the spaces of directions Σ_p for countably many points $p \in X$. Since the points p are only needed to locally form a maximal ε-discrete net in X, they can be chosen to be regular points of X. Thus, $\mathrm{Sing}(X)$ is contained in countably many Lipschitz images from subsets of $\mathbb{S}^{n-1}$ and is therefore $(n-1)$-rectifiable. □

Let us then recall a well-known consequence of the nonbranching property of Alexandrov spaces. For its proof, we need the notion of an angle. Let $\alpha, \beta\colon [0,1] \to X$ be two constant speed geodesics emanating from the same point $p = \alpha(0) = \beta(0)$. Let us denote by $\theta_k(t,s) := \tilde{\measuredangle}_k(\alpha(t), p, \beta(s))$ the angle at $\tilde{p}$ of the comparison triangle $\Delta(\tilde{p}, \tilde{\alpha}(t), \tilde{\beta}(s))$ in M_k of $\Delta(p, \alpha(t), \beta(s))$. In Alexandrov spaces the angle

$$\measuredangle(\alpha, \beta) := \lim_{t,s \searrow 0} \theta_k(t,s)$$

is well-defined for every pair of geodesics α, β emanating from the same point. Moreover, by Alexandrov convexity (see for instance [34, Section 2.2]) the quantity $\theta_k(t,s)$ is monotone non-increasing in both variables t and s.

Lemma 2.6. *Let* $\gamma_1, \gamma_2\colon [0,1] \to X$ *be be two constant speed geodesics with* $\gamma_1(0) = \gamma_2(0)$ *and* $\gamma_1(1) \neq \gamma_2(1)$*. Then*

$$\lim_{t \searrow 0} \frac{d(\gamma_1(t), \gamma_2(t))}{t} > 0.$$

Proof. We may assume $\ell(\gamma_1) \geq \ell(\gamma_2)$. If $\ell(\gamma_1) > \ell(\gamma_2)$, then by triangle inequality $d(\gamma_1(t), \gamma_2(t)) \geq t(\ell(\gamma_1) - \ell(\gamma_2))$, giving the claim. If $\ell(\gamma_1) = \ell(\gamma_2)$, then $\theta_k(1,1) = \tilde{\measuredangle}_k(\gamma_1(1), x, \gamma_2(1)) > 0$. Then by Alexandrov convexity, $\measuredangle(\gamma_1, \gamma_2) \geq \theta_k(1,1) > 0$, and thus by the cosine law

$$\frac{d(\gamma_1(t), \gamma_2(t))}{t} \to \ell(\gamma_1)\sqrt{2 - 2\cos(\measuredangle(\gamma_1, \gamma_2))} > 0,$$

as $t \to 0$. □

Our aim is to arrive at a contradiction with cyclical monotonicity at a small scale near a regular point. We will transfer the Euclidean argument to the Alexandrov space X using the following standard charts φ. Since we need the existence of directions of geodesics in these charts, we write the existence down explicitly inside the following theorem.

Theorem 2.7. *For every $p \in \mathrm{Reg}(X)$ and every $\varepsilon > 0$ there exist a neighbourhood U of p and a $(1+\varepsilon)$-biLipschitz map $\varphi\colon U \to \mathbb{R}^n$ with $\varphi(U)$ open so that for every constant speed geodesic $\gamma\colon [0,1] \to U$ the limit*

$$\lim_{t\searrow 0} \frac{\varphi(\gamma(t)) - \varphi(\gamma(0))}{d(\gamma(t),\gamma(0))}$$

exists.

Proof. We recall (see [29] or [9, Theorem 10.8.4]) that the local $(1+\varepsilon)$-biLipschitz chart $\varphi\colon U \to \mathbb{R}^n$ can be obtained as

$$\varphi(x) = (d(a_1, x), d(a_2, x), \ldots, d(a_n, x)),$$

where $(a_i, b_i)_{i=1}^n$ is a δ-strainer for p, for some $\delta > 0$. Now, the first variation formula (see [29, Theorem 3.5] or [9, Theorem 4.5.6, Corollary 4.5.7]) implies that

$$\lim_{t\searrow 0} \frac{d(a_i,\gamma(t)) - d(a_i,\gamma(0))}{d(\gamma(t),\gamma(0))} = -\cos(\alpha),$$

where $\alpha = \measuredangle(\gamma, \beta)$, with β a geodesic from $\gamma(0)$ to a_i. Thus, the required limit exists for each i. □

2.3. **Optimal mass transportation.** In this section we recall a few basic things in optimal mass transportation.

The Monge-Kantorovich formulation of optimal mass transportation problem (with quadratic cost) is to investigate for two Borel probability measures μ_0 and μ_1 the following infimum

$$\inf \int_{X\times X} d^2(x,y)\,\mathrm{d}\pi(x,y),$$

where the infimum is taken over all Borel probability measures $\pi \in \mathcal{P}(X \times X)$ which has μ_0 and μ_1 as marginals, that is, $\pi(A\times X) = \mu_0(A)$ and $\pi(X \times A) = \mu_1(A)$ for all Borel sets $A \in \mathcal{B}(X)$. In order to guarantee that the above infimum is finite, it is standard to assume the measures μ_0 and μ_1 to have finite second moments. The set of all Borel probability measures in X with finite second moments is denoted by $\mathcal{P}_2(X)$.

An admissible measure that minimizes the above infimum is called an optimal (transport) plan, and the set of optimal plans between μ_0 and μ_1 is denoted by $\mathrm{Opt}(\mu_0, \mu_1)$. We say that an optimal plan π is induced by a map, if there exists a Borel measurable function $T\colon X \to X$ so that $\pi = (\mathrm{id} \times T)_{\#}\mu_0$. Such a map is called an optimal (transport) map. While optimal plans exist under fairly general assumptions [38], the existence of optimal maps is not true in general.

Optimality of a given transport plan depends only on the c-cyclical monotonicity of the support of the plan. Let us recall this notion.

Definition 2.8 (cyclical monotonicity)**.** A set $\Gamma \subset X \times X$ is called c-cyclically monotone, if for all finite sets of points $\{(x_i, y_i)\}_{i=1}^N \subset \Gamma$ the inequality

$$\sum_{i=1}^{N} d^2(x_i, y_i) \leq \sum_{i=1}^{N} d^2(x_{\sigma(i)}, y_i)$$

holds for all permutations $\sigma \in S_N$ of $\{1, \dots, N\}$.

A characterization of optimality using c-cyclical monotonicity of the support that is sufficient for us is the following result proven in [31] which holds for continuous cost functions.

Lemma 2.9 ([31, Theorem B])**.** *Let X be a Polish space and $\mu_0, \mu_1 \in \mathcal{P}_2(X)$. Then a transport plan π between μ_0 and μ_1 is optimal if and only if its support is c-cyclically monotone set.*

In the following lemma we recall a well-known fact which allows us to localize the problem. One way to prove this is to use the result of Lisini in [24] about Wasserstein geodesics and their lifts to the space of probability measures on geodesics of X, see [14] for the proof.

Lemma 2.10. *Let (X, d) be a complete and separable geodesic metric space, and let $\Gamma \subset X \times X$ be a c-cyclically monotone. Then, the set*

$$\Gamma_t := \{(\gamma(0), \gamma(t)) \in X \times X : \gamma \in \mathrm{Geo}(X) \text{ with } (\gamma(0), \gamma(1)) \in \Gamma\}$$

is c-cyclically monotone for all $t \in [0, 1]$.

In order to arrive at a contradiction with cyclical monotonicity, we will use the following lemma.

Lemma 2.11. *For each $C > 1$ there exists $\delta > 0$ so that*

$$\frac{1}{2}|y_1 + y_2|^2 < (1 - \delta)(|y_1|^2 + |y_2|^2)$$

for all

$$y_1, y_2 \in K := \left\{(y_1, y_2) \in \mathbb{R}^{2n} \ : \ |y_2| = 1 \text{ and } |y_2 - y_1| \in \left[\frac{1}{C}, C\right]\right\}.$$

Proof. Let us first observe that for $y_1, y_2 \in \mathbb{R}^n$, with $y_1 \neq y_2$ we have

$$0 < |y_1 - y_2|^2 = |y_1|^2 - 2\langle y_1, y_2\rangle + |y_2|^2$$

and thus

$$|y_1 + y_2|^2 = |y_1|^2 + 2\langle y_1, y_2\rangle + |y_2|^2 < 2(|y_1|^2 + |y_2|^2). \tag{2.1}$$

The quantitative claim then follows by compactness of K: first of all notice that $K \subset \bar{B}(0, 2 + C)$ and thus K is bounded. The set K is also closed and hence it is compact. The function

$$(y_1, y_2) \mapsto \frac{|y_1 + y_2|^2}{|y_1|^2 + |y_2|^2}$$

is continuous as a function $K \to \mathbb{R}$. Therefore, the maximum of the above function is achieved in K. By (2.1), this maximum is strictly less than two and hence there exists $\delta > 0$ as in the claim. □

3. Proof of Theorem 1.1

In order to prove the uniqueness of optimal transport plans it suffices to show that any optimal transport plan is induced by a map. Indeed, if there were two different optimal plans π_1 and π_2, then their convex combination $\frac{1}{2}(\pi_1 + \pi_2)$ would also be optimal and not given by a map. We will prove Theorem 1.1 by assuming that there exists an optimal plan that is not induced by a map, then localizing to a chart and using an Euclidean argument to find a contradiction.

Step 1: initial uniform bounds and measurable selections
Let $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with μ_0 purely $(n-1)$-unrectifiable. Let π be an optimal plan from μ_0 to μ_1. Towards a contradiction, we assume that π is not induced by a map, that is, there does not exist a Borel map $T\colon X \to X$ so that $\pi = (\mathrm{id}, T)_\sharp \mu_0$. Consider the set

$$A \coloneqq \{x \in X : \text{ there exist } y^1, y^2 \in X \text{ such that } (x, y^1), (x, y^2) \in \operatorname{spt}(\pi), y^1 \neq y^2\}.$$

Since A is a projection of a Borel set

$$\{(x, y, z, w) \in \operatorname{spt}(\pi) \times \operatorname{spt}(\pi) : d(x, z) = 0,\ d(y, w) > 0\},$$

it is a Souslin set and thus μ_0-measurable. (Actually, as a projection of a σ-compact set, A is Borel.) We will show that A has positive μ_0 measure.

For that we will first show that there exists a Borel selection $T\colon \mathtt{p}_1(\operatorname{spt}(\pi)) \to X$ of $\operatorname{spt}(\pi)$, where $\mathtt{p}_1\colon X \times X \to X$ is the projection to the first coordinate. Define

$$(\operatorname{spt}(\pi))_x \coloneqq \{y \in X : (x, y) \in \operatorname{spt}(\pi)\}.$$

Then $(\operatorname{spt}(\pi))_x = (\{x\} \times X) \cap \operatorname{spt}(\pi)$ and thus it is closed. Furthermore, as a proper space, X is also σ-compact, and thus so is $(\operatorname{spt}(\pi))_x$. Hence, by the Arsenin-Kunugui Theorem [21, Theorem 35.46] there exists a Borel selection of $\operatorname{spt}(\pi)$, in other words, there exists a Borel map $T\colon \mathtt{p}_1(\operatorname{spt}(\pi)) \to X$ so that $T(x) \in (\operatorname{spt}(\pi))_x$ for all $x \in \mathtt{p}_1(\operatorname{spt}(\pi))$.

Suppose now that $\mu_0(A) = 0$. We will show that in this case π would be induced by the map T. Indeed, for $E \subset X \times X$ we have that

$$\begin{aligned}
(\mathrm{id}, T)_{\#}\mu_0(E) &= \mu_0((\mathrm{id}, T)^{-1}(E) \setminus A) = \mu_0(\mathtt{p}_1(E \cap \operatorname{Graph}(T))) \\
&= \pi((\mathtt{p}_1(E \cap \operatorname{Graph}(T)) \times X) \cap \operatorname{spt}(\pi)) \\
&= \pi((E \cap \operatorname{spt}(\pi)) \setminus (A \times X)) = \pi(E \cap \operatorname{spt}(\pi)) = \pi(E).
\end{aligned}$$

Thus $\mu_0(A) > 0$.

Since X is geodesic, for all $x \in A$ there exist $\gamma_x^1, \gamma_x^2 \in \operatorname{Geo}(X)$ such that $\gamma_x^1(0) = x = \gamma_x^2(0)$, $\gamma_x^1(1) \neq \gamma_x^2(1)$, and $(\gamma_x^i(0), \gamma_x^i(1)) \in \operatorname{spt}(\pi)$ for $i \in \{1, 2\}$. We will need to choose the geodesics γ_x^1 and γ_x^2 in a measurable way. We will also make the selection so that

$$d(x, \gamma_x^1(1)) \leq d(x, \gamma_x^2(1)) \neq 0. \tag{3.1}$$

By now, we have a Borel selection T of $\operatorname{spt}(\pi)$. Since $\mathtt{p}_1(\operatorname{spt}(\pi))$ is a Borel set, we can extend T to a Borel map $T\colon X \to X$. Consider now the set $\operatorname{spt}(\pi) \setminus \operatorname{Graph}(T)$. Since T is a Borel map, the graph of T is a Borel set and thus the set $\operatorname{spt}(\pi) \setminus \operatorname{Graph}(T)$ is a Borel set. Since $X \setminus T(x)$ is σ-compact by the properness and separability of X, we have that $(\operatorname{spt}(\pi) \setminus \operatorname{Graph}(T))_x$ is σ-compact as a closed subset of $X \setminus T(x)$. Thus again by the

Arsenin-Kunugui Theorem there exists a Borel selection $S\colon \mathtt{p}_1(\mathrm{spt}(\pi)\setminus \mathrm{Graph}(T)) \to X$ that we can further extend to a Borel map $S\colon X\to X$ for which we have that $T(x)=S(x)$ for $x\notin A$, and $T(x)\neq S(x)$ for $x\in A$.

To have (3.1) we will define two auxiliary maps $\tilde T^1, \tilde T^2\colon X\to X\times X$ as

$$\tilde T^1(x) := \begin{cases} (x,T(x)), & x\in h^{-1}(-\infty,0) \\ (x,S(x)), & x\in h^{-1}[0,\infty), \end{cases}$$

where $h(x) := d(x,T(x)) - d(x,S(x))$, and similarly

$$\tilde T^2(x) := \begin{cases} (x,S(x)), & x\in h^{-1}(-\infty,0) \\ (x,T(x)), & x\in h^{-1}[0,\infty). \end{cases}$$

The maps $\tilde T^1$ and $\tilde T^2$ are Borel maps since T,S and h are Borel maps.

It remains to select the geodesics between points x and $T^i(x)$. For that, we consider the set

$$G := \{(x,y,\gamma)\in X\times X\times \mathrm{Geo}(X) \,:\, \gamma(0)=x, \gamma(1)=y\}.$$

The set G is Borel as the preimage of zero under the Borel map

$$(x,y,\gamma)\mapsto \sup\{d(x,\gamma(0)), d(y,\gamma(1))\}.$$

Furthermore, we have by the Arzelà-Ascoli Theorem that

$$G_{(x,y)} := \{\gamma\in \mathrm{Geo}(X) \,:\, \gamma(0)=x, \gamma(1)=y\}$$

is compact. Thus, by the Arsenin-Kunugui Theorem there exists a Borel selection $F\colon X\times X\to G_{(x,y)}$. With this we may finally define $T^1,T^2\colon X\to \mathrm{Geo}(X)$ as

$$T^1 := F\circ \tilde T^1 \quad \text{and}$$
$$T^2 := F\circ \tilde T^2.$$

From now on, we will denote $\gamma_x^1 = T^1(x)$ and $\gamma_x^2 = T^2(x)$ for all $x\in A$. Notice that γ_x^1 and γ_x^2 satisfy (3.1).

By Lemma 2.6, we have for all $x\in A$ that

$$\lim_{t\searrow 0} \frac{d(\gamma_x^1(t),\gamma_x^2(t))}{d(x,\gamma_x^2(t))} \in (0,\infty).$$

Thus, we may write A as a countable union of sets

$$A_i := \left\{ x\in A \,:\, d(x,\gamma_x^2(1))\in[1/i,i] \text{ and } \frac{d(\gamma_x^1(t),\gamma_x^2(t))}{d(x,\gamma^2(t))}\in[1/i,i] \text{ for all } t\le \frac{1}{i}\right\},$$

and therefore there exists $k\in\mathbb{N}$ so that $\mu_0(A_k)>0$. Notice that the sets A_i are measurable, since we can write A_i as the intersection of

$$\left\{x\in A : d(x,\gamma_x^2(1))\in[1/i,i]\right\}$$

and

$$\bigcap_{\substack{t\le \frac{1}{i}\\ t\in\mathbb{Q}}} \left\{ x\in X : \frac{d(\gamma_x^1(t),\gamma_x^2(t))}{d(x,\gamma_x^2(t))} \in [1/i, i] \right\}.$$

We now consider $k \in \mathbb{N}$ fixed so that $\mu_0(A_k) > 0$.

Step 2: localization to a chart

Now we are ready to localize the problem so that we may use properties of the Euclidean space to arrive to the contradiction. We will need to choose $\varepsilon > 0$ sufficiently small to arrive to a contradiction with c-cyclical monotonicity in a $(1+\varepsilon)$-chart given by Theorem 2.7. We define

$$\varepsilon := \frac{\delta}{100} \in (0, 1/200),$$

where $\delta = \delta(2k) \in (0, 1/2)$ is the constant given by Lemma 2.11 for the k fixed above. Since μ_0 is purely $(n-1)$-unrectifiable and $\mathrm{Sing}(X)$ is $(n-1)$-rectifiable by Theorem 2.5, we have $\mu_0(A_k \cap \mathrm{Reg}(X)) = \mu_0(A_k)$. By Theorem 2.7 we can cover the set $\mathrm{Reg}(X)$ with open sets U for which the associated maps $\varphi\colon U \to \mathbb{R}^n$ are $(1+\varepsilon)$-biLipschitz, and the limit

$$\lim_{t\searrow 0} \frac{\varphi(\gamma(t)) - \varphi(\gamma(0))}{d(\gamma(t),\gamma(0))}$$

exists for all geodesics $\gamma \subset U$. Since X is a proper metric space, it is in particular hereditarily Lindelöf. Therefore, there exists a countable subcover $\mathcal{F}$ of such open sets U. Hence, there exists $U \in \mathcal{F}$ for which $\mu_0(U \cap A_k) > 0$. Let $\varphi\colon U \to \mathbb{R}^n$ be as in Theorem 2.7.

Step 3: discretization and choice of points for the contradiction

Next we take a subset of $A_k \cap U$ where the direction of the two selected geodesics is independent of the point, up to a small error

$$\hat{\varepsilon} := \frac{\varepsilon}{80k^4} > 0. \tag{3.2}$$

This is done by covering the set $\mathbb{R}^n$ by sets $\{B(y_i, \hat{\varepsilon})\}_{i\in\mathbb{N}}$. Then there exist i, j and $t_0 > 0$ so that the set

$$B := \left\{ x \in A_k \cap U \ : \frac{\varphi(\gamma_x^1(t)) - \varphi(x)}{t} \in B(y_i, \hat{\varepsilon}), \frac{\varphi(\gamma_x^2(t)) - \varphi(x)}{t} \in B(y_j, \hat{\varepsilon}), \right.$$
$$\left. \varphi(\gamma_x^1(t)), \varphi(\gamma_x^2(t)) \in U \text{ for all } t \le t_0 \right\}$$

has positive μ_0-measure. Notice that B is seen to be measurable by a similar argument than A_i. By relabeling, we may assume that $i = 1$ and $j = 2$.

Since φ is biLipschitz, the measure $\varphi_{\#}\mu_0$ is purely $(n-1)$-unrectifiable on $\mathbb{R}^n$. Hence, by Lemma 2.2 there exist points $x_1, x_2 \in B$ such that

$$\varphi(x_2) \in C\left(\varphi(x_1), \frac{y_2 - y_1}{|y_2 - y_1|}, \hat{\varepsilon}\right) \cap B(\varphi(x_1), r), \tag{3.3}$$

where $r \leq \hat{\varepsilon}$ is such that $r \leq \frac{t_0}{2}|y_2 - y_1|$. Now that we have selected the initial points x_1 and x_2 for the contradiction argument, we still need to bring the target points close enough to x_1 and x_2 by contracting along the geodesics $\gamma^2_{x_1}$ and $\gamma^1_{x_2}$. Since $|\varphi(x_1) - \varphi(x_2)| < r$, there exists the desired contraction parameter $t \leq t_0$ for which

$$2|\varphi(x_2) - \varphi(x_1)| = |ty_2 - ty_1|. \tag{3.4}$$

We will now use as target points the points $\gamma^2_{x_1}(t)$ and $\gamma^1_{x_2}(t)$.

Step 4: verifying the bounds for Lemma 2.11

In the remainder of the proof we verify that the four selected points $x_2, x_1, \gamma^2_{x_1}(t)$ and $\gamma^1_{x_2}(t)$ give a contradiction with c-cyclical monotonicity. Towards this goal we first check that we may apply Lemma 2.11 with the selected δ.

First of all, we have by the definition of A_k that

$$\frac{|\varphi(\gamma^2_{x_1}(t)) - \varphi(\gamma^1_{x_1}(t))|}{|\varphi(\gamma^2_{x_1}(t)) - \varphi(x_1)|} \in \left[\frac{1}{(1+\varepsilon)^2 k}, (1+\varepsilon)^2 k\right].$$

Since

$$\hat{\varepsilon} \leq \frac{\varepsilon}{2(1+\varepsilon)k^2},$$

we have by the fact that $x_1 \in A_k$ and φ is $(1+\varepsilon)$-biLipschitz, that

$$2t\hat{\varepsilon} \leq \frac{\varepsilon}{(1+\varepsilon)} \frac{d(\gamma^2_{x_1}(t), x_1)}{k} \leq \frac{\varepsilon d(\gamma^2_{x_1}(t), \gamma^1_{x_1}(t))}{(1+\varepsilon)} \leq \varepsilon|\varphi(\gamma^2_{x_1}(t)) - \varphi(\gamma^1_{x_1}(t))|.$$

Similarly, since $\hat{\varepsilon} \leq \frac{\varepsilon}{(1+\varepsilon)k}$, we have that

$$t\hat{\varepsilon} \leq \varepsilon|\varphi(\gamma^2_{x_1}(t)) - \varphi(x_1)|.$$

Therefore, we have by the fact that $x_1 \in B$, the triangle inequality and the choice of ε and $\hat{\varepsilon}$ that

$$\begin{aligned}
\frac{|ty_2 - ty_1|}{|ty_2|} &\leq \frac{|\varphi(\gamma^2_{x_1}(t)) - \varphi(\gamma^1_{x_1}(t))| + 2t\hat{\varepsilon}}{|\varphi(\gamma^2_{x_1}(t)) - \varphi(x_1)| - t\hat{\varepsilon}} \\
&\leq \frac{(1+\varepsilon)}{(1-\varepsilon)} \frac{|\varphi(\gamma^2_{x_1}(t)) - \varphi(\gamma^1_{x_1}(t))|}{|\varphi(\gamma^2_{x_1}(t)) - \varphi(x_1)|} \\
&\leq \frac{(1+\varepsilon)}{(1-\varepsilon)}(1+\varepsilon)^2 k < 2k.
\end{aligned}$$

By similar arguments, we have that

$$\frac{|ty_2 - ty_1|}{|ty_2|} > \frac{1}{2k}.$$

Thus, by Lemma 2.11 with the $\delta = \delta(2k)$ already chosen accordingly, we have

$$\frac{\frac{1}{2}|t(y_1 + y_2)|^2}{|ty_2|^2} < (1-\delta)\frac{(|ty_2|^2 + |ty_1|^2)}{|ty_2|^2},$$

that is,

$$\frac{1}{2}|t(y_1+y_2)|^2 < (1-\delta)(|ty_2|^2+|ty_1|^2). \tag{3.5}$$

Step 5: the contradiction
We will then use the inequality (3.5) to get to a contradiction with the c-cyclical monotonicity guaranteed by Lemma 2.10. Let us first estimate the terms on the right-hand side of (3.5).

By the definition of y_1 and A_k we have that

$$\begin{aligned}|ty_1| &\le |ty_1-\varphi(\gamma^1_{x_1}(t))+\varphi(x_1)|+|\varphi(\gamma^1_{x_1}(t))-\varphi(x_1)|\\ &\le t\hat{\varepsilon}+(1+\varepsilon)d(\gamma_{x_1}(t),x_1)\le tk+(1+\varepsilon)tk\le 3tk.\end{aligned}$$

Similarly,

$$|ty_2|\le 3tk.$$

Therefore, we have that

$$|\frac{1}{2}t(y_1+y_2)|,|\frac{1}{2}t(y_2-y_1)|\le 3tk. \tag{3.6}$$

Using the definition of the set B, and (3.4), (3.3) and (3.6), we have

$$\begin{aligned}\frac{1}{(1+\varepsilon)^2}&d^2(x_2,\gamma^2_{x_1}(t))\le|\varphi(\gamma^2_{x_1}(t))-\varphi(x_2)|^2\\ &=|\frac{1}{2}t(y_1+y_2)+(\varphi(\gamma^2_{x_1}(t))-\varphi(x_1)-ty_2)-(\varphi(x_2)-\varphi(x_1)-\frac{1}{2}t(y_2-y_1))|^2\\ &\le(|\frac{1}{2}t(y_1+y_2)|+|\varphi(\gamma^2_{x_1}(t))-\varphi(x_1)-ty_2|+|\varphi(x_2)-\varphi(x_1)-\frac{1}{2}t(y_2-y_1)|)^2\\ &\le(|\frac{1}{2}t(y_1+y_2)|+t\hat{\varepsilon}+\frac{1}{2}|t(y_2-y_1)|\hat{\varepsilon})^2\le(|\frac{1}{2}t(y_1+y_2)|+(3k+1)t\hat{\varepsilon})^2\\ &\le|\frac{1}{2}t(y_1+y_2)|^2+6tk(3k+1)t\hat{\varepsilon}+((3k+1)t\hat{\varepsilon})^2\le|\frac{1}{2}t(y_1+y_2)|^2+40t^2k^2\hat{\varepsilon}\end{aligned}$$

and similarly

$$\frac{1}{(1+\varepsilon)^2}d^2(x_1,\gamma^2_{x_2}(t))\le|\frac{1}{2}t(y_1+y_2)|^2+40t^2k^2\hat{\varepsilon}.$$

Thus, by summing the two terms, using (3.2) and the fact that $x_1\in A_k$,

$$\begin{aligned}\frac{1}{(1+\varepsilon)^2}[d^2(x_2,\gamma^2_{x_1}(t))+d^2(x_1,\gamma^1_{x_2}(t))]&\le 2|\frac{1}{2}t(y_1+y_2)|^2+80t^2k^2\hat{\varepsilon}\\ &\le\frac{1}{2}|t(y_1+y_2)|^2+\frac{t^2}{k^2}\varepsilon\le\frac{1}{2}|t(y_1+y_2)|^2+\varepsilon d^2(\gamma^2_{x_1}(t),x_1).\end{aligned} \tag{3.7}$$

Again, by the definition of the set B and the choice of $\hat{\varepsilon}$

$$
\begin{aligned}
|ty_1|^2 &\le \big((1+\varepsilon)d(\gamma^1_{x_2}(t),x_2)+t\hat{\varepsilon}\big)^2 \le \big((1+\varepsilon)d(\gamma^1_{x_2}(t),x_2)+\varepsilon d(\gamma^2_{x_1}(t),x_1)\big)^2\\
&\le ((1+\varepsilon)^2+2(1+\varepsilon)\varepsilon)d^2(\gamma^1_{x_2}(t),x_2)+(\varepsilon^2+2(1+\varepsilon)\varepsilon)d^2(\gamma^2_{x_1}(t),x_1)\\
&\le (1+7\varepsilon)d^2(\gamma^1_{x_2}(t),x_2)+5\varepsilon d^2(\gamma^2_{x_1}(t),x_1)
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
|ty_2|^2 &\le \big((1+\varepsilon)d(\gamma^2_{x_1}(t),x_1)+t\hat{\varepsilon}\big)^2\\
&\le (1+2\varepsilon)^2d^2(\gamma^2_{x_1}(t),x_1)\le(1+8\varepsilon)d^2(\gamma^2_{x_1}(t),x_1).
\end{aligned}
\tag{3.9}
$$

Using the inequalities (3.5), (3.8) and (3.9), we get that

$$
\begin{aligned}
\frac{1}{2}|t(y_1+y_2)|^2 &< (1-\delta)(|ty_2|^2+|ty_1|^2)\\
&\le (1-\delta)(1+13\varepsilon)(d^2(\gamma^2_{x_1(t)},x_1)+d^2(\gamma^1_{x_2(t)},x_2)).
\end{aligned}
\tag{3.10}
$$

Hence, by (3.7), (3.10), the fact that $\delta \le \frac{1}{2}$ and the choice of ε, we have that

$$
\begin{aligned}
&d^2(x_2,\gamma^2_{x_1}(t))+d^2(x_1,\gamma^1_{x_2}(t))\\
&\quad\le (1+\varepsilon)^2\left(\frac{1}{2}|t(y_1+y_2)|^2+\varepsilon d^2(\gamma^2_{x_1}(t),x_1)\right)\\
&\quad\le (1+\varepsilon)^2(1-\delta)(1+15\varepsilon)(d^2(\gamma^2_{x_1}(t),x_1)+d^2(\gamma^1_{x_2}(t),x_2))\\
&\quad\le (1-\delta)(1+100\varepsilon)(d^2(\gamma^2_{x_1}(t),x_1)+d^2(\gamma^1_{x_2}(t),x_2))\\
&\quad< d^2(x_2,\gamma^1_{x_2}(t))+d^2(x_1,\gamma^2_{x_1}(t)).
\end{aligned}
$$

However, since $(x_2,\gamma^1_{x_2}(1)),(x_1,\gamma^2_{x_1}(1))\in \mathrm{spt}(\pi)$ we have by Lemma 2.10 that

$$
d^2(x_2,\gamma^1_{x_2}(t))+d^2(x_1,\gamma^2_{x_1}(t))\le d^2(x_2,\gamma^2_{x_1}(t))+d^2(x_1,\gamma^1_{x_2}(t))
$$

which is a contradiction. Therefore, the plan π is induced by a map.

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Equivalent definitions of very strict $CD(K,N)$ -spaces

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Preprint

EQUIVALENT DEFINITIONS OF VERY STRICT $CD(K, N)$ -SPACES

TIMO SCHULTZ

Abstract. We show the equivalence of the definitions of very strict $CD(K, N)$ -condition defined, on one hand, using (only) the entropy functionals, and on the other, the full displacement convexity class $\mathcal{DC}_N$. In particular, we show that assuming the convexity inequalities for the critical exponent implies it for all the greater exponents. We also establish the existence of optimal transport maps in very strict $CD(K, N)$ -spaces with finite N.

1. Introduction

Synthetic notions of curvature (bounds) have established their position in geometric analysis both as a tool to study geometric and analytic properties of non-smooth spaces, and as a new approach to attack problems even in the smooth setting. The framework present in this paper is the generalisation of Ricci curvature lower bounds to metric measure spaces, more precisely the setting of $CD(K, N)$-spaces introduced in the seminal papers of Lott–Villani [9] and Sturm [16, 17] based on a concept of displacement convexity of certain entropy functionals introduced by McCann [10].

The definitions of Sturm and Lott–Villani of $CD(K, N)$-spaces both share two notable properties, namely they are true generalisations of the notion of Ricci curvature lower bounds of (weighted) Riemannian manifolds, and, keeping in mind the Gromov's pre-compactness theorem for Riemannian manifolds sharing a common Ricci lower bound, they are stable under suitable convergence of metric measure spaces. The definitions of $CD(K, N)$-spaces by Sturm, and by Lott and Villani are different, but under an additional (essential) non-branching assumption of the spaces in question, these two notions of $CD(K, N)$-spaces agree. However, the non-branching property, while giving many desired results for $CD(K, N)$-spaces [17, 4, 7, 14, 6, 5, 11], is not stable under any reasonable convergence even when coupled with the $CD(K, N)$-condition.

In this paper, we study convexity properties of a pointwise density of transport plans in (possibly) branching $CD(K, N)$-spaces giving an equivalent definition (Proposition 4.2) for the so-called *very strict* $CD(K, N)$ -condition introduced in [15] (see also [2]), analogous to the known characterisation of essentially non-branching $CD(K, N)$-spaces, see [4]. Having the pointwise definition in hand, we prove Theorem 4.4, the equivalence of very strict $CD(K, N)$ -condition and its Lott–Villani type analogue (see Section 2.2 for the precise definitions).

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The main difference in the definitions by Sturm and by Lott–Villani is that while Sturm requires convexity to hold only for certain specific entropy functionals, namely the Rényi entropies, Lott and Villani require it to hold for *all* functionals in the so-called displacement convexity class. Using the defining convexity properties of the functionals in the displacement convexity class, we deduce easily the equivalence of the two definitions of very strict $CD(K,N)$ -spaces from the pointwise convexity inequality.

To obtain the pointwise condition, we use Theorem 3.1, the existence of optimal transport maps between two measures absolutely continuous with respect to the reference measure proven in [15] in the infinite dimensional case. For completeness, we present here the proof in the finite dimensional case. In fact, we need a bit more than just the existence of transport map. We need the plan to be given by a map not only from the endpoints, but also from the intermediate points.

As a byproduct, we prove Theorem 3.3, the existence of optimal transport map from a (boundedly supported) absolutely continuous measure to a singular one. We construct the plan given by a map by gluing together plans obtained between (absolutely continuous) intermediate points of the endpoints. We prove, in similar fashion to what is done in [13], that the resulting plan satisfies the convexity inequality of reduced curvature dimension condition between any three points of the unit interval.

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2. Preliminaries

Standing assumptions of this paper for a metric measure space $(X,d,\mathfrak{m})$ are completeness and separability for the metric d, and local finiteness for the Borel measure $\mathfrak{m}$.

A metric space (X,d) is said to be a length space, if the distance between any two points x and y is obtained by infimising the length of curves connecting x and y. A constant speed curve parametrised on the unit interval with length equal to the distance between the endpoints is called a (constant speed) geodesic. The set of all constant speed geodesics endowed with the supremum metric is denoted by $\mathrm{Geo}(X)$.

2.1. Optimal mass transportation.

We consider the Monge–Kantorovich formulation of the optimal transport problem with quadratic cost. Denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X. We define the Wasserstein 2-distance W_2 between two Borel probability measures $\mu,\nu\in\mathcal{P}(X)$ as the infimum

$$W_2(\mu,\nu) := \left(\inf_{\sigma\in\mathcal{A}(\mu,\nu)}\int_{X\times X} d^2(x,y)\,\mathrm{d}\sigma(x,y)\right)^{\frac{1}{2}},$$

where $\mathcal{A}(\mu,\nu) := \{\sigma\in\mathcal{P}(X\times X) : P^1_{\#}\sigma=\mu, P^2_{\#}\sigma=\nu\}$ is the set of admissible transport plans between μ and ν. The existence of an admissible plan that realises the infimum is true in rather general setting, including ours [18]. Such a minimising admissible plan is called an optimal plan, and the set of optimal plans between measures μ and ν is denoted by $\mathrm{Opt}(\mu,\nu)$.

Denote by $\mathcal{P}_2(X)$ the set of all Borel probability measures with finite second moment, that is, those $\mu \in \mathcal{P}(X)$ which are of finite W_2-distance from a Dirac mass. Moreover, denote by $\mathcal{P}_2^{ac}(X)$ a further subset of $\mathcal{P}_2(X)$ of measures absolutely continuous with respect to the reference measure $\mathfrak{m}$.

We recall, that the Wasserstein distance W_2 defines an actual metric on the set $\mathcal{P}_2(X)$. The space $(\mathcal{P}_2(X), W_2)$ inherits also some properties from the base space X, namely the space $(\mathcal{P}_2(X), W_2)$ is complete and separable length space, if (X,d) is. In the case of length spaces, we have the following useful characterisation of Wasserstein geodesics. A curve $t \mapsto \mu_t \in \mathcal{P}_2(X)$ is geodesic, if and only if there exists a measure $\pi \in \mathcal{P}(\mathrm{Geo}(X))$ so that $(e_0,e_1)_\#\pi \in \mathrm{Opt}(\mu_0,\mu_1)$, and $\mu_t = (e_t)_\#\pi$ for all $t \in [0,1]$, where $\gamma \mapsto e_t(\gamma) := \gamma_t$ is the evaluation map [8]. Such a probability measure π is called an optimal dynamical plan, or just an optimal plan for short, and the set of all optimal dynamical plans from μ_0 to μ_1 is denoted by $\mathrm{OptGeo}(\mu_0,\mu_1)$.

Recall, that for $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$, we have that $(\mathrm{restr}_{t_1}^{t_2})_\#(F\pi)$ is still an optimal plan for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and for all F with $\int F \,\mathrm{d}\pi = 1$, where $\mathrm{restr}_{t_1}^{t_2} \colon \mathrm{Geo}(X) \to \mathrm{Geo}(X)$, $(\mathrm{restr}_{t_1}^{t_2})(\gamma)(t) = \gamma(tt_2 + (1-t)t_1)$. For $\pi \in \mathcal{P}(\mathrm{Geo}(X))$, we denote by π^{-1} the pushforward measure of π under the map $\gamma \mapsto \gamma^{-1}$, $\gamma^{-1}(t) := \gamma(1-t)$.

2.2. **Synthetic Ricci curvature lower bounds.** Based on the notion of displacement convexity, introduced by McCann [10], of suitable entropy functionals, Sturm [16], and independetly Lott and Villani [9] introduced notions of Ricci curvature lower bounds for general (non-smooth) metric measure spaces.

We recall the definition of a more restrictive version of curvature dimension condition – the so-called *very strict CD(K,N)* -condition – and, motivated by the existence result for optimal maps in the context of such spaces, we introduce a Lott–Villani type analogue of the very strict $CD(K,N)$ -condition.

For the definitions, we need to introduce some auxiliary notation. As building blocks, we define, for $K \in \mathbb{R}$ and $N \in (0,\infty]$, coefficients $[0,1] \times \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$, $(t,\theta) \mapsto \sigma_{K,N}^{(t)}(\theta)$ as

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} t, & \text{if } N = \infty \\ \infty, & \text{if } K\theta^2 \geq N\pi^2 \\ \frac{\sin(t\theta\sqrt{\frac{K}{N}})}{\sin(\theta\sqrt{\frac{K}{N}})}, & \text{if } 0 < K\theta^2 < N\pi^2 \\ t, & \text{if } K = 0 \\ \frac{\sinh(t\theta\sqrt{\frac{-K}{N}})}{\sinh(\theta\sqrt{\frac{-K}{N}})}, & \text{if } K < 0. \end{cases}$$

Using these coefficients we further define, for $N \in (1,\infty]$, coefficients $\beta^{(t)}_{K,N}(\theta)$ and $\tau^{(t)}_{K,N}(\theta)$ as

$$\begin{aligned}\beta^{(t)}_{K,N}(\theta) &\coloneqq t^{1-N}(\sigma^{(t)}_{K,N-1}(\theta))^{N-1}, \quad \text{and}\\ \tau^{(t)}_{K,N}(\theta) &\coloneqq t^{\frac{1}{N}}(\sigma^{(t)}_{K,N-1}(\theta))^{\frac{N-1}{N}}.\end{aligned}$$

To be precise, we define for $t > 0, N > 1$

$$\beta^{(t)}_{K,N}(\theta) \coloneqq \begin{cases} e^{\frac{K}{6}(1-t^2)\theta^2}, & \text{if } N=\infty\\ \infty, & \text{if } N<\infty,\ K\theta^2 > (N-1)\pi^2\\ \left(\frac{\sin(t\theta\sqrt{\frac{K}{N-1}})}{t\sin(\theta\sqrt{\frac{K}{N-1}})}\right)^{N-1}, & \text{if } 0 < K\theta^2 \le (N-1)\pi^2\\ 1, & \text{if } N<\infty,\ K=0\\ \left(\frac{\sinh(t\theta\sqrt{\frac{-K}{N-1}})}{t\sinh(\theta\sqrt{\frac{-K}{N-1}})}\right)^{N-1}, & \text{if } N<\infty,\ K<0,\end{cases}$$

and $\beta^{(0)}_{K,N} \equiv 1$.

For $N \in (1,\infty]$, define the entropy functionals $\mathrm{Ent}_N \colon \mathcal{P}_2(X) \to \mathbb{R}\cup\{\pm\infty\}$ as

$$\mathrm{Ent}_N(\mu) \coloneqq -\int \rho^{-\frac{1}{N}}\,\mathrm{d}\mu,$$

for $N < \infty$, and

$$\mathrm{Ent}_\infty(\mu) \coloneqq \int \log\rho\,\mathrm{d}\mu + \int \infty\,\mathrm{d}\mu^\perp.$$

Here $\mu = \rho\mathfrak{m} + \mu^\perp$ with $\mu^\perp \perp \mathfrak{m}$, and $\mu^\perp(\{\rho > 0\}) = 0$. Further, for a transport plan $\pi \in \mathcal{P}(\mathrm{Geo}(X))$ with $(e_0)_\#\pi = \mu_0 \in \mathcal{P}_2(X)$, and for $t \in [0,1],\ K \in \mathbb{R}$, define the distorted entropy

$$\mathrm{Ent}^{(t)}_{N,\pi}(\mu_0) \coloneqq -\int \left(\beta^{(t)}_{K,N}(d(\gamma_0,\gamma_1))\right)^{\frac{1}{N}} \rho_0(\gamma_0)^{-\frac{1}{N}}\,\mathrm{d}\pi(\gamma),$$

for $N < \infty$, and

$$\mathrm{Ent}^{(t)}_{\infty,\pi}(\mu_0) \coloneqq \int \log\left(\frac{\rho_0(\gamma_0)}{\beta^{(t)}_{K,\infty}(d(\gamma_0,\gamma_1))}\right)\mathrm{d}\pi(\gamma) + \int \infty\,\mathrm{d}\mu_0^\perp.$$

Definition 2.1. We say that a metric measure space $(X,d,\mathfrak{m})$ is a very strict $CD(K,N)$ -space, if for all $\mu_0,\mu_1 \in \mathcal{P}_2^{ac}(X)$ with bounded supports, there exists $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ such that for all non-negative and bounded Borel functions $F\colon \mathrm{Geo}(X) \to \mathbb{R}$ with $\int F\,\mathrm{d}\pi = 1$, and for all $t_1,t_2 \in [0,1]$, $t_1 < t_2$, we have

$$\mathrm{Ent}_N(\tilde\mu_t) \le (1-t)\mathrm{Ent}^{(1-t)}_{N,\tilde\pi}(\tilde\mu_0) + t\mathrm{Ent}^{(t)}_{N,\tilde\pi^{-1}}(\tilde\mu_1) \tag{1}$$

for all $t \in [0,1]$, where $\tilde\mu_t \coloneqq (e_t)_\#\tilde\pi \coloneqq (e_t)_\#(\mathrm{restr}^{t_2}_{t_1})_\# F\pi$.

Remark 2.2. The definition would make sense also without the assumption on the boundedness of the supports. In that case, the functionals Ent_∞ and $\mathrm{Ent}_{\infty,\pi}$ are not a priori well-defined for all $\mu \in \mathcal{P}_2(X)$, due to the fact that $\int(\rho\log\rho)_-\,\mathrm{d}\mathfrak{m}$ might be $-\infty$. However, after requiring (1) to hold for μ_i, $i \in \{0,1\}$, with $(\rho_i \log\rho_i)_+ \in \mathrm{L}^1(\mathfrak{m})$, we know by [16, Theorem 4.24], that (for fixed $x_0 \in X$) $\mathfrak{m}(B(x_0,r)) \le Ae^{(Br^2)}$ holds for all $r > 1$, and thus $(\rho\log\rho)_- \in \mathrm{L}^1(\mathfrak{m})$ for all $\mu = \rho\mathfrak{m} \in \mathcal{P}_2(X)$, see [1].

We will also use the definition of *very strict* $CD^*(K,N)$ *-condition*, which one gets by modifying the above definitions (see [3] for the definition of reduced curvature dimension condition). More precisely, one replaces the convexity inequality (1) by the inequality

$$\mathrm{Ent}_N(\tilde{\mu}_t) \le -\int \sigma_{K,N}^{(1-t)}(d(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0) + \sigma_{K,N}^{(t)}(d(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\tilde{\pi}. \tag{2}$$

Definition 2.1 is a (possibly) more restrictive version of the strict $CD(K,N)$ -condition introduced in [2], and is given in the spirit of Sturm's original definition for curvature dimension condition. To define Lott–Villani type analogue of the condition, we need to introduce the so-called *displacement convexity classes*, introduced by McCann in [10].

We say, that a continuous and convex function $U\colon \mathbb{R}_+ \to \mathbb{R}$ is in the displacement convexity class $\mathcal{DC}_N$ (of dimension $N \in (1,\infty]$), if $U(0) = 0$, and if the function $s \mapsto u(s)$ is convex, where u is defined as

$$u\colon (0,\infty) \to \mathbb{R}, \quad s \mapsto s^N U(s^{-N}),$$

if $N < \infty$, and

$$u\colon \mathbb{R} \to \mathbb{R}, \quad s \mapsto e^s U(e^{-s}),$$

if $N = \infty$.

Remark 2.3. We recall, that the displacement convexity classes are nested. Indeed, if $N < N'$, we have that $\mathcal{DC}_{N'} \subset \mathcal{DC}_N$. This can be seen for example by writing

$$u_N(s) := s^N U(s^{-N}) = (s^{\frac{N}{N'}})^{N'} U((s^{\frac{N}{N'}})^{-N'}) =: u_{N'}(s^{\frac{N}{N'}})$$

as the composition of a convex and decreasing function $u_{N'}$ and a concave function $s \mapsto s^{\frac{N}{N'}}$. If $N' = \infty$, one writes

$$u_N(s) = e^{N\log s} U(e^{-N\log s}),$$

and concludes again, by the concavity of $s \mapsto \log s$, that u_N is convex.

For $U \in \mathcal{DC}_N$, define the (entropy) functional $U_\mathfrak{m}\colon \mathcal{P}_2(X) \to \mathbb{R}\cup\{\infty\}$ as

$$U_\mathfrak{m}(\mu) := \int U \circ \rho \,\mathrm{d}\mathfrak{m} + \int U'(\infty)\,\mathrm{d}\mu^\perp,$$

where $U'(\infty) := \lim_{s\to\infty} \frac{U(s)}{s} \in \mathbb{R} \cup \{\infty\}$. Furthermore, for $\pi \in \mathcal{P}(\mathrm{Geo}(X))$, $K \in \mathbb{R}$ and $t \in [0,1]$, define the functional $U^{(t)}_{\pi,\mathfrak{m}} \colon \mathcal{P}_2(X) \to \mathbb{R} \cup \{\infty\}$ as

$$U^{(t)}_{\pi,\mathfrak{m}}(\mu) := \int_X \int_{\mathrm{Geo}(X)} U\left(\frac{\rho(\gamma_0)}{\beta^{(t)}_{K,N}(\gamma_0,\gamma_1)}\right) \beta^{(t)}_{K,N}(\gamma_0,\gamma_1)\,\mathrm{d}\pi_x(\gamma)\,\mathrm{d}\mathfrak{m}(x) + \int_X U'(\infty)\,\mathrm{d}\mu^{\perp},$$

where $\{\pi_x\}$ is a disintegration of π with respect to the evaluation map e_0.

Remark 2.4. The functional $U^{(t)}_{\pi,\mathfrak{m}}$ is not well-defined in general due to the non-uniqueness of the disintegration. However, the definition will be used only for $\pi \in \mathrm{OptGeo}(\mu,\nu)$, in which case the disintegration is unique up to μ-measure zero set. Another cause of being ill-defined is the possible integrability issue, which may appear both for the positive and for the negative part of $U \circ \rho$ (and $\beta U(\rho/\beta)$), creating $\infty - \infty$ situations. This can be seen by taking $U(s) = s\log s - s^{1-\frac{1}{N}}$ in the hyperbolic space. Because of these issues, we will use the above definitions only for measures with bounded support, in which case the functionals are well-defined, see e.g. [18, Theorem 17.28] for the proof.

Definition 2.5. A metric measure space is said to satisfy the very strict $CD(K,N)$ condition *in the spirit of Lott–Villani*, if for all $\mu_0,\mu_1 \in \mathcal{P}_2^{ac}(X)$ with bounded supports, there exists $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ such that for all bounded non-negative Borel functions $F\colon \mathrm{Geo}(X) \to \mathbb{R}$ with $\int F\,\mathrm{d}\pi = 1$, and for all $t_1,t_2 \in [0,1]$, $t_1 < t_2$, we have

$$U_{\mathfrak{m}}(\tilde{\mu}_t) \le (1-t)U^{(1-t)}_{\tilde{\pi},\mathfrak{m}}(\tilde{\mu}_0) + tU^{(t)}_{\tilde{\pi}^{-1},\mathfrak{m}}(\tilde{\mu}_1)$$

for all $t \in [0,1]$ and for all $U \in \mathcal{DC}_N$, where $\tilde{\mu}_t := (e_t)_\#\tilde{\pi} := (e_t)_\#(\mathrm{restr}_{t_1}^{t_2})_\# F\pi$.

Remark 2.6. By choosing $U_N(s) = -s^{1-\frac{1}{N}}$, for $N < \infty$, and $U_\infty(s) = s\log s$, one immediately sees that spaces satisfying Definition 2.5 also satisfy Definition 2.1.

3. Existence of optimal maps

In proving our main results in Section 4, we will use the fact that the plan given by the definition of very strict $CD(K,N)$ -spaces is induced by a map. The case $N = \infty$ is covered in [15], and the proof of the finite dimensional case follows along the same lines. For completeness, we will outline the proof of the finite dimensional case here. It should be pointed out, that with our definition of very strict $CD(K,N)$ -spaces, we do not a priori know that very strict $CD(K,N)$ -condition for finite N implies the very strict $CD(K,\infty)$ -condition.

Theorem 3.1 (Existence of optimal maps)**.** *Let $(X,d,\mathfrak{m})$ be a very strict $CD^*(K,N)$ ($CD(K,N)$) -space, and let $\mu_0,\mu_1 \in \mathcal{P}_2^{ac}(X)$ with bounded supports. Let $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ be the optimal plan given by the very strict $CD^*(K,N)$ ($CD(K,N)$) -condition. Then π is induced by a Borel map $T\colon X \to \mathrm{Geo}(X)$, i.e. $\pi = T_\#\mu_0$ with $e_0 \circ T = \mathrm{id}$.*

Remark 3.2. If we remove in Definition 2.1 the assumption of the boundedness of the supports of μ_0 and μ_1, we may remove it also from Theorem 3.1.

Proof. Let $N < \infty$, and $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$. Furthermore, let $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ be the optimal plan given by the definition of very strict $CD^*(K,N)$ -space. Suppose that π is not induced by a map. Towards a contradiction, we will show that there exist plans $\pi^1, \pi^2 \ll \pi$, and times t_1 and t_2 sufficiently close to each other so that $\mu^1_{t_i} = \mu^2_{t_i}$ and $\mu^1_{t_{i+1}} \perp \mu^2_{t_{i+1}}$.

We begin by doing some reductions. First of all, by writing the whole space X as a union of bounded sets, we may assume that the length of the geodesics in the support of π is bounded by some constant C, and since $\operatorname{spt}\mathfrak{m}$ is proper, we may also assume that $\operatorname{spt}\mu_0$ is compact. Furthermore, by dividing the interval $[0,1]$ into sufficiently small subintervals I_j, and looking at the restriction measures $(\mathrm{restr}_{I_j})_{\#}\pi$, we may assume that

$$\sigma^{(t)}_{K,N}(\theta) \in [(1-\varepsilon)t, (1-\varepsilon)^{-1}t] \tag{3}$$

for all $t \in [0,1]$ and $\theta \leq C$. Here $\varepsilon > 0$ is chosen so that $(1-\varepsilon)^4 2^{\frac{1}{N}} > 1$.

Next, as was done in [15], we find times $T, S \in (0,1)$, $T < S$, and optimal plans $\pi^1, \pi^2 \ll \pi$ so that $\mu^1_T = \mu^2_T$ and $\mu^1_S \perp \mu^2_S$, where $\mu_t := (e_t)_{\#}\pi$ for all $t \in [0,1]$. We refer to [15] for the arguments and the construction. Let then $n \in \mathbb{N}$ be such that

$$\frac{t}{t+\frac{1}{n}}\left(\frac{1-(t+\frac{1}{n})}{1-t}\right) \geq \frac{1}{(1-\varepsilon)^4}2^{-\frac{1}{N}}, \tag{4}$$

for $t \in [T,S]$. Again, by the arguments used in [15], we find times $t_1, t_2 \in [T,S]$, $t_1 < t_2$, with $|t_2 - t_1| < \frac{1}{n}$, and optimal plans $\bar\pi^1, \bar\pi^2$ such that $\bar\mu^1_{t_1} = \bar\mu^1_{t_2}$ and $\mu^1_{t_2} \perp \mu^2_{t_2}$.

Now we are ready to arrive to a contradiction by similar computations as was done in [12]. We first use the convexity of the entropy along $\frac{1}{2}(\bar\pi^1 + \bar\pi^2)$ between points 0, t_1 and t_2, then along $\bar\pi^1$ and $\bar\pi^2$ separately between points t_1, t_2 and 1. Also the inequality (4) is used both times with the convexity inequality. Then we use the bound (3) and finally arrive to a contradiction by

$$\begin{aligned}
\int (\bar\rho^1_{t_1})^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} &\geq (1-\varepsilon)^2 \frac{t_2-t_1}{t_2} 2^{\frac{1}{N}-1} \int ((\bar\rho^1_0)^{1-\frac{1}{N}} + (\bar\rho^2_0)^{1-\frac{1}{N}})\,\mathrm{d}\mathfrak{m} \\
&\quad + (1-\varepsilon)^2 \frac{t_1}{t_2} 2^{\frac{1}{N}-1}\left(\int (\bar\rho^1_{t_2})^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} + \int (\bar\rho^2_{t_2})^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}\right) \\
&> (1-\varepsilon)^4 \frac{t_1}{t_2}\frac{(1-t_2)}{(1-t_1)} 2^{\frac{1}{N}} \int (\bar\rho^1_{t_1})^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} \geq \int (\bar\rho^1_{t_1})^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Here $\bar\rho^i_t$ is the density of $(e_t)_{\#}\bar\pi^i$ with respect to $\mathfrak{m}$. In the case of very strict $CD(K,N)$ -space, the proof is exactly the same after replacing $\sigma^{(t)}_{K,N}$ by $\tau^{(t)}_{K,N}$ in the condition (3). □

As a corollary, we get the existence of an optimal map from absolutely continuous measure to singular one, by approaching the singular endpoint with absolutely continuous intermediate points. Combined with construction similar to the one used in [13], we arrive to the following theorem.

Theorem 3.3. *Let $(X, d, \mathfrak{m})$ be a very strict $CD^*(K,N)$ -space with $N < \infty$, and $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, $\operatorname{spt}\mu_1 \subset \operatorname{spt}\mathfrak{m}$, probability measures with bounded support. Then there exists $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ along which the convexity inequality (2) holds between any points $t_1 < t_2 < t_3$ (with $\tilde{\mu}_t = \mu_t = (e_t)_\# \pi$) for the entropy Ent_N. Moreover, π is induced by a map from μ_0.*

Remark 3.4. We do not claim, that the convexity would hold along $F\pi$, where F is an arbitrary bounded non-negative Borel function with $\int F \,\mathrm{d}\pi = 1$. In fact, the proof below will in some cases produces a geodesic (μ_t) such that for any lift π of (μ_t) this is known to be false.

The idea of the proof of the above theorem is fairly simple. First of all, by approximating the possibly singular measure μ_1 by absolutely continuous ones, one obtains a geodesic μ_t with $\mu_t \ll \mathfrak{m}$ due to the lower semi-continuity of the entropy Ent_N. Then, by compactness of midpoints, there exist t-intermediate points μ_t, $t \in \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\}$, that are absolutely continuous, and minimise the entropy Ent_N among all midpoints of the previous point and 1. Now taking $\pi^i \in \mathrm{OptGeo}(\mu_{\frac{2^i-1}{2^i}}, \mu_{\frac{2^{i+1}-1}{2^{i+1}}})$ given by Theorem 3.1, and concatenating them, one obtains in the limit a plan with desired properties.

In the proof we will use the following lemma.

Lemma 3.5. *Let $(X, d, \mathfrak{m})$ be a very strict $CD^*(K,N)$ -space with $N < \infty$, and $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, $\operatorname{spt}\mu_1 \subset \operatorname{spt}\mathfrak{m}$, probability measures with bounded support, and let $\mu_{\frac{1}{2}}$ be a midpoint of μ_0 and μ_1 minimising the entropy among all midpoints. Then $\mu_{\frac{1}{2}} \in \mathcal{P}_2^{ac}(X)$.*

Proof. Clearly, we may assume that $K < 0$. Let μ_1^i be a sequence of absolutely continuous measures converging to μ_1, and having (uniformly) bounded support. Let $\pi_i \in \mathrm{OptGeo}(\mu_0, \mu_1^i)$ be a sequence satisfying the convexity inequality (2) and (sub)converging to some $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$. Then, by lower semi-continuity of the entropy, we have

$$\begin{aligned}\mathrm{Ent}_N(\mu_{\frac{1}{2}}) &\le \varliminf_i \mathrm{Ent}_N(\mu_{\frac{1}{2}}^i) \le \varliminf_i -\int \sigma_{K,N}^{\frac{1}{2}}(d(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0)\,\mathrm{d}\pi^i \\ &\le -\sigma_{K,N}^{\frac{1}{2}}(D)\mathrm{Ent}_N(\mu_0) < 0,\end{aligned}$$

where D is a bound for the diameters of the supports. Thus, we know that $\mu_{\frac{1}{2}}$ is not purely singular. Let now $\mu_{\frac{1}{2}} = \rho_{\frac{1}{2}}\mathfrak{m} + \mu_{\frac{1}{2}}^{\perp}$ be the Lebesgue decomposition of $\mu_{\frac{1}{2}}$, and let A be a Borel set on which $\mu_{\frac{1}{2}}^{\perp}$ is concentrated and with $(\rho_{\frac{1}{2}}\mathfrak{m})(A) = 0$. We want to show that $\mu_{\frac{1}{2}}^{\perp}(A) = 0$. Suppose that this is not the case, and define $\tilde{\mu}_j := (e_j)_\#\pi|_{e_{\frac{1}{2}}^{-1}(A)}$ for $j \in \{0,1\}$. Since $\tilde{\mu}_0$ is absolutely continuous with respect to $\mathfrak{m}$, there exists, by taking the minimiser of the entropy (which exists by compactness of midpoints), a midpoint $\tilde{\mu}_{\frac{1}{2}}$ of $\tilde{\mu}_0$ and $\tilde{\mu}_1$ which is not purely singular. Hence, $\hat{\mu}_{\frac{1}{2}} := \rho_{\frac{1}{2}}\mathfrak{m} + \tilde{\mu}_{\frac{1}{2}}$ is a midpoint of μ_0 and μ_1 with

$$\mathrm{Ent}_N(\hat{\mu}_{\frac{1}{2}}) < \mathrm{Ent}_N(\mu_{\frac{1}{2}}),$$

which contradicts the assumption of $\mu_{\frac{1}{2}}$ realising the minimum of the entropy. □

Proof of Theorem 3.3. Let $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, $\operatorname{spt}\mu_1 \subset \operatorname{spt}\mathfrak{m}$, be probability measures with bounded support. Since the space is boundedly compact, we know that the set of midpoints $\mathcal{M}(\mu_0,\mu_1)$ of μ_0 and μ_1 is compact. Moreover, the entropy Ent_N is lower semi-continuous on $\mathcal{M}(\mu_0,\mu_1)$ due to the finiteness of $\mathfrak{m}$ on bounded sets. Thus, there exists a midpoint $\mu_{\frac{1}{2}} \in \mathcal{M}(\mu_0,\mu_1)$ that minimises the entropy among midpoints. By induction we get a sequence of t_i-intermediate points $(\mu_{t_i})_{i\in\mathbb{N}}$, where $t_i = (1-2^{-i})$, and μ_{t_i} minimises the entropy among all midpoints of μ_{t_i} and μ_1.

By Lemma 3.5 we have that $\mu_{t_i} \ll \mathfrak{m}$. Thus, for each $i \in \mathbb{N}$, there exists $\pi_i \in \mathrm{OptGeo}(\mu_{t_{i-1}},\mu_{t_i})$ satisfying the very strict $CD^*(K,N)$ -condition, hence is induced by a map T_i from $\mu_{t_{i-1}}$. Consider now the decreasing sequence (A_i) of sets

$$A_i \coloneqq \{\pi \in \mathrm{OptGeo}(\mu_0,\mu_1) : (\mathrm{restr}_{t_{j-1}}^{t_j})_\#\pi = \pi_j \text{ for all } j \le i\}.$$

We will show that the intersection $A \coloneqq \cap A_i$ is singleton, and that the unique element of A satisfies the desired conditions. Since the sequence is nested, to show that A is non-empty, it suffices to show that each A_i is compact. Each A_i is tight, since the set $(\mathrm{restr}_0^{\frac{1}{2}})^{-1}(\operatorname{spt}\pi^1)$ is compact due to the continuity of the map $(\mathrm{restr})_0^{\frac{1}{2}}$ and Arzelà-Ascoli theorem. To see that A_i is closed, take a converging sequence $\tilde{\pi}_n \in A_i$, $\tilde{\pi}_n \longrightarrow \tilde{\pi}$. Then $(\mathrm{restr}_{t_j}^{t_{j+1}})_\#\tilde{\pi}_n \longrightarrow (\mathrm{restr}_{t_j}^{t_{j+1}})_\#\tilde{\pi}$, and hence $(\mathrm{restr}_{t_j}^{t_{j+1}})_\#\tilde{\pi} = \pi_j$. Therefore A_i is compact, and A is non-empty.

Let now $\pi \in A$. Then, for all $i \in \mathbb{N}$, $(\mathrm{restr}_0^{t_i})_\#\pi$ is induced by a map T_i due to the fact that $(\mathrm{restr}_{t_{i-1}}^{t_i})_\#\pi$ is induced by a map (see, e.g. [15, Lemma 4]). When $i < j$, we have that $T_i = \mathrm{restr}_0^{t_i/t_j} \circ T_j$. Thus, we have by completeness of X that T_i converges pointwise to some T. Indeed, for any $x \in X$, the sequence $T_i(x)$ is a Cauchy sequence. Hence, by dominated convergence we have for any continuous and bounded function $f\colon \mathrm{Geo}(X) \to \mathbb{R}$, that

$$\int f \,\mathrm{d}(T_i)_\#\mu_0 = \int f \circ T_i \,\mathrm{d}\mu_0 \longrightarrow \int f \circ T \,\mathrm{d}\mu_0 = \int f \,\mathrm{d}T_\#\mu_0$$

giving the weak convergence $(\mathrm{restr}_0^{t_i})_\#\pi \longrightarrow T_\#\mu_0$. On the other hand, we know that $(\mathrm{restr}_0^{t_i})_\#\pi \longrightarrow \pi$. Hence, the plan π is induced by a map.

Let us now prove the convexity of the entropy along π. The steps are similar to the ones in [13]. We will first prove, that the convexity holds between points δ, $\frac{1}{2}$ and 1, where δ is arbitrarily small. Let $\delta \in (0,\frac{1}{2})$. Suppose now that the claim is not true. Then there exists an interval $I = (a,b) \subset (0,\infty)$ so that

$$\int_{l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi < \int_{l^{-1}(I)} \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_\delta^{-\frac{1}{N}}(\gamma_\delta) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi, \quad (5)$$

where $l\colon \mathrm{Geo}(X) \to \mathbb{R}$ is the map sending a geodesic to its length. By continuity of the distortion coefficients we may assume, by subdividing the interval further, that

$$(1-\varepsilon)\sigma^\alpha((1-\delta)a) \le \sigma^\alpha((1-\delta)b), \quad (6)$$

where $\alpha \in \{\frac{\frac{1}{2}}{1-\delta}, \frac{\frac{1}{2}-\delta}{1-\delta}\}$, and ε is chosen so that

$$\begin{aligned}\int_{l^{-1}(I)} & \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \\ & < (1-\varepsilon)\int_{l^{-1}(I)} \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_\delta^{-\frac{1}{N}}(\gamma_\delta) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi.\end{aligned} \tag{7}$$

Let $\pi^I := \pi|_{l^{-1}(I)}$, and let $\mu_j^I = (e_j)_\#\pi^I$ for $j \in \{0,\delta,1\}$. Let $\mu_1^i \longrightarrow \mu_1^I$ be a sequence of absolutely continuous measures (with equibounded support) for which

$$\int (\rho_1^i)^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} \longrightarrow \int (\rho_1^I)^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}.$$

This can be done simply by approximating separately the singular part of μ_1^I, due to the lower semi-continuity of the entropy. Let now $\pi^i \in \mathrm{OptGeo}(\mu_0^I,\mu_1^i)$ be such that the converse of (5) holds for π^i between points δ_i, $\frac{1}{2}$ and 1, where $\delta_i \longrightarrow \delta$ with $\tilde{\mu}_{\delta_i} = \mu_\delta^I$. Finally, define

$$\tilde{\pi}^i := \pi|_{\mathrm{Geo}(X)\setminus l^{-1}(I)} + \pi^i.$$

We may assume, that $\tilde{\pi}^i \longrightarrow \tilde{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ weakly. By c-cyclical monotonicity (see [13, Proposition 1]), and by weak convergence, we know that $\tilde{\pi}^i(l^{-1}(I)) \longrightarrow 1$. Thus,

$$\begin{aligned}\mathrm{Ent}_N(\tilde{\mu}_{\frac{1}{2}}) &\le \varliminf_{i\to\infty} \mathrm{Ent}_N(\tilde{\mu}_1^i) = \varliminf_{i\to\infty}\left[-\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} (\tilde{\rho}_{\frac{1}{2}}^i)^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi - \int_{\mathrm{Geo}(X)} (\tilde{\rho}_{\frac{1}{2}}^i)^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi^i\right] \\ &\le \varliminf_{i\to\infty}\left[-\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi - \int (\rho_{\frac{1}{2}}^i)^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi^i\right] \\ &\le \varliminf_{i\to\infty}\left[-\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \right. \\ &\quad \left. - \int \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta_i}}(d(\gamma_{\delta_i},\gamma_1))(\rho_{\delta_i}^i)^{-\frac{1}{N}}(\gamma_{\delta_i}) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta_i}{1-\delta_i}}(d(\gamma_{\delta_i},\gamma_1))(\rho_1^i)^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi^i\right] \\ &\le \varliminf_{i\to\infty}\left[-\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \right. \\ &\quad \left. - \int_{l^{-1}(I)} \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta_i}}(d(\gamma_{\delta_i},\gamma_1))(\rho_{\delta_i}^i)^{-\frac{1}{N}}(\gamma_{\delta_i})\,\mathrm{d}\pi^i - \int_{l^{-1}(I)} \sigma_{K,N}^{\frac{\frac{1}{2}-\delta_i}{1-\delta_i}}(d(\gamma_{\delta_i},\gamma_1))(\rho_1^i)^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi^i\right]\end{aligned} \tag{8}$$

due to the lower semi-continuity of the entropy, the fact $\tilde{\rho}_{\frac{1}{2}} \le \rho_{\frac{1}{2}}^i$ everywhere, $\tilde{\rho}_{\frac{1}{2}}(\gamma_{\frac{1}{2}}) \le \rho_{\frac{1}{2}}(\gamma_{\frac{1}{2}})$ in $\mathrm{Geo}(X) \setminus l^{-1}(I)$, and the convexity of the entropy along π^i. To arrive to a contradiction, we will need the following observation, which follows by the disintegration theorem, Hölder's inequality, and Jensen's inequality:

$$\begin{aligned}
&\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} (\rho_t^i)^{-\frac{1}{N}}(\gamma_t)\,\mathrm{d}\pi^i = \iint \chi_{\mathrm{Geo}(X)\setminus l^{-1}(I)} (\rho_t^i)^{-\frac{1}{N}} \circ e_t \,\mathrm{d}\pi_x^i \,\mathrm{d}\mu_t^i(x) \\
&= \int (\rho_t^i)^{-\frac{1}{N}}(x) \int \chi_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \,\mathrm{d}\pi_x^i \,\mathrm{d}\mu_t^i(x) = \int (\rho_t^i)^{1-\frac{1}{N}}(x) \int \chi_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \,\mathrm{d}\pi_x^i \,\mathrm{d}\mathfrak{m}(x) \\
&\le \left(\int \rho_t^i(x) \left(\int \chi_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \,\mathrm{d}\pi_x^i \right)^{\frac{N}{N-1}} \mathrm{d}\mathfrak{m}(x) \right)^{\frac{N-1}{N}} (\mathfrak{m}(\operatorname{spt}\mu_1^i))^{\frac{1}{N}} \qquad (9) \\
&\le C \left(\int \rho_t^i(x) \int \chi_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \,\mathrm{d}\pi_x^i \,\mathrm{d}\mathfrak{m}(x) \right)^{\frac{N-1}{N}} = C\pi_i(\mathrm{Geo}(X)\setminus l^{-1}(I))^{\frac{N-1}{N}} \longrightarrow 0,
\end{aligned}$$

when $i \longrightarrow \infty$. Hence, by (8)

$$\begin{aligned}
&\mathrm{Ent}_N(\tilde{\mu}_{\frac{1}{2}}) \\
&\le \varliminf_{i\to\infty} \Bigg[-\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \\
&\quad - \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta_i}}((1-\delta_i)b) \int_{l^{-1}(I)} (\rho_{\delta_i}^i)^{-\frac{1}{N}}(\gamma_{\delta_i})\,\mathrm{d}\pi^i - \sigma_{K,N}^{\frac{\frac{1}{2}-\delta_i}{1-\delta_i}}((1-\delta_i)b) \int_{l^{-1}(I)} (\rho_1^i)^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi^i \Bigg] \\
&= \varliminf_{i\to\infty} \Bigg[-\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi + \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta_i}}((1-\delta_i)b)\mathrm{Ent}_N(\mu_\delta^I) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta_i}{1-\delta_i}}((1-\delta_i)b)\mathrm{Ent}_N(\mu_1^i) \\
&\quad + \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta_i}}((1-\delta_i)b) \int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} (\rho_{\delta_i}^i)^{-\frac{1}{N}}(\gamma_{\delta_i})\,\mathrm{d}\pi^i + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta_i}{1-\delta_i}}((1-\delta_i)b) \int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} (\rho_1^i)^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi^i \Bigg] \\
&\overset{(9)}{=} -\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi + \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)b)\mathrm{Ent}_N(\mu_\delta^I) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}((1-\delta)b)\mathrm{Ent}_N(\mu_1^I) \\
&\overset{(6)}{\le} -\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi - (1-\varepsilon) \int_{l^{-1}(I)} \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_\delta^{-\frac{1}{N}}(\gamma_\delta) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_\delta)\,\mathrm{d}\pi \\
&\overset{(7)}{<} -\int_{\mathrm{Geo}(X)\setminus l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi - \int_{l^{-1}(I)} \rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi = \mathrm{Ent}_N(\mu_{\frac{1}{2}}),
\end{aligned}$$

which is a contradiction, since $\mu_{\frac{1}{2}}$ was the minimiser of the entropy. Notice, that we actually proved a stronger version of the convexity between points δ, $\frac{1}{2}$ and 1, namely that the convexity holds whenever the plan π is restricted to $l^{-1}(I)$ for any open interval I.

To show that the convexity holds between 0, $\frac{1}{2}$ and 1, we use the convexity first between δ, $\frac{1}{2}$ and 1, then between 0, δ, and $\frac{1}{2}$, and then conclude by letting $\delta \longrightarrow 0$. Write

$$[0,\infty) = \cup_{i\in\mathbb{N}} I_i,$$

where $I_i = [s_i, s_{i+1}]$ are intervals with equal length $\varepsilon > 0$. Since the functions $\sigma_{K,N}^{(t)}$ are Lipschitz continuous with uniform Lipschitz constant L, we have

$$\begin{aligned}
\mathrm{Ent}_N(\mu_{\frac{1}{2}}) \leq &- \int \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_\delta^{-\frac{1}{N}}(\gamma_\delta) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi \\
&\leq - \sum_i \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)s_{i+1}) \int_{l^{-1}(I_i)} \rho_\delta^{-\frac{1}{N}}(\gamma_\delta)\,\mathrm{d}\pi - \int \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi \\
&\leq - \sum_i \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)s_{i+1}) \int_{l^{-1}(I_i)} \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_0^{-\frac{1}{N}}(\gamma_0)\,\mathrm{d}\pi \\
&\quad - \sum_i \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)s_{i+1}) \int_{l^{-1}(I_i)} \sigma_{K,N}^{\frac{\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \\
&\quad - \int \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi \\
&\leq - \sum_i (1-L|s_{i+1}-s_i|)\sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)s_i) \int_{l^{-1}(I_i)} \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_0^{-\frac{1}{N}}(\gamma_0)\,\mathrm{d}\pi \\
&\quad - \sum_i (1-L|s_{i+1}-s_i|)\sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)s_i) \int_{l^{-1}(I_i)} \sigma_{K,N}^{\frac{\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \\
&\quad - \int \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi \\
&\leq - \sum_i (1-L|s_{i+1}-s_i|) \int_{l^{-1}(I_i)} \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)d(\gamma_0,\gamma_1))\sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_0^{-\frac{1}{N}}(\gamma_0)\,\mathrm{d}\pi \\
&\quad - \sum_i (1-L|s_{i+1}-s_i|) \int_{l^{-1}(I_i)} \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)d(\gamma_0,\gamma_1))\sigma_{K,N}^{\frac{\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \\
&\quad - \int \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi \\
&\leq (1-\varepsilon) \int \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)d(\gamma_0,\gamma_1))\sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_0^{-\frac{1}{N}}(\gamma_0) + \sigma_{K,N}^{\frac{\frac{1}{2}-\delta}{1-\delta}}(d(\gamma_\delta,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\varepsilon) \int \sigma_{K,N}^{\frac{\frac{1}{2}}{1-\delta}}((1-\delta)d(\gamma_0,\gamma_1))\sigma_{K,N}^{\frac{\delta}{\frac{1}{2}}}(d(\gamma_0,\gamma_{\frac{1}{2}}))\rho_{\frac{1}{2}}^{-\frac{1}{N}}(\gamma_{\frac{1}{2}})\,\mathrm{d}\pi \\
\longrightarrow &\int \sigma_{K,N}^{\frac{1}{2}}(d(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0) + \sigma_{K,N}^{\frac{1}{2}}(d(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N}}(\gamma_1)\,\mathrm{d}\pi,
\end{aligned}
$$

where we first let $\delta \longrightarrow 0$, and then $\varepsilon \longrightarrow 0$. In the first limit, we used dominated convergence with $C\rho_i^{1-\frac{1}{N}}$ as a dominant, and the explicit form of the distortion coefficients.

To show that the convexity holds between points 0, t, and 1, where $t \in (0, \frac{1}{2})$, one uses analogous computations as above, now using the convexity first between points 0, t, and $\frac{1}{2}$, and then between δ, $\frac{1}{2}$, and 1, again letting $\delta \to 0$.

Finally, the case for general $t \in (0,1)$ follows inductively – after the observation that the convexity between t_i, t_{i+1}, and 1 (and thus, between points $\frac{1}{2}$, t_i, and 1 by yet another induction) is of the stronger form, more precisely, the convexity holds when restricted to curves with length in an interval $[a,b]$ (converse inequality of (5) with $\delta = t_i$, and t_{i+1} in place of $\frac{1}{2}$).

We have now shown that the convexity holds between points 0, t, and 1 for any $t \in (0,1)$. Next, we will turn into the proof of convexity between any three points $r < s < t$. It will follow analogously to the previous case after a couple of simple observations. First of all, if $r \in [t_i, t_{i+1})$, and $t \in [t_k, t_{k+1})$, $k > i$, then $\mu_{t_{i+1}}$ minimises the entropy among all $(t_{i+1} - r)/(t - r)$-intermediate points of μ_r and μ_t. Furthermore, μ_j minimises the entropy among all $(t_j - t_{j-1})/(t - t_{j-1})$-intermediate points of $\mu_{t_{j-1}}$ and μ_t for all $j \in \{i+2, \ldots, k\}$. The second observation needed is that the pushforward of a plan given by the definition of very strict $CD^*(K,N)$-space under the restriction map still satisfies the requirements of the very same definition. The only difference in the argument is that now instead of infinitely many steps in the induction argument, one only has a finite number of steps, and one special case, namely when $s \in (t_k, t)$. This special case, however, follows easily with the same arguments. □

4. Equivalent definitions of very strict $CD(K,N)$ -condition

In this section we will prove that the definition of very strict $CD(K,N)$ -spaces is equivalent to an analogous pointwise convexity requirement for the density of a Wasserstein geodesic along optimal plan. This pointwise definition is then used to prove the equivalence of the definition of very strict $CD(K,N)$ and Lott–Villani-type analogous of the definition.

We will need the following simple lemma.

Lemma 4.1. *Let $(X,d,\mathfrak{m})$ be a very strict $CD(K,N)$ -space, $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$ absolutely continuous measures with respect to the reference measure and with bounded supports, and $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ given by the definition of very strict $CD(K,N)$ -condition. Then $\mu_t \in \mathcal{P}_2^{ac}(X)$ for all $t \in (0,1)$.*

Proof. Suppose the claim is not true. Then there exists $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ as in Definition 2.1 with $\mu_t := (e_t)_\# \pi = \rho\mathfrak{m} + \mu^\perp$, $\mu^\perp \perp \mathfrak{m}$. Thus, there exists a Borel set $A \subset X$ so that $\mu^\perp(A) > 0$ and $\mathfrak{m}(A) = 0$. Let $\mathcal{A} := e_t^{-1}(A)$, and define $\tilde{\pi} := \pi|_{\mathcal{A}}$.

In the case $N=\infty$ we get a contradiction after restricting the plan π further so that ρ_0 and ρ_1 are bounded, and hence the entropies $\mathrm{Ent}_\infty(\mu_0)$ and $\mathrm{Ent}_\infty(\mu_1)$ are finite.

In the case $N<\infty$ the argument goes as follows. For π-a.e. $\gamma\in\mathcal{A}$, we have that $d(\gamma_0,\gamma_1)>0$ and thus $\tau^{(t)}_{K,N}(d(\gamma_0,\gamma_1))>0$. Thus,

$$0<\int \tau^{(1-t)}_{K,N}(d(\gamma_0,\gamma_1))\rho_0(\gamma_0)+\tau^{(t)}_{K,N}(d(\gamma_0,\gamma_1))\rho_1(\gamma_1)\,\mathrm{d}\tilde{\pi}(\gamma)\le -\mathrm{Ent}_N(\mu^\perp)=0$$

giving the contradiction. □

Proposition 4.2. *Let $(X,d,\mathfrak{m})$ be a metric measure space. Then $(X,d,\mathfrak{m})$ is very strict $CD(K,N)$ -space, if and only if for all absolutely continuous measures $\mu_0,\mu_1\in\mathcal{P}_2^{ac}(X)$ with bounded support, there exists an optimal plan $\pi\in\mathrm{OptGeo}(\mu_0,\mu_1)$, with $\mu_t:=(e_t)_\#\pi\in\mathcal{P}_2^{ac}(X)$, for which the following two conditions hold:*

(i) For all $t\in(0,1)$, there exists a Borel map $T_t\colon X\to\mathrm{Geo}(X)$ for which $\pi=(T_t)_\#\mu_t$, and $e_t\circ T_t=\mathrm{id}$.

(ii) If $N=\infty$, then for every $t_1<t_2<t_3$, the inequality

$$\begin{aligned}\log\rho_{t_2}(\gamma_{t_2})\le\frac{(t_3-t_2)}{(t_3-t_1)}\log\rho_{t_1}(\gamma_{t_1})+\frac{(t_2-t_1)}{(t_3-t_1)}\log\rho_{t_3}(\gamma_{t_3})\\ -\frac{K}{2}\frac{(t_3-t_2)}{(t_3-t_1)}\frac{(t_2-t_1)}{(t_3-t_1)}d^2(\gamma_{t_1},\gamma_{t_3})\end{aligned}\tag{10}$$

holds for π-almost every γ, where ρ_t is the density of μ_t with respect to the reference measure $\mathfrak{m}$.

If $N<\infty$, then for every $t_1<t_2<t_3$, the inequality

$$\rho_{t_2}^{-\frac{1}{N}}(\gamma_{t_2})\ge\tau_{K,N}^{\frac{(t_3-t_2)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_1}^{-\frac{1}{N}}(\gamma_{t_1})+\tau_{K,N}^{\frac{(t_2-t_1)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_3}^{-\frac{1}{N}}(\gamma_{t_3})\tag{11}$$

holds for π-almost every γ.

Moreover, if π is the plan given by the definition of very strict $CD(K,N)$ -space, then for π-almost every γ, the inequality (10) *($N=\infty$) or* (11) *($N<\infty$) holds for $\mathcal{L}^3$-almost every $(t_1,t_2,t_3)\in[0,1]$ with $t_1<t_2<t_3$.*

Remark 4.3. If we remove in Definition 2.1 the assumption of the boundedness of the supports of μ_0 and μ_1, we may remove it also from Proposition 4.2.

Proof. We will prove only the case $N=\infty$. The proof of the finite dimensional case is the same with obvious modifications. Let $(X,d,\mathfrak{m})$ be a very strict $CD(K,\infty)$ -space. Let $\mu_0,\mu_1\in\mathcal{P}_2^{ac}(X)$, and let π be the optimal plan given by the definition of very strict $CD(K,\infty)$ -space. We will prove that the conditions (i) and (ii) hold for π. By Lemma 4.1 we have that μ_t is absolutely continuous with respect to $\mathfrak{m}$. Moreover, the plan $(\mathrm{restr}_0^t)_\#\pi\in\mathrm{OptGeo}(\mu_0,\mu_t)$ is such as in the definition of very strict $CD(K,\infty)$. Thus, by Theorem 3.1, it is induced by a map T from the intermediate measure μ_t. Hence we have that $\pi=(S\circ e_0\circ T)_\#\mu_t=:(T_t)_\#\mu_t$, where S is the map given by Theorem 3.1 for which $\pi=S_\#\mu_0$, proving the claim (i).

For (ii), suppose to the contrary, that there exist $t_1, t_2, t_3 \in [0,1]$, $t_1 < t_2 < t_3$, and a set $\mathcal{A} \subset \mathrm{Geo}(X)$ with $\pi(\mathcal{A}) > 0$, so that the inequality (10) does not hold for any $\gamma \in \mathcal{A}$. Define $\tilde{\pi} := \pi|_{\mathcal{A}}$, and further define $\tilde{\mu}_t := (e_t)_\# \tilde{\pi} = \tilde{\rho}_t \mathfrak{m}$, for all $t \in [0,1]$. Writing $\mathcal{A}$ as union

$$\mathcal{A} = \bigcup_{i \in \mathbb{N}} \{\gamma \in \mathcal{A} : \max_{j=1,2,3} \rho_{t_j}(\gamma_{t_j}) \leq i\},$$

we may assume that $\tilde{\rho}_{t_j}$ is bounded from above, and so in particular that $(\tilde{\rho}_{t_j} \log \tilde{\rho}_{t_j})_+$ is integrable for $j \in \{1,2,3\}$. Let $\{\pi_x^t\}$ be the disintegration of π with respect to the evaluation map e_t. Then we have, for all non-negative Borel functions $f \colon X \to \mathbb{R}$, that

$$\begin{aligned}
\int_X f(x)\tilde{\rho}_t(x) \,\mathrm{d}\mathfrak{m}(x) &= \int_{\mathrm{Geo}(X)} f(\gamma_t)\chi_{\mathcal{A}}(\gamma) \,\mathrm{d}\pi(\gamma) \\
&= \int_X \int_{\mathrm{Geo}(X)} f(\gamma_t)\chi_{\mathcal{A}}(\gamma) \,\mathrm{d}\pi_x^t(\gamma) \,\mathrm{d}\mu_t(x) \\
&= \int_X f(x) \int_{\mathrm{Geo}(X)} \chi_{\mathcal{A}}(\gamma) \,\mathrm{d}\pi_x^t(\gamma) \,\mathrm{d}\mu_t(x) \\
&= \int_X f(x) \left(\int_{\mathrm{Geo}(X)} \chi_{\mathcal{A}}(\gamma) \,\mathrm{d}\pi_x^t(\gamma) \right) \rho_t(x) \,\mathrm{d}\mathfrak{m}(x),
\end{aligned}$$

where ρ_t is the density of $\mu_t := (e_t)_\# \pi$ with respect to the reference measure $\mathfrak{m}$. Thus $\tilde{\rho}_t(x) = \chi_{\mathcal{A}}(T_t(x))\rho_t(x)$ for $\mathfrak{m}$-almost every $x \in X$. In particular, we have that

$$\tilde{\rho}_t(\gamma_t) = \chi_{\mathcal{A}}(T_t(\gamma_t))\rho_t(\gamma_t) = \chi_{\mathcal{A}}(\gamma)\rho_t(\gamma_t),$$

for π-almost every γ, and for all $t \in \{t_1, t_2, t_3\}$. Hence, we get

$$\begin{aligned}
\mathrm{Ent}_\infty(\tilde{\mu}_{t_2}) &= \int_X \tilde{\rho}_{t_2} \log \tilde{\rho}_{t_2} \,\mathrm{d}\mathfrak{m} = \int_X \log \tilde{\rho}_{t_2} \,\mathrm{d}\tilde{\mu}_{t_2} = \int_{\mathrm{Geo}(X)} \log \tilde{\rho}_{t_2}(\gamma_{t_2}) \,\mathrm{d}\tilde{\pi} \\
&= \int_{\mathcal{A}} \log \rho_{t_2}(\gamma_{t_2}) \,\mathrm{d}\pi \\
&> \frac{(t_3 - t_2)}{(t_3 - t_1)} \int_{\mathcal{A}} \log \rho_{t_1}(\gamma_{t_1}) \,\mathrm{d}\pi + \frac{(t_2 - t_1)}{(t_3 - t_1)} \int_{\mathcal{A}} \log \rho_{t_3}(\gamma_{t_3}) \,\mathrm{d}\pi \\
&\quad - \frac{K}{2} \frac{(t_3 - t_2)}{(t_3 - t_1)} \frac{(t_2 - t_1)}{(t_3 - t_1)} \int_{\mathcal{A}} d^2(\gamma_{t_1}, \gamma_{t_3}) \,\mathrm{d}\pi \\
&= \frac{(t_3 - t_2)}{(t_3 - t_1)} \mathrm{Ent}_\infty(\tilde{\mu}_{t_1}) + \frac{(t_2 - t_1)}{(t_3 - t_1)} \mathrm{Ent}_\infty(\tilde{\mu}_{t_3}) \\
&\quad - \frac{K}{2} \frac{(t_3 - t_2)}{(t_3 - t_1)} \frac{(t_2 - t_1)}{(t_3 - t_1)} W_2^2(\tilde{\mu}_{t_1}, \tilde{\mu}_{t_3}),
\end{aligned}$$

which contradicts the assumption of π being the plan given by the definition of very strict $CD(K,\infty)$ -space. Hence (ii) holds.

For the other direction, suppose that $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is such that conditions (i) and (ii) hold. Let $F \colon \mathrm{Geo}(X) \to \mathbb{R}$ be a bounded non-negative Borel function for which

$\int F \, \mathrm{d}\pi = 1$, and let $t_1, t_2, t_3 \in [0,1]$, $t_1 < t_2 < t_3$. Denote $\mu_t^F := (e_t)_\# F\pi$. As previously, by (i), we get that

$$\rho_t^F(x) := F(T_t(x))\rho_t(x),$$

is the density of μ_t^F with respect to $\mathfrak{m}$. Here ρ_t is the density of μ_t with respect to the reference measure $\mathfrak{m}$. In particular, we have that along geodesics the density is, up to a multiplicative constant, the same as the original density. More precisely, we have

$$\rho_t^F(\gamma_t) = F(\gamma)\rho_t(\gamma_t),$$

for π-almost every γ, and for every $t \in \{t_1, t_2, t_3\}$. Thus, by (ii) we have that

$$\begin{aligned}
\int \rho_{t_2}^F \log \rho_{t_2}^F \, \mathrm{d}\mathfrak{m} &= \int \log \rho_{t_2}^F \, \mathrm{d}\mu_{t_2}^F = \int \log \rho_{t_2}^F(\gamma_{t_2}) F(\gamma) \, \mathrm{d}\pi \\
&= \int \log \rho_{t_2}(\gamma_{t_2}) F(\gamma) \, \mathrm{d}\pi + \int \log F(\gamma) F(\gamma) \, \mathrm{d}\pi \\
&\leq \frac{(t_3 - t_2)}{(t_3 - t_1)} \int \log \rho_{t_1}(\gamma_{t_1}) F(\gamma) \, \mathrm{d}\pi + \frac{(t_2 - t_1)}{(t_3 - t_1)} \int \log \rho_{t_3}(\gamma_{t_3}) F(\gamma) \, \mathrm{d}\pi \\
&\quad - \frac{K}{2} \frac{(t_3 - t_2)}{(t_3 - t_1)} \frac{(t_2 - t_1)}{(t_3 - t_1)} \int d^2(\gamma_{t_1}, \gamma_{t_3}) F(\gamma) \, \mathrm{d}\pi + \int \log F(\gamma) F(\gamma) \, \mathrm{d}\pi \\
&= \frac{(t_3 - t_2)}{(t_3 - t_1)} \int \rho_{t_1}^F \log \rho_{t_1}^F \, \mathrm{d}\mathfrak{m} + \frac{(t_2 - t_1)}{(t_3 - t_1)} \int \rho_{t_3}^F \log \rho_{t_3}^F \, \mathrm{d}\mathfrak{m} \\
&\quad - \frac{K}{2} \frac{(t_3 - t_2)}{(t_3 - t_1)} \frac{(t_2 - t_1)}{(t_3 - t_1)} W_2^2(\mu_{t_1}^F, \mu_{t_3}^F),
\end{aligned}$$

giving the claim.

For the last claim, define for all $\gamma \in \mathrm{Geo}(X)$

$$I_\gamma := \{(t_1, t_2, t_3) \in J : \text{(ii) fails along } \gamma \text{ at } (t_1, t_2, t_3)\},$$

where $J := \{(t_1, t_2, t_3) \in [0,1] : t_1 < t_2 < t_3\}$, and the set

$$\mathcal{I} := \bigcup_\gamma \{\gamma\} \times I_\gamma.$$

Then by (ii)

$$\begin{aligned}
0 &= \int_J \pi(\{\gamma : t \in I_\gamma\}) \, \mathrm{d}\mathcal{L}^3(t) = \int \chi_{\mathcal{I}} \, \mathrm{d}(\pi \otimes \mathcal{L}^3) \\
&= \int \mathcal{L}^3(I_\gamma) \, \mathrm{d}\pi(\gamma).
\end{aligned}$$

Hence I_γ has Lebesgue measure zero for π-almost every $\gamma \in \mathrm{Geo}(X)$. □

Theorem 4.4. *Let $(X, d, \mathfrak{m})$ be a metric measure space. Then the following are equivalent:*

(i) The space $(X, d, \mathfrak{m})$ is a very strict $CD(K, N)$ -space (see Definition 2.1).

(ii) The space $(X, d, \mathfrak{m})$ is a very strict $CD(K, N)$ -space in the spirit of Lott–Villani (see Definition 2.5).

Proof. Clearly condition (ii) implies condition (i). For the other implication, assume that $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$, and $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ given by the definition of very strict $CD(K,N)$ -condition. Let $U \in \mathcal{DC}_N$, and $F \colon \mathrm{Geo}(X) \to \mathbb{R}$ non-negative, bounded Borel function with $\int F \, \mathrm{d}\pi = 1$.

We first prove the claim, when $N < \infty$. Define $u(s) := s^N U(s^{-N})$. Then u is a decreasing and convex function, since $U \in \mathcal{DC}_N$. Hence, by Proposition 4.2 condition (ii),

$$
\begin{aligned}
U(\mu_{t_2}^F) &= \int U \circ \rho_{t_2}^F \, \mathrm{d}\mathfrak{m} = \int u((\rho_{t_2}^F)^{-\frac{1}{N}}) \rho_{t_2}^F \, \mathrm{d}\mathfrak{m} = \int u((\rho_{t_2}^F)^{-\frac{1}{N}}(\gamma_{t_2})) F(\gamma) \, \mathrm{d}\pi \\
&= \int u(F(\gamma) \rho_{t_2}^{-\frac{1}{N}}(\gamma_{t_2})) F(\gamma) \, \mathrm{d}\pi \\
&\leq \int u(F(\gamma)(\tau_{K,N}^{\frac{(t_3-t_2)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_1}^{-\frac{1}{N}}(\gamma_{t_1}) + \tau_{K,N}^{\frac{(t_2-t_1)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_3}^{-\frac{1}{N}}(\gamma_{t_3}))) F(\gamma) \, \mathrm{d}\pi \\
&\leq \frac{(t_3-t_2)}{(t_3-t_1)} \int u(F(\gamma) \frac{(t_3-t_1)}{(t_3-t_2)} \tau_{K,N}^{\frac{(t_3-t_2)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_1}^{-\frac{1}{N}}(\gamma_{t_1})) F(\gamma) \, \mathrm{d}\pi \\
&+ \frac{(t_2-t_1)}{(t_3-t_1)} \int u(F(\gamma) \frac{(t_3-t_1)}{(t_2-t_1)} \tau_{K,N}^{\frac{(t_2-t_1)}{(t_3-t_1)}}(d(\gamma_{t_1},\gamma_{t_3}))\rho_{t_3}^{-\frac{1}{N}}(\gamma_{t_3})) F(\gamma) \, \mathrm{d}\pi \\
&= \frac{(t_3-t_2)}{(t_3-t_1)} U_{\pi,\mathfrak{m}}^{\beta_{(K,N)}^{\frac{(t_3-t_2)}{(t_3-t_1)}}}(\mu_{t_1}^F) + \frac{(t_2-t_1)}{(t_3-t_1)} U_{\pi^{-1},\mathfrak{m}}^{\beta_{(K,N)}^{\frac{(t_2-t_1)}{(t_3-t_1)}}}(\mu_{t_3}^F),
\end{aligned}
$$

giving the claim.

If $N = \infty$, we have that the function $u \colon s \mapsto e^s U(e^{-s})$ is convex and decreasing by assumption. Hence, by Proposition 4.2 condition (ii)

$$
\begin{aligned}
U(\mu_{t_2}^F) &= \int u(-\log(F(\gamma)\rho_{t_2}(\gamma_{t_2}))) F(\gamma) \, \mathrm{d}\pi \\
&\leq \frac{(t_3-t_2)}{(t_3-t_1)} \int u\left(-\log\left(\frac{F(\gamma)\rho_{t_1}(\gamma_{t_1})}{\beta_{\frac{(t_3-t_2)}{(t_3-t_1)}}(\gamma_0,\gamma_1)}\right)\right) F(\gamma) \, \mathrm{d}\pi \\
&\quad + \frac{(t_2-t_1)}{(t_3-t_1)} \int u\left(-\log\left(\frac{F(\gamma)\rho_{t_3}(\gamma_{t_3})}{\beta_{\frac{(t_2-t_1)}{(t_3-t_1)}}(\gamma_0,\gamma_1)}\right)\right) F(\gamma) \, \mathrm{d}\pi \\
&= \frac{(t_3-t_2)}{(t_3-t_1)} U_{\pi,\mathfrak{m}}^{\beta_{(K,N)}^{\frac{(t_3-t_2)}{(t_3-t_1)}}}(\mu_{t_1}^F) + \frac{(t_2-t_1)}{(t_3-t_1)} U_{\pi^{-1},\mathfrak{m}}^{\beta_{(K,N)}^{\frac{(t_2-t_1)}{(t_3-t_1)}}}(\mu_{t_3}^F),
\end{aligned}
$$

which completes the proof. □

Recall, that in our definition of very strict $CD(K,N)$-spaces, we only require the convexity of the entropy to hold for the critical exponent N, opposed to the definition of general $CD(K,N)$ -spaces. Therefore, the following immediate corollary is a non-trivial fact in this setting.

Corollary 4.5. *A metric measure space satisfying very strict $CD(K,N)(CD^*(K,N))$ -condition, satisfies very strict $CD(K,N')(CD^*(K,N'))$ -condition for any $N' > N$.*

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On one-dimensionality of metric measure spaces

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ON ONE-DIMENSIONALITY OF METRIC MEASURE SPACES

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Abstract. In this paper, we prove that a metric measure space which has at least one open set isometric to an interval, and for which the (possibly non-unique) optimal transport map exists from any absolutely continuous measure to an arbitrary measure, is a one-dimensional manifold (possibly with boundary). As an immediate corollary we obtain that if a metric measure space is a very strict $CD(K, N)$ -space or an essentially non-branching $MCP(K, N)$-space with some open set isometric to an interval, then it is a one-dimensional manifold. We also obtain the same conclusion for a metric measure space which has a point in which the Gromov-Hausdorff tangent is unique and isometric to the real line, and for which the optimal transport maps not only exist but are unique. Again, we obtain an analogous corollary in the setting of essentially non-branching $MCP(K, N)$-spaces.

1. Introduction

The strong interplay between optimal mass transportation and (metric) geometry has been acknowledged in the last few decades leading to a great number of applications for example in the study of geometric and analytic inequalities, in describing and defining curvature bounds, and in the regularity theory of partial differential equations. The optimal transport theory is useful both in generalising classical results from the theory of smooth manifolds to possibly singular spaces, and in obtaining new results even in the smooth setting.

In the present paper we will use tools from optimal transport theory to obtain global topological/geometric information about a metric (measure) space from information near a single point in the space. More precisely, we will use the existence – and in some cases uniqueness – of an optimal transport map together with one-dimensionality at (Theorem 3.10 and Corollary 3.11), or near (Theorem 3.1, Corollary 3.2 and Theorem 3.5), a point in the space to prove that the space in question is a one-dimensional manifold. Here, by one-dimensionality *at* a point, we mean that the Gromov-Hausdorff tangent at that point is unique and isometric to the real line, and by one-dimensionality *near* a point, we mean that the point has an open neighbourhood isometric to an open interval.

Such a result was first proven in the setting of Ricci limit spaces in [9] by Honda and generalised to the setting of $RCD^*(K, N)$-spaces in [12] by Kitabeppu and Lakzian. In both papers, it is proven that the underlying space satisfying the synthetic Ricci curvature lower

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bound in question, is a one-dimensional manifold if it is one-dimensional at a single point. These two papers share a common (and natural) viewpoint coming from the structure theory: in both settings one could write the metric measure space up to a zero measure set as a union of sets $\mathcal{R}_k$ in which one has the existence and uniqueness of tangents isomorphic to k-dimensional Euclidean space [5, 16]. In our setting such a decomposition of the space cannot be true in general, not least because of the allowance of Finslerian (type) structures. Note that it is still meaningful to ask what can be concluded from the existence of a one-dimensional part even if we would impose Finslerian type behaviour, since a priori the dimension of the space needs not be constant.

The study of the present paper was partially motivated by the results concerning the existence of optimal transport maps on the spaces having synthetic Ricci curvature bounded from below. A notion of Ricci curvature lower bound for (possibly singular) metric measure spaces – the so-called $CD(K, N)$-condition (*curvature dimension condition*) – was introduced in the seminal works of Sturm [21, 22] and of Lott and Villani [14]. The existence and uniqueness of the optimal map in the setting of spaces with synthetic Ricci curvature lower bounds was first proven by Gigli in [7] for non-branching $CD(K, N)$-spaces, and then generalised to strong $CD(K, N)$-spaces by Rajala and Sturm in [18] and by Gigli, Rajala and Sturm in [8]. In their paper, Rajala and Sturm introduced the notion called *essential non-branchingness*, which turned out to be a useful generalisation of the non-branching assumption on metric measure spaces. In [4], Cavalletti and Mondino proved the existence and uniqueness of optimal transport maps in $MCP(K, N)$-spaces if one assumes that the underlying metric measure space is essentially non-branching. Then in [10], Kell generalised the result to spaces satisfying even weaker version of curvature lower bound, namely to the setting of *qualitatively non-degenerate* spaces (studied by Cavalletti and Huesmann in the non-branching case in [2]) – still under the essential non-branching assumption.

Heuristically, the non-branching assumption prevents the geodesics of an optimal plan to intersect at intermediate times, while the curvature lower bound assumption forces them to intersect when the plan is assumed not to be induced by a map, hence the existence and uniqueness of optimal maps is obtained by combining these two. Therefore, while the uniqueness of the optimal map is lost if there exists an essential amount of branching geodesics, one might still pursue the existence of such a map. This approach was taken in [19] (and continued in [20]), where the author proved the existence of optimal transport maps in the setting of so-called very strict $CD(K, N)$-spaces. We remark that while in general (branching) $MCP(K, N)$-space the existence of an optimal transport map might fail by the example in [11], it is still not known whether optimal maps exist in general $CD(K, N)$-spaces.

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2. Preliminaries

For the purposes of this paper, we will always assume that $(X, d, \mathfrak{m})$ is a metric measure space which is a complete, locally compact and separable length space (X, d) equipped with a locally finite measure $\mathfrak{m}$. We will also assume that $\operatorname{spt}\mathfrak{m} = X$. The space of (constant speed, length minimising) geodesics parametrised by $[0,1]$ is denoted by $\mathrm{Geo}(X)$, and it is equipped with the supremum distance.

2.1. **Optimal mass transportation.** In this section, we introduce the basic notions of optimal transport theory which set the basis for the paper. In addition, we establish in Proposition 2.1 a subtle detail about the existence and uniqueness of optimal transport maps in the case of non-geodesic spaces.

In the main results of the paper we are assuming the existence of an optimal transport map for the Monge–Kantorovich problem with quadratic cost. The reason for such a choice for the cost function lies in the connection between Wasserstein geodesics and optimal dynamical transport plans. Note that one could obtain similar results by considering cost functions of the form d^p for $p \in (1,\infty)$ different from 2, since the representation of Wasserstein geodesics by measures on the space of geodesics, and the corresponding existence results of transport maps remain true in this case.

The quadratic Monge–Kantorovich problem reads as follows. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ be Borel probability measures on X. Consider the minimisation problem

$$W_2^2(\mu_0,\mu_1) \coloneqq \inf \int d^2(x,y)\,\mathrm{d}\sigma(x,y),$$

where the infimum is taken over all *transport plans* σ, that is, over all Borel probability measures $\sigma \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as marginals ($\mathrm{P}^1_\#\sigma = \mu_0, \mathrm{P}^2_\#\sigma = \mu_1$). It is a standard fact in optimal transport theory that in the setting of complete and separable metric spaces, the above infimum is in fact a minimum. We will call a minimiser of the problem an *optimal (transport) plan*, and denote the set of all optimal plans by $\mathrm{Opt}(\mu_0,\mu_1)$.

The optimality of a plan can be characterised by the so-called c-cyclical monotonicity in the following way. Let σ be a transport plan between measures μ_0 and μ_1 for which $W_2^2(\mu_0,\mu_1) < \infty$. Then σ is optimal if and only if it is concentrated on a c-cyclically monotone set, that is, if there exists a set $\Gamma \subset X \times X$ so that $\sigma(\Gamma) = 1$, and

$$\sum_i d^2(x_i,y_i) \le \sum_i d^2(x_i,y_{\tau(i)})$$

for any finite set $\{(x_i,y_i)\}_i \subset \Gamma$ and for any permutation τ.

The function $W_2(\cdot,\cdot)$ defines a metric on the subset $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ of probability measures with finite second moment, that is $\mu \in \mathcal{P}(X)$ with $\int d^2(x,x_0)\,\mathrm{d}\mu(x) < \infty$ for some $x_0 \in X$. The distance W_2 is the so-called *Wasserstein distance* (or more precisely the 2-Wasserstein distance). Since X is a complete, separable and – by Hopf–Rinow theorem – geodesic space, so is the Wasserstein space $(\mathcal{P}_2(X), W_2)$. Moreover, a curve $(\mu_t) \subset \mathcal{P}_2(X)$ is a geodesic if, and only if, there exists a measure $\pi \in \mathcal{P}(\mathrm{Geo}(X))$ so that $\mu_t = (e_t)_\#\pi$ and $(e_0,e_1)_\#\pi \in \mathrm{Opt}(\mu_0,\mu_1)$ [13]. Here $e_t\colon \mathrm{Geo}(X) \to \mathbb{R}$ is the evaluation map $\gamma \mapsto \gamma_t$. Such a π is called an *optimal (geodesic) plan*, and the set of all optimal geodesic plans is denoted by $\mathrm{OptGeo}(\mu_0,\mu_1)$.

We say that the optimal plan $\sigma \in \mathrm{Opt}(\mu_0, \mu_1)$ is induced by a map if there exists a Borel map $T\colon X \to X \times X$ such that $\sigma = T_\# \mu_0$ and $\mathrm{P}^1 \circ T = \mathrm{id}$. Analogously, $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is induced by a map $T\colon X \to \mathrm{Geo}(X)$, if $\pi = T_\# \mu_0$ and $e_0 \circ T = \mathrm{id}$.

In the setting of geodesic spaces, one can always lift the optimal plan $\sigma \in \mathrm{Opt}(\mu_0, \mu_1)$ to an optimal geodesic plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ by making a measurable selection of geodesics for each pair of points in X. Therefore, the question of existence of optimal maps in the level of plans in $\mathrm{Opt}(\mu_0, \mu_1)$ and of geodesic plans in $\mathrm{OptGeo}(\mu_0, \mu_1)$ are equivalent. Furthermore, if the optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is unique, so is $\sigma \in \mathrm{Opt}(\mu_0, \mu_1)$. Since the framework for this paper is that of complete and locally compact metric spaces, being a length space is equivalent to being geodesic. However, sometimes it is more natural to drop the assumption of local compactness and still impose conditions implying the length structure for the space (this is the case for example in the $CD(K, \infty)$-setting). We remark the following connection between existence of optimal maps for transport plans and for geodesic transport plans in the above-mentioned non-geodesic case, which may be of independent interest.

Proposition 2.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space (possibly non-geodesic and non-locally-compact). Assume that for all $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$ the set $\mathrm{OptGeo}(\mu_0, \mu_1)$ is non-empty, and that each optimal dynamical plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is induced by a map. Then, for any $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$, every optimal plan $\sigma \in \mathrm{Opt}(\mu_0, \mu_1)$ is induced by a map, granting also the uniqueness of the optimal plan.*

In particular, every optimal plan $\sigma \in \mathrm{Opt}(\mu_0, \mu_1)$, $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$, can be lifted to a unique dynamical plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ for which $\sigma = (e_0, e_1)_\# \pi$.

Proof. Let $\sigma \in \mathrm{Opt}(\mu_0, \mu_1) \subset \mathcal{P}(X^2)$. Suppose that σ is not induced by a map. Then there exists a μ_0-positive measure Borel set $A \subset X$ such that σ_x is not a Dirac mass for any $x \in A$, where $\{\sigma_x\}_{x \in X}$ is a disintegration of σ with respect to the projection P^1. Write

$$A = \bigcup_{i,j \in \mathbb{N}} A_{ij},$$

where $A_{ij} \coloneqq \{x \in A : \sigma_x(B(\xi_i, 1/j)) \in (0,1)\}$, and $\{\xi_i\}_{i \in \mathbb{N}}$ is dense in X. Sets A_{ij} are measurable, since the maps $x \mapsto \sigma_x(B(\xi_i, 1/j))$ are measurable for all $i, j \in \mathbb{N}$ by the disintegration theorem. Since $\mu_0(A) > 0$, there exist i_0 and j_0, such that $\mu_0(A_{i_0 j_0}) > 0$.

Define now σ^1 and σ^2 as

$$\sigma^1 \coloneqq \sigma|_{X \times B(\xi_{i_0}, \frac{1}{j}_0)}, \quad \sigma^2 \coloneqq \sigma|_{X \times (X \setminus B(\xi_{i_0}, \frac{1}{j}_0))}.$$

For $k \in \{1, 2\}$, define $\hat{\sigma}^k$ as the measure for which

$$\int f \, \mathrm{d}\hat{\sigma}^k \coloneqq \int f \frac{\min\{\rho_0^1, \rho_0^2\}}{\rho_0^k} \circ \mathrm{P}^1 \, \mathrm{d}\sigma^k,$$

for all positive Borel functions f, where ρ_0^k is the density of $\mathrm{P}^1_\# \sigma^k$ with respect to the reference measure $\mathfrak{m}$. Here we use convention $\frac{\min\{\rho_0^1, \rho_0^2\}}{\rho_0^k} = 0$, when $\rho_0^k = 0$. Since the function $\frac{\min\{\rho_0^1, \rho_0^2\}}{\rho_0^k} \in [0,1]$, we have that $\hat{\sigma}^k$ is a well-defined and finite measure. By the definition of $A_{i_0 j_0}$, we have that $\rho_0^k \neq 0$ for μ_0-almost every $x \in A_{i_0 j_0}$. In particular, by the absolute continuity of μ_0, there exists an $\mathfrak{m}$-positive measure set where $\rho_0^k \neq 0$, and thus $\hat{\sigma}^k$ is non-trivial for $k \in \{1, 2\}$.

For $k \in \{1, 2\}$ and $j \in \{0, 1\}$, write $\hat{\mu}_j^k := \mathrm{P}_\#^{j+1} \hat{\sigma}^k$. Then we have that $\hat{\mu}_0^1 = \hat{\mu}_0^2$, $\hat{\mu}_1^1 \perp \hat{\mu}_1^2$, and $\hat{\sigma}^k \in \mathrm{Opt}(\hat{\mu}_0^k, \hat{\mu}_1^k)$ (or more precisely, $\frac{1}{N}\hat{\sigma}^k \in \mathrm{Opt}(\frac{1}{N}\hat{\mu}_0^k, \frac{1}{N}\hat{\mu}_1^k)$, where N is the normalisation constant $N = \hat{\mu}_0^1(X) = \hat{\mu}_0^2(X)$). Since $\hat{\sigma}^k \ll \sigma$, we have that $\hat{\mu}_j^k \ll \mathfrak{m}$. Thus by assumption, for $k \in \{1, 2\}$, there exists an optimal dynamical plan $\pi^k \in \mathrm{OptGeo}(\hat{\mu}_0^k, \hat{\mu}_1^k)$. On the other hand, since $(\hat{\sigma}^1 + \hat{\sigma}^2) \ll \sigma$ and since

$$\int d^2(x, y)\, \mathrm{d}(\hat{\sigma}^1 + \hat{\sigma}^2)(x, y) = \int d^2(x, y)\, \mathrm{d}((e_0 + e_1)_\#(\pi^1 + \pi^2))(x, y),$$

we have that $\pi^1 + \pi^2$ is an optimal dynamical plan between absolutely continuous measures $2\hat{\mu}_0^1$ and $\hat{\mu}_1^1 + \hat{\mu}_1^2$ that is not induced by a map. Hence, we arrive to a contradiction with the assumption. □

Remark 2.2. In the above proof the full existence and uniqueness of optimal geodesic plans is not needed, but instead existence and uniqueness of optimal geodesic plans inside some linearly convex subset of $\mathcal{P}(\mathrm{Geo}(X))$. In particular, the proof can be adapted to prove the uniqueness of the optimal plan in $\mathrm{Opt}(\mu_0, \mu_1)$ between absolutely continuous measures in the essentially non-branching $CD(K, \infty)$-spaces by using the result [10, Corollary 5.22] of the existence and uniqueness of optimal geodesic plans among all plans π for which $\mu_t = (e_t)_\# \pi$ is absolutely continuous for all $t \in [0, 1]$.

2.2. **(Measured) Gromov–Hausdorff tangents.** There are different notions of blow-ups for metric (measure) spaces. The one that we will use is based on a convergence of pointed metric spaces in the Gromov–Hausdorff sense:

Definition 2.3 (pointed Gromov–Hausdorff convergence)**.** Let $(X_i)_{i\in\mathbb{N}} = (X_i, d_i, x_i)_{i\in\mathbb{N}}$ be a sequence of pointed, complete, separable and locally compact geodesic spaces. Then $X_i \longrightarrow X_\infty = (X_\infty, d_\infty, x_\infty)$, if for all $R > 0$ there exists a sequence $\varepsilon_i \to 0$ and $(1, \varepsilon_i)$-quasi-isometries $f_i \colon \bar{B}(x_i, R) \to \bar{B}(x_\infty, R)$ with $x_i \mapsto x_\infty$.

Here f being $(1, \varepsilon)$-quasi-isometry is defined by requiring that f is 1-biLipschitz up to an additive constant $\varepsilon > 0$ with an ε-dense image.

Tangents of a metric space X at a point $x \in X$ are then obtained by looking at sequences of the form $(X, \lambda_i d, x)$ with $\lambda_i \to \infty$.

Definition 2.4. Let X be a complete, separable and locally compact geodesic space, and let $x \in X$. A pointed metric space (Y, d_Y, y) is a Gromov–Hausdorff tangent of X at x, $(Y, d_Y, y) \in \mathrm{Tan}(X, x)$, if there exists a sequence $\lambda_i \to \infty$ so that $(X, \lambda_i d, x) \longrightarrow (Y, d_Y, y)$ in the pointed Gromov–Hausdorff topology.

For doubling metric spaces the set of tangents at each point is non-empty. On the other hand, in general tangents are not unique.

We point out that there exist notions of tangents of metric measure spaces that take into account the convergence of the (normalised) measure, which are in many cases more suitable for the study of metric measure spaces. However, for our purposes it is enough to consider the convergence as a metric concept (keeping in mind that we are assuming $\operatorname{spt}\mathfrak{m} = X$).

3. One-dimensionality of metric measure spaces

In this section we provide a generalisation of the following theorem

Theorem ([12, Theorem 3.7])**.** *Let $(X, d, \mathfrak{m})$ be an $RCD^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume that there exists a point $x_0 \in X$ such that there exists a unique (up to an isomorphism) measured Gromov-Hausdorff tangent of $(X, d, \mathfrak{m})$ at x_0 isomorphic (as a pointed metric measure space) to $(\mathbb{R}, |\cdot|, c\mathcal{L}^1, 0)$. Then for any $x \in X$, there exists a positive number $\varepsilon > 0$ such that $B(x, \varepsilon)$ is isometric to $(-\varepsilon, \varepsilon)$ or to $[0, \varepsilon)$.*

In Theorem 3.10 we state the result implying one-dimensional manifold structure from assumptions on the one-dimensionality at a point at infinitesimal level, analogously to the original result, when imposing existence and uniqueness of optimal transport maps. In Theorem 3.1 we give a local counterpart of the result for the case where uniqueness of transport maps is lost. Proofs presented here take advantage of the existence of optimal transport maps (assumption which may be justified by the results in [8, 18, 4, 10, 19]), and by that simplify the ones given for Theorem 3.7 (and hence for Theorem 1.1) in [12].

Theorem 3.1. *Let $(X, d, \mathfrak{m})$ be a metric measure space with the following properties:*

(1) For every $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, there exists $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ that is induced by a map from μ_0.
(2) There exists a point $x \in X$, and a neighbourhood $B(x, r)$ isometric to an open interval in $\mathbb{R}$.

Then X is a one-dimensional manifold, possibly with boundary.

Due to the existence of optimal transport maps in very strict $CD(K, N)$ -spaces [20], and in essentially non-branching $MCP(K, N)$-spaces [4], we get the following immediate corollary of Theorem 3.1.

Corollary 3.2. *Let $(X, d, \mathfrak{m})$ be a very strict $CD(K, N)$ -space ($N \in (1, \infty)$), or essentially non-branching $MCP(K, N)$-space (or ess. nb., qualitatively non-degenerate space, see* [10]*). Suppose that there exists a point $x \in X$, and a neighbourhood $B(x, r)$ isometric to an open interval in $\mathbb{R}$. Then X is a one-dimensional manifold, possibly with boundary.*

It is worth noticing, that while Corollary 3.2 is known to be false in general $MCP(K, N)$-space by the example given by Ketterer and Rajala in [11], it remains still open in general $CD(K, N)$-space:

Question 1. Let (X, d, m) be a $CD(K, N)$-space, and $B(x, r) \subset X$ isometric to an open interval. Is X a manifold (possibly with boundary)?

Proof of Theorem 3.1. The beginning of the proof goes just as that of Theorem 3.7 in [12]. Denote by $\mathcal{F}$ the set of all the points that have a neighbourhood isometric to an open interval. Clearly, $\mathcal{F}$ is an open set in X. Suppose now that $X \setminus \mathcal{F} \neq \varnothing$. Since $\mathcal{F}$ is assumed to be non-empty, we deduce that $\mathcal{F}$ is not closed, in particular it is not a circle.

Let $\gamma\colon (-a,b) \to X$, $a,b \in (0,\infty]$, be a locally minimising, unit speed curve for which $\gamma_0 = x$, x being as in the assumption (2), and $\gamma(-a,b)$ is the maximal connected subset of $\mathcal{F}$ containing x, and let $\varepsilon > 0$ be such that $\gamma|_{(-\varepsilon,\varepsilon)}$ is an isometry.

If $a = b = \infty$, we have that $\gamma(-a,b) = X$ by the following argument. Suppose that there exists a limit point $y \in \overline{\operatorname{Im}\gamma} \setminus \operatorname{Im}\gamma$ of γ. Let $(t_i) \subset \mathbb{R}$ be a sequence so that $\gamma_{t_i} \longrightarrow y$. We may assume that t_i is increasing. If (t_i) is bounded, then there exists t_∞ so that $t_i \longrightarrow t_\infty$, and hence by continuity we would have that $y = \gamma_{t_\infty}$. Hence $t_i \longrightarrow \infty$. Let now α be a geodesic from x to y, and let $s := \inf\{t : \alpha_t \notin \operatorname{Im}\gamma\}$. We know that $s > 0$, since the neighbourhood of x is isometric to an open interval. On the other hand, since α is a geodesic and $l(\gamma) = \infty$, we know that there exists a sequence (s_i) so that $s_i \longrightarrow s_\infty < \infty$, and $\gamma_{s_i} \longrightarrow \alpha_s$. Hence, $\gamma_{s_\infty} = \alpha_s$. This is in contradiction with the definition of s, since γ_{s_∞} has a neighbourhood of the form $\gamma(s_\infty - \delta, s_\infty + \delta)$. Thus, $\gamma(-a,b)$ is open and closed, and hence $\gamma(-a,b) = X$ giving a contradiction.

By the above, we may assume that $b < \infty$. Let $y \in X \setminus \operatorname{Im}\gamma$, α be a unit speed geodesic from x to y, and, as above, $s := \inf\{t : \alpha_t \notin \operatorname{Im}\gamma\}$. Since by the arguments above, $\gamma_t \neq \alpha_s$ for any $t \in (-a,b)$, there exists a sequence (t_i) that (we may assume without loss of generality to) converge to b so that $\gamma_{t_i} \longrightarrow \alpha_s$. By the maximality of $\operatorname{Im}\gamma$, we know that $B(\alpha_s, r) \setminus (\operatorname{Im}\gamma \cup \operatorname{Im}\alpha) \neq \varnothing$ for any $r > 0$. Let now $z \in B(\alpha_s, r) \setminus (\operatorname{Im}\gamma \cup \operatorname{Im}\alpha)$, where $l(\alpha) > r > 0$ is chosen small enough so that any geodesic from x to z goes through the point α_s. We are ready to arrive to a contradiction with (1). Let β be a unit speed geodesic from x to z. Take $t_1 > s$ so that $d(\alpha_s, \alpha_{t_1}) = d(\alpha_s, z)$, and define measures $\mu_0 := \frac{1}{\mathfrak{m}(\gamma([0,\varepsilon)))}\mathfrak{m}|_{\gamma([0,\varepsilon))} \in \mathcal{P}_2^{ac}(X)$, and $\mu_1 := \frac{1}{2}((\alpha \circ \operatorname{restr}_{t_0}^{t_1})_\# \mathcal{L}^1 + (\beta \circ \operatorname{restr}_{t_0}^{t_1})_\# \mathcal{L}^1) \in \mathcal{P}_2(X)$, where t_0 is chosen so that $\alpha_t \neq \beta_t$ for every $t \in [t_0, t_1]$.

Then, by using c-cyclical monotonicity as follows, we deduce that any plan from μ_0 to μ_1 cannot be given by a map, giving a contradiction with the assumption (1). Indeed, if $\sigma \in \operatorname{Opt}(\mu_0, \mu_1)$ is induced by a map, there exists a c-cyclically monotone set $\Gamma \subset X \times X$ so that $\sigma(\Gamma) = 1$, and $\Gamma_{\tilde{x}} := \{y \in X : (\tilde{x}, y) \in \Gamma\}$ is a singleton for all $\tilde{x} \in \mathrm{P}^1(\Gamma)$. By the definition of μ_1, and the fact that $\sigma(\Gamma) = 1$, we have that there exist $(x_1, \alpha(t)), (x_2, \beta(t)) \in \Gamma$ with some $t \in [t_0, t_1]$. Since $\Gamma_{\tilde{x}}$ is a singleton for all $\tilde{x}$, we deduce that $x_1 \neq x_2$. Then by the definitions of μ_0 and μ_1, there exists $\tilde{x}$ between x_1 and x_2 so that $(\tilde{x}, y) \in \Gamma$ for some $y \in X$, and so that $\beta(t) \neq y \neq \alpha(t)$. Suppose that $y = \alpha(\tilde{t})$, with $\tilde{t} < t$ (the other cases are analogous). Then

$$\begin{aligned} d^2(x_1, \alpha(t)) + d^2(\tilde{x}, y) &= [d(\tilde{x}, \alpha(t)) + d(\tilde{x}, x_1)]^2 + [d(x_1, y) - d(\tilde{x}, x_1)]^2 \\ &> d^2(\tilde{x}, \alpha(t)) + d^2(x_1, y), \end{aligned}$$

which contradicts the c-cyclical monotonicity of the set Γ. □

Remark 3.3. The assumption that there exists an optimal map for *all* μ_1, instead of only the absolutely continuous ones, is crucial in Theorem 3.1. This can be seen by taking three non-atomic, mutually singular probability measures with full supports on the interval $[0,1]$, and pushing them to different branches of a tripod (see [10, Example in Section 3] for more details). The metric measure space obtained in this way satisfies the assumption of existence of transport maps between absolutely continuous measures, but does not satisfy the conclusion of Theorem 3.1.

One might wonder, whether after relaxing the assumption (1) to concern only absolutely continuous measures μ_1 one could still conclude that the space is one-dimensional in some appropriate sense (as in the case of the above-mentioned tripod). The following example shows that this is not the case in general.

Example 3.4. We will construct a metric measure space having – as a metric space – a one-dimensional part (a line segment), and a two-dimensional part (a circular sector) which has the property that the optimal plan between any two absolutely continuous Borel probability measures is unique and induced by a map (see Figure 1). Let us first define

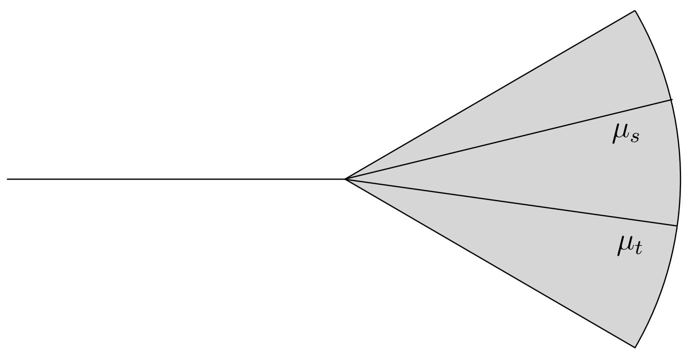

FIGURE 1. The space satisfying the weakened assumptions of Theorem 3.1, but failing the conclusion.

auxiliary measures on the unit square $I \times I$. Let $f\colon 2^{\mathbb{N}} \to I$ be a map defined as

$$(x_i)_{i\in\mathbb{N}} \mapsto \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i.$$

Define a family of measures $\{\mu_t\}_{t\in I}$ on $I \times I$ as a pushforward of the measures

$$\nu_t \coloneqq \bigotimes_{\mathbb{N}} [(1-t)\delta_0 + t\delta_1]$$

under the graph map (ι, f), where $\iota\colon 2^{\mathbb{N}} \to I$ is the Borel isomorphism (up to removing a countable set) $(x_i)_{i\in\mathbb{N}} \mapsto \sum_i x_i 2^{-i}$.

Finally, define a measure $\tilde{\mathfrak{m}}$ on $I \times I$ by setting

$$\int g \,\mathrm{d}\tilde{\mathfrak{m}} \coloneqq \int_I \int_{I\times I} g \,\mathrm{d}\mu_t \,\mathrm{d}\mathcal{L}^1(t) = \int_I \int_{I\times\{t\}} g \,\mathrm{d}\mu_t \,\mathrm{d}\mathcal{L}^1(t)$$

for every positive Borel function g. Here the second equality is due to the fact that $\nu_t(f^{-1}(t)) = 1$ by the strong law of large numbers. To see that the definition makes sense, we need to show that the map $t \mapsto \mu_t(A)$ is Borel for every $A \in \mathcal{B}(I \times I)$. It suffices to prove the Borel measurability of the map $t \mapsto \nu_t(A)$ in the case where A is of the form

$$A = \{x_1\} \times \cdots \times \{x_n\} \times 2^{\mathbb{N}}.$$

Indeed, by setting

$$P \coloneqq \{U \subset 2^{\mathbb{N}} : U = \{x_1\} \times \cdots \times \{x_n\} \times 2^{\mathbb{N}} \text{ for some } x_1, \ldots, x_n \in \{0,1\}\}$$

and

$$D \coloneqq \{V \subset 2^{\mathbb{N}} : t \mapsto \nu_t(V) \text{ is Borel}\},$$

we deduce from Dynkin's $\pi - \lambda$ theorem that $\mathcal{B}(2^{\mathbb{N}}) \subset \sigma(P) \subset D$, if $P \subset D$. Thus, we obtain that $t \mapsto \mu_t(B \times C)$ is Borel measurable for $B, C \in \mathcal{B}(I)$, if $P \subset D$. Applying the $\pi - \lambda$ theorem again, we deduce that $t \mapsto \mu_t(A)$ is Borel, if $P \subset D$.

Let now $A = \{x_1\} \times \cdots \times \{x_n\} \times 2^{\mathbb{N}}$. Then the map $t \mapsto \nu_t(A) = \Pi_{i=1}^n [(1-t)\delta_0(x_i) + t\delta_1(x_i)]$ is continuous, thus Borel. Hence, the measure $\tilde{\mathfrak{m}}$ is a well-defined Borel probability measure on $I \times I$.

With $\tilde{\mathfrak{m}}$ defined, we may define a metric measure space $(X, d, \mathfrak{m})$ in the following way. Let

$$X = ([-1, 0] \times \{0\}) \cup T(I \times I) \subset \mathbb{R}^2,$$

where T is the map $(t, \theta) \mapsto te^{i(\theta - \frac{1}{2})} \in \mathbb{R}^2$. Define the metric d on X as the length metric of X as a subset of $\mathbb{R}^2$, and the measure $\mathfrak{m}$ as the pushforward of $\tilde{\mathfrak{m}}$ under the map T on $T(I \times I)$, and as the Lebesgue measure on the interval $[-1, 0] \times \{0\}$. Then $\mathfrak{m}$ is a well-defined, non-trivial and finite measure on X with $\operatorname{spt} \mathfrak{m} = X$.

We will show that the metric measure space $(X, d, \mathfrak{m})$ satisfies the condition that for all $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$ there exists a unique plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$, and it is induced by a map. To do so, we will use the fact, that in $\mathbb{R}^n$ it suffices to check that the starting measure gives zero measure to Lipschitz graph to deduce the existence and uniqueness of the optimal transport map [6]. First of all, if $G \subset I \times I$ is a Lipschitz graph, then the intersection of G with $\mathcal{L}^1$-almost every line parallel to x-axis has only finitely many points (e.g. [15, Theorem 10.10]). Thus $\tilde{\mathfrak{m}}(G) = 0$ due to the fact that each μ_t is non-atomic. Furthermore, since T is biLipschitz on every set of the form $[\varepsilon, 1] \times I$, $\varepsilon > 0$, we have that $\mathfrak{m}|_{T(I \times I)}(G) = 0$ for every Lipschitz graph $G \subset X$.

Let now $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$, and $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$. Write $\pi = \pi^1 + \pi^2$, where $\pi^1 \coloneqq \pi|_{e_0^{-1}([-1,0] \times \{0\})}$ and $\pi^2 \coloneqq \pi|_{e_0^{-1}(T(I \times I))}$. Since $\mu_0|_{[-1,0] \times \{0\}} \ll \mathcal{L}^1$, we have by the following argument that π^1 is induced by a map. Due to the fact that the measures $\{\nu_t\}$ are concentrated on the pairwise disjoint sets $f^{-1}(t)$, we have that the measure $\tilde{\mathfrak{m}}$ is concentrated on the graph of the Borel function $f \circ \iota^{-1}$. Thus, the measure $\mathfrak{m}|_{T(I \times I)}$ is concentrated on a set $F = T(\mathrm{Graph}(f \circ \iota^{-1}))$ having the property that if $te^{i\theta_1}, te^{i\theta_2} \in F$, then $\theta_1 = \theta_2$. Then by absolute continuity of μ_1, the measure $(e_1)_\# \pi^1$ is concentrated on F. Moreover, by the definition of the metric d we have that for points $x = (s, 0) \in [-1, 0] \times \{0\}$ and $y = te^{i\theta} \in T(I \times I)$, the distance is $d(x, y) = |t - s|$. Hence, $\sigma \coloneqq G_\# (e_0, e_1)_\# \pi^1 \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ is an optimal plan, where $G((s, 0), (t, 0)) = (s, t)$ for $(t, 0) \in [-1, 0] \times \{0\}$, and $G((s, 0), te^{i\theta}) = (s, t)$ for $te^{i\theta} \in F$. By the absolute continuity of $\mu_0|_{[-1,0] \times \{0\}}$ and by the definition of G we have that

$P^1_\# \sigma$ is absolutely continuous with respect to the Lebesgue measure. Therefore, σ is induced by a map $S\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. Thus, $(e_0, e_1)_\# \pi^1$ is induced by the map $(x, 0) \mapsto G^{-1} \circ S(x)$, and so by the uniqueness of geodesics in X, the plan π^1 is also induced by a map.

Thus, it remains to show that π^2 is induced by a map. Since geodesics starting from $T(I \times I)$ do not branch (i.e. $\gamma^1|_{[0,t]} = \gamma^2|_{[0,t]}$ implies $\gamma^1 = \gamma^2$), we may assume after contracting the plan towards the starting point, that $\pi^2(\mathrm{Geo}(T(I \times I))) = 1$. Hence, by the fact that $\mathfrak{m}|_{T(I\times I)}$ gives zero measure to Lipschitz graphs, we conclude that π^2, and therefore also π, is induced by a map.

In spite of Example 3.4, one may try to find sufficient conditions for the metric measure space $(X, d, \mathfrak{m})$ so that Theorem 3.1 would still hold after weakening the assumptions to merely consider measures μ_1 that are absolutely continuous with respect to the reference measure $\mathfrak{m}$. Naively mimicking the proof of Theorem 3.1 one gets a sufficient condition for the reference measure $\mathfrak{m}$, namely the conclusion of Lemma 3.6, leading to the following theorem.

Theorem 3.5. *Let $(X, d, \mathfrak{m})$ be an $MCP(K, N)$-space with the following properties:*

(1) For every $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X)$, there exists $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ that is induced by a map from μ_0.

(2) There exists a point $x \in X$, and a neighbourhood $B(x, r)$ isometric to an open interval in $\mathbb{R}$.

Then X is a one-dimensional manifold, possibly with boundary.

In the proof of Theorem 3.5, we will use the following lemma.

Lemma 3.6. *Let $(X, d, \mathfrak{m})$ be an $MCP(K, N)$-space. Then*

$$\int_X f \, \mathrm{d}\mathfrak{m} = \int_{(0,\infty)} \int_{\partial B(x_0,r)} f \, \mathrm{d}\mathfrak{m}_r \, \mathrm{d}\mathcal{L}^1(r), \tag{1}$$

for every non-negative Borel function f, where $\mathfrak{m}_r$ is a finite Borel measure on $\partial B(x_0, r)$. Moreover, for any three points $x_0, y, z \in X$ with $d(x_0, y) = d(x_0, z)$, and for any $r > 0$ for which $B(y, r) \cap B(z, r) = \varnothing$, there exists a set $E \subset (0, \infty)$ of positive Lebesgue measure such that $\mathfrak{m}_r(B(y, r)), \mathfrak{m}_r(B(z, r)) > 0$ for all $r \in E$.

The identity (3.6) of Lemma 3.6 follows directly from Lemma 3.7 and the Bishop-Gromov theorem in $MCP(K, N)$-spaces proven by Ohta in [17]. The rest of the proof of Lemma 3.6 goes exactly as the proof of [12, Lemma 2.13], after proving Lemma 3.9 (see [12, Claim 2.16]).

Lemma 3.7. *Let $(X, d, \mathfrak{m})$ be a metric measure space satisfying the following local Bishop–Gromov volume comparison property: for every $x_0 \in X$ and $R > 0$ there exists a locally absolutely continuous $w := w_{x_0,R}\colon (0, R) \to (0, \infty)$ such that the map*

$$r \mapsto \frac{\mathfrak{m}(B(x, r))}{w(r)}, \quad B(x, r) \subset B(x_0, R) \tag{2}$$

is decreasing. Then we can write $\mathfrak{m}$ *as*

$$\int_X f\,\mathrm{d}\mathfrak{m} = \int_{(0,\infty)} \int_{\partial B(x_0,r)} f\,\mathrm{d}\mathfrak{m}_r\,\mathrm{d}\mathcal{L}^1(r),$$

for every non-negative Borel function f*.*

Proof. First of all, notice that by the Bishop-Gromov inequality (3.7) we have that the measure is boundedly finite (since it is locally finite by assumption). Thus, by the disintegration theorem we have that

$$\int_X f\,\mathrm{d}\mathfrak{m} = \int_{(0,\infty)} \int_{\partial B(x_0,r)} f\,\mathrm{d}\mathfrak{m}_r\,\mathrm{d}\varpi(r),$$

where ϖ is the pushforward of $\mathfrak{m}$ with respect to the function $x \mapsto d(x_0, x)$, and $\mathfrak{m}_r$ is a finite Borel measure. We need to show that ϖ is absolutely continuous with respect to Lebesgue measure. Observe first, that for $r < r_1 < r_2 < R$ we have that

$$\begin{aligned}\varpi([r_1, r_2]) &= \mathfrak{m}(B(x, r_2)) - \mathfrak{m}(B(x, r_1))\\ &\le (w(r_2) - w(r_1))\frac{\mathfrak{m}(B(x, r_1))}{w(r_1)}\\ &\le C_{r,R}(w(r_2) - w(r_1)).\end{aligned}$$

Thus, by the absolute continuity we have that for any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\varpi\left(\bigsqcup_{i\in\mathbb{N}}[r_i, s_i]\right) \le C_{r,R}\sum_{i\in\mathbb{N}}(w_{s_i} - w_{r_i}) < \varepsilon,$$

whenever $\mathcal{L}^1\left(\bigsqcup_{i\in\mathbb{N}}[r_i, s_i]\right) < \delta$. Hence, by the definition of Lebesgue measure, we have that $\varpi \ll \mathcal{L}^1$. □

Definition 3.8 (Non-degenerate measure [3])**.** A measure $\mathfrak{m}$ is called *non-degenerate*, if for every Borel subset A with $\mathfrak{m}(A) > 0$ we have that $\mathfrak{m}(A_{t,x}) > 0$ for every $x \in X$ and $t \in [0, 1)$, where $A_{t,x} := \{\gamma_t : \gamma \in \mathrm{Geo}(X), \gamma_0 \in A, \gamma_1 = x\}$.

Lemma 3.9. *Let* $(X, d, \mathfrak{m})$ *be a metric measure space with* $\mathfrak{m}$ *non-degenerate and satisfying the Bishop-Gromov inequality* (3.7)*. Let* $x, y \in X$ *with* $l := d(x, y)$*, let* $0 < r_0 < l$ *and let* $E := \{r \in (l - r_0, l) : \mathfrak{m}_r(B(y, r_0)) = 0\}$*. Then for every* $t \in (0, 1]$*, we have that*

$$\mathcal{L}^1\Big(\Big((\frac{1}{t}E) \cap \big(l - r_0, l)\Big) \setminus E\Big) = 0.$$

Proof. Suppose the contrary. Then there exists t so that

$$\mathcal{L}^1(F) > 0,$$

where $F := ((\frac{1}{t}E) \cap (l - r_0, l)) \setminus E$. Define $B_F := \{z \in B(y, r_0) : d(x, z) \in F\}$. Then we have that

$$\mathfrak{m}(B_F) = \int_F \mathfrak{m}_r(B(y, r_0))\,\mathrm{d}\mathcal{L}^1(r) > 0.$$

By the non-degeneracy assumption we have that $\mathfrak{m}((B_F)_{t,x}) > 0$. Therefore, we have that
$$\mathcal{L}^1(tF \setminus E) > 0.$$
On the other hand, by the definition of the set F, we have that $tF \subset E$, which is a contradiction. □

With Lemma 3.6 at our disposal, we are ready to prove Theorem 3.5.

Proof of Theorem 3.5. Suppose that the claim is not true. As in the proof of Theorem 3.1, we find a pair of branching geodesics α and β with equal length, which start from x and end up to two distinct points. Let r_0 be such that $B(\alpha(1), r_0) \cap B(\beta(1), r_0) = \varnothing$. Let E be the set given by Lemma 3.6, and $A \coloneqq \{z \in B(\alpha(1), r_0) \cup B(\beta(1), r_0) : d(x, z) \in E\}$. Let $\varepsilon > 0$ be such that $B(x, \varepsilon)$ is isometric to an interval. Define $\mu_0 \coloneqq \frac{1}{\mathfrak{m}(\gamma([0,\varepsilon)))}\mathfrak{m}|_{\gamma([0,\varepsilon))}$ and $\mu_1 \coloneqq \frac{1}{\mathfrak{m}(A)}\mathfrak{m}|_A$. Now, due to the definition of the set E, we conclude exactly with the same arguments as in the proof of Theorem 3.1 that none of the optimal plans from μ_0 to μ_1 is given by a map, which is a contradiction with the assumption. □

In Theorem 3.1 and Theorem 3.5 we gained a global one-dimensionality from a local one-dimensionality near a single point in the space. In the next theorem we will weaken the assumption on the local one-dimensionality to assumption on the infinitesimal one-dimensionality at a single point. However, at the same time we need to assume not only the existence of optimal maps but also the uniqueness of the plan. Notice that the existence of optimal maps is quite often proven in such a way that the uniqueness is achieved at the same time.

Theorem 3.10. *Let $(X, d, \mathfrak{m})$ be a locally metrically doubling metric measure space with the following properties:*

(1) For every $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, there exists a unique $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ and it is induced by a map from μ_0.

(2) There exists a point $x \in X$ such that $\mathrm{Tan}(X, x) = \{(\mathbb{R}, 0)\}$.

Then X is a one-dimensional manifold, possibly with boundary.

In particular, we get the following stronger version of Corollary 3.2 in the case of essentially non-branching spaces.

Corollary 3.11. *Let $(X, d, \mathfrak{m})$ be an essentially non-branching $MCP(K, N)$-space (or ess. nb., qualitatively non-degenerate space). Suppose that there exists a point $x \in X$ for which $\mathrm{Tan}(X, x) = \{(\mathbb{R}, 0)\}$. Then X is a one-dimensional manifold, possibly with boundary.*

It is worth mentioning that the case of branching very strict $CD(K, N)$ -spaces is more difficult due to the non-uniqueness of the optimal transport map, and that it is not clear how to overcome this issue to obtain the same result in that case.

Proof of Theorem 3.10. Let $x \in X$ be such that $\mathrm{Tan}(X, x) = \{(\mathbb{R}, 0)\}$. By Theorem 3.1 it suffices to prove that x has a neighbourhood isometric to an open interval. Let $\varepsilon_n \searrow 0$. Then by the fact that $\mathbb{R}$ is a tangent of X at x, we find $\tilde{r}_n \searrow 0$ and points $y_n, z_n \in X$ with

$d(y_n, z_n) \geq 2\tilde{r}_n - \tilde{r}_n\varepsilon_n$, and $\tilde{r}_n + \tilde{r}_n\varepsilon_n \geq d(y_n, x), d(z_n, x) \geq \tilde{r}_n - \tilde{r}_n\varepsilon_n$. Let $\delta_n := d(y_n, x)$. We may assume that $B(y_n, \frac{\delta_n}{2})$ is not isometric to an open interval (otherwise we are done by Theorem 3.1).

Suppose that x has no neighbourhood isometric to an open interval. Hence, by the following argument, there exist two geodesics α^n and β^n from x to $B(y_n, \frac{\delta_n}{2})$ with $l(\alpha^n) = l(\beta^n)$ and $\alpha_1^n \neq \beta_1^n$. Let $y_n' \in B(y_n, \delta_n)$ be a point lying in the interior of a geodesic γ connecting x to y_n. Let B be a ball centered at y_n' so that $B \subset B(y_n, \frac{\delta_n}{2}) \setminus \{y_n\}$. The ball B is not isometric to an interval (again by Theorem 3.1). Thus, it contains a point y_n'' which does not lie in the geodesic γ. Then, by triangle inequality, there exists a point w_n lying in γ with $d(x, y_n'') = d(x, w_n)$. The geodesics α^n and β^n connecting x to y_n'' and w_n, respectively, satisfy the desired conditions.

Define measures $\mu_0^n := \frac{1}{\mathfrak{m}(B(z_n, \frac{\delta_n}{2}))}\mathfrak{m}|_{B(z_n, \frac{\delta_n}{2})}$ and $\mu_1^n := \frac{1}{2}[\delta_{\alpha_1^n} + \delta_{\beta_1^n}]$. Let $\pi_n \in \mathrm{OptGeo}(\mu_0^n, \mu_1^n)$ be the unique optimal plan. Then there exists a π_n- positive measure set $\mathcal{A}_n$ such that any $\eta \in \mathcal{A}_n$ does not go through x, since otherwise the uniqueness of the plan could be easily violated by simply sending any point to both of the points α_1^n and β_1^n. More precisely, suppose that π_n- almost every γ goes through x. This implies that for μ_0^n- almost every $\tilde{x}$ we have that

$$d(\tilde{x}, \alpha_1^n) = d(\tilde{x}, x) + d(x, \alpha_1^n) = d(\tilde{x}, x) + d(x, \beta_1^n) = d(\tilde{x}, \beta_1^n). \tag{3}$$

Let $M \subset X$ be the set of full μ_0-measure whose points satisfy (3). Then $M \times \{\alpha_1^n, \beta_1^n\}$ is c-cyclically monotone set. Therefore, the measure $\mu_0^n \otimes \mu_1^n$ is concentrated on a c-cyclically monotone set, and thus $\mu_0^n \otimes \mu_1^n \in \mathrm{Opt}(\mu_0^n, \mu_1^n)$. Clearly, $\mu_0^n \otimes \mu_1^n$ is not induced by a map, but $(e_0, e_1)_{\#}\pi_n$ is, which contradicts the uniqueness of the optimal plan.

Let $\eta^n \in \mathcal{A}_n$ be a geodesic not going via x. Consider the sequence (X, d_{δ_n}, x), $d_{\delta_n} := \frac{d}{\delta_n}$, which converges (up to subsequence) to $\mathbb{R}$. Since $d(\eta_t^n, x) \leq 8\delta_n$ for all $t \in [0, 1]$, we know that η^n converges (again, up to subsequence) to a geodesic $\eta^\infty \in \mathrm{Geo}(\mathbb{R})$. Morever, since $d(y_n, \eta_0^n), d(z_n, \eta_1^n) < \frac{1}{2}\delta_n$, and since $z_n \longrightarrow -1$ and $y_n \longrightarrow 1$ (or vice versa), we have that there exists a sequence (s_n) so that $\eta_{s_n}^n \longrightarrow 0$. In particular, if we take s_n so that $d(\eta_{s_n}^n, x) = d(\mathrm{Im}\, \eta^n, x)$, we have that $\eta_{s_n}^n \longrightarrow 0$ implying that $\frac{d(\eta_{s_n}^n, x)}{\delta_n} \longrightarrow 0$. Thus, we find $s_n^1 < s_n^2$ such that $d(\eta_{s_n^1}^n, \eta_{s_n}^n) = d(x, \eta_{s_n}^n) = d(\eta_{s_n^2}^n, \eta_{s_n}^n) =: r_n$ and $d(\eta_{s_n^2}^n, \eta_{s_n^1}^n) = 2d(x, \eta_{s_n}^n)$ (see Figure 2).

Finally, consider a sequence (X, d_{r_n}, x), $d_{r_n} := \frac{d}{r_n}$, converging by the local doubling property (up to subsequence) to $\mathbb{R}$. Then we have again that η^n (sub)converges to a limit geodesic in $\mathbb{R}$. Now by the choice of r_n we have that $\eta_{s_n}^n$ converges either to 1 or to -1. We may assume without loss of generality that it converges to 1. Then, by the choice of s_n^1 and s_n^2, we have that $\mathrm{restr}_{s_n^1}^{s_n^2}(\eta^n)$ converges to an interval $[0, 2]$, implying that $\eta_{s_n^i}^n$ converges to 0 for either $i \in \{1, 2\}$. This contradicts the fact that $d(\eta_{s_n^i}^n, x) \geq d(\eta_{s_n}^n, x) = r_n$ for both $i \in \{1, 2\}$.

□

The following example shows that the uniqueness of the optimal map is crucial in Theorem 3.10.

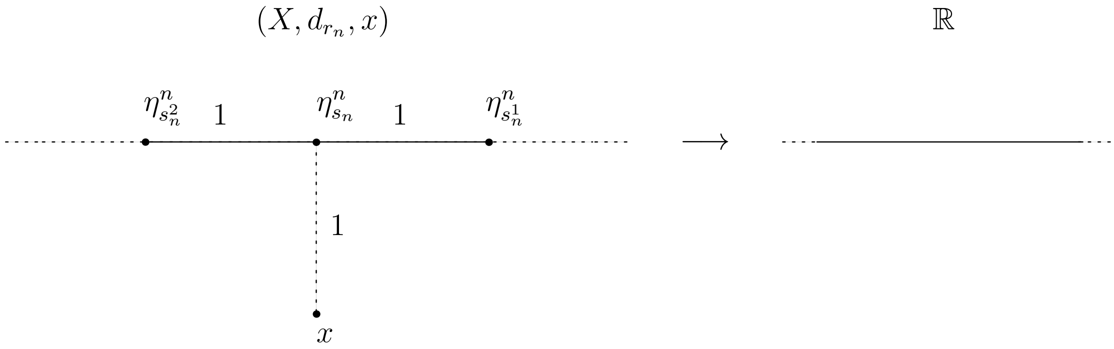

FIGURE 2. Tripod type configuration leading to a contradiction at the blow-up.

Example 3.12 (Necessity of the uniqueness of the map in Theorem 3.10)**.** Let $X \subset \mathbb{R}^2$ be a weakly (geodesically) convex, two-sided cusp on the plane endowed with supremum norm, for example $A := \{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{1}{2}, |y| \leq x^2\}$. Let d be the distance induced by the supremum norm restricted to X, and $\mathfrak{m}$ a locally finite measure absolutely continuous with respect to $\mathcal{L}^2$. Then $(X, d, \mathfrak{m})$ is a geodesic (since $|(x^2)'| \leq 1$) metric measure space satisfying

(1) For every $\mu_0 \in \mathcal{P}_2^{ac}(X)$ and $\mu_1 \in \mathcal{P}_2(X)$, there exists $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ that is induced by a map from μ_0[1],
(2) $\mathrm{Tan}(X, 0) = \{(\mathbb{R}, 0)\}$,

but does not satisfy the conclusion of Theorem 3.10.

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[1]The existence of such π is due to the existence of optimal maps on $(\mathbb{R}^2, |\cdot|_{\sup})$, see for example [1], and the weak convexity of X.

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