The case of equality in the dichotomy of Mohammadi-Oh

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Abstract

If $n \geq 3$ and Γ is a convex-cocompact Zariski-dense discrete subgroup of $\mathbf{SO}^o(1, n{+}1)$ such that $\delta_\Gamma = n{-}m$ where m is an integer, $1 \leq m \leq n{-}1$, we show that for any m-dimensional subgroup U in the horospheric group N, the Burger-Roblin measure associated to Γ on the quotient of the frame bundle is U-recurrent.

1 Introduction

1.1 Notations

We fix once and for all an integer $n \geq 2$. Let $G = \mathbf{SO}^o(1, n+1)$, this is the group of direct isometries of the real $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}$. Its acts conformally on the boundary $\partial\mathbf{H}^{n+1}$.

Recall the Busemann function

$$b_\xi(x, y) = \lim_{t\to\infty} d(x, \xi_t) - d(y, \xi_t) \qquad \xi \in \partial\mathbf{H}^{n+1}, \quad x, y \in \mathbf{H}^{n+1}$$

where $t \mapsto \xi_t$ is some geodesic with positive endpoint ξ.

Fix an Iwasawa decomposition $G = KAN$; recall that the maximal compact subgroup K is isomorphic to $\mathbf{SO}(n+1)$, whereas the Cartan subgroup A is isomorphic to $\mathbf{R}$ (since G has rank 1) and the maximal unipotent subgroup N is isomorphic to $\mathbf{R}^n$.

Denote by M the centralizer of A in K; M is isomorphic to $\mathbf{SO}(n)$. Recall that M normalizes N and there are isomorphisms $M \simeq \mathbf{SO}(n)$, $N \simeq \mathbf{R}^n$ such that the operation of M on N by conjugation identifies with the natural operation of $\mathbf{SO}(n)$ on $\mathbf{R}^n$.

We will always tacitly endow N with the corresponding Euclidean metric.

Let Γ be a discrete non-elementary subgroup of G. Throughout this paper we make the standing assumptions that

Γ is Zariski-dense and has finite Bowen-Margulis-Sullivan measure.

In fact except in the last paragraph we will always assume that Γ is convex-cocompact (this is stronger than finiteness of the Bowen-Margulis-Sullivan measure).

As usual, we denote by δ_Γ the growth exponent (also called Poincaré exponent) of Γ

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{\log \mathrm{Card}\{\gamma \in \Gamma \ ; \ d(x, \gamma x) \leq R\}}{R}$$

which does not depend on the fixed point $x \in \mathbf{H}^{n+1}$. This is the Hausdorff dimension (with respect to the spherical metric on the boundary) of the limit set

$$\Lambda_\Gamma = \overline{\Gamma \cdot x} \cap \partial \mathbf{H}^{n+1}$$

(which also does not depend on x). Bear in mind that $0 < \delta_\Gamma \leq n$; in this paper we will be interested in the case when δ_Γ is an integer strictly less than n.

The boundary $\partial \mathbf{H}^{n+1}$ is endowed with the Patterson-Sullivan density $(\mu_x)_{x \in \mathbf{H}^{n+1}}$. This is the (essentially unique since Γ has finite Bowen-Margulis-Sullivan measure) family of finite Borel measures on $\partial \mathbf{H}^{n+1}$ satisfying

1. Γ-equivariance : $\mu_{\gamma x}$ is the push-forward of μ_x through the mapping induced by γ on $\partial \mathbf{H}^{n+1}$;
2. δ_Γ-conformality: for any $x, y \in \mathbf{H}^{n+1}$, μ_x and μ_y are equivalent and the Radon-Nikodym cocycle is given by

$$\frac{\mathrm{d}\mu_y}{\mathrm{d}\mu_x}(\xi) = e^{-\delta_\Gamma b_\xi(y,x)}$$

almost everywhere.

This is the Patterson-Sullivan density associated to Γ. If a base point $o \in \mathbf{H}^{n+1}$ is fixed, the boundary $\partial \mathbf{H}^{n+1}$ may be identified canonically with the n-sphere S^n and thus endowed with the usual spherical metric. When Γ is convex-cocompact, μ_o is proportional to the δ_Γ-dimensional Hausdorff measure on δ_Γ with respect to the spherical metric (see [16] or [1]).

We now recall the definition of the Bowen-Margulis-Sullivan (BMS) measure – first on the unit tangent bundle, then on the frame bundle. Let $T^1\mathbf{H}^{n+1}$ be the unit tangent bundle over $\mathbf{H}^{n+1}$. The Hopf isomorphism is the bijective mapping from $T^1\mathbf{H}^{n+1}$ to $\partial^2\mathbf{H}^{n+1} \times \mathbf{R}$ that maps the unit tangent vector u with base point x to the triple

$$(\xi, \eta, s) = (u^-, u^+, b_{u^-}(x, o))$$

where u^-, u^+ respectively are the negative and positive endpoints of the geodesic whose derivative at $t = 0$ is u. The notation $\partial^2\mathbf{H}^{n+1}$ stands for the set of all $(\xi, \eta) \in \partial\mathbf{H}^{n+1} \times \partial\mathbf{H}^{n+1}$ such that $\xi \neq \eta$.

In these coordinates, the BMS measure on $T^1\mathbf{H}^{n+1}$ is given by

$$\mathrm{d}\tilde{m}_{\mathrm{BMS}}(u) = e^{\delta_\Gamma(b_\xi(x,u)+b_\eta(x,u))}\mathrm{d}\mu_x(\xi)\mathrm{d}\mu_x(\eta)\mathrm{d}s$$

(it does not depend on the choice of $x \in \mathbf{H}^{n+1}$).

The BMS measure is a Radon measure that is invariant under the geodesic flow as well as under the natural operation of Γ. The quotient of this measure with respect to Γ is a Radon measure m_{BMS} on $\Gamma \backslash T^1\mathbf{H}^{n+1}$ that is still invariant with respect to the geodesic flow. This quotient measure may be finite or infinite; we will always assume that is is finite and in fact we will usually assume that it is compactly supported, which is equivalent to Γ being convex-cocompact ([12], [16]).

The Burger-Roblin (BR) measure is defined in a similar fashion:

$$\mathrm{d}\tilde{m}_{\mathrm{BR}}(u) = e^{\delta_\Gamma b_\xi(x,u)+n b_\eta(x,u)}\mathrm{d}\mu_x(\xi)\mathrm{d}\nu_x(\eta)\mathrm{d}s$$

where ν_x is the unique Borel probability measure on $\partial\mathbf{H}^{n+1}$ that is invariant under the stabilizer of x in G; if $\partial\mathbf{H}^{n+1}$ is identified with S^n accordingly, this is just the Lebesgue measure on S^n.

Likewise, the Burger-Roblin measure is Γ-invariant and thus defines a Radon measure on $\Gamma\backslash T^1\mathbf{H}^{n+1}$. This Radon measure is always infinite, unless Γ is a lattice.

Both these measures lift to the frame bundle over $\Gamma\backslash\mathbf{H}^{n+1}$ in the following way. The hyperbolic space $\mathbf{H}^{n+1}$ identifies with the quotient space G/M so that G identifies with the $(n+1)$-frame bundle over $\mathbf{H}^{n+1}$. The quotient space $\Gamma\backslash G$ accordingly identifies with the $(n+1)$-frame bundle over the orbifold $\Gamma\backslash\mathbf{H}^{n+1}$. There is a unique measure on $\Gamma\backslash G$ that is (right) invariant with respect to M and projects onto the BMS measure in $\Gamma\backslash G/M$, we denote it by m_{BMS} as well. Same thing for the BR measure. The lift of the geodesic flow to $\Gamma\backslash G$ is called the frame flow.

The point in doing this is we can now let N act by translation (to the right) on $\Gamma\backslash G$. Let us agree that $A = \{a_t \ ; \ t \in \mathbf{R}\}$ where $(a_t)_t$ parametrizes the frame flow over $\mathbf{H}^{n+1}$, in such a way that N parametrizes the *unstable* horospheres.

We then have, for every $h \in N$,

$$a_{-t}ha_t = S_t(h) \tag{1}$$

where S_t is the homothety $N \to N$ with ratio e^t.

We summarize the important points in the following

Lemma 1. *Assume that Γ has finite BMS measure and is Zariski-dense.*

1. *The BMS measure on $\Gamma\backslash G$ is mixing with respect to the ergodic flow.*
2. *The BR measure on $\Gamma\backslash G$ is invariant and ergodic with respect to N.*
3. *If $\Omega \subset \Gamma\backslash G$ has full BMS measure, then ΩN has full BR measure.*

Proof. For 1 and 2 see [17]. For 3 compare the definitions of BMS and BR measure, taking into account the fact that N parametrizes the unstable horospheres in the frame bundle. □

1.2 Background

The basic motivation for this paper is the following

Theorem (Mohammadi-Oh, [11], Theorem 1.1)**.** *Assume that Γ is convex-cocompact and Zariski-dense. Let m be an integer, $1 \leq m \leq n-1$, and U be an m-plane in N. If $\delta_\Gamma > n - m$, then m_{BR} is U-ergodic.*

This result was also obtained by Maucourant and Schapira [9] under the weaker hypothesis that Γ has finite BMS measure. The case when $\delta_\Gamma < n - m$ has also been settled by these authors:

Theorem (Maucourant-Schapira, [9])**.** *Assume that Γ is convex-cocompact and Zariski-dense. Let m be an integr, $1 \leq m \leq n-1$, and U be an m-plane in N. If $\delta_\Gamma < n - m$, then m_{BR} is totall U-dissipative. In particular, it is not ergodic.*

Mohammadi-Oh and Maucourant-Schapira use Marstrand's projection Theorem to look at the geometry of the BMS measure along U and N. For more on this, see [4].

In this paper, we use Besicovitch-Federer's projection theorem to study the case $\delta_\Gamma = n - m$. Our main result is the following

Theorem. *Assume that Γ is convex-cocompact and Zariski-dense. Let m be an integer, $1 \leq m \leq n-1$. If $\delta_\Gamma = n - m$, then the Burger-Roblin measure is recurrent with respect to any m-plane U in N.*

Whether the BR measure is ergodic with respect to U under these hypotheses remains an open question. We will see that the return rate of U-orbits is quite low (*i.e.* subexponential) but this does not contradict ergodicity since BR is not finite.

Let us mention that the Theorem is not empty; indeed it is possible to construct some Zariski-dense convex-cocompact group $\Gamma \subset \mathbf{SO}^o(1,3)$ with $\delta_\Gamma = 1$. Start with the Apollonian gasket associated to 4 mutually tangent circles on the boundary of $\mathbf{H}^3$; the limit set has dimension $\delta_\Gamma > 1$. Now shrink continuously the radii of the circles, thus lowering continuously δ_Γ. The deformed group will remain Zariski-dense because the centers of the circles are not aligned. For details see [10].

With this result for $\delta_\Gamma = n - m$, the situation is summarized in the following table. We assume that Γ is Zariski-dense, has finite BMS measure, and we fix some m-plane U in N with $1 \leq m \leq n-1$. With respect to U, the BMS and BR measures are:

	$\delta_\Gamma < n - m$	$\delta_\Gamma = n - m$	$\delta_\Gamma > n - m$
BMS	dissipative [4]	dissipative [4]	recurrent and ergodic [9]
BR	totally dissipative if Γ convex-cocompact [9]	recurrent if Γ convex-cocompact	recurrent and ergodic [9]

Note that it follows immediately from the definitions that if the BMS measure is recurrent, so is the BR measure. The other implications are not so obvious.

We now sketch briefly our argument. In order to prove that the BR measure is U-recurrent (where U is some m-plane), we need to show that the U-orbit of m_{BR}-almost every $x \in \Gamma\backslash G$ will pass through some compact set K infinitely often. If is enough to construct some sequence h_k in N that goes to infinity while staying uniformly close from U, such that $xh_k \in K$; indeed, if u_k is the orthogonal projection of h_k onto U, the sequence u_k still goes to infinity and xu_k will belong to some compact K' that is just slightly bigger than K.

To show that such a sequence $(h_k)_k$ exists, our strategy is to prove that any ρ-neighbourhood of U in N has infinite measure with respect to the conditional measure of m_{BMS} along N; we then use the fact that the support of m_{BMS} is a compact set. This is the main reason why we need Γ to be convex-cocompact.

In order to prove that any "strip" along U has infinite measure, we argue by contradiction: if some ρ-neighbourhood has finite measure with respect to the conditional measure of m_{BMS} along N, then this must hold almost surely for any neighbourhood as large as we like (because of the self-similarity of the conditional measures). In particular we can project these conditional measures onto N/U and end up with a family of Radon measures. These "transversal" Radon measures must still have dimension $\delta_\Gamma = n-m$ (this was shown in [4]), and this implies in turn that they must be the Lebesgue measure of N/U. On the other hand, the Besicovitch-Federer projection Theorem implies that the projection of the conditional measures onto N/U must be singular with respect to the Lebesgue measure, because the conditional measure are purely unrectifiable. Hence our Theorem is proved.

The push-forward of the Borel measure μ through the Borel function f is denoted by $f\mu$; thus $f\mu(A) = \mu(f^{-1}(A))$ for any Borel set A.

For any set E, we denote by $\mathbf{1}_E$ the characteristic function:

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E \end{cases}$$

2 Proof of the main theorem

2.1 Preliminary setup

In order to study the BR measure with respect to some m-plane U in N, it is useful to look at the geometry of the BMS measure with respect to the foliation induced by U in the N-orbits (more precisely, with respect to the projection along this foliation).

The technical tool that allows this is disintegration of measures.

Since we are going to apply tools from classical geometric measure theory, we want to work with measures living on N (recall that N identifies with the Euclidean space $\mathbf{R}^n$). To m_{BMS}-almost every $x \in \Gamma\backslash G$ we are going to associate a measure (more precisely, a *projective* measure, *i.e.* a measure *modulo* a positive scalar) $\sigma(x)$ on N that reflects the geometry of m_{BMS} along the unstable horosphere passing through x.

We now set up the needed formalism. The operation of N on G (on the right) is smooth (*i.e.* the quotient Borel space G/N is a standard Borel space). Lift m_{BMS} (which lives on $\Gamma\backslash G$) to G; the measure we get is a Γ-invariant Radon measure $\tilde{m}_{\mathrm{BMS}}$. Disintegrate this measure along N; for almost every $g \in G$ we thus get a measure m_{gN} supported on gN (see [12] section 3.9 for a description of this measure).

In general when disintegrating an infinite measure, the conditional measures are canonically defined only up to a (non-zero) scalar; in fact here there is a way to normalize them in a canonical way (by introducing an appropriate measure on the space of horospheres, more precisely this space lifted by M) but this would not be useful for our purpose. See *e.g.* [13].

We now want to look at measures on N instead of measures on G. For any $g \in G$, there is a mapping $\phi_g : N \to G$ which parametrizes the "unstable horosphere" $H^+(g) = gN$ in the usual way: $\phi_g(h) = gh$ for any $h \in N$.

Since $\tilde{m}_{\mathrm{BMS}}$ is Γ-invariant, the pull-back measures

$$(\phi_g)^{-1}(m_{gN}), \quad (\phi_{\gamma g})^{-1}(m_{\gamma gN})$$

(which live on N) are equal up to a scalar multiple, for $\tilde{m}_{\mathrm{BMS}}$-almost every $g \in G$ and every $\gamma \in \Gamma$.

Let $\mathcal{M}_{\mathrm{rad}}(N)$ be the space of positive Radon measures on N and $\mathcal{M}^1_{\mathrm{rad}}(N)$ be the space of projective classes of Radon measures on N, that is, the quotient of $\mathcal{M}_{\mathrm{rad}}(N)$ by the equivalence relation

$$\mu \sim \nu \Leftrightarrow \nu = t\mu, \quad t > 0.$$

We define a mapping $\sigma : \Gamma\backslash G \to \mathcal{M}^1_{\mathrm{rad}}(N)$ by letting $\sigma(x)$ be the projective class of

$$(\phi_g)^{-1}(m_{gN})$$

if $x = \Gamma g$. This is well-defined m_{BMS}-almost everywhere.

We say that σ is obtained by *disintegrating* m_{BMS} *along* N.

This is a particular instance of the general theory of conditional measures along a group operation, see [3] or [2] (Chapter 2).

We record the following facts which we will use freely throughout this paper:

Lemma 2. *1. If some Borel subset $\Omega \subset \Gamma\backslash G$ has full m_{BMS}-measure, then for m_{BMS}-almost every x, the set*

$$\{h \in N \ ; \ xh \in \Omega\}$$

has full $\sigma(x)$-measure.

2. *There is a Borel subset $X \subset \Gamma\backslash G$ of full m_{BMS}-measure such that if $x \in X$ and $h_0 \in H$ are such that $xh_0 \in X$, then $\sigma(xh_0)$ is the push-forward of $\sigma(x)$ through left translation by h_0 in N,*

$$h \mapsto h_0 h.$$

3. *For m_{BMS}-almost every $x \in \Gamma\backslash G$, the origin of N belongs to the support of $\sigma(x)$.*

4. *For any $t \in \mathbf{R}$ and m_{BMS}-almost every $x \in \Gamma\backslash G$,*

$$\sigma(xa_t) = S_t\sigma(x)$$

i.e. *$\sigma(xa_t)$ is the push-forward of $\sigma(x)$ through the ghomothety $S_t : N \to N$.*

5. *For any $m \in M$, and m_{BMS}-almost every $x \in \Gamma\backslash G$, $\sigma(xm)$ is the push-forward of $\sigma(x)$ through the mapping $h \mapsto mhm^{-1}$. (Recall that the operation of M by conjugation on N identifies with the canonical operation of $\mathbf{SO}(n)$ on $\mathbf{R}^n$.)*

6. *For m_{BMS}-almost every $x \in \Gamma\backslash G$ and $\sigma(x)$-almost every $h \in N$,*

$$0 < \liminf_{\rho\to 0} \frac{\sigma(x)(B(h,\rho))}{\rho^{\delta_\Gamma}} \leq \limsup_{\rho\to 0} \frac{\sigma(x)(B(h,\rho))}{\rho^{\delta_\Gamma}} < \infty.$$

Proof. Statements 1, 2 and 3 are clear. Statement 4 holds because of invariance of m_{BMS} with respect to the geodesic flow and formula (1). Statement 5 holds because m_{BMS} is M-invariant by definition. Statement 6 holds because Γ is convex-cocompact and $\sigma(x)$ is equivalent to the Patterson-Sullivan measure; see [1], Proposition 7.4 and [12], section 3.9 □

Notation. *If μ is a Borel measure or projective measure on N, the support of which contains the origin on N, we let*

$$\mu^* = \frac{\mu}{\mu(B_1)}$$

i.e. *μ^* is the measure colinear to μ that gives measure 1 to the unit ball B_1.*

We also denote by $S_t^*\mu$ the measure $(S_t\mu)^*$.

In particular, since for m_{BMS}-almost every $x \in \Gamma\backslash G$, the origin of N belongs to the support of $\sigma(x)$, we denote by $\sigma^*(x)$ the Radon measure on N that belongs to the projective class $\sigma(x)$ and such that the unit ball $B_1 \subset N$ has measure 1:

$$\sigma^*(x)(B_1) = 1.$$

We denote by $\mathrm{Dirac}(x)$ the Dirac mass at x, *i.e.* the probability measure giving measure 1 to $\{x\}$. Associated to m_{BMS} is the following probability measure on the space of Radon measures on N:

$$P = \int_{\Gamma\backslash G} \mathrm{d}m_{\mathrm{BMS}}(x)\ \mathrm{Dirac}(\sigma^*(x)). \tag{2}$$

Recall that we assume that Γ is Zariski-dense and has finite BMS measure, so that P is an Ergodic Fractal Distribution (EFD) in the sense of Hochman (see [5], Definition 1.2, and [4], Lemma 5.3 for a proof that P is indeed an EFD).

2.2 Unrectifiability of the limit set

Recall that a Radon measure μ on the Euclidean space $\mathbf{R}^n$ is said to be purely m-unrectifiable if for any Lipschitz mapping $f : \mathbf{R}^m \to \mathbf{R}^n$, the range $f(\mathbf{R}^m)$ has measure zero with respect to μ.

Assume that the growth exponent δ_Γ is an integer $< n$. The fact that the limit set of Γ is purely δ_Γ-unrectifiable when Γ is convex-cocompact and Zariski-dense (the latter hypothesis is obviously necessary) is probably well-known, and certainly very intuitive. We give a full proof of this fact as it is pivotal in our argument.

Proposition 3. *Assume that Γ is convex-cocompact and Zariski-dense. If δ_Γ is an integer strictly smaller than n, the conditional measure $\sigma(x)$ is almost surely purely δ_Γ-unrectifiable.*

Proof. Let Ω be the set of all $x \in \Gamma\backslash G$ such that

$$\frac{1}{T}\int_0^T \mathrm{Dirac}(S_t^*\sigma(x))\mathrm{d}t$$

converges weakly to P (recall equation (2)) as $T \to +\infty$. This set has full BMS measure ([4], Lemma 5.4). Now fix some $x_0 \in \Omega$ such that for $\sigma(x_0)$-almost every $h \in N$, $x_0 h \in \Omega$ (see Lemma 2).

We argue by contradiction. Assume that some subset $L \in N$ is the image of a Lipschitz mapping $\mathbf{R}^{\delta_\Gamma} \to N$ and satisfies

$$\sigma(x_0)(L) > 0.$$

Note that the restriction $\sigma(x_0)|L$, which we denote by $\sigma_L(x_0)$, is δ_Γ-rectifiable, and satisfies

$$0 < \liminf_{\rho\to 0} \frac{\sigma_L(x_0)(B(h,\rho))}{\rho^{\delta_\Gamma}} \leq \limsup_{\rho\to 0} \frac{\sigma_L(x_0)(B(h,\rho))}{\rho^{\delta_\Gamma}} < \infty$$

for $\sigma_L(x_0)$-almost every h (Lemma 2). By virtue of [8], Theorem 16.7 and Lemma 14.5, for $\sigma_L(x_0)$-almost every h, there is a δ_Γ-plane $V(h)$ such that

$$S_t^*\sigma(x_0 h)$$

converges weakly to the Haar measure on $V(h)$ as $t \to \infty$

Recall that for $\sigma(x_0)$-almost every h,

$$\frac{1}{T}\int_0^T \mathrm{Dirac}(S_t^*\sigma(x_0 h))\ \mathrm{d}t$$

also converges weakly to P as T goes to infinity.

We thus see that P-almost every μ is the Haar measure on some δ_Γ-plane. In other words, for m_{BMS}-almost every x the conditional measure at x, $\sigma(x)$, is concentrated on some δ_Γ-plane of N; this contradicts the fact that the support of $\sigma(x)$ must be Zariski-dense, since Γ is Zariski-dense. Hence the proposition. ∎

Corollary 4. *Under the same hypotheses, the limit set Λ_Γ is purely δ_Γ-unrectifiable.*

Recall that the limit set Λ_Γ is the set of accumulation points of Γ in $\mathbf{H}^{n+1} \cup \partial \mathbf{H}^{n+1}$. It is locally bilipschitz equivalent to the support of $\sigma(x)$ for m_{BMS}-almost every x, so that the corollary follows readily from the proposition.

2.3 The conditional measures are transversally singular

Proposition 5. *Assume that Γ is Zariski-dense and convex-cocompact and that $\delta_\Gamma = n - m$ where m is an integer, $1 \leq m \leq n-1$. Fix some m-plane U in N.*

For m_{BMS}-almost every $x \in \Gamma \backslash G$, the push-forward of the conditional measure $\sigma(x)$ through the canonical projection $N \to N/U$ is singular with respect to the Lebesgue measure on N/U.

Recall that a measure μ is singular with respect to a measure ν if it gives full measure to a ν-negligible set.

Proof. For any m-plane V, denote by π_V the canonical projection $N \to N/V$.

We will show that there exists an m-plane U_0 such that for almost every x, the push-forward of $\sigma(x)$ through $N \to N/U_0$ is singular with respect to the Lebesgue measure on N/U. Since the BMS measure is M-invariant, this implies that the same statement holds for any other m-plane U.

According to Lemma 6 and the previous Propostion, for m_{BMS}-almost every x there is a sequence of Borel sets $(A_k)_k$ such that

- $\cup_k A_k$ has full $\sigma(x)$-measure,
- each A_k has finite $(n-m)$-dimensional Hausdorff measure,
- and each A_k is purely $(n-m)$-unrectifiable.

By virtue of the Besicovitch-Federer projection theorem ([8], Theorem 18.1 (2)), the image of $\cup_k A_k$ in N/V is Lebesgue-negligible for almost every m-plane V (with respect to the Haar measure on the Grassmannian of m-planes in N). This shows that for almost every m-plane V, the push-forward of $\sigma(x)$ through π_V is singular with respect to the Lebesgue measure.

This holds for almost every x. A standard application of Fubini's theorem now yields that there exists an m-plane U_0 such that for almost every x, the push-forward of $\sigma(x)$ through π_{U_0} is singular with respect to the Lebesgue measure. The proposition is thus proved. ☐

Lemma 6. *Assume that Γ is convex-cocompact. For m_{BMS}-almost every $x \in \Gamma \backslash G$, $\sigma(x)$ is supported by a countable union of δ_Γ-sets.*

Recall that E is a δ-set if its δ-dimensional Hausdorff measure is finite and non-zero.

Proof. It is well-known (see [15], Theorem 7) that the limit set Λ_Γ is a δ_Γ-set. Since it is (almost surely) locally bilipschitz-equivalent to the support of $\sigma(x)$, the lemma follows. ☐

2.4 Conditional measure of strips

If U is any m-plane in N ($1 \leq m \leq n-1$), we denote by $B^T_\rho(U)$ the ρ-neighbourhood of U in N, that is the set of all $h \in N$ such that

$$d(h, U) < \rho.$$

When it is clear from the context which m-plane we are talking about, we dispense ourselves with the letter U in the notation.

Proposition 7. *Assume that Γ is convex-cocompact and Zariski-dense and that $\delta_\Gamma = n - m$ where m is an integer, $1 \leq m \leq n-1$. Fix some m-plane U in N. For m_{BMS}-almost every $x \in \Gamma\backslash G$ and any $\rho > 0$,*

$$\sigma(x)(B^T_\rho) = \infty.$$

Proof. It is enough to show that for any $\rho > 0$, and almost every $x \in \Gamma\backslash G$, $\sigma(x)(B^T_\rho) = \infty$ (see lemma 2). We argue by contradiction and assume that the set of those x such that

$$\sigma(x)(B^T_\rho) < \infty$$

has positive BMS measure; it must then have full measure since m_{BMS} is mixing and because of Lemma 2.4.

It is easy to see then that for m_{BMS}-almost every $x \in \Gamma\backslash G$,

$$\sigma(x)(B^T_\rho) < \infty$$

for any $\rho > 0$.

This implies that the push-forward of $\sigma(x)$ through the projection $\pi_U : N \to N/U$ is a projective *Radon* measure.

Now consider the distribution

$$P^T = \int \mathrm{d}m(x)\ \mathrm{Dirac}((\pi_U \sigma(x))^*)$$

on the space of Radon measures on N/U. It is straight-forward to check that P^T is an Ergodic Fractal Distribution (see [4], Lemma 5.3). Since P^T has dimension $n - m$ (see [4], Theorem 4.1) this is possible only if

$$P^T = \mathrm{Dirac}(\mathrm{Haar}_{N/U})$$

i.e. P^T is the Dirac mass at the Haar measure of N/U.

We are using the fact that a Fractal Distribution of dimension d on some Euclidean space $\mathbf{R}^d$ has to be the only one we can think of, *i.e.* $\mathrm{Dirac}(\mathrm{Haar}_{\mathbf{R}^d})$. In essence, this fact goes back to Ledrappier-Young ([6], Corollary G). In the setting of Fractal Distributions it was proved by Hochman in [5], Proposition 6.4 (see also [7]).

Now we end up with the conclusion that for m_{BMS}-almost every $x \in \Gamma\backslash G$, the push-forward of $\sigma(x)$ through π_U is the Haar measure on N/U; this contradicts Proposition 5. Hence the proposition is proved. □

Remark. *Propositions 3, 5 and 7 admit obvious counter-examples when Γ is not Zariski-dense: take some lattice $\Gamma \subset \mathbf{SO}^o(1, m+1)$ and look at the image of Γ through the embedding*

$$\mathbf{SO}^o(1, m+1) \to \mathbf{SO}^o(1, n+1).$$

2.5 Recurrence of the Burger-Roblin measure

We are now ready to prove our main theorem. We use the following consequence of proposition 7.

Lemma 8. *Assume that Γ is Zariski-dense and convex-cocompact and thatand $\delta_\Gamma = n - m$. Fix an m-plane U in N. Let Ω_Γ be the support of the Bowen-Margulis-Sullivan measure in $\Gamma\backslash G$. For almost every $x \in \Gamma\backslash G$, and any $\rho > 0$, the set of all $h \in B_\rho^T(U)$ such that $xh \in \Omega_\Gamma$ is unbounded.*

Proof. By construction of the disintegration mapping σ, the support of $\sigma(x)$, $\mathrm{supp}(\sigma(x))$, is almost surely the set of all $h \in N$ such that xh belongs to Ω_Γ. Since the *Radon* measure $\sigma(x)$ gives infinite measure to $B_\rho^T(U)$, the intersection $B_\rho^T \cap \mathrm{supp}(\sigma(x))$ must be unbounded; hence the lemma. □

Proposition 9. *Assume that Γ is Zariski-dense and convex-cocompact and that $\delta_\Gamma = n - m$. Fix some m-plane U in N. For BMS-almost every x, there is a compact $K \subset \Gamma\backslash G$ such that*

$$\int_U \mathbf{1}_K(xu)\mathrm{d}u = \infty.$$

Furthermore, if W is any neighbourhood of Ω_Γ, K may be chosen inside W.

Of course U is endowed with the Haar measur in this formula.

Proof. First of all, recall that Ω_Γ is a compact subset of $\Gamma\backslash G$ since Γ is convex-cocompact.

For any $\rho > 0$, let K_ρ be the set of all xh where $x \in \Omega_\Gamma$ and h belongs to the closed ρ-ball centered at the origin in N. This is again a compact set. If ρ is small enough, K_ρ is a subset of W. Fix such a ρ.

By lemma 8, we may find a sequence $(h_k)_k$ of elements of $B_\rho^T(U)$ that goes to infinity and such that $xh_k \in \Omega_\Gamma$ for any k; if we let $h_k = u_k v_k$ where $u_k \in U$ and v_k is orthogonal to U, we have

$$xu_k \in K_\rho$$

for any k, and the sequence $(u_k)_k$ goes to infinity.

According to lemma 11, we may thicken K_ρ to get a compact set $K \subset W$, such that the conclusion of the proposition holds. □

Remark. *It is necessary to consider a compact set K that is slightly bigger than Ω_Γ in this lemma, since by virtue of Proposition 3, one has*

$$\int_U \mathbf{1}_{\Omega_\Gamma}(xu)\mathrm{d}u = 0$$

for BMS-almost every x.

Corollary 10. *Under the same hypothesis, for BR-almost every x there is a compact K such that*

$$\int_U \mathbf{1}_K(xu)\mathrm{d}u = \infty.$$

In particular, the BR measure is recurrent with respect to U.

Proof. The set of all $x \in \Gamma\backslash G$ that satisfy the conclusion is obviously N-invariant; since it has full BMS measure, it must have full BR measure as well. □

The following lemma is well-known but I have not been able to pinpoint a proof in the literature. We need it only when G is some $\mathbf{R}^m$ but there is no reason not to prove it in full generality.

Lemma 11. *Let X be some second countable locally compact space where a second countable locally compact topological group G acts continuously. Assume that we are given some fixed $x_0 \in X$ and a sequence $(g_n)_n$ in G that goes to infinity, such that $g_n x_0$ belongs to a fixed compact subset K for every n. Then for any neighbourhood W of K, there is a compact subset L of W such that*

$$\int \mathbf{1}_K(gx_0)\ \mathrm{d}g = \infty.$$

Here G is endowed with some *right-invariant* Haar measure.

Proof. Endow X with some compatible metric; endow G with some compatible metric that is also right invariant and proper (which means that closed balls are compact), see [14].

Fix some $\delta > 0$ small enough that the set

$$L = \{x \in X\ ;\ d(x, K) \leq \delta\}$$

is compact and contained in W.

For any $n \geq 1$, let ε_n be the lower bound of the set of all $\varepsilon > 0$ for which the closed ball $B(g_n, \varepsilon)$ contains some h such that $d(g_n x_0, hx_0) = \delta$. If there is no such ε, then the whole orbit Gx_0 is contained in L and the proof is over. We may thus assume that $\varepsilon_n < \infty$ for every n. It is clear also that $\varepsilon_n > 0$.

We now prove that $\inf_n \varepsilon_n > 0$. The mapping from $B_1 \times K$ to $\mathbf{R}$ (where B_1 is the closed unit ball in G)

$$(g, y) \mapsto d(y, gy)$$

is uniformly continuous because it is continuous and $B_1 \times K$ is compact. In particular there is some $\eta \in]0, 1[$ such that the relation $d(g, e) < \eta$ (where e is the unit of G) implies

$$d(gy, y) < \delta$$

for any $y \in K$.

We are going to show that $\varepsilon_n \geq \eta$ for any n. Let h be any element of G such that $d(g_n, h) < \eta$; then $d(hg_n^{-1}, e) < \eta$ (because the distance on G is right-invariant) which implies

$$d(hx_0, g_n x_0) = d(hg_n^{-1} g_n x_0, g_n x_0) < \delta.$$

By definition of ε_n, this is means that $\varepsilon_n \geq \eta$. Hence $\inf_n \varepsilon_n > 0$.

Now pick some positive ε smaller than $\inf_n \varepsilon_n$. If $h \in G$ is such that $d(h, g_n) \leq \frac{\varepsilon}{2}$, then $d(hx_0, gx_0) < \delta$, so that $hx_0 \in L$.

This shows that the orbital mapping $\rho_{x_0} : g \mapsto gx_0$ maps each $B(g_n, \varepsilon/2)$ inside L. As g goes to infinity in G, we may, passing to a subsequence, assume that these balls are pair-wise disjoint. Their union has infinite Haar measure because the metric on G is right-invariant. Whence the lemma. $\square$

2.6 Return rate

In the following proposition we let B_R be the R-ball centered at the origin in N and as previously B_ρ^T is the ρ-neighbourhood of the m-plane U in N.

We do not assume that Γ is convex-cocompact nor that δ_Γ is an integer.

Proposition 12. *Assume that Γ has finite BMS measure and is Zariski-dense. Let m be some integer, $1 \leq m \leq n-1$. Fix an m-plane U in N. For all $\rho > 0$ and almost every $x \in \Gamma \backslash G$,*

$$\liminf_{R \to \infty} \frac{\log \sigma(x)(B_\rho^T \cap B_R)}{\log R} \leq \sup\{0, \delta_\Gamma - (n-m)\}.$$

Remark. *It is not clear whether one should expect the lower limit in this proposition to be a genuine limit.*

Proof. Recall the following:

- for almost every x and every fixed $R > 0$,

$$\lim_{\rho \to 0} \frac{\log \sigma(x)(B_\rho^T \cap B_R)}{\log \rho} = \inf\{n-m, \delta_\Gamma\}$$

- for almost every x,

$$\lim_{R \to +\infty} \frac{\log \sigma(x)(B_R)}{\log R} = \delta_\Gamma.$$

The first limit comes from the fact that the projection of $\sigma(x)|B_R$ onto N/U has exact dimension $\inf\{n-m, \delta_\Gamma\}$ (see [4], Theorem 4.1). The second limit holds because m_{BMS} is ergodic with respect to the automorphism a_t for $t > 0$ as well as for $t < 0$; thus,

$$\lim_{R \to \infty} \frac{\log \sigma(x)(B_R)}{\log R} = \lim_{r \to 0} \frac{\log \sigma(x)(B_r)}{\log r} = \delta_\Gamma$$

see [2], Lemme 2.2.1.

Let us denote by θ the number $\inf\{n-m, \delta_\Gamma\}$. Fix some $\varepsilon > 0$.

For m_{BMS}-almost every x, there is some $\rho_0(x) > 0$ such that the relation $\rho \leq \rho_0(x)$ implies that

$$\sigma^*(x)(B_\rho^T \cap B_1) \leq \rho^{\theta - \varepsilon}.$$

Choose $\rho_0 > 0$ small enough that the set E_{ρ_0} of all x such that $\rho_0(x) > \rho_0$ has positive BMS measure. Let $a = a_t$ for some $t > 0$.

For m_{BMS}-almost every x, one can find arbitrarily big integers k such that $a^k x \in E_{\rho_0}$ (because m_{BMS} is a-ergodic). If k is such an integer, we have

$$\frac{\sigma^*(x)(B_{e^k \rho}^T \cap B_{e^k})}{\sigma^*(x)(B_{e^k})} \leq \rho^{\theta - \varepsilon}$$

for any $\rho \leq \rho_0$ (Lemma 2.4).

Assume, furthermore, that k is so large that $\sigma^*(x)(B_{e^k}) \leq e^{k(\delta_\Gamma + \varepsilon)}$, and that $e^{-k} < \rho_0$. Letting $\rho = e^{-k}$, we get

$$\sigma^*(x)(B_1^T \cap B_{e^k}) \leq e^{-k(\theta - \varepsilon)} e^{k(\delta_\Gamma + \varepsilon)} = e^{k(\delta_\Gamma - \theta + 2\varepsilon)}.$$

Since k can be as large as we like, this shows that

$$\liminf_{k \to \infty} \frac{\log \sigma(x)(B_1^T \cap B_{e^k})}{k} \leq \sup\{0, \delta_\Gamma - (n-m)\} + 2\varepsilon$$

for any $\varepsilon > 0$. The lemma follows.

$\square$

Corollary 13. *Assume that Γ is convex-cocompact and Zariski-dense. Let m be an integer, $1 \leq m \leq n-1$. For any m-plane U in N, and any compact K in $\Gamma\backslash G$,*

$$\liminf_{R\to\infty} \frac{\log\left(\mathrm{Haar}_U(\{u \in B_R\ ;\ xu \in K\})\right)}{\log R} \leq \sup\{0, \delta_\Gamma - (n-m)\}$$

for m_{BMS}-almost every x and also for m_{BR}-almost every x.

We skip the straight-forward proof.

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