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# POINCAÉ TYPE INEQUALITIES FOR VECTOR FUNCTIONS WITH ZERO MEAN NORMAL TRACES ON THE BOUNDARY AND APPLICATIONS TO INTERPOLATION METHODS

S. REPIN

*Dedicated to Professor Yuri Kuznetsov on the occasion of his 70th birthday*

ABSTRACT. In the paper, we consider inequalities of the Poincaré–Steklov type for subspaces of  $H^1$ -functions defined in a bounded domain  $\Omega \in \mathbb{R}^d$  with Lipschitz boundary  $\partial\Omega$ . For scalar valued functions, the subspaces are defined by zero mean condition on  $\partial\Omega$  or on a part of  $\partial\Omega$  having positive  $d - 1$  measure. For vector valued functions, zero mean conditions are imposed on components (e.g., normal components) of the function on certain  $d - 1$  dimensional manifolds (e.g., on plane or curvilinear faces of  $\partial\Omega$ ). We find explicit and simply computable bounds of the respective constants for domains typically used in finite element methods (triangles, quadrilaterals, tetrahedrons, prisms, pyramids, and domains composed of them). The second part of the paper discusses applications of the estimates to interpolation of scalar and vector valued functions.

Key words: Poincaré type inequalities, interpolation of functions, estimates of constants in functional inequalities

## 1. INTRODUCTION

**1.1. Classical Poincaré inequality.** H. POINCARÉ [22] proved that  $L^2$  norms of functions with zero mean defined in a bounded domain  $\Omega$  with smooth boundary  $\partial\Omega$  are uniformly bounded by the  $L^2$  norm of the gradient, i.e.,

$$(1.1) \quad \|w\|_{2,\Omega} \leq C_P(\Omega) \|\nabla w\|_{2,\Omega}, \quad \forall w \in \tilde{H}^1(\Omega),$$

where

$$\tilde{H}^1(\Omega) := \left\{ w \in H^1(\Omega) \mid \{w\}_\Omega := \frac{1}{|\Omega|} \int_\Omega w \, dx = 0 \right\}.$$

Poincaré also deduced the very first estimates of  $C_P$ :

$$(1.2) \quad C_P(\Omega) \leq \frac{3}{4}d_\Omega, \quad d_\Omega := \text{diam } \Omega \quad \text{for } d = 3$$

$$(1.3) \quad C_P(\Omega) \leq \sqrt{\frac{7}{24}}d_\Omega \approx 0.5401d_\Omega \quad \text{for } d = 2.$$

For piecewise smooth domains the inequality (1.1) (and a similar inequality for functions vanishing on the boundary) was independently established by

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V. STEKLOV [28], who proved that  $C_P = \lambda^{-\frac{1}{2}}$ , where  $\lambda$  is the smallest positive eigenvalue of the problem

$$(1.4) \quad -\Delta u = \lambda u \quad \text{in } \Omega;$$

$$(1.5) \quad \partial_{\mathbf{n}} u = 0 \quad \text{on } \partial\Omega.$$

Easily computable estimates of  $C_P(\Omega)$  are known for *convex domains* in  $\mathbb{R}^d$ . An upper bound

$$(1.6) \quad C_P(\Omega) \leq \frac{d_\Omega}{\pi} \approx 0.3183 d_\Omega$$

was established in L. E. PAYNE and H. F. WEINBERGER [23] (notice that for  $d = 2$  the upper bound (1.3) found by Poincaré is not far from the sharp estimate (1.6)).

A lower bound of  $C_P(\Omega)$  was derived in S. Y. CHENG [6] (for  $d = 2$ ):

$$(1.7) \quad C_P(\Omega) \geq \frac{d_\Omega}{2j_{0,1}} \approx 0.2079 d_\Omega.$$

Here  $j_{0,1} \approx 2.4048$  is the smallest positive root of the Bessel function  $J_0$ .

For *isosceles triangles* an improvement of the upper bound (1.6) is presented in R. S. LAUGESEN and B. A. SIUDEJA [17]

$$(1.8) \quad C_P(\Omega) \leq \frac{d_\Omega}{j_{1,1}},$$

where  $j_{1,1} \approx 3.8317$  is the smallest positive root of the Bessel function  $J_1$ .

Poincaré type inequalities also hold for  $L^q$  norms if  $1 \leq q < +\infty$ . In G. ACOSTA AND R. DURAN (2003), it was shown that for convex domains the constant in  $L_1$  Poincaré type inequality satisfies the estimate

$$(1.9) \quad \inf_{c \in \mathbb{R}} \|w - c\|_{L_1} \leq \frac{d_\Omega}{2} \|\nabla w\|_{L_1}.$$

Estimates of the constant for other  $q$  can be found in S.-K. SHUA and R. L. WHEEDEN (2006) (also for convex domains).

**1.2. Poincaré type inequalities for functions with zero mean boundary traces.** Inequalities similar to (1.1) also hold for functions with zero mean traces on the boundary (or on a measurable part  $\Gamma \subset \partial\Omega$ ) such that  $\text{meas}_{(d-1)}\Gamma > 0$ . For any

$$w \in \tilde{H}_\Gamma^1(\Omega) = \left\{ w \in H^1(\Omega) \mid \{w\}_\Gamma := \frac{1}{|\Gamma|} \int_\Gamma w ds = 0 \right\},$$

we have two estimates for the  $L^2(\Omega)$  norm of  $w$

$$(1.10) \quad \|w\|_{2,\Omega} \leq C_\Gamma(\Omega) \|\nabla w\|_{2,\Omega}$$

and for its trace on the  $\Gamma$

$$(1.11) \quad \|w\|_{2,\Gamma} \leq C_\Gamma^{\text{Tr}}(\Omega) \|\nabla w\|_{2,\Omega}.$$

Existence of positive constants  $C_\Gamma(\Omega)$  and  $C_\Gamma^{\text{Tr}}(\Omega)$  is proved by standard compactness arguments. Inequality (1.10) arises in analysis of certain physical phenomena (the so called "sloshing" frequencies, see D. W. FOX and J. R. KUTTLER [8], V. KOZLOV et al. [9, 10] and references therein). In the paper by I. BABUSKA and A. K. AZIZ [3] it was used in proving sufficiency

of the maximal angle condition for finite element meshes with triangular elements. Inequalities (1.10) and (1.11) can be useful in many other cases, e.g., for nonconforming approximations, a posteriori error estimates (see [19, 26, 18, 24]), and advanced interpolation methods for scalar and vector valued functions. In this paper, we are mainly interested in the inequality (1.10) for functions with zero mean on  $\Gamma$ . For the sake of brevity, we will call it the *boundary Poincaré inequality*.

Exact constants  $C_\Gamma$  and  $C_\Gamma^{\text{Tr}}$  are known only for a restricted number of "simple" domains. Table 1 summarises some of the results presented in A. NAZAROV AND S. REPIN [21], which are related to such domains as rectangle  $\Pi_{h_1 \times h_2} := (0, h_1) \times (0, h_2)$ , parallelepiped  $\Pi_{h_1 \times h_2 \times h_3} := (0, h_1) \times (0, h_2) \times (0, h_3)$ , and right triangle  $\bar{T}_h := \text{conv}\{(0, 0), (h, 0), (0, h)\}$ .

$d$	$\Omega$	$\Gamma$	$C_\Gamma(\Omega)$
2	$\Pi_{h_1 \times h_2}$	face $x_1 = 0$	$c_1 \max\{2h_1; h_2\}$ , $c_1 = 1/\pi$
2	$\Pi_{h_1 \times h_2}$	$\partial\Omega$	$c_1 \max\{h_1; h_2\}$
3	$\Pi_{h_1 \times h_2 \times h_3}$	face $x_1 = 0$	$c_1 \max\{2h_1; h_2; h_3\}$
2	$T_h$	leg	$c_2 h$ , $c_2 = 1/\zeta$ , $\zeta \approx 2.02876$
2	$T_h$	two legs	$c_1 h$
2	$T_h$	hypotenuse	$\sqrt{2}c_2 h$

TABLE 1. Sharp constants

In Section 2 we deduce easily computable majorants of  $C_\Gamma$  for *triangles, rectangles, tetrahedrons, polyhedrons, pyramids and prismatic type domains*. These results yield interpolation estimates (and respective constants) for interpolation of scalar valued functions on macrocells based on mean values on faces. As a result, we can deduce interpolation estimates for functions defined on meshes with very complicated (e.g., nonconvex) cells.

Section 3 is concerned with boundary Poincaré inequalities for vector valued functions. Certainly, (1.10) admits a straightforward extension to vector fields. We consider more sophisticated forms where zero mean conditions are imposed on mean values of different components of a vector valued function  $\mathbf{v}$  on different  $d - 1$  dimensional manifolds (which are assumed to be sufficiently regular). In particular, it suffices to impose zero mean conditions on normal components of  $\mathbf{v}$  on  $d$  Lipschitz manifolds (e.g., on  $d$  faces lying on  $\partial\Omega$ ). Then,

$$(1.12) \quad \|\mathbf{v}\|_\Omega \leq \mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d) \|\nabla \mathbf{v}\|_\Omega.$$

Theorem 3.1 proves (1.12) by compactness arguments. After that, we consider the case where the conditions are imposed on normal components of a vector field on  $d$  different faces of polygonal domains in  $\mathbb{R}^d$  and deduce (1.12) directly by applying (1.10) to normal components of the vector field. This method also yields easily computable majorants of the constant  $\mathbb{C}$ .

The last part of the paper is devoted to interpolation of functions defined in a bounded Lipschitz domain  $\Omega \in \mathbb{R}^d$ , which are based on mean values of the function (or of mean values of normal components) on some set  $\Gamma \in \mathbb{R}^{d-1}$ . It should be noted that interpolation methods based on normal

components of vector fields defined on edges of finite elements are widely used in numerical analysis of PDEs (see, e.g., [5, 27]). Raviart–Thomas (RT) type interpolation operators and their properties for approximations on polyhedral meshes has been deeply studied in papers of D. ARNOLD, D. BOFFI AND R. FALK [1, 2], A. BERMUDEZ et. all [4] and other publications. The respective interpolants belong to the space  $H(\Omega, \text{div})$ . Approximations of this type are often used in mixed and hybrid finite element methods (see, e.g., F. BREZZI and M. FORTIN [5], J. E. ROBERTS and J.-M. THOMAS [27], V. GIRAULT and P. A. RAVIART [7]).

This paper is concerned with coarser interpolation methods, which provide only  $L^2$  approximation of fluxes (and  $H^{-1}$  approximation for the divergence what is sufficient for treating balance equations in a weak sense!). Hopefully this type interpolation methods could be useful for numerical analysis of PDEs on highly distorted meshes. This challenging problem has been studying for many years by YU. KUZNETSOV and coauthors (see [11, 12, 13, 14, 15, 16] and other publications cited therein). Smooth (high order) methods are probably too difficult for the interpolation of vector valued functions on very irregular (distorted) meshes. Moreover, in the majority of cases smooth interpolants seem to be not really natural because exact solutions often have a very restricted regularity and because efficient numerical procedures (offered, e.g., by the above mentioned dual mixed and hybrid methods) operate with low order approximations for fluxes. If meshes are very irregular, then it is convenient to apply approximations of the lowest possible order and respective numerical methods with minimal regularity requirements. Poincaré type estimates for functions with zero mean conditions on manifolds of the dimension  $d - 1$  yield interpolants of exactly this type.

In Section 4 it is proved that in  $\Omega$  the difference between  $u$  and its interpolant  $\mathbb{I}_\Gamma u$  is controlled by the norm of  $\nabla u$  with a constant, which depends on the maximal diameter of the cell (due to results of previous sections, realistic estimates the interpolation constants are known for "typical" cells). Finally, we shortly discuss interpolation on meshes when a (global) domain  $\Omega$  is decomposed into a collection of local subdomains (cells)  $\Omega_i$ . Using cell interpolation operators, we define the global interpolation operator  $\mathbb{I}_{\mathcal{T}_h}$  and prove the respective interpolation estimates for scalar and vector valued functions. The interpolation method operates with minimal amount of interpolation parameters related to mean values on a certain amount of faces and preserves mean values on faces (for scalar valued functions) and mean values of normal components (for vector valued functions).

## 2. ESTIMATES OF $C_\Gamma$ FOR TYPICAL MESH CELLS

**2.1. Triangles.** Consider a nondegenerate triangle ABC (Fig. 1 left) where  $\Gamma$  coincides with the side AC.

**2.1.1. Majorant of  $C_\Gamma$ .** Our analysis is based upon the estimate

$$(2.1) \quad C_\Gamma^2 \leq C_P^2 + \frac{|\Omega|}{|\Gamma|^2} \inf_{\tau \in Q(\Omega)} \|\boldsymbol{\tau}\|_{2,\Omega}^2,$$

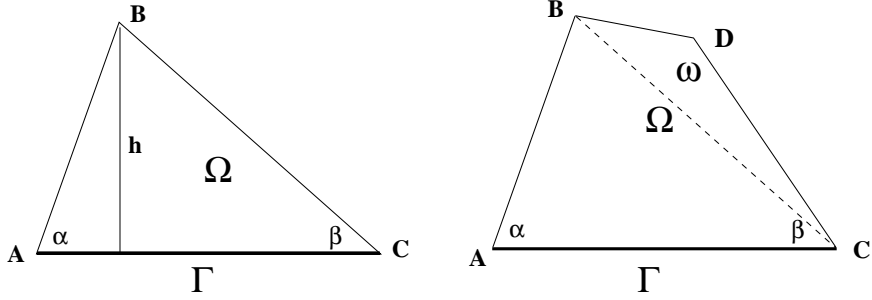


FIGURE 1. Triangle and quadrilateral

which is a special form of the upper bound of  $C_\Gamma$  derived in S. REPIN [25]. Here  $Q(\Omega)$  is a subset of  $H(\Omega, \text{div})$  containing vector functions such that  $\text{div} \boldsymbol{\tau} = \frac{|\Gamma|}{|\Omega|}$ ,  $\boldsymbol{\tau} \cdot \mathbf{n} = 1$  on  $\Gamma$ , and  $\boldsymbol{\tau} \cdot \mathbf{n} = 0$  on  $\partial\Omega \setminus \Gamma$ . We set  $\boldsymbol{\tau}$  as an affine field with values at the nodes A, B, and C  $(-\cot \alpha, -1)$ ,  $(0, 0)$ , and  $(\cot \beta, -1)$ , respectively. In this case,

$$\|\boldsymbol{\tau}\|_{2,\Omega}^2 = \frac{1}{3}|\Omega| \left( \frac{3}{2} + \frac{1}{4} \cot^2 \alpha + \frac{1}{4} \cot^2 \beta + \frac{1}{4} (\cot \beta - \cot \alpha)^2 \right) = \frac{|\Omega|}{6} \Sigma_{\alpha\beta}.$$

where

$$\Sigma_{\alpha\beta} = \cot^2 \alpha + \cot^2 \beta - \cot \alpha \cot \beta + 3.$$

Since  $|\Omega| = \frac{1}{2}h|\Gamma|$ , we see that  $\frac{|\Omega|^2}{|\Gamma|^2} = \frac{h^2}{4}$ . In view of (1.6), the constant  $C_P$  is bounded from above by  $\frac{d_\Omega}{\pi}$ , where  $d_\Omega = \max\{|AB|, |BC|, |CD|\}$ , and we deduce an easily computable bound

$$(2.2) \quad C_\Gamma^2 \leq C_P^2 + \frac{h^2 \Sigma_{\alpha\beta}}{24} \leq \frac{d_\Omega^2}{\pi^2} + \frac{h^2 \Sigma_{\alpha\beta}}{24}.$$

We can represent  $\Sigma_{\alpha\beta}$  in a somewhat different form

$$\Sigma_{\alpha\beta} = \frac{|AB|^2 + |BC|^2 + \vec{AB} \cdot \vec{BC}}{h^2},$$

which yields the estimate

$$(2.3) \quad C_\Gamma^2 \leq \frac{d_\Omega^2}{\pi^2} + \frac{|AB|^2 + |BC|^2 + \vec{BA} \cdot \vec{BC}}{24}.$$

Example. If  $\alpha = \frac{\pi}{2}$ , then  $d_\Omega^2 = h^2 + |\Gamma|^2$ ,  $|\Gamma| = h \cot \beta$ ,  $d_\Omega^2 = h^2(1 + \cot^2 \beta)$  and we obtain

$$C_\Gamma \leq h \sqrt{\frac{1}{\pi^2} + \frac{1}{8} + \cot^2 \beta \left( \frac{1}{\pi^2} + \frac{1}{24} \right)} \approx 0.4757 h \sqrt{1 + 0.6354 \cot^2 \beta}$$

In particular, for  $\beta = \frac{\pi}{4}$ , we obtain  $C_\Gamma \leq 0.6083h$  (exact constant for the riht triangle is 0.4929h).

2.1.2. *Minorant of  $C_\Gamma$ .* A lower bound for  $C_\Gamma$  follows from (1.7) and relations between  $C_P(\Omega)$  and  $C_\Gamma(\Omega)$ . Any function in  $\tilde{H}_\Gamma^1(\Omega)$  can be represented as  $w - \{w\}_\Gamma$ , where  $w \in H^1(\Omega)$ . Hence,

$$(C_\Gamma(\Omega))^{-2} = \inf_{w \in H^1(\Omega)} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega |w - \{w\}_\Gamma|^2 dx}$$

and the constant  $C_\Gamma(\Omega)$  can be defined as maximum of  $\|w - \{w\}_\Gamma\|_{2,\Omega}$  for all  $w \in H^1(\Omega)$  such that  $\|\nabla w\|_{2,\Omega} = 1$ . Analogously,  $C_P$  can be defined as maximum of  $\|w - \{w\}_\Omega\|_{2,\Omega}$  over the same set of functions. Since

$$\|w - \{w\}_\Gamma\|_{2,\Omega} \geq \inf_{c \in \mathbb{R}} \|w - c\|_{2,\Omega} = \|w - \{w\}_\Omega\|_{2,\Omega},$$

we conclude that for any selection of  $\Gamma$

$$(2.4) \quad C_P(\Omega) \leq C_\Gamma(\Omega).$$

From (1.7) and (2.4), it follows that  $C_\Gamma \geq \frac{1}{2} \frac{d_\Omega}{j_{0,1}}$ . In particular, for  $\alpha = \frac{\pi}{2}$  we have  $C_\Gamma \geq 0.2079 h \sqrt{1 + \cot^2 \beta}$ .

2.2. **Quadrilaterals.** Using previous results, we deduce an estimate of  $C_\Omega$  for a quadrilateral ABCD (Fig. 2.1 right). On  $\Omega_1$  we set the same field  $\tau$  as in the previous case and set  $\tau = 0$  on  $\Omega_2$ . Let  $\kappa^2 = \frac{|\Omega_2|}{|\Omega_1|}$ . Then,

$$(2.5) \quad C_\Gamma^2 \leq C_P^2 + \left( \kappa C_P + \frac{\Sigma_{\alpha\beta}^{1/2} |\Omega|}{\sqrt{6} |\Gamma|} \right)^2.$$

Note that (2.5) also holds for more general cases in which  $\Omega_2$  is a bounded Lipschitz domain having only one common boundary with  $\Omega_1$ , which is  $BC$ .

2.3. **Tetrahedrons.** Consider a tetrahedron OABC (Fig. 2 left), where  $\Gamma$  is the triangle ABC which lies in the plane  $Ox_1x_2$ .

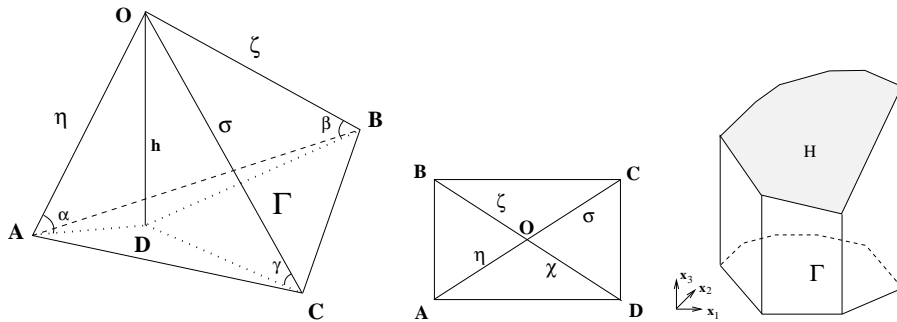


FIGURE 2. Tetrahedron, pyramide, and prizm

At vertexes A, B, and C, we define three constant vectors

$$\hat{\tau}_A = \frac{\eta}{|\eta| \sin \alpha}, \quad \hat{\tau}_B = \frac{\zeta}{|\zeta| \sin \beta}, \quad \text{and} \quad \hat{\tau}_C = \frac{\sigma}{|\sigma| \sin \gamma}.$$

The vector field  $\boldsymbol{\tau}(x_1, x_2, x_3)$  is the affine field in  $\Omega$  with zero value at the vertex O. We compute

$$\int_{\Omega} |\boldsymbol{\tau}|^2 dx = \int_0^h \left( \int_{\omega(x_3)} |\boldsymbol{\tau}(x_1, x_2, x_3)|^2 dx_1 dx_2 \right) dx_3.$$

Notice that the cross section  $\omega(x_3)$  associated with the height  $x_3$  has the measure  $|\omega(x_3)| = \left(1 - \frac{x_3}{h}\right)^2 |\Gamma|$  and at the respective point  $A'$  on  $OA$  (which third coordinate is  $x_3$ ) by linear proportion we have  $\boldsymbol{\tau}_{A'} = \left(1 - \frac{x_3}{h}\right) \widehat{\boldsymbol{\tau}}_A$ . Similar relations hold for the points  $B'$  and  $C'$  associated with the cross section on the height  $x_3$ . For the internal integral we apply the Gaussian quadrature for  $|\boldsymbol{\tau}|^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$  and obtain

$$(2.6) \quad C_{\Gamma}^2 \leq \frac{d_{\Omega}^2}{\pi^2} + \frac{|\boldsymbol{\eta}|^2 + |\boldsymbol{\zeta}|^2 + |\boldsymbol{\sigma}|^2 + \boldsymbol{\eta} \cdot \boldsymbol{\zeta} + \boldsymbol{\eta} \cdot \boldsymbol{\sigma} + \boldsymbol{\zeta} \cdot \boldsymbol{\sigma}}{90}$$

In particular, for the equilateral tetrahedron with all edges equal to  $h$  we have

$$\boldsymbol{\eta} \cdot \boldsymbol{\zeta} = \boldsymbol{\eta} \cdot \boldsymbol{\sigma} = \boldsymbol{\zeta} \cdot \boldsymbol{\sigma} = \frac{1}{2}h^2, \quad d_{\Omega} = h,$$

and, therefore,  $C_{\Gamma} \leq h\sqrt{\frac{1}{\pi^2} + \frac{1}{20}} \approx 0.39h$ .

For the right tetrahedron with nodes  $(0, 0, 0)$ ,  $(h, 0, 0)$ ,  $(0, h, 0)$ ,  $(0, 0, h)$  and face  $\Gamma = \{x \in \overline{\Omega}, x_3 = 0\}$ , we have  $d_{\Omega} = h\sqrt{2}$ ,  $|\boldsymbol{\eta}| = h$ ,  $|\boldsymbol{\zeta}| = |\boldsymbol{\sigma}| = h\sqrt{2}$ , scalar products are equal to  $h^2$  and (2.6) yields  $C_{\Gamma} \leq h\sqrt{\frac{2}{\pi^2} + \frac{4}{45}} \approx 0.54h$ . Sharp constants  $C_{\Gamma}$  for triangle and tetrahedrons has been recently evaluated in [20]. For the right tetrahedron, the constant computed in [20] numerically is  $C_{\Gamma} \approx 0.3756h$ .

**2.4. Pyramide.** We can apply (2.6) in order to evaluate  $C_{\Gamma}$  for a pyramid OABCD, which can be divided into two tetrahedrons OABC and OACD (Fig. 2 middle, view from above). Assume that the triangles ABC and ACD have equal areas and  $\Gamma$  is the pyramid basement ABCD. Then, we can use (2.1) with  $\boldsymbol{\tau}$  defined in each tetrahedron as in 2.3. We obtain

$$(2.7) \quad C_{\Gamma}^2 \leq \frac{d_{\Omega}^2}{\pi^2} + \frac{2|\boldsymbol{\eta}|^2 + |\boldsymbol{\zeta}|^2 + 2|\boldsymbol{\sigma}|^2 + |\boldsymbol{\chi}|^2 + 2\boldsymbol{\eta} \cdot \boldsymbol{\sigma} + (\boldsymbol{\eta} + \boldsymbol{\sigma}) \cdot (\boldsymbol{\chi} + \boldsymbol{\zeta})}{180}.$$

**2.5. Prismatic cells.** Consider domains of the form (Fig. 2 right).

$$\Omega = \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \Gamma, \quad 0 \leq x_3 \leq H(x_1, x_2), \quad H(x_1, x_2) \geq H_{min}\}.$$

By the same method as in 2.1 we find that

$$(2.8) \quad C_{\Gamma}^2 \leq \overline{C}_{\Gamma}^2 := C_{\text{P}}^2 + \left( \frac{\{H\}_{\Gamma}}{\sqrt{3}} + C_{\text{P}} \kappa \right)^2.$$

where  $\kappa = \left(\frac{\{H\}_{\Gamma}}{H_{min}} - 1\right)^{1/2}$  characterises variations of the mean height.

In particular, if  $H = \text{const}$  (so that  $\kappa = 0$ ) and  $\Gamma$  is a convex domain in  $\mathbb{R}^{d-1}$ , then

$$(2.9) \quad C_{\Gamma}^2 \leq \frac{d_{\Gamma}^2 + H^2}{\pi^2} + \frac{H^2}{3} = \frac{1}{\pi^2} \left( d_{\Gamma}^2 + \left(1 + \frac{\pi^2}{3}\right) H^2 \right).$$



For a parallelepiped with  $\Gamma = (0, a) \times (0, b)$ , we know that the exact value of  $C_\Gamma$  is  $\frac{1}{\pi} \max\{2H, a, b\}$ . In this case  $d_\Gamma^2 = a^2 + b^2$  and we can compare it with the upper bound that follows from (2.8):

$$(2.10) \quad \frac{\overline{C}_\Gamma}{C_\Gamma} = \frac{\sqrt{a^2 + b^2 + 4.29H^2}}{\max\{2H, a, b\}} \geq 1.$$

For the cases where one dimension of  $\Omega$  dominates,  $\overline{C}_\Gamma$  is a good approximation of  $C_\Gamma^2$ . If  $a = b = H$  (cube), then we have  $\frac{\overline{C}_\Gamma}{C_\Gamma} = \frac{\sqrt{6.29}}{2} \approx 1.25$ . The largest ratio is for  $a = b = 2H$  ( $\approx 1.75$ ).

### 3. BOUNDARY POINCARÉ INEQUALITIES FOR VECTOR VALUED FUNCTIONS

Estimates (1.10) and (1.11) yield analogous estimates for vector valued functions in  $H^1(\Omega, \mathbb{R}^d)$ . Let  $\Omega \in \mathbb{R}^d$  ( $d \geq 1$ ) be a connected domain with  $N$  plane faces  $\Gamma_i \in \mathbb{R}^{d-1}$ . Assume that we have  $d$  unit vectors  $\mathbf{n}^{(k)}$ , (associated with some faces) that form a linearly independent system in  $\mathbb{R}^d$ , i.e.,

$$(3.1) \quad \det \mathbf{N} \neq 0, \quad \mathbf{N} := \left\{ n_j^{(i)} \right\} \in \mathbb{M}^{d \times d}, \quad i, j = 1, 2, \dots, d,$$

where  $n_j^{(i)} = \mathbf{n}^{(i)} \cdot \mathbf{e}_j$  and  $\mathbf{e}_j$  denote the Cartesian ords. Then,  $\mathbf{v} \in H^1(\Omega, \mathbb{R}^d)$  satisfies a Poincaré type estimate provided that it satisfies zero mean conditions (3.2).

**Theorem 3.1.** *If (3.1) holds and*

$$(3.2) \quad \left\{ \mathbf{v} \cdot \mathbf{n}^{(i)} \right\}_{\Gamma_i} = 0 \quad i = 1, 2, \dots, d,$$

then

$$(3.3) \quad \|\mathbf{v}\|_\Omega \leq \mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d) \|\nabla \mathbf{v}\|_\Omega,$$

where  $\mathbb{C} > 0$  depends only on geometrical properties of the cell.

*Proof.* Assume the opposite. Then, there exists a sequence  $\{\mathbf{v}_k\}$  such that  $\left\{ \mathbf{v}_k \cdot \mathbf{n}^{(i)} \right\}_{\Gamma_i} = 0$  and

$$(3.4) \quad \|\mathbf{v}_k\| \geq k \|\nabla \mathbf{v}_k\|.$$

Without a loss of generality we can operate with a sequence of normalised functions, so that

$$(3.5) \quad \|\mathbf{v}_k\| = 1.$$

Hence,

$$(3.6) \quad \|\nabla \mathbf{v}_k\| \leq \frac{1}{k} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

We conclude that there exists a subsequence (for simplicity we omit additional subindexes and keep the notation  $\{\mathbf{v}_k\}$ ) such that

$$(3.7) \quad \mathbf{v}_k \rightharpoonup \mathbf{w} \quad \text{in } H^1(\Omega, \mathbb{R}^d),$$

$$(3.8) \quad \mathbf{v}_k \rightarrow \mathbf{w} \quad \text{in } L^2(\Omega, \mathbb{R}^d).$$

In view of (3.7),

$$0 = \liminf_{k \rightarrow +\infty} \|\nabla \mathbf{v}_k\| \geq \|\nabla \mathbf{w}\|,$$

we see that  $\mathbf{w} \in P^0(\Omega, \mathbb{R}^d)$ . For any face  $\Gamma_i$  we have (in view of the trace theorem)

$$(3.9) \quad \|\mathbf{v}_k - \mathbf{w}\|_{2, \Gamma_i} \leq C (\|\mathbf{v}_k - \mathbf{w}\|_{2, \Omega} + \|\nabla \mathbf{v}_k\|_{2, \Omega}).$$

We recall (3.6) and (3.8) and conclude that the traces of  $\mathbf{v}_k$  on  $\Gamma_i$  converge to the trace of  $\mathbf{w}$ . Since  $\mathbf{v}_k \cdot \mathbf{n}^{(i)}$  have zero means,

$$(3.10) \quad \mathbf{w} \cdot \mathbf{n}_i |\Gamma_i| = \int_{\Gamma_i} \mathbf{w} \cdot \mathbf{n}_i d\Gamma = 0$$

and  $\mathbf{w}$  is orthogonal to  $d$  linearly independent vectors, i.e.,  $\mathbf{w} = 0$ . On the other hand,  $\|\mathbf{w}\| = 1$ . We obtain a contradiction, which shows that the assumption is not true.  $\square$

We notice that conditions of the Theorem are very flexible with respect to choosing  $\Gamma_i$  and vectors  $\mathbf{n}^{(i)}$  entering the integral type conditions (3.2). Probably the most interesting case is where  $\mathbf{n}^{(i)}$  are defined as unit outward normals to faces  $\Gamma_s$ . If  $d = 2$ , then we can also define  $\mathbf{n}^{(i)}$  as unit tangential vectors. Moreover, in the proof it is not essential that  $\mathbf{n}^{(i)}$  is strictly related to one face  $\Gamma_i$  (only the condition (3.1) is essential). For example, if  $d = 3$  then we can define two vectors as two mutually orthogonal tangential vectors of one face and the third one as a normal vector to another face. Theorem holds for this case as well. Henceforth, for the sake of definiteness we assume that  $\mathbf{n}^{(i)}$  are normal vectors or mean normal vectors (for curvilinear faces) associated with faces  $\Gamma_i$ ,  $i = 1, 2, \dots, d$ . Possible modifications of the results to other cases are rather obvious.

**3.1. Value of the constant for  $d = 2$ .** Estimates of the constant  $\mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d)$  follow from (1.10) and depend on the constants  $C_{\Gamma_i}(\Omega)$ . Now, our goal is to deduce explicit and easily computable bounds of  $\mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d)$ .

First, we consider a special, but important case where  $\Omega$  is a polygonal domain in  $\mathbb{R}^2$ . Let  $\Gamma_1$  and  $\Gamma_2$  be two faces selected for the interpolation of  $\mathbf{v}$ . The respective normals  $\mathbf{n}^{(1)} = (n_1^{(1)}, n_2^{(1)})$  and  $\mathbf{n}^{(2)} = (n_1^{(2)}, n_2^{(2)})$  must satisfy the condition (3.1), which means that

$$(3.11) \quad \angle(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}) = \beta \in (0, \pi).$$

Let the conditions (3.2) hold. Then

$$(3.12) \quad \|n_1^{(1)} v_1 + n_2^{(1)} v_2\|^2 \leq C_{\Gamma_1}^2(\Omega) \|n_1^{(1)} \nabla v_1 + n_2^{(1)} \nabla v_2\|^2,$$

$$(3.13) \quad \|n_1^{(2)} v_1 + n_2^{(2)} v_2\|^2 \leq C_{\Gamma_2}^2(\Omega) \|n_1^{(2)} \nabla v_1 + n_2^{(2)} \nabla v_2\|^2.$$

Introduce the matrix

$$\mathbf{T} := \mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + \mathbf{n}^{(2)} \otimes \mathbf{n}^{(2)} = \begin{pmatrix} (n_1^{(1)})^2 + (n_1^{(2)})^2 & n_1^{(1)} n_2^{(1)} + n_1^{(2)} n_2^{(2)} \\ n_1^{(1)} n_2^{(1)} + n_1^{(2)} n_2^{(2)} & (n_2^{(1)})^2 + (n_2^{(2)})^2 \end{pmatrix}.$$

Here and later on  $\otimes$  denotes the diadic product of vectors. Summation of (3.12) and (3.13) yields

$$(3.14) \quad \int_{\Omega} \mathbf{T}\mathbf{v} \cdot \mathbf{v} dx_1 dx_2 \leq C^2 \int_{\Omega} (T_{11}|\nabla v_1|^2 + 2T_{12}\nabla v_1 \cdot \nabla v_2 + T_{22}|\nabla v_2|^2) dx_1 dx_2,$$

where

$$C = \max\{C_{\Gamma_1}(\Omega); C_{\Gamma_2}(\Omega)\}.$$

It is easy to see that  $\mathbf{T}$  is a positive definite matrix. Indeed,

$$\begin{aligned} \det(\mathbf{T} - \lambda\mathbf{E}) &= ((n_1^{(1)})^2 + (n_1^{(2)})^2 - \lambda)((n_2^{(1)})^2 + (n_2^{(2)})^2 - \lambda) - (n_1^{(1)}n_2^{(1)} + n_1^{(2)}n_2^{(2)})^2 \\ &= \lambda^2 - 2\lambda + (n_1^{(1)}n_2^{(2)} - n_1^{(2)}n_2^{(1)})^2 = \lambda^2 - 2\lambda + (\det \mathbf{N})^2, \end{aligned}$$

where

$$\mathbf{N} := \begin{pmatrix} \mathbf{n}^{(1)} \\ \mathbf{n}^{(2)} \end{pmatrix}.$$

Hence for any vector  $\mathbf{b}$ , we have  $\lambda_1|\mathbf{b}|^2 \leq \mathbf{T}\mathbf{b} \cdot \mathbf{b} \leq \lambda_2|\mathbf{b}|^2$ , and

$$(3.15) \quad \lambda_{1,2} = 1 \mp \sqrt{1 - (\det \mathbf{N})^2}.$$

If  $\mathbf{n}^{(1)}$  and  $\mathbf{n}^{(2)}$  are orthogonal, then  $\det \mathbf{N} = 1$  and the unique eigenvalue of  $\mathbf{N}$  is  $\lambda = 1$ . In this case, the left hand side of (3.14) coincides with  $\|\mathbf{v}\|^2$ . In all other cases  $\det \mathbf{N} < 1$  and  $\lambda_1 < \lambda_2$ .

We can always select the coordinate system such that

$$n_1^{(1)} = 1, n_2^{(1)} = 0, n_1^{(2)} = -\cos \beta, n_2^{(2)} = \sin \beta.$$

Then,

$$T_{11} = 1 + \cos^2 \beta, T_{22} = 1 - \cos^2 \beta, T_{12} = -\sin \beta \cos \beta,$$

and the matrix is

$$\mathbf{N} := \begin{pmatrix} 1 & 0 \\ -\cos \beta & \sin \beta \end{pmatrix}.$$

We see that  $\det \mathbf{N} = \sin \beta$ , and  $\lambda_1 = 1 - |\cos \beta|$ .

Consider the right hand side of (3.14). It is bounded from above by the quantity

$$I(\mathbf{v}) := C^2 \int_{\Omega} ((T_{11} + \gamma|T_{12}|)|\nabla v_1|^2 + (T_{22} + \gamma^{-1}|T_{12}|)|\nabla v_2|^2) dx,$$

where  $\gamma$  is any positive number. We define  $\gamma$  by means of the relation  $T_{11} - T_{22} = (\gamma^{-1} - \gamma)|T_{12}|$ , which yields  $\gamma = \frac{1 - |\cos \beta|}{\sin \beta}$ . Then,

$$(3.16) \quad I(\mathbf{v}) \leq (1 + |\cos \beta|)\|\nabla \mathbf{v}\|^2.$$

From (3.14) and (3.16), we find that

$$(3.17) \quad \boxed{\|\mathbf{v}\| \leq \max_{i=1,2} \{C_{\Gamma_i}(\Omega)\} \sqrt{\frac{1+|\cos\beta|}{1-|\cos\beta|}} \|\nabla\mathbf{v}\|.$$

This is the Poincaré type inequality for the vector valued function  $\mathbf{v}$  with zero mean normal traces on  $\Gamma_1$  and  $\Gamma_2$ . It is worth noting that for small  $\beta$  (and for  $\beta$  close to  $\pi$ ) the constant blows up. Therefore, interpolation operators (considered in Sect. 4) should avoid such situations.

**3.2. Value of the constant for  $d \geq 3$ .** Now we are concerned with the general case and deduce the estimate valid for any dimension  $d$ .

In view of (3.2) we have

$$\sum_{k=1}^d \|\mathbf{n}^{(k)} \cdot \mathbf{v}\|_{L^2(\Omega)}^2 \leq C^2 \sum_{k=1}^d \int_{\Omega} \left( \sum_{i=1}^d n_i^{(k)} \nabla v_i \right)^2 dx, \quad 2(\Omega, \Gamma_3) \|n_1^{(3)} \nabla v_1 + n_2^{(3)} \nabla v_2 + n_3^{(3)} \nabla v_3\|^2$$

where

$$C = \max_{k=1,2,\dots,d} \{C_{\Gamma_k}(\Omega)\}.$$

In view of the relation

$$(\mathbf{n}^{(k)} \otimes \mathbf{n}^{(k)}) \mathbf{v} \cdot \mathbf{v} = (\mathbf{n}^{(k)} (\mathbf{n}^{(k)} \cdot \mathbf{v})) \cdot \mathbf{v} = (\mathbf{n}^{(k)} \cdot \mathbf{v})^2,$$

the left hand side of (3.18) is  $\int_{\Omega} \mathbf{T} \mathbf{v} \cdot \mathbf{v}$ , where

$$(3.19) \quad \mathbf{T} := \sum_{k=1}^d \mathbf{n}^{(k)} \otimes \mathbf{n}^{(k)}.$$

If  $\mathbf{n}^{(k)}$  form a linearly independent system, then  $\mathbf{T}$  is a positive definite matrix. Indeed,  $\mathbf{T} \mathbf{b} \cdot \mathbf{b} = \sum_{k=1}^d (\mathbf{n}^{(k)} \cdot \mathbf{b})^2$ . Hence,  $\mathbf{T} \mathbf{b} \cdot \mathbf{b} = 0$  if and only if  $\mathbf{b}$  has zero projections to  $d$  linearly independent vectors  $\mathbf{n}^{(k)}$ , i.e.,  $\mathbf{T} \mathbf{b} \cdot \mathbf{b} = 0$  if and only if  $\mathbf{b} = 0$ . Therefore,

$$(3.20) \quad \lambda_1 \|\mathbf{v}\|^2 \leq \int_{\Omega} \mathbf{T} \mathbf{v} \cdot \mathbf{v} dx,$$

where  $\lambda_1 > 0$  is the minimal eigenvalue of  $\mathbf{T}$ .

Consider the right hand side of (3.18). We have

$$\begin{aligned} \int_{\Omega} \left( \sum_{i=1}^d n_i^{(k)} \nabla v_i \right)^2 dx &= \int_{\Omega} \sum_{i,j=1}^d n_i^{(k)} n_j^{(k)} \nabla v_i \cdot \nabla v_j dx \\ &= \sum_{i,j=1}^d n_i^{(k)} n_j^{(k)} \int_{\Omega} \nabla v_i \cdot \nabla v_j dx = \mathbf{n}^{(k)} \otimes \mathbf{n}^{(k)} : \mathbf{G}, \end{aligned}$$

where

$$\mathbf{G}(\mathbf{v}) := \{G_{ij}\}, \quad G_{ij}(\mathbf{v}) = \int_{\Omega} \nabla v_i \cdot \nabla v_j dx.$$

Hence,

$$(3.21) \quad \sum_{k=1}^d \int_{\Omega} \left( \sum_{i=1}^d n_i^{(k)} \nabla v_i \right)^2 d\mathbf{x} = \mathbf{T} : \mathbf{G}(\mathbf{v}) \leq |\mathbf{T}| |\mathbf{G}(\mathbf{v})|.$$

Now (3.18), (3.19), (3.20), and (3.21) yield the estimate

$$\|\mathbf{v}\|^2 \leq C^2 \frac{1}{\lambda_1} |\mathbf{T}| |\mathbf{G}(\mathbf{v})| \leq C^2 \frac{d}{\lambda_1} |\mathbf{G}(\mathbf{v})|.$$

Since  $|\mathbf{G}(\mathbf{v})| \leq \|\nabla \mathbf{v}\|^2$ , for any  $\mathbf{v} \in H^1(\Omega, \mathbb{R}^d)$  satisfying (3.2) we have

$$(3.22) \quad \boxed{\|\mathbf{v}\| \leq C \sqrt{\frac{d}{\lambda_1}} \|\nabla \mathbf{v}\|}.$$

In other words, the constant in (3.22) can be defined as follows:

$$(3.23) \quad \mathbb{C}(\Omega, \Gamma_1, \Gamma_2, \dots, \Gamma_d) = \max_{k=1,2,\dots,d} \{C_{\Gamma_k}(\Omega)\} \sqrt{\frac{d}{\lambda_1}},$$

where  $\lambda_1$  is the minimal eigenvalue of  $\mathbf{T}$ .

For  $d = 2$  this estimate exposes a slightly worse constant than (3.17) with the factor  $\sqrt{\frac{2}{1-|\cos \beta|}}$  instead of  $\sqrt{\frac{1+|\cos \beta|}{1-|\cos \beta|}}$ .

#### 4. INTERPOLATION OF FUNCTIONS

The classical Poincaré inequality (1.1) yields a simple interpolation operator  $\mathbb{I}_{\Omega} : H^1(\Omega) \rightarrow P^0(\Omega)$  defined by the relation  $\mathbb{I}_{\Omega} w := \{w\}_{\Omega}$ . In view of (1.1), we know that

$$(4.1) \quad \|w - \mathbb{I}_{\Omega} w\|_{2,\Omega} \leq C_P(\Omega) \|\nabla w\|_{2,\Omega},$$

which means that the interpolation operator is stable and  $C_P(\Omega)$  is the respective constant.

Above discussed estimates for functions with zero mean traces yield somewhat different interpolation operators for scalar and vector valued functions. For a scalar valued function  $w \in H^1(\Omega)$ , we set  $\mathbb{I}_{\Gamma}(w) := \{w\}_{\Gamma}$ , i.e., the interpolation operator uses mean values of  $w$  a  $d - 1$  - dimensional set  $\Gamma$ . Since  $\{w - \mathbb{I}_{\Gamma} w\}_{\Gamma} = 0$ , we use (1.10) and obtain the interpolation estimate

$$(4.2) \quad \|w - \mathbb{I}_{\Gamma} w\|_{2,\Omega} \leq C_{\Gamma}(\Omega) \|\nabla w\|_{2,\Omega},$$

where the constant  $C_{\Gamma}$  appears as the interpolation constant. Analogously, (1.11) yields an interpolation estimate for the boundary trace

$$(4.3) \quad \|w - \mathbb{I}_{\Gamma} w\|_{2,\Gamma} \leq C_{\Gamma}^{\text{Tr}} \|\nabla w\|_{2,\Omega}.$$

Applying these estimates to cells of meshes we obtain analogous interpolation estimates for mesh interpolation of scalar functions with explicit constants depending on character diameter of cells.

For the interpolation of vector valued functions we use (3.22) and generalise this idea.

4.1. **Cells with plane faces.** Define the operator

$$\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} : H^1(\Omega, \mathbb{R}^d) \rightarrow P^0(\Omega, \mathbb{R}^d)$$

that performs zero order interpolation of a vector valued function  $\mathbf{v}$  using mean values of normal components on the faces  $\Gamma_i$ ,  $i = 1, 2, \dots, d$ . In this case, we set

$$(4.4) \quad \int_{\Gamma_i} (\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}) \cdot \mathbf{n}^{(i)} d\Gamma = \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n}^{(i)} d\Gamma \quad i = 1, 2, \dots, d.$$

This condition means that *the interpolant must preserve integral values of normal flux through  $d$  selected faces*. In general we may define several different operators associated with different collections of faces. However, once the set of  $\Gamma_1, \Gamma_2, \dots, \Gamma_d$  satisfying (3.1) has been defined, the operator  $\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d}$  uniquely defines the vector  $\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}$ . In view of (4.4) and the identity

$$(\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}) \cdot \mathbf{n}^{(i)} = (\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v})_j \mathbf{e}_j \cdot \mathbf{n}^{(i)},$$

we conclude that the components of the interpolant are uniquely defined by the system

$$(4.5) \quad \sum_{j=1}^d n_j^{(i)} (\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v})_j = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n}^{(i)} d\Gamma \quad i = 1, 2, \dots, d.$$

Define  $\mathbf{w} := \mathbf{v} - \mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}$ . From (4.4), it follows that

$$\left\{ \mathbf{w} \cdot \mathbf{n}^{(i)} \right\}_{\Gamma_i} = 0 \quad i = 1, 2, \dots, d.$$

Therefore, we can apply Theorem 3.1 to  $\mathbf{w}$  and find that

$$(4.6) \quad \|\mathbf{w}\|_{\Omega} \leq \mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d) \|\nabla \mathbf{w}\|_{\Omega}.$$

Since  $\nabla \mathbf{w} = \nabla \mathbf{v}$ , (4.6) yields the estimate

$$(4.7) \quad \boxed{\|\mathbf{v} - \mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}\|_{\Omega} \leq \mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d) \|\nabla \mathbf{v}\|_{\Omega},}$$

where  $\mathbb{C}(\Omega, \Gamma_1, \dots, \Gamma_d)$  depends on the constants  $C_{\Gamma_i}$  (see section 3.2).

4.2. **Cells with curvilinear faces.** Let  $\Omega$  be a Lipschitz domain with a piecewise smooth boundary consisting of smooth parts  $\Gamma_1, \Gamma_2, \dots, \Gamma_N$  (see Fig. 4.2). In order to avoid complicated topological structures (which may lead to difficulties with definitions of "mean normals"), we assume that all the faces are such that normal vectors can be defined at almost all points and impose an additional condition

$$\mathbf{n}_i(x^{(1)}) \cdot \mathbf{n}_i(x^{(2)}) > 0 \quad \forall x^{(1)}, x^{(2)} \in \Gamma_i, \quad i = 1, 2, \dots, d.$$

Then, we can define the mean normal vector associated with  $\Gamma_i$ :

$$\hat{\mathbf{n}}(i) := \left\{ \frac{1}{|\Gamma_i|} \int_{\Gamma_i} n_1^{(i)} d\Gamma, \frac{1}{|\Gamma_i|} \int_{\Gamma_i} n_2^{(i)} d\Gamma, \dots, \frac{1}{|\Gamma_i|} \int_{\Gamma_i} n_d^{(i)} d\Gamma \right\}.$$

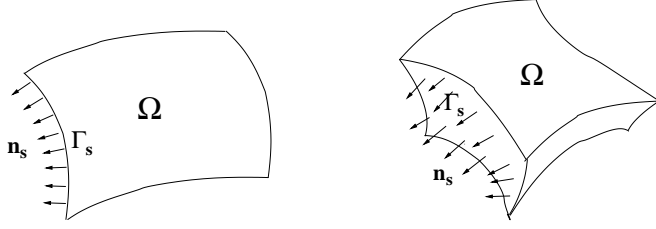


FIGURE 3. Cells with curvilinear faces in 2D and 3D

It is not difficult to verify that Theorem 3.1 holds if  $\mathbf{N}$  is replaced by  $\widehat{\mathbf{N}}$  formed by mean normal vectors, i.e.,

$$(4.8) \quad \det \widehat{\mathbf{N}} \neq 0, \quad \text{where} \quad \widehat{n}_j^{(i)} := \widehat{\mathbf{n}}^{(i)} \cdot \mathbf{e}_j,$$

and (3.2) is replaced by the condition

$$(4.9) \quad \left\{ \mathbf{v} \cdot \widehat{\mathbf{n}}^{(i)} \right\}_{\Gamma_i} = 0 \quad i = 1, 2, \dots, d.$$

In other words, for cells with curvilinear faces the necessary interpolation condition reads as follows: *mean values of normal vectors averaged on faces must form a linearly independent system satisfying (4.8).*

The operator  $\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}$  is defined by modifying the condition (4.4). Since

$$\int_{\Gamma_i} \mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v} \cdot \mathbf{n}^{(i)} d\Gamma = \mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v} \cdot \widehat{\mathbf{n}}^{(i)} |\Gamma_i|,$$

the interpolant  $\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}$  is defined by the system

$$(4.10) \quad \sum_{j=1}^d \widehat{n}_j^{(i)} (\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v})_j = \frac{1}{|\Gamma_i|} \int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n}^{(i)} d\Gamma \quad i = 1, 2, \dots, d.$$

By repeating the same arguments, we obtain the estimate (4.7) for the interpolant  $\mathbb{I}_{\Gamma_1, \Gamma_2, \dots, \Gamma_d} \mathbf{v}$ .

### 4.3. Comparison of interpolation constants for $\mathbb{I}_\Omega$ and $\mathbb{I}_\Gamma$ .

4.3.1. *Triangles.* First, we compare five different interpolation operators for the right triangle with equal legs (see Fig. 1). For the interpolation operator  $\mathbb{I}_\Omega$  (Fig. 1a) we have (1.9), where (1.6) yields the upper bound of the respective interpolation constant  $C_P(\Omega) \leq \sqrt{2} \frac{h}{\pi} \approx 0.4502h$ .

Four different operators  $\mathbb{I}_\Gamma$  are generated by setting zero mean values on one leg (b), two legs (c), median (d), and hypotenuse (e)

$$\|w - \mathbb{I}_\Gamma(w)\|_{2,\Omega} \leq C_\Gamma(\Omega) h \|\nabla w\|_{2,\Omega}.$$

The respective constants follow from Tab. 1. For (b),  $C_\Gamma(\Omega) = \frac{h}{\zeta} \approx 0.4929h$ , for (c)  $C_\Gamma(\Omega) = \frac{h}{\pi} \approx 0.3183h$ , for (d) and (e)  $C_\Gamma(\Omega) = \frac{h}{\zeta\sqrt{2}} \approx 0.3485h$ .

We can use these data and compare the efficiency of  $\mathbb{I}_\Gamma$  and  $\mathbb{I}_\Omega$  for uniform meshes which cells are right equilateral triangles (Fig. 1 f). For a mesh with  $2nm$  cells, the operator  $\mathbb{I}_\Omega$  uses  $2nm$  parameters (mean values on triangles) and provides interpolation with the constant  $C_P$ . The operator  $\mathbb{I}_\Gamma$  using

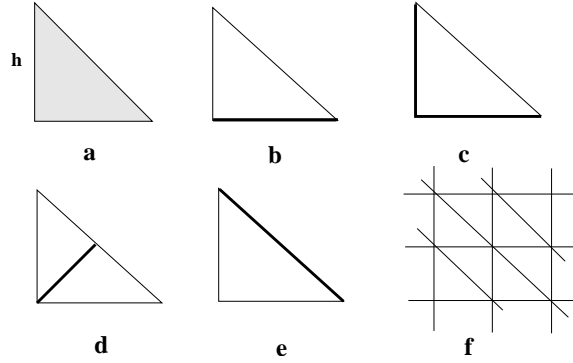


FIGURE 4. Triangular cells

mean values on diagonals (see (e)) has almost the same constant but needs only  $mn$  parameters.

4.3.2. *Squares.* Similar results hold for square cells. For the interpolation operator  $\mathbb{I}_\Omega$  (Fig. 2a) we have the exact constant  $C_P = \frac{\pi}{h}$ . The constants for  $\mathbb{I}_\Gamma$  are as follows. For (b),  $C_\Gamma = \frac{h}{\pi}$ , for (c) and (d)  $C_\Gamma = \frac{2h}{\pi}$ , and for (e)  $C_\Gamma = \frac{h}{2.869}$ . We see that for a uniform mesh with square cells  $\mathbb{I}_\Gamma$  and  $\mathbb{I}_\Omega$  have the same efficiency if  $\Gamma$  is selected as on (d) or (e).

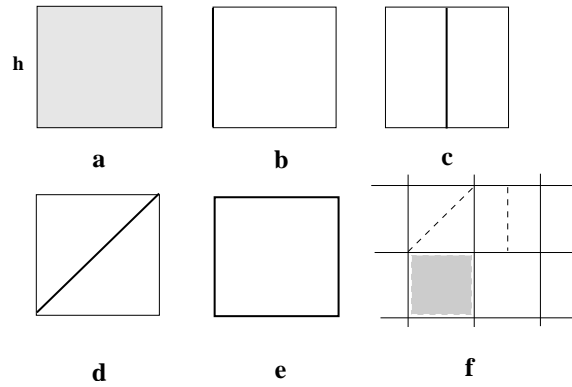


FIGURE 5. Square cells

4.4. **Interpolation on macrocells.** Advanced numerical approximations often operate with macrocells. Let  $\Omega$  be a macrocell consisting of  $N$  simple subdomains  $\omega_i$  (e.g., simplexes). Let the boundary  $\Gamma$  consist of faces  $\Gamma_i$  (each  $\Gamma_i$  is a part of some subdomain boundary  $\partial\omega_i$ ). For  $w \in H^1(\Omega)$  we define  $\mathbb{I}_\Gamma w$  as a piecewise constant function that satisfies the conditions

$$(4.11) \quad \{w - \mathbb{I}_\Gamma w\}_{\Gamma_i} = 0 \quad i = 1, 2, \dots, N.$$



Then, we can apply interpolation operators  $\mathbb{I}_{\gamma_i}$  to any subdomain  $\omega_i$  and find that for the whole cell

$$(4.12) \quad \begin{aligned} \|w - \mathbb{I}_{\Gamma} w\|_{2,\Omega}^2 &= \sum_{i=1}^N \|w - \mathbb{I}_{\Gamma} w\|_{2,\omega_i}^2 \\ &\leq \sum_{i=1}^N C_{\gamma_i}^2 \|\nabla w\|_{2,\omega_i}^2 \leq C_{\Gamma}^2 \|\nabla w\|_{2,\Omega}^2, \end{aligned}$$

where  $C_{\Gamma} = \max_i \{C_{\Gamma_i}\}$ .

Estimates for vector valued functions are derived quite similarly. For example, let  $d = 2$  and  $\Omega$  be a polygonal domain with  $N$  faces. If  $N$  is an odd number, then we form out of  $\Gamma_i$  a set of  $K$  pairs  $\{\Gamma_1^{(l)}, \Gamma_2^{(l)}\}$ ,  $l = 1, 2, \dots, K$  such that the respective subdomains cover  $\Omega$  and for each pair  $\mathbf{n}_1^{(l)}$  and  $\mathbf{n}_2^{(l)}$  satisfy (3.1). Then, the interpolant  $\mathbb{I}_{\Gamma} \mathbf{v}$  can be defined as a piece wise constant field in each pair of subdomains  $\omega_1^{(l)} \cup \omega_2^{(l)}$  that satisfies

$$(4.13) \quad \{(\mathbf{v} - \mathbb{I}_{\Gamma} \mathbf{v}) \cdot \mathbf{n}_i\}_{\Gamma_i} = 0 \quad i = 1, 2, \dots, N.$$

Analogously to (4.12), we obtain

$$(4.14) \quad \|\mathbf{v} - \mathbb{I}_{\Gamma} \mathbf{v}\|_{2,\Omega} \leq \mathbf{C} \|\nabla \mathbf{v}\|_{2,\Omega} \quad \mathbf{v} \in H^1(\Omega, \mathbb{R}^2),$$

where  $\mathbf{C} = \max_{l=1,2,\dots,K} \mathbf{C}_{\Gamma_1^{(l)}, \Gamma_2^{(l)}}(\omega_1^{(l)} \cup \omega_2^{(l)})$ .

**4.5. Interpolation on meshes.** Finally, we shortly discuss applications to mesh interpolation. It is clear that analogous operators  $\mathbb{I}_{\Gamma}$  can be constructed for scalar and vector valued functions defined in a bounded Lipschitz domain  $\Omega$ , which is covered by a mesh  $\mathcal{T}_h$  with sells  $\Omega_i$ ,  $i = 1, 2, \dots, M_h$ .

Let  $\Omega_i$  be Lipschitz domains such that  $\Omega_i \cup \Omega_j = \emptyset$  if  $i \neq j$  and

$$(4.15) \quad \bar{\Omega} = \bigcup_{i=1}^{M_h} \bar{\Omega}_i.$$

We assume that  $c_1 h \leq \text{diam} \Omega_i \leq c_2 h$  for all  $i = 1, 2, \dots, M_h$ , where  $c_2 \geq c_1 > 0$  and  $h$  is a small parameter. The intersection of  $\bar{\Omega}_i$  and  $\bar{\Omega}_j$  is either empty or a face  $\Gamma_{ij}$  (which is a Lipschitz domain in  $\mathbb{R}^{d-1}$ ). By  $\mathcal{E}_h$  we denote the collection of all faces in  $\mathcal{T}_h$ .

It is easy to see that a function  $w \in H^1(\mathcal{D})$  can be interpolated by a piece wise constant function on cells of  $\mathcal{T}_h$  if we set

$$(4.16) \quad \mathbb{I}_{\mathcal{T}_h}(w)(x) = \mathbb{I}_{\Gamma_i} w(x) = \{w\}_{\Gamma_i} \quad \text{if } x \in \Omega_i.$$

Here  $\Gamma_i$  is a face of  $\Omega_i$  selected for the local interpolation operator. Then,

$$(4.17) \quad \|w - \mathbb{I}_{\mathcal{T}_h}(w)\|_{2,\Omega} \leq C(\mathcal{T}_h) \|\nabla w\|_{2,\Omega},$$

where  $C(\mathcal{T}_h)$  is the maximal constant in inequalities (1.10) associated with  $\Omega_i$ ,  $i = 1, 2, \dots, M_h$ . We note that the amount of parameters used in such type interpolation is essentially smaller than the amount of faces in  $\mathcal{T}_h$ .

If  $\mathbb{I}_{\mathcal{T}_h}$  is constructed by means of averaging on each face  $\Gamma_{ij}$  then (4.17) holds with a better constant and the interpolant  $\mathbb{I}_{\mathcal{T}_h} w$  possesses an important property: *it preserves mean values of  $w$ .*

Similar consideration is valid for vector valued functions. If we define the interpolation operator  $\mathbb{I}_{\mathcal{T}_h}(\mathbf{v})(x)$  on  $\mathcal{T}_h$  by the conditions

$$(4.18) \quad \mathbb{I}_{\mathcal{T}_h} \mathbf{v} \cdot \mathbf{n}_{ij} = \{\mathbf{v} \cdot \mathbf{n}_{ij}\}_{\Gamma_{ij}} \quad \forall \Gamma_{ij} \in \mathcal{E}_h,$$

then

$$(4.19) \quad \|\mathbf{v} - \mathbb{I}_{\mathcal{T}_h} \mathbf{v}\|_{2,\Omega} \leq \mathbf{C}(\mathcal{T}_h) \|\nabla \mathbf{v}\|_{2,\Omega},$$

where  $\mathbf{C}(\mathcal{T}_h)$  is the maximal constant in inequalities (4.14) used for  $\Omega_i$ ,  $i = 1, 2, \dots, N(\mathcal{T}_h)$ . The interpolant  $\mathbb{I}_{\mathcal{T}_h} \mathbf{v}$  possesses an important property: *it preserves mean values of  $\mathbf{v} \cdot \mathbf{n}_{ij}$  on all the faces of  $\mathcal{T}_h$ .*

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