Harmonic Balance Method and Stability of Discontinuous Systems

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Abstract The development of the theory of discontinuous dynamical systems and differential inclusions was not only due to research in the field of abstract mathematics but also a result of studies of particular problems in mechanics. One of first methods, used for the analysis of dynamics in discontinuous mechanical systems, was the harmonic balance method developed in the thirties of the 20th century. In our work the results of analysis obtained by the method of harmonic balance, which is an approximate method, are compared with the results obtained by rigorous mathematical methods and numerical simulation.

1 Introduction

The development of the theory of discontinuous dynamical systems and differential inclusions was not only due to research in the field of abstract mathematics in the thirties of the last 20th century but also a result of studies of particular problems in mechanics. In the thirties and forties of the 20th century J. Hartog, A. Andronov, N. Bautin, M. Keldysh were among the first who rigorously treated the mathematical peculiarities of discontinuous dynamical models [1, 2, 3] on the examples of mechanical models. One of first methods, used for the analysis of stability and oscillations in discontinuous dynamical models, was the harmonic balance method (or the describing function method) developed in the thirties of the 20th century

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[4]. This method is not strictly mathematically justified and is one of approximate methods of analysis of oscillation in nonlinear systems. Nowadays we can apply various rigorous analytical and reliable numerical methods, which have been developed from that time till now: mathematical theory of differential inclusions (see, e.g. [5, 6, 7, 8, 9] and others), direct Lyapunov method and frequency methods (see, e.g. [6, 10]), special numerical approaches for solving differential inclusions (see, e.g. [11, 12, 13]).

In our work for the Hartog, Keldysh and modified Fitss models we compare the results of analysis obtained by the method of harmonic balance with the results obtained by rigorous mathematical methods and numerical simulation.

2 Hartog model

In 1930, J. Hartog studied vibrations in a mechanical model with dry friction[1] described by the following equation [1].

$$m\ddot{x} + kx = -\varphi(\dot{x}), \quad \varphi(\dot{x}) = F_0 \mathrm{sign}(\dot{x}) \tag{1}$$

where $m > 0$ is a mass, $k > 0$ is spring stiffness, $F_0 > 0$ is the dry friction coefficient. Following the mechanical sense, Hartog defined $\mathrm{sign}(0)$ as a value from $[-F_0, F_0]$ and, thus, the discontinuous differential equation (1) has a segment of equilibria (rest segment).

Follow the theory of differential inclusion, for the model (1) we consider the discontinuity manifold: $S = \{\dot{x} : \ \dot{x} = 0\}$ on the phase space $(x, \dot{x})$, define $\varphi(\dot{x})$ on S as the set $[-F_0, +F_0]$, and get differential inclusion

$$m\ddot{x} + kx \in -\hat{\varphi}(\dot{x}), \quad \hat{\varphi}(\dot{x}) = \begin{cases} \varphi(\dot{x}), & \text{if } \dot{x} \neq 0, \\ [-F_0, +F_0], & \text{if } \dot{x} = 0. \end{cases} \tag{2}$$

The solutions of (2) are considered in the sense of Filippov [5]. Remark that here solutions cannot slide on the discontinuity manifold S, but can tend to the rest segment:

$$\Lambda = \{ \ -F_0/k \leq x \leq F_0/k, \dot{x} = 0\} \subset S,$$

or pierce the manifold $S \backslash \Lambda$. The phase portrait of (2) is shown in Fig. 1.

For equation (2), the harmonic balance methods states that there is no periodic oscillations for any values of the parameters. This result can be rigorously justified by the analog direct Lyapunov method for differential inclusions [6, Lemma 1.5, p.58]. Consider Lyapunov function

$$V(x, \dot{x}) = \frac{1}{2}(m\dot{x}^2 + kx^2). \tag{3}$$

[1] The history of the dry friction law can be found, e.g, in [14].

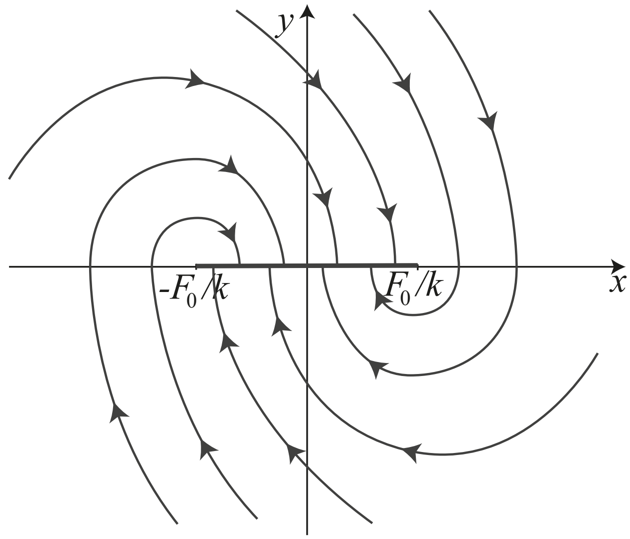

Fig. 1 Phase portrait of system (1): trajectories tend toward the rest segment $\{|x| \le F_0/k,\, y = 0\}$.

Then we have

$$\dot{V}(x,\dot{x}) = -F_0\dot{x}\operatorname{sign}(\dot{x}) < 0, \quad \forall \dot{x} \notin S$$

and the equality $V(x(t),\dot{x}(t)) \equiv \text{const}$ can hold only for $x \in \Lambda$. Thus, any solution of (2) converges to the rest segment Λ.

3 Two-dimensional Keldysh model

M. Keldysh, in 1944, studied a two-dimensional model of damping flutter in aircraft control systems with dry friction [3]

$$J\ddot{x} + kx = -\mu\dot{x} - \varphi(\dot{x}), \quad \mu = \lambda - h,\ \varphi(\dot{x}) = (F_0 + \kappa\dot{x}^2)\operatorname{sign}(\dot{x}), \tag{4}$$

where $J > 0$ is the moment of inertia, $k > 0$ is sprig stiffness, $h\dot{x}$ is an excitation force proportional to the angular velocity $\dot{x}$, $f(\dot{x}) = \lambda\dot{x} + \varphi(\dot{x})$ is the nonlinear characteristic of hydraulic damper with dry friction, $F_0 > 0$ is the dry friction coefficient, $\lambda > 0$ and $\kappa > 0$ are parameters of the hydraulic damper.

Using the harmonic balance method, Keldysh formulated the following result: *If*

$$-2.08\sqrt{F_0\kappa} = \delta_K < \mu$$

then all trajectories of (4) converge to the rest segment; If $\mu < -2.08\sqrt{F_0\kappa}$ *then there are two periodic trajectories (limit cycles)* $\approx a_\pm \cos(\omega t)$ *with amplitudes*

$$a_{\pm}(\mu) = \frac{3}{8\kappa}\sqrt{\frac{J}{k}}\left(\pi\mu \pm \sqrt{\pi^2\mu^2 - \frac{32}{3}\kappa F_0}\right); \tag{5}$$

Other trajectories behave as follows. The trajectories, emerging from infinity, tend to the external limit cycle. The domain between two limit cycles is filled with trajectories unwinding from the internal (unstable) limit cycle and winding onto external (stable) limit cycle. The stability domain bounded by the internal limit cycle is filled with trajectories tending to one of the possible equilibrium on the rest segment.

By analogy with the above consideration of the Hartog model, we transform the Keldysh model to the differential inclusion

$$J\ddot{x} + kx + \mu\dot{x} \in -\hat{\varphi}(\dot{x}), \quad \hat{\varphi}(\dot{x}) = \begin{cases} \varphi(\dot{x}) & \dot{x} \neq 0, \\ [-F_0, +F_0] & \dot{x} = 0, \end{cases} \tag{6}$$

consider Lyapunov function (3) with $m = J$, and get

$$\dot{V}(x, \dot{x}) = -\mu\dot{x}^2 - \dot{x}\varphi(\dot{x}) < 0, \quad \forall \dot{x} \notin S.$$

Thus, if $\dot{x}\varphi(\dot{x}) > 0$ for $\dot{x} \neq 0$, i.e.

$$-2\sqrt{F_0\kappa} < \mu,$$

then any solution of (6) converges to the rest segment Λ [15]. Here the estimate obtained by direct Lyapunov method is close to the Keldysh estimate obtained by the harmonic balance method.

To check the second part of Keldysh's result we use numerical simulation [13]. The qualitative behavior of trajectories in the case of two coexisting limit cycles is shown in Fig. 2.

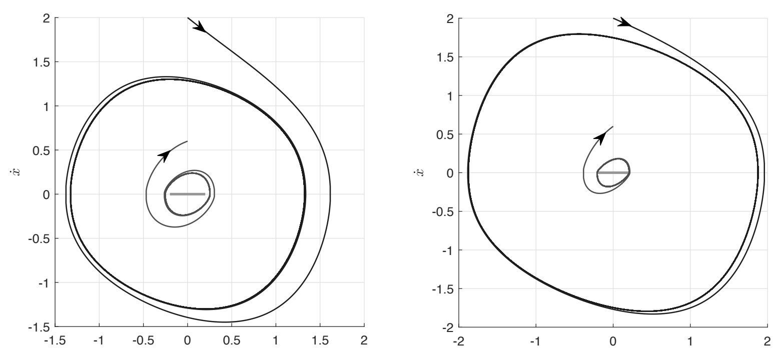

Fig. 2 Numerical experiment with $F_0 = 0.2, J = 1, k = 1, \kappa = 1$. Outer trajectory winds onto stable limit cycle, inner trajectory unwinds from unstable limit cycle and winds onto the stable limit cycle (hidden attractor). Left subfigure: $\mu = -1.3967\delta_K$: $a_+(\mu) >> a_-(\mu) > F_0$. Right subfigure: $\mu = -1.7847\delta_K$: $a_+(\mu) >> F_0 > a_-(\mu)$.

Here the largest limit cycle is a hidden attractor [16, 17, 18, 19, 20, 21, 22, 23] and corresponds to the flutter. Fig. 3 shows the bifurcation of collision of the limit cycles and the rest segment. In the right subfigure of Fig. 3, both limit cycles have disappeared and trajectories tend to the rest segment while the second part of the Keldysh estimate is valid.

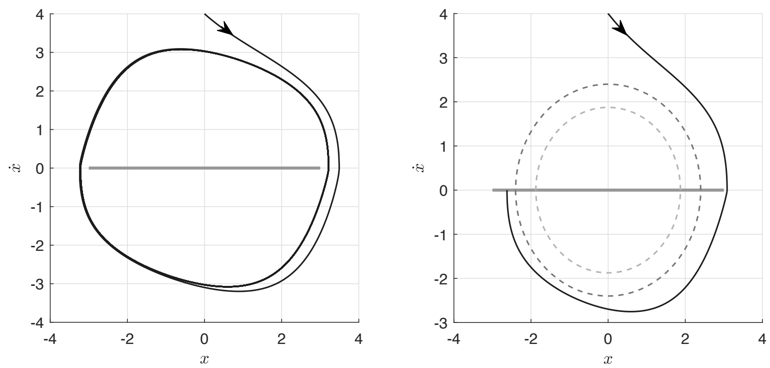

Fig. 3 Numerical experiment with $F_0 = 3, J = 1, k = 1, \kappa = 1$. Left subfigure: $\mu = -1.0713\delta_K$, $a_+(\mu) \gtrapprox F_0 > a_-(\mu)$; outer trajectory winds onto stable limit cycle, internal unstable limit cycle is not revealed numerically (due to stiffness). Right subfigure: $\mu = -1.0076\delta_K$, $F_0 \gtrapprox a_+(\mu) > a_-(\mu)$ (dash circles); outer trajectory approaches the stationary segment, both limit cycles have disappeared.

4 Discontinuous modification of the Fitts counterexample

It is known that the harmonic balance method may lead to wrong conclusion on the global stability. For example, it states that the Aizerman and Kalman conjectures on the global stability of nonlinear control systems are valid, while various counterexamples with hidden attractors have been found (see, e.g. [24, 25, 26, 27, 28, 29, 30, 16, 31, 32]). Consider a modification of one of first counterexamples to the Kalman conjecture [33]

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4, \\ &\dot{x}_4 \in -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + \hat{\varphi}(-x_3), \quad \hat{\varphi}(\dot{x}) = \begin{cases} \mathrm{sign}(-x_3) & x_3 \neq 0, \\ [-1,1] & x_3 = 0, \end{cases} \end{aligned} \tag{7}$$

where $a_i > 0$.

The sliding mode surface for the system (7) is given by

$$D = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 = x_4 = 0, -1 \leq a_0x_1 + a_1x_2 \leq 1\}$$

and the rest segment is

$$\Lambda = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_2 = x_3 = x_4 = 0, -\tfrac{1}{a_0} \le x_1 \le \tfrac{1}{a_0}\}.$$

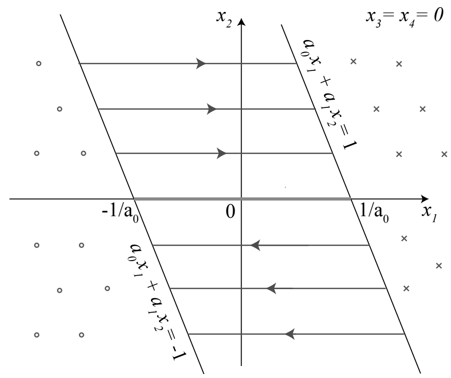

Fig. 4 Sliding mode surface $\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_3 = x_4 = 0, -1 \le a_0x_1 + a_1x_2 \le 1\}$ for system (7). Arrowed lines define the motion on the surface, thick line defines the rest segment.

This system has infinite sector of the linear stability and, thus, the harmonic balance method can not reveal any periodic solutions [16]. However, for parameters $a_0 = 0.981919$, $a_1 = 0.121308$, $a_2 = 2.0254$, $a_3 = 0.12$ it can be found numerically periodic solution (see Fig. 5) with initial data [34, 33]

$$\begin{aligned}(x_1^0,x_2^0,x_3^0,x_4^0) = (&-0.62520516260693109534342362490723,\\ &-3.7324097072650610465825278562594,\ 0,\\ &\qquad 3.4754169728697120793989274111636)\end{aligned}$$

Using the continuation procedure and passing from parameters $a_0 = 0.981919$, $a_1 = 0.121308$, $a_2 = 2.0254$, $a_3 = 0.12$ to parameters $a_0 = 1.0004$, $a_1 = 4.08$, $a_2 = 2.08$, $a_3 = 0.4$ it is possible to localize non-periodic oscillating solution (see Fig. 6).

Conclusions

While harmonic balance method is widely used for study of stability and oscillations of nonlinear dynamical systems, it may lead to wrong results. Some limitations of the use of harmonic balance method for the study of systems with dry friction and rest segment are demonstrated.

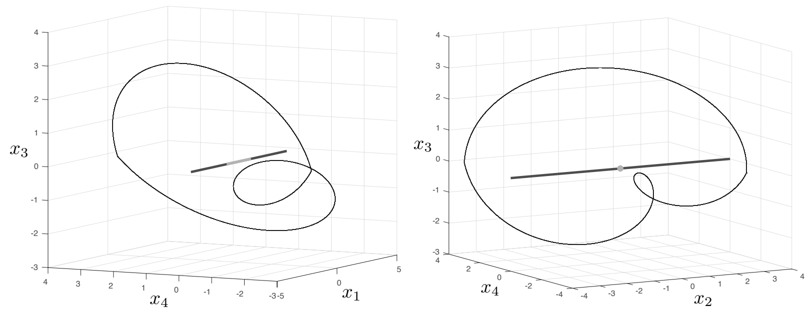

Fig. 5 Periodic solution of system (7) for parameters $a_0 = 0.981919$, $a_1 = 0.121308$, $a_2 = 2.0254$, $a_3 = 0.12$. Thick dark gray line defines the projection of the sliding mode surface on the corresponding three-dimensional hyperspace (x_1, x_3, x_4) (or (x_2, x_3, x_4)), light gray line (or dot) defines the projection of the rest segment.

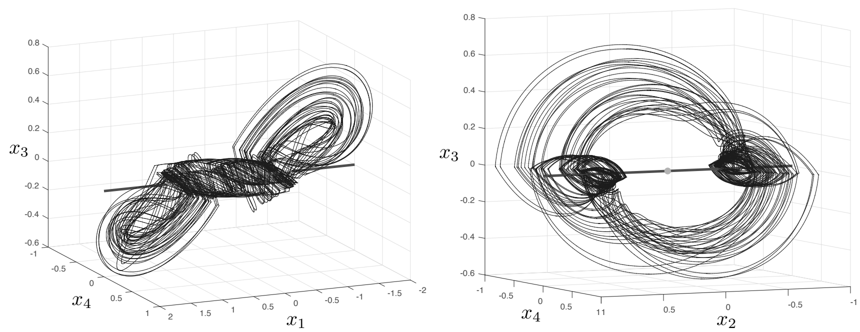

Fig. 6 Non-periodic oscillating solution of system (7) for parameters $a_0 = 1.0004$, $a_1 = 4.08$, $a_2 = 2.08$, $a_3 = 0.4$ Thick dark gray line defines the projection of the sliding mode surface on the corresponding three-dimensional hyperspace (x_1, x_3, x_4) (or (x_2, x_3, x_4)), light gray line (or dot) defines the projection of the rest segment.

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