A DENSITY RESULT FOR HOMOGENEOUS SOBOLEV SPACES ON PLANAR DOMAINS

DEBANJAN NANDI, TAPIO RAJALA, AND TIMO SCHULTZ

Abstract. We show that in a bounded simply connected planar domain Ω the smooth Sobolev functions $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ are dense in the homogeneous Sobolev spaces $L^{k,p}(\Omega)$.

1. Introduction

By the result of Meyers-Serrin [16] it is known that $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for every open set Ω in $\mathbb{R}^d$. The space $C^\infty(\mathbb{R}^d)$ is not always dense in $W^{k,p}(\Omega)$, for example when Ω is a slit disk. However, a slit disk is not a very appealing example as it is not the interior of its closure. Counterexamples for the density satisfying $\Omega = \mathrm{int}(\overline{\Omega})$ were given by Amick [1] and Kolsrud [9]. In fact, in these examples even $C(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$. Going further in counterexamples, O'Farrell [18] constructed a domain satisfying $\Omega = \mathrm{int}(\overline{\Omega})$ where $W^{k,\infty}(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for any k and p. The domain constructed by O'Farrell was infinitely connected. From the recent results of Koskela-Zhang [14] and Koskela-Rajala-Zhang [13] we can conclude that this is necessary for such constructions in the plane, since $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for all finitely connected bounded planar domains (see also the earlier work by Giacomini-Trebeschi [4]). Further examples of domains where $W^{1,p}(\Omega)$ is not dense in $W^{1,q}(\Omega)$ were constructed by Koskela [11] and Koskela-Rajala-Zhang [13].

In this note we continue the study of density of $W^{k,\infty}(\Omega)$ in $W^{k,p}(\Omega)$. Let us remark that such density clearly holds in the case where the Sobolev functions in $W^{k,p}(\Omega)$ can be extended to Sobolev functions defined on the whole $\mathbb{R}^2$. By work of Jones [8], this is true when $\partial\Omega$ is a quasi-circle. (See also the works [6, 5, 7].) Geometric characterizations of Sobolev extension domains are known, especially in the planar simply connected domains when $k = 1$, see [3, 10, 19, 12].

Being an extension domain is only a sufficient condition for the density. For example, there are Jordan domains Ω and functions $f \in W^{1,p}(\Omega)$ that cannot be extended to a function in $W^{1,p}(\mathbb{R}^2)$. However, global smooth functions are dense in $W^{1,p}(\Omega)$ for any Jordan domain and any $p \in [1, \infty]$, see Lewis [15] and Koskela-Zhang [14]. For $W^{k,p}(\Omega)$ with $k \geq 2$ this is still unknown.

In [20] Smith-Stanoyevitch-Stegenga studied the density of $C^\infty(\mathbb{R}^2)$ as well as the density of functions in $C^\infty(\Omega)$ with bounded derivatives, in $W^{k,p}(\Omega)$. For the latter class they

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obtained a density result assuming Ω to be starshaped or to satisfy an interior segment condition. For the smaller class of functions $C^\infty(\mathbb{R}^2)$ they also required an extra assumption on the boundary points to be m_2-limit points. (See also Bishop [2] for a counterexample on a related question.)

The result of Koskela-Zhang [14] showing that $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for every bounded simply connected planar domain was generalized to higher dimensions by Koskela-Rajala-Zhang [13]. They showed that simply connectedness is not sufficient to give such a density result, but Gromov hyperbolicity in the hyperbolic distance is. In this paper we provide another generalization to the Koskela-Zhang result by going to higher order Sobolev spaces. We show that if we restrict attention to the homogenous norm, then being simply connected is sufficient for domains in the plane.

For a domain $\Omega \subset \mathbb{R}^2$ and $p \in [1, \infty)$, by homogenous Sobolev space $L^{k,p}(\Omega)$ we mean functions with p-integrable distributional derivatives of order k;

$$L^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : \nabla^\alpha u \in L^p(\Omega), \text{ if } |\alpha| = k\},$$

with semi-norm $\sum_{|\alpha|=k} \|\nabla^\alpha u\|_{L^p(\Omega)}$, where α is any 2-vector of non-negative integers and $|\alpha|$ is its ℓ_1-norm. The (non-homogenous) Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : \nabla^\alpha u \in L^p(\Omega), \text{ if } |\alpha| \le k\},$$

with norm $\sum_{|\alpha|\le k} \|\nabla^\alpha u\|_{L^p(\Omega)}$.

Theorem 1.1. *Let $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then the subspace $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in the space $L^{k,p}(\Omega)$.*

The approach in [13] differs from ours in that there the approximating functions are defined via shifting matters to the disk via the Riemann mapping. Instead, we directly make a Whitney decomposition of the domain and a rough reflection to define our approximating sequence. We achieve this via an elementary use of simply connectedness in the plane. In both of these approaches the values of the function in a suitable compact set are used to define a smooth function in the entire domain which approximates the original function in Sobolev norm. For this we employ similar tools as used by Jones in [8].

In p-Poincaré domains, that is domains Ω where a p-Poincaré inequality

$$\int_\Omega |u - u_D|^p \,\mathrm{d}x \le C \int_\Omega |\nabla u|^p \,\mathrm{d}x$$

holds, we can bound the integrals of the lower order derivatives by the integrals of the higher order ones and thus we obtain the following corollary to our Theorem 1.1.

Corollary 1.2. *Let $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^2$ be a bounded simply connected p-Poincaré domain. Then $W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ is dense in the space $W^{k,p}(\Omega)$.*

For instance Hölder-domains are p-Poincaré domains for $p \ge 2$, see Smith-Stegenga [21]. It still remains an open question whether Corollary 1.2 holds if one drops the assumption of being a p-Poincaré domain.

Next we come to the question of density of $C^\infty(\mathbb{R}^2)$ functions in $L^{k,p}(\Omega)$ in our setting of bounded simply connected domains. We have the following corollary which is analogous to

[14, Corollary 1.2], where it is shown that Ω being Jordan is sufficient. A small modification of the argument there applies to our situation as well. See the end of Section 4 for the proof.

Corollary 1.3. *Let $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then $C^\infty(\mathbb{R}^2)$ is dense in the space $L^{k,p}(\Omega)$.*

In Section 2, we collect the necessary ingredients which will be used for defining the approximating sequence; these include a suitable Whitney-type decomposition of a simply connected domain and a local polynomial approximation of Sobolev functions. In Section 3 we describe a partition of the domain using the Whitney-type decomposition of Section 2, which is needed for obtaining a suitable partition of unity. Then in Section 4, we define the approximating sequence and present the necessary estimates for proving Theorem 1.1 and Corollary 1.3.

2. Preliminaries

For sets $A, B \subset \mathbb{R}^2$ we denote the diameter of A by $\operatorname{diam}(A)$ and the distance between A and B by $\operatorname{dist}(A, B)$. We denote by $B(x, r)$ the open ball with center $x \in \mathbb{R}^2$ and radius $r > 0$ and more generally, by $B(A, r)$ the open r-neighbourhood of a set $A \subset \mathbb{R}^2$. Given a connected set $E \subset \mathbb{R}^2$ and points $x, y \in E$, we define the inner distance $\mathrm{d}_E(x, y)$ between x and y in E to be the infimum of lengths of curves in E joining x to y. (Notice that in general the infimum might have value ∞.) We write the inner distance in E between sets $A, B \subset E$ as $\operatorname{dist}_E(A, B)$.

With a slight abuse of notation, by a curve γ we refer to both, a continuous mapping $\gamma\colon [0, 1] \to \mathbb{R}^2$ and its image $\gamma([0, 1])$. Given two curves $\gamma_1, \gamma_2\colon [0, 1] \to \mathbb{R}^2$ such that $\gamma_1(1) = \gamma_2(0)$, we denote by $\gamma_1 * \gamma_2\colon [0, 1] \to \mathbb{R}^2$ the concatenated curve $\gamma_1 * \gamma_2(t) = \gamma_1(2t)$ for $t \le 1/2$ and $\gamma_1 * \gamma_2(t) = 2t - 1$ for $t \ge 1/2$. We denote the length of a curve γ by $L(\gamma)$.

We will use the following facts in plane topology whose proofs can be found in the book of Newman [17, Chapter VI, Theorem 5.1 and Chapter V, Theorem 11.8].

Lemma 2.1. *Let Ω be a simply connected domain in $\mathbb{R}^2$ and $\gamma\colon [0, 1] \to \mathbb{R}^2$ a continuous curve that is injective on $(0, 1)$, whose endpoints $\gamma(0)$ and $\gamma(1)$ are in $\partial\Omega$ and interior $\gamma((0, 1))$ in Ω. Then $\Omega \setminus \gamma$ has two connected components, both of which are simply connected.*

In the case where Ω is Jordan and γ is homeomorphic to a closed interval, the two connected components of $\Omega \setminus \gamma$ have boundaries $\gamma \cup J_1$ and $\gamma \cup J_2$, where J_1 and J_2 are the two connected components of $\partial\Omega \setminus \gamma$.

2.1. **A dyadic decomposition.**

Although it is standard to consider a Whitney decomposition of a domain in $\mathbb{R}^d$ (see for instance Whitney [23] or the book of Stein [22, Chapter VI]), we will use a precise construction of such a decomposition. We present this construction below. Here and later on we denote the sidelength of a square Q by $l(Q)$.

For notational convenience we start the Whitney decomposition below from squares with sidelength 2^{-1}. Formally, by rescaling, we may consider all bounded domains $\Omega \subset \mathbb{R}^2$ to have $\operatorname{diam}(\Omega) \leq 1$ in which case no Whitney decomposition would have squares larger than the ones used below regardless of the starting scale.

Definition 2.2 (Whitney decomposition)**.** Let $\Omega \subset \mathbb{R}^2$ be a bounded (simply connected) open set. Let $\mathcal{Q}_n$ be the collection of all closed dyadic squares of sidelength 2^{-n}. Define a Whitney decomposition as $\tilde{\mathcal{F}} := \bigcup_{n\in\mathbb{N}} \tilde{\mathcal{F}}_n$ where the sets $\tilde{\mathcal{F}}_n$ are defined recursively as follows. Define

$$\tilde{\mathcal{F}}_1 := \left\{ Q \in \mathcal{Q}_1 : \bigcup_{\substack{Q'\in\mathcal{Q}_1\\ Q'\cap Q\neq\emptyset}} Q' \subset \Omega \right\}$$

and

$$\tilde{\mathcal{F}}_{n+1} := \left\{ Q \in \mathcal{Q}_{n+1} : Q \not\subset \tilde{F}_n \text{ and } \bigcup_{\substack{Q'\in\mathcal{Q}_{n+1}\\ Q'\cap Q\neq\emptyset}} Q' \subset \Omega \right\},$$

where $\tilde{F}_n = \bigcup_{j\leq n} \bigcup_{Q\in\tilde{\mathcal{F}}_j} Q$.

Lemma 2.3. *A Whitney decomposition given by Definition 2.2 has the following properties.*

(W1) $\Omega = \bigcup_{Q\in\tilde{\mathcal{F}}} Q$
(W2) $l(Q) < \operatorname{dist}(Q,\Omega^c) \leq 3\sqrt{2}l(Q) = 3\operatorname{diam}(Q)$ *for all* $Q \in \tilde{\mathcal{F}}$
(W3) $\operatorname{int} Q_1 \cap \operatorname{int} Q_2 = \emptyset$ *for all* $Q_1, Q_2 \in \tilde{\mathcal{F}}, Q_1 \neq Q_2$
(W4) If $Q_1, Q_2 \in \tilde{\mathcal{F}}$ *and* $Q_1 \cap Q_2 \neq \emptyset$*, then* $\frac{l(Q_1)}{l(Q_2)} \leq 2$.

Proof. Although the proof is very elementary, we give it here for completeness.

For (W1), take any $x \in \Omega$ and $n \in \mathbb{N}$ such that $x \in Q \in \mathcal{Q}_n$, where $2^{-n+2}\sqrt{2} < \operatorname{dist}(x,\Omega^c) \leq 2^{-n+3}\sqrt{2}$. Then for any $Q' \in \mathcal{Q}_n$ with $Q' \cap Q \neq \emptyset$ we have $Q' \subset \Omega$. Hence by definition either $Q \in \tilde{F}_n$ or $x \in Q \subset Q'' \in \tilde{\mathcal{F}}_i$ for some $i < n$.

In order to see (W2), let $Q \in \tilde{\mathcal{F}}_n$. Then all $Q' \subset \Omega$ for all $Q' \in \mathcal{Q}_n$ with $Q' \cap Q \neq \emptyset$. Consequently, $\operatorname{dist}(Q,\Omega^c) > 2^{-n} = l(Q)$. For the upper bound, suppose $\operatorname{dist} Q, \Omega^c > 3\sqrt{2}2^{-n}$. Let $Q_2 \in \mathcal{Q}_{n-1}$ be such that $Q \subset Q_2$. Then $\operatorname{dist}(Q_2,\Omega^c) > \sqrt{2}2^{-n+1}$ and so $Q_3 \subset \Omega$ for all $Q_3 \in \mathcal{Q}_{n-1}$ for which $Q_2 \cap Q_3 \neq \emptyset$. Thus $Q_2 \in \tilde{F}_{n-1}$ or $Q_2 \subset Q_4 \in \tilde{F}_i$ for some $i < n-1$. In either case, $Q \notin \tilde{\mathcal{F}}_n$ giving a contradiction.

Property (W3) holds by the recursion in the definition and the fact that the dyadic squares are nested.

Suppose (W4) is not true. Then there exist $Q_1 \in \tilde{\mathcal{F}}_n$ and $Q_2 \in \tilde{\mathcal{F}}_m$ with $n < m-1$ and $Q_1 \cap Q_2 \neq \emptyset$. Let $Q_3 \in \tilde{F}_{n+1}$ be such that $Q_2 \subset Q_3$. Then

$$\bigcup_{\substack{Q'\in\mathcal{Q}_{n+1}\\ Q'\cap Q_3\neq\emptyset}} Q' \subset \bigcup_{\substack{Q'\in\mathcal{Q}_{n}\\ Q'\cap Q_1\neq\emptyset}} Q' \subset \Omega$$

and so either $Q_3 \in \tilde{\mathcal{F}}_{n+1}$ or $Q_3 \subset \tilde{F}_n$. In both cases $Q_2 \subset \tilde{F}_{n+1}$ and so $Q_2 \notin \tilde{\mathcal{F}}_m$. □

By a chain of dyadic squares $\{Q_i\}_{i=1}^m$ we mean a collection of sets $Q_i \in \tilde{\mathcal{F}}$ such that $Q_i \cap Q_{i+1}$ is a non-degenerate line segment for all $i \in \{1, \dots, m-1\}$. We say that the chain connects Q_1 and Q_m.

2.2. **Approximating polynomials.** We record here the following two Lemmas from [8] which will be used when estimating the approximation in Section 4. By $|E|$ we denote the Lebesgue measure of a set $E \subset \mathbb{R}^2$.

Lemma 2.4 (Lemma 2.1, [8]). *Let Q be any square in $\mathbb{R}^2$ and P be a polynomial of degree k defined in $\mathbb{R}^2$. Let $E, F \subset Q$ be such that $|E|, |F| > \eta|Q|$ where $\eta > 0$. Then*

$$\|P\|_{L^p(E)} \leq C(\eta, k)\|P\|_{L^p(F)}.$$

Given a function $u \in C^\infty(\Omega)$ and a bounded set $E \subset \Omega$ with $|E| > 0$, we define (see [8]) the polynomial approximation of u in E , $P_k(u, E)$ to be the polynomial of order $k-1$ which satisfies

$$\int_E \nabla^\alpha(u - P_k(u, E)) = 0$$

for each $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| = \alpha_1 + \alpha_2 \leq k-1$. Once k is fixed, we denote the polynomial approximation of u in a dyadic square Q as P_Q

The next lemma is a consequence of Poincaré inequality for Lipschitz domains.

Lemma 2.5 (Lemma 3.1, [8]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain and $\tilde{\mathcal{F}}$ a Whitney decomposition of Ω. Fix α such that $|\alpha| \leq k$. Let $\{Q_i\}_{i=1}^m$ in $\tilde{\mathcal{F}}$ be a chain of dyadic squares in $\tilde{\mathcal{F}}$.Then we have*

$$\|\nabla^\alpha(P_{Q_1} - P_{Q_m})\|_{L^p(Q_1)} \leq Cl(Q_1)^{k-|\alpha|}\|\nabla^k u\|_{L^p(\bigcup_{i=1}^m Q_i)},$$

where $\nabla^k u$ is the vector $(\nabla^\alpha u)_{|\alpha|=k}$ normed by the ℓ_2-norm and $C = C(m)$.

In what follows, given $\beta = (\beta_1, \beta_2)$ and $\alpha = (\alpha_1, \alpha_2)$, we write $\beta \leq \alpha$ if the inequality holds coordinate-wise.

3. Decomposition of the domain

From now on we fix a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ and a Whitney decomposition $\tilde{\mathcal{F}}$ of Ω given by Definition 2.2. For our purposes we need to choose at each level a nice enough subcollection of $\tilde{\mathcal{F}}_n$, namely we take connected components of the Whitney decomposition (see Figure 1). More precisely we fix $Q_0 \in \tilde{\mathcal{F}}_1$ and for each $n \in \mathbb{N}$ let C_n be the connected component of the interior of $\tilde{F}_n$ that has $\operatorname{int} Q_0$ as a subset. We define

$$\mathcal{F}_{n,j} := \{Q \in \tilde{\mathcal{F}}_j : \operatorname{int} Q \subset C_n\}$$

and using this the families of squares

$$\mathcal{F}_n := \mathcal{F}_{n,n}, \qquad \mathcal{D}_n := \bigcup_{j \leq n} \mathcal{F}_{n,j}$$

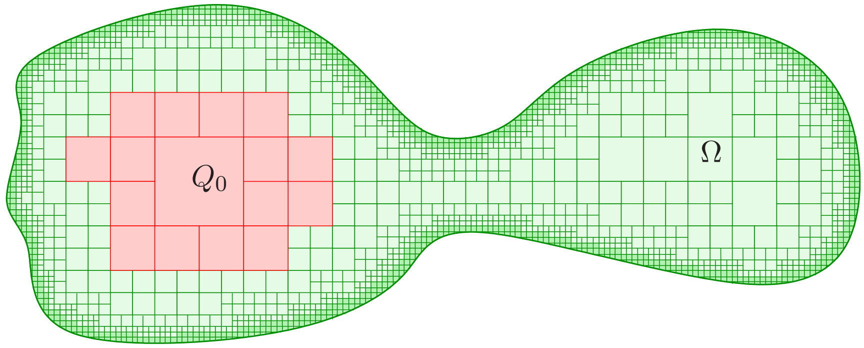

FIGURE 1. A core part D_n is selected from the Whitney decomposition of Ω by taking the connected component containing Q_0 of the interior of the union of Whitney squares with sidelength at least 2^{-n}.

and the corresponding sets for two of the above collections by

$$F_n := \bigcup_{Q \in \mathcal{F}_n} Q \quad \text{and} \quad D_n := \bigcup_{Q \in \mathcal{D}_n} Q = \overline{C}_n.$$

The collection of boundary layer squares in $\mathcal{D}_n$ is denoted by

$$\partial\mathcal{D}_n := \left\{ Q \in \mathcal{D}_n : Q \cap \overline{(\Omega \setminus D_n)} \neq \emptyset \right\}.$$

With this notation we have the following lemma.

Lemma 3.1. *The above collections have the properties:*

(i) $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ *for all* $k \in \mathbb{N}$.
(ii) $\Omega = \bigcup_{n\in\mathbb{N}} D_n$.
(iii) If $Q_1, Q_2 \in \mathcal{F}_n$ *and* $Q_1 \cap Q_2$ *is a singleton, then there exists* $Q_3 \in \mathcal{D}_n$ *for which* $Q_1 \cap Q_2 \cap Q_3 \neq \emptyset$.
(iv) If $Q \in \partial\mathcal{D}_n$, *then* $Q \in \mathcal{F}_n$.
(v) If $Q \in \partial\mathcal{D}_n$, *then* $Q \cap \overline{(\Omega \setminus \tilde{F}_n)} \neq \emptyset$.
(vi) The set C_n *is simply connected.*

Proof. The property (i) is obvious by the definitions of $\mathcal{F}_{n,j}$ and $\mathcal{D}_n$ since $C_n \subset C_{n+1}$.

For (ii) it suffices to prove that for every $Q \in \tilde{\mathcal{F}}$ there exists $n \in \mathbb{N}$ so that $Q \in \mathcal{D}_n$. Let $Q \in \mathcal{F}_n$. Since Ω is connected and open, there exists a path γ in Ω joining Q to Q_0. By the fact that $\tilde{F}_j \subset \operatorname{int} \tilde{F}_{j+1}$ and the property (W1) of the decomposition $\tilde{\mathcal{F}}$ we have that $\Omega = \cup_{j\in\mathbb{N}} \operatorname{int} \tilde{F}_j$. Then by the compactness of γ there exists $m \geq n$ so that $\gamma \subset \operatorname{int} \tilde{F}_m$. Hence $Q \in \mathcal{D}_m$.

For (iii) let $Q_1, Q_2 \in \mathcal{F}_n$ be so that $Q_1 \cap Q_2$ is a singleton $\{q\}$. Assume that the claim is false. Then for the two squares $Q \in \mathcal{Q}_n$ that intersect both Q_1 and Q_2 it is true that $Q \notin \tilde{\mathcal{F}}_n$ and $Q \not\subset Q'$ for all $Q' \in \tilde{\mathcal{F}}_{n-1}$. Let q_1 and q_2 be the centres of the squares Q_1 and Q_2 respectively. Consider a curve $\gamma' \colon [0,1] \to \Omega$ for which $\gamma'_0 = q_1$, $\gamma'_1 = q_1$ and $\gamma' \subset C_n$. Such a curve exists by the definition of C_n. We may also assume that γ' is an injective

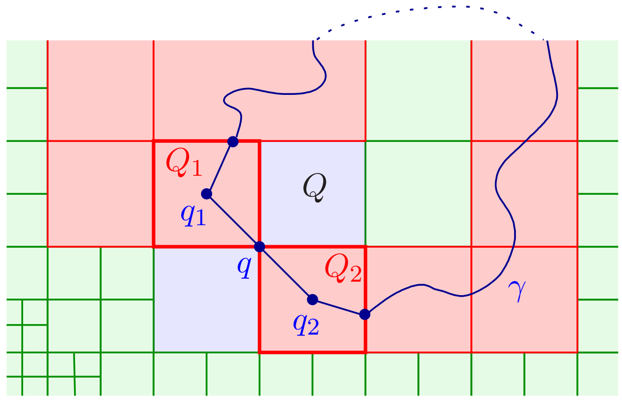

FIGURE 2. The constructed Jordan curve γ in the proof of Lemma 3.1 ((*iii*)) has in its interior domain a dyadic square Q that also has to be an element of $\mathcal{D}_n$.

curve. Let $t_0 \coloneqq \sup\{t : \gamma'(t) \in Q_1\}$ and $t_1 \coloneqq \inf\{t \geq t_0 : \gamma'(t) \in Q_2\}$. Define a Jordan curve

$$\gamma \coloneqq \gamma^1 * \gamma^2 * \gamma'|_{[t_0,t_1]} * \gamma^3 * \gamma^4,$$

where γ^1, γ^2, γ^3 and γ^4 correspond to the line segments $[q, q_1]$, $[q_1, \gamma'_{t_0}]$, $[\gamma'_{t_1}, q_2]$ and $[q_2, q]$ respectively. By Jordan curve theorem γ divides $\mathbb{R}^2$ into two components, one of which is precompact (see Figure 2). Denote the precompact component by A.

For small enough ball B around q we have by the definition of γ that $B \setminus \gamma$ has exactly two components. Since γ is a Jordan curve one of those components has to contain an interior point of A and thus the whole component lies inside A. On the other hand that component has to intersect with one of the dyadic squares in $\mathcal{Q}_n$ touching both Q_1 and Q_2 (but being different from Q_1 and Q_2). Let $Q \in \mathcal{Q}_n$ be that square. Now for all the neighbouring squares $\tilde{Q} \in \mathcal{Q}_n$ (except the opposite one) of Q either $\tilde{Q} \cap \gamma([0,1]) \neq \emptyset$ implying that $\tilde{Q} \in \mathcal{D}_n$ or $\tilde{Q}$ is in the precompact component of $\mathbb{R}^2 \setminus \gamma([0,1])$ and thus by simply connectedness $\tilde{Q} \subset \Omega$. Since $Q_1 \in \mathcal{F}_n$, also the opposite square of Q is a subset of Ω. Hence $Q \in \tilde{\mathcal{F}}_n$ or $Q \subset Q' \in \mathcal{F}_{n-1}$ which is a contradiction. Thus we have proven (*iii*).

In order to see (iv), suppose that there exists $Q \in \partial\mathcal{D}_n$ such that $Q \notin \mathcal{F}_n$. Then $Q \in \mathcal{F}_{n,i} \subset \tilde{\mathcal{F}}_i$ for some $i < n$. By Property (W4), for all the $Q' \in \tilde{\mathcal{F}}$ with $Q' \cap Q \neq \emptyset$ we have $Q' \in \tilde{\mathcal{F}}_j$ for $j \leq i+1 \leq n$. Thus, $Q' \subset D_n$ and $Q \notin \partial\mathcal{D}_n$ giving a contradiction.

If property (v) fails for some $Q \in \partial\mathcal{D}_n$, then for every $Q' \in \tilde{\mathcal{F}}$ with $Q' \cap Q \neq \emptyset$ we have $Q' \in \tilde{\mathcal{F}}_i$ for some $i \leq n$. Thus, again $Q' \subset D_n$ and $Q \notin \partial\mathcal{D}_n$ giving a contradiction.

Finally, we prove property (vi). Since C_n is open it suffices to prove that every Jordan curve is loop homotopic to a constant loop. Suppose this is not the case. Then there exists a Jordan curve γ that is not homotopic to a constant loop, and a point $x \in \Omega \setminus C_n$ that lies inside γ. In particular there exists $Q \in \mathcal{Q}_n$ such that $Q \not\subset D_n$ which lies inside γ and for which $Q \cap D_n$ is an edge of a square. Now by similar argument as in (iii) we conclude that $Q \in \mathcal{D}_n$, which is a contradiction. $\square$

The next lemma shows that we can connect the boundary of D_n to the boundary of Ω with a short curve in the complement of D_n.

Lemma 3.2. *For each point $x \in \partial D_n$, there exists an injective curve $\gamma\colon [0,1] \to \mathbb{R}^2$ so that $\gamma(0) = x$, $\gamma(1) \in \partial\Omega$, $\gamma(0,1) \subset \Omega \setminus \operatorname{int} D_n$ and $L(\gamma) \le 2\sqrt{2}l(Q)$.*

Proof. Let $Q \in \partial\mathcal{D}_n$ be such that $x \in Q \cap \partial D_n$. By Lemma 3.1 (v) we have that there exists a square $Q' \in \mathcal{Q}_n$ touching Q at x so that $Q' \notin \tilde{\mathcal{F}}_n$ and $Q' \not\subset \tilde{Q}$ for every $\tilde{Q} \in \tilde{F}_j$, $j < n$. Thus, there exists a neighbouring square $Q'' \in \mathcal{Q}_n$ of Q' and a point $y \in \partial\Omega \cap Q''$. Let γ^1 be a curve corresponding to a line segment connecting x to a point $z \in Q' \cap Q''$ and let γ^2 be a curve corresponding to a line segment connecting z to y. Moreover, let $t_0 := \inf\{t : \gamma^2_t \in \partial\Omega\}$. Since $\partial\Omega$ is closed, we have that $\gamma^2_{t_0} \in \partial\Omega$. Define a curve $\gamma := \gamma^1 * \gamma^2|_{[0,t_0]}$. For γ we have that $\gamma(0,1) \subset (\Omega \setminus \operatorname{int} D_n) \cap (Q' \cup Q'')$, $\gamma(0) = x$, $\gamma(1) \in \partial\Omega$ and $L(\gamma) \le d(Q') + d(Q'') = 2\sqrt{2}l(Q)$. □

Observe that by Lemma 3.1 (iv) we have $\partial D_n = \bigcup_{Q \in \partial\mathcal{D}_n}(Q \cap \partial D_n)$. Thus, by Lemma 3.1 (iii) we have that ∂D_n is locally homeomorphic to the real line. Since by Lemma 3.1 (vi) C_n is simply connected, we have that $\partial D_n = \partial C_n$ is connected. Hence, ∂D_n is a Jordan curve. Thus, we may write

$$\partial D_n = \bigcup_{i=1}^{L_n} I_i, \tag{3.1}$$

where $I_i = [y_i, y_{i+1}]$ is an edge of a square in $\mathcal{F}_n$ with vertices y_i and y_{i+1}, and $y_1 = y_{L_n+1}$.

For the rest of the paper we fix a constant $M > (4\sqrt{2} + 2)$. However, the following lemma is true for any $M > 0$ and with C depending on M.

Lemma 3.3. *There exists $C \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ and $x, y \in \partial D_n$ with $d_{\partial D_n}(x, y) \ge 2^{-n}C$, and for any γ in $\Omega \setminus \operatorname{int} D_n$ connecting x to y we have that $\gamma \cap (\Omega \setminus B(x, M2^{-n})) \ne \emptyset$. In particular, $L(\gamma) \ge M2^{-n}$.*

Proof. By taking a slightly larger C, namely $C + 2$, we may assume that $x = y_i$ and $y = y_j$ for some i and j, where y_i, y_j are two endpoints of intervals from the collection $\{I_i\}$ forming the boundary as noted above. Moreover, by symmetry we may assume that $i < j$ and $j - i \le n + 1 - j$. Since each I_i is a side for two squares in $\mathcal{Q}_n$, by taking C large enough, we obtain

$$\left|B(x, 2(M+1)2^{-n})\right| = \pi(2(M+1)2^{-n})^2 < \frac{1}{2}C(2^{-n})^2 \le \left|\bigcup Q\right|,$$

where the union is taken over all $Q \in \mathcal{Q}_n$ having I_m as one of it sides for some $i < m \le j - 1$, and $|\cdot|$ denotes the 2-dimensional Lebesgue measure. Therefore, one of the intervals I_{m_1}, for $i < m_1 \le j - 1$, has to intersect with the complement of the ball $B(x, 2M2^{-n})$. Let $Q'_1 \in \partial\mathcal{D}_n$ be the boundary square corresponding to that interval and let $q_1 \in I_{m_1} \setminus B(x, 2M2^{-n})$. By symmetry, there also exists $Q'_2 \in \partial\mathcal{D}_n$ whose side is some I_{m_2} with $m_2 \notin \{i+1, i+1, \ldots, j-1\}$ such that there is $q_2 \in I_{m_2} \setminus B(x, 2M2^{-n})$.

Suppose now that there exists a curve γ in $\Omega \setminus \operatorname{int} D_n$ joining x to y with $\gamma \subset B(x, M2^{-n})$. We may assume that γ is injective, and by compactness that $\gamma(t) \in \Omega \setminus D_n$ for every

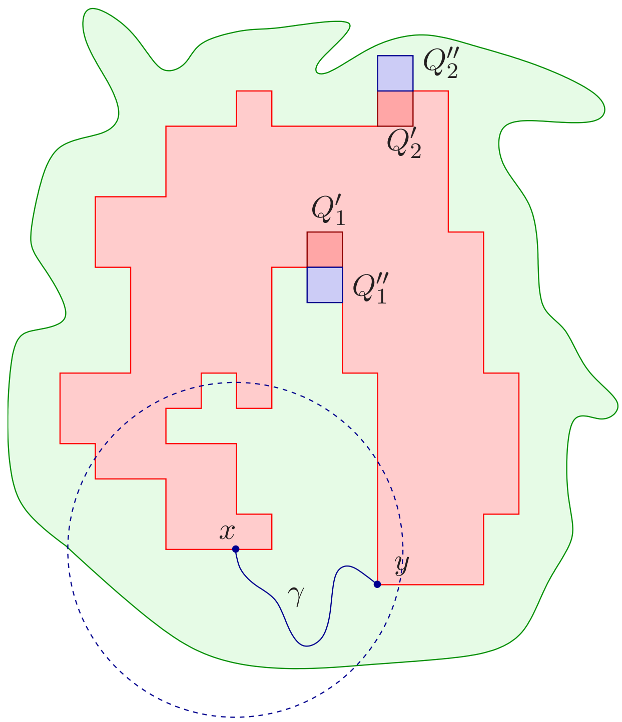

FIGURE 3. In the proof of Lemma 3.3 we assume towards a contradiction that x and y can be connected by a short curve γ in $\Omega \setminus D_n$. This will imply that one more square in $\mathcal{Q}_n$ (here Q_1'') will be a subset of D_n.

$t \in (0,1)$. Then, for $i = 1,2$ we have that $B(Q_i', 2\sqrt{2}l(Q_i')) \subset B(q, M2^{-n})$ and hence $B(Q_i', 2\sqrt{2}l(Q')) \cap \gamma = \emptyset$. Now by definition of Q_i' there is a neighbouring square $Q_i'' \in \mathcal{Q}_n$ of Q_i' which is not a subset of D_n, see Figure 3. We claim that either Q_1'' or Q_2'' lies inside the Jordan curve γ' obtained by concatenating the curve γ and the part of the boundary, denoted by γ'', obtained from the intervals $\{I_h\}_{h=i}^{j-1}$, or by concatenating γ and $\partial D_n \setminus \gamma''$.

This can be seen in the following way. Consider $\Omega \xrightarrow{h} \mathbb{R}^2 \hookrightarrow \mathbb{S}^2$, where h is a homeomorphism and the inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{S}^2$ is the inverse of the stereographic projection. Under this composite map $\mathbb{S}^2 \setminus D_n$ is a simply connected domain. Hence, by Lemma 2.1 $(\mathbb{S}^2 \setminus D_n) \setminus \gamma$ has exactly two components whose boundaries are the two connected components of $\partial D_n \setminus \gamma$ together with γ. Thus, $(\Omega \setminus D_n) \setminus \gamma = (\mathbb{S}^2 \setminus D_n) \setminus \gamma$ has exactly two components. Since $\partial Q_1'' \cap \partial D_n$ and $\partial Q_2'' \cap \partial D_n$ are in two different connected components of $\partial D_n \setminus \gamma$, we conclude that Q_1'' and Q_2'' are in different components of $(\Omega \setminus D_n) \setminus \gamma$. We denote the Q_i'' that lies inside the Jordan curve by Q''.

Since $Q'' \subset B(Q', \sqrt{2}l(Q'))$, we have that every neighbouring square of Q'' either lies inside γ' or is an element of $\partial \mathcal{D}_n$. In particular, by the simply connectedness of Ω they all are subsets of Ω. Hence, $Q'' \subset D_n$ which is a contradiction. Thus, we have proven that $\gamma \cap (\Omega \setminus B(x, M2^{-n})) \neq \emptyset$. □

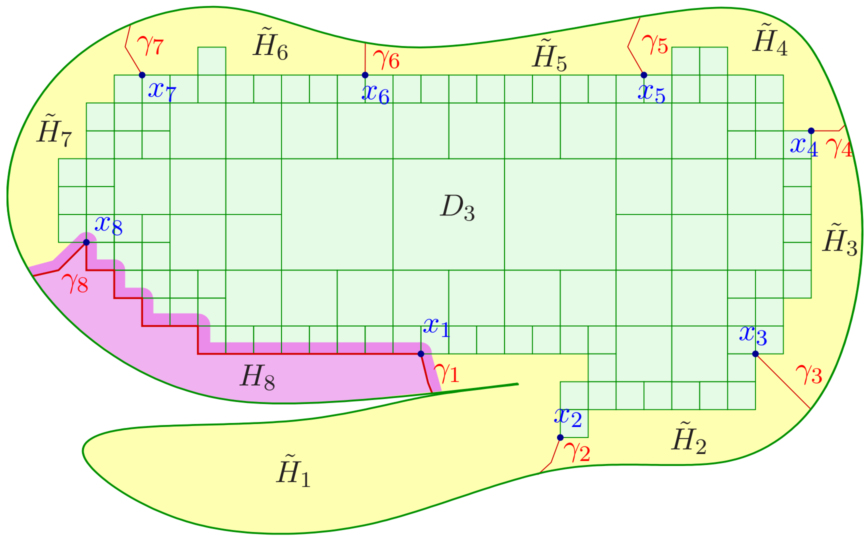

FIGURE 4. Here the domain Ω is decomposed into the core part D_3 and eight boundary parts $\tilde{H}_i$. A neighbourhood H_8 of $\tilde{H}_8$ is also illustrated.

Let us now partition $\Omega \setminus D_n$ in the following way. Recall (3.1). Notice that for large enough n we have that $L_n \geq 2C$. Define $x_1 := y_1$ and then $x_m := y_{(m-1)C}$ until $L_n+1-(m-1)C < 2C$. Notice that for every $i \neq j$ we have $d_{\partial D_n}(x_i, x_j) \geq 2^{-n}C$. We now partition the set $\Omega \setminus D_n$ up to Lebesgue measure zero into connected sets $\{\tilde{H}_j\}_{j=1}^m$ where $\tilde{H}_j$ is the open set bounded by γ_j, γ_{j+1} given by Lemma 3.2 for points x_j and x_{j+1}, and $J_j := \bigcup_{i=Cj}^{C(j+1)} I_i$ (with interior in $\Omega \setminus D_n$). This partition is well defined by Lemma 2.1. Notice that since $L(\gamma_i) \leq M$ for all i, we have that $\gamma_i \cap \gamma_j = \emptyset$ for all $i \neq j$. Let us define H_j as the connected component containing $\tilde{H}_j$ of the set $\Omega \cap \left(\tilde{H}_j \cup B_{\mathbb{R}^2}(\gamma_j \cup \gamma_{j+1} \cup J_j, \delta)\right)$, where $\delta = 2^{-n-3}$. See Figure 4 for an illustration of the decomposition. Although the decomposition depends on n, for simplicity we do not display the dependence in the notation. A crucial property of our decomposition is the following lemma.

Lemma 3.4. *We have $H_j \cap H_i \neq \emptyset$, if and only if $|i-j| \leq 1$ in a cyclic manner.*

Proof. Trivially $\gamma_{i+1} \in H_i \cap H_{i+1}$. Thus, we only need to show that $H_j \cap H_i \neq \emptyset$ implies $|i-j| \leq 1$. We may assume that $i \neq j$. Let $x \in H_i \cap H_j$.

Suppose first that $x \in \tilde{H}_i$. Then, by (path) connectedness of H_j there exists a path γ in H_j from x to $\tilde{H}_j$. Let

$$t_0 := \inf\{t \in [0,1] : \gamma(t) \notin \tilde{H}_i\}.$$

Then, $\gamma(t_0) \notin \tilde{H}_i$ but $\gamma(t) \in H_i \cap H_j$. Thus it suffices to consider the case when $x \notin \tilde{H}_i \cup \tilde{H}_j$.

Suppose now that $x \in D_n$. Since $\delta < \frac{2^{-n}}{2}$, we have that $x \in Q$ for some $Q \in \partial\mathcal{D}_n$. Then, there are neighbouring squares $Q_i, Q_j \in \mathcal{Q}_n$ of Q for which $Q_i \cap \tilde{H}_i \neq \emptyset$ and $Q_j \cap \tilde{H}_j \neq \emptyset$. Since δ is small, we may choose the Q_i, Q_j so that $Q_i \cap Q_j \neq \emptyset$. If $Q_i = Q_j$ or if Q_i and Q_j

have a common edge, then there is a curve γ' in $Q_i \cup Q_j$ from $\tilde{H}_i$ to $\tilde{H}_j$ with $L(\gamma') < 2\delta$. If $Q_i \cap Q_j$ is a singleton, then by Lemma 3.1 (iii) the neighbouring square $Q' \neq Q$ of both Q_i and Q_j lies in $\Omega \setminus \operatorname{int} D_n$. Indeed, if this were not the case, then $Q', Q \in \mathcal{F}_n$ and $Q' \cap Q$ is a singleton, implying that $Q_i \in \mathcal{D}_n$ or $Q_j \in \mathcal{D}_n$. Thus, there exists a curve γ' in $\Omega \setminus D_n$ joining Q_i and Q_j with $L(\gamma') < 4\delta$.

Now, we have $Q_i \cap J_i \neq \emptyset$ or $Q_i \cap (\gamma_i \cup \gamma_{i+1}) \neq \emptyset$. Notice that $\gamma_i \cap J_i \neq \emptyset \neq \gamma_{i+1} \cap J_i$. By Lemma 3.2 we have $\max(l(\gamma_i), l(\gamma_{i+1})) \leq 2\sqrt{2} \cdot 2^{-n}$. Combining these observations with the analogous ones for Q_j, we have that J_i and J_j can be connected by a curve in $\Omega \setminus D_k$ with length less than $4\delta + 4\sqrt{2} \cdot 2^{-n} < 2^{-n}M$. Hence, we have by Lemma 3.3 that $\operatorname{dist}_{\partial D_n}(J_i, J_j) \leq C$. Thus, $|i - j| \leq 1$ in cyclical manner.

We are left with the case where $x \in \Omega \setminus (D_n \cup \tilde{H}_i \cup \tilde{H}_j)$. By definition we have that $B(D_n, 2\delta) \subset \Omega$. Thus, if $\operatorname{dist}(x, J_i) < \delta$, we may join x to J_i by a curve in $\Omega \setminus \operatorname{int} D_n$ with length less than δ. If $\operatorname{dist}(x, J_i) \geq \delta$, then $x \in B(\gamma_m, \delta)$, where $m \in \{i, i+1\}$. By path connectedness of H_i there is a curve γ in H_i connecting x to $\gamma_i \cup \gamma_{i+1} \cup J_i$. We want to prove that x can be joined to γ_m in the δ-neighbourhood of γ_m. If (a subcurve of) γ is not such a curve, then we may define

$$t_0 := \inf\{t \in [0,1] : \gamma(t) \in B(D_n, \delta)\}.$$

Then, $\gamma|_{[0,t_0]} \subset B(\gamma_m, \delta)$. Therefore, there exists a point $y \in \gamma_m$ with $d(\gamma(t_0), y) < \delta$. In particular, the line segment $[\gamma(t_0), y]$ lies in $(\Omega \setminus D_n) \cap B(\gamma_m, \delta)$ and thus we have proven that there exists a curve γ' in $(\Omega \setminus D_n) \cap B(\gamma_m, \delta)$ connecting x to γ_m. By the definition of γ_m we have that $\gamma' \subset B(\gamma_m(0), 2\sqrt{2} \cdot 2^{-n} + \delta)$. By the same argument for j we conclude that J_i and J_j can actually be connected by a curve γ in $(\Omega \setminus \operatorname{int} D_n) \cap B(\gamma(0), 4\sqrt{2} \cdot 2^{-n} + 2\delta)$. Hence, by Lemma 3.3 $\operatorname{dist}_{\partial D_n}(J_i, J_j) < C$, and thus $|i - j| \leq 1$ in cyclical manner. □

4. Approximation

In this section we finish the proof of Theorem 1.1 by making a partition of unity using the decomposition of Ω constructed in Section 3 and by approximating a given function by polynomials in this decomposition. Recall that our aim is to show that for any $u \in L^{k,p}(\Omega)$ and $\epsilon > 0$ there exists a function $u_\epsilon \in W^{k,\infty}(\Omega) \cap C^\infty(\Omega)$ with $\|\nabla^k u - \nabla^k u_\epsilon\|_{L^p(\Omega)} \lesssim \epsilon$. By noting that $L^{k,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $L^{k,p}(\Omega)$ we may assume that function $u \in L^{k,p}(\Omega) \cap C^\infty(\Omega)$. From now on, let u and $\epsilon > 0$ be fixed.

Using the notation from Section 3, we write the domain Ω as the union of the core part D_n and the boundary regions $\{H_i\}_{i=1}^l$. For each $\tilde{H}_i$ we let $\mathcal{I}_i$ be the collection of squares Q in $\partial\mathcal{D}_n$ such that $Q \cap \tilde{H}_i \neq \emptyset$, and note that the cardinality of $\mathcal{I}_i$ is bounded in number independently of n. We need to decide what polynomial to attach to each set H_i. For this purpose, for each $1 \leq i \leq l$ we assign a square $Q_i \in \mathcal{I}_i$. We call Q_i the associated square of H_i.

Given $Q \in \mathcal{I}_i$ we set $\mathcal{P}_Q := \bigcup_{j=i-1}^{i+1}\{Q' \in \mathcal{I}_j\}$, which is a collection of squares from a suitable neighbourhood of Q.

Recall the approximating polynomials P_Q introduced in Section 2.2. We abbreviate $P_i = P_{Q_i}$ for the associated squares Q_i.

We make a smooth partition of unity by using a Euclidean mollification. (Compare to [14] where the inner distance in Ω was used for the mollification.) Let ρ_r denote a standard Euclidean mollifier supported in $B(0,r)$. We start with a collection of functions $\{\tilde{\psi}_i\}_{i=0}^l$, where $\tilde{\psi}_0 = \chi_{D_n} * \rho_{2^{-n-5}}$ and $\tilde{\psi}_i = \left(\chi_{\tilde{H}_i} * \rho_{2^{-n-5}}\right)|_{H_i}$ for $i \geq 1$. Using this we obtain a partition of unity $\{\psi_i\}_{i=0}^l$ by setting $\psi_i = \tilde{\psi}_i / \sum_{j=0}^l \tilde{\psi}_j$.

Now the partition of unity $\{\psi_i\}_{i=0}^l$ satisfies the following.

(1) The function ψ_0 is supported in $B(D_n, \frac{2^{-n}}{10})$.
(2) For $i \geq 1$ the function ψ_i is supported in H_i.
(3) For all i, $0 \leq \psi_i \leq 1$.
(4) $\sum \psi_i \equiv 1$ on Ω.
(5) For all i, $|\nabla^\alpha \psi_i| \leq C_\alpha 2^{-n|\alpha|}$ for all multi-indeces α.

We will fix n later such that for the function u_ϵ defined as

$$u_\epsilon(x) := u(x)\psi_0(x) + \sum_{i=1}^{l} \psi_i(x) P_i(x)$$

for $x \in \Omega$, we have

$$\|\nabla^k u - \nabla^k u_\epsilon\|_{L^p(\Omega)} < C\epsilon.$$

Note that $u_\epsilon = u$ on D_{n-1}; indeed $D_{n-1} \cap \psi_i = \emptyset$ for $i \geq 1$, see Lemma 3.1 (iv).

First of all, we consider only n large enough so that

$$\|\nabla^k u\|_{L^p(\Omega \setminus D_{n-1})} \leq \epsilon. \tag{4.1}$$

Now, we need to show that n can actually be chosen large enough so that also

$$\|\nabla^k u_\epsilon\|_{L^p(\Omega \setminus D_{n-1})} \leq C\epsilon.$$

So, we compute for $Q \in \mathcal{I}_i$ and $|\alpha| = k$

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(Q)} \leq & \sum_{\beta \leq \alpha} \left(\int_Q |\nabla^\beta u - \nabla^\beta P_i(x)|^p |\nabla^{\alpha-\beta} \psi_0(x)|^p \, \mathrm{d}x \right)^{1/p} \\ & + \sum_{\beta \leq \alpha} \sum_j \left(\int_Q |\nabla^\beta P_j(x) - \nabla^\beta P_i(x)|^p |\nabla^{\alpha-\beta} \psi_j(x)|^p \, \mathrm{d}x \right)^{1/p} \\ & =: A_1 + A_2, \end{aligned} \tag{4.2}$$

where A_1 and A_2 are the first and second terms on the right hand side of the inequality and we used that for $\beta < \alpha$, $\sum_j \nabla^{\alpha-\beta} \psi_j = 0$ and order of P_i is at most $k-1$. We first estimate A_1 as

$$\begin{aligned} A_1 &\lesssim \sum_{\beta\le\alpha} 2^{n(|\alpha|-|\beta|)}\|\nabla^\beta u-\nabla^\beta P_i\|_{L^p(Q)}\\ &\lesssim \sum_{\beta\le\alpha} 2^{n(|\alpha|-|\beta|)}(\|\nabla^\beta P_i-\nabla^\beta P_Q\|_{L^p(Q)}+\|\nabla^\beta u-\nabla^\beta P_Q\|_{L^p(Q)})\\ &\lesssim \sum_{\beta\le\alpha} 2^{n(|\alpha|-|\beta|)}2^{n(|\beta|-k)}\|\nabla^k u\|_{L^p(\cup\tilde{Q})}\\ &\lesssim \|\nabla^k u\|_{L^p(\cup\tilde{Q})}, \end{aligned} \tag{4.3}$$

where in the third inequality we used that Q_i (associated square of $\tilde{H}_i$) and Q may be joined by a chain of bounded number of squares from $\mathcal{I}_i$ by our construction, and therefore we may apply Lemma 2.5. Similarly we estimate A_2 as

$$\begin{aligned} A_2 &\lesssim \sum_{\beta\le\alpha}\sum_{j=i-1}^{i+1}\left(\int_Q |\nabla^\beta P_j(x)-\nabla^\beta P_i(x)|^p|\nabla^{\alpha-\beta}\psi_j(x)|^p\,\mathrm{d}x\right)^{1/p}\\ &\lesssim \sum_{\beta\le\alpha} 2^{n(|\alpha|-|\beta|)}\sum_{j=i-1}^{i+1}(\|\nabla^\beta P_j-\nabla^\beta P_Q\|_{L^p(Q)}+\|\nabla^\beta P_i-\nabla^\beta P_Q\|_{L^p(Q)})\\ &\lesssim \sum_{\beta\le\alpha} 2^{n(|\alpha|-|\beta|)}2^{n(|\beta|-k)}\|\nabla^k u\|_{L^p(\cup\tilde{Q})}\\ &\lesssim \|\nabla^k u\|_{L^p(\cup\tilde{Q})}, \end{aligned} \tag{4.4}$$

where again in the second inequality we used that if $\psi_j(x)\neq 0$ for $x\in Q\in\mathcal{I}_i$ then by our construction Q_j and Q can be joined by a chain of bounded number of squares as j is either $i-1, i$ or $i+1$ (cyclically); and therefore we can apply Lemma 2.5.

For $Q\in\tilde{\mathcal{F}}\setminus\mathcal{D}_n$ such that $Q\cap\operatorname{spt}(\psi_0)\neq\emptyset$, we assign to Q a square $Q'\in\mathcal{I}_i$, such that $Q\cap Q'\neq\emptyset$. Note that such a square Q' exists by our construction. Then Q and Q' can be joined by a chain of bounded (by an absolute constant) number of squares from $\mathcal{D}_{n+1}$. We choose such a chain for Q and denote it by $\mathcal{B}_Q$. We also set

$$\mathcal{J}_n := \{Q\in\tilde{\mathcal{F}}\setminus\mathcal{D}_n : Q\cap\operatorname{spt}(\psi_0)\neq\emptyset\}.$$

We estimate using Lemma 2.5 exactly as above (see (4.2)) to obtain for $|\alpha|=k$

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(Q)} &\le \sum_{\beta\le\alpha}\left(\int_Q |\nabla^\beta u-\nabla^\beta P_Q(x)|^p|\nabla^{\alpha-\beta}\psi_0(x)|^p\,\mathrm{d}x\right)^{1/p}\\ &+\sum_{\beta\le\alpha}\sum_j\left(\int_Q |\nabla^\beta P_j(x)-\nabla^\beta P_Q(x)|^p|\nabla^{\alpha-\beta}\psi_j(x)|^p\,\mathrm{d}x\right)^{1/p}\\ &=: B_1+B_2. \end{aligned} \tag{4.5}$$

Again, we estimate separately,

$$\begin{aligned} B_1 &\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)} \|\nabla^\beta u - \nabla^\beta P_Q\|_{L^p(Q)} \\ &\lesssim \|\nabla^k u\|_{L^p(Q)} \end{aligned}$$

and

$$\begin{aligned} B_2 &\lesssim \sum_{\beta\leq\alpha}\sum_{j=i-1}^{i+1}\left(\int_Q |\nabla^\beta P_j(x) - \nabla^\beta P_Q(x)|^p |\nabla^{\alpha-\beta}\psi_j(x)|^p \,\mathrm{d}x\right)^{1/p} \\ &\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)} \sum_{j=i-1}^{i+1} (\|\nabla^\beta P_j - \nabla^\beta P_{Q'}\|_{L^p(Q)} + \|\nabla^\beta P_{Q'} - \nabla^\beta P_Q\|_{L^p(Q)}) \\ &\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)} 2^{n(|\beta|-k)} (\|\nabla^k u\|_{L^p(\bigcup_{Q''\in\mathcal{P}_{Q'}} Q'')} + \|\nabla^k u\|_{L^p(\bigcup_{Q''\in\mathcal{B}_{Q'}} Q'')}) \\ &\lesssim \|\nabla^k u\|_{L^p(\bigcup_{Q''\in\mathcal{P}_{Q'}\cup\mathcal{B}_{Q'}} Q'')}. \end{aligned}$$

Next we note that $\nabla^k u_\epsilon \equiv 0$ in $\tilde{H}_i \setminus \cup_{j\neq i} \operatorname{spt}(\psi_j)$ and we compute for $|\alpha| = k$

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(H_i)} \leq &\sum_{Q\in\tilde{Q}_i} \|\nabla^\alpha u_\epsilon\|_{L^p(Q)} + \sum_{Q\in\mathcal{J}_n, Q\cap H_i\neq\emptyset} \|\nabla^\alpha u_\epsilon\|_{L^p(Q)} \\ &+ \sum_{j=i-1}^{i+1} \|\nabla^\alpha u_\epsilon\|_{L^p((\operatorname{spt}(\psi_j)\cap\operatorname{spt}(\psi_i))\setminus \bigcup_{Q''\in\partial\mathcal{D}_n\bigcap\mathcal{J}_n} Q'')} \end{aligned} \tag{4.6}$$

The terms in the first and second summands have been estimated earlier. Denoting $H_i' := (\cup_{j=i-1}^{i+1}\operatorname{spt}(\psi_j)\cap\operatorname{spt}(\psi_i)) \setminus \bigcup_{Q''\in\partial\mathcal{D}_n\bigcap\mathcal{J}_n} Q''$, we estimate now the third one;

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(H_i')} &\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)} \sum_{j=i-1}^{i+1} \|\nabla^\beta P_j - \nabla^\beta P_i\|_{L^p(H_i')} \\ &\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)} (\|\nabla^\beta P_j - \nabla^\beta P_i\|_{L^p(Q_i)} \\ &\lesssim \sum_{\beta\leq\alpha} 2^{n(|\alpha|-|\beta|)} 2^{n(|\beta|-k)} \|\nabla^k u\|_{L^p(\bigcup_{Q'\in\mathcal{P}_{Q_i}} Q')} \\ &\lesssim \|\nabla^k u\|_{L^p(\bigcup_{Q'\in\mathcal{P}_{Q_i}} Q')}, \end{aligned} \tag{4.7}$$

where we used the facts that for $\beta < \alpha$, $\nabla^{\alpha-\beta}\sum_j(\psi_j) = 0$ and $\psi_0 \equiv 0$ in H_i' in the first inequality, Lemma 2.4 in the second inequality since $H_i' \subset CQ_i$ for some absolute constant C coming from Lemma 3.2 and in the third inequality we used Lemma 2.5.

Remark 4.1. Note that for each $Q \in \mathcal{I}_i$ we have $\mathcal{P}_Q = \mathcal{P}_{Q_i}$ where Q_i is the associated square of $\tilde{H}_i$. We note that any $Q' \in \partial\mathcal{D}_n$ occurs in at most three distinct collections $\mathcal{P}_{Q_i}$. Moreover any $Q \in \mathcal{D}_{n+1}$ appears in only a bounded number of the collections $\mathcal{B}_{Q''}$, where $Q'' \in \mathcal{J}_n$. In particular, any $Q' \in \partial\mathcal{D}_n$ appears in only a bounded number of the collections $\mathcal{B}_{Q''}$, where $Q'' \in \mathcal{J}_n$. The bounds are provided by absolute constants coming from volume comparison.

Now it follows from equations (4.3), (4.4), (4.5), (4.6) and (4.7) that

$$\begin{aligned} \|\nabla^\alpha u_\epsilon\|_{L^p(\Omega\setminus C_n)} &\lesssim \sum_i \|\nabla^k u\|_{L^p(H_i)} + \|\nabla^k u\|_{L^p(\bigcup_{Q\in\partial\mathcal{D}_n} Q)} \\ &\lesssim \|\nabla^k u\|_{L^p(\bigcup_{Q\in\partial\mathcal{D}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{Q\in\mathcal{J}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{\substack{Q\in\mathcal{J}_n\\ Q'\in\mathcal{B}_Q}} Q')} \end{aligned} \tag{4.8}$$

when $|\alpha| = k$. By Remark 4.1 we may choose n such that

$$\|\nabla^k u\|_{L^p(\bigcup_{Q\in\partial\mathcal{D}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{Q\in\mathcal{J}_n} Q)} + \|\nabla^k u\|_{L^p(\bigcup_{\substack{Q\in\mathcal{J}_n\\ Q'\in\mathcal{B}_Q}} Q')} < \epsilon.$$

Then, the claim follows from (4.1) and (4.8).

Remark 4.2. We note that when $k = 1$ we may take the function to be smooth as well as bounded for showing the density of $W^{1,\infty}(\Omega)$ in $W^{1,p}(\Omega)$. This is because truncations approximate the functions in $W^{1,p}(\Omega)$. This allows us to also approximate the L^p norm of u. Indeed, let $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ such that $\|u\|_{L^\infty} \le M$. Decompose the domain as in the above construction; then choose n large enough such that $\|u\|_{W^{1,p}(\Omega\setminus D_{n-1})} \le \epsilon$ and $M|\Omega \setminus D_{n-1}| < \epsilon$. Then it follows from estimates in the proof that the function u_ϵ defined as above approximates u in $W^{1,p}(\Omega)$ with error given by ϵ. This conclusion is the content of [14].

Finally, let us show how the smooth approximation in Jordan domains is done.

Proof of Corollary 1.3. The argument we need follows the one used to prove [14, Corollary 1.2]. As in [14], given a bounded Jordan domain we approximate it from outside by a nested sequence of Lipschitz and simply connected domains G_s which are obtained for example by taking the complement of the unbounded connected component of the union of Whitney squares larger than 2^{-s} from the Whitney decomposition of the complementary Jordan domain of Ω.

Then, we note that for given n, taking s_n large enough, we have that the squares in $\partial\mathcal{D}_n$ are Whitney type sets in G_{s_n}, meaning they have diameters comparable to the distance from the boundary of G_{s_n}.

Note that $G_{s_n} \subset B(\Omega, 2^{-s_n+5})$ are simply connected. Now the set $G_{s_n} \setminus \bar{C}_n$ (recall that C_n is a suitable connected component of the interior of the union of the Whitney squares of scale less than 2^{-n}) can be decomposed in the same way as $\Omega \setminus \bar{C}_n$ was decomposed into the sets $\tilde{H}_i$ in Section 3.

We may then follow the argument used in the proof of Theorem 1.1 to obtain an approximating sequence of functions u_n in G_{s_n} which are in the space $W^{k,\infty}(G_{s_n}) \cap L^{k,p}(G_{s_n}) \cap C^\infty(G_{s_n})$. By multiplying with a smooth cut-off function that is 1 on Ω and compactly supported in G_{s_n}, we obtain a sequence of global smooth functions having the desired properties. □

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