JYU DISSERTATIONS 161

Jesse Railo

Geodesic Tomography Problems on Riemannian Manifolds

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS
AND SCIENCE

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Foreword

I wish to thank my advisor, Mikko Salo, for his support and help during my PhD studies at the University of Jyväskylä. He has been the best teacher and academic role model that I could have hoped for. I owe him a great debt of gratitude. I thank the Department of Mathematics and Statistics for giving me a friendly working environment and support in 2015–2019.

I wish to thank my collaborators, Joonas Ilmavirta, Olli Koskela, and Jere Lehtonen, for many productive and instructive discussions. I have learned many good ways of thinking and working from you. I also thank any other colleagues I have interacted with and whom are not mentioned here by name.

I wish to thank François Monard who has agreed to be the opponent at the public examination of my dissertation. I wish to thank Todd Quinto and Hanming Zhou for their preliminary examinations of my dissertation, which I have already received when writing this.

Finally, I thank my wife, Heli, for supporting and loving me during these years. She has been very compassionate for my, sometimes, comical working hours and habit to work at home. I thank my family and friends for offering me many great opportunities to take a break from mathematics to do something completely different and fun.

Jyväskylä, November 11, 2019
Department of Mathematics and Statistics
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List of included articles

This dissertation consists of an introductory part and the following four articles:

(A) Jere Lehtonen, Jesse Railo and Mikko Salo. *Tensor tomography on Cartan-Hadamard manifolds.* Inverse Problems 34 (2018), special issue: 100 years of the Radon transform, no. 4, 044004.

(B) Joonas Ilmavirta and Jesse Railo. *Geodesic ray transform with matrix weights for piecewise constant functions.* Preprint (2019), arXiv:1901.03525.

(C) Joonas Ilmavirta, Olli Koskela and Jesse Railo. *Torus computed tomography.* Preprint (2019), arXiv:1906.05046.

(D) Jesse Railo. *Fourier analysis of periodic Radon transforms.* Preprint (2019), arXiv:1909.00495.

The author of this dissertation has actively taken part in the research of the joint articles (A), (B) and (C).

Tiivistelmä

Väitöskirjassa tutkitaan integraaligeometriaan liittyviä inversio-ongelmia. Geodeettinen sädemuunnos on operaattori, joka laskee funktion polkuintegraalin geodeesia pitkin. Väitöskirjassa määritetään monia ehtoja, joilla tällainen tieto määrää funktion yksikäsitteisesti ja vakaasti. Lisäksi osana väitöskirjan työtä on toteutettu numeerinen malli, jota voidaan käyttää tietokonetomografiassa.

Väitöskirjan johdannossa esitetään inversio-ongelmien peruskäsitteitä ja tietokonetomografiaan läheisesti liittyviä matemaattisia malleja. Johdannon pääpaino on integraaligeometriaan liittyvien mallien määrittelyssä, tutkimusaiheen kirjallisuuskatsauksessa ja väitöskirjan tutkimustulosten esittelyssä. Lisäksi annetaan lista integraaligeometrian tärkeistä avoimista matemaattisista ongelmista.

Väitöskirjan ensimmäisessä artikkelissa osoitetaan, että symmetrinen solenoidaalinen tensorikenttä voidaan määrätä yksikäsitteisesti sen geodeettisesta sädemuunnoksesta Cartan-Hadamard monistolla, kun tietyt geometriasta riippuvat vähenemisehdot täyttyvät. Tutkittu integraalimuunnos esiintyy sirontaan liittyvissä käänteisongelmissa kvanttifysiikassa ja yleisessä suhteellisuusteoriassa.

Väitöskirjan toisessa artikkelissa näytetään, että paloittain vakio vektoriarvoinen funktio voidaan määrittää yksikäsitteisesti sen matriisipainotetusta geodeettisesta sädemuunnoksesta reunallisella Riemannin monistolla, jos geometria sallii aidosti konveksin funktion olemassaolon ja epäsingulaarinen matriisipaino riippuu jatkuvasti sen sijainnista moniston yksikköpallokimpulla. Tällaista integraalimuunnosta voidaan käyttää mallintamaan attenuoitua sädemuunnosta sekä inversio-ongelmia konnektiolle ja Higgsin kentälle.

Väitöskirjan kolmannessa ja neljännessä artikkelissa tutkitaan geodeettista sädemuunnosta suljettujen geodeesien yli toruksella, kun funktioiden säännöllisyys on alhainen. Neljännessä artikkelissa tarkastellaan lisäksi tällaisen muunnoksen yleistystä, kun funktion integraalit tunnetaan isometrisesti upotettujen alempiasteisten toruksien yli. Artikkeleissa todistetaan uusia rekonstruktiokaavoja, regularisointistrategioita ja vakausestimaatteja tällaisille integraalimuunnoksille. Saaduilla tutkimustuloksilla on sovelluskohteita erilaisissa laskennallisissa tomografiamenetelmissä.

Abstract

This dissertation is concerned with integral geometric inverse problems. The geodesic ray transform is an operator that encodes the line integrals of a function along geodesics. The dissertation establishes many conditions when such information determines a function uniquely and stably. A new numerical model for computed tomography imaging is created as a part of the dissertation.

The introduction of the dissertation contains an introduction to inverse problems and mathematical models associcated to computed tomography. The main focus is in definitions of integral geometry problems, survey of the related literature, and introducing the main results of the dissertation. A list of important open problems in integral geometry is given.

In the first article of the dissertation, it is shown that a symmetric solenoidal tensor field can be determined uniquely from its geodesic ray transform on Cartan-Hadamard manifolds, when certain geometric decay conditions are satisfied. The studied integral transforms appear in inverse scattering theory in quantum physics and general relativity.

In the second article of the dissertation, it is shown that a piecewise constant vector-valued function can be determined uniquely from its geodesic ray transform with a continuous and non-singular matrix weight on Riemannian manifolds that admit a strictly convex function and have a strictly convex boundary. These integral transforms can be used to model attenuated ray transforms and inverse problems for connections and Higgs fields.

The third and fourth articles of the dissertation study the geodesic ray transform over closed geodesics on flat tori when the functions have low regularity assumptions. The fourth article studies a generalization of the geodesic ray transform when the integrals of a function are known over lower dimensional isometrically embedded flat tori. New inversion formulas, regularization strategies and stability estimates are proved in the articles. The new results have applications in different computational tomography methods.

Contents

1. Introduction

One of the most fundamental inverse problems asks if an unknown function is determined uniquely from the knowledge of the values of its line integrals over all possible lines in Euclidean space. This is in fact the mathematical model used for X-ray computed tomography (CT). This can be viewed as an integral transform acting on functions. Its many possible generalizations model other tomographic methods such as computerized axial tomography (CAT), positron-emission tomography (PET) and single-photon emission tomography (SPECT). It also has close connection to other inverse problems and applications such as seismic imaging, electrical impedance tomography, polarization tomography, quantum state tomography, inverse spectral problems and inverse scattering problems. This thesis studies generalizations of X-ray tomography on Riemannian manifolds.

This introductory part of the thesis is organized as follows. We discuss inverse problems and X-ray computed tomography in general in sections 1.1 and 1.2 respectively. We shortly describe the articles (A)–(D) in section 1.3. Preliminaries on Riemannian manifolds are given in section 2. We define different geodesic tomography models and corresponding inverse problems in section 3. We also survey related solved and unsolved problems in section 3. We introduce the main results of this thesis in section 4. The results are proved in the included articles (A)–(D).

1.1. **Inverse problems.** Inverse problems is a field of mathematics where one typically measures data outside or on the boundary of an object and wants to recover knowledge of its internal structure. Such mathematical problems occur often in medical, engineering and physical applications. In some inverse problems, measurements are done very far from an object. Such problems can be naturally studied using noncompact spaces in mathematical models.

Typical mathematical questions that one studies in inverse problems include:

i) *(Forward problem)* What is a good mathematical model that captures the physical phenomenon which relates measurement data to physical parameters of an unknown object? Does the mathematical model define data uniquely?
ii) *(Uniqueness)* Do measurements determine the unknown physical parameters uniquely? If not, can non-uniqueness be characterized?
iii) *(Reconstruction)* How can the unknown physical parameters be computed from measurement data?
iv) *(Stability)* Do the unknown physical parameters depend continuously on measurement data? Does there exists a quantitative stability estimate?
v) *(Simulations and regularization)* How can reconstruction methods be implemented into numerical algorithms? How to overcome instability caused by ill-posedness and measurement noise, finiteness of measurements, and numerical approximations?

The questions i)–ii) are encountered in (A), the question ii) in (B), and the questions iii)–v) in (C) and (D). The textbooks [42, 45, 57] and the survey [80] can be used to find more details and references on inverse problems in general.

1.2. **X-ray tomography and its generalizations.** Let f be a function $\mathbb{R}^n \to \mathbb{R}$. One defines the *X-ray transform* in $\mathbb{R}^n$ as

$$\mathcal{R}f(x,v) = \int_{\mathbb{R}} f(x+tv)dt \tag{1}$$

where $(x,v) \in \mathbb{R}^n \times S^{n-1}$ whenever the integral is well-defined and finite. This is the standard mathematical model for X-ray tomography measurements, and it is also known as the *Radon transform* if $n=2$. In higher dimensions, the X-ray transform and the Radon transform are different operators [28]. The corresponding uniqueness problem asks if $\mathcal{R}f = \mathcal{R}g$ implies that $f = g$. The other questions i)–v) of section 1.1 could be asked as well.

The inverse problems associated to the X-ray transform were first studied by Johann Radon in 1917 [73]. Fritz John characterized the range of the X-ray transform in $\mathbb{R}^3$ in terms of ultrahyperbolic equations (called *John's equations*) in 1938 [41]. Later, the mathematical problem was restudied independently by Allan Cormack in 60s [12, 13]. Godfrey Hounsfield studied practical CT imaging a few years later.

For their seminal works on CT imaging, Cormack and Hounsfield won the 1979 Nobel Prize in Physiology or Medicine. The monographs [59, 43, 28] and the surveys [72, 29, 46] are recommended references on the mathematics of the X-ray and Radon transforms.

The X-ray transform can be generalized many ways:

i) Instead of integrating over straight lines, suppose one knows integrals of f over other families of curves. For example, data could be measured over geodesics of a Riemannian manifold.

ii) Instead of integrating against the measure dt, suppose one knows integrals of f against the weighted measure $w(x, v)dt$ where $w(x, v) > 0$ is a continuous function on $\mathbb{R}^n \times S^{n-1}$.

iii) Instead of integrating over straight lines, suppose one knows integrals of f over other families of sets. For example, data could be measured over hyperplanes.

iv) Instead of integrating a function, suppose one knows integrals of a tensor field so that the value of f depends also on the direction of an X-ray, not only on a point in $\mathbb{R}^n$.

v) Some combination of the above cases.

The case i) corresponds to the *geodesic X-ray transform*, ii) to the *X-ray transform with weights*, iii) to the *Radon transform*, and iv) to *tensor tomography*. These different generalizations of the X-ray transform are studied in this thesis. One of the fundamental properties is that all of these integral transforms are linear. This reduces the uniqueness problem to studying kernels of the transforms.

The field of inverse problems that studies these integral transforms, among other problems of similar nature, is often called *integral geometry*. For example, the boundary rigidity problem asks if the knowledge of distances between any two boundary points determines the geometrical shape of a compact connected object with boundary uniquely (see section 3.4 for a rigor formulation). This is an example of a nonlinear integral geometry problem. We give a more detailed introduction to integral geometry problems in section 3. More references and recent developments in integral geometry can be found from the textbook [76] and the surveys [66, 38].

1.3. **On the articles in this thesis.** The first article (A) with Lehtonen and Salo considers tensor tomography on Cartan-Hadamard manifolds. Tensors can be used for modeling physical parameters that have spatial and directional dependence. In this work, we characterize the kernel of the geodesic ray transform for symmetric tensor fields of any order under sufficient decay conditions. This generalizes injectivity results of the geodesic ray transform from compact manifolds with boundary to noncompact manifolds.

The second article (B) with Ilmavirta considers the geodesic ray transform with matrix weights on manifolds that admit a strictly convex function. In this work, we restrict our study to the class of piecewise constant vector-valued functions. We show injectivity of this transform under the assumption that the weight is continuous and invertible at any point. This assumption on weights is very mild, and counterexamples for injectivity on smooth functions exist even in Euclidean case. The geometric assumption is equivalent to a manifold being nontrapping in dimension two. Injectivity of the geodesic ray transform (without a weight) for smooth functions on nontrapping manifolds is one of the most important unanswered geometric inverse problem at the moment.

The third article (C) with Ilmavirta and Koskela studies the geodesic X-ray transform over periodic geodesics on the flat 2-torus. In this work, reconstruction methods, including regularization and numerical implementations, drive theoretical considerations. We prove new reconstruction formulas for integrable functions, solve a minimization problem associated to Tikhonov regularization in Sobolev spaces, and prove that the unique minimizer provides a regularization strategy. We have also computed and analyzed the adjoint and the normal operators. Regularization of reconstructions is important since measurement noise is amplified in practice due to ill-posedness of the problem. Another reason for regularization is that one can collect only finitely many measurements in practice. We created Matlab codes, performed numerical tests and demonstrated how the developed methods can be applied in practical CT imaging.

The fourth article (D) studies the d-plane Radon transforms on the flat n-tori $\mathbb{T}^n$. The main results in (D) extend theorems in (C) to higher dimensions. In addition, new stability estimates in Bessel potential norms and inversion formulas for periodic distributions are proved. It is shown that the d-plane Radon transforms maps the Bessel potential spaces continuously into the weighted Bessel potential spaces on $\mathbb{T}^n \times \mathbf{Gr}(d,n)$ where $\mathbf{Gr}(d,n)$ is the collection of d-dimensional subspaces of $\mathbb{Q}^n$. The use and analysis of such structures is the main methodological advance compared to (C). Quite surprisingly, one of the inversion formulas in (D) implies that a compactly supported function on the plane with zero average is a sum of its X-ray data.

2. Preliminaries on Riemannian manifolds

Let (M,g) denote a Riemannian manifold with or without boundary. We assume always that M is complete and $\dim(M) \geq 2$. We define the following notations:

- The unit tangent bundle is denoted by

$$SM = \{ (x, v) \in TM \, ; \, |v|_g = 1 \}. \tag{2}$$

- If $(x, v) \in SM$, then $\gamma_{x,v}$ denotes the unique unit-speed geodesic such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. The set of maximal unit-speed geodesics of M is denoted by Γ.
- We denote the boundary of M by ∂M and by $\nu(x)$ the inward pointing unit normal of ∂M at $x \in \partial M$.
- We say that M has a *strictly convex boundary* if the second fundamental form of ∂M is positive definite or, equivalently, principal curvatures of ∂M are positive.
- We denote the covariant derivative by ∇ and the Riemannian curvature tensor by R.
- We denote the sectional curvature of a two-plane $\Pi \subset T_xM$ by $K_x(\Pi)$ and $\mathcal{K}(x) = \sup\{ |K_x(\Pi)| \, ; \, \Pi \subset T_xM \text{ is a two-plane} \}$.
- We write $K \leq 0$ if $K_x(\Pi) \leq 0$ for any $x \in M$ and any two-plane $\Pi \subset T_xM$. In this case, we say that M has *non-positive (sectional) curvature.*

2.1. **Definitions related compact Riemannian manifolds with boundary.** Let (M, g) be a compact Riemannian manifold with a strictly convex boundary. We define some useful geometric terminology in this sections. In the following sections, we give results on geodesic ray transforms using these different geometric definitions.

We say that M is *simple* if the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism from its maximal domain for any $p \in M$. This, in particular, implies that there are no conjugate points, any two points are connected by a unique geodesic, and M is diffeomorphic to the Euclidean ball of dimension $\dim(M)$ [70]. We say that M is *nontrapping* if $\gamma_{x,v}(t)$ meets ∂M in finite time for any $(x, v) \in SM$. In particular, simple manifolds are nontrapping.

We say that $f : M \to \mathbb{R}$ is a *strictly convex function* if $f \in C^\infty(M)$ so that $\mathrm{Hess}_x(f)$ is positive definite for any $x \in M$ or, equivalently, $(f \circ \gamma)''(t) > 0$ for every geodesic $\gamma \in \Gamma$. A manifold M satisfies the *foliation condition* if there exists a strictly convex function [81, 68].

Remark 2.1. The level sets of a strictly convex function are strictly convex hypersurfaces besides the special case of the minimum whose level set is a single point [68]. The corresponding level sets form layers that foliate the whole manifold. The tangential geodesics of a strictly convex hypersurface do not locally travel inside the hypersurface. Using the foliation condition, this type of behavior can be made global.

In turn, this allows one to use a layer stripping argument for proving injectivity of the geodesic ray transform if local injectivity can be shown [81].

The *trapped set of* M, denoted by $K \subset SM$, consists of points $(x, v) \in SM$ such that $\gamma_{x,v}(t)$ does not meet the boundary ∂M for any $t \in \mathbb{R}$. In particular, if M is nontrapping (in the sense of above), then $K = \emptyset$. The trapped set is said to be *hyperbolic* if there is a certain orthogonal splitting to geodesic, stable and unstable parts of $T_{(x,v)}(SM)$ for any $(x, v) \in K$. For exact definitions, see [61, 23].

Let $\beta \geq 0$. We say that J is *β-Jacobi field* along $\gamma \in \Gamma$ if it satisfies

$$D_t^2 J(t) + \beta R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0. \tag{3}$$

We say that two distinct points along γ are *β-conjugate* if there exists a non-trivial β-Jacobi field which vanishes at the points. The *β-terminator value* β_{Ter} is the supremum of the numbers β so that M is free of β-conjugate points. In particularly $\beta_{Ter} = \infty$ if and only if $K \leq 0$, and M has no conjugate points if and only if $\beta_{Ter} \geq 1$. For more details see [67].

Remark 2.2. We do not study manifolds that have trapped geodesics in this thesis, but this condition is included as we will give references to other works on the geodesic ray transform where a hyperbolic trapped set is a part of the geometrical assumptions. Our reason for introducing β-Jacobi fields here is similar and they are not applied in this thesis.

2.2. **Cartan-Hadamard manifolds.** We say that a Riemannian manifold (M, g) without boundary is a *Cartan-Hadamard manifold* if (M, g) is complete, simply connected and $K \leq 0$. The classical Cartan-Hadamard theorem states that $\exp_p : T_pM \to M$ is a diffeomorphism for any $p \in M$ (see e.g. [70, Chapter 6] or [49, Chapter 11]). In particular, M with $\dim(M) = n$ is diffeomorphic to Euclidean space $\mathbb{R}^n$. This implies that Cartan-Hadamard manifolds are noncompact.

The model spaces of Cartan-Hadamard manifolds are the hyperbolic space $\mathbb{H}^n$ ($K \equiv -1$) and Euclidean space $\mathbb{R}^n$ ($K \equiv 0$). Many other examples can be constructed using warped products with radial metrics [9, 44, 22, 70]. A discussion on such constructions, related to the theorems of the article (A), is given in [(A), Section 2].

3. Geodesic tomography problems

3.1. **Geodesic tensor tomography.** We denote by $C^1(T^m M)$ the set of C^1-smooth covariant m-tensor fields of M and by $C^1(S^m M) \subset C^1(T^m M)$ the set of symmetric covariant m-tensor fields. Each $f \in$

$C^1(T^mM)$ can be written in local coordinates as

$$f = f_{j_1\cdots j_m}(x)dx^{j_1} \otimes \cdots \otimes dx^{j_m} \tag{4}$$

using the Einstein summation convention. Let Π_M denote the permutation group of $\{1,\dots,m\}$. Tensors in $f \in C^1(S^mM)$ are symmetric in the sense that

$$f = f_{j_{\sigma(1)}\cdots j_{\sigma(m)}}(x)dx^{j_1} \otimes \cdots \otimes dx^{j_m} \tag{5}$$

for any $\sigma \in \Pi_m$.

If every maximal geodesic of M has finite length, then one defines the *geodesic ray transform of symmetric m-tensor fields* by the formula

$$I_m f(\gamma) = \int_\gamma \lambda_m f(\gamma(t), \dot\gamma(t))dt \tag{6}$$

where $\gamma \in \Gamma$ and $\lambda_m f(x,v) = f_{j_1\cdots j_m}(x)v^{j_1}\cdots v^{j_m}$ is a mapping $SM \to \mathbb{R}$. In fact, λ_m maps $C^1(S^mM) \to C^1(SM)$ so that the spherical harmonics decomposition with respect to v of $\lambda_m f$ is of degree m. A more detailed exposition of symmetric tensors and λ_m are given in [76, 14]. There is also a brief discussion in [(A), Section 3.3].

In general, the geodesic ray transform I can be straightforwardly defined for every $f \in C(SM)$ if every maximal geodesic of M has finite length. However, this transform always has a non-trivial kernel on manifolds with boundary, even in the case of symmetric m-tensor fields with $m \geq 1$, as we will explain later. This motivates to study functions of $C(SM)$ that have a special form.

In the article (A), we study the kernel of the geodesic X-ray transform on Cartan-Hadamard manifolds for functions that arise from symmetric tensors. In this case, any maximal geodesic of M has infinite length. Therefore, the integrals (6) are finite only if the tensors decay sufficiently fast along every geodesic.

3.1.1. *On the kernel of I_m and solenoidal injectivity.* We define the *symmetrization* of a tensor $\sigma_m : T^mM \to S^mM$ by

$$\sigma_m(f) = \frac{1}{m!}\sum_{\sigma\in\Pi_m} f_{j_{\sigma(1)}\cdots j_{\sigma(m)}}(x)dx^{j_1} \otimes \cdots \otimes dx^{j_m}. \tag{7}$$

Let $\varphi_t(x,v) = (\gamma_{x,v}(t), \dot\gamma_{x,v}(t))$ be the *geodesic flow* on SM. One defines the *geodesic vector field* X for functions in $C^1(SM)$ as

$$Xf(x,v) := \frac{d}{dt} f(\varphi_t(x,v))|_{t=0}. \tag{8}$$

Suppose now that M is a nontrapping Riemannian manifold. One can show that $IXf = 0$ for any $f \in C^1(SM)$ with $f|_{\partial M} = 0$ by the fundamental theorem of calculus. Another calculation shows that

$$X(\lambda_m f) = \lambda_m(\sigma_m \nabla f) \tag{9}$$

for any $f \in C^1(S^m M)$. Therefore, if $m \geq 1$, the kernel of I_m contains all symmetric m-tensors of the form $\sigma_m \nabla f$ where $f \in C^1(S^{m-1}M)$ and $f|_{\partial M} = 0$. We say that f is a *potential* of the tensor $\sigma_m \nabla f$.

We identify the space $C^1(S^{-1}M)$ as the space of the zero function. We say that I_m is *s-injective* if the kernel of I_m contains only tensors that arise from a potential described above. This implies that the solenoidal part of a symmetric tensor can be uniquely determined from its geodesic ray transform (see [76] for details about the Helmholtz decomposition of symmetric tensors). We list next some known injectivity results for smooth tensor fields on compact Riemannian manifolds with a strictly convex boundary:

- If M is a simple manifold, then I_m is s-injective for $m = 0, 1$ [58, 4].
- If M is a simple manifold whose metric is from a generic class (including real analytic metrics), then I_2 is s-injective [77].
- If M is a simple manifold of $\dim(M) = 2$, then I_m is s-injective for every $m \geq 0$ [64].
- If M is a nontrapping manifold of $\dim(M) = 2$, I_m is s-injective for $m = 0, 1$ and the adjoint of I_0 is surjective, then I_m is s-injective for every $m \geq 0$ [64].
- If M is a simple manifold with $n = \dim(M) \geq 2$ and $\beta_{Ter} \geq \frac{m(m+n-1)}{2m+n-2}$, then I_m is s-injective [67].
- If M is a nontrapping manifold of $\dim(M) \geq 3$ with a strictly convex foliation, then I_m is s-injective for $m = 0, 1, 2, 4$ [81, 78, 15].
- If M is a compact Riemannian manifold with no conjugate points and hyperbolic trapped set, then I_m is s-injective for $m = 0, 1$. If moreover $K \leq 0$, then I_m is s-injective for every $m \geq 0$ [23].
- If M is a compact Riemannian manifold of $\dim(M) = 2$ with no conjugate points and hyperbolic trapped set, then I_m is s-injective for every $m \geq 0$ [50].
- If M is a simple manifold with real analytic metric, then $I_m, m \in \mathbb{N}$, admit a certain local support theorem [47, 1]. (These results are partly contained in the results of [81, 78, 15].)

We state some of the related open problems in section 3.4.

3.2. **Geodesic ray transform with matrix weights.** Suppose that $W : SM \to \mathbb{C}^{m\times m}$ is continuous and $W(x, v) : \mathbb{C}^m \to \mathbb{C}^m$ is injective for any $(x,v) \in SM$. Let $f : SM \to \mathbb{C}^m$ be a continuous function. One can then define *the geodesic ray transform with the weight* W as

$$I_W f(x,v) := \int_{a_{x,v}}^{b_{x,v}} W(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) f(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) dt \qquad (10)$$

where $[a_{x,v}, b_{x,v}]$ is the maximal domain of $\gamma_{x,v} \in \Gamma$ (possibly infinite).

The corresponding uniqueness problem asks if the knowledge of $I_W f$ and W determine f uniquely. There exist counterexamples and positive results to the uniqueness problem. Clearly, if W does not depend on the coordinate v, then injectivity of I_W is equivalent to injectivity of I without a weight (i.e. $W \equiv 1$).

An important special case of the geodesic ray transforms with weights is the *attenuated geodesic ray transform*. The attenuated geodesic ray transforms is studied very recently for example in [74, 63, 6, 30, 56, 55, 8]. In the simplest model for the attenuated ray transform (with $m = 1$), the weight has a special form

$$w_a(x,v) = \exp\left(\int_{t_{x,v}}^{0} a(\gamma_{x,v}(s)) ds\right), \quad a \in C(M), \qquad (11)$$

where $t_{x,v}$ is the maximal backward time for the geodesic $\gamma_{x,v}$ (possibly infinite). The attenuated ray transform is the mathematical basis for the medical imaging method SPECT [16, 60, 17]. Other applications of matrix weighted ray transforms are described in the introduction of the article (B). More details and references can be found from [38].

We list some positive injectivity results next:

- If (M,g) is a compact Riemannian manifold of $\dim(M) \geq 3$ with a strictly convex boundary and admits a smooth strictly convex function, and $W \in C^\infty(SM; GL(k,\mathbb{C}))$, then I_W is injective for smooth functions [68].
- Let (M,g) be a simple manifold of $\dim(M) = 2$. Let $a \in C^\infty(M)$ be a complex function and $I^a = I_{w_a}$ the attenuated ray transform with the weight w_a. Suppose that $F(x,v) = f(x) + \alpha(x,v)$ is the sum of a function $f \in C^\infty(M)$ and a 1-form $\alpha \in C^\infty(T^1 M)$. If $I^a F = 0$, then $F(x,v) = ap(x) + \nabla p(x,v)$ for some $p \in C^\infty(M)$ with $p|_{\partial M} = 0$ [74]. The result generalizes to the matrix weighted case where the matrix weight is the sum of a smooth unitary connection and a smooth skew-Hermitian matrix function [63], and to higher dimensions if $K \leq 0$ [24, 62].
- If M has a strictly convex boundary and $w \in C(SM)$, then $I_w f$ determines the boundary jet of a smooth function [31]. Hence, I_w is injective for analytic functions. This result is based on

a local argument and generalizes to the matrix weighted case straightforwardly even though it is not stated in [31].

- Many positive results are known in Euclidean spaces. If $n \geq 2$, w is smooth, and has a rotation invariance [71] or w is real analytic [11], then I_w is injective. If $n \geq 3$ and the weight is regular enough ($C^{1,\alpha}$ is sufficient for example), then I_w is injective [52, 16, 33].

There are two important counterexamples for uniqueness in Euclidean spaces [10, 20]. The counterexample in [10] gives a construction of a smooth weight so that the kernel of I_w is nontrivial on the unit disk of the plane. The counterexample in [20] gives a construction of a α-Hölder continuous rotation invariant weight (in the sense of [71]) in $\mathbb{R}^n$, $n \geq 2$, for some small $\alpha > 0$, so that the kernel of I_w is nontrivial. This also gives a counterexample to the result of [71] if the weight is not regular enough.

In the article (B), we restrict our study to the class of piecewise constant functions. We show that under this assumption continuity of a matrix weight is sufficient for showing that $I_W f = 0$ implies $f = 0$. This result is valid for manifolds of $\dim(M) \geq 2$ that admit a strictly convex function.

3.3. **Geodesic ray transform on closed manifolds.** Suppose that (M, g) is a closed Riemannian manifold with $\dim(M) \geq 2$. Let $\Gamma_c \subset \Gamma$ be the set of closed unit speed geodesics. Let τ_γ be the smallest period of $\gamma \in \Gamma_c$. The *geodesic ray transform on a closed manifold* is defined by

$$If(\gamma) = \int_0^{\tau_\gamma} f(\gamma(t))dt. \tag{12}$$

This definition can be generalized to the functions on SM as well.

There is again a vast literature on the geodesic ray transforms of this type in general. A lot is known for flat tori, Lie groups and other symmetric spaces [32, 34, 28, 29]. More generally, the geodesic ray transform has been studied on Anosov surfaces and manifolds of negative curvature [67, 24, 65]. It has applications to the *spectral rigidity problem* which asks if the spectrum of the Laplace-Beltrami operator determines the metric up to a natural gauge [25, 26].

A historically interesting fact is that the geodesic ray transform of S^2, called the *Funk transform*, was studied for the first time by Hermann Minkowski in the early 1900s [54] and by Paul Funk a few years later [18, 19], about a decade before the first studies of Radon on $\mathbb{R}^2$. The injectivity result on S^2 states that a symmetric function can be uniquely determined from its line integrals over great circles [19].

In the article (C), we study the ray transform of closed geodesics in the special case of the flat torus $(\mathbb{T}^2, g_E)$. Our arguments in (C) are specialized to the case of the flat tori and based on rather simple analysis of Fourier series. The work (C) has applications in computational reconstructions from practical X-ray data since the geometry is flat. These results are further generalized to the periodic d-plane Radon transforms on $(\mathbb{T}^n, g_E)$ in the article (D). These generalizations require suitable weighted Sobolev spaces on the image side, and give another view of the theorems in (C) in terms of weighted Sobolev spaces.

3.4. **Related open problems.** We list here some important open problems in integral geometry [66, 38]:

i) Is I_m s-injective for $m \geq 2$ if (M, g) is a simple manifold and $\dim(M) \geq 3$?
ii) Is I_m s-injective for $m \geq 0$ if (M, g) is a nontrapping manifold and $\dim(M) \geq 2$?
iii) If (M, g) is a simple or a nontrapping manifold and $\dim(M) \geq 3$, does there exists a strictly convex function?
iv) Is I_m s-injective for $m \geq 0$ if (M, g) has a strictly convex boundary and a strictly convex function, and $\dim(M) = 2$?
v) Is the attenuated geodesic ray transform I^a injective if (M, g) is a simple manifold and $\dim(M) \geq 3$?
vi) Is the attenuated geodesic ray transform I^a injective if (M, g) is a nontrapping manifold and $\dim(M) \geq 2$?
vii) Is the class of simple metrics of M with $\dim(M) \geq 3$ *boundary distance rigid*: Suppose that g and h are simple metrics on M. Does $d_g|_{\partial M \times \partial M} = d_h|_{\partial M \times \partial M}$ imply that $g = \varphi_* h$ for some diffeomorphism $\varphi : M \to M$ with $\varphi|_{\partial M} = \mathrm{Id}$?

If one can solve one of the corresponding problems for nontrapping manifolds with a positive answer, then this would solve the corresponding problem for simple manifolds. Vice versa, counterexamples for simple manifolds would serve as counterexamples for nontrapping manifolds. The positive answer to the question ii) was conjectured in [64] when $\dim(M) = 2$, and the problem iv) is equivalent to ii) in this case [68]. A positive answer to iii) in the case of simple manifolds would imply a positive answer to the boundary rigidity problem vii) [81, 78] and the injectivity problem v) [68]. The positive answer to the problem vii) was conjectured by Michel in 1981 [53], and was proved when $\dim(M) = 2$ by Pestov and Uhlmann in 2005 [69]. As far as the author knows, there do not exist positive theorems or counterexamples to the precise statements of the problems in this list.

Injectivity of the geodesic ray transform with a smooth weight is also open on simple manifolds of $\dim(M) \geq 3$. In $\dim(M) = 2$, a positive answer cannot be obtained due to the smooth counterexample of Boman [10] on Euclidean plane. Minimal regularity assumptions of the weights for which injectivity of I_w holds is also an open question in $\mathbb{R}^n$, $n \geq 3$ [20, 33]. For example, is I_w injective on smooth functions of the closed unit ball of $\mathbb{R}^n$, $n \geq 3$, if w is Lipschitz continuous?

If a Riemannian manifold M is assumed to be noncompact, then there are many results in symmetric geometries, but several questions of integral geometry are yet unstudied in more general geometries. The article (A) and the work [21] contain the only s-injectivity results, that the author is aware of, when special symmetries such as a constant curvature is not assumed. A further discussion on the geodesic ray transform on noncompact manifolds is given in section 4.1 of the thesis.

4. Main results

4.1. **S-injectivity of the geodesic ray transform on Cartan-Hadamard manifolds, (A).** We begin by introducing some notations and definitions. We then state our main results in the article (A) and discuss earlier works in tensor tomography on noncompact manifolds. We finish this section by giving an outline of the used methods and arguments.

Let (M, g) be a Cartan-Hadamard manifold. Fix a point $o \in M$. If $\eta > 0$ and $f \in C(M)$, we say that f *decays exponentially* and denote that $f \in E_\eta(M)$ if

$$|f(x)| \leq Ce^{-\eta d(x,o)} \quad \text{for some } C > 0, \tag{13}$$

and f *decays polynomially* and denote that $f \in P_\eta(M)$ if

$$|f(x)| \leq C(1 + d(x, o))^{-\eta} \quad \text{for some } C > 0. \tag{14}$$

Let $f \in C^1(M)$. We denote $f \in E^1_\eta(M)$ if $|f(x)| + |\nabla f(x)| \in E_\eta(M)$, and $f \in P^1_\eta(M)$ if $|f(x)| \in P_\eta(M)$ and $|\nabla f(x)| \in P_{\eta+1}(M)$.

Let $f, h \in C^1(T^m M)$. We define the standard inner product for m-tensors on $T_x M$ by

$$g_x(f, h) := g^{j_1k_1}(x) \cdots g^{j_mk_m}(x) f_{j_1\cdots j_m}(x) h_{k_1\cdots k_m}(x). \tag{15}$$

The norm is defined by $|f|_g := \sqrt{g(f, f)}$ and defines a mapping $M \to \mathbb{R}$. If $f \in C^1(S^m M)$, then we write $f \in E_\eta(M)$ if $|f|_g \in E_\eta(M)$, and $f \in E^1_\eta(M)$ if $|f|_g \in E_\eta(M)$ and $|\nabla f|_g \in E_\eta(M)$. We define analogously the sets $P_\eta(M)$ and $P^1_\eta(M)$ for tensors.

In [(A), Lemma 4.1], we show that $I_m f$ is well defined if $f \in P_\eta$ for some $\eta > 1$. Since M is noncompact and every geodesic has infinite length, this must be shown. It is also straightforward to argue that the kernel of I_m contains symmetric tensors of the form $\sigma_m(\nabla f)$ such that

$f \in C(S^{m-1}M)$ and f has suitable decay at infinity. We are ready to state our main results on s-injectivity of I_m on Cartan-Hadamard manifolds.

Theorem 4.1 ((A), Theorem 1.1)**.** *Let (M, g) be a Cartan-Hadamard manifold of dimension $n \geq 2$ with $-K_0 \leq K \leq 0$ for some $K_0 > 0$. Let $f \in E^1_\eta(M)$ be a symmetric m-tensor field for some $\eta > \frac{n+1}{2}\sqrt{K_0}$. If $I_m f = 0$, then $f = \sigma_m(\nabla h)$ for some symmetric $(m-1)$-tensor field h such that $h \in E_{\eta-\epsilon}(M)$ for any $\epsilon > 0$. (If $m = 0$, then $f \equiv 0$.)*

Theorem 4.2 ((A), Theorem 1.2)**.** *Let (M, g) be a Cartan-Hadamard manifold of dimension $n \geq 2$ and assume that $\mathcal{K} \in P_\kappa(M)$ for some $\kappa > 2$. Let $f \in P^1_\eta(M)$ be a symmetric m-tensor field for some $\eta > \frac{n+2}{2}$. If $I_m f = 0$, then $f = \sigma_m(\nabla h)$ for some symmetric $(m-1)$-tensor field h such that $h \in P_{\eta-1}(M)$. (If $m = 0$, then $f \equiv 0$.)*

These theorems extend the earlier results in [51] where the same problem was studied in the case of functions ($m = 0$) and $n = 2$. We remark that the proof of [51, Lemma 4.6] is incomplete, and hence, the theorems cannot be used as stated in [51]. Theorems 4.1 and 4.2 here are proved by a different method and thus the corresponding lemma is not required. However, there might be a possibility to find better lower bounds for η in theorems 4.1 and 4.2 by combining arguments of [51] and (A) carefully.

The geodesic ray transform for functions on noncompact manifolds has been studied before in Euclidean and hyperbolic spaces [27, 28, 40], and for vector fields in [7]. In these works, the regularity and decay conditions are sharper than those in theorems 4.1 and 4.2. Differentiability is not needed but similar decay conditions for the function itself is required with slightly better lower bounds for η. There exist counterexamples if one does not assume a decay condition [82, 5]. Theorem 4.1 resembles the hyperbolic results and theorem 4.2 the Euclidean. Our differentiability assumption comes from the method of proof that is based on the Pestov identity.

There are also works in noncompact spaces of constant curvature and noncompact homogeneous spaces [28, 29]. Theorems 4.1 and 4.2 are the first results on the geodesic ray transform of noncompact manifolds without special symmetries, which the author is aware of.

There is a recent related work [21] where s-injectivity for I_m for $m = 0, 1$ was shown in the case of asymptotically hyperbolic manifolds without conjugate points and with hyperbolic trapped set. It was also shown there that if additionally $K \leq 0$, then I_m is s-injective for any $m \geq 0$. The results in (A) are not included in [21], and vice versa. Of course, there are geometries which satisfy the assumptions of the both works.

Outline of the proof of theorems 4.1 and 4.2. Let $f \in P_\eta(M)$ be a symmetric m-tensor field. One defines the function

$$u^f(x,v) := \int_0^\infty \lambda_m f(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))dt. \tag{16}$$

A simple calculation shows that

$$u^f(x,v) + (-1)^m u^f(x,-v) = I_m f(x,v) < \infty. \tag{17}$$

We write here $f = \lambda_m f$ to keep notation shorter. It can be calculated that $Xu^f = -f$ where X is the geodesic vector field. Now one needs to understand the system $Xu = -f$ when f is a symmetric m-tensor such that $I_m f = 0$ and f satisfies the assumptions of theorem 4.1 or theorem 4.2.

We list the main ideas next:

i) The goal is to show that $f = -Xu^f = \lambda_m(\sigma_m \nabla U)$ for some $U \in C(S^{m-1}M)$ with right decay properties.
ii) If M is a compact manifold with boundary and $K \leq 0$, then the Pestov identity can be used to show i). This follows from a contraction property of the Beurling transform on manifolds of nonpositive sectional curvature [67].
iii) The energy estimates of the step ii) in compact manifolds that involve only terms up to the first order derivatives can be extended to $H^1(SM)$ when M is a complete manifold with $K \leq 0$. These $H^1(SM)$ extensions of the energy estimates and the final argument to show i) are done in [(A), Section 5].
iv) Hence, we need to show that $u^f \in H^1(SM)$ under the assumptions of theorems 4.1 and 4.2. The core part of this is done in [(A), Section 4] by estimating growths of Jacobi fields on Cartan-Hadamard manifolds.

We next explain some of the details. Showing that $u^f \in H^1(SM)$ is a bit tricky and our argument uses geometric estimates for growths of Jacobi fields and the decay assumptions of f. The idea could be summarized as follows: the faster the geodesics spread the faster the functions (and derivatives) should decay to make L^2 estimates work because of the growth rate of volumes of balls (cf. [(A), Lemmas 4.8 and 5.4]).

One can orthogonally split the gradient of SM as

$$\nabla_{SM} u = (Xu)X + \overset{h}{\nabla} u + \overset{v}{\nabla} u \tag{18}$$

where X and $\overset{h}{\nabla}$ represents *horizontal derivatives* with respect to x and $\overset{v}{\nabla}$ *vertical derivatives* with respect to v. These and other geometric preliminaries are given in [(A), Section 3]. The most technical part is the proof of [(A), Lemma 4.7]. In that lemma, we first show that u^f is locally Lipschitz and then estimate the components (18) of the gradient $\nabla_{SM}u^f$ for a.e. $(x,v) \in SM$ based on our Jacobi field estimates. This implies that $u^f \in H^1(SM)$ [(A), Lemma 5.4].

The rest of the argument uses estimates and methods developed in [67]. Details of the spherical harmonics decomposition of $L^2(SM)$ are given in [26, 14]. Let $H_k(SM)$ be the eigenspace for the eigenvalue $k(k+n-2)$ of the spherical Laplacian. One can split the geodesic vector field $X = X_+ + X_-$ into two parts so that $X_+ : \Omega_k \to H_{k+1}(SM)$ and $X_- : \Omega_k \to H_{k-1}(SM)$ where $\Omega_k = H_k(SM) \cap H^1(SM)$. We can show this by proving the estimate

$$\|X_+u\|^2 + \|X_-u\|^2 \leq \|Xu\|^2 + \|\overset{h}{\nabla}u\|^2 \tag{19}$$

for $u \in H^1(SM)$ [(A), Lemma 5.1]. This part of the proof requires the Pestov identity and estimates based on the contraction property of the Beurling transformation from [67]. If $u \in H^1(SM)$, it follows that the spherical harmonics decomposition has the form

$$u = \sum_{k=0}^{\infty} u_k, u_k \in \Omega_k, \tag{20}$$

where the series converges in $L^2(SM)$. We can now conclude that if $u \in H^1(SM)$, then $\|X_+u_k\| \to 0$ as $k \to \infty$ [(A), Corollary 5.2].

Since symmetric m-tensors have only terms up to degree m in their spherical harmonic decomposition, we get

$$-\sum_{k=0}^{m} f_k = -f = Xu^f = X_+u^f + X_-u^f. \tag{21}$$

The rest of the proof follows from the formula (21) and [(A), Corollary 5.2 and Lemma 5.3] by following arguments from [64, 67]. The final step is to straightforwardly estimate decay of the elements of the kernel. These details are given in [(A), Proof of theorems 1.1 and 1.2.]. □

4.2. **On the geodesic ray transform with matrix weights for piecewise constant functions, (B).** The geodesic ray transform for piecewise constant functions was studied on the manifolds that admit a strictly convex function in [37]. The work [37] was motivated by the fact that injectivity of the geodesic ray transform is an open problem for nontrapping manifolds. If $n = 2$, then a manifold with strictly convex boundary is nontrapping if and only if it has a strictly convex function (see [68, Section 2] for details and references). The main result of [37] was to show that $If = 0$ implies $f \equiv 0$ if f is a piecewise constant function on M. Reconstruction of a piecewise constant function from If was studied recently in [48].

Piecewise constant functions are defined according to the definition of [37]. We recall this definition next. A *regular tiling* of a manifold is a collection of regular n-simplices which cover the manifold, whose interiors are disjoint, and whose boundaries intersect nicely [37, Section 2.1]. A function $f : M \to \mathbb{C}^k$ is called *piecewise constant* if there exists a regular tiling $\{\Delta_1, \ldots, \Delta_N\}$ such that $f|_{\mathrm{Int}(\Delta_i)}$ is constant for any $i \in \{1, \ldots, N\}$ and $f \equiv 0$ elsewhere.

The main result of the article (B) generalizes the main result of [37] to the matrix weighted case, analogous to the problem studied in [68] for smooth functions and weights in dimensions $n \geq 3$. We denote by $\mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^m)$ the space of injective linear maps $\mathbb{C}^k \to \mathbb{C}^m$.

Theorem 4.3 ((B), Theorem 1.1). *Let (M, g) be a compact nontrapping Riemannian manifold with strictly convex smooth boundary and $W \in C(SM; Mon(\mathbb{C}^k, \mathbb{C}^m))$. Let either*

(a) $\dim(M) = 2$, *or*

(b) $\dim(M) \geq 3$ *and (M, g) admits a smooth strictly convex function.*

If $f : M \to \mathbb{C}^k$ is a piecewise constant vector-valued function and $I_W f = 0$, then $f \equiv 0$.

Remark 4.4. Piecewise constant functions do not form a vector space under the definition used in the study [37, Remark 2.7]. Hence, injectivity follows only if the tiling of the piecewise constant function are known beforehand. It is an open problem how to determine the tiling of a piecewise constant function from the data If.

The proof of theorem 4.3 is strongly based on the method developed in [37]. We show that locally the matrix weighted geodesic ray transform data can be reduced to the data of the geodesic ray transform without weight [(B), Lemma 2.4 and Lemma 2.5]. We remark that this reduction does not work for general functions but it works for piecewise constant functions. Local injectivity of the geodesic ray transform for piecewise constant functions was shown in [37]. The layer stripping argument of [37], using a strictly convex function, allows one to go from the local uniqueness result to the global uniqueness result [(B), Theorem 2.6].

4.3. **Theory of Tikhonov regularized reconstructions from the X-ray transform data on the flat 2-torus, (C).** The geodesic ray transform on the flat torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ is defined for the closed geodesics. A geodesic is closed on $\mathbb{T}^2$ if and only if its directional vector is a multiple of an integer vector. Instead of unit-speed parametrization of geodesics, we parametrize geodesics so that each closed geodesic has the period 1. This is convenient since the 1-periodic geodesics are of the form

$$\gamma_{x,v}(t) = \pi(x + tv), \quad x \in \mathbb{R}^2, v \in \mathbb{Z}^2 \setminus 0, t \in [0, 1] \tag{22}$$

where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the quotient mapping. Clearly, if $\pi(x) = \pi(y)$, then $\gamma_{x,v} = \gamma_{y,v}$ for any $v \in \mathbb{Z}^2 \setminus 0$.

Hence, the (geodesic) X-ray transform on $\mathbb{T}^2$ can be defined by

$$If(x, v) = \int_0^1 f(\gamma_{x,v}(t))dt \tag{23}$$

for continuous functions. We remark that this definition actually scales the data (12) by the factor $|v|^{-1}$. However, there is one-to-one correspondence between the both definitions of I on $\mathbb{T}^2$. This definition extends to the dual space of smooth functions, called *distributions* and denoted by $\mathcal{T}'$, since $If(\cdot, v)$ is formally $L^2(\mathbb{T}^2)$ self-adjoint for every fixed $v \in \mathbb{Z}^2 \setminus 0$. For further details see [32, 35] or [(C), Section 2.1].

Injectivity of I on tori is well understood and it has been studied earlier in [79, 3, 2, 32]. The main contributions of (C) are related to reconstruction, better understanding of functional properties, and numerical simulations that demonstrate applicability of the method in CT imaging. It is described in [(C), Section 2.3] and [35, Chapter 3] how practical X-ray data of a compactly supported object on $\mathbb{R}^2$ can be mapped into X-ray data on $\mathbb{T}^2$.

One has the *Fourier series decomposition*

$$f(x) = \sum_{k \in \mathbb{Z}^2} \hat{f}(k) e^{2\pi i k \cdot x}, \quad \hat{f}(k) := f(e^{-2\pi i k \cdot x}), k \in \mathbb{Z}^2, \tag{24}$$

for any $f \in \mathcal{T}'$. It was shown in [32, Eq. (9)] that for any $f \in \mathcal{T}'$ the identity

$$\widehat{If}(k, v) = \begin{cases} \hat{f}(k) & k \cdot v = 0 \\ 0 & k \cdot v \neq 0 \end{cases} \tag{25}$$

holds. This gives a reconstruction formula for f from the data If and shows injectivity. In the work (C), we have studied consequences of this formula further and implemented a reconstruction algorithm based on our new findings.

We state and describe our main theorems in (C) next. Our first theorem simplifies the reconstruction formula (25) for integrable functions. This simplification results better computational efficiency since the dimension of the integrals (25) are reduced by one.

Theorem 4.5 ((C), Theorem 1)**.** *Suppose that* $f \in L^1(\mathbb{T}^2)$*. Let* $k \in \mathbb{Z}^2$*. If* $k, v \neq 0$ *and* $v \perp k$*, then*

$$\hat{f}(k) = \begin{cases} \int_0^1 I_v f(0,y) \exp(-2\pi i k_2 y) dy, & k_2 \neq 0 \\ \int_0^1 I_v f(x,0) \exp(-2\pi i k_1 x) dx, & k_1 \neq 0. \end{cases} \tag{26}$$

If $k = 0$*, then*

$$\hat{f}(k) = \int_0^1 I_{(1,0)} f(0,y) dy = \int_0^1 I_{(0,1)} f(x,0) dx. \tag{27}$$

This theorem can be proved by a change of coordinates and Fubini's theorem. We gave two proofs in [(C), Section 2.2]. The first proof gives a new proof of injectivity of I on $\mathbb{T}^2$. The second proof uses the formula (25) directly. A slightly more general statement is actually proved in [(C), Theorem 8].

Our next two theorems are about regularization. We need to first introduce a suitable Sobolev space structure on the image side. Let $Q \subset \mathbb{Z}^2$ be such that every nonzero $v \in \mathbb{Z}^2$ is an integer multiple of a unique element in Q. This set can be naturally identified with the rational projective space $\mathbb{P}^1$. The X-ray transform takes a function on $\mathbb{T}^2$ to a function on $\mathbb{T}^2 \times Q$.

Remark 4.6. There is a connection between X-ray tomography with partial data and Schanuel's theorem [75] on heights of projective spaces [(C), Section 2.6.2]. In particular, the number of directions $v \in \mathbb{Z}^2 \setminus 0$ needed in the reconstruction of the Fourier coefficients of f in $B_{\ell^\infty}(0,R)$ from $I_v f$ can be estimated using Schanuel's theorem.

We use the standard Sobolev scale of spaces $H^s(\mathbb{T}^2)$ with the norms

$$\|f\|^2_{H^s(\mathbb{T}^2)} = \sum_{k \in \mathbb{Z}^2} \langle k \rangle^{2s} \left| \hat{f}(k) \right|^2, \tag{28}$$

where $\langle k \rangle = (1 + |k|^2)^{1/2}$ as usual. On $\mathbb{T}^2 \times Q$, we define the spaces $H^s(\mathbb{T}^2 \times Q)$ to be the set of functions $g \colon \mathbb{T}^2 \times Q \to \mathbb{C}$ for which

(i) $g(\cdot, v) \in H^s(\mathbb{T}^2)$ for every $v \in Q$,
(ii) the average of every $g(\cdot, v)$ over $\mathbb{T}^2$ is the same, and
(iii) the norm

$$\|g\|^2_{H^s(\mathbb{T}^2 \times Q)} = |\hat{g}(0,0)|^2 + \sum_{k \in \mathbb{Z}^2 \setminus 0} \sum_{v \in Q} \langle k \rangle^{2s} |\hat{g}(k,v)|^2 \tag{29}$$

is finite. We set $v = 0$ for the Fourier term $k = 0$ to emphasize that it is the same for every $v \in Q$. We remind the reader that $0 \notin Q$.

Now, we can consider a *Tikhonov minimization problem*: given some data $g \in H^r(\mathbb{T}^2 \times Q)$, find

$$\arg\min_{f \in H^r(\mathbb{T}^2)} \left(\|If - g\|^2_{H^r(\mathbb{T}^2 \times Q)} + \alpha \|f\|^2_{H^s(\mathbb{T}^2)} \right). \tag{30}$$

Let us define the post-processing operator P^s_α to be the Fourier multiplier $(1 + \alpha \langle k \rangle^{2s})^{-1}$ and denote by I^* the adjoint of I. Formulas that define the adjoint and normal operators are proved in [(C), Proposition 11]. In fact, the X-ray transform is unitary as a mapping $H^s(\mathbb{T}^2) \to H^s(\mathbb{T}^2 \times Q)$ for any $s \in \mathbb{R}$.

Theorem 4.7 ((C), Theorem 2)**.** *Let $r \in \mathbb{R}$, $s \geq r$, and $\alpha > 0$. Suppose $g \in H^r(\mathbb{T}^2 \times Q)$. The unique minimizer f of the minimization problem* (30) *corresponding to Tikhonov regularization is $f = P^{s-r}_\alpha I^* g \in H^{2s-r}(\mathbb{T}^2) \subset H^s(\mathbb{T}^2)$.*

Theorem 4.8 ((C), Theorem 3)**.** *Suppose $r, t, s, \delta \in \mathbb{R}$ are such that $2s + t \geq r$, $\delta \geq 0$, and $s > 0$. We assume that $f \in H^{r+\delta}(\mathbb{T}^2)$ and $g \in H^t(\mathbb{T}^2 \times Q)$.*

Then our regularized reconstruction operator $P^s_\alpha I^*$ gives a regularization strategy in the sense that

$$\lim_{\varepsilon \to 0} \sup_{\|g\|_{H^t(\mathbb{T}^2 \times Q)} \leq \varepsilon} \left\| P^s_{\alpha(\varepsilon)} I^*(If + g) - f \right\|_{H^r(\mathbb{T}^2)} = 0, \tag{31}$$

where $\alpha(\varepsilon) = \sqrt{\varepsilon}$.

Moreover, if $\|g\|_{H^t(\mathbb{T}^2 \times Q)} \leq \varepsilon$, $0 < \delta < 2s$ and $0 < \alpha \leq 2s/\delta - 1$, we have

$$\|P^s_\alpha I^*(If + g) - f\|_{H^r(\mathbb{T}^2)} \leq \alpha^{\delta/2s} C(\delta/2s) \|f\|_{H^{r+\delta}(\mathbb{T}^2)} + \frac{\varepsilon}{\alpha}, \tag{32}$$

where $C(x) = x(x^{-1} - 1)^{1-x}$.

A simple calculation shows that the optimal rate of convergence is obtained if the regularization parameter is chosen so that $\alpha = \epsilon^\lambda$ where $\lambda = (1 + \delta/2s)^{-1}$.

The proofs of the theorems are based on quite straightforward computations on the Fourier side and the formula (25). It seems that the key theoretical finding in (C) was the right structure on the image side. It is quite easy to see that I is non-surjective between the Sobolev spaces $H^s(\mathbb{T}^2)$ and $H^s(\mathbb{T}^2 \times Q)$. Hence, the choices made for the image side Sobolev norms do not fully trivialize the problem and, instead of that, those choices describe the behavior of $I|_{H^s(\mathbb{T}^2)}$.

Numerical implementation, simulations and conclusions are described in [(C), Sections 3–5]. A short discussion of typical numerical methods in CT imaging is given in [(C), Section 1.2]. We do not repeat the details or discussions here. We did not perform tests with measured

X-ray laboratory data. This would be the next step towards practical CT imaging based on the reconstruction method on the flat torus.

4.4. **Fourier analysis of periodic Radon transforms, (D).** The article (D) studies the periodic d-plane Radon transforms on $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ when $1 \leq d \leq n-1$ and $n \geq 2$. If $n = 2$ and $d = 1$, then the d-plane Radon transform is the X-ray transform studied in the article (C). The periodic Radon transforms have been applied in other mathematical tomography problems earlier: the broken ray transform on boxes [32], the geodesic ray transform on Lie groups [34], tensor tomography on periodic slabs [39], and the ray transforms on Minkowski tori [36].

We generalize the main theorems in (C) into higher dimensions [(D), Theorems 1.4 and 1.5, Proposition 3.1]. We do not restate these statements here. We state here results on the adjoint and normal operators and the stability estimates. We also introduce a new inversion formula which might be of a practical interest due to its simplicity.

We begin by introducing necessary mathematical preliminaries. Suppose that $f \in \mathcal{T} := C^\infty(\mathbb{T}^n)$, then we define the *$d$-plane Radon transform* of f by

$$R_d f(x, A) := \int_{[0,1]^d} f(x + t_1 v_1 + \cdots + t_d v_d) dt_1 \dots dt_d \tag{33}$$

where $A = \{v_1, \dots, v_d\}$ is a set of d linearly independent integer vectors $v_i \in \mathbb{Z}^n$.

It can be shown that A spans a periodic d-plane on $\mathbb{T}^n$. On the other hand, if A and B span the same periodic d-plane on $\mathbb{T}^n$, then $R_d f(x, A) = R_d f(x, B)$ for any $x \in \mathbb{T}^n$. Let $\mathbf{Gr}(d, n)$ denote the collection of d-dimensional subspaces of $\mathbb{Q}^n$. These spaces are called *Grassmannians*. For any element in $\mathbf{Gr}(d, n)$ there exists a basis of integer vectors. Hence, we may define $R_d f : \mathbf{Gr}(d, n) \to \mathcal{T}$ using bases of integer vectors as representatives of elements in $\mathbf{Gr}(d, n)$. The definition of R_d extends to the periodic distributions $\mathcal{T}'$ using the duality and the fact that $R_d(\cdot, A) : \mathcal{T} \to \mathcal{T}$ is formally L^2 self-adjoint for any fixed $A \in \mathbf{Gr}(d, n)$. Let us denote $R_{d,A} f = R_d f(\cdot, A)$ for any $f \in \mathcal{T}'$.

Next, we define suitable structures for the data spaces such that the images of the Bessel potential spaces $L^p_s(\mathbb{T}^n)$ under R_d are contained into the data spaces. Let $p, l \in [1, \infty]$ and $s \in \mathbb{R}$. We define the *Bessel* *potential norms* as

$$\|f\|_{L^p_s(\mathbb{T}^n)} = \| \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s \hat{f}(k) e^{2\pi i k\cdot x} \|_{L^p(\mathbb{T}^n)},$$
$$\|f\|_{H^s(\mathbb{T}^n)} = \sqrt{\sum_{k\in\mathbb{Z}^k} \langle k\rangle^{2s} \left|\hat{f}(k)\right|^2} \tag{34}$$

where $\langle k\rangle = (1+|k|^2)^{1/2}$ as usual. The space $L^p_s(\mathbb{T}^n) \subset \mathcal{T}'$ consists of all $f \in \mathcal{T}'$ with $\|f\|_{L^p_s(\mathbb{T}^n)} < \infty$. If $p = 2$, then $H^s(\mathbb{T}^n) = L^p_s(\mathbb{T}^n)$. One has equivalently that $f \in L^p_s(\mathbb{T}^n)$ if and only if $(1-\Delta)^{s/2} f \in L^p(\mathbb{T}^n)$ and $f \in \mathcal{T}'$.

Let us denote $X_{d,n} := \mathbb{T}^n \times \mathbf{Gr}(d,n)$ to keep our notation shorter. Let $w : \mathbb{Z}^n \times \mathbf{Gr}(d,n) \to (0,\infty)$ be a weight function such that $w(\cdot, A)$ is at most of polynomial decay for any fixed $A \in \mathbf{Gr}(d,n)$ (see [(D), Section 2.2] for the definition). We say that a function $g : X_{d,n} \to \mathbb{C}$ belongs to $L^{p,l}_s(X_{d,n}; w)$ with $1 \le l < \infty$ if the norm

$$\|g\|^l_{L^{p,l}_s(X_{d,n};w)} := \sum_{A\in\mathbf{Gr}(d,n)} \|g(\cdot, A)\|^l_{L^p_s(\mathbb{T}^n; w(\cdot,A))} \tag{35}$$

is finite and $g(\cdot, A) \in \mathcal{T}'$ when $A \in \mathbf{Gr}(d,n)$. Similarly, if $l = \infty$, we define

$$\|g\|_{L^{p,\infty}_s(X_{d,n};w)} := \sup_{A\in\mathbf{Gr}(d,n)} \|g(\cdot, A)\|_{L^p_s(\mathbb{T}^n; w(\cdot,A))} \tag{36}$$

If $p, l = 2$, then the norm is generated by the corresponding inner product. The spaces $L^{p,l}_s(X_{d,n}; w)$ are Banach spaces [(D), Lemma 2.1].

We have introduced weighted structures since most of the theorems in (D) would have been unreachable without such structures when $d < n-1$. If $d = n-1$, then the analysis of (C) using slightly different data spaces generalizes nicely without weights. It is explained in the article (D) how the results in (C) can be obtained from the results in (D). We construct weights that satisfy the assumptions of our theorems in [(D), Section 2.3].

We state some of the main results in (D) next.

Theorem 4.9 ((D), Theorem 1.1)**.** *Let $s \in \mathbb{R}$ and suppose that there exists $C_w > 0$ such that*

$$\sum_{A\in\Omega_k} w(k,A)^2 \le C_w^2, \quad \Omega_k := \{\, A \in \mathbf{Gr}(d,n)\, ;\, k \perp A \,\} \tag{37}$$

for any $k \in \mathbb{Z}^n$. Then the adjoint of $R_d : H^s(\mathbb{T}^n) \to L^{2,2}_s(X_{d,n}; w)$ is given by

$$\widehat{R^*_d g}(k) = \sum_{A\in\Omega_k} w(k,A)^2 \hat{g}(k,A) \tag{38}$$

and the normal operator $R_d^* R_d : H^s(\mathbb{T}^n) \to H^s(\mathbb{T}^n)$ *is the Fourier multiplier* $W_k := \sum_{A\in\Omega_k} w(k,A)^2$. *In particular, the mapping* $F_{W_k^{-1}} R_d^* : R_d(\mathcal{T}') \to \mathcal{T}'$ *is the inverse of* R_d.

Theorem 4.9 generalizes [(C), Proposition 11] into higher dimensions and implies the following results on stability.

Corollary 4.10 ((D), Corollary 1.2)**.** *Suppose that the assumptions of theorem 4.9 hold, and that there exists* $c_w > 0$ *such that* $W_k \geq c_w^2$ *for any* $k \in \mathbb{Z}^n$.

(i) Then $F_{W_k^{-1}} R_d^* : L_s^{2,2}(X_{d,n}; w) \to H^s(\mathbb{T}^n)$ *is* $1/c_w$*-Lipschitz.*

(ii) Let $f \in \mathcal{T}'$. *Then*

$$\|f\|_{H^s(\mathbb{T}^n)} \leq \frac{1}{c_w} \|R_d f\|_{L_s^{2,2}(X_{d,n};w)}. \tag{39}$$

(iii) Let $\tilde{w}(k,A) = \frac{w(k,A)}{\sqrt{W_k}}$ *and* $p \in [1,\infty]$. *Then* $R_d^{*,\tilde{w}} R_d f = f$ *and* $\|f\|_{L_s^p(\mathbb{T}^n)} = \|R_d^{*,\tilde{w}} R_d f\|_{L_s^p(\mathbb{T}^n)}$ *for any* $f \in \mathcal{T}'$.

Other stability estimates on $L_s^p(\mathbb{T}^n)$ are given in terms of $R_d f$ in [(D), Proposition 4.3]. Those stability estimates follow from corollary 4.10 and the Sobolev inequality on $\mathbb{T}^n$. This method requires additional smoothness of $R_d f$ in order to control the norm of f due to the use of the Sobolev inequality. The stability estimates in (D) are new in any dimension, and different than the stability estimates in [32].

Theorem 4.11 ((D), Theorem 1.3)**.** *Suppose that* $f \in \mathcal{T}'$. *Let* $w : \mathbb{Z}^n \times \mathbf{Gr}(d,n) \to \mathbb{R}$ *be a weight so that*

$$\sum_{A\in\Omega_k} w(k,A) = 1, \quad \Omega_k := \{\, A \in \mathbf{Gr}(d,n)\,;\, k \perp A \,\} \tag{40}$$

and the series is absolutely converging for any $k \in \mathbb{Z}^n$ *(the weight does not have to generate a norm or have at most of polynomial decay). Then*

$$(f,h) = \sum_{A\in\mathbf{Gr}(d,n)} (F_{w(\cdot,A)} R_{d,A} f, h), \quad \forall h \in \mathcal{T}. \tag{41}$$

Moreover, if f *has zero average and* $d = n-1$, *then*

$$f = \sum_{A\in\mathbf{Gr}(d,n)} R_{d,A} f. \tag{42}$$

Theorem 4.11 gives a new reconstructive formula for the inverse of R_d. The case $d = n-1$ is especially interesting since it does not involve any filtering, and averages are simple to reconstruct and filter out from $R_d f$. The proof of theorem 4.11 follows easily from the higher dimensional version of the formula (25) proved in [32].

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Included articles

(A)

Tensor tomography on Cartan-Hadamard manifolds

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Tensor tomography on Cartan–Hadamard manifolds

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Abstract

We study the geodesic x-ray transform on Cartan–Hadamard manifolds, generalizing the x-ray transforms on Euclidean and hyperbolic spaces that arise in medical and seismic imaging. We prove solenoidal injectivity of this transform acting on functions and tensor fields of any order. The functions are assumed to be exponentially decaying if the sectional curvature is bounded, and polynomially decaying if the sectional curvature decays at infinity. This work extends the results of Lehtonen (2016 arXiv:1612.04800) to dimensions $n \geqslant 3$ and to the case of tensor fields of any order.

Keywords: differential geometry, tensor tomography, Cartan–Hadamard manifolds, x-ray transform

1. Introduction

1.1. Motivation

This article considers the geodesic x-ray transform on noncompact Riemannian manifolds. This transform encodes the integrals of a function f, where f satisfies suitable decay conditions at infinity, over all geodesics. In the case of Euclidean space the geodesic x-ray transform is just the usual x-ray transform involving integrals over all lines, and in two dimensions it coincides with the Radon transform introduced in the seminal work of Radon in 1917 [Rad17].

X-ray and Radon type transforms in Euclidean space are widely used as mathematical models for medical and industrial imaging methods, such as CT, PET, SPECT and MRI (see [Nat01]). In these applications one is interested in reconstructing unknown coefficients in a bounded region. However, it is often convenient to model the problems in terms of compactly supported functions in the noncompact space $\mathbb{R}^n$, which makes it possible to use Fourier transform based methods for instance.

Another important class of imaging problems arises in geophysics, when determining interior properties of the Earth from acoustic scattering or earthquake measurements. In these

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problems one encounters x-ray transforms over general families of curves, which often correspond to geodesic curves of a sound speed profile within the Earth. Moreover, if the sound speed is anisotropic (depends on direction), then one needs to consider geodesic x-ray transforms of tensor fields [Sha94]. A typical feature is that rays originating near the Earth surface eventually curve back to the surface. A simple mathematical model, which has been used as a first approximation for this behaviour, is to think of the domain as embedded in hyperbolic space $\mathbb{H}^n$ and to consider the geodesic x-ray transform in $\mathbb{H}^n$ [Bal05]. The hyperbolic geodesic x-ray transform also appears in Electrical Impedance Tomography in connection with the method of Barber and Brown [BCT96] and in partial data problems [KS14].

Another setting where x-ray transforms on noncompact manifolds appear is inverse scattering theory (for instance in quantum mechanics, acoustics, or electromagnetics). The connection between scattering theory and Radon type transforms goes back at least to Lax and Phillips [LP89], and the x-ray transform of a scattering potential can be determined from measurements of the full scattering amplitude at high frequencies (see e.g. [Wed14]). The x-ray transforms that appear in these contexts are often Euclidean. However, in inverse scattering applications related to general relativity and black holes one encounters more general manifolds that resemble asymptotically hyperbolic ones [JSB00], and in recent results on phaseless inverse scattering problems more general geodesic x-ray transforms also arise (see [Kli17] and references therein). We remark that both in quantum mechanics and general relativity, the functions that one would like to reconstruct are often not compactly supported and thus it is important to deal with noncompact manifolds.

In this article we will study the invertibility of geodesic x-ray transforms on noncompact Riemannian manifolds. Our results will include Euclidean and hyperbolic space as special cases, but will apply to more general manifolds with nonpositive curvature (Cartan–Hadamard manifolds). This work also follows the long tradition of integral geometry problems as discussed for instance in [GGG03, Hel99, Hel13]. Here one of the main points is that our results apply to manifolds that do not need to have special symmetries (see the recent preprint [GGSU17] for related results).

1.2. Results

For Euclidean or hyperbolic space in dimensions $n \geqslant 2$, one has the following basic theorems on the injectivity of this transform (see [Hel94, Hel99, Jen04]):

Theorem A. *If f is a continuous function in $\mathbb{R}^n$ satisfying $|f(x)| \leqslant C(1+|x|)^{-\eta}$ for some $\eta > 1$, and if f integrates to zero over all lines in $\mathbb{R}^n$, then $f \equiv 0$.*

Theorem B. *If f is a continuous function in the hyperbolic space $\mathbb{H}^n$ satisfying $|f(x)| \leqslant C\mathrm{e}^{-d(x,o)}$, where $o \in \mathbb{H}^n$ is some fixed point, and if f integrates to zero over all geodesics in $\mathbb{H}^n$, then $f \equiv 0$.*

We remark that some decay conditions for the function f are required, since there are examples of nontrivial functions in $\mathbb{R}^2$ which decay like $|x|^{-2}$ on every line and whose x-ray transform vanishes [Arm94, Zal82]. Related results on the invertibility of Radon type transforms on constant curvature spaces or noncompact homogeneous spaces may be found in [Hel99, Hel13].

The purpose of this article is to give analogues of the above theorems on more general, not necessarily symmetric Riemannian manifolds. We will work in the setting of Cartan–Hadamard manifolds, i.e. complete simply connected Riemannian manifolds with nonpositive sectional curvature. Euclidean and hyperbolic spaces are special cases of Cartan–Hadamard manifolds, and further explicit examples are recalled in section 2. It is well known that any Cartan–Hadamard manifold is diffeomorphic to $\mathbb{R}^n$, the exponential map at any point is a diffeomorphism, and the map $x \mapsto d(x,p)^2$ is strictly convex for any $p \in M$ (see e.g. [Pet06]).

Definition. Let (M,g) be a Cartan–Hadamard manifold, and fix a point $o \in M$. If $\eta > 0$, define the spaces of exponentially and polynomially decaying continuous functions by

$$E_\eta(M) = \{f \in C(M);\ |f(x)| \leqslant C\mathrm{e}^{-\eta d(x,o)} \text{ for some } C > 0\},$$
$$P_\eta(M) = \{f \in C(M);\ |f(x)| \leqslant C(1+d(x,o))^{-\eta} \text{ for some } C > 0\}.$$

Also define the spaces

$$E^1_\eta(M) = \{f \in C^1(M);\ |f(x)| + |\nabla f(x)| \leqslant C\mathrm{e}^{-\eta d(x,o)} \text{ for some } C > 0\},$$
$$P^1_\eta(M) = \{f \in C^1(M);\ |f(x)| \leqslant C(1+d(x,o))^{-\eta} \text{ and } |\nabla f(x)| \leqslant C(1+d(x,o))^{-\eta-1} \text{ for some } C > 0\}.$$

Here $\nabla = \nabla_g$ is the total covariant derivative in (M,g) and $|\cdot| = |\cdot|_g$ is the g-norm on tensors.

It follows from lemma 4.1 that if $f \in P_\eta(M)$ for some $\eta > 1$, then the integral of f over any maximal geodesic in M is finite. For such functions f we may define the geodesic x-ray transform $I_0 f$ of f by

$$I_0 f(\gamma) = \int_{-\infty}^{\infty} f(\gamma(t))\,\mathrm{d}t, \qquad \gamma \text{ is a geodesic.}$$

The inverse problem for the geodesic x-ray transform is to determine f from the knowledge of $I_0 f$. By linearity, uniqueness for this inverse problem reduces to showing that $I_0 f = 0$ implies $f = 0$.

More generally, suppose that f is a C^1-smooth symmetric covariant m-tensor field on M, written in local coordinates (using the Einstein summation convention) as

$$f = f_{j_1\ldots j_m}(x)\,\mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_m}.$$

We say that $f \in P_\eta(M)$ if $|f|_g \in P_\eta(M)$, and $f \in P^1_\eta(M)$ if $|f|_g \in P_\eta(M)$ and $|\nabla f|_g \in P_{\eta+1}(M)$, etc. We recall that, in terms of local coordinates,

$$\left|f(x)\right|_g = \left(g^{j_1k_1}(x)\cdots g^{j_mk_m}(x) f_{j_1\ldots j_m}(x) f_{k_1\ldots k_m}(x)\right)^{1/2}$$

where (g^{jk}) is the inverse matrix of (g_{jk}).

Now if $f \in P_\eta(M)$ for some $\eta > 1$, then the geodesic x-ray transform $I_m f$ of f is well defined by the formula

$$I_m f(\gamma) = \int_{-\infty}^{\infty} f_{\gamma(t)}(\dot\gamma(t),\ldots,\dot\gamma(t))\,\mathrm{d}t, \qquad \gamma \text{ is a geodesic.}$$

This transform always has a kernel when $m \geqslant 1$: if h is a symmetric $(m-1)$-tensor field satisfying $h \in P^1_\eta(M)$ for some $\eta > 0$, then $I_m(\sigma\nabla h) = 0$ where σ denotes symmetrization of a tensor field (see section 3.3). We say that I_m is solenoidal injective if $I_m f = 0$ implies $f = \sigma\nabla h$ for some $(m-1)$-tensor field h.

Our first theorem proves solenoidal injectivity of I_m for any $m \geqslant 0$ on Cartan–Hadamard manifolds with bounded sectional curvature, assuming exponential decay of the tensor field and its first derivatives. We will denote the sectional curvature of a two-plane $\Pi \subset T_xM$ by $K_x(\Pi)$, and we write $-K_0 \leqslant K \leqslant 0$ if $-K_0 \leqslant K_x(\Pi) \leqslant 0$ for all $x \in M$ and for all two-planes $\Pi \subset T_xM$.

Theorem 1.1. *Let (M, g) be a Cartan–Hadamard manifold of dimension $n \geqslant 2$, and assume that*

$$-K_0 \leqslant K \leqslant 0, \qquad \text{for some } K_0 > 0.$$

If f is a symmetric m-tensor field in $E^1_\eta(M)$ for some $\eta > \frac{n+1}{2}\sqrt{K_0}$, and if $I_m f = 0$, then $f = \sigma\nabla h$ for some symmetric $(m-1)$-tensor field h such that $h \in E_{\eta-\varepsilon}(M)$ for any $\varepsilon > 0$. (If $m = 0$, then $f \equiv 0$.)

The second theorem considers the case where the sectional curvature decays polynomially at infinity, and proves solenoidal injectivity if the tensor field and its first derivatives also decay polynomially.

Theorem 1.2. *Let (M, g) be a Cartan–Hadamard manifold of dimension $n \geqslant 2$, and assume that the function*

$$\mathcal{K}(x) = \sup\{|K_x(\Pi)|;\, \Pi \subset T_xM \text{ is a two-plane}\}$$

satisfies $\mathcal{K} \in P_\kappa(M)$ for some $\kappa > 2$. If f is a symmetric m-tensor field in $P^1_\eta(M)$ for some $\eta > \frac{n+2}{2}$, and if $I_m f = 0$, then $f = \sigma\nabla h$ for some symmetric $(m-1)$-tensor field $h \in P_{\eta-1}(M)$. (If $m = 0$, then $f \equiv 0$.)

The second theorem is mostly of interest in two dimensions because of the following rigidity phenomenon: any manifold of dimension $\geqslant 3$ that satisfies the conditions of the theorem is isometric to Euclidean space [GW82]. See section 2 for a discussion. We will give the proof in any dimension since this may be useful in subsequent work.

We remark that theorems 1.1 and 1.2 correspond to theorems A and B above, but the manifolds considered in theorems 1.1 and 1.2 can be much more general and include many examples with nonconstant curvature (see section 2). The results will be proved by using energy methods based on Pestov identities, which have been studied extensively in the case of compact manifolds with strictly convex boundary. We refer to [Kni02, Muk77, PS88, PSU14, Sha94] for some earlier results. In fact, theorems 1.1 and 1.2 can be viewed as an extension of the tensor tomography results in [PS88] from the case of compact nonpositively curved manifolds with boundary to the case of certain noncompact manifolds. We remark that one of the main points in our theorems is that the functions and tensor fields are not compactly supported (indeed, the compactly supported case would reduce to known results on compact manifolds with boundary).

More recently, the work [PSU13] gave a particularly simple derivation of the basic Pestov identity for x-ray transforms and proved solenoidal injectivity of I_m on simple 2D manifolds. Some of these methods were extended to all dimensions in [PSU15] and to the case of attenuated x-ray transforms in [GPSU16]. Following some ideas in [PSU13], the work [Leh16] proved versions of theorems 1.1 and 1.2 for the case of 2D Cartan–Hadamard manifolds.

In this paper we combine the main ideas in [Leh16] with the methods of [PSU15] and prove solenoidal injectivity results on Cartan–Hadamard manifolds in any dimension $n \geqslant 2$.

However, instead of using the Pestov identity in its standard form (which requires two derivatives of the functions involved), we will use a different argument from [PSU15] related to the L^2 contraction property of a Beurling transform on nonpositively curved manifolds. This argument dates back to [GK80a, GK80b], it only involves first order derivatives and immediately applies to tensor fields of arbitrary order. The C^1 assumption in theorems 1.1 and 1.2 is due to this method of proof, and the decay assumptions are related to the growth of Jacobi fields. We mention that theorems 1.1 and 1.2 also extend the 2D results of [Leh16] by assuming slightly weaker conditions.

This article is organized as follows. Section 1 is the introduction, and section 2 contains examples of Cartan–Hadamard manifolds. In section 3 we review basic facts related to geodesics on Cartan–Hadamard manifolds, geometry of the sphere bundle and symmetric covariant tensors fields, following [DS10, Leh16, PSU15]. Section 4 collects some estimates concerning the growth of Jacobi fields and related decay properties for solutions of transport equations. Finally, section 5 includes the proofs of the main theorems based on L^2 inequalities for Fourier coefficients.

2. Examples of Cartan–Hadamard manifolds

In this section we recall some facts and examples related to Cartan–Hadamard manifolds. Most of the details can be found in [BO69, GW79, GW82, KW74, Pet06]. We first discuss the case of 2D manifolds, which is quite different compared to manifolds of higher dimensions.

2.1. Dimension two

Let $K \in C^\infty(\mathbb{R}^2)$. A theorem of Kazdan and Warner [KW74] states that a necessary and sufficient condition for existence of a complete Riemannian metric on $\mathbb{R}^2$ with Gaussian curvature K is

$$\lim_{r\to\infty} \inf_{|x|\geqslant r} K(x) \leqslant 0. \tag{2.1}$$

This provides a wide class of Riemannian metrics satisfying the assumptions of theorem 1.1 in dimension two. However, this does not directly give an example of a manifold satisfying the assumptions of theorem 1.2 since the condition (2.1) is given with respect to the Euclidean metric of $\mathbb{R}^2$.

Examples of manifolds satisfying the assumptions of theorem 1.2 can be constructed using warped products. Let (r, θ) be the polar coordinates in $\mathbb{R}^2$ and consider a warped product

$$\mathrm{d}s^2 = \mathrm{d}r^2 + f^2(r)\mathrm{d}\theta^2, \tag{2.2}$$

where f is a smooth function that is positive for $r > 0$ and satisfies $f(0) = 0$ and $f'(0) = 1$. This is a Riemannian metric on $\mathbb{R}^2$ having Gaussian curvature

$$K(x) = -\frac{f''(|x|)}{f(|x|)}, \tag{2.3}$$

which depends only on the Euclidean distance $|x| := r(x)$ to the origin. We remark that distances to the origin in the Euclidean metric and in the warped metric coincide. It is shown in [GW79, proposition 4.2] that for every $k \in C^\infty([0,\infty))$ with $k \leqslant 0$ there exists a unique warped metric of the form (2.2) such that $k(|x|) = K(x)$. Hence warped products provide many examples of 2D manifolds for which $\mathcal{K}(x) \leqslant C(1+|x|)^{-\kappa}$ with $\kappa > 0$, i.e. $\mathcal{K} \in P_\kappa(M)$.

2.2. Higher dimensions

Warped products can also be used to construct examples of higher dimensional Cartan–Hadamard manifolds satisfying the assumptions of theorem 1.1, see e.g. [BO69].

In the case of theorem 1.2 it turns out that the decay condition for curvature is very restrictive in higher dimensions: the only possible geometry is the Euclidean one. This follows directly from a theorem by Greene and Wu in [GW82]. If M is a Cartan–Hadamard manifold with $n = \dim(M) \geqslant 3$, $k(s) = \sup\{\mathcal{K}(x)\,;x \in M, d(x,o) = s\}$, where o is a fixed point, and one of the following holds:

(1) n is odd and $\liminf_{s\to\infty} s^2 k(s) \to 0$ or
(2) n is even and $\int_0^\infty sk(s)\,\mathrm{d}s$ is finite,

then M is isometric to $\mathbb{R}^n$.

3. Geometric facts

Throughout this work we will assume (M,g) to be an n-dimensional Cartan–Hadamard manifold with $n \geqslant 2$ unless otherwise stated. We also assume unit speed parametrization for geodesics.

In this section we collect some preliminary facts on geodesics on Cartan–Hadamard manifolds, derivatives on the unit tangent bundle and related Jacobi fields, and tensor fields. These facts will be used in the subsequent sections.

3.1. Behaviour of geodesics

By the Cartan–Hadamard theorem the exponential map $\exp_x$ is defined on all of T_xM and is a diffeomorphism for every $x \in M$. Hence every pair of points can be joined by a unique geodesic. Let $SM = \{(x,v) \in TM\,;\,|v| = 1\}$ be the unit sphere bundle, and if $(x,v) \in SM$ denote by $\gamma_{x,v}$ the unique geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = v$. The triangle inequality implies that

$$d_g(\gamma_{x,v}(t), o) \geqslant |t| - d_g(x,o) \tag{3.1}$$

for all $t \in \mathbb{R}, o \in M$.

We say that a geodesic γ is escaping with respect to the point o if the function $t \mapsto d_g(\gamma(t),o)$ is strictly increasing on the interval $[0,\infty)$. The set of all such geodesics is denoted by $\mathcal{E}_o$. For $\gamma_{x,v} \in \mathcal{E}_o$ the triangle inequality gives

$$d_g(\gamma_{x,v}(t), o) \geqslant \begin{cases} d_g(x,o), & \text{if } 0 \leqslant t \leqslant 2d_g(x,o), \\ t - d_g(x,o), & \text{if } 2d_g(x,o) < t. \end{cases} \tag{3.2}$$

However, since (M,g) is a Cartan–Hadamard manifold, Jacobi field estimates give a stronger bound. For $\gamma_{x,v} \in \mathcal{E}_o$ one has (see [Jos08, corollary 4.8.5] or [Pet06, section 6.3])

$$d_g(\gamma_{x,v}(t), o) \geqslant \sqrt{d_g(x,o)^2 + t^2}, \qquad t \geqslant 0. \tag{3.3}$$

The following lemma is proved in [Leh16] in two dimensions. The proof in higher dimensions is identical, but we include a short argument for completeness.

Lemma 3.1. *Suppose $o \in M$. At least one of the geodesics $\gamma_{x,v}$ and $\gamma_{x,-v}$ is in $\mathcal{E}_o$.*

Proof. Since (M, g) is a Cartan–Hadamard manifold, the function $h(t) = d_g(\gamma_{x,v}(t), o)^2$ is strictly convex, $h'' > 0$, on $\mathbb{R}$. If $h'(0) \geqslant 0$ then $\gamma_{x,v}$ is escaping, and if $h'(0) \leqslant 0$ then $\gamma_{x,-v}$ is escaping. □

3.2. On the geometry of the unit tangent bundle

We first briefly explain the splitting of the tangent bundle of SM into horizontal and vertical bundles. Then we give a short discussion on geodesics of SM. Finally, we include a proof that SM is complete when M is.

3.2.1. The structure of the tangent bundle. The following discussion is based on [Pat99, PSU15], where these topics are considered in more detail. We denote by $\pi\colon TM \to M$ the usual base point map $\pi(x, v) = x$. The connection map $K_\nabla\colon T(TM) \to TM$ of the Levi-Civita connection ∇ of M is defined as follows. Let $\xi \in T_{x,v}TM$ and $c\colon (-\varepsilon, \varepsilon) \to TM$ be a curve such that $\dot{c}(0) = \xi$. Write $c(t) = (\gamma(t), Z(t))$, where $Z(t)$ is a vector field along the curve γ, and define

$$K_\nabla(\xi) := D_t Z(0) \in T_x M.$$

The maps K_∇ and $\mathrm{d}\pi$ yield a splitting

$$T_{x,v}TM = \tilde{\mathcal{H}}(x, v) \oplus \tilde{\mathcal{V}}(x, v) \tag{3.4}$$

where $\tilde{\mathcal{H}}(x, v) = \ker K_\nabla$ is the horizontal bundle and $\tilde{\mathcal{V}}(x, v) = \ker d_{x,v}\pi$ is the vertical bundle. Both are n-dimensional subspaces of $T_{x,v}TM$.

On TM we define the Sasaki metric g_{s} by

$$\langle v, w\rangle_{g_{\mathrm{s}}} = \langle K_\nabla(v), K_\nabla(w)\rangle_g + \langle \mathrm{d}\pi(v), \mathrm{d}\pi(w)\rangle_g,$$

which makes (TM, g_{s}) a Riemannian manifold of dimension $2n$. The maps $K_\nabla\colon \tilde{\mathcal{V}}(x, v) \to T_x M$ and $\mathrm{d}\pi\colon \tilde{\mathcal{H}}(x, v) \to T_x M$ are linear isomorphisms. Furthermore, the splitting (3.4) is orthogonal with respect to g_s. Using the maps K_∇ and $\mathrm{d}\pi$, we will identify vectors in the horizontal and vertical bundles with corresponding vectors on $T_x M$.

The unit sphere bundle SM was defined as

$$SM := \bigcup_{x\in M} S_x M, \qquad S_x M := \{(x, v) \in T_x M\,;\, |v|_g = 1\}.$$

We will equip SM with the metric induced by the Sasaki metric on TM. The geodesic flow $\phi_t(x, v)\colon \mathbb{R} \times SM \to SM$ is defined as

$$\phi_t(x, v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)).$$

The associated vector field is called the geodesic vector field and denoted by X.

For SM we obtain an orthogonal splitting

$$T_{x,v}SM = \mathbb{R}X(x, v) \oplus \mathcal{H}(x, v) \oplus \mathcal{V}(x, v) \tag{3.5}$$

where $\mathbb{R}X \oplus \mathcal{H}(x,v) = \tilde{\mathcal{H}}(x,v)$ and $\mathcal{V}(x,v) = \ker d_{x,v}(\pi|_{SM})$. Both $\mathcal{H}(x,v)$ and $\mathcal{V}(x,v)$ have dimension $n-1$ and can be canonically identified with elements in the codimension one subspace $\{v\}^{\perp} \subset T_xM$ via $\mathrm{d}\pi$ and K_{∇}, respectively. We will freely use this identification.

Following [PSU15], if $u \in C^1(SM)$, then the gradient $\nabla_{SM}u$ has the decomposition

$$\nabla_{SM}u = (Xu)X + \overset{\mathtt{h}}{\nabla} u + \overset{\mathtt{v}}{\nabla} u,$$

according to (3.5). The quantities $\overset{\mathtt{h}}{\nabla} u$ and $\overset{\mathtt{v}}{\nabla} u$ are called the horizontal and the vertical gradients, respectively. It holds that $\langle \overset{\mathtt{v}}{\nabla} u(x,v), v\rangle_g = 0$ and $\langle \overset{\mathtt{h}}{\nabla} u(x,v), v\rangle_g = 0$ for all $(x,v) \in SM$.

As discussed in [PSU15], on 2D manifolds the horizontal and vertical gradients reduce to the horizontal and vertical vector fields $X_{\perp}$ and V via

$$\overset{\mathtt{h}}{\nabla} u(x,v) = -(X_{\perp}u(x,v))v^{\perp} \quad \text{and} \quad \overset{\mathtt{v}}{\nabla} u(x,v) = (Vu(x,v))v^{\perp}$$

where $v^{\perp}$ is such that $\{v, v^{\perp}\}$ is a positive orthonormal basis of T_xM. In Leh16 the flows associated with $X_{\perp}$ and V were used to derive estimates for $X_{\perp}u$ and Vu. We will proceed in a similar manner in the higher dimensional case.

Let $(x,v) \in SM$ and $w \in S_xM$, $w \perp v$. We define $\phi^{\mathtt{h}}_{w,t}\colon \mathbb{R} \to SM$ by $\phi^{\mathtt{h}}_{w,t}(x,v) = (\gamma_{x,w}(t), V(t))$, where $V(t)$ is the parallel transport of v along $\gamma_{x,w}$. It holds that

$$K_{\nabla}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^{\mathtt{h}}_{w,t}(x,v)\Big|_{t=0}\right) = 0 \quad \text{and} \quad \mathrm{d}\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^{\mathtt{h}}_{w,t}(x,v)\Big|_{t=0}\right) = w. \tag{3.6}$$

We define $\phi^{\mathtt{v}}_{w,t}\colon \mathbb{R} \to SM$ by $\phi^{\mathtt{v}}_{w,t}(x,v) = (x, (\cos t)v + (\sin t)w)$. It holds that

$$K_{\nabla}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^{\mathtt{v}}_{w,t}(x,v)\Big|_{t=0}\right) = w \quad \text{and} \quad \mathrm{d}\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi^{\mathtt{v}}_{w,t}(x,v)\Big|_{t=0}\right) = 0. \tag{3.7}$$

The following lemma states the relation between $\phi^{\mathtt{h}}_{w,t}$ and $\phi^{\mathtt{v}}_{w,t}$ and the horizontal and the vertical gradients of a function.

Lemma 3.2. *Suppose u is differentiable at $(x,v) \in SM$. Fix $w \in S_xM, w \perp v$. Then it holds that*

$$\langle \overset{\mathtt{h}}{\nabla} u(x,v), w\rangle_g = \frac{\mathrm{d}}{\mathrm{d}t}u(\phi^{\mathtt{h}}_{w,t}(x,v))\Big|_{t=0}$$

and

$$\langle \overset{\mathtt{v}}{\nabla} u(x,v), w\rangle_g = \frac{\mathrm{d}}{\mathrm{d}t}u(\phi^{\mathtt{v}}_{w,t}(x,v))\Big|_{t=0}.$$

Proof. Using the chain rule and the equations (3.6) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\phi^{\mathtt{h}}_{w,t}(x,v))\Big|_{t=0} = \langle \nabla_{SM}u(\phi^{\mathtt{h}}_{w,t}(x,v)), \frac{\mathrm{d}}{\mathrm{d}t}\phi^{\mathtt{h}}_{w,t}(x,v)\rangle_{g_s}\Big|_{t=0} = \langle \overset{\mathtt{h}}{\nabla} u(x,v), w\rangle_g.$$

For $\overset{\mathtt{v}}{\nabla}$ we use the equations (3.7) in a similar fashion. □

The maps $\phi^{\mathtt{h}}_{w,t}$ and $\phi^{\mathtt{v}}_{w,t}$ are related to normal Jacobi fields along geodesics. We can define

$$J^{\mathtt{h}}_w(t) := \frac{\mathrm{d}}{\mathrm{d}s}\pi\left(\phi_t(\phi^{\mathtt{h}}_{w,s}(x,v))\right)\Big|_{s=0} = d_{\phi_t(x,v)}\pi\left(\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^{\mathtt{h}}_{w,s}(x,v))\Big|_{s=0}\right).$$

Since $\Gamma(s,t) = \pi\left(\phi_t(\phi^{\mathrm{h}}_{w,s}(x,v))\right)$ is a variation of $\gamma_{x,v}$ along geodesics, $J^{\mathrm{h}}_w(t)$ is a Jacobi field along $\gamma_{x,v}$. It has the initial conditions $J^{\mathrm{h}}_w(0) = w$ and $D_t J^{\mathrm{h}}_w(0) = 0$ by the symmetry lemma (see e.g. [Lee97]).

Replacing $\phi^{\mathrm{h}}_{w,s}$ with $\phi^{\mathrm{v}}_{w,s}$ gives a Jacobi field $J^{\mathrm{v}}_w(t)$ with the initial conditions $J^{\mathrm{v}}_w(t)(0) = 0$ and $D_t J^{\mathrm{v}}_w(t)(0) = w$. In the both cases the Jacobi field is normal because $\langle v, w\rangle_g = 0$.

By the symmetry lemma

$$K_\nabla\left(\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^{\mathrm{h}}_{w,s}(x,v))\Big|_{s=0}\right) = D_s\partial_t\gamma_{\phi^{\mathrm{h}}_{w,s}(x,v)}(t)\Big|_{s=0} = D_t\partial_s\gamma_{\phi^{\mathrm{h}}_{w,s}(x,v)}(t)\Big|_{s=0} = D_t J^{\mathrm{h}}_w(t).$$

From the definition of the Sasaki metric we then see that

$$\left\langle\nabla_{SM}u(\phi_t(x,v)), \frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^{\mathrm{h}}_{w,s}(x,v))\Big|_{s=0}\right\rangle_{g_s} = \left\langle \overset{\mathrm{h}}{\nabla} u(\phi_t(x,v)), J^{\mathrm{h}}_w(t)\right\rangle_g + \left\langle \overset{\mathrm{v}}{\nabla} u(\phi_t(x,v)), D_t J^{\mathrm{h}}_w(t)\right\rangle_g.$$

and

$$\left\langle\nabla_{SM}u(\phi_t(x,v)), \frac{\mathrm{d}}{\mathrm{d}s}\phi_t(\phi^{\mathrm{v}}_{w,s}(x,v))\Big|_{s=0}\right\rangle_{g_s} = \left\langle \overset{\mathrm{h}}{\nabla} u(\phi_t(x,v)), J^{\mathrm{v}}_w(t)\right\rangle_g + \left\langle \overset{\mathrm{v}}{\nabla} u(\phi_t(x,v)), D_t J^{\mathrm{v}}_w(t)\right\rangle_g.$$

Remark 1. The constructions in this subsection remain valid at a.e. $(x,v) \in SM$ if one assumes that u is in the space $W^{1,\infty}_{\mathrm{loc}}(SM)$. Functions in $W^{1,\infty}_{\mathrm{loc}}(SM)$ are characterized as locally Lipschitz functions, and further by Rademacher's theorem, differentiable almost everywhere and weak gradients equal to gradients almost everywhere (see e.g. [Eva98, chapters 5.8.2 and 5.8.3]).

3.2.2. Geodesics on the unit tangent bundle. Next we describe some facts related to geodesics on SM (see e.g. [BBNV03] and references therein). Let $R(U,V)$ denote the Riemannian curvature tensor. A curve $\Gamma(t) = (x(t), V(t))$ on SM is a geodesic if and only if

$$\begin{cases} \nabla_{\dot x}\dot x &= -R(V, \nabla_{\dot x}V)\dot x \\ \nabla_{\dot x}\nabla_{\dot x}V &= -|\nabla_{\dot x}V|_g^2 V, \quad |\nabla_{\dot x}V|_g^2 \text{ is a constant along } x(t) \end{cases} \tag{3.8}$$

holds for every t in the domain of Γ (see [Sas62, equations (5.2)]). Given $(x,v) \in SM$, the horizontal lift of $w \in T_xM$ is denoted by w^{h}, i.e. the unique vector $w^{\mathrm{h}} \in T_{x,v}(SM)$ such that $d(\pi|_{SM})(w^{\mathrm{h}}) = w$ and $K_\nabla(w^{\mathrm{h}}) = 0$, and the vertical lift w^{v} is defined similarly. Initial conditions for $x, \dot x, V$ and $\nabla_{\dot x}V$ at $t=0$ with $g(V(0), \nabla_{\dot x(0)}V(0)) = 0$ and $|V(0)|_g = 1$ determine a unique geodesic $\Gamma = (x, V)$, by (3.8), which satisfies the initial conditions $\Gamma(0) = (x(0), V(0))$ and $\dot\Gamma(0) = \dot x(0)^{\mathrm{h}} + (\nabla_{\dot x(0)}V(0))^{\mathrm{v}}$ where the lifts are done with respect to $(x(0), V(0)) \in SM$. The geodesics of SM are of the following three types:

(1) If $\nabla_{\dot x(0)}V(0) = 0$, then Γ is a parallel transport of $V(0)$ along the geodesic x on M (horizontal geodesics).
(2) If $\dot x(0) = 0$, then Γ is a great circle on the fibre $\pi^{-1}(x(0))$ and $x(t) = x(0)$ (vertical geodesics, in this case one interprets the system (3.8) via $\nabla_{\dot x} = D_t$).
(3) All the rest, i.e. solutions of (3.8) with initial conditions $\dot x(0) \neq 0$ and $\nabla_{\dot x(0)}V(0) \neq 0$ (oblique geodesics).

We state the following lemma for the sake of clarity.

Lemma 3.3. *Fix $(x, v) \in SM$ and $w \in S_xM$, $w \perp v$. Then $\phi_t(x, v)$ and $\phi^{\mathrm{h}}_{w,t}(x, v)$ are horizontal unit speed geodesics and $\phi^{\mathrm{v}}_{w,t}(x, v)$ is a vertical unit speed geodesic with respect to t.*

Proof. The fact that $\phi_t(x, v)$ and $\phi^{\mathrm{h}}_{w,t}(x, v)$ are horizontal geodesics and $\phi^{\mathrm{v}}_{w,t}(x, v)$ is a vertical geodesic follows immediately from their definitions and the above discussion based on the system of differential equations (3.8). The fact that $\phi_t(x, v)$, $\phi^{\mathrm{h}}_{w,t}(x, v)$ and $\phi^{\mathrm{v}}_{w,t}(x, v)$ are unit speed follows from the equations (3.6) and (3.7) and the definition of the Sasaki metric. □

Lemma 3.3 allows us to derive the following formulas which are used in the proof of lemma 4.7.

Corollary 3.4. *Let $(x, v) \in SM$. Assume that $Y \in T_{x,v}(SM)$ has the decomposition*

$$Y = aX(x, v) + H + V, \quad H \in \mathcal{H}(x, v), V \in \mathcal{V}(x, v), a \in \mathbb{R}.$$

Then

$$\begin{aligned}(D\phi_t)_{x,v}(aX(x, v)) &= aX(\phi_t(x, v)),\\ (D\phi_t)_{x,v}(H) &= |H|_{g_{\mathrm{s}}}\left[(J^{\mathrm{h}}_{w_{\mathrm{h}}}(t))^{\mathrm{h}} + (D_tJ^{\mathrm{h}}_{w_{\mathrm{h}}}(t))^{\mathrm{v}}\right],\\ (D\phi_t)_{x,v}(V) &= |V|_{g_{\mathrm{s}}}\left[(J^{\mathrm{v}}_{w_{\mathrm{v}}}(t))^{\mathrm{h}} + (D_tJ^{\mathrm{v}}_{w_{\mathrm{v}}}(t))^{\mathrm{v}}\right],\end{aligned}$$

where $D\phi_t$ is the differential of ϕ_t, $w_{\mathrm{h}} = \mathrm{d}\pi(H)/|\mathrm{d}\pi(H)|_g$ and $w_{\mathrm{v}} = K_\nabla(V)/|K_\nabla(V)|_g$. Moreover, $(D\phi_t)_{x,v}(X(x, v))$ is orthogonal to $(D\phi_t)_{x,v}(H)$ and $(D\phi_t)_{x,v}(V)$.

Proof. Lemma 3.3 gives that $\phi_s(x, v)$, $\phi^{\mathrm{h}}_{w_{\mathrm{h}},s}(x, v)$ and $\phi^{\mathrm{v}}_{w_{\mathrm{v}},s}(x, v)$ are unit speed geodesics on *SM*. If $\Gamma(s) = \phi_s(x, v)$, then $\Gamma(s)$ is a unit speed geodesic on *SM*, $\dot\Gamma(0) = X(x, v)$, and

$$(D\phi_t)_{x,v}(X(x, v)) = D\phi_t(\dot\Gamma(0)) = (\phi_t \circ \Gamma)'(0) = X(\phi_t(x, v)).$$

Moreover, using the unit speed geodesic $\Gamma(s) = \phi^{\mathrm{h}}_{w_{\mathrm{h}},s}(x, v)$ on *SM*, and using the formulas after lemma 3.2, gives

$$\begin{aligned}(D\phi_t)_{x,v}(H) &= D\phi_t(|H|_{g_{\mathrm{s}}}\dot\Gamma(0)) = |H|_{g_{\mathrm{s}}}(\phi_t \circ \Gamma)'(0)\\ &= |H|_{g_{\mathrm{s}}}\left[(J^{\mathrm{h}}_{w_{\mathrm{h}}}(t))^{\mathrm{h}} + (D_tJ^{\mathrm{h}}_{w_{\mathrm{h}}}(t))^{\mathrm{v}}\right]\end{aligned}$$

which is orthogonal to $X(\phi_t(x, v))$. Finally, the unit speed geodesic $\Gamma(s) = \phi^{\mathrm{v}}_{w_{\mathrm{v}},s}(x, v)$ on *SM* gives

$$\begin{aligned}(D\phi_t)_{x,v}(V) &= D\phi_t(|V|_{g_{\mathrm{s}}}\dot\Gamma(0)) = |V|_{g_{\mathrm{s}}}(\phi_t \circ \Gamma)'(0)\\ &= |V|_{g_{\mathrm{s}}}\left[(J^{\mathrm{v}}_{w_{\mathrm{v}}}(t))^{\mathrm{h}} + (D_tJ^{\mathrm{v}}_{w_{\mathrm{v}}}(t))^{\mathrm{v}}\right]\end{aligned}$$

which is also orthogonal to $X(\phi_t(x, v))$. □

3.2.3. Completeness of the unit tangent bundle. We will need the fact that SM is complete when M is complete. This need arises from theory of Sobolev spaces on manifolds (see section 5). We could not find a reference so a proof is included.

Lemma 3.5. *Let M be a complete Riemannian manifold with or without boundary. Then SM is complete.*

Proof. Let $(y^{(j)})$ be a Cauchy sequence in (SM, d_{g_s}). We show that it converges in the topology induced by g_s. The definition of the Sasaki metric implies that

$$L_{g_s}(\Gamma) \geqslant \int_0^\tau \left| d\pi_{\Gamma(t)}(\dot{\Gamma}(t)) \right|_g dt = L_g(\pi \circ \Gamma) \geqslant d_g(\pi(\Gamma(0)), \pi(\Gamma(\tau)))$$

where $\Gamma : [0, \tau] \to SM$ is any piecewise C^1-smooth curve. Hence

$$d_{g_s}(a, b) \geqslant d_g(\pi(a), \pi(b)) \tag{3.9}$$

for all $a, b \in SM$. The above inequality implies that $(\pi(y^{(j)}))$ is a Cauchy sequence in (M, g) and converges, say to $p \in M$, by completeness of M.

Consider a coordinate neighborhood U of p in M, so that $\pi^{-1}(U)$ is diffeomorphic to $U \times S^{n-1}$. Choose an open set V and a compact set K so that $p \in V \subset K \subset U$. Now $\pi^{-1}(K)$ is homeomorphic to $K \times S^{n-1}$ which is compact as a product of two compact sets. Since $\pi(y^{(j)}) \to p$, there exists N such that $\pi(y^{(j)}) \in V$ for all $j \geqslant N$, and this implies $y^{(j)} \in \pi^{-1}(K)$ for all $j \geqslant N$. Hence $(y^{(j)})$ has a limit in $(\pi^{-1}(K), d_{g_s}|_{\pi^{-1}(K)})$ since it is a Cauchy sequence, and thus $(y^{(j)})$ converges also in (SM, d_{g_s}). □

3.3. Symmetric covariant tensor fields

We denote by $S^m(M)$ the set of C^1-smooth symmetric covariant m-tensor fields and by $S^m_x(M)$ the symmetric covariant m-tensors at point x. Following [DS10] (where more details are also given), we define the map $\lambda_x\colon S^m_x(M) \to C^\infty(S_xM)$,

$$\lambda_x(f)(v) = f_x(v, \ldots, v)$$

which is given in local coordinates by

$$\lambda_x(f_{i_1 \ldots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m})(v) = f_{i_1 \ldots i_m}(x) v^{i_1} \ldots v^{i_m}.$$

The map λ smoothly depends on x and hence we get an embedding $\lambda\colon S^m(M) \to C^1(SM)$. The map λ identifies symmetric *trace-free* covariant m-tensor fields with spherical harmonics (with respect to v) of degree m on SM. More precisely, if $S^m_x(M)$ and $C^\infty(S_xM)$ are endowed with their usual L^2-inner products, then λ_x is an isomorphism, and even an isometry up to a factor, from the set of trace-free symmetric m-tensors at x onto the set of spherical harmonics (with respect to v) of degree m on S_xM (see [DS10, lemma 2.4 and subsequent remarks]). We will use this identification and do not always write λ explicitly.

The symmetrization of a tensor is defined by

$$\sigma(\omega_1 \otimes \cdots \otimes \omega_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} \omega_{\pi(1)} \otimes \cdots \otimes \omega_{\pi(m)},$$

where Π_m is the permutation group of $\{1,\dots,m\}$. From the above expression we see that if a covariant m-tensor field f is in $E^1_\eta(M)$ or $P^1_\eta(M)$ for some $\eta>0$, then so is σf too. Furthermore, for $f\in S^m(M)$ one has

$$\lambda(\sigma\nabla f)=X\lambda(f). \tag{3.10}$$

It follows from the last identity and the fundamental theorem of calculus that if $f\in P^1_\eta(M)$ for some $\eta>0$, then $I_m(\sigma\nabla f)=0$. This shows that I_m always has a nontrivial kernel for $m\geqslant 1$, as described in the introduction.

The next lemma states how the decay properties of a tensor field carry over to functions on SM.

Lemma 3.6. *Suppose $f\in S^m(M)$ and $\eta>0$.*

(a) If $f\in E^1_\eta(M)$, then

$$\sup_{v\in S_xM}\left|Xf(x,v)\right|_g\in E_\eta(M),\quad \sup_{v\in S_xM}\left|\overset{\mathtt{h}}{\nabla}f(x,v)\right|_g\in E_\eta(M)\quad\text{and}\quad \sup_{v\in S_xM}\left|\overset{\mathtt{v}}{\nabla}f(x,v)\right|_g\in E_\eta(M).$$

(b) If $f\in P^1_\eta(M)$, then

$$\sup_{v\in S_xM}\left|Xf(x,v)\right|_g\in P_{\eta+1}(M),\quad \sup_{v\in S_xM}\left|\overset{\mathtt{h}}{\nabla}f(x,v)\right|_g\in P_{\eta+1}(M)\quad\text{and}\quad \sup_{v\in S_xM}\left|\overset{\mathtt{v}}{\nabla}f(x,v)\right|_g\in P_{\eta}(M).$$

Proof. (a) The result for Xf follows from (3.10). To prove the other statements we take $x\in M$ and use local normal coordinates $(x^1,\dots,x^n)$ centered at x and the associated coordinates $(v^1,\dots,v^n)$ for T_xM. In these coordinates $f(x)=f_{i_1\dots i_m}(x)\,\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_m}$ and $\nabla f(x)=\partial_{x_j}f_{i_1\dots i_m}(x)\,\mathrm{d}x^j\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_m}$. We see that

$$\left|f(x)\right|_g=\left(\sum_{i_1,\dots,i_m}\left|f_{i_1\dots i_m}(x)\right|^2\right)^{1/2}\quad\text{and}\quad\left|\nabla f(x)\right|_g=\left(\sum_{j,i_1,\dots,i_m}\left|\partial_{x_j}f_{i_1\dots i_m}(x)\right|^2\right)^{1/2}.$$

For Xf, $\overset{\mathtt{h}}{\nabla}f$ and $\overset{\mathtt{v}}{\nabla}f$ at x we have coordinate representations (see [PSU15, appendix A])

$$\begin{aligned}Xf(x,v)&=v^j\partial_{x_j}f,\\ \overset{\mathtt{h}}{\nabla}f(x,v)&=\left(\partial^{x_j}f-(v^k\partial_{x_k}f)v^j\right)\partial_{x_j},\\ \overset{\mathtt{v}}{\nabla}f(x,v)&=\partial^{v_j}f\partial_{x_j}.\end{aligned}$$

We get that

$$Xf(x,v)X(x,v)+\overset{\mathtt{h}}{\nabla}f(x,v)=\partial^{x_j}f\partial_{x_j}=\partial^{x_j}f_{i_1\dots i_m}(x)v^{i_1}\dots v^{i_m}\partial_{x_j}$$

and, using the orthogonality of $Xf(x,v)X(x,v)$ and $\overset{\mathtt{h}}{\nabla}f(x,v)$ and the Cauchy–Schwarz inequality,

$$\sup_{v\in S_xM}\left|\overset{\mathtt{h}}{\nabla}f(x,v)\right|_g\leqslant\left(\sum_{j,i_1,\dots,i_m}\left|\partial_{x_j}f_{i_1\dots i_m}(x)\right|^2\right)^{1/2}=\left|\nabla f(x)\right|_g.$$

This implies that $\sup_{v\in S_xM}\left|\overset{\mathrm{h}}{\nabla} f(x,v)\right|_g \in E_\eta(M)$.

For $\overset{\mathrm{v}}{\nabla} f$, the identity $\partial_{v_j} v^k = \delta_j^k - v_j v^k$ (see [PSU15]) implies that

$$\overset{\mathrm{v}}{\nabla} f(x,v) = \sum_{j=1}^{n}(f_{ji_2\ldots i_m} v^{i_2}\ldots v^{i_m} - f(x,v)v_j)\partial_{x_j} + \ldots + \sum_{j=1}^{n}(f_{i_1\ldots i_{m-1}j} v^{i_1}\ldots v^{i_{m-1}} - f(x,v)v_j)\partial_{x_j}$$
$$= m\sum_{j=1}^{n}(f_{ji_2\ldots i_m} v^{i_2}\ldots v^{i_m} - f(x,v)v_j)\partial_{x_j}$$

Thus orthogonality and expanding the squares gives

$$\left|\overset{\mathrm{v}}{\nabla} f(x,v)\right|_g^2 = m^2\sum_{j=1}^{n}|f_{ji_2\ldots i_m}(x)v^{i_2}\ldots v^{i_m}|^2 \leqslant m^2\sum_{i_1,\ldots,i_m}|f_{i_1\ldots i_m}(x)^2 = m|^2|f(x)|_g^2$$

which in turn implies that $\sup_{v\in S_xM}\left|\overset{\mathrm{v}}{\nabla} f(x,v)\right|_g \in E_\eta(M)$. The proof for (b) is the same. □

4. Growth estimates

Throughout this section we assume that f is a symmetric covariant m-tensor field in $P_\eta(M)$ for some $\eta > 1$. The main results in this section are lemmas 4.3 and 4.7. They state that if f is such a tensor field, possibly with some additional decay at infinity, then the corresponding solution u^f of the transport equation will have decay at infinity.

We begin by observing that the geodesic x-ray transform is well defined for such f.

Lemma 4.1. *Let $f \in P_\eta(M)$ for some $\eta > 1$. For any $(x,v) \in SM$ one has*

$$\int_{-\infty}^{\infty}\left|f_{\gamma_{x,v}(t)}(\dot\gamma_{x,v}(t),\ldots,\dot\gamma_{x,v}(t))\right|\,\mathrm{d}t < \infty.$$

Proof. The assumption implies that $\left|f_{\gamma_{x,v}(t)}(\dot\gamma_{x,v}(t),\ldots,\dot\gamma_{x,v}(t))\right| \leqslant C(1+d(\gamma_{x,v}(t),o))^{-\eta}$. One can then change variables so that $t = 0$ corresponds to the point on the geodesic that is closest to o, split the integral over $t \geqslant 0$ and $t \leqslant 0$, and use the fact that the integrands are $\leqslant C(1+|t|)^{-\eta}$ by the estimate (3.3). □

If $f \in P_\eta(M)$ for some $\eta > 1$, we may now define

$$u^f(x,v) := \int_0^{\infty} f_{\gamma_{x,v}(t)}(\dot\gamma_{x,v}(t),\ldots,\dot\gamma_{x,v}(t))\,\mathrm{d}t.$$

It is straigthforward to see that

$$u^f(x,v) + (-1)^m u^f(x,-v) = I_m f(x,v)$$

for all $(x,v) \in SM$.

We have the usual reduction to the transport equation.

Lemma 4.2. *Let $f \in P_\eta(M)$ for some $\eta > 1$. Then $Xu^f = -f$.*

Proof. By definition

$$Xu^f(x,v) = \lim_{s\to 0} -\frac{1}{s}\int_0^s f_{\gamma_{x,v}(t)}(\dot\gamma_{x,v}(t),\ldots,\dot\gamma_{x,v}(t))\,\mathrm{d}t = -f_x(v,\ldots,v).$$ □

Next we derive decay estimates for u^f under the assumption that $I_m f = 0$.

Lemma 4.3. *Suppose that $I_m f = 0$.*

(a) If $f \in E_\eta(M)$ for $\eta > 0$, then

$$|u^f(x,v)| \leqslant C(1+d_g(x,o))\mathrm{e}^{-\eta d_g(x,o)}$$

for all $(x,v) \in SM$.

(b) If $f \in P_\eta(M)$ for $\eta > 1$, then

$$|u^f(x,v)| \leqslant \frac{C}{(1+d_g(x,o))^{\eta-1}}$$

for all $(x,v) \in SM$.

Proof. Since $I_m f = 0$, one has $|u^f(x,v)| = |u^f(x,-v)|$. By lemma 3.1, possibly after replacing (x,v) by $(x,-v)$, we may assume that $\gamma_{x,v}$ is escaping. We have

$$|u^f(x,v)| = \left|\int_0^\infty f(\gamma_{x,v}(t))(\dot\gamma_{x,v}(t),\ldots,\dot\gamma_{x,v}(t))\,\mathrm{d}t\right| \leqslant \int_0^\infty \left|f(\gamma_{x,v}(t))\right|_g \mathrm{d}t.$$

The rest of the proof is as in [Leh16, lemma 3.2]. □

Lemma 4.4. *Let $f \in P_\eta(M)$ for some $\eta > 1$. If $I_m f = 0$ and u^f is differentiable at $(x,v) \in SM$, then*

$$\overset{\mathtt{h}}{\nabla} u^f(x,-v) = (-1)^{m-1}\overset{\mathtt{h}}{\nabla} u^f(x,v) \quad \text{and} \quad \overset{\mathtt{v}}{\nabla} u^f(x,-v) = (-1)^m \overset{\mathtt{v}}{\nabla} u^f(x,v).$$

Proof. From $I_m f = 0$ it follows that

$$u^f(x,v) + (-1)^m u^f(x,-v) = 0.$$

Fix $w \in S_x M$, $w \perp v$. We note that

$$u^f(\phi^{\mathtt{h}}_{w,s}(x,-v)) + (-1)^m u^f(\phi^{\mathtt{h}}_{-w,-s}(x,v)) = 0$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}s}u^f(\phi^{\mathtt{h}}_{w,s}(x,-v))\Big|_{s=0} = -(-1)^m\frac{\mathrm{d}}{\mathrm{d}s}(u^f(\phi^{\mathtt{h}}_{-w,-s}(x,v)))\Big|_{s=0} = (-1)^m\frac{\mathrm{d}}{\mathrm{d}s}(u^f(\phi^{\mathtt{h}}_{-w,s}(x,v)))\Big|_{s=0}.$$

By lemma 3.2

$$\langle\overset{\mathtt{h}}{\nabla} u^f(x,-v),w\rangle = (-1)^m\langle\overset{\mathtt{h}}{\nabla} u^f(x,v),-w\rangle = -(-1)^m\langle\overset{\mathtt{h}}{\nabla} u^f(x,v),w\rangle.$$

For $\overset{\mathrm{v}}{\nabla} u^f$ we use that

$$u^f(\phi^{\mathrm{v}}_{w,s}(x,-v)) + (-1)^m u^f(\phi^{\mathrm{v}}_{-w,s}(x,v)) = 0$$

and by lemma 3.2 we get that

$$\langle \overset{\mathrm{v}}{\nabla} u^f(x,-v), w\rangle = (-1)^{m-1}\langle \overset{\mathrm{v}}{\nabla} u^f(x,v), -w\rangle = (-1)^m\langle \overset{\mathrm{v}}{\nabla} u^f(x,v), w\rangle.$$

□

We move on to prove growth estimates for Jacobi fields. These estimates will be used to derive estimates for $\overset{\mathrm{h}}{\nabla} u^f$ and $\overset{\mathrm{v}}{\nabla} u^f$.

Lemma 4.5. *Suppose $J(t)$ is a normal Jacobi field along a geodesic γ.*

(a) If all sectional curvatures along $\gamma([0,\tau])$ are $\geqslant -K_0$ for some constant $K_0 > 0$, and if $J(0) = 0$ or $D_tJ(0) = 0$, then

$$\left|J(t)\right|_g \leqslant \left|J(0)\right|_g \cosh(\sqrt{K_0}t) + \left|D_tJ(0)\right|_g \frac{\sinh(\sqrt{K_0}t)}{\sqrt{K_0}}$$

for $t \in [0,\tau]$.

(b) If $t_0 \in (0,\tau)$, then

$$\left|D_tJ(t)\right|_g + \left|\frac{J(t)}{t} - D_tJ(t)\right|_g \leqslant \left[\left|D_tJ(t_0)\right|_g + \left|\frac{J(t_0)}{t_0} - D_tJ(t_0)\right|_g\right] \mathrm{e}^{2\int_{t_0}^t s\mathcal{K}(\gamma(s))\,\mathrm{d}s}$$

for $t \in [t_0,\tau]$, where $\mathcal{K}$ is as defined in theorem 1.2.

Proof. (a) follows from the Rauch comparison theorem [Jos08, theorem 4.5.2]. For (b), we follow the argument in [Leh16]. Consider an orthonormal frame $\{\dot\gamma(t), E_1(t), \ldots, E_{n-1}(t)\}$ obtained by parallel transporting an orthonormal basis of $T_{\gamma(0)}M$ along γ. Write $J(t) = u^j(t)E_j(t)$, so that the Jacobi equation becomes

$$\ddot u(t) + R(t)u(t) = 0 \tag{4.1}$$

where $u(t) = (u^1(t), \ldots, u^{n-1}(t))$ and $R_{jk} = R(E_j, \dot\gamma, \dot\gamma, E_k)$. We wish to estimate $v(t) = \frac{u(t)}{t}$, and we do this by writing $v(t) = A(t) + \frac{B(t)}{t}$ where

$$A(t) = \dot u(t), \qquad B(t) = u(t) - t\dot u(t).$$

By using the equation (4.1), we see that

$$A(t) - A(t_0) = -\int_{t_0}^t sR(s)v(s)\,\mathrm{d}s,$$

$$B(t) - B(t_0) = \int_{t_0}^t s^2R(s)v(s)\,\mathrm{d}s.$$

Write $g(t) = |A(t)| + \left|\frac{B(t)}{t}\right|$. If $t \geqslant t_0$ one has

$$g(t) = \left|A(t_0) - \int_{t_0}^t sR(s)v(s)\,\mathrm{d}s\right| + \frac{1}{t}\left|B(t_0) + \int_{t_0}^t s^2R(s)v(s)\,\mathrm{d}s\right| \leqslant g(t_0) + 2\int_{t_0}^t s\|R(s)\|g(s)\,\mathrm{d}s.$$

The Gronwall inequality implies that

$$g(t) \leqslant g(t_0) \mathrm{e}^{2\int_{t_0}^{t} s\left\|R(s)\right\| \, \mathrm{d}s}.$$

The result follows from this, since $\left\|R(s)\right\| = \sup_{|\xi|=1} R(s)\xi \cdot \xi = \sup_{\dot{\gamma}(s)\in\Pi} K(\Pi) \leqslant \mathcal{K}(\gamma(s))$. □

Corollary 4.6. *Suppose that* (M, g) *is a Cartan–Hadamard manifold. Let* γ *be a geodesic and J a normal Jacobi field along it, satisfying either* $J(0) = 0$ *and* $|D_t J(0)| \leqslant 1$ *or* $|J(0)| \leqslant 1$ *and* $D_t J(0) = 0$.

(a) If $-K_0 \leqslant K \leqslant 0$ *and* $K_0 > 0$, *then*

$$\left|J(t)\right|_g \leqslant C\mathrm{e}^{\sqrt{K_0}t} \quad \text{and} \quad \left|D_t J(t)\right|_g \leqslant C\mathrm{e}^{\sqrt{K_0}t}$$

for $t \geqslant 0$ *where the constants do not depend on the geodesic* γ.

(b) If $\mathcal{K} \in P_\kappa(M)$ *for some* $\kappa > 2$, *then*

$$\left|J(t)\right|_g \leqslant C(t+1) \quad \text{and} \quad \left|D_t J(t)\right|_g \leqslant C$$

for $t \geqslant 0$. *If in addition* $\gamma \in \mathcal{E}_o$, *then the constants do not depend on the geodesic* γ.

Proof.

(a) The estimate for $\left|J(t)\right|_g$ follows directly from lemma 4.5. Using the same notations as in the proof of that lemma we have $\left|D_t J(t)\right|_g = |\dot{u}(t)|$ and by integrating (4.1) from 0 to t we get

$$\begin{aligned} |\dot{u}(t)| &\leqslant |\dot{u}(0)| + \int_0^t \left\|R(s)\right\| |u(s)| \, \mathrm{d}s \\ &\leqslant |D_t J(0)| + \int_0^t K_0 |J(s)| \, \mathrm{d}s \\ &\leqslant C\mathrm{e}^{\sqrt{K_0}t}. \end{aligned}$$

(b) For a fixed geodesic, the estimates follow from lemma 4.5. If $\mathcal{K} \in P_\kappa(M)$ for $\kappa > 2$, then

$$A := \sup_{\gamma\in\mathcal{E}_o} \int_0^\infty s\mathcal{K}(\gamma(s)) \, \mathrm{d}s \leqslant C \sup_{\gamma\in\mathcal{E}_o} \int_0^\infty s(1 + d_g(\gamma(s), o))^{-\kappa} \, \mathrm{d}s < \infty$$

by using (3.3). Let us fix $t_0 = 1$ and suppose that J is a Jacobi field along a geodesic in $\mathcal{E}_o$ whose initial values satisfy the given assumptions. From lemma 4.5 and (a) we then get that

$$\begin{aligned} \left|J(t)\right|_g &\leqslant \mathrm{e}^{2A} \left(2\left|D_t J(1)\right|_g + \left|J(1)\right|_g\right) t \\ &\leqslant \mathrm{e}^{2A} C\mathrm{e}^{\sqrt{K_0}} t \end{aligned}$$

for $t \geqslant 1$, where $K_0 = \sup_{x \in M} \mathcal{K}(x)$.

For $t \in [0,1]$ we can estimate $\left|J(t)\right|_g \leqslant Ce^{\sqrt{K_0}}$. By combining these two estimates we get

$$\left|J(t)\right|_g \leqslant C(1 + e^{2A} t) \leqslant Ce^{2A}(1+t)$$

for $t \geqslant 0$, and the constants do not depend on $\gamma \in \mathcal{E}_o$.

For $\left|D_t J(t)\right|_g$, lemma 4.5 gives the estimate

$$\left|D_t J(t)\right|_g \leqslant e^{2A}\left(2\left|D_t J(1)\right|_g + \left|J(1)\right|_g\right)$$

for $t \geqslant 1$, and for $t \in [0,1]$ we get a bound from (a). Neither of these bounds depends on $\gamma \in \mathcal{E}_o$. □

Lemma 4.7. *Suppose that $I_m f = 0$.*

(a) If $-K_0 \leqslant K \leqslant 0$, $K_0 > 0$ and $f \in E^1_\eta(M)$ for some $\eta > \sqrt{K_0}$, then u^f is differentiable along every geodesic on SM, $u^f \in W^{1,\infty}(SM)$ and

$$\left|\overset{\mathrm{h}}{\nabla} u^f(x,v)\right|_g \leqslant Ce^{-(\eta - \sqrt{K_0}) d_g(x,o)}$$

for a.e. $(x,v) \in SM$.

(b) If $\mathcal{K} \in P_\kappa(M)$ for some $\kappa > 2$ and $f \in P^1_\eta(M)$ for some $\eta > 1$, then u^f is differentiable along every geodesic on SM, $u^f \in W^{1,\infty}(SM)$ and

$$\left|\overset{\mathrm{h}}{\nabla} u^f(x,v)\right|_g \leqslant \frac{C}{(1 + d_g(x,o))^{\eta - 1}}$$

for a.e. $(x,v) \in SM$.

The same estimates hold for $\overset{\mathrm{v}}{\nabla} u^f$ with the same assumptions.

Proof of $u^f \in W^{1,\infty}_{\mathrm{loc}}(SM)$. We show that u^f is locally Lipschitz continuous. Fix $(x_0, v_0) \in SM$, and suppose that $\Gamma(s)$ is a unit speed geodesic on SM through (x_0, v_0). We have for any $r > 0$

$$\begin{aligned}\frac{u^f(\Gamma(r)) - u^f(\Gamma(0))}{r} &= \int_0^\infty \frac{f(\phi_t(\Gamma(r))) - f(\phi_t(\Gamma(0)))}{r}\,\mathrm{d}t \\ &= \int_0^\infty \frac{1}{r}\int_0^r \frac{\partial}{\partial s}\left[f(\phi_t(\Gamma(s)))\right]\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\infty \frac{1}{r}\int_0^r \langle \nabla_{SM} f(\phi_t(\Gamma(s))), D\phi_t(\Gamma(s))\dot{\Gamma}(s)\rangle_{g_s}\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{4.2}$$

We write

$$\dot{\Gamma}(s) = \langle \dot{\Gamma}(s), X(\Gamma(s))\rangle_{g_s} X(\Gamma(s)) + H_{\dot{\Gamma}}(s) + V_{\dot{\Gamma}}(s)$$

where $H_{\dot{\Gamma}}(s) \in \mathcal{H}(\Gamma(s))$ and $V_{\dot{\Gamma}}(s) \in \mathcal{V}(\Gamma(s))$. When we apply corollary 3.4 to the right hand side of (4.2) (and omit the identifications), we find that

$$\frac{u^f(\Gamma(r)) - u^f(\Gamma(0))}{r} = \int_0^\infty \frac{1}{r} \int_0^r \Bigg[Xf(\phi_t(\Gamma(s)))\langle \dot{\Gamma}(s), X(\Gamma(s))\rangle_{g_s}$$

$$+ \langle \overset{\mathtt{h}}{\nabla} f(\phi_t(\Gamma(s))), \left|H_{\dot{\Gamma}}(s)\right|_{g_s} J^{\mathtt{h}}_{w_{\mathtt{h}}(s)}(t) + \left|V_{\dot{\Gamma}}(s)\right|_{g_s} J^{\mathtt{v}}_{w_{\mathtt{v}}(s)}(t)\rangle_g$$

$$+ \langle \overset{\mathtt{v}}{\nabla} f(\phi_t(\Gamma(s))), \left|H_{\dot{\Gamma}}(s)\right|_{g_s} D_t J^{\mathtt{h}}_{w_{\mathtt{h}}(s)}(t) + \left|V_{\dot{\Gamma}}(s)\right|_{g_s} D_t J^{\mathtt{v}}_{w_{\mathtt{v}}(s)}(t)\rangle_g \Bigg] \mathrm{d}s\, \mathrm{d}t \tag{4.3}$$

where $w_{\mathtt{h}}(s) = H_{\dot{\Gamma}}(s)/\left|H_{\dot{\Gamma}}(s)\right|_{g_s}$ and $w_{\mathtt{v}}(s) = V_{\dot{\Gamma}}(s)/\left|V_{\dot{\Gamma}}(s)\right|_{g_s}$. Here the Jacobi fields are along the geodesic $\gamma_{\Gamma(s)}(t) := \pi(\phi_t(\Gamma(s)))$. By definition their initial values fulfil the assumptions of corollary 4.6.

From this point on we will work under assumptions of (b). The proof under assumptions of (a) is similar but simpler. We fix a small $\varepsilon > 0$. We show that the integral (4.3) has a uniform upper bound for every $r \in (0, 1]$ and every geodesic Γ through a point in $B_{(x_0,v_0)}(\varepsilon) \subset SM$. For $(x, v) \in SM$ we denote by $\mathcal{G}(x, v)$ the set of unit speed geodesics on SM through (x, v), and define

$$J(x_0, v_0, \varepsilon) := \{\Gamma \in \mathcal{G}(x, v)\,;\, (x, v) \in B_{(x_0,v_0)}(\varepsilon)\}.$$

For all $\Gamma \in J(x_0, v_0, \varepsilon)$, $\Gamma(0) = (x, v)$, and $s \in (0, r]$ the estimate (3.9) gives that $d_g(x, x_0) \leqslant \varepsilon$ and

$$d_g(\gamma_{\Gamma(s)}(0), x) = d_g(\pi(\Gamma(s)), x) \leqslant d_{g_s}(\Gamma(s), (x, v)) \leqslant s.$$

The estimate (3.1) implies that

$$\begin{aligned} d_g(\pi(\phi_t(\Gamma(s))), o) &= d_g(\gamma_{\Gamma(s)}(t), o) \geqslant t - d_g(\gamma_{\Gamma(s)}(0), x_0) \\ &\geqslant t - \sup_{s\in(0,r]} d_g(\gamma_{\Gamma(s)}(0), o) \geqslant t - d_g(x, o) - r \\ &\geqslant t - d_g(x_0, o) - \varepsilon - r \end{aligned} \tag{4.4}$$

for all $t \geqslant t_0$ where $t_0 := d_g(x_0, o) + r + \varepsilon$. We can use a trivial estimate $d_g(\pi(\phi_t(\Gamma(s))), o) \geqslant 0$ on the interval $[0, t_0]$. Further, the estimate (4.4) gives

$$\mathcal{K}(\gamma_{\Gamma(s)}(t)) \leqslant \frac{C}{(1 + d_g(\gamma_{\Gamma(s)}(t), o))^\eta} \leqslant \frac{C}{(1 + t - d_g(x_0, o) - \varepsilon - r)^\eta} \tag{4.5}$$

for all $t \geqslant t_0$ where the constant C does not depend on $s \in (0, r]$ or the geodesic $\Gamma \in J(x_0, v_0, \varepsilon)$, and hence

$$\sup_{\substack{\Gamma\in J(x_0,v_0,\varepsilon),\\ s\in(0,r]}} \int_0^\infty t\mathcal{K}(\gamma_{\Gamma(s)}(t))\, \mathrm{d}t < \infty. \tag{4.6}$$

Using the proof of corollary 4.6 together with (4.6), we can find a constant C which does not depend on $s \in (0, r]$ so that one has

$$\left|J^{\mathtt{h}}_{w_{\mathtt{h}}(s)}(t)\right|_g \leqslant Ct, \quad \left|D_t J^{\mathtt{h}}_{w_{\mathtt{h}}(s)}(t)\right|_g \leqslant C$$

for all $t \geqslant 0$ and $\Gamma \in J(x_0, v_0, \varepsilon)$. Similar estimates hold also uniformly for $J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)$ and $D_t J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)$.

Recall that $|H_{\dot\Gamma}(s)|_{g_s}, |V_{\dot\Gamma}(s)|_{g_s} \leqslant |\dot\Gamma(s)|_{g_s} = 1$, and that $w_{\mathrm{h}}(s), w_{\mathrm{v}}(s)$ depend on Γ. By combining the above estimates for Jacobi fields with estimate (4.4) and lemma 3.6 we get for the integrand in (4.3) that

$$\begin{aligned}
&\big|Xf(\phi_t(\Gamma(s)))\langle\dot\Gamma(s), X(\Gamma(s))\rangle_{g_s} \\
&\quad + \langle \overset{\mathrm{h}}{\nabla} f(\phi_t(\Gamma(s))), |H_{\dot\Gamma}(s)|_{g_s} J^{\mathrm{h}}_{w_{\mathrm{h}}(s)}(t) + |V_{\dot\Gamma}(s)|_{g_s} J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)\rangle_g \\
&\quad + \langle \overset{\mathrm{v}}{\nabla} f(\phi_t(\Gamma(s))), |H_{\dot\Gamma}(s)|_{g_s} D_t J^{\mathrm{h}}_{w_{\mathrm{h}}(s)}(t) + |V_{\dot\Gamma}(s)|_{g_s} D_t J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)\rangle_g\big| \\
&\leqslant |Xf(\gamma_{\Gamma(s)}(t))|_g + |\overset{\mathrm{h}}{\nabla} f(\gamma_{\Gamma(s)}(t))|_g \big| |H_{\dot\Gamma}(s)|_{g_s} J^{\mathrm{h}}_{w_{\mathrm{h}}(s)}(t) + |V_{\dot\Gamma}(s)|_{g_s} J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)\big|_g \\
&\quad + |\overset{\mathrm{v}}{\nabla} f(\gamma_{\Gamma(s)}(t))|_g \big| |H_{\dot\Gamma}(s)|_{g_s} D_t J^{\mathrm{h}}_{w_{\mathrm{h}}(s)}(t) + |V_{\dot\Gamma}(s)|_{g_s} D_t J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)\big|_g \\
&\leqslant |Xf(\gamma_{\Gamma(s)}(t))|_g + |\overset{\mathrm{h}}{\nabla} f(\gamma_{\Gamma(s)}(t))|_g \left(|J^{\mathrm{h}}_{w_{\mathrm{h}}(s)}(t)|_g + |J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)|_g \right) \\
&\quad + |\overset{\mathrm{v}}{\nabla} f(\gamma_{\Gamma(s)}(t))|_g \left(|D_t J^{\mathrm{h}}_{w_{\mathrm{h}}(s)}(t)|_g + |D_t J^{\mathrm{v}}_{w_{\mathrm{v}}(s)}(t)|_g \right) \\
&\leqslant \frac{Ct}{(1+t-d_g(x_0,o)-\varepsilon-r)^{\eta+1}} + \frac{C}{(1+t-d_g(x_0,o)-\varepsilon-r)^{\eta}} \qquad (4.7)
\end{aligned}$$

for all $t \in [t_0, \infty)$, $s \in (0, r]$ and $\Gamma \in J(x_0, v_0, \varepsilon)$. On the interval $[0, t_0]$ we also get a uniform upper bound since f, its covariant derivative and sectional curvatures are all bounded.

We can conclude that integral on the right hand side of (4.3) converges absolutely with some uniform bound $C < \infty$ over $r \in (0, 1]$ and the set $J(x_0, v_0, \varepsilon)$. This shows that u^f is locally Lipschitz, i.e. $u^f \in W^{1,\infty}_{\mathrm{loc}}(SM)$ (see remark 1). Moreover, the uniform estimate together with the dominated convergence theorem guarantees that the limit $r \to 0$ of (4.2) exists for all geodesics Γ on SM. This finishes the first part of the proof. □

Proof of the gradient estimates. By Rademacher's theorem u^f is differentiable almost everywhere, and thus we can assume that u^f is differentiable at $(x, v) \in SM$. By lemmas 3.1 and 4.4 we can assume that (x, v) satisfies $\gamma = \gamma_{x,v} \in \mathcal{E}_o$. We may also assume that $\overset{\mathrm{h}}{\nabla} u^f(x, v) \neq 0$. Since $\langle \overset{\mathrm{h}}{\nabla} u^f(x, v), v\rangle_g = 0$, we can take $w = \overset{\mathrm{h}}{\nabla} u^f(x, v)/|\overset{\mathrm{h}}{\nabla} u^f(x, v)|_g$ in lemma 3.2 and get that

$$\begin{aligned}
|\overset{\mathrm{h}}{\nabla} u^f(x, v)|_g &= \frac{\mathrm{d}}{\mathrm{d}s} u^f(\phi^{\mathrm{h}}_{w,s}(x, v))\Big|_{s=0} \\
&= \int_0^\infty \langle \overset{\mathrm{h}}{\nabla} f(\phi_t(x, v)), J^{\mathrm{h}}(t)\rangle_g + \langle \overset{\mathrm{v}}{\nabla} f(\phi_t(x, v)), D_t J^{\mathrm{h}}(t)\rangle_g \,\mathrm{d}t \qquad (4.8)
\end{aligned}$$

where J^{h} is again a Jacobi field along γ fulfilling the assumptions of corollary 4.6. Under the conditions in part (a), the estimate (3.3) implies

$$|\overset{\mathrm{h}}{\nabla} u^f(x, v)|_g \leqslant C \int_0^\infty \mathrm{e}^{-\eta d_g(\gamma(t), o)} \mathrm{e}^{\sqrt{K_0}\, t} \,\mathrm{d}t \leqslant \int_0^\infty \mathrm{e}^{-\eta\sqrt{d_g(x,o)^2 + t^2}} \mathrm{e}^{\sqrt{K_0}\, t} \,\mathrm{d}t.$$

Writing $r = d_g(x,o)$ and splitting the integral over $[0, r)$ and $[r, \infty)$ gives

$$\big|\overset{\mathrm{h}}{\nabla} u^f(x,v)\big|_g \leqslant C\left[\int_0^r \mathrm{e}^{-\eta r}\mathrm{e}^{\sqrt{K_0}\,t}\,\mathrm{d}t + \int_r^\infty \mathrm{e}^{-\eta t}\mathrm{e}^{\sqrt{K_0}\,t}\,\mathrm{d}t\right] \leqslant C\mathrm{e}^{-(\eta-\sqrt{K_0})d_g(x,o)}.$$

The above estimate also shows that $\big|\overset{\mathrm{h}}{\nabla} u^f\big|_g$ is bounded. Similarly, under the conditions in part (b), lemma 3.6, corollary 4.6 and (3.3) imply

$$\begin{aligned}\big|\overset{\mathrm{h}}{\nabla} u^f(x,v)\big|_g &\leqslant C\int_0^\infty \frac{1+t}{(1+d_g(\gamma(t),o))^{\eta+1}}\,\mathrm{d}t + C\int_0^\infty \frac{C}{(1+d_g(\gamma(t),o))^{\eta}}\,\mathrm{d}t \\ &\leqslant C\left[\int_0^r \frac{1+t}{(1+r)^{\eta+1}}\,\mathrm{d}t + \int_r^\infty \frac{1+t}{(1+t)^{\eta+1}}\,\mathrm{d}t\right] \leqslant C(1+r)^{-(\eta-1)}\end{aligned}$$

where $r = d_g(x,o)$. The same arguments apply to $\overset{\mathrm{v}}{\nabla} u^f$. Hence $u^f \in W^{1,\infty}(SM)$ in the both cases, (a) and (b). □

Lemma 4.8.

(a) If $-K_0 \leqslant K \leqslant 0$ *and* $K_0 > 0$, *then*

$$\mathrm{Vol}\ S_o(r) \leqslant C\mathrm{e}^{(n-1)\sqrt{K_0}r}$$

for all $r \geqslant 0$.

(b) If $\mathcal{K} \in P_\kappa(M)$ *for* $\kappa > 2$, *then*

$$\mathrm{Vol}\ S_o(r) \leqslant Cr^{n-1}$$

for all $r \geqslant 0$.

Proof. We define the mapping $f\colon S_oM \to S_o(r)$,

$$f(v) = (\pi\circ\phi_r)(o,v) = \exp_o(rv).$$

We denote by $\mathrm{d}\Sigma$ the volume form on $S_o(r)$ and have that

$$\mathrm{Vol}\ S_o(r) = \int_{S_o(r)} \mathrm{d}\Sigma = \int_{S_oM} f^*(\mathrm{d}\Sigma) = \int_{S_oM} \mu\,\mathrm{d}S,$$

where $\mathrm{d}S$ denotes the volume form on S_oM (induced by Sasaki metric) and $\mu\colon S_oM \to \mathbb{R}$.

Let $v \in S_oM$ and $\{w_i\}_{i=1}^{n-1}$ be an orthonormal basis for T_vS_oM with respect to Sasaki metric. By the Gauss lemma $\{d_vf(w_i)\}_{i=1}^{n-1}$ is an orthonormal basis for $T_{f(v)}S_o(r)$ and

$$f^*(\mathrm{d}\Sigma)_v(w_1,\ldots,w_{n-1}) = \mathrm{d}\Sigma_{f(v)}(d_vf(w_1),\ldots,d_vf(w_{n-1})).$$

It holds that $d_vf(w_i) = J_i(r)$ where J_i is a Jacobi field along the geodesic $\gamma_{o,v}$ with initial values $J_i(0) = d_v\pi(w_i)$ and $D_tJ_i(0) = K_\nabla(w_i)$. We get that

$$|\mu(v)| \leqslant \prod_{i=1}^{n-1}\big|d_vf(w_i)\big|_g = \prod_{i=1}^{n-1}\big|J_i(r)\big|_g.$$

Since the tangent vectors w_i lie in $\mathcal{V}(o,v)$ we have $\big|J_i(0)\big|_g = 0$ and $\big|D_tJ_i(0)\big|_g = \big|w_i\big|_{g_s} = 1$, and the estimates for the volume of $S_o(r)$ then follow from corollary 4.6. □

5. Proof of the main theorems

In this section we will combine the facts above to prove theorems 1.1 and 1.2. We begin by introducing some useful notation related to operators on the sphere bundle and spherical harmonics. One can find more details in [DS10, GK80b] and [PSU15]. We prove the main theorems of this work in the end of this section.

The norm $\|\cdot\|$ in this section will always be the $L^2(SM)$-norm. We define the Sobolev space $H^1(SM)$ as the set of all $u \in L^2(SM)$ for which $\|u\|_{H^1(SM)} < \infty$, where

$$\|u\|_{H^1(SM)} = \left(\|u\|^2 + \|\nabla_{SM}u\|^2\right)^{1/2} = \left(\|u\|^2 + \|Xu\|^2 + \|\overset{\mathrm{h}}{\nabla} u\|^2 + \|\overset{\mathrm{v}}{\nabla} u\|^2\right)^{1/2}.$$

Let $C_c^\infty(SM)$ denote the smooth compactly supported functions on SM. It is well known that if N is complete Riemannian manifold, then $C_c^\infty(N)$ is dense in $H^1(N)$ (see [Eic88, satz 2.3]). By lemma 3.5 SM is complete when M is complete. Hence $C_c^\infty(SM)$ is dense in $H^1(SM)$.

For the following facts see [PSU15]. The vertical Laplacian $\Delta : C^\infty(SM) \to C^\infty(SM)$ is defined as the operator

$$\Delta := -\overset{\mathrm{v}}{\mathrm{div}}\,\overset{\mathrm{v}}{\nabla}.$$

Here $\overset{v}{\mathrm{div}}$ denotes the vertical divergence which is the adjoint of $-\overset{\mathrm{v}}{\nabla}$ (see [PSU15, appendix A]). The Laplacian Δ has eigenvalues $\lambda_k = k(k+n-2)$ for $k = 0, 1, 2, \ldots$, and its eigenfunctions are homogeneous polynomials in v. One has an orthogonal eigenspace decomposition

$$L^2(SM) = \bigoplus_{k \geqslant 0} H_k(SM),$$

where $H_k(SM) := \{f \in L^2(SM)\,;\, \Delta f = \lambda_k f\}$. We define $\Omega_k = H_k(SM) \cap H^1(SM)$. In particular, by lemma 5.1 below any $u \in H^1(SM)$ can be written as

$$u = \sum_{k=0}^{\infty} u_k, \quad u_k \in \Omega_k,$$

where the series converges in $L^2(SM)$.

One can split the geodesic vector field in two parts, $X = X_+ + X_-$, so that (by lemma 5.1) $X_+ : \Omega_k \to H_{k+1}(SM)$ and $X_- : \Omega_k \to H_{k-1}(SM)$. The next lemma gives an estimate for $X_\pm u$ in terms of Xu and $\overset{\mathrm{h}}{\nabla} u$.

Lemma 5.1. *Suppose $u \in H^1(SM)$. Then $X_\pm u \in L^2(SM)$ and*

$$\|X_+u\|^2 + \|X_-u\|^2 \leqslant \|Xu\|^2 + \|\overset{\mathrm{h}}{\nabla} u\|^2.$$

Moreover, for each $k \geqslant 0$ one has $u_k \in H^1(SM)$, and there is a sequence $(u_k^{(j)})_{j=1}^\infty \subset C_c^\infty(SM) \cap H_k(SM)$ with $u_k^{(j)} \to u_k$ in $H^1(SM)$ as $j \to \infty$.

Proof. Let $u \in C_c^\infty(SM)$. By [PSU15, lemma 4.4] one has the decomposition

$$\overset{\mathrm{h}}{\nabla} u = \overset{\mathrm{v}}{\nabla}\left[\sum_{l=1}^{\infty}\left(\frac{1}{l}X_+u_{l-1} - \frac{1}{l+n-2}X_-u_{l+1}\right)\right] + Z(u)$$

where $Z(u)$ is such that $\overset{\mathrm{v}}{\mathrm{div}}\, Z(u) = 0$. Hence

$$\|\overset{\mathrm{h}}{\nabla} u\|^2 = \sum_{l=1}^{\infty} \left(l(l+n-2) \left\| \frac{1}{l} X_+ u_{l-1} - \frac{1}{l+n-2} X_- u_{l+1} \right\|^2 \right) + \|Z(u)\|^2$$

$$= \sum_{l=1}^{\infty} \left(\frac{l+n-2}{l} \|X_+ u_{l-1}\|^2 - 2\langle X_+ u_{l-1}, X_- u_{l+1} \rangle + \frac{l}{l+n-2} \|X_- u_{l+1}\|^2 \right) + \|Z(u)\|^2.$$

We also have

$$\|Xu\|^2 = \|X_- u_1\|^2 + \sum_{l=1}^{\infty} \left(\|X_+ u_{l-1} + X_- u_{l+1}\|^2 \right)$$

$$= \|X_- u_1\|^2 + \sum_{l=1}^{\infty} \left(\|X_+ u_{l-1}\|^2 + 2\langle X_+ u_{l-1}, X_- u_{l+1} \rangle + \|X_- u_{l+1}\|^2 \right)$$

by the definition of X_+ and X_-. Adding up these estimates gives that

$$\|Xu\|^2 + \|\overset{\mathrm{h}}{\nabla} u\|^2 = \|Z(u)\|^2 + \|X_- u_1\|^2 + \sum_{l=1}^{\infty} \left(A(n,l) \|X_+ u_{l-1}\|^2 + B(n,l) \|X_- u_{l+1}\|^2 \right)$$

where $A(n,l) = 2 + \frac{n-2}{l}$ and $B(n,l) = 1 + \frac{l}{l+n-2}$. Since $A(n,l) \geqslant 1$ and $B(n,l) \geqslant 1$ for all $l = 1, 2, \ldots$ and $n \geqslant 2$, the estimate for $\|X_+ u\|^2 + \|X_- u\|^2$ follows when $u \in C_c^\infty(SM)$, and it extends to $H^1(SM)$ by density and completeness.

Moreover, if $u \in C_c^\infty(SM)$ and if $k \geqslant 0$, then the triangle inequality $\|Xu_k\| \leqslant \|X_+ u_k\| + \|X_- u_k\|$ and orthogonality imply that

$$\|u_k\| + \|Xu_k\| + \|\overset{\mathrm{v}}{\nabla} u_k\| \leqslant \|u\| + \|X_+ u\| + \|X_- u\| + \|\overset{\mathrm{v}}{\nabla} u\|.$$

We may also estimate $\overset{\mathrm{h}}{\nabla} u_k$ by [PSU15, proposition 3.4] and orthogonality to obtain

$$\|\overset{\mathrm{h}}{\nabla} u_k\|^2 \leqslant (2k+n-1)\|X_+ u_k\|^2 + (\sup_M K)\|\overset{\mathrm{v}}{\nabla} u_k\|^2 \leqslant C_k(\|X_+ u\|^2 + \|\overset{\mathrm{v}}{\nabla} u\|^2).$$

It follows from the first part of this lemma that

$$\|u_k\|_{H^1(SM)} \leqslant C_k \|u\|_{H^1(SM)}, \qquad u \in C_c^\infty(SM).$$

This extends to $u \in H^1(SM)$ by density and completeness. Finally, if $u \in H^1(SM)$ and the sequence $(u^{(j)}) \subset C_c^\infty(SM)$ satisfies $u^{(j)} \to u$ in $H^1(SM)$, then also $u_k^{(j)} \to u_k$ in $H^1(SM)$ by the above inequality. □

Corollary 5.2. *Suppose $u \in H^1(SM)$. Then*

$$\lim_{k\to\infty} \|X_+ u_k\|_{L^2(SM)} = 0.$$

Proof. By lemma 5.1 one has

$$\|X_+ u\|^2 = \sum_{k=0}^{\infty} \|X_+ u_k\|^2 < \infty$$

which implies the claim. □

Lemma 5.3. *Let $u \in H^1(SM)$ and $k \geqslant 1$. Then one has that*

$$\|X_- u_k\| \leqslant D_n(k)\|X_+ u_k\|$$

where

$$D_2(k) = \begin{cases} \sqrt{2}, & k = 1 \\ 1, & k \geqslant 2, \end{cases}$$

$$D_3(k) = \left[1 + \frac{1}{(k+1)^2(2k-1)}\right]^{1/2}$$

$$D_n(k) \leqslant 1 \quad \text{for } n \geqslant 4.$$

Proof. This result was shown for smooth compactly supported functions in [PSU15, lemma 5.1]. The result follows for $u \in H^1(SM)$ by an approximation argument using lemma 5.1. □

The estimates from section 4 allow us to prove the following result:

Lemma 5.4. *Suppose that f is a symmetric m-tensor field and either of the following holds:*

(a) $-K_0 \leqslant K \leqslant 0$, $K_0 > 0$ and $f \in E^1_\eta(M)$ for $\eta > \frac{(n+1)\sqrt{K_0}}{2}$
(b) $\mathcal{K} \in P_\kappa(M)$ for $\kappa > 2$ and $f \in P^1_\eta(M)$ for $\eta > \frac{n+2}{2}$.

Then $u^f \in H^1(SM)$.

Proof. We prove only (a), the proof for (b) is similar. By lemma 4.7 we have that $u^f \in W^{1,\infty}(SM)$. Lemma 4.3 gives that

$$|u^f(x,v)| \leqslant C(1 + d_g(x,o))\mathrm{e}^{-\eta d_g(x,o)}$$

on SM. By using the coarea formula with lemma 4.8 we get

$$\begin{aligned} \int_{SM} |u^f(x,v)|^2 \,\mathrm{d}V_{g_s} &\leqslant C\int_M (1 + d_g(x,o))^2 \mathrm{e}^{-2\eta d_g(x,o)} \,\mathrm{d}V_g \\ &= C\int_0^\infty (1+r)^2 \mathrm{e}^{-2\eta r}\left(\int_{S_o(r)} \mathrm{d}S\right)\mathrm{d}r \\ &\leqslant C\int_0^\infty (1+r)^2 \mathrm{e}^{-2\eta r}\mathrm{e}^{(n-1)\sqrt{K_0}r}\mathrm{d}r. \end{aligned}$$

The last integral above is finite and hence $u^f \in L^2(SM)$. Similar calculations using lemmas 4.2 and 4.7 show that $Xu^f, \overset{\mathrm{h}}{\nabla} u^f$ and $\overset{\mathrm{v}}{\nabla} u^f$ all have finite L^2-norms under the assumption $\eta > \frac{(n+1)\sqrt{K_0}}{2}$, and therefore the H^1-norm of u^f is finite. □

We are ready to prove our main theorems.

Proof of theorems 1.1 and 1.2. Suppose that the m-tensor field f and the sectional curvature K satisfy the assumptions of theorem 1.1 or 1.2. Recall that we identify f with a function on SM as described in section 3.3. Then $u = u^f$ is in $H^1(SM)$ by lemma 5.4, and lemma 4.2 states that $Xu = -f$ on SM. Note also that $f \in H^1(SM)$, which follows as in the proof of lemma 5.4.

Since f is of degree m it has a decomposition

$$f = \sum_{k=0}^{m} f_k, \quad f_k \in \Omega_k,$$

and u has a decomposition

$$u = \sum_{k=0}^{\infty} u_k, \quad u_k \in \Omega_k.$$

We first show that $u_k = 0$ for $k \geqslant m$. From $Xu = -f$ it follows that for $k \geqslant m$ we have

$$X_+ u_k + X_- u_{k+2} = 0.$$

This implies that

$$\|X_+ u_k\| \leqslant \|X_- u_{k+2}\|, \qquad k \geqslant m. \tag{5.1}$$

Fix $k \geqslant m$. We apply lemma 5.3 and the inequality (5.1) iteratively to get

$$\begin{aligned} \|X_- u_k\| &\leqslant D_n(k)\|X_+ u_k\| \\ &\leqslant D_n(k)\|X_- u_{k+2}\| \\ &\leqslant D_n(k)D_n(k+2)\|X_+ u_{k+2}\| \\ &\leqslant \left[\prod_{l=0}^{N} D_n(k+2l)\right] \|X_+ u_{k+2N}\|. \end{aligned}$$

By corollary 5.2

$$\lim_{l\to\infty} \|X_+ u_{k+2l}\| = 0.$$

Moreover, as stated in [PSU15, theorem 1.1], one has

$$\prod_{l=0}^{\infty} D_n(k+2l) < \infty.$$

Thus we obtain that

$$\|X_- u_k\| = \|X_+ u_k\| = 0.$$

This gives $Xu_k = 0$, which implies that $t \mapsto u_k(\phi_t(x, v))$ is a constant function on $\mathbb{R}$ for any $(x, v) \in SM$. Since u decays to zero along any geodesic we must have $u_k = 0$, and this holds for all $k \geqslant m$.

It remains to verify that the equation $Xu = -f$ on SM together with the fact $u = \sum_{k=0}^{m-1} u_k$ imply the conclusions of theorems 1.1 and 1.2. This is done as in [PSU13, end of section 2]. Suppose that m is odd (the case where m is even is similar). The function f is a homogeneous polynomial of order m in v and hence its Fourier decomposition has only odd terms, i.e.

$$f = f_m + f_{m-2} + \cdots + f_1.$$

It follows that the decomposition of u has only even terms,

$$u = u_{m-1} + u_{m-3} + \cdots + u_0.$$

By taking tensor products with the metric g and symmetrizing it is possible to raise the degree of a symmetric tensor: if $F \in S^m(M)$, then $\alpha F := \sigma(F \otimes g) \in S^{m+2}(M)$. Functions

$\lambda(\alpha F)$ and $\lambda(F)$ have the same restriction to SM, since $\lambda(g)$ has a constant value 1 on SM.

We define $h \in S^{m-1}(M)$ by

$$h := -\sum_{j=0}^{(m-1)/2} \alpha^j(U_{m-1-2j}),$$

where $U_{m-1-2j}(x)$ is the unique symmetric trace-free $(m-1-2j)$-tensor field which satisfies $\lambda_x(U_{m-1-2j}(x)) = u_{m-1-2j}(x,\,\cdot\,)$, see section 3.3.

Then $\lambda(h) = -u$ on SM. Equation (3.10) gives $\lambda(\sigma\nabla h) = X(\lambda h) = -Xu = \lambda(f)$ on SM. Since both f and $\sigma\nabla h$ are symmetric we get $f = \sigma\nabla h$. To show the decay condition for h, we assume the conditions of theorem 1.1 and observe that lemma 4.3 implies that for any fixed $\varepsilon > 0$,

$$|u(x,v)| \leqslant C(1+d_g(x,o))\mathrm{e}^{-\eta d_g(x,o)} \leqslant C_\varepsilon \mathrm{e}^{-(\eta-\varepsilon)d_g(x,o)}. \tag{5.2}$$

We next observe that $|\sigma F| \leqslant |F|$ for any tensor F (this can be seen by using an orthonormal basis $\{\varepsilon^{i_1} \otimes \ldots \otimes \varepsilon^{i_m}\}$ for m-tensors, Cauchy–Schwarz and the definitions), and $|F \otimes g| = n^{1/2}|F|$ (which also follows from the definitions). Thus $|\alpha F| \leqslant n^{1/2}|F|$. Consequently, using that the map λ_x in section 3.3 is an isometry up to a factor depending on n and m,

$$|h(x)|^2 \leqslant C_{n,m} \sum_{j=0}^{(m-1)/2} |U_{m-1-2j}(x)|^2 \leqslant C_{n,m} \sum_{j=0}^{(m-1)/2} \|u_{m-1-2j}(x,\,\cdot\,)\|^2_{L^2(S_xM)}.$$

The orthogonality of spherical harmonics and the estimate (5.2) imply that

$$|h(x)|^2 \leqslant C_{n,m} \int_{S_xM} |u(x,v)|^2 \, dS \leqslant C_{\varepsilon,n,m} \mathrm{e}^{-2(\eta-\varepsilon)d_g(x,o)}.$$

This shows that $h \in E_{\eta-\varepsilon}(M)$ as required. The proof in the case of theorem 1.2 follows similarly by replacing (5.2) with the estimate in lemma 4.3(b). □

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(B)

Geodesic ray transform with matrix weights for piecewise constant functions

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GEODESIC RAY TRANSFORM WITH MATRIX WEIGHTS FOR PIECEWISE CONSTANT FUNCTIONS

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Abstract. We show injectivity of the geodesic X-ray transform on piecewise constant functions when the transform is weighted by a continuous matrix weight. The manifold is assumed to be compact and nontrapping of any dimension, and in dimension three and higher we assume a foliation condition. We make no assumption regarding conjugate points or differentiability of the weight. This extends recent results for unweighted transforms.

1. Introduction

This article studies the weighted geodesic X-ray transform with injective matrix weights on nontrapping Riemannian manifolds with strictly convex boundary. This operator arises in many applications, and one of the basic questions is if the weighted line integrals over all maximal geodesics determine an unknown function. We show an injectivity result for a class of piecewise constant functions under the assumptions that the manifold admits a strictly convex function and the weight depends continuously on its coordinates on the unit sphere bundle. In two dimensions, the result follows for nontrapping manifolds with strictly convex boundary.

Let (M, g) be a compact nontrapping Riemannian manifold with strictly convex boundary. We say that the boundary ∂M is *strictly convex* if its second fundamental form is positive definite at any $x \in \partial M$. A smooth function $f\colon M \to \mathbb{R}$ is said to be *strictly convex* if its Hessian $\nabla^2_x f\colon T_xM \times T_xM \to \mathbb{R}$ is positive definite at any $x \in M$. We denote by SM the unit sphere bundle and by Γ the set of maximal unit speed geodesics. We say that M is *nontrapping* if every geodesic in Γ has finite length, and we make this assumption. We denote the unique unit speed geodesic through $(x, v) \in SM$ by $\gamma_{x,v}$, that is, $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$.

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The *geodesic X-ray transform with matrix weights* is defined as follows. Fix some integers $m, k \geq 1$ and denote the set of linear injections (monomorphisms) $\mathbb{C}^k \to \mathbb{C}^m$ by $\mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^m)$, which is a subset of the space $\mathrm{Lin}(\mathbb{C}^k, \mathbb{C}^m)$ of all linear maps. If $k = m$, we have $\mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^m) = GL(\mathbb{C}, k)$, and for $m < k$ we have $\mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^m) = \emptyset$. The geodesic X-ray transform with weight $W \in C(SM, \mathrm{Lin}(\mathbb{C}^k, \mathbb{C}^m))$ is defined so that it maps a function $f\colon M \to \mathbb{C}^k$ to $I_W f\colon \Gamma \to \mathbb{C}^m$ defined by

$$I_W f(\gamma) = \int_0^\tau W(\gamma(t), \dot{\gamma}(t)) f(\gamma(t)) \mathrm{d}t \tag{1.1}$$

for any maximal geodesic $\gamma\colon [0, \tau] \to M$ whenever the integral is defined.

Injectivity of I_W for smooth functions was established by Paternain, Salo, Uhlmann, and Zhou [PSUZ19] if $\dim(M) \geq 3$, (M, g) admits a smooth strictly convex function, and $W \in C^\infty(SM; GL(k, \mathbb{C}))$. The result in [PSUZ19] is based on the methods developed in the work of Uhlmann and Vasy [UV16]. In this paper we consider a special case of the matrix weighted X-ray transform for the piecewise constant vector-valued functions. We gain more flexibility on geometrical assumptions and the proof is considerably simpler, but at the expense of only having the result for a restricted class of functions. Injectivity was shown recently in the case of piecewise constant functions without weights by Ilmavirta, Lehtonen, and Salo [ILS18], and reconstruction was studied in [Leb19]. Our main theorem is the following.

Theorem 1.1. *Let (M, g) be a compact nontrapping Riemannian manifold with strictly convex smooth boundary and $W \in C(SM; \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^m))$. Suppose that either*
(a) $\dim(M) = 2$*, or*
(b) $\dim(M) \geq 3$ *and (M, g) admits a smooth strictly convex function.*
If $f\colon M \to \mathbb{C}^k$ is a piecewise constant function and $I_W f = 0$, then $f \equiv 0$.

There is also a local version of theorem 1.1; see theorem 2.6.

Remark 1.2. The result can be generalized by replacing $\mathbb{C}^k$ and $\mathbb{C}^m$ with two Banach spaces and letting W be an invertible linear map depending continuously on the coordinates on the sphere bundle SM.

Remark 1.3. The functions $f\colon M \to \mathbb{C}^k$ are vector-valued in the sense that they are sections of the trivial bundle $M \times \mathbb{C}^k$. We do not study geodesic X-ray tomography of vector fields or higher order tensor fields.

Theorem 1.1 generalizes results of [ILS18] for the matrix weighted X-ray transform similar to the one studied in [PSUZ19]. Our theorem holds if $\dim(M) \geq 2$, $W \in C(SM; GL(k, \mathbb{C}))$, and functions are piecewise constant in comparison to [PSUZ19] where it is assumed that $\dim(M) \geq 3$, $W \in C^\infty(SM; GL(k, \mathbb{C}))$ and functions are smooth. In the Euclidean space $\mathbb{R}^n$ with $n \geq 3$ injectivity is known for $C^{1,\alpha}$ weights [Ilm16] but there is an example of non-injectivity for $W \in C^\alpha$ by Goncharov and Novikov [GN17]. Boman constructed an example of a smooth nonvanishing weight on the plane for which the weighted X-ray transform for smooth functions is non-injective [Bom93]. Theorem 1.1 shows that there are no such counterexamples for piecewise constant functions. The known results — including the new ones obtained here — are summarized on Table 1.

We will prove Theorem 1.1 in Section 2. We remark that Theorem 1.1 is based on an generalization of [ILS18, Lemma 4.2] whereas the rest of the proof is almost identical to the one in [ILS18]. The method in [ILS18] relies on existence of a strictly convex foliation as in the works of Stefanov, Uhlmann, and Vasy [UV16, SUV17], but the method of proof is far simpler. For a further discussion on the foliation condition see [PSUZ19] and references therein. We say that M satisfies the *foliation condition* if M admits a strictly convex function. We define the precise meaning of a strictly convex foliation in Section 2.3.

The matrix weighted X-ray transform is related to recovering matrix valued connection from its parallel transport [FU01, Nov02a, PSU12, GPSU16, PSUZ19]. It has also applications in polarization tomography [Sha94, NS07, Hol13] and quantum state tomography [Ilm16].

One source of weights is pseudolinearization, a procedure where a nonlinear problem is reduced to a linear problem with weights depending on the unknown. For a more detailed description of the idea, first appearing in [SU98, SUV16], see e.g. [IM19, Section 8]. Pseudolinearization also leads to an iterative inversion algorithm [Ilm16, SU98, SUV16].

A boundary reconstruction of the normal derivatives of a function from the broken ray transform reduces to a certain weighted geodesic ray transform on the boundary [Ilm14]. Some weights can be realized as attenuation, but we make no such assumptions on W. The attenuated X-ray transform (see e.g. [ABK97, Nov02b, SU11, AMU18]) is a well-known special case of the matrix weighted X-ray transform and it is the mathematical basis for the medical imaging method SPECT (see e.g. the survey [Fin03]).

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Regularity	**Dimension**	$W = \mathrm{Id}$	$W \in C^\infty$	$W \in C$
PWC	$= 2$	Yes. [ILS18]	Yes!	Yes!
L^2 or C^∞	$= 2$	Unknown.	No. [Bom93]	No. [Bom93]
PWC	≥ 3	Yes. [UV16]	Yes. [PSUZ19]	Yes!
L^2 or C^∞	≥ 3	Yes. [UV16]	Yes. [PSUZ19]	No. [GN17]

Table 1. Is the X-ray transform injective on manifolds that admit a strictly convex function? The answers in various different cases are summarized below. Here "PWC" stands for piecewise constant. A "Yes!" with an exclamation mark is a new result proven here. In two dimensions injectivity is known on simple manifolds, but the foliation condition does not imply simplicity.

2. Proof

2.1. Definitions.

We follow the notation of [ILS18], and any details omitted here can be found there. We review the main concepts here in a somewhat informal manner.

The *standard m-dimensional simplex* is the convex hull of the standard base of $\mathbb{R}^{m+1}$. A *regular m-simplex on a manifold M* is a C^1-smoothly embedded standard m-dimensional simplex. The boundary of a regular m-simplex is a union of $m+1$ regular $(m-1)$-simplices.

We define the *depth* of a point x in a regular m-simplex as follows. We say that x has depth 0 if x belongs to the interior of the simplex. We say that x has depth 1 if x belongs to the interior of a boundary simplex of the simplex. Other depths are defined similarly up to depth m at the $m+1$ corner points of the original simplex.

If Δ_1 and Δ_2 are two regular m-simplices, we say that their boundaries align nicely if $x \in \Delta_1 \cap \Delta_2$ implies that x has the same depth in both, Δ_1 and Δ_2.

We denote $n = \dim(M)$. A *regular tiling of a manifold* is a collection of regular n-simplices which cover the manifold, whose interiors are disjoint, and whose boundaries align nicely. An example is given in Figure 1. A piecewise constant function is such that the values are constant in the interior of every simplex and zero on their boundaries. The geometry of corners of simplices is important for our argument, and we review the crucial definitions in more depth.

Definition 2.1 (Tangent cone)**.** Let Δ be a regular m-simplex in M with $0 \leq m \leq n = \dim(M)$, and let $x \in \Delta$. Let $\Gamma = \Gamma(x, \Delta)$ be the set of all C^1-curves starting at x and staying in Δ. The *tangent cone* of Δ at x is the set

$$C_x\Delta := \{\, \dot\gamma(0) \mid \gamma \in \Gamma \,\}. \tag{2.1}$$

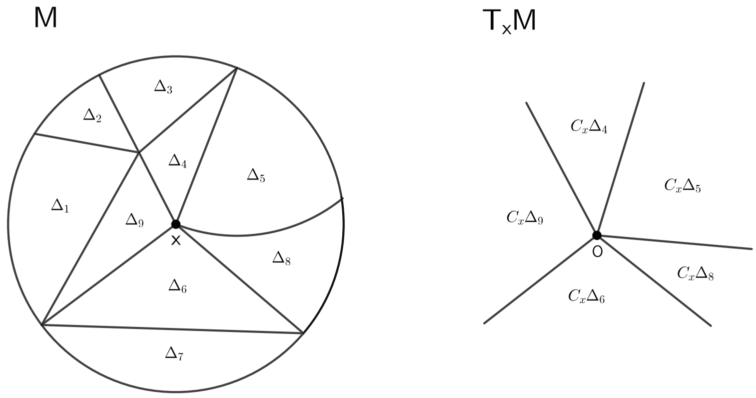

FIGURE 1. An example of a regular tiling and the tangent space at the point x. The simplices touching x have their corresponding tangent cones on T_xM. See Definition 2.1.

Definition 2.2 (Tangent function)**.** Let $f\colon M \to \mathbb{C}^k$ be a piecewise constant function and $x \in M$ with respect to a regular tiling. Let $\Delta_1, \dots, \Delta_N$ be the simplices of the regular tiling that contain x. Denote by $v_1, \dots, v_N \in \mathbb{C}^k$ the constant values of f in the interior of these simplices. The *tangent function* $T_xf\colon T_xM \to \mathbb{C}^k$ of f at x is defined so that for each $i \in \{1, \dots, N\}$ the function T_xf takes the constant value v_i in the interior of the tangent cone $C_x\Delta_i$. The tangent function takes the value zero in $T_xM \setminus \bigcup_{i=1}^N \mathrm{Int}(C_x\Delta_i)$.

We stress that the tangent function is not a derivative, as a piecewise constant function is typically not differentiable at the points of interest. Instead of linearizing the function, we linearize the geometry of the simplices and keep the constant values of the function.

2.2. **Lemmas.** In this subsection we recall a key lemma proved in [ILS18, Section 4] and use it to prove a new lemma.

Let M be a C^2-smooth Riemannian surface with C^2-boundary. Suppose the boundary ∂M is strictly convex at $x \in \partial M$. Let $\gamma_i, i = 1, 2$, be two unit speed C^1-curves in M starting nontangentially at x so that $|\langle \dot\gamma_1(0), \dot\gamma_2(0)\rangle| < 1$.

Let the radius $r > 0$ be small enough such that the geodesic ball $B(x, r) \subset M$ is split by the curves $\gamma_i, i = 1, 2$, into three parts. Let A be the middle one. Let $\sigma_i, i = 1, 2$, be the curves on T_xM with constant speed $\dot\gamma_i(0)$ respectively. Let S be the sector in T_xM laying between σ_1 and σ_2.

If $h > 0$ and if $v \in T_xM$ is an inward pointing unit vector, let the geodesic γ_v^h be constructed as follows: Take a unit vector w normal to v at x — which is unique up to sign — and let $w(h)$ be the parallel transport (with respect to the Levi–Civita connection) of w along the geodesic $\gamma_{x,v}$ by distance h. Let γ_v^h be the maximal geodesic in the direction of $w(h)$ at $\gamma_{x,v}(h)$. One could denote $\gamma_v^h = \gamma_{\gamma_{x,v}(h),w(h)}$ to be more precise, but we have chosen to keep the notation lighter.

We denote by σ_v^h the corresponding line $hv + w\mathbb{R}$ in T_xM. The correspondence is not by the exponential map $\exp_x$ as typically $\gamma_v^h \neq \exp_x(\sigma_v^h)$, but in the sense of lemma 2.3 below. We denote by $\nu \in \partial(SM)$ the inward pointing unit vector of ∂M at x.

We restate a lemma from [ILS18] for convenience.

Lemma 2.3 ([ILS18, Lemma 4.1])**.** *Let M be a C^2-smooth Riemannian surface with C^2-boundary, which is strictly convex at $x \in \partial M$. There exists an open neighborhood U of ν such that for every $v \in U$ we have*

$$\lim_{h\to 0} \frac{1}{h} \int_{\gamma_v^h \cap A} ds = \int_{\sigma_v^1 \cap S} ds. \tag{2.2}$$

We prove the global result of Theorem 1.1 by way of proving a local version near a boundary point. The relevant local version is given below in Lemma 2.5. A crucial step in its proof is Lemma 2.3, which allows conversion of the local problem on the manifold into a problem on the tangent space. However, Lemma 2.3 as stated is not sufficient in the weighted situation, but is used to prove the weighted analogue in Lemma 2.4 below. Lemma 2.4 can be seen as a generalization of [ILS18, Lemma 4.2] for the class of piecewise constant vector-valued functions with matrix weighted integrals.

Lemma 2.4. *Let M be a C^2-smooth Riemannian surface with C^2 boundary, which is strictly convex at $x \in \partial M$. Let $\tilde{M}$ be such an extension of M that $x \in \mathrm{Int}(\tilde{M})$. Let $\Delta \subset M$ be a regular 2-simplex so that $C_x\Delta \cap T_x\partial M = \{0\}$. Let $W \in C(\tilde{M}; \mathrm{Lin}(\mathbb{C}^k, \mathbb{C}^m))$ and $f\colon \tilde{M} \to \mathbb{C}^k$ be a piecewise constant function supported in Δ. Then there exists an open neighborhood U of ν such that for every $v \in U$ we have*

$$\lim_{h\to 0} \frac{1}{h} \int_{\gamma_v^h} W(\gamma_v^h(s), \dot{\gamma}_v^h(s)) f(\gamma_v^h(s)) \mathrm{d}s = W(x, v^\perp) \int_{\sigma_v^1} T_x f(\sigma_v^1(s)) \mathrm{d}s. \tag{2.3}$$

Proof. By linearity we can assume that f is constant in Δ. A piecewise constant function is a linear combination of characteristic functions of interiors of simplices.

Fix $v \in U$ given by Lemma 2.3. Let $s_h \in \gamma_v^h$ be any maximizer of

$$s \mapsto \left\| W(\gamma_v^h(s), \dot{\gamma}_v^h(s)) - W(x, v^\perp) \right\|, \tag{2.4}$$

where — as throughout this proof — we use the operator norm of matrices. We have

$$\sup_{s\in\gamma_v^h} d((\gamma_v^h(s), \dot{\gamma}_v^h(s)), (x, v^\perp)) \to 0 \tag{2.5}$$

as $h \to 0$, and so $(\gamma_v^h(s_h), \dot{\gamma}_v^h(s_h)) \to (x, v^\perp)$ as $h \to 0$. We have $\lim_{h\to 0} \frac{l(\gamma_v^h \cap \Delta)}{h} = |\sigma_v^1|$, and in particular the fraction $\frac{l(\gamma_v^h \cap \Delta)}{h}$ is uniformly bounded for all small $h > 0$.

We are ready to compare the weighted integral on the left-hand side of (2.3) to the corresponding integral with the weight frozen to its limit value $W(x, v^\perp)$ (as $h \to 0$). Straightforward estimates give

$$\begin{aligned}&\left\| \frac{1}{h} \int_{\gamma_v^h} (W(\gamma_v^h(s), \dot{\gamma}_v^h(s)) - W(x, v^\perp) f ds \right\| \\ \le& \frac{l(\gamma_v^h \cap \Delta)}{|h|} \sup_{s\in\gamma_v^h} \|f(s)\| \sup_{s\in\gamma_v^h} \left\| W(\gamma_v^h(s), \dot{\gamma}_v^h(s)) - W(x, v^\perp) \right\|.\end{aligned} \tag{2.6}$$

As f is bounded, the quotient $\frac{l(\gamma_v^h \cap \Delta)}{h}$ is bounded, and the matrix norm tends to zero as $h \to 0$, the left-hand side of (2.6) tends to zero as well.

We may thus conclude that the limit on the left-hand side of (2.3) is the same as

$$W(x, v^\perp) \lim_{h\to 0} \frac{1}{h} \int_{\gamma_v^h} f(\gamma_v^h(s)) \mathrm{d}s. \tag{2.7}$$

That is, the weight can be frozen to its limiting value. The function f is constant, so up to that constant the integral is just the length of the geodesic segment in Δ. Lemma 2.3 shows that the characteristic function of a simplex satisfies (2.3) in the absence of weight. This concludes the proof. □

Suppose Σ is a hypersurface containing the point $x \in \mathrm{Int}(M)$ and Σ is strictly convex in a neighbourhood of x. Let V be a small neighbourhood of x such that $V \setminus \Sigma$ consists of two open sets which are denoted by V_+ and V_-. We choose V_+ to be the one for which the boundary section $\partial V_+ \cap \Sigma$ is strictly convex. Next we state Lemma 2.5 that allows one to build a layer stripping argument that is used to prove Theorem 1.1.

Lemma 2.5. *Let M be a C^2-smooth Riemannian manifold, $W \in C(M; \mathrm{Mon}(\mathbb{C}^k, \mathbb{C}^m))$ and $f\colon M \to \mathbb{C}^k$ be a piecewise constant function. Fix $x \in \mathrm{Int}(M)$ and let Σ be an $(n-1)$-dimensional hypersurface through x. Suppose that V is a neighbourhood of x so that*

- *V intersects only simplices containing x,*
- *Σ is strictly convex in V,*
- *$f|_{V_-} = 0$, and*
- *$I_W f = 0$ over every maximal geodesic in V having endpoints on Σ.*

Then $f|_V = 0$.

Proof. The lemma follows from Lemma 2.4 using ideas developed in [ILS18, Lemma 5.1 and Lemma 6.2]. We summarize the idea briefly; details can be found in the cited paper.

Consider two dimensions first. Take a unit vector $v \in T_xM$ pointing towards V_+ and $h > 0$. Define the geodesic γ_v^h as above. The weighted integrals of f over these geodesics vanish by assumption. By Lemma 2.4 and injectivity of W everywhere on the sphere bundle, we find that T_xf integrates to zero over σ_v^1.

This argument reduces the X-ray tomography problem on M to the corresponding problem on T_xM. We have the freedom to vary the direction v, and any open set is sufficient. The Euclidean problem is unweighted and can be solved by explicit calculation; see [ILS18, Lemma 3.1]. The calculation is based on describing the direction v by a parameter and computing derivatives of high orders with respect to that parameter. That these derivatives determine the values of T_xf in the cones uniquely boils down to the invertibility of a Vandermonde matrix.

In higher dimensions one can proceed as follows. Take a unit tangent vector w tangential to Σ and an unit vector v pointing towards V_+. As above, we can define the geodesics $\gamma_{v,w}^h(t)$, where we now keep the dependence on w explicit. Near the point x the function $(h,t) \mapsto \gamma_{v,w}^h(t)$ defines a smooth two-dimensional submanifold $S_{v,w} \subset M$. Now the geodesics $\gamma_{v,w}^h(t)$ are geodesics on both M and $S_{v,w}$ although the submanifold is not totally geodesic in general.

Now for almost all choices of v and w this submanifold $S_{v,w}$ we are in the setting of our two-dimensional result. Issues can arise when boundaries of the simplices are tangent to $S_{v,w}$ at x, but this is rare. In such cases f has to vanish on $S_{v,w}$ and therefore on all simplices that meet this submanifold near x. For any simplex containing x there are such w and v (see [ILS18]), and therefore the claim holds. We point out that for different pairs (v,w) we get a different submanifold. □

2.3. **Proof of Theorem 1.1.** We are now ready to prove our main theorem. We begin with a local version.

Following [PSUZ19], we say that a subset $U \subset M$ has a *strictly convex foliation* if there is a strictly convex function $\phi\colon U \to \mathbb{R}$ so that the sets $\{x \in U; \phi(x) \geq c\}$ for all $c > \inf_U \phi$ are compact.

Theorem 2.6. *Let M be a C^2-smooth Riemannian manifold with strictly convex boundary and* $\dim(M) \geq 2$. *Suppose that a subset $U \subset M$ has a strictly convex foliation. Let $W \in C(M; GL(k,\mathbb{C}))$ and $f\colon M \to \mathbb{C}^k$ be a piecewise constant function. If $I_W f = 0$ for all geodesics in U, then $f|_U = 0$.*

Proof. The proof is very similar to that of [ILS18, Theorem 5.3 and Theorem 6.4], and we only give an outline.

The set U can be foliated by strictly convex hypersurfaces, and we are interested at the times when the foliation meets a new simplex. It suffices to prove local injectivity in the neighborhood of a strictly convex boundary point — a point on the leaf of a foliation — whenever a new simplex is met. If the point meets only one new simplex, then one can use a sequence of short geodesics that pass the simplex and argue as in Lemma 2.4 to see that f has to vanish in the simplex. If the point meets more simplices, we are in the setting of Lemma 2.5, and we can conclude that the function vanishes on each new simplex. □

Theorem 2.6 has also corollaries analogous to [ILS18, Corollaries 6.5–6.7] but we omit them here.

Proof of Theorem 1.1. Under the assumptions there is a global foliation: we may choose $U = M$ in Theorem 2.6. See [PSUZ19, Section 2] for details. The proof is complete. □

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Torus computed tomography

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TORUS COMPUTED TOMOGRAPHY

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Abstract. We present a new computed tomography (CT) method for inverting the Radon transform in 2D. The idea relies on the geometry of the flat torus, hence we call the new method Torus CT. We prove new inversion formulas for integrable functions, solve a minimization problem associated to Tikhonov regularization in Sobolev spaces and prove that the solution operator provides an admissible regularization strategy with a quantitative stability estimate. This regularization is a simple post-processing low-pass filter for the Fourier series of a phantom. We also study the adjoint and the normal operator of the X-ray transform on the flat torus. The X-ray transform is unitary on the flat torus. We have implemented the Torus CT method using Matlab and tested it with simulated data with promising results. The inversion method is meshless in the sense that it gives out a closed form function that can be evaluated at any point of interest.

1. Introduction

We present a new computed tomography (CT) method for X-ray tomography in 2D. The method reconstructs the Fourier series of a phantom via the projection of X-ray data into X-ray data on the flat torus which has a remarkably simple inverse X-ray transform. Therefore we call the new method *Torus CT*. We have developed new mathematical theory and computational implementations. The numerical implementation was used to demonstrate the potential of Torus CT method in various simulations and tests, including data simulation in torus geometry and traditional experimental projections. Torus CT provided an efficient basis for inverse solution and its efficacy is shown in this work.

The article is organized as follows. In section 1.1 we give an overview of computed tomography and regularization, in section 1.2 we discuss works related to X-ray tomography on torus, and in section 1.3 we state the main theoretical results in this paper. Section 2 includes mathematical preliminaries, proofs of theorems and numerical analysis for Torus CT method. Section 3 contains mathematical formulation of computational forward and inverse models. Section 4 presents numerical experiments and their analysis. Conclusions are given in section 5. We have included a short note on supplementary material in the end of the article.

1.1. **Overview of computed tomography and regularization methods.**

We give here an overview of X-ray tomography. Practical CT imaging was first introduced by Cormack and Hounsfield in 1970s based on the theoretical work of Cormark [3, 4] in 1960s. The mathematical theory itself was in fact earlier studied by Radon [26] in 1917. We give here only a narrow sample of topics and references in X-ray computed tomography. More references can be found in the cited works.

CT has many applications in medical imaging and engineering utilizing computerized axial tomography (CAT), positron-emission tomography (PET) and single-photon emission computed tomography (SPECT) [21]. Possible applications include imaging of patients in medicine and nondestructive testing in engineering. The most common inversion method for CT imaging is based on the filtered back-projection (FBP) algorithms [23, 14].

The FBP algorithms work well if there is sufficiently dense set of measurements, and otherwise regularization is often required. Another reason for regularization comes from the need of controlling errors in reconstructions cased by a measurement noise. See for example [22, 21].

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Usually a regularization method is applied for a discretized X-ray tomography model as in the examples listed next. The most common regularization methods include Tikhonov regularization and truncated singular value decomposition (TSVD) which promote smoothness of reconstructions [22]. Other common regularization approaches include total variation (TV) regularization which promotes sparsity of reconstructions [31, 8, 24, 7]. Another approach is to encode a priori information as a probability distribution and think the reconstruction problem as an Bayesian inverse problem for finding a posterior distribution [31, 17, 13, 8, 7].

The main difference of our proposed Tikhonov regularization approach, stated in theorem 2, compared to the usual regularization methods is that we do not discretize a phantom and regularization takes a form of a simple low-pass filter on the Fourier side. This also reflects the fact that Torus CT method is meshless (or meshfree) method. Theorem 3 states that the proposed regularization method is an admissible regularization strategy. Details are given in the subsequent sections.

1.2. **The X-ray transform on torus, the Radon transform and the geodesic X-ray transform.** In this paper we consider application of the X-ray transform on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ to the usual CT in the case when $n = 2$. In this section we give an account of theoretical works on the X-ray transforms on tori. As expected, the *d-plane Radon transform* of a function f on $\mathbb{T}^n$ encodes the integrals of f over all periodic d-planes. The X-ray transform corresponds to the case when $d = 1$ and is in fact the *geodesic X-ray transform* on $\mathbb{T}^n$ over closed geodesics. It is described in section 2.3 how the usual CT reconstruction on $\mathbb{R}^2$ can be reduced to a reconstruction on $\mathbb{T}^2$.

Injectivity, reconstruction and certain stability estimates of the d-plane Radon transform on $\mathbb{T}^n$ were proved for distributions by Ilmavirta in [10]. The first injectivity result for the geodesic X-ray transform on $\mathbb{T}^2$ was obtained by Strichartz in [32], and generalized to $\mathbb{T}^n$ by Abouelaz and Rouvière in [2] if the Fourier transform is $\ell^1(\mathbb{Z}^n)$. Abouelaz proved uniqueness under the same assumption for the d-plane Radon transform in [1]. A more general view and references on the Radon transform and the geodesic X-ray transform are given in [30, 9, 25, 12].

1.3. **Inversion formulas and Tikhonov regularization.** We state here our main theorems regarding the X-ray transform on $\mathbb{T}^2$. We write the X-ray transform on $\mathbb{T}^2$ as $\mathcal{I}$ and denote $\mathcal{I}f(x, v) = \mathcal{I}_v f(x)$. In our proofs, we subsequently apply the fundamental (but simple) property of the X-ray transform on $\mathbb{T}^2$, stated in the formula (9), that was found in [10]. The exact definitions are given in section 2.

Our first theorem gives new inversion formulas for the X-ray transform. We give two proofs of theorem 1 in section 2.2. The first one does not rely to the inversion formula of [10] whereas the second simpler proof does.

Theorem 1. *Suppose that $f \in L^1(\mathbb{T}^2)$. Let $k \in \mathbb{Z}^2$. If $k, v \neq 0$ and $v \perp k$, then*

$$\hat{f}(k) = \begin{cases} \int_0^1 \mathcal{I}_v f(0, y) \exp(-2\pi i k_2 y) dy, & k_2 \neq 0 \\ \int_0^1 \mathcal{I}_v f(x, 0) \exp(-2\pi i k_1 x) dx, & k_1 \neq 0. \end{cases} \tag{1}$$

If $k = 0$, then

$$\hat{f}(k) = \int_0^1 \mathcal{I}_{(1,0)} f(0, y) dy = \int_0^1 \mathcal{I}_{(0,1)} f(x, 0) dx. \tag{2}$$

The function f can be reconstructed by the Fourier series (8) *and the formulas* (1) *and* (2).

Let Q denote the set of all integer directions; a more detailed description will be given later. We consider a Tikhonov minimization problem: given some data $g \in H^r(\mathbb{T}^2 \times Q)$, find

$$\operatorname*{arg\,min}_{f \in H^r(\mathbb{T}^2)} \left(\|\mathcal{I}f - g\|^2_{H^r(\mathbb{T}^2 \times Q)} + \alpha \|f\|^2_{H^s(\mathbb{T}^2)} \right). \tag{3}$$

Let us define the post-processing operator P_α^s to be the Fourier multiplier $(1+\alpha \langle k \rangle^{2s})^{-1}$ and denote by $\mathcal{I}^*$ the adjoint of $\mathcal{I}$. We have the following theorems on regularization. The proofs are given in sections 2.4 and 2.5 respectively.

Theorem 2. *Let $r \in \mathbb{R}$, $s \geq r$, and $\alpha > 0$. Suppose $g \in H^r(\mathbb{T}^2 \times Q)$. The unique minimizer f of the minimization problem (3) corresponding to Tikhonov regularization is $f = P_\alpha^{s-r}\mathcal{I}^* g \in H^{2s-r}(\mathbb{T}^2) \subset H^r(\mathbb{T}^2)$.*

Theorem 3. *Suppose $r, t, s, \delta \in \mathbb{R}$ are such that $2s + t \geq r$, $\delta \geq 0$, and $s > 0$. We assume that $f \in H^{r+\delta}(\mathbb{T}^2)$ and $g \in H^t(\mathbb{T}^2 \times Q)$.*

Then our regularized reconstruction operator $P_\alpha^s \mathcal{I}^*$ gives a regularization strategy in the sense that

$$\lim_{\varepsilon\to 0} \sup_{\|g\|_{H^t(\mathbb{T}^2\times Q)}\leq\varepsilon} \left\| P^s_{\alpha(\varepsilon)}\mathcal{I}^*(\mathcal{I}f+g) - f \right\|_{H^r(\mathbb{T}^2)} = 0, \tag{4}$$

where $\alpha(\varepsilon) = \sqrt{\varepsilon}$.

Moreover, if $\|g\|_{H^t(\mathbb{T}^2\times Q)} \leq \varepsilon$, $0 < \delta < 2s$ and $0 < \alpha \leq 2s/\delta - 1$, we have

$$\|P_\alpha^s \mathcal{I}^*(\mathcal{I}f+g) - f\|_{H^r(\mathbb{T}^2)} \leq \alpha^{\delta/2s} C(\delta/2s) \|f\|_{H^{r+\delta}(\mathbb{T}^2)} + \frac{\varepsilon}{\alpha}, \tag{5}$$

where $C(x) = x(x^{-1}-1)^{1-x}$.

Remark 4. If we choose the regularization parameter as $\alpha = \varepsilon^\gamma$, the optimal asymptotic rate of convergence is obtained when $\gamma = (1+\delta/2s)^{-1}$. Then the terms $\alpha^{\delta/2s}$ and ε/α are of equal order.

We have also studied mapping properties, the adjoint and the normal operator of $\mathcal{I}$ in propositions 10 and 11; these results are stated in section 2. For example, it turns out that $\mathcal{I}|_{H^s(\mathbb{T}^2)}$ is unitary to its range for any $s \in \mathbb{R}$ (see proposition 11).

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2. Torus CT method

In this section we will lay out the theory of the Torus CT method. The reconstruction method is based on the Fourier series and properties of the geodesic X-ray transform on $\mathbb{T}^2$. There is a natural projection operator from the X-ray transform data of a compactly supported function on the plane to the X-ray transform data on $\mathbb{T}^2$. This so called torus-projection operator plays the role of the back-projection operator. For more details on the geodesic X-ray transform on tori see [10] and [11, Chapter 3].

2.1. The geodesic X-ray transform on $\mathbb{T}^2$.

We define the *flat torus* as the quotient $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ and denote the *quotient mapping* $[\cdot]\colon \mathbb{R}^2 \to \mathbb{T}^2$. A function $f\colon \mathbb{T}^2 \to \mathbb{C}$ can be equivalently thought as a periodic function on $\mathbb{R}^2$ via the quotient mapping $[\cdot]$. We may thus consider a function $f\colon \mathbb{T}^2 \to \mathbb{C}$ as a periodic function on the whole $\mathbb{R}^2$.

On closed Riemannian manifolds one defines the geodesic X-ray transform as a collection of line integrals of a function over periodic geodesics. The all geodesics of $\mathbb{T}^2$ are given by the parametrizations $\gamma_{x,v}(t) := [x+tv]$, $(x,v) \in [0,1]^2 \times \mathbb{R}^2 \setminus 0$. The geodesic $\gamma_{x,v}$ is periodic with the period 1 (with respect to the parameter t) if and only if $(x,v) \in [0,1]^2 \times (\mathbb{Z}^2 \setminus 0)$ (see e.g.

[11, Exercise 23]). In general, a geodesic is periodic on $\mathbb{T}^2$ if and only if its direction vector is a multiple of a rational vector.

We denote the space of test functions by $\mathcal{T} := C^\infty(\mathbb{T}^2)$ and the set of all mappings $X \to Y$ by Y^X. We define the *(geodesic) X-ray transform on* $\mathbb{T}^2$ as an operator $\mathcal{I}\colon \mathcal{T} \to \mathbb{R}^{\mathbb{T}^2\times(\mathbb{Z}^2\setminus 0)}$ by

$$\mathcal{I}f(x,v) := \int_0^1 f(\gamma_{x,v}(t))\mathrm{d}t, \quad f \in \mathcal{T}, \quad x \in [0,1]^2 \quad v \in \mathbb{Z}^2 \setminus 0. \tag{6}$$

A simple calculation shows that that $f \mapsto \mathcal{I}f(\cdot,v)$ is a formally self-adjoint operator on $\mathcal{T}$ for any fixed $v \in \mathbb{Z}^2 \setminus 0$. We denote the dual space of $\mathcal{T}$ by $\mathcal{T}'$, i.e. the space of *distributions*. By formal self-adjointness of $\mathcal{I}$, we may define the *X-ray transform on distributions* $f \in \mathcal{T}'$ by

$$[\mathcal{I}f(\cdot,v)](\eta) := (f, \mathcal{I}\eta(\cdot,v)), \quad \eta \in \mathcal{T} \tag{7}$$

where $(\cdot,\cdot)$ is the duality pairing.

If $f \in \mathcal{T}'$, then we denote the Fourier coefficients of f as $\hat{f}(k) := f(e^{-2\pi i k\cdot x}), k \in \mathbb{Z}^2$, and the Fourier series

$$f(x) = \sum_{k\in\mathbb{Z}^2} \hat{f}(k)e^{2\pi i k\cdot x} \tag{8}$$

converges in the sense of distributions. We are now ready to recall the inversion formula in [10]:

Theorem 5 (Eq. (9) in [10])**.** *If* $f \in \mathcal{T}'$, *then*

$$\widehat{\mathcal{I}f}(k,v) = \begin{cases} \hat{f}(k) & k\cdot v = 0 \\ 0 & k \cdot v \neq 0. \end{cases} \tag{9}$$

Theorem 5 gives a constructive formula (9) for the inverse of the X-ray transform on $\mathbb{T}^2$.

2.2. **Inversion formula for integrable functions.** In this section we simplify the formula (9) for functions in $L^1(\mathbb{T}^2)$. It turns out that the dimension of the integral defining $\widehat{\mathcal{I}f}(k,v)$ can decreased by one using a change of coordinates, which enables a computationally faster implementation.

Recall that $f \in \mathcal{T}'$ is in $L^1(\mathbb{T}^2)$ if there exists a function $\tilde{f} \in L^1(\mathbb{T}^2)$ such that

$$(f,\varphi) = \int_{\mathbb{T}^2} \tilde{f}\varphi dm, \quad \forall\varphi \in C^\infty(\mathbb{T}^2). \tag{10}$$

It holds that $L^1(\mathbb{T}^2) \subset \mathcal{T}'$. If $\mathcal{I}f(\cdot,v) \in L^1(\mathbb{T}^2)$ for some $f \in \mathcal{T}'$, then we simply denote that $\mathcal{I}_v f = \mathcal{I}f(\cdot,v)$.

We define a family of coordinates which will be used repeatedly in this subsection. Suppose that $v \in \mathbb{Z}^2 \setminus 0$ and $v_1, v_2 \neq 0$. Let $w_m := \frac{m}{|v_1|}v, m \in \mathbb{Z}$, and define the coordinates $\varphi_{v,m}$ on $\mathbb{T}^2$ as

$$\varphi_{v,m}(a,b) \mapsto a\frac{v}{|v_2|} + (0,b) + w_m, \quad a \in [0, \left|\frac{v_2}{v_1}\right|), b \in [0,1). \tag{11}$$

Notice that $a\frac{v}{|v_2|} + (0,b) = (a\frac{v_1}{|v_2|}, \frac{v_2}{|v_2|}(a+b))$ and $w_m = m(\frac{v_1}{|v_1|}, \frac{v_2}{|v_1|})$ in the Cartesian coordinates. It easily follows that the Lebesgue measure on $\mathbb{T}^2$ transforms as $dm = \left|\frac{v_1}{v_2}\right| d(a,b)$ where $d(a,b)$ denotes the Lebesgue measure on $X := [0, \left|\frac{v_2}{v_1}\right|) \times [0,1)$.

Remark 6. The coordinates $\varphi_{v,m}$ parametrize $\mathbb{T}^2$ as parallelograms which are located on $\mathbb{R}^2$. Moreover, the parallelograms associated with $\varphi_{v,m}, m \in \mathbb{Z}$ are disjoint for a fixed $v \in \mathbb{Z}^2 \setminus 0$ when looked on $\mathbb{R}^2$. An example is given on Figure 1.

The next lemma states that the geodesic X-ray transform of $L^1(\mathbb{T}^2)$ function can be defined geodesic-wise for almost every closed geodesic. Furthermore, the X-ray data for any fixed direction is also $L^1(\mathbb{T}^2)$, and this definition agrees with the distributional definition.

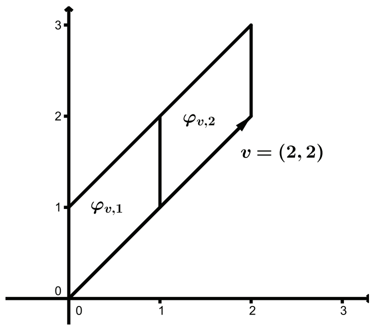


FIGURE 1. Parallelograms associated to the coordinates $\varphi_{v,m}$ when $v = (2, 2)$ and $m = 0, 1$.

Lemma 7. *Suppose that $v \in \mathbb{Z}^2 \setminus 0$. Then the X-ray transform $\mathcal{I}_v \colon L^1(\mathbb{T}^2) \to L^1(\mathbb{T}^2)$ can be defined by the formula*

$$\mathcal{I}_v f(p) := \int_0^1 f(p + tv)dt \quad \text{for a.e. } p \in \mathbb{T}^2. \tag{12}$$

Moreover, we have:

(1) *This definition coincides with the distributional definition; for every $f \in L^1(\mathbb{T}^2)$ and $g \in L^\infty(\mathbb{T}^2)$ it holds that $(\mathcal{I}_v f, g) = (f, \mathcal{I}_v g)$.*
(2) *$\mathcal{I}_v \colon L^1(\mathbb{T}^2) \to L^1(\mathbb{T}^2)$ is Lipschitz continuous with Lipschitz constant 1.*
(3) *For almost every $p \in \mathbb{T}^2$ and every $v \in \mathbb{Z}^2 \setminus 0$ and $t \in \mathbb{R}$ it holds that $\mathcal{I}_v f(p) = \mathcal{I}_v f(p + tv)$.*

Proof. This follows from the Fubini's theorem and straightforward calculations using the coordinates $\varphi_{v,m}$. We omit the details. □

We will give two proofs for theorem 1. The first proof is based on the assumption that $f \in L^1(\mathbb{T}^2)$ and straightforward computation of the Fourier coefficients. The first proof proves the injectivity of the X-ray transform on $\mathbb{T}^2$ for $L^1(\mathbb{T}^2)$ functions independently of [10]. The second proof is based on the formula (9) and the assumption that $\mathcal{I}f(\cdot, v) \in L^1(\mathbb{T}^2)$. Both of the proofs involve the coordinates $\varphi_{v,k}$.

First proof of theorem 1. Recall that

$$\hat{f}(k) = \int_0^1 \int_0^1 f(x, y) \exp(-2\pi i k \cdot (x, y))dxdy. \tag{13}$$

If $k_1 = 0$ or $k_2 = 0$, then the formulas (1) and (2) follow trivially from (13).

The case $k_1, k_2 \neq 0$. We can use the coordinates $\varphi_{v,m}, m \in \mathbb{Z}$, defined by the formula (11). Using these coordinates we can calculate

$$\hat{f}(k) = \int_0^1 \int_0^{|v_2/v_1|} f(\varphi_{v,m}(a, b)) \exp\left(-2\pi i k \cdot \left(\frac{a}{|v_2|}v + (0, b) + w_m\right)\right) \left|\frac{v_1}{v_2}\right| dadb. \tag{14}$$

Notice that $k \cdot \left(\frac{a}{|v_2|}v + (0, b) + w_m\right) = k_2 b$ since $v \cdot k = w_m \cdot k = 0$.

Hence, we have

$$\hat{f}(k) = \left|\frac{v_1}{v_2}\right| \int_0^1 \int_0^{|v_2/v_1|} f\left(a\frac{v}{|v_2|} + (0, b) + w_m\right) da \exp(-2\pi i k_2 b)db. \tag{15}$$

We sum the formula (15) for values $m = 0, \ldots, |v_1| - 1$, which gives

$$|v_1| \hat{f}(k) = |v_1| \int_0^1 \mathcal{I}_v f(0, y) \exp(-2\pi i k_2 y)dy. \tag{16}$$

This completes the proof. □

We will next prove a more general version of theorem 1.

Theorem 8. *Suppose that $f \in \mathcal{T}'$ and $\mathcal{I}f(\cdot,v) \in L^1(\mathbb{T}^2)$ for any $v \in \mathbb{Z}^2 \setminus 0$. Then the formulas* (1) *and* (2) *are true.*

Proof. We only show how to argue if $k_1, k_2 \neq 0$ since the other special cases are trivial. Recall that the inversion formula (9) states that $\widehat{\mathcal{I}_v f}(k) = \hat{f}(k)$ for any $v \in \mathbb{Z}^2 \setminus 0$ such that $k \perp v$. We apply the coordinates $\varphi_{v,0}$.

Using the Fubini's theorem and calculations similar to the first proof of theorem 1, we get

$$\widehat{\mathcal{I}_v f}(k) = \left|\frac{v_1}{v_2}\right| \int_0^1 \int_0^{|v_2/v_1|} \mathcal{I}_v f(\varphi_{v,0}(a,b))da \exp(-2\pi i k_2 b)db. \tag{17}$$

Now, the formula (1) follows from the property (3) of lemma 7. This completes the proof. □

Second proof of theorem 1. Lemma 7 implies that $\mathcal{I}_v f \in L^1(\mathbb{T}^2)$ if $f \in L^1(\mathbb{T}^2)$. Hence, theorem 8 implies the inversion formulas. □

2.3. **The torus-projection operator.** We denote the X-ray transform of $f \colon \mathbb{R}^2 \to \mathbb{C}$ by $\mathcal{R}_v f(p)$ for any $(p,v) \in \mathbb{R}^2 \times S^1$. We parametrize the lines of the plane so that

$$\mathcal{R}_v f(p) = \int_{\mathbb{R}} f(p+tv)dt. \tag{18}$$

Suppose that f is a compactly supported function on $\mathbb{R}^2$. We may then consider f as a function defined on $\mathbb{T}^2$ after rescaling and periodizing. Let us denote the periodic extension of f into $\mathbb{T}^2$ by the same symbol f.

Suppose further that $f \in C(\mathbb{T}^2)$. As described in [11, Lemma 3.1], for any $p \in \mathbb{T}^2$ and $v \in \mathbb{Z}^2 \setminus 0$ one can write $\mathcal{I}_v f(p)$ as a finite sum of terms $\mathcal{R}_{v/|v|} f(p_i)$, $i = 1, \dots, m$. One simply has to write any periodic geodesic γ of $\mathbb{T}^2$ as a finite disjoint union of line segments that are supported in $[0,1) \times [0,1)$ and travel from the boundary to the boundary in the fundamental domain of $\mathbb{T}^2$. However, such unions are tedious to write down rigorously. This procedure defines the torus-projection operator $\mathcal{R}f \mapsto \mathcal{I}f$ for compactly supported continuous functions $f \colon \mathbb{R}^2 \to \mathbb{C}$. For further details, see [11, Chapter 3]. Using duality, this operator extends to distributions. See also the description of our numerical implementation in section 3.1.2.

2.4. **Sobolev spaces, adjoint, normal operator and regularization.** Let $Q \subset \mathbb{Z}^2$ be such that every nonzero $v \in \mathbb{Z}^2$ is an integer multiple of a unique element in Q. We can simply take Q to be the set of those vectors (a,b) for which a and b are coprime with $a > 0$ and $b \neq 0$ and the vectors $(0,1)$ and $(1,0)$. The set Q is the set of all periodic directions on the torus, with all multiple counts removed. This set can be naturally identified with the projective space $\mathbb{P}^1$ defined later.

The X-ray transform we study takes a function on $\mathbb{T}^2$ to a function on $\mathbb{T}^2 \times Q$. To set things up properly, we need to define function spaces and norms on both sides. On $\mathbb{T}^2$, we use the standard Sobolev scale of spaces $H^s(\mathbb{T}^2)$ with the norms

$$\|f\|^2_{H^s(\mathbb{T}^2)} = \sum_{k \in \mathbb{Z}^2} \langle k \rangle^{2s} \left|\hat{f}(k)\right|^2, \tag{19}$$

where $\langle k \rangle = (1 + |k|^2)^{1/2}$ as usual. On $\mathbb{T}^2 \times Q$, we define the spaces $H^s(\mathbb{T}^2 \times Q)$ to be the set of functions $g \colon \mathbb{T}^2 \times Q \to \mathbb{C}$ for which

(i) $g(\cdot,v) \in H^s(\mathbb{T}^2)$ for every $v \in Q$;
(ii) the average of every $g(\cdot,v)$ is the same; and
(iii) the norm

$$\|g\|^2_{H^s(\mathbb{T}^2 \times Q)} = |\hat{g}(0,0)|^2 + \sum_{k \in \mathbb{Z}^2 \setminus 0} \sum_{v \in Q} \langle k \rangle^{2s} |\hat{g}(k,v)|^2 \tag{20}$$

is finite. We set $v = 0$ for the Fourier term $k = 0$ to emphasize that it is the same for every $v \in Q$. We remind the reader that $0 \notin Q$.

We emphasize that the regularity parameter s can be any real number in the theory presented in this section. By setting $s = 0$ one obtains a theory in $H^0 = L^2$. We point out that the spaces considered here are different from [10].

Remark 9. The norm of $g \in H^s(\mathbb{T}^2 \times Q)$ is essentially an $\ell^2(Q)$ norm of the $H^s(\mathbb{T}^2)$ norms of the functions $g(\cdot, v)$. This ℓ^2 can be replaced with any ℓ^p without much effect to the theory, as the different functions in the family indexed by Q have disjointly supported Fourier series apart from the origin. The case $p = \infty$ is particularly convenient because then special considerations are not needed at $k = 0$. We choose $p = 2$ to stay in a Hilbert space setting.

We denote $v^\perp = (-v_2, v_1)$ for any $v = (v_1, v_2) \in \mathbb{Z}^2$. For $v \in \mathbb{Z}^2 \setminus 0$, we denote by $\hat{v}$ the unique point in Q that is parallel to v. We can define $\hat{0}$ to be any point in Q; this choice will not matter. To keep notation neater, we will write $\hat{v}^\perp$ instead of $\widehat{v^\perp}$.

Proposition 10. *The X-ray transform is continuous $H^s(\mathbb{T}^2) \to H^s(\mathbb{T}^2 \times Q)$ for any $s \in \mathbb{R}$.*

Proof. For any $v \in Q$, the Fourier transform of function $\mathcal{I}f(\cdot, v)$ is supported on the line $v^\perp\mathbb{Z}$ by theorem 5. In fact, it is the restriction of $\hat{f}$ to this line. It then follows easily from the definition of the Sobolev norm on the Fourier side that $\mathcal{I}f(\cdot, v) \in H^s(\mathbb{T}^2)$ whenever $f \in H^s(\mathbb{T}^2)$.

It follows from the same theorem that $\widehat{\mathcal{I}f}(0, v) = \hat{f}(0)$ for all $v \in Q$, and so all the averages agree as required.

Since $\mathbb{Z}^2$ is a disjoint union of the origin and the punctured lines $v\mathbb{Z} \setminus 0$ with $v \in Q$, one can easily verify that $\|\mathcal{I}f\|_{H^2(\mathbb{T}^2 \times Q)} = \|f\|_{H^s(\mathbb{T}^2)}$. □

Proposition 11. *Fix any $s \in \mathbb{R}$. The adjoint of $\mathcal{I}\colon H^s(\mathbb{T}^2) \to H^s(\mathbb{T}^2 \times Q)$ is $\mathcal{I}^*\colon H^s(\mathbb{T}^2 \times Q) \to H^s(\mathbb{T}^2)$ given by*

$$\widehat{\mathcal{I}^* g}(k) = \hat{g}(k, \hat{k}^\perp). \tag{21}$$

The normal operator $\mathcal{I}^*\mathcal{I}\colon H^s(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ is the identity, so that $\mathcal{I}$ is unitary to its range.

Remark 12. We emphasize that there is a striking difference to the usual Euclidean X-ray transform, where the normal operator is a convolution. In our setup the X-ray transform is directly inverted by its normal operator without any filtering or post-processing.

Proof of proposition 11. Let us take any two functions $f \in H^s(\mathbb{T}^2)$ and $g \in H^s(\mathbb{T}^2 \times Q)$. We denote the complex conjugate of $z \in \mathbb{C}$ as z^*. Theorem 5 shows that $\widehat{\mathcal{I}f}(k, v) = \hat{f}(k)\delta_{0,k\cdot v}$, and so the H^s inner products satisfy

$$\begin{aligned} (\mathcal{I}f, g) &= \widehat{\mathcal{I}f}(0,0)^* \hat{g}(0,0) + \sum_{k \in \mathbb{Z}^2 \setminus 0} \sum_{v \in Q} \langle k \rangle^{2s} \widehat{\mathcal{I}f}(k, v)^* \hat{g}(k, v) \\ &= \sum_{k \in \mathbb{Z}^2} \langle k \rangle^{2s} \hat{f}(k)^* \hat{g}(k, \hat{k}^\perp) \\ &= (f, \mathcal{I}^* g)\,. \end{aligned} \tag{22}$$

Therefore, the operator $\mathcal{I}^*$ defined above is the adjoint of $\mathcal{I}$.

It follows directly from the formula of theorem 5 that $\mathcal{I}^*$ is a left inverse of $\mathcal{I}$. □

Remark 13. The X-ray transform or its normal operator have no effect on regularity. In the usual formulation, the normal operator does increase the smoothness index s, but when everything is set up on $\mathbb{T}^2$ the operators leave the regularity level intact.

We now turn to regularized inversion, and solve the Tikhonov minimization problem (3). We will make use of the post-processing operator P^s_α, which is the Fourier multiplier $(1 + \alpha \langle k \rangle^{2s})^{-1}$. It is evident that P^s_α maps continuously $H^r(\mathbb{T}^2) \to H^{r+2s}(\mathbb{T}^2)$ for any $s, r \in \mathbb{R}$.

Proof of theorem 2. We begin with expanding the norms along lines given by Q on the Fourier side. We have

$$\|f\|^2_{H^s(\mathbb{T}^2)} = \left|\hat{f}(0)\right|^2 + \sum_{v \in Q} \sum_{p \in \mathbb{Z} \setminus 0} \langle pv \rangle^{2s} \left|\hat{f}(pv^\perp)\right|^2 \tag{23}$$

and

$$\|\mathcal{I}f - g\|^2_{H^r(\mathbb{T}^2\times Q)} = \left|\widehat{\mathcal{I}f}(0,0) - \hat{g}(0,0)\right|^2 + \sum_{v\in Q} A_v, \tag{24}$$

where

$$\begin{aligned} A_v = &\sum_{p\in\mathbb{Z}\setminus 0} \langle pv\rangle^{2r} \left|\widehat{\mathcal{I}f}(pv^\perp, v) - \hat{g}(pv^\perp, v)\right|^2 \\ &+ \sum_{w\in\mathbb{Z}^2\setminus v\mathbb{Z}} \langle w\rangle^{2r} \left|\widehat{\mathcal{I}f}(w,v) - \hat{g}(w,v)\right|^2. \end{aligned} \tag{25}$$

Each $\widehat{\mathcal{I}f}(w,v)$ vanishes in the last sum by theorem 5. Therefore, the second sum of A_v is independent of f and can be left out of the minimization problem. Furthermore, $\widehat{\mathcal{I}f}(pv^\perp, v) = \hat{f}(pv^\perp)$.

Thus, we are left with minimizing

$$\begin{aligned} &\left|\widehat{\mathcal{I}f}(0,0) - \hat{g}(0,0)\right|^2 + \alpha\left|\hat{f}(0)\right|^2 \\ &+ \sum_{v\in Q}\sum_{p\in\mathbb{Z}\setminus 0}\left(\langle pv\rangle^{2r}\left|\hat{f}(pv^\perp) - \hat{g}(pv^\perp, v)\right|^2 + \alpha\langle pv\rangle^{2s}\left|\hat{f}(pv^\perp)\right|^2\right). \end{aligned} \tag{26}$$

The notation introduced above allows us to rewrite the minimized quantity as

$$\sum_{k\in\mathbb{Z}^2}\langle k\rangle^{2r}\left(\left|\hat{f}(k) - \hat{g}(k,\hat{k}^\perp)\right|^2 + \alpha\langle k\rangle^{2(s-r)}\left|\hat{f}(k)\right|^2\right), \tag{27}$$

and this can be minimized explicitly.

It suffices to choose each $\hat{f}(k)$ so that the term in the parentheses of (27) is minimized. A straightforward computation shows that the minimal $\hat{f}(k)$ is

$$\hat{f}(k) = (1 + \alpha\langle k\rangle^{2(s-r)})^{-1}\hat{g}(k,\hat{k}^\perp). \tag{28}$$

That is, the minimizer we sought is $f = P_\alpha^{s-r}\mathcal{I}^* g$. Finally, by the mapping properties of P_α^{s-r} and $\mathcal{I}^*$ we have $f \in H^{2s-r}(\mathbb{T}^2)$. This implies that f is in the correct space $H^r(\mathbb{T}^2)$ since we assumed $s \geq r$. □

Remark 14. Choosing $r = 0$ and $s = 1$, we reconstruct a function in L^2 with an H^1 penalty term. If we want the penalty to be the L^2 norm of the gradient without the L^2 norm of the function, the Fourier multiplier in the penalty term is changed from $\langle k\rangle^2$ to $|k|^2$. This corresponds to changing the Sobolev norm to a homogeneous Sobolev norm. Such changes lead to similar results but with slightly different postprocessing operator.

2.5. **Regularization strategy.** We define the concept of a regularization strategy according to [6, 15]. Let X and Y be subsets of Banach spaces and $F\colon X \to Y$ a continuous mapping. A family of continuous maps $\mathcal{R}_\alpha\colon Y \to X$ with $\alpha \in (0, \alpha_0]$ is called a *regularization strategy* if $\lim_{\alpha\to 0}\mathcal{R}_\alpha(F(x)) = x$ for every $x \in X$. A choice of regularization parameter $\alpha(\epsilon)$ with $\lim_{\epsilon\to 0}\alpha(\epsilon) = 0$ is called *admissible* if

$$\lim_{\epsilon\to 0}\sup_{y\in Y}\left\{\|\mathcal{R}_{\alpha(\epsilon)}y - x\|_X\,;\,\|y - F(x)\|_Y \leq \epsilon\right\} = 0 \tag{29}$$

holds for every $x \in X$. Regularization strategies have been found for other inverse problems including, for example, electrical impedance tomography (EIT) [16] and inverse problem for the $1+1$ dimensional wave equation [18, 19].

We will next prove that the regularized inversion operator $P_\alpha^s\mathcal{I}^*$ obtained in theorem 2 actually provides an admissible regularization strategy with a quantitative stability estimate.

Proof of theorem 3. Using proposition 11, we write

$$P^s_\alpha \mathcal{I}^*(\mathcal{I}f + g) - f = (P^s_\alpha - \mathrm{id})f + P^s_\alpha \mathcal{I}^* g \tag{30}$$

and aim to estimate these two terms. In this proof, we denote the norm of $H^r(\mathbb{T}^2)$ simply by $\|\cdot\|_r$.

Since $\|g\|_{H^t(\mathbb{T}^2\times Q)} \le \varepsilon$ and $\|\mathcal{I}^*\| = \|\mathcal{I}\| = 1$, we have $\|\mathcal{I}^* g\|_t \le \varepsilon$. Applying the definitions of the norms and the operator P^s_α, we find

$$\|P^s_\alpha \mathcal{I}^* g\|^2_r \le \varepsilon^2 \sup_{k\in\mathbb{Z}^2} (1 + \alpha \langle k\rangle^{2s})^{-2} \langle k\rangle^{2r-2t}. \tag{31}$$

Estimating $1 + \alpha \langle k\rangle^{2s} \ge \alpha \langle k\rangle^{2s}$ and using $-4s + 2r - 2t \le 0$ shows that the supremum is at most α^{-2}. Therefore

$$\|P^s_\alpha \mathcal{I}^* g\|_r \le \alpha^{-1}\varepsilon, \tag{32}$$

which converges to zero as $\varepsilon \to 0$ with $\alpha = \sqrt{\varepsilon}$.

A calculation shows that $P^s_\alpha - \mathrm{id} = -\frac{\alpha\langle k\rangle^{2s}}{1+\alpha\langle k\rangle^{2s}}$ as a Fourier multiplier. Unfortunately, this implies that

$$\|P^s_\alpha - \mathrm{id}\|_{H^r(\mathbb{T}^2)\to H^r(\mathbb{T}^2)} = \sup_{k\in\mathbb{Z}^2} \frac{\alpha \langle k\rangle^{2s}}{1 + \alpha \langle k\rangle^{2s}} = 1 \tag{33}$$

whenever $s > 0$ and $\alpha > 0$. Therefore, a uniform estimate is impossible when $\delta = 0$, but it follows from the dominated convergence theorem that $\|(P^s_\alpha - \mathrm{id})f\|^2_r \to 0$ as $\alpha \to 0$ when $f \in H^r(\mathbb{T}^2)$. The first claim of the theorem follows.

If $\delta > 0$, the additional regularity of f can be used to our advantage. It follows from the definitions of the norms that

$$\|(P^s_\alpha - \mathrm{id})f\|^2_r \le \left(\sup_{k\in\mathbb{Z}^2} \left(\frac{\alpha \langle k\rangle^{2s}}{1 + \alpha \langle k\rangle^{2s}} \right)^2 \langle k\rangle^{-2\delta} \right) \|f\|^2_{r+\delta}, \tag{34}$$

and thus

$$\|P^s_\alpha - \mathrm{id}\|_{H^{r+\delta}(\mathbb{T}^2)\to H^r(\mathbb{T}^2)} = \sup_{k\in\mathbb{Z}^2} \frac{\alpha \langle k\rangle^{2s-\delta}}{1 + \alpha \langle k\rangle^{2s}}. \tag{35}$$

Estimating this norm is crucial for the proof.

The supremum of (35) can be studied using the function $F\colon (0,\infty) \to (0,\infty)$ given by

$$F(x) = \frac{\alpha x^{2s-\delta}}{1 + \alpha x^{2s}}. \tag{36}$$

Simple calculus shows that if $2s > \delta$, then the maximum is attained at $x^{2s} = \alpha^{-1}(2s/\delta - 1)$ and the maximal value on $(0,\infty)$ is

$$\alpha^{\delta/2s} \frac{\delta}{2s} \left(\frac{2s}{\delta} - 1 \right)^{1-\delta/2s}. \tag{37}$$

We are interested in the maximum of F on $[1,\infty)$. If $2s/\delta - 1 < \alpha$, then the maximum is reached at $x \in (0,1)$, and so the maximum on the relevant interval is $F(1) = \alpha/(1+\alpha)$. (One can also verify that the two maxima coincide when $2s/\delta - 1 = \alpha$, as they should.) We assumed that $2s/\delta > 1$, so $\alpha \in (0, 2s/\delta - 1]$ for small enough α.

For $\alpha \le 2s/\delta - 1$, the maximum value of F is

$$\alpha^{\delta/2s} \frac{\delta}{2s} \left(\frac{2s}{\delta} - 1 \right)^{1-\delta/2s} = \alpha^{\delta/2s} C(\delta/2s). \tag{38}$$

We conclude that

$$\|P^s_\alpha - \mathrm{id}\|_{H^{r+\delta}(\mathbb{T}^2)\to H^r(\mathbb{T}^2)} \le \alpha^{\delta/2s} C(\delta/2s), \tag{39}$$

and so

$$\|(P^s_\alpha - \mathrm{id})f\|_r \le \alpha^{\delta/2s} C(\delta/2s)\, \|f\|_{r+\delta} \,. \tag{40}$$

The estimate (5) now follows easily from the estimates (32) and (40). □

If α is bigger than assumed in the proof, then we may use the simpler estimate $F(x) \le \alpha/(1+\alpha) \le \alpha$ for all $x \ge 1$, which would lead to replacing $\alpha^{\delta/2s}C(\delta/2s)$ in estimate (5) by simply α. However, we are only interested in the limit of small α.

We point out that $C(\delta/2s) \to 1$ and $\alpha^{\delta/2s} \to 1$ when $\delta \to 0$, matching the norm in the limiting case of (33).

The noise g in theorem 3 can be in any function space so that $\mathcal{I}^* g$, the reconstruction from pure noise, is in a suitable Sobolev space.

2.6. **Numerical and asymptotic analysis for discretized problem.** In this section, we consider questions arising from discrete practice. We analyze errors caused by a discretization of data in section 2.6.1. In section 2.6.2, we study how to choose a minimal set of X-ray directions in order to reconstruct all Fourier coefficients of a phantom in a given box.

Another source of errors in practice comes from the fact that we can only calculate finitely many coefficients of the Fourier series. The error caused by the cutoff of the Fourier series can be estimated with knowledge of asymptotic behavior of the Fourier coefficients. We do not consider this matter here further since it is a general question about convergence rates of Fourier series.

2.6.1. *On convergence rates for discretization.* Let $\mathbf{f} \in \mathbb{C}^N$ be written as $\mathbf{f} = (\mathbf{f}_0, \dots, \mathbf{f}_{N-1})$. We define the discrete Fourier transform (DFT) of $\mathbf{f}$ by

$$\mathrm{DFT}(\mathbf{f})_k := \frac{1}{N} \sum_{l=0}^{N-1} \mathbf{f}_l \exp(-2\pi i k l/N), \quad k = 0, \dots, N-1. \tag{41}$$

The following corollary of theorem 8 discretizes the inverse problem and reduces it to calculations of 1-dimensional DFTs. It is elementary and included here for completeness.

Corollary 15. *Let $f \in \mathcal{T}'$, $\mathcal{I}_v f \in L^1(\mathbb{T}^2)$, $k \in \mathbb{Z}^2 \setminus 0$. Denote $g_v(y) := \mathcal{I}_v f(0,y)$ and $h_v(x) := \mathcal{I}_v f(x,0)$.*

(1) If $v \perp k$, then $\hat{f}(0,0) = \hat{g}_{(1,0)}(0) = \hat{h}_{(0,1)}(0)$ and

$$\hat{f}(k_1,k_2) = \begin{cases} \hat{g}_v(k_2), & k_2 \ne 0 \\ \hat{h}_v(k_1), & k_1 \ne 0. \end{cases}$$

(2) (Left-point rule and DFT) Let $N \in \mathbb{Z}_+$. We denote $\boldsymbol{g}_l = g_v(l/N)$ and $\boldsymbol{h}_l = h_v(l/N)$ for $l = 0, \dots, N-1$. If $\mathcal{I}_v f$ is Riemann integrable along vertical and horizontal lines, then

$$\mathrm{DFT}(\boldsymbol{g})_{k_2} \to \hat{g}_v(k_2) \text{ as } N \to \infty.$$

Moreover, if $\mathcal{I}_v f \in C^1(\mathbb{T}^2)$, then $|\hat{g}_v(k_2) - \mathrm{DFT}(\boldsymbol{g})_{k_2}| \le C_{f,k_2}/N$ where $C_{f,k_2} > 0$ does not depend on N. Similar statements hold for h_v as well.

(3) (Mid-point rule and DFT) Let $N \in \mathbb{Z}_+$. We denote $\boldsymbol{g}_l = g_v(l/N + 1/2N)$ and $\boldsymbol{h}_l = h_v(l/N + 1/2N)$ for $l = 0, \dots, N-1$. If $\mathcal{I}_v f$ is Riemann integrable along vertical and horizontal lines, then

$$\exp(-\pi i k_2/N)\mathrm{DFT}(\boldsymbol{g})_{k_2} \to \hat{g}_v(k_2) \text{ as } N \to \infty.$$

Moreover, if $\mathcal{I}_v f \in C^2(\mathbb{T}^2)$, then $|\hat{g}_v(k_2) - \exp(-\pi i k_2/N)\mathrm{DFT}(\boldsymbol{g})_{k_2}| \le C_{f,k_2}/N^2$ where $C_{f,k_2} > 0$ does not depend on N. Similar statements hold for h_v as well.

Proof. The statement (1) is a rephrased version of theorem 8. We only prove the statement (3). The proof of the statement (2) is similar and thus omited. Let $N \in \mathbb{Z}_+$ be fixed. By the definition of the DFT

$$\begin{aligned} &\exp(-\pi i k_2/N)\mathrm{DFT}(\mathbf{g})_{k_2} \\ &= \frac{1}{N}\sum_{l=0}^{N-1} \mathbf{g}_l \exp(-\pi i k_2/N)\exp(-2\pi i k_2 l/N) \\ &= \frac{1}{N}\sum_{l=0}^{N-1} g_v(l/N + 1/2N)\exp(-2\pi i k_2(l/N + 1/2N)). \end{aligned} \tag{42}$$

The statement follows since this the mid-point approximation of $\hat{g}_v(k_2)$. The convergence rate is just a standard result on the mid-point rule (see e.g. [5]). □

2.6.2. *Choosing directions for X-ray data.* Let us define the set

$$A_N := \{v \in \mathbb{Z}^2 \setminus 0\,;\, v \in k^\perp \text{ for some } k \in \mathcal{Z}_N\} \tag{43}$$

where $\mathcal{Z}_N = [-N, N]^2 \cap \mathbb{Z}^2$. It is known that the data $(\mathcal{I}f(\cdot, v))_{v\in A_N}$ determines $(\hat{f}(k))_{k\in\mathcal{Z}_N}$. Thus, we define

$$\varphi(N) := \min\{|B|\,;\, B \subset A_N, (\mathcal{I}f(\cdot, v))_{v\in B} \text{ determines } (\hat{f}(k))_{k\in\mathcal{Z}_N}\}. \tag{44}$$

Define the set $V_N := X_+ \cup X_- \cup \{(1, 0), (0, 1)\}$ where

$$\begin{aligned} X_+ &= \{\, (v_1, v_2) \in \mathcal{Z}_N \setminus 0\,;\, \gcd(v_1, v_2) = 1, v_1, v_2 \geq 1 \,\}, \\ X_- &= \{\, (-v_1, v_2)\,;\, v \in X_+ \,\}. \end{aligned} \tag{45}$$

Now, it is an elementary observation that the data $(\mathcal{I}f(\cdot, v))_{v\in V_N}$ determines $(\hat{f}(k))_{k\in\mathcal{Z}_N}$ and $|V_N| = \varphi(N)$.

We then turn to studying the asymptotic behavior of $\varphi(N)$. We denote by $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{Q})$ the collection of equivalence classes $(a : b)$, $(a, b) \in \mathbb{Z}^2 \setminus 0$, such that $(x, y) \in (a : b)$ if and only if $c(x, y) = (a, b)$ for some $c \neq 0$ and $(x, y) \in \mathbb{Z}^2 \setminus 0$. The *height* is defined as $H(a : b) := \max\{|a|, |b|\}$ using the unique representative (up to a sign) of $(a : b)$ with $\gcd(a, b) = 1$. One of the simplest special cases of the Schanuel's theorem [27, Theorem 1] states that

$$\left|\{\, (a : b) \in \mathbb{P}^1\,;\, H(a : b) \leq N \,\}\right| = \frac{2}{\zeta(2)}N^2 + O(N \log N) \tag{46}$$

as $N \to \infty$. More detailed exposition is given in the book of Serre [28, Chapter 2.5].

We conclude with the following proposition.

Proposition 16. *It holds that* $\varphi(N) = \frac{2}{\zeta(2)}N^2 + O(N \log N)$.

Proof. If we want to reconstruct $\hat{f}(k)$, then we need at least one $v \in k^\perp$ by theorem 5 and, on the other hand, just one $v \in k^\perp$ is enough. It follows from the definition of height that

$$\varphi(N) = \left|\{\, (a : b) \in \mathbb{P}^1\,;\, H(a : b) \leq N \,\}\right|. \tag{47}$$

The estimate follows now from the Schanuel's theorem. □

Remark 17. The trivial estimate for directions needed in reconstruction of the Fourier coefficients $(\hat{f}(k))_{k\in\mathcal{Z}_N}$ would be $\varphi(N) \leq (2N + 1)^2$. In comparison, proposition 16 implies that one needs to use asymptotically about $3/\pi^2 \approx 30$ % of the data $(\mathcal{I}f(\cdot, v))_{v\in\mathcal{Z}_N}$.

3. Computational forward and inverse models

We have implemented two forward models for the X-ray transform on $\mathbb{T}^2$. The first forward model is based on direct integration over periodic geodesics on $\mathbb{T}^2$ (two different numerical integration schemes are implemented), and the second forward model on the usual Radon transform and the torus-projection operator. The regularized inverse model is based on theorems 1 and 2.

3.1. **Computational forward models.**

3.1.1. *Forward models on the torus.* We have two different numerical integration schemes for the forward integration. The first one is analytical integration of a phantom which is discretized into square pixels of equal size. In this case, the forward operator, denoted by $\mathcal{A}_1$, is

$$\mathcal{A}_1 f(x,v) := \frac{1}{|v|}\sum_{i=1}^{N} d_i f_i \approx \int_0^1 f(x+tv)dt = \mathcal{I}f(x,v) \tag{48}$$

where d_i is the length of the geodesic $\gamma_{x,v}$ and f_i is the value of the discretized phantom in the i'th pixel, and N is the size of the grid. The lengths d_i are calculated by solving the intersection points of the line $\{\, x+tv\,;\, t\in[0,1]\,\}$ and the edges of the pixels when the pixels are periodically extended to $\mathbb{R}^2$.

In the second one, the integral is based on the use of global adaptive quadature [29] which is implemented into the Matlab's `integral` function. In this case, a phantom is given in an analytical form. We denote this forward model by $\mathcal{A}_2$.

3.1.2. *Forward model using the torus-projection and Radon data.* This forward model corresponds to converting conventional X-ray data sets on $\mathbb{R}^2$ into X-ray data sets on $\mathbb{T}^2$. The forward model has two steps. The first step is to calculate Radon transform data using the Matlab's `radon` function. The second step is to calculate the torus-projection (see Section 2.3) of the Radon data. The directions for the Radon transform are chosen so that they contain all directions generated by integer vectors (see Section 2.6.2).

The X-ray beams on the `radon` function are parametrized by the distance between the line of a X-ray beam and the center of a domain O, and the angle of a X-ray beam measured from the y-axis into the counterclockwise direction. We denote simply that $Rf(v) = \texttt{radon}(f,\alpha_v,M)$ where α_v is the angle defined above and M is the number of X-rays taken into direction v. We index the rays as $k=1,\ldots,M$. Further, denote the distances of rays to O by $c_{k,v}$ and the projection values with the respective rays by $Rf(v)_k$.

We split each geodesic $\gamma_{x,v}$ into segments in which it travels from the boundary to the boundary when looked at the fundamental domain $[0,1]\times[0,1]$ of $\mathbb{T}^2$. Let $d_{x,v,i}$ be the distance of the i'th segment of the geodesic $\gamma_{x,v}$ and O, and N the number of distinct segments. Finally, we can define the forward model $\mathcal{A}_{\mathbb{T}^2}$ as

$$\mathcal{A}_{\mathbb{T}^2} f(x,v) = \frac{1}{|v|}\sum_{i=1}^{N}\big(w_{1,i}Rf(v)_{k_{1,i}} + w_{2,i}Rf(v)_{k_{2,i}}\big) \tag{49}$$

where

$$k_{1,i} = \operatorname*{arg\,min}_{k\in\{1,\ldots,M\}} |c_{k,v}-d_{x,v,i}|, \quad k_{2,i} = \operatorname*{arg\,min}_{k\in\{1,\ldots,M\}\setminus\{k_{1,i}\}} |c_{k,v}-d_{x,v,i}|,$$

$$w_{1,i} = \left|\frac{c_{k_{2,i},v}-d_{x,v,i}}{c_{k_{1,i},v}-c_{k_{2,i},v}}\right|, \quad w_{2,i} = \left|\frac{c_{k_{1,i},v}-d_{x,v,i}}{c_{k_{1,i},v}-c_{k_{2,i},v}}\right|,$$

if $\big|c_{k_{1,i},v}-d_{x,v,i}\big| + \big|c_{k_{2,i},v}-d_{x,v,i}\big| < \big|c_{k_{1,i},v}-c_{k_{2,i},v}\big|$, and $w_{1,i}=w_{2,i}=0$ otherwise.

The last condition ensures that the rays, corresponding to the data in interpolation, are on the different sides of the geodesic segment. Vise versa, if the condition does not hold, the geodesic segment is outside the projection width. In other words, this condition is the zero extension of the data near boundaries of the domain. In short, $\mathcal{A}_{\mathbb{T}^2}$ is the sum of weighted averages of two closest projection values with respect to their distances to the corresponding geodesic segments.

3.2. **Computational inverse model.**

In the inverse model, we calculate the Fourier series coefficients of a phantom and reconstruct its Fourier series up to a finite radius $r>0$. The Fourier coefficients are calculated using the inversion formulas (1) and (2) of theorem 1. Furthermore, we Tikhonov regularize reconstructions using the filter P^s_α on the Fourier side according to theorems 2 and 3.

Let us write $B_r = B(0,r) \cap \mathbb{Z}^2$. The inverse model is

$$(50) \quad \begin{aligned} f_{\text{rec}}^{\alpha,s}(x) &= \sum_{k \in B_r} P_\alpha^s \widehat{f}_{\text{rec}}(k) \exp(2\pi i k \cdot x) \\ &= \sum_{k \in B_r} (1 + \alpha \langle k \rangle^{2s})^{-1} \widehat{f}_{\text{rec}}(k) \exp(2\pi i k \cdot x) \end{aligned}$$

where $\hat{f}_{\text{rec}}(k)$ is calculated from data using the left-point rule and the DFT according to (1) and (2) of corollary 15. We remark that the inverse model is meshless, its output is a trigonometric polynomial, and thus, completely avoids the so called inverse crime.

4. Numerical experiments

4.1. **Phantoms, convergence rates of Fourier series and discretization.**

4.1.1. *Phantoms.* We have used two phantoms in the numerical experiments, the Shepp–Logan phantom based on the Matlab's function `phantom` and the Flag phantom which is a piece-wise constant function representing a Nordic flag. The Flag phantom $f_F \colon [0,1] \times [0,1] \to \mathbb{R}$ was defined as

$$(51) \qquad f_F(x,y) = \begin{cases} g_F(x,y), & x \in (0.14, 0.86) \text{ and } y \in (0.28, 0.72) \\ 0, & \text{otherwise} \end{cases}$$

where

$$(52) \qquad g_F(x,y) = \begin{cases} 0.3, & x \in (0.34, 0.46) \text{ or } y \in (0.44, 0.56) \\ 0.9, & \text{otherwise.} \end{cases}$$

That is, f_F describes the outer boundaries of the flag, and g_F returns the background unless x or y is on the horizontal or vertical stripe, respectively.

4.1.2. *Cutoff errors of Fourier series of phantoms.* We analyzed the cutoff errors of Fourier series of the phantoms in order to determine a good, practical value of $r > 0$ for the reconstructions. The squared cutoff error of Fourier series can be calculated via the formula

$$(53) \qquad \epsilon_r = \|f\|^2_{L^2(\mathbb{T}^2)} - \sum_{k \in B_r} \hat{f}(k)^2$$

using the Parseval's identity.

We computed ϵ_r for the Shepp–Logan phantom, the Flag phantom and the Flag phantom with a 45° rotation. All the three phantoms were studied without noise and with salt-and-pepper (S&P) type noise applied to the phantoms using the Matlab's `imnoise` function with 0.02 noise density. The phantoms were discretized into 4000×4000 pixel grid and the Fourier coefficients $\hat{f}(k)$ were computed using the Matlab's `fft2` and `fftshift` functions.

The squared cutoff errors ϵ_r are shown in Figure 2. The squared cutoff errors saturate at around $r = 50$, though some improvement might be gained up to $r = 200$. In our forward and inverse simulations, we have mainly used $r = 50$ as it practically seems to be a sufficiently good choice.

4.1.3. *Discretizations of phantoms and geodesics.* The starting points of the used geodesics were chosen to be the equispaced points $\{(0,0), (1/n_d, 0), (2/n_d, 0), \ldots, (1-1/n_d, 0)\}$ on the x-axis, except for geodesics in direction $v = (1,0)$ where the sampling was $\{(0,0), (0, 1/n_d), (0, 2/n_d), \ldots, (0, 1-1/n_d)\}$ on the y-axis. In our experiments, we set $n_d = 128$ when the cutoff radius of the Fourier series was $r \in \{50, 100\}$, and $n_d = r$ when $r \in \{150, 200\}$.

The phantoms were discretized with 512×512 pixel grid when used for forward simulations with the forward models $\mathcal{A}_1$ and $\mathcal{A}_{\mathbb{T}^2}$. When we used the forward model $\mathcal{A}_2$, the Flag phantom was not discretized. The values of reconstructions were evaluated at equispaced points in 256×256 pixel grid, and when compared to the ground truth, the Shepp–Logan and the Flag phantoms were discretized for the same grid.

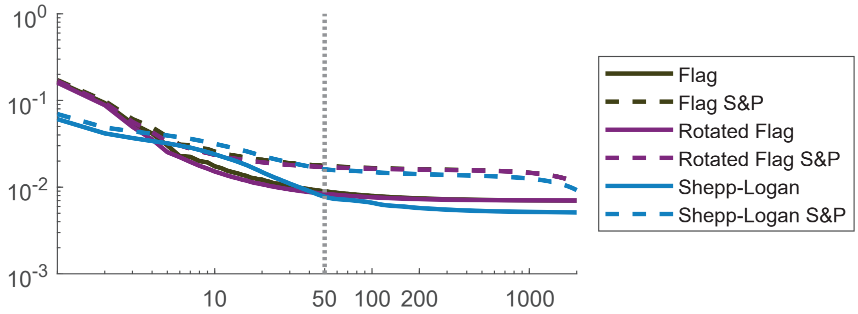


FIGURE 2. The graphs (r, ϵ_r) in a logarithmic scale for the different phantoms. Vertical, dashed line marks $r = 50$ where ϵ_r saturates.

4.2. Numerical analysis of forward models $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_{\mathbb{T}^2}$.

4.2.1. *Forward models $\mathcal{A}_1$ and $\mathcal{A}_2$ on the torus.* We tested Torus CT using the Shepp–Logan phantom with simulated data

$$y = \mathcal{A}_1 f + \mathcal{E}, \quad \mathcal{E} \sim \mathcal{N}(0, \sigma^2), \quad \sigma = \frac{2}{100}. \tag{54}$$

We made reconstructions with cutoff radii $r \in \{50, 100, 150, 200\}$ of the Fourier series.

In the case of $r = 50$, we experimented with Tikhonov regularization. The reconstruction errors with different regularization parameters are shown in Figure 3. We have calculated the (relative) reconstruction errors using the formula

$$\epsilon_p^{\alpha,s} = \frac{\|f - f_{\text{rec}}^{\alpha,s}\|_{L^p(\mathbb{T}^2)}}{\|f\|_{L^p(\mathbb{T}^2)}}. \tag{55}$$

The optimal regularization parameter values yielding the smallest error are given in Table 1. The plotted errors Figure 3 share some similarities in shape and the resulting regularization parameter values are close to each other.

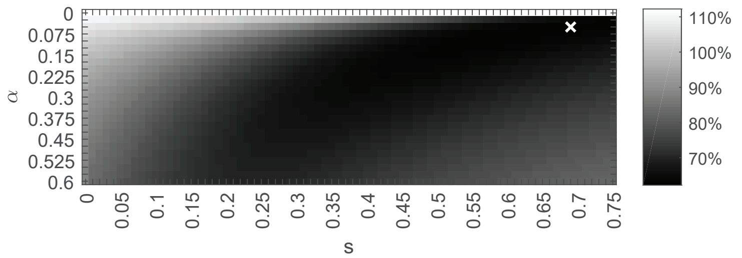
TABLE 1. The regularization parameters (α, s) that give the best reconstructions with respect to the L^p-norms with $p = 1, 2, \infty$; respective error $\epsilon_p^{\alpha,s}$; and error $\epsilon_p^{0,0}$ of non-regularized reconstruction.

norm	Shepp–Logan				Flag			
p	α	s	$\epsilon_p^{\alpha,s}$	$\epsilon_p^{0,0}$	α	s	$\epsilon_p^{\alpha,s}$	$\epsilon_p^{0,0}$
1	0.050	0.69	62%	112%	0.025	0.71	41%	69%
2	0.025	0.61	48%	70%	0.025	0.68	29%	45%
∞	0.025	0.56	75%	112%	0.025	0.78	73%	106%

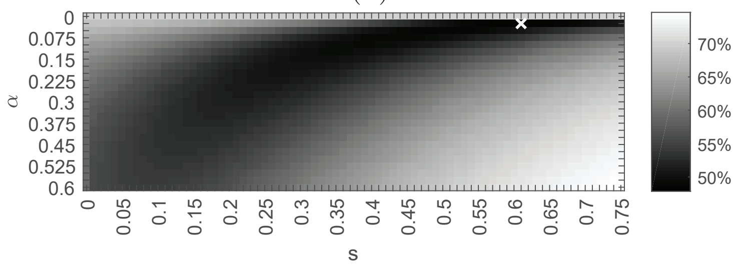
The Shepp–Logan phantom is shown in Figure 4a and its non-regularized solution in Figure 4b. The regularized solutions with $p = 2$ and $p = \infty$ based regularization parameter values (Figures 4d and 4e) are similar, and $p = 1$ based values yield slightly smoother reconstruction (Figure 4c).

We tested the effect of increasing the Fourier coefficient by computing the forward data required for reconstruction of the Fourier coefficients up to radii $r = 100$, $r = 150$ and $r = 200$, and reconstructions are shown in Figures 4f, 4g, and 4h respectively. The constant regions in the phantom become a bit more smoother, but overall dynamical range is increased and the impact of noise in reconstructions remains relatively high.

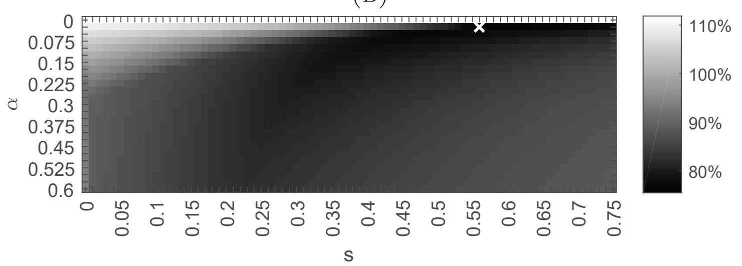
Similar analysis was also performed with the Flag phantom. We simulated noisy data using the model $y = \mathcal{A}_2 f + \mathcal{E}$ with the noise model of (54). The case $r = 50$ was used to test regularization. The reconstruction errors $\epsilon_p^{\alpha,s}$ are shown in Figure 5 and the regularization values yielding the minimum error are given in Table 1. The regions close to the minimum



(A)



(B)



(C)

FIGURE 3. Error surfaces from Shepp–Logan phantom data using (A) L^1-norm, (B) L^2-norm and (C) L^∞-norm. The values of the regularization parameters are $\alpha \in \{0, 0.025, 0.050, \ldots, 0.600\}$ and $s \in \{0, 0.01, 0.02, \ldots, 0.75\}$.

of $\epsilon_p^{\alpha,s}$ are more distinct than in the case of the Shepp–Logan phantom, but similar shape is seen.

The Flag phantom is shown in Figure 6a and the non-regularized reconstruction in Figure 6b. The regularized reconstructions with the optimal regularization parameters yielding the minimum errors with $p = 1$, $p = 2$ and $p = \infty$ are shown in Figures 6c, 6d and 6e, respectively. The regularization parameter values yielding the minimum were close to each other, and with the Flag phantom, no significant difference is seen in the regularized reconstructions.

Increasing the radius of the Fourier coefficients again increases the dynamical range, plotted in Figures 6f, 6g and 6h for $r \in \{100, 150, 200\}$, respectively. However, unlike with the Shepp–Logan phantom, the details become more distinct, especially the details of the corners in the Flag phantom.

4.2.2. *Forward model $\mathcal{A}_{\mathbb{T}^2}$ using the torus-projection and Radon data.* To test how a Torus CT algorithm would work with experimental data acquisition, we computed Radon transform of the phantoms and projected it to $\mathbb{T}^2$ using the model $\mathcal{A}_{\mathbb{T}^2}$ with noise on each data point on $Rf(v)_k$. More precisely, we simulated data according to the formula (49) where each $Rf(v)_k$ was replaced by noisy data $Rf(v)_k + \mathcal{E}$ where $\mathcal{E} \sim \mathcal{N}(0, \sigma^2)$ with $\sigma = 0.02$. This setup modeled experimental X-ray tomography as the starting point was Radon transform data with additive noise.

The projection directions for Radon transform were computed such that they determined the Fourier coefficients up to radius $r = 50$. An illustration of how the projection directions in $(0, 90)^\circ$ are distributed is shown in Figure 7, and the remaining projection directions are reflections of the projection directions in $(0, 90)^\circ$ about the y-axis. In total, with $r = 50$, there

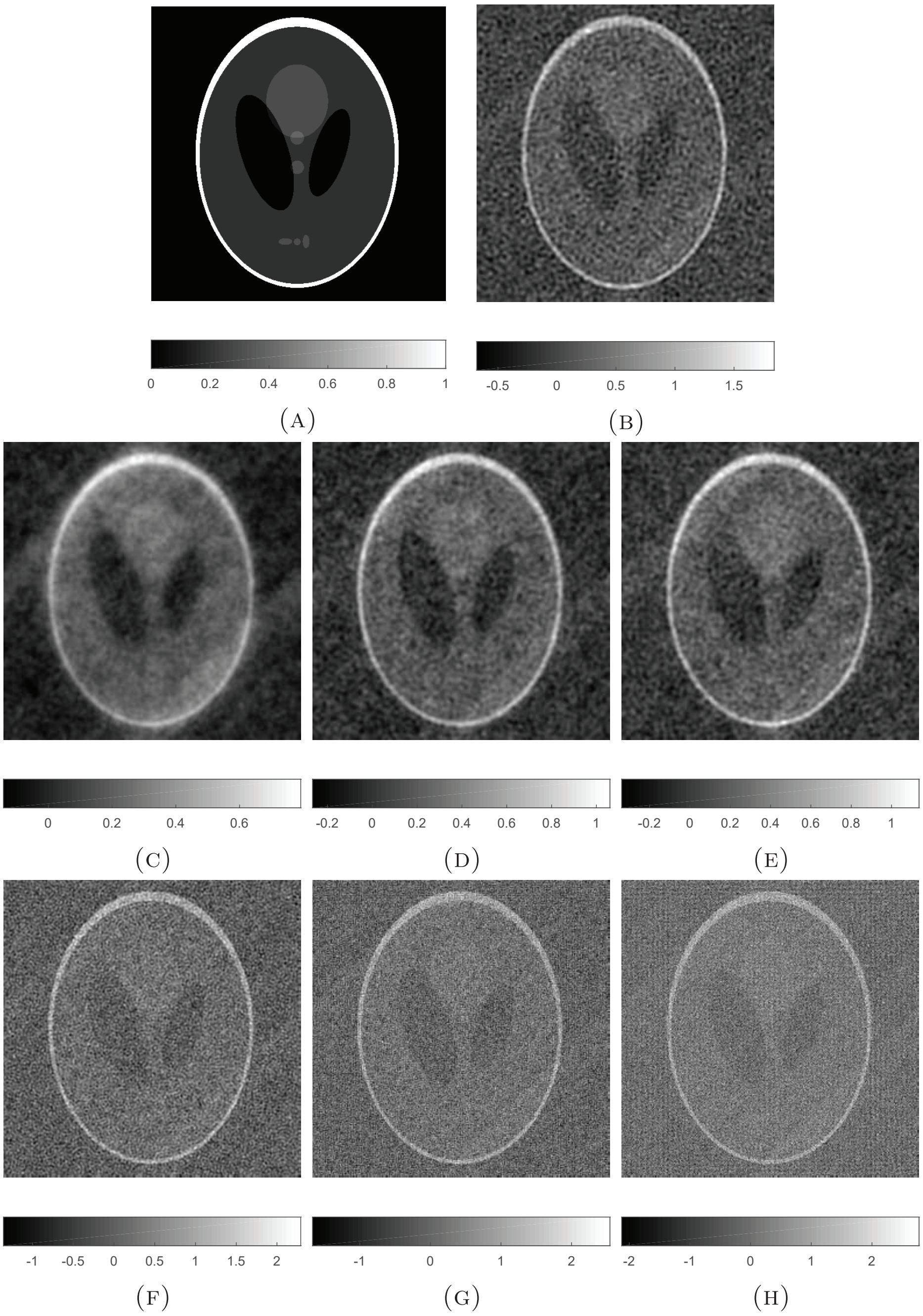


FIGURE 4. (A) Shepp–Logan phantom, (B) non-regularized reconstruction and (C-E) regularized reconstructions respectively with L^1-, L^2- and L^∞-norm based choice of reconstruction values. (F-H) Non-regularized reconstruction with increased cutoff radii of the Fourier series, $r = 100, 150, 200$, respectively.

are 3097 unique projection directions. Two major concentrations of the directions are close to 45°, both above and below, but also smaller concentrations are found elsewhere, e.g., close to 22.5°.

The reconstructions from data computed with $\mathcal{A}_{\mathbb{T}^2}$ are shown in Figure 8. Shepp–Logan (Figures 8a and 8d) and 30° rotated Flag (Figures 8c and 8f) are reconstructed well even with the noisy data but, surprisingly, the non-rotated Flag phantom (Figures 8b and 8e) is rather poor. Especially with the Shepp–Logan phantom, the features are clearly detected in the noise-free case (Figure 8a) indicating potential in the technique. Regularized solutions are shown in

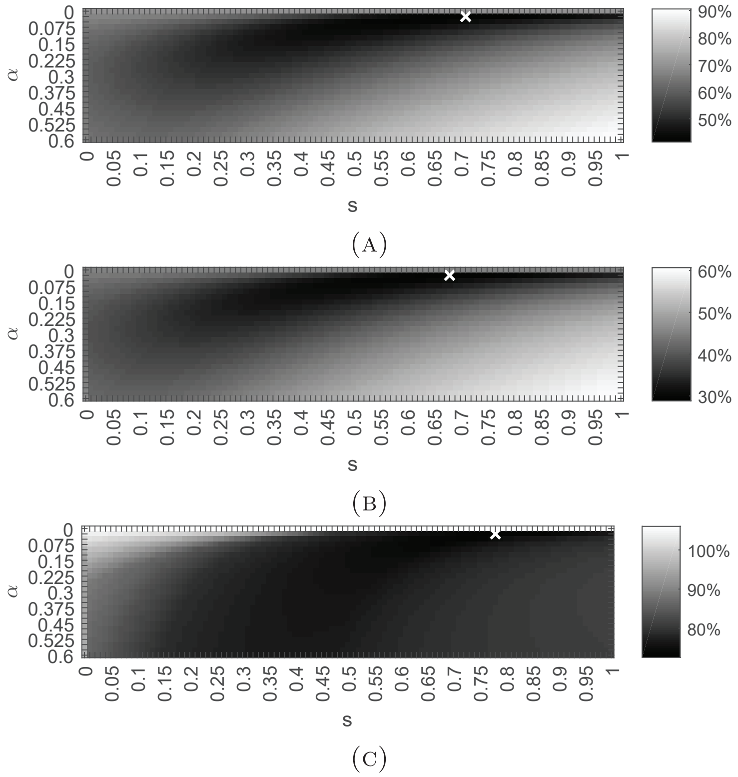


FIGURE 5. Error surfaces from Flag phantom data using (A) L^1-norm, (B) L^2-norm and (C) L^∞-norm. Regularization parameters values are $\alpha \in \{0, 0.025, 0.050, \ldots, 0.600\}$ and $s \in \{0, 0.01, 0.02, \ldots, 1.0\}$.

Figures 8g), 8h and 8i from the Shepp–Logan, the non-rotated and the rotated Flag phantoms, repectively. The regularization smoothed the reconstructions, decreased their dynamic range and no additional features were revealed from the noise. The regularization parameter values were $\alpha = 0.75$ and $s = 0.5$, chosen with manual experimentation.

TABLE 2. Errors in reconstructions computed with $\mathcal{A}_{\mathbb{T}^2}$ and the FBP.

	Shepp–Logan	Flag	Rotated Flag	Shepp–Logan	Flag	Rotated Flag
	$\mathcal{A}_{\mathbb{T}^2}$ with noiseless data ($\mathcal{E} = 0$)			$\mathcal{A}_{\mathbb{T}^2}$ with noisy data ($\mathcal{E} \sim \mathcal{N}(0, \sigma^2)$)		
$\epsilon_1^{0,0}$	313%	298%	302%	310%	300%	301%
$\epsilon_2^{0,0}$	161%	171%	168%	162%	172%	167%
$\epsilon_\infty^{0,0}$	75%	108%	121%	79%	125%	125%
				Regularized reconstruction from noisy data		
$\epsilon_1^{\alpha,s}$				331%	305%	303%
$\epsilon_2^{\alpha,s}$				170%	174%	173%
$\epsilon_\infty^{\alpha,s}$				53%	99%	100%
	FBP with torus optimized angles			FBP with evenly distributed angles		
ϵ_1	73%	59%	67%	64%	55%	56%
ϵ_2	59%	41%	51%	54%	45%	45%
ϵ_∞	155%	87%	127%	129%	93%	120%

For comparison, we computed the respective FBP reconstructions (shown in Figure 9) with Matlab's `iradon` function using default settings. The projection data $\mathbb{R}_{\texttt{radon}} f(v)_k + \mathcal{E}$ was down sampled by factor of 2 with `imresize` to match reconstruction resolution 256×256. It seems, that the uneven distribution of projection angles creates errors in reconstruction, since similar

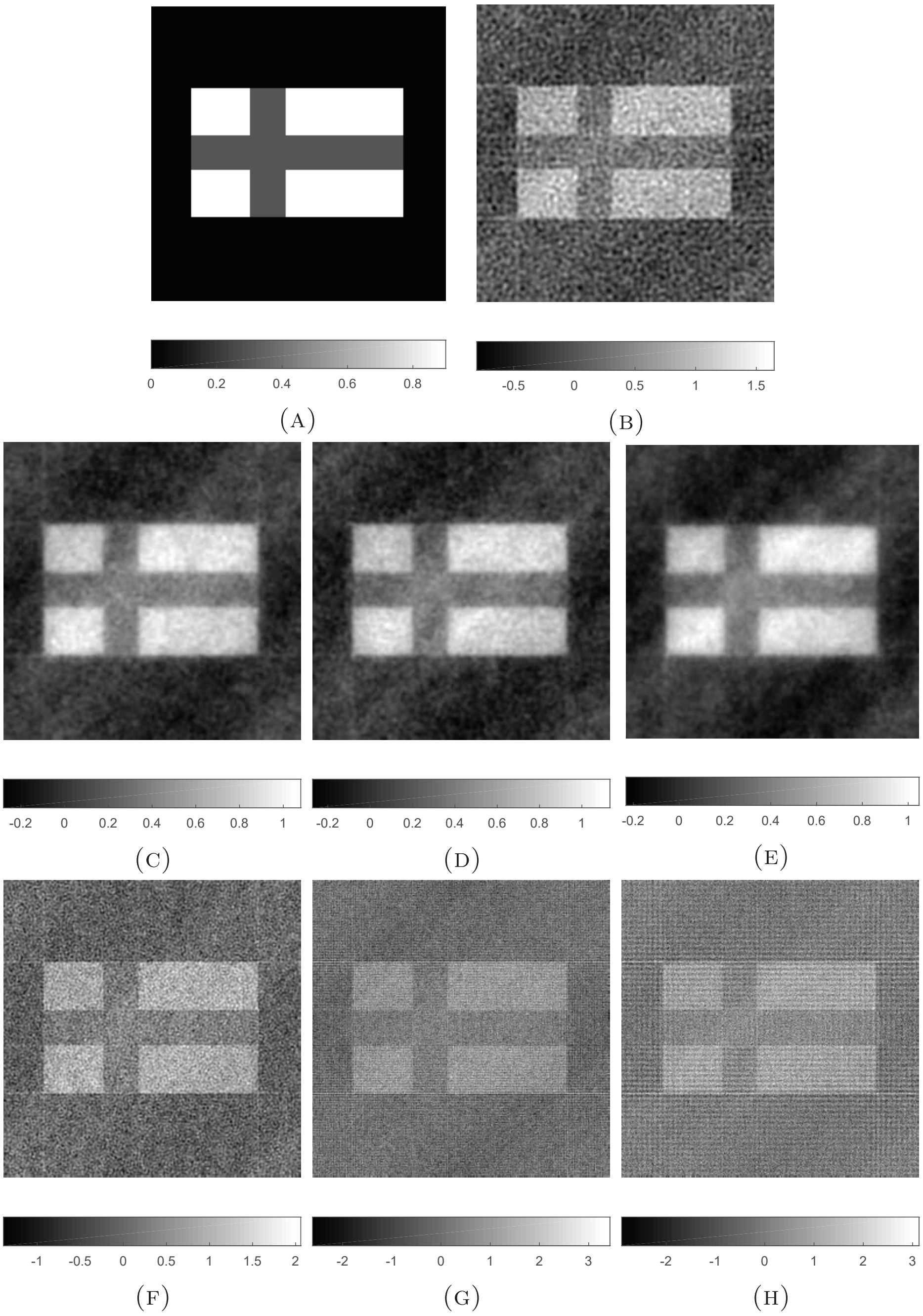
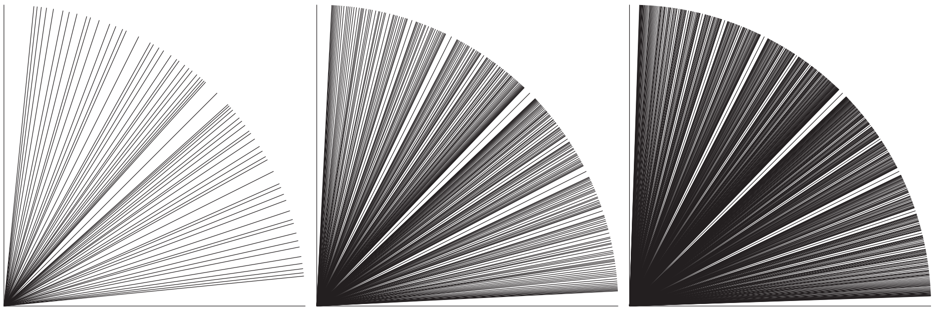


FIGURE 6. (A) Flag phantom, (B) non-regularized reconstruction and (C-E) regularized reconstructions respectively with L^1-, L^2- and L^∞-norm based choice of reconstruction values. (F-H) Non-regularized reconstruction with increased cutoff radii of the Fourier series, $r = 100, 150, 200$, respectively.

artefacts in horizontal, vertical and diagonal directions are seen also in the FBP reconstruction as in the ones computed with the Torus CT method in Figure 8. From the FBP this was expected as it is prone to streaking. In general, the FBP reconstructions are of good quality, since there is a lot of data available. With the same number of projections, 3097, but evenly distributed as they normally are, the FBP reconstruction are better quality than any other presented in this paper.

The error $\epsilon_p = \|f - f_{\text{rec}}^{\text{FBP}}\|_{L^p(\mathbb{R}^2)}/\|f\|_{L^p(\mathbb{R}^2)}$ between the FBP reconstruction $f_{\text{rec}}^{\text{FBP}}$ and the phantom f is tabulated in Table 2. When compared with $\mathcal{A}_1$ and $\mathcal{A}_2$ and Shepp–Logan and



(A) (B) (C)

FIGURE 7. Visualization of the Radon projection angles from 0° to 90° that are required in the reconstruction of the Fourier series coefficients up to radii $r = 10, 20, 30$, respectively. Each line represents a projection direction.

TABLE 3. Reconstruction errors $\epsilon_p^{0,0}$ of (non-regularized) reconstructions from rotational data sets.

Shepp–Logan	Θ_1^{SL}	Θ_2^{SL}	Θ_3^{SL}	Θ_4^{SL}	Θ_5^{SL}	Θ_6^{SL}	Θ_7^{SL}	Θ_8^{SL}	Θ_9^{SL}
$p = 1$	113%	82%	69%	61%	56%	52%	49%	46%	44%
$p = 2$	72%	54%	46%	42%	40%	38%	36%	34%	33%
$p = \infty$	97%	89%	84%	77%	77%	73%	70%	69%	68%
Flag	Θ_1^{F}	Θ_2^{F}	Θ_3^{F}	Θ_4^{F}	Θ_5^{F}	Θ_6^{F}			
$p = 1$	22%	18%	16%	14%	12%	12%			
$p = 2$	25%	21%	18%	16%	15%	14%			
$p = \infty$	96%	86%	79%	66%	68%	69%			

Flag phantoms, with all values of p, the errors ϵ_p are higher than errors of regularized Torus CT reconstructions $\epsilon_p^{\alpha,s}$ shown in Table 1. When compared with the errors of non-regularized reconstructions $\epsilon_1^{0,0}$ and $\epsilon_2^{0,0}$, the FBP and Torus CT are relatively close, but the error $\epsilon_\infty^{0,0}$ of Torus CT is lower with the Shepp–Logan phantom and higher with the Flag phantom.

For use with practical data acquisition, Torus CT requires more work to handle the increased additive noise in $\mathcal{A}_{\mathbb{T}^2}$, since the reconstructions have more noise outside of the support of the phantom than in the FBP reconstructions as seen in the Figure 8. The respective errors $\epsilon_1^{\alpha,s}$ and $\epsilon_2^{\alpha,s}$ are higher than those of the FBP, presented in Table 2. Nevertheless, in terms of reconstructing the correct dynamical range of the objects, measured with $\epsilon_\infty^{\alpha,s}$ and ϵ_∞, the Torus CT method is equivalent or better than with the FBP.

4.3. **Rotating phantom on torus.** The theoretical formulation allows to place a phantom inside $\mathbb{T}^2$ in many different positions. Such choice of an orientation of a phantom leads to different choice of projection directions, and thus results different reconstructions when only finitely many Fourier coefficients are recovered. This motivated to test, whether the reconstructions improve when data is acquired from several rotated phantoms. We verified this by computing the data of the Shepp–Logan phantom from nine different rotational orientations with 20° interval and the Flag phantom with six different rotational orientations with 30° interval with Fourier coefficient radius $r = 50$. Denote the angles of rotational data sets of the Shepp–Logan phantom with $\Theta_i^{SL} = \{(k-1)\cdot 20^\circ\,;\, k \in \{1,\dots,i\}\}$ and of the Flag phantom with $\Theta_i^F = \{(k-1)\cdot 30^\circ\,;\, k \in \{1,\dots,i\}\}$, $i \in \mathbb{N}$. The forward solutions were simulated as described in subsection 4.2 by taking count to the rotations Θ_i^{SL} and Θ_i^F. For the Shepp–Logan phantom, the rotation was computed using Matlab's `imrotate` with `crop` option for the Shepp–Logan phantom. For the Flag phantom, rotation about the point $(0.5, 0.5)$ was made by changing the coordinates in equations (51) and (52).

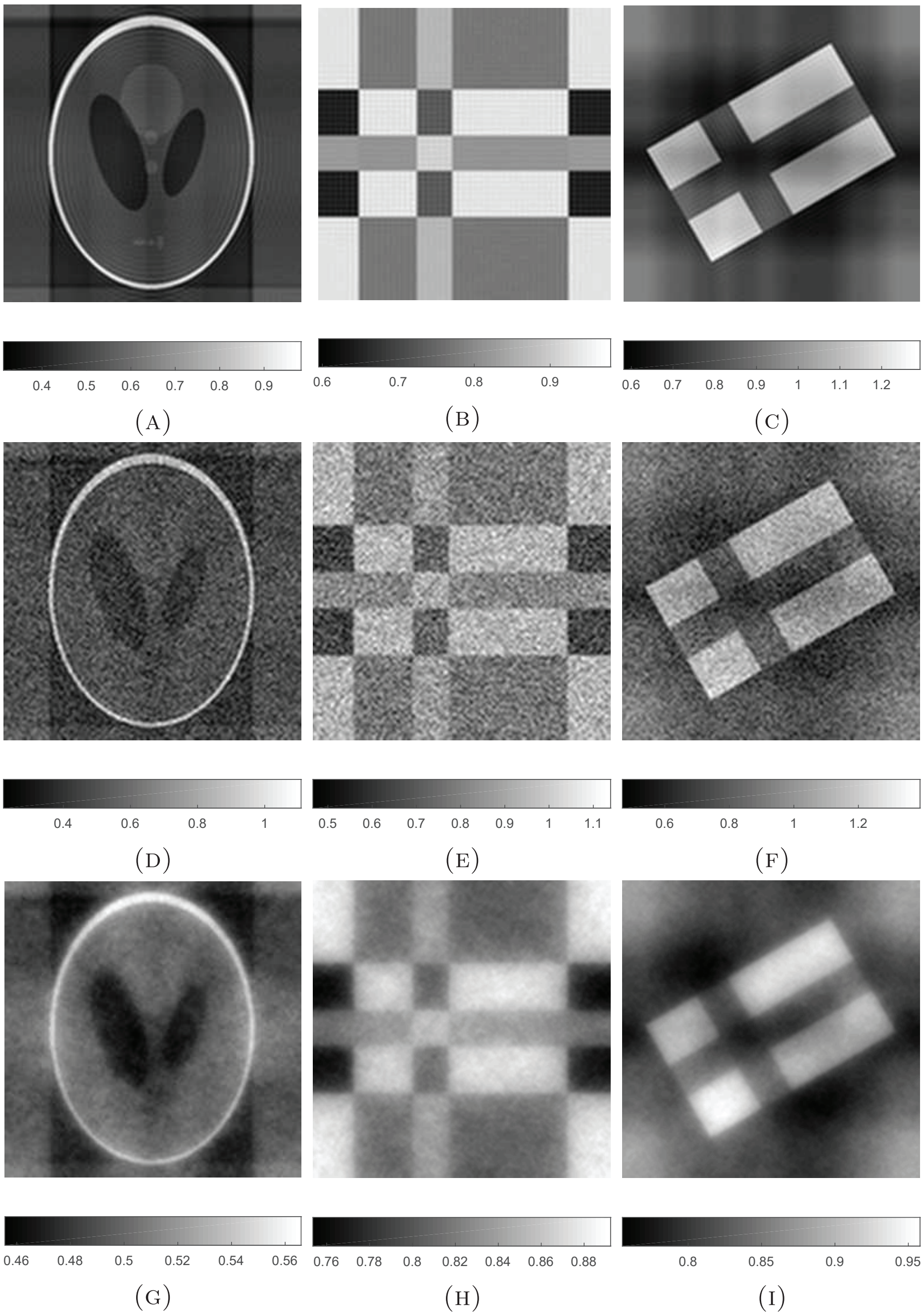


Figure 8. Reconstruction from (A)-(C) noise-less and (D)-(F) noisy Radon data mapped to torus of Shepp–Logan, Flag and Flag rotated 30°. (G)-(I) regularized reconstructions from respective noisy data.

We reconstructed the phantoms by calculating the independent reconstructions for each rotation in Θ_i^{SL} and Θ_i^{F}, and then by taking the average of the reconstructions $1, \ldots, i$ to be the final reconstruction. The reconstructions from the rotational data sets are shown in Figure 10 for the Shepp–Logan phantom and in Figure 11 for the Flag phantom. The reconstruction errors $\epsilon_p^{0,0}$ (eq. 55) are tabulated in Table 3 for both phantoms. With the Flag phantom, the reconstruction errors is computed only on the support of the flag phantom and the errors are ruled out at the points where the Flag phantom vanish. With the Shepp–Logan, $\epsilon_p^{0,0}$ was computed on the whole grid. The reconstructions were evaluated in a grid of 256×256 pixels.

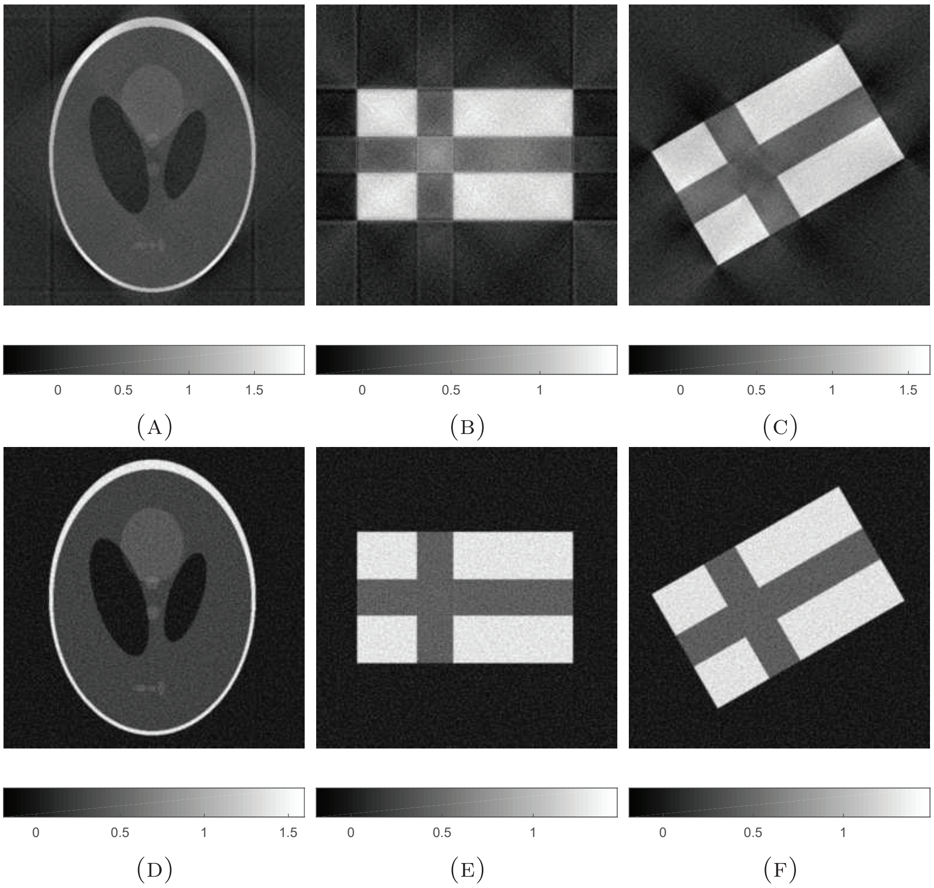
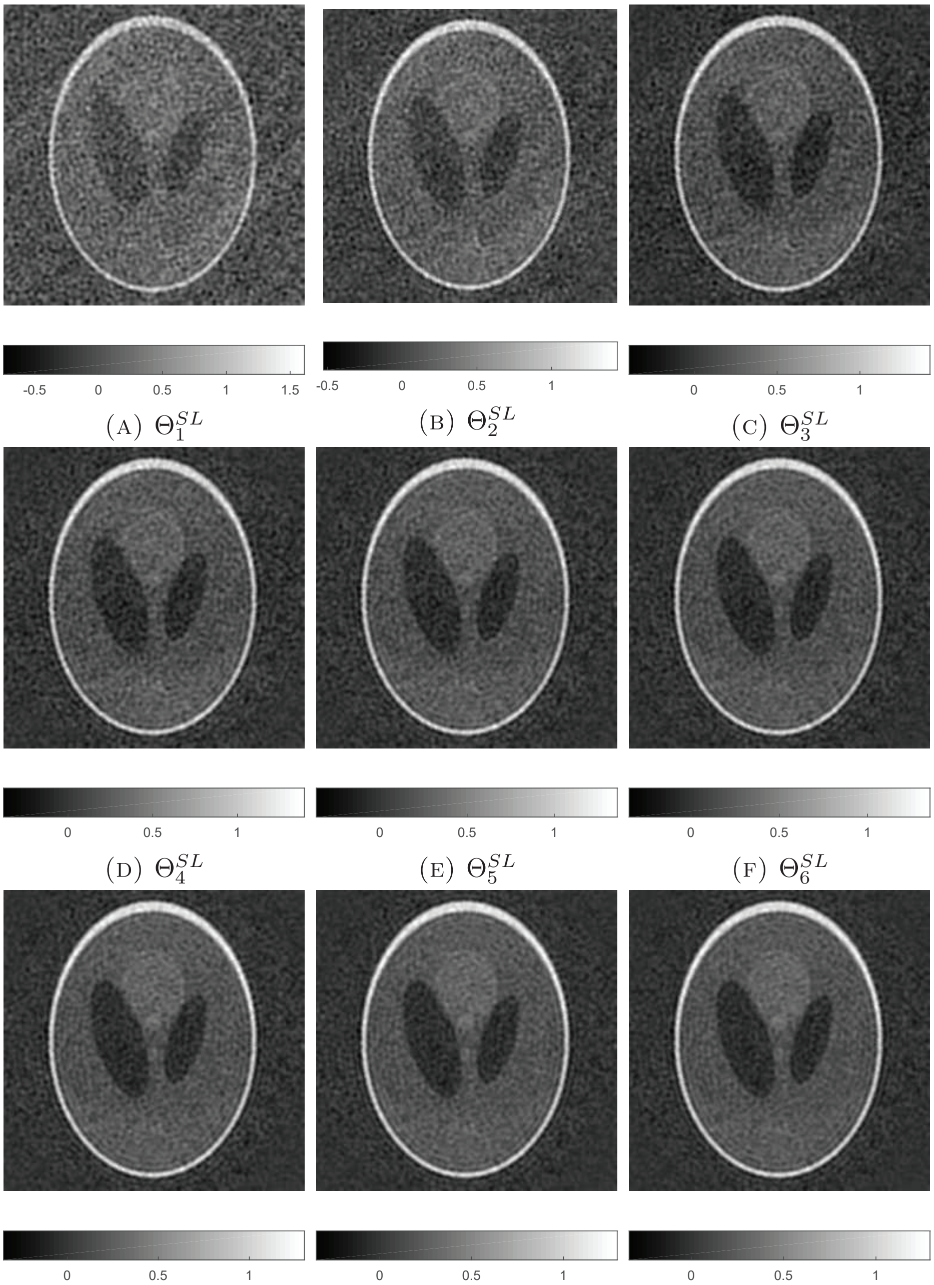


FIGURE 9. (A)-(C) Filtered backprojection reconstructions from Radon data with torus optimized angles. (D)-(F) Filtered backprojection reconstruction from equispaced projection angles with same amount of projections as in (A)-(C).

With the Shepp–Logan phantom there is clear visual improvement with the increase of rotationally acquired data and also decreasing trend in the errors. With the simpler Flag phantom, reconstruction quality seems to saturate as the decrease of error norms is not as clear as with the Shepp–Logan phantom. Nevertheless, the smallest error norm is given with the highest number of rotational data (shown in Table 3) and the visual evaluations support this.

The rotation of the phantom does contribute to the improvement of reconstructions. In other words, the improvement is not merely due to averaging out the zero mean noise. We simulated data from the same rotational orientation of the Shepp–Logan phantom nine times, and the errors were as follows: $\epsilon_1^{0,0} = 52\%$, $\epsilon_2^{0,0} = 37\%$ and $\epsilon_\infty^{0,0} = 66\%$. For the Flag phantom with six times from same rotational orientation, the errors were: $\epsilon_1^{0,0} = 43\%$, $\epsilon_2^{0,0} = 28\%$ and $\epsilon_\infty^{0,0} = 76\%$. In both cases the errors were higher than what was gained with different rotational orientations, except for $\epsilon_\infty^{0,0}$ with Shepp–Logan phantom where the error was almost equal. It should also be noted that the use `imrotate` induces some blurring during the rotation of reconstructions and the Shepp–Logan phantom and nonetheless rotational reconstructions performed better.

4.4. **Computing times.** The computing time of the forward system, i.e., data, depends mainly on the cutoff radius r of the Fourier series and on the number used geodesics in each direction (see discretization of x in 4.1.3). In terms of this paper, the radius r was more of the interest. Discretization relates to the numerical accuracy and the data acquisition accuracy of experimental setup. Example computing times t_r for data on Lenovo P51 laptop with Intel i7-7820HQ CPU and 32 GB of RAM having MATLAB R2017a (The MathWorks, Inc.) with the Shepp–Logan phantom and $\mathcal{A}_1$ are $t_{50} = 5.5$ min and $t_{100} = 51$ min; and with the Flag phantom and $\mathcal{A}_2$, $t_{50} = 100$ min and $t_{100} = 62$ min. On Lenovo P910 high-end workstation with



(A) Θ_1^{SL} (B) Θ_2^{SL} (C) Θ_3^{SL}

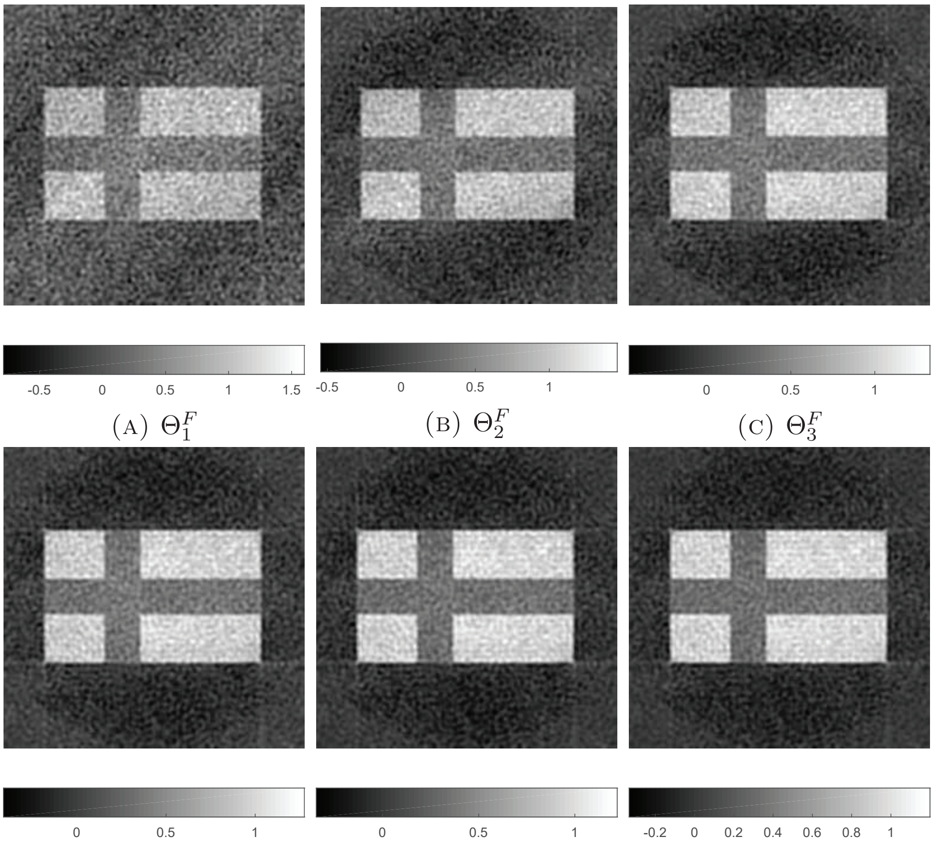
(D) Θ_4^{SL} (E) Θ_5^{SL} (F) Θ_6^{SL}

(G) Θ_7^{SL} (H) Θ_8^{SL} (I) Θ_9^{SL}

FIGURE 10. Reconstructions based on data sets of Shepp–Logan phantom rotated with respective angles.

two Intel Xeon E5-2697 processors and 256 GB RAM having MATLAB version R2016b 64-bit (The MathWorks, Inc.), the computation times were of order $t_{50} = 2.2$ min, $t_{100} = 15$ min, $t_{150} = 60$ min and $t_{200} = 188$ min with the Shepp–Logan phantom and $\mathcal{A}_1$; and $t_{50} = 1.6$ min, $t_{100} = 21$ min, $t_{150} = 79$ min and $t_{200} = 242$ min with the Flag phantom $\mathcal{A}_2$. The analytical integration applied when using the Shepp–Logan phantom with $\mathcal{A}_1$ explains its faster computations times.

The projection of Radon transform sinogram to the torus $\mathcal{A}_{\mathbb{T}^2}$ and its reconstruction lasted approximately eight minutes on Lenovo P51. However, the current implementation was not optimized at all and included, among other, three nested for-loops. Hence, here the computational efficiency will increase during further development.



(A) Θ_1^F (B) Θ_2^F (C) Θ_3^F

(D) Θ_4^F (E) Θ_5^F (F) Θ_6^F

FIGURE 11. Reconstructions based on data sets of the Flag phantom rotated with respective angles.

5. Conclusions

We have developed a theory for the X-ray transform on the flat torus for the purpose of implementation. Our theoretical results were strongly motivated by practical requirements, including new and computationally fast reconstruction formulas from X-ray data in theorem 1, and rigorous mathematical theory for Tikhonov regularized reconstructions from X-ray data on the flat torus in theorems 2 and 3. We further derived mathematical formulations of discretized forward and inverse models in section 3, and considered numerical analysis in section 2.6. We implemented a numerical Torus CT algorithm and performed simulation tests in Matlab which verified that the new theory could be applied in practice in section 4.

The numerical implementation demonstrated the efficacy of Torus CT. Torus CT is computationally relatively efficient compared with iterative techniques, though still slower than current implementations of the FBP. An interesting feature of Torus CT is its meshless nature: Once the Fourier coefficients are computed, the reconstruction can be evaluated in any desired grid points. Currently, theory and the implementation are established in 2 dimensions, which is suitable for slice-wise reconstructions of 3-dimensional structures. One future research direction could be the development of algorithms and theory in higher dimensional settings.

Data simulation was also computed with the traditional Radon transform corresponding to experimental image acquisition with projection angles preferred by the Torus CT. Here, reconstruction quality was promising. Some initial work has been conducted with conventional, evenly distributed projection angles, in which case there are various ways to interpolate projection data to directions preferred by Torus CT. It seems that rotations of a phantom could result sharp reconstructions and might allow reduction in the number of projection directions. This question should be studied more and with experimental X-ray data. We admit that at this stage the method cannot compete with the existing methods in visual accuracy and sharpness, though Torus CT reconstructs better than the FBP with respect to L^p norms. This improvement is due to a choice of optimal regularization parameters. In the future, one should study efficient rules of choosing regularization parameters from data without knowing a phantom a priori.

Supplementary material

Matlab code. We provide Creative Commons 4.0 licensed Matlab code files that implement forward model $\mathcal{A}_1$ (section 3.1.1) and inverse solutions on torus (section 3.2). The code package comprises of three files: `TorusCTrun.m` is the main script, `DFT.m` implements discrete Fourier transform (section 2.6), and `LineIntegralOnGrid.m` computes the exact line integral (48) over periodically extended, pixelized phantom. Files are available at [20].

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Fourier analysis of periodic Radon transforms

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FOURIER ANALYSIS OF PERIODIC RADON TRANSFORMS

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Abstract. We study reconstruction of an unknown function from its d-plane Radon transform on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ when $1 \leq d \leq n-1$. We prove new reconstruction formulas and stability results with respect to weighted Bessel potential norms. We solve the associated Tikhonov minimization problem on H^s Sobolev spaces using the properties of the adjoint and normal operators. One of the inversion formulas implies that a compactly supported distribution on the plane with zero average is a weighted sum of its X-ray data.

1. Introduction

We study reconstruction of an unknown function from its d-plane Radon transform on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ when $1 \leq d \leq n-1$. The d-plane Radon transform of a function f on $\mathbb{T}^n$ encodes the integrals of f over all periodic d-planes. The usual d-plane Radon transform of compactly supported objects on $\mathbb{R}^n$ can be reduced into the periodic d-plane Radon transform, but not vice versa. This was demonstrated for the geodesic X-ray transform in the recent work of Ilmavirta, Koskela and Railo [10]. As general references on the Radon transforms, we point to [5, 15, 6, 14].

Reconstruction formulas for integrable functions and a family of regularization strategies considered in this article were derived in [10] for the geodesic X-ray transform ($d = 1$) on $\mathbb{T}^2$. We extend these methods to the d-plane Radon transforms of higher dimensions, study new types of reconstruction formulas for distributions, and prove new stability estimates on the Bessel potential spaces. This article considers only the mathematical theory of Radon transforms on $\mathbb{T}^n$, whereas numerical algorithms (Torus CT) were implemented in [10, 13].

Injectivity, a reconstruction method and certain stability estimates of the d-plane Radon transform on $\mathbb{T}^n$ were proved for distributions by Ilmavirta in [7]. Our reconstruction formulas and stability estimates in this article are different than the ones in [7]. The first injectivity result for the geodesic X-ray transform on $\mathbb{T}^2$ was obtained by Strichartz in [19], and generalized to $\mathbb{T}^n$ by Abouelaz and Rouvière in [2] if the Fourier transform is $\ell^1(\mathbb{Z}^n)$. Abouelaz proved uniqueness under the same assumption for the d-plane Radon transform in [1].

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The X-ray transform and tensor tomography on $\mathbb{T}^n$ has been applied to other integral geometry problems. These examples include the broken ray transform on boxes [7], the geodesic ray transform on Lie groups [8], tensor tomography on periodic slabs [11], and the ray transforms on Minkowski tori [9]. We expect that the d-plane Radon transform on $\mathbb{T}^n$ has applications in similar and generalized geometric problems as well, but have not studied this possibility any further.

This article is organized as follows. The main results are stated in section 1.1. We recall preliminaries and prove some basic properties in section 2. We prove new inversion formulas in section 3. We prove our stability estimates and theorems on Tikhonov regularization in section 4.

1.1. **Results.** We describe our results next. Here we only briefly introduce the used notation, and more details are given in subsequent sections. One can also find more details in [7, 10]. Let $n, d \in \mathbb{Z}$ be such that $n \geq 2$ and $1 \leq d \leq n-1$. We define the *d-plane Radon transform* of $f \in C^\infty(\mathbb{T}^n)$ as

$$R_d f(x, A) := \int_{[0,1]^d} f(x + t_1 v_1 + \cdots + t_d v_d) dt_1 \cdots dt_d \tag{1}$$

where $A = \{v_1, \ldots, v_d\}$ is any set of linearly independent integer vectors $v_i \in \mathbb{Z}^n$.

It can be shown that A spans a periodic d-plane on $\mathbb{T}^n$, and on the other hand, any periodic d-plane on $\mathbb{T}^n$ has a basis of integer vectors. We can identify all periodic d-planes on $\mathbb{T}^n$ by the elements in the Grassmannian space $\mathbf{Gr}(d, n)$ which is the collection of all d-dimensional subspaces of $\mathbb{Q}^n$. We redefine the d-plane Radon transform on $\mathbb{T}^n$ as $R_d f : \mathbf{Gr}(d, n) \to \mathbb{C}^\infty(\mathbb{T}^n)$ without a loss of data. The definition of R_d extends to the periodic distributions $f \in \mathcal{T}'$ such that $R_d f(\cdot, A) \in \mathcal{T}'$ for any $A \in \mathbf{Gr}(d, n)$. We use the shorter notations $R_{d,A} f = R_d f(\cdot, A)$ and $X_{d,n} = \mathbb{T}^n \times \mathbf{Gr}(d, n)$. More details are given in section 2.1.

Let $w : \mathbb{Z}^n \times \mathbf{Gr}(d, n) \to (0, \infty)$ be a weight function such that $w(\cdot, A)$ is at most of polynomial decay (17) for any fixed $A \in \mathbf{Gr}(d, n)$. If not said otherwise, then a weight w is always assumed to be of this form. The associated Fourier multipliers on distributions are denoted by F_w. We denote the weighted Bessel potential space on the image side by $L^{p,l}_s(X_{d,n}; w)$ where $s \in \mathbb{R}$, $p, l \in [1, \infty]$. The usual Bessel potential spaces on $\mathbb{T}^n$ are denoted by $L^p_s(\mathbb{T}^n)$, and $H^s(\mathbb{T}^n) = L^2_s(\mathbb{T}^n)$ is the fractional L^2 Sobolev space. The $L^{p,l}_s(X_{d,n}; w)$ norms are ℓ^l norms over $\mathbf{Gr}(d, n)$ of the w-weighted Bessel potential norms of $L^p_s(\mathbb{T}^n; w(\cdot, A))$ with $A \in \mathbf{Gr}(d, n)$. More details are given in section 2.2.

We show that $L^{p,l}_s(X_{d,n}; w)$ are Banach spaces in lemma 2.1. Many of our results consider the Hilbert spaces with $p = l = 2$. Most of the theorems in this article would have been unreachable for R_d when $d < n-1$ if we do not include weights in the data spaces. We construct weights which satisfy the assumptions of our theorems in section 2.3.

Remark 1.1. If $d = n-1$, then weights are not that important for the analysis of R_d as demonstrated in the case of $n = 2, d = 1$ in [10], or for example in the special case of theorem 1.3.

Our first theorem considers the adjoint and the normal operators of $R_d : H^s(\mathbb{T}^n) \to L^{2,2}_s(X_{d,n}; w)$. This generalizes [10, Proposition 11] into higher dimensions. Theorem 1.1 and corollary 1.2 are proved in section 2.4.3.

Theorem 1.1 (Adjoint and normal operators)**.** *Let $s \in \mathbb{R}$ and suppose that there exists $C_w > 0$ such that*

$$\sum_{A\in\Omega_k} w(k,A)^2 \le C_w^2, \quad \Omega_k := \{\, A \in \mathbf{Gr}(d,n)\,;\, k \perp A \,\} \tag{2}$$

for any $k \in \mathbb{Z}^n$. Then the adjoint of $R_d : H^s(\mathbb{T}^n) \to L^{2,2}_s(X_{d,n}; w)$ is given by

$$\widehat{R_d^* g}(k) = \sum_{A\in\Omega_k} w(k,A)^2 \hat{g}(k,A) \tag{3}$$

and the normal operator $R_d^* R_d : H^s(\mathbb{T}^n) \to H^s(\mathbb{T}^n)$ is the Fourier multiplier $W_k := \sum_{A\in\Omega_k} w(k,A)^2$. In particular, the mapping $F_{W_k^{-1}} R_d^* : R_d(\mathcal{T}') \to \mathcal{T}'$ is the inverse of R_d.

Theorem 1.1 gives a new inversion formula in terms of the adjoint and a Fourier multiplier. Its corollary 1.2 gives new stability estimates on $H^s(\mathbb{T}^n)$. The stability estimates of R_1 on $H^s(\mathbb{T}^2)$ were not explicitly written down in [10] but they can be found between the lines. We denote by $R_d^{*,w}$ the adjoint of R_d associated to the weight w when the weight needs to be specified.

Corollary 1.2 (Stability estimates)**.** *Suppose that the assumptions of theorem 1.1 hold, and that there exists $c_w > 0$ such that $W_k \ge c_w^2$ for any $k \in \mathbb{Z}^n$.*

(i) Then $F_{W_k^{-1}} R_d^* : L^{2,2}_s(X_{d,n}; w) \to H^s(\mathbb{T}^n)$ is $1/c_w$-Lipschitz.

(ii) Let $f \in \mathcal{T}'$. Then

$$\|f\|_{H^s(\mathbb{T}^n)} \le \frac{1}{c_w} \|R_d f\|_{L^{2,2}_s(X_{d,n};w)}. \tag{4}$$

(iii) Let $\tilde{w}(k,A) = \frac{w(k,A)}{\sqrt{W_k}}$ and $p \in [1,\infty]$. Then $R_d^{*,\tilde{w}} R_d f = f$ and $\|f\|_{L^p_s(\mathbb{T}^n)} = \|R_d^{*,\tilde{w}} R_d f\|_{L^p_s(\mathbb{T}^n)}$ for any $f \in \mathcal{T}'$.

In order to prove $L^p_s \lesssim L^p_s$ type stability (iii) for more general weights in terms of the normal operator, one would have to show that $F_{W_k^{-1}}$ is a bounded L^p multiplier. Other stability estimates on $L^p_s(\mathbb{T}^n)$ are given in terms of $R_d f$ in proposition 4.3. These stability estimates follow from corollary 1.2 and the Sobolev inequality on $\mathbb{T}^n$. This method requires additional smoothness of $R_d f$ in order to control the norm of f due to the use of the Sobolev inequality.

We have proved three other new inversion formulas for R_d as well. The first two inversion formulas are given in proposition 3.1 and its corollary 3.3. Proposition 3.1 generalizes the inversion formula [10, Theorem 1] into higher dimensions. Its corollary 3.3 generalizes the formula for all periodic distributions using the structure theorem. We state the third inversion formula here since we find it to be the most interesting one.

Theorem 1.3 (The third inversion formula)**.** *Suppose that* $f \in \mathcal{T}'$*. Let* $w : \mathbb{Z}^n \times \mathbf{Gr}(d,n) \to \mathbb{R}$ *be a weight so that*

$$\sum_{A\in\Omega_k} w(k,A) = 1, \quad \Omega_k := \{\, A \in \mathbf{Gr}(d,n)\,;\, k \perp A \,\} \tag{5}$$

and the series is absolutely converging for any $k \in \mathbb{Z}^n$*. (The weight does not have to generate a norm or have at most of polynomial decay.) Then*

$$(f,h) = \sum_{A\in\mathbf{Gr}(d,n)} (F_{w(\cdot,A)} R_{d,A} f, h), \quad \forall h \in C^\infty(\mathbb{T}^n). \tag{6}$$

Moreover, if f *has zero average and* $d = n-1$*, then*

$$f = \sum_{A\in\mathbf{Gr}(d,n)} R_{d,A} f. \tag{7}$$

Remark 1.2. The author is not aware of a similar formula for the inverse Radon transform in earlier literature. We emphasize that this new result implies that a clever sum of the $(n-1)$-plane Radon transform data is the target function. If $n = 2$, this holds true for the X-ray transform of compactly supported functions on the plane $\mathbb{R}^2$. We further remark that it is easy to recover the average of a function and filter it out from $R_{n-1}f$.

Finally, we state our results on regularization. These results generalize [10, Theorems 2 and 3] into higher dimensions. Let $g \in L^{2,l}_r(X_{d,n}; w)$. We consider the *Tikhonov minimization problem*

$$\operatorname*{arg\,min}_{f\in H^t(\mathbb{T}^n)} \left(\|R_d f - g\|^l_{L^{2,l}_r(X_{d,n};w)} + \alpha \|f\|^2_{H^s(\mathbb{T}^n)} \right). \tag{8}$$

for any $n \geq 2$, $1 \leq d \leq n-1$, $\alpha > 0$, $l = 2$, and $r, s, t \in \mathbb{R}$. We do not fix the regularity of f a priori but the space $H^t(\mathbb{T}^n)$ will be found after solving the minimization problem for distributions in general.

Let us define

$$P^\alpha_{w,z} := \frac{1}{W_k + \alpha \langle k \rangle^{2z}} \tag{9}$$

as a Fourier multiplier associated to a weight w, $z \in \mathbb{R}$, and $\alpha > 0$.

Theorem 1.4 (Tikhonov minimization problem)**.** *Let* w *be a weight such that* $c_w^2 \leq W_k \leq C_w^2$ *for some uniform constants* $c_w, C_w > 0$*. Suppose that* $\alpha > 0$*, and* $s \geq r$*. Then the unique minimizer of the Tikhonov minimization problem (8) with* $g \in L^{2,2}_r(X_{d,n}; w)$ *is given by* $f = P^\alpha_{w,s-r} R^*_d g \in H^{2s-r}(\mathbb{T}^n)$*.*

The last theorem we state in the introduction generalizes the result [10, Theorem 3] on regularization strategies to higher dimensions.

Theorem 1.5 (Regularization strategy)**.** *Let* w *be a weight such that* $c_w^2 \leq W_k \leq C_w^2$ *for some uniform constants* $c_w, C_w > 0$*. Suppose* $r, t, s, \delta \in \mathbb{R}$ *are constants such that* $2s + t \geq r$*,* $\delta \geq 0$*, and* $s > 0$*. Let* $g \in L^{2,2}_t(X_{d,n}; w)$ *and* $f \in H^{r+\delta}(\mathbb{T}^n)$*.*

Then the Tikhonov regularized reconstruction operator $P^\alpha_{w,s} R^*_d$ *is a regularization strategy in the sense that*

$$\lim_{\epsilon\to 0} \sup_{\|g\|_{L^{2,2}_t(X_{d,n};w)} \leq \epsilon} \|P^{\alpha(\epsilon)}_{w,s} R^*_d (R_d f + g) - f\|_{H^r(\mathbb{T}^n)} = 0 \tag{10}$$

where $\alpha(\epsilon) = \sqrt{\epsilon}$ is an admissible choice of the regularization parameter.

Moreover, if $\|g\|_{L_t^{2,2}(X_{d,n};w)} \leq \epsilon$, $0 < \delta < 2s$, and $0 < \alpha \leq c_w^2(2s/\delta - 1)$, we have a quantitative convergence rate

$$
\begin{aligned}
&\|P^{\alpha}_{w,s} R_d^*(R_d f + g) - f\|_{H^r(\mathbb{T}^n)} \\
&\leq \alpha^{\delta/2s} c_w^{-\delta/s} C(\delta/2s) \|f\|_{H^{r+\delta}(\mathbb{T}^n)} + C_w^3 c_w^{-2} \frac{\epsilon}{\alpha}
\end{aligned} \tag{11}
$$

where $C(x) = x(x^{-1} - 1)^{1-x}$.

Remark 1.3. The optimal rate of convergence with respect to $\epsilon > 0$ can be found by choosing the regularization parameter $\alpha(\epsilon)$ so that the terms on the right hand side of (11) are of the same order.

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2. Preliminaries

2.1. **Periodic Radon transforms and Grassmannians.** We denote by $\mathcal{T}$ the set $C^\infty(\mathbb{T}^n)$ and $\mathcal{T}'$ its dual space, i.e. the space of periodic distributions. Denote by G_d^n the set of linearly independent unordered d-tuples in $\mathbb{Z}^n \setminus 0$. We may write any element $A \in G_d^n$ as $A = \{v_1, \dots, v_d\}$. The elements in the set G_d^n span all periodic d-planes on $\mathbb{T}^n$.

Suppose that $f \in \mathcal{T}$. We define the *d-plane Radon transform* of f as

$$
R_d f(x, A) := \int_{[0,1]^d} f(x + t_1 v_1 + \cdots + t_d v_d) dt_1 \cdots dt_d. \tag{12}
$$

We remark that $R_d : \mathcal{T} \to \mathcal{T}^{G_d^n}$, $R_d f : \mathbb{T}^n \times G_d^n \to \mathbb{C}$ and $R_d f(\cdot, A) : \mathbb{T}^n \to \mathbb{C}$.

Denote the duality pairing between $\mathcal{T}'$ and $\mathcal{T}$ by $(\cdot,\cdot)$. If $f, g \in \mathcal{T}$, it follows easily from Fubini's theorem that

$$
(f, R_d g(\cdot, A)) = (R_d f(\cdot, A), g). \tag{13}
$$

We define the d-plane Radon transform for any $f \in \mathcal{T}'$ and $A \in G_d^n$ simply as

$$
(R_d f(\cdot, A))(g) = (f, R_d g(\cdot, A)) \quad \forall g \in \mathcal{T}. \tag{14}
$$

This is the unique continuous extension of $R_d(\cdot, A)$ to the periodic distributions. The Fourier series coefficients of $R_d f(\cdot, A)$ are defined as usual.

We denote the *Grassmannian* of d-dimensional subspaces of $\mathbb{Q}^n$ by $\mathbf{Gr}(d, n)$. If $A, B \in G_d^n$ span the same subspace of $\mathbb{Q}^n$, then A and B represent the same element in $\mathbf{Gr}(d, n)$ and $R_d f(\cdot, A) = R_d f(\cdot, B)$ for any $f \in \mathcal{T}'$. On the other hand, for every $A \in \mathbf{Gr}(d, n)$ there exists $\tilde{A} \in G_d^n$ that spans A. This allows one to define the Radon transform as $R_d f : \mathbf{Gr}(d, n) \to \mathcal{T}'$ without data redundancy by setting $R_d f(\cdot, A) := R_d f(\cdot, \tilde{A})$ where $\tilde{A} \in G_d^n$ spans $A \in \mathbf{Gr}(d, n)$. This connection to the Grassmannians was mentioned earlier in [7] but was not directly used.

Remark 2.1. Let us denote the *projective space* $\mathbb{P}^{n-1} := \mathbf{Gr}(1,n)$. The *height* of $P \in \mathbb{P}^{n-1}$ is defined by $H(P) = \gcd(p)^{-1} |p|_{\ell^\infty}$ using any representative p of P. The projective space $\mathbb{P}^1$ and the height were used in [10] to analyze the number of projection directions required to reconstruct the Fourier series coefficients of a phantom up to a fixed radius. This question reduces to Schanuel's theorem [17] in algebraic number theory. This analysis in [10] extends to higher dimensions when $d = n - 1$.

2.2. **Bessel potential spaces and data spaces.** We define the *Bessel potential norms* for any $p \in [1,\infty]$ and $s \in \mathbb{R}$ by

$$\begin{aligned} \|f\|_{L^p_s(\mathbb{T}^n)} &= \| \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s \hat{f}(k) e^{2\pi i k\cdot x} \|_{L^p(\mathbb{T}^n)}, \\ \|f\|_{H^s(\mathbb{T}^n)} &= \sqrt{\sum_{k\in\mathbb{Z}^k} \langle k\rangle^{2s} \left|\hat{f}(k)\right|^2} \end{aligned} \tag{15}$$

where $\langle k\rangle = (1+|k|^2)^{1/2}$ as usual (see e.g. [3]). The space $L^p_s(\mathbb{T}^n) \subset \mathcal{T}'$ consists of all $f \in \mathcal{T}'$ with $\|f\|_{L^p_s(\mathbb{T}^n)} < \infty$. If $p = 2$, then $H^s(\mathbb{T}^n) = L^p_s(\mathbb{T}^n)$ is the fractional L^2 Sobolev space. One has equivalently that $f \in L^p_s(\mathbb{T}^n)$ if and only if $(1-\Delta)^{s/2} f \in L^p(\mathbb{T}^n)$ and $f \in \mathcal{T}'$. The Bessel potential spaces are used as domains of R_d in this work, which extends studies of the case $p = 2$ in [7, 10].

If $\omega : \mathbb{Z}^n \to (0,\infty)$ and $f \in \mathcal{T}'$, then we define the ω-weighted norms by

$$\|f\|_{L^p_s(\mathbb{T}^n;\omega)} := \|F_\omega f\|_{L^p_s(\mathbb{T}^n)} \tag{16}$$

where F_ω is the Fourier multiplier of ω. We say that a weight $\omega : \mathbb{Z}^n \to (0,\infty)$ is *at most of polynomial decay* if there exists $C, N > 0$ such that

$$\omega(k) \geq C \langle k\rangle^{-N} \quad \forall k \in \mathbb{Z}^n. \tag{17}$$

We next define suitable data spaces that contain ranges of R_d when its domains are restricted to the Bessel potential spaces. Let us denote $X_{d,n} := \mathbb{T}^n \times \mathbf{Gr}(d,n)$ to keep our notation shorter. We generalize the data space given in [10] to all $n \geq 2$, $1 \leq d \leq n-1$, and $p \in [1,\infty]$, using the Grassmannians, the Bessel potential spaces and weights.

Let $1 \leq d \leq n-1$ and $w : \mathbb{Z}^n \times \mathbf{Gr}(d,n) \to (0,\infty)$ be a weight function such that $w(\cdot, A)$ is at most of polynomial decay for any fixed $A \in \mathbf{Gr}(d,n)$. We always assume in this work that the weight is at most of polynomial decay. We say that a (generalized) function $g : X_{d,n} \to \mathbb{C}$ belongs to $L^{p,l}_s(X_{d,n}; w)$ with $1 \leq l < \infty$ if the norm

$$\|g\|^l_{L^{p,l}_s(X_{d,n};w)} := \sum_{A\in\mathbf{Gr}(d,n)} \|g(\cdot,A)\|^l_{L^p_s(\mathbb{T}^n;w(\cdot,A))} \tag{18}$$

is finite and $g(\cdot, A) \in \mathcal{T}'$ for any fixed $A \in \mathbf{Gr}(d,n)$. Similarly, if $l = \infty$, we define

$$\|g\|_{L^{p,\infty}_s(X_{d,n};w)} := \sup_{A\in\mathbf{Gr}(d,n)} \|g(\cdot,A)\|_{L^p_s(\mathbb{T}^n;w(\cdot,A))} \tag{19}$$

In the above definition, one can replace $\mathbf{Gr}(d,n)$ by any countable set Y (cf. lemma 2.1).

If $p, l = 2$, then the norm is generated by the inner product

$$(h, g)_{L_s^{2,2}(X_{d,n};w)} := \sum_{A \in \mathbf{Gr}(d,n)} (F_{w(\cdot,A)} h, F_{w(\cdot,A)} g)_{H^s(\mathbb{T}^n)} \tag{20}$$

which makes $L_s^{2,2}(X_{d,n}; w)$ a Hilbert space. We prove that the spaces $L_s^{p,l}(X_{d,n}; w)$ are Banach spaces in lemma 2.1. We emphasize that a weight does not have to have uniform coefficients for its at most of polynomial decay with respect to $\mathbf{Gr}(d, n)$.

There is a connection to the norms used in [10]. Let w be any weight such that $\sum_{A \in \mathbf{Gr}(1,2)} w(0, A)^2 = 1$, and $w(k, A) \equiv 1$ if $k \neq 0$. Now the results in [10] follow from the results of this article using the norm $L_s^{2,2}(X_{1,2}; w)$ as the image side spaces in [10] are contained in $L_s^{2,2}(X_{1,2}; w)$.

Yet another norm was used for the stability estimates in [7]. In the cases $d = n - 1$ and $l = \infty$, our analysis of R_d would not require weights, and can be performed similarly to [7, 10]. The analysis of $R_d|_{L_s^p(\mathbb{T}^n)}$ has not been done before if $p \neq 2$. The Bessel potential norms on the domain side are used to understand better the mapping properties of R_d.

We state and prove the following lemma for the sake of completeness. We remark that without the decay condition on weights these weighted spaces would not be complete.

Lemma 2.1. *Let Y be a countable set. Let $w : \mathbb{Z}^n \times Y \to (0, \infty)$ be a weight that is at most of polynomial decay for any fixed $y \in Y$. Suppose that $s \in \mathbb{R}, p, l \in [1, \infty]$, and $n \geq 1$. Then $L_s^{p,l}(\mathbb{T}^n \times Y; w)$ is a Banach space. In particular, $L_s^{2,2}(\mathbb{T}^n \times Y; w)$ is a Hilbert space.*

Proof. Suppose that $1 \leq l < \infty$. (If $l = \infty$, the proof is similar.) We first show that $L_s^{p,l}(\mathbb{T}^n \times Y; w)$ is a vector space. Let $c \in \mathbb{C}$ and $f, g \in L_s^{p,l}(\mathbb{T}^n \times Y; w)$. We have trivially that

$$\|cf\|^l_{L_s^{p,l}(\mathbb{T}^n \times Y;w)} = |c|^l \sum_{y \in Y} \|f(\cdot, y)\|^l_{L_s^p(\mathbb{T}^n;w)}. \tag{21}$$

The Minkowski and triangle inequalities imply

$$\begin{aligned} \|f + g\|_{L_s^{p,l}(\mathbb{T}^n \times Y;w)} &= \left(\sum_{y \in Y} \|F_{w(\cdot,y)} f(\cdot, y) + F_{w(\cdot,y)} g(\cdot, y)\|^l_{L_s^p(\mathbb{T}^n)} \right)^{1/l} \\ &\leq \|f\|_{L_s^{p,l}(\mathbb{T}^n \times Y;w)} + \|g\|_{L_s^{p,l}(\mathbb{T}^n \times Y;w)}. \end{aligned} \tag{22}$$

This shows that $L_s^{p,l}(\mathbb{T}^n \times Y; w)$ is a vector subspace of all collections of distributions $\{f(\cdot, y)\}_{y \in Y}$ with $f(\cdot, y) \in \mathcal{T}'$.

We show next that $L_s^{p,l}(\mathbb{T}^n \times Y; w)$ is a complete space. Let $f_i \in L_s^{p,l}(\mathbb{T}^n \times Y; w)$ be a Cauchy sequence. It follows from the definition of the norm in $L_s^{p,l}(\mathbb{T}^n \times Y; w)$ that $f_i(\cdot, y) \in L_s^p(\mathbb{T}^n; w(\cdot, y))$ is a Cauchy sequence for any $y \in Y$. Suppose that each $L_s^p(\mathbb{T}^n; w(\cdot, y))$ is complete. It follows that $f_i(\cdot, y) \to f_y \in L_s^p(\mathbb{T}^n; w(\cdot, y))$ as $i \to \infty$. This implies that there exists a limit of f_i in $L_s^{p,l}(\mathbb{T}^n \times Y; w)$ by standard arguments.

Let us prove that $L^p_s(\mathbb{T}^n; w(\cdot, y))$ is complete for any $y \in Y$. Take a Cauchy sequence $f_i \in L^p_s(\mathbb{T}^n; w(\cdot, y))$. Now it follows that the functions

$$g_i(x) = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s w(k, y)\hat{f}_i(k)e^{2\pi ik\cdot x} \tag{23}$$

are in $L^p(\mathbb{T}^n)$ and form a Cauchy sequence. Therefore $\lim_{i\to\infty} g_i =: g$ exists. We claim that the distribution defined on the Fourier side as $\hat{f}(k) := \frac{\hat{g}(k)}{\langle k\rangle^s w(k,y)}$ is the limit of f_i in $L^p_s(\mathbb{T}^n; w(\cdot, y))$.

We need to show two things, that $f \in \mathcal{T}'$ and $\|f_i - f\|_{L^p_s(\mathbb{T}^n;w(\cdot,y))} \to 0$ as $i \to \infty$. We have that

$$\begin{aligned} \|f_i - f\|_{L^p_s(\mathbb{T}^n;w(\cdot,y))} &= \| \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s w(k, y)(\widehat{f_i - f})(k)e^{2\pi ik\cdot x}\|_{L^p(\mathbb{T}^n)} \\ &= \| \sum_{k\in\mathbb{Z}^n} [\langle k\rangle^s w(k, y)\hat{f}_i(k) - \hat{g}(k)]e^{2\pi ik\cdot x}\|_{L^p(\mathbb{T}^n)} \end{aligned} \tag{24}$$

for any $i \in \mathbb{N}$. Therefore, $\|f_i - f\|_{L^p_s(\mathbb{T}^n;w(\cdot,x))} \to 0$ as $i \to \infty$.

It is enough that the Fourier coefficients of f have polynomial growth by the structure theorem of periodic distributions [18, Chapter 3.2.3]. We have $|\hat{g}(k)| \leq C_1 \langle k\rangle^\alpha$ for some $\alpha, C_1 > 0$ since $g \in L^p(\mathbb{T}^n) \subset \mathcal{T}'$. On the other hand, we assumed that $w(k, y) \geq C_2 \langle k\rangle^{-N}$ for some $C_2, N > 0$. Hence, we obtain that

$$\left|\hat{f}(k)\right| = \left|\frac{\hat{g}(k)}{\langle k\rangle^s w(k, y)}\right| \leq (C_1/C_2) \langle k\rangle^{\alpha+N-s} . \tag{25}$$

This shows that $f \in \mathcal{T}'$. □

Remark 2.2. One uses the fact that weights have at most of polynomial decay only to show that the limits of Cauchy sequences are in $\mathcal{T}'$. One could also allow more rapid decay for weights but in that case, the objects of the completion would not be distributions but ultra-distributions [18]. In the analysis of R_d, such generality seems to be unnecessary and our assumptions avoid this.

2.3. **On constructions of weights.** In this section, we discuss how to construct weights that satisfy the assumptions of our theorems. The weights of this paper are of the form $w : \mathbb{Z}^n \times \mathbf{Gr}(d, n) \to (0, \infty)$ with the following properties.

(i) For any $A \in \mathbf{Gr}(d, n)$ there exists $C, N > 0$ such that $w(k, A) \geq C \langle k\rangle^{-N}$ for every $k \in \mathbb{Z}^n$.
(ii) There exists $C > 0$ such that $W_k \leq C$ for every $k \in \mathbb{Z}^n$ where $W_k = \sum_{A\in\Omega_k} w(k, A)^2$ and $\Omega_k = \{ A \in \mathbf{Gr}(d, n) ; k\perp A \}$.
(iii) There exists $c > 0$ such that $c \leq W_k$ for every $k \in \mathbb{Z}^n$.

The property (i) is assumed for any weight in this article to guarantee that $L^{p,l}_s(X_{d,n}; w)$ are Banach spaces. The property (ii) is assumed for most of the weights to guarantee that $R_d : L^p_s(\mathbb{T}^n) \to L^{p,l}_s(X_{d,n}; w)$ is continuous (with some restrictions if $p, l \neq 2$). The property (iii) is additionally assumed to prove the stability estimates and the theorems on regularization.

First of all, it is very easy to construct weights that satisfy (i) alone. It is not hard to construct weights that satisfy (i) and (ii). Since the set $\mathbf{Gr}(d, n)$

is countable, we may write it with an enumeration $\varphi : \mathbf{Gr}(d,n) \to \mathbb{N}$. For example, we construct a weight $w(k,A) = 2^{-\varphi(A)} \langle k \rangle^{-N}$ with large enough $N > 0$ chosen such that $\sum_{k\in\mathbb{Z}^n} \langle k \rangle^{-2N} < \infty$. Then $\sum_{A\in\mathbf{Gr}(d,n)} \sum_{k\in\mathbb{Z}^n} w(k,A)^2 < C$ for some $C > 0$. This shows that both conditions (i) and (ii) hold.

We give next a nontrivial example of a weight satisfying (ii) and (iii) but not (necessarily) (i). Let $\varphi_k : \Omega_k \to \mathbb{N}$ be an enumeration. Let $Q := \{ (k,A) \in \mathbb{Z}^n \times \mathbf{Gr}(d,n)\, ;\, A \in \Omega_k \}$. For any $(k,A) \in Q$, we define the weight $w(k,A) := \frac{h(k)}{\varphi_k(A)^{1/2+\epsilon}}$ with some mapping $h : \mathbb{Z}^n \to (a,b)$ with $0 < a \le b < \infty$ and $\epsilon > 0$. If $(k,A) \notin Q$, we set $w(k,A) = 1$. One has that $|\Omega_k| = \infty$ if $1 \le d < n-1$ or $k = 0$, and $|\Omega_k| = 1$ if $d = n-1$ and $k \ne 0$. Now

$$\sum_{A\in\Omega_k} w(k,A)^2 = h^2(k) \sum_{i=1}^{|\Omega_k|} i^{-1-2\epsilon}. \tag{26}$$

Hence, we get that $a^2 \le W_k \le Cb^2$ where $C = \sum_{i=1}^{\infty} i^{-1-2\epsilon}$.

The problem gets more difficult if the all three conditions must be satisfied at the same time. We solve this problem now by combining ideas from the both constructions above. We make a proposition about a concrete example, and more general methods are summarized in remarks 2.3 and 2.4.

Proposition 2.2. *Let $\varphi_k : \Omega_k \to \mathbb{N}$ be an enumeration for any $k \in \mathbb{Z}^n$, and let $\varphi : \mathbf{Gr}(d,n) \to \mathbb{N}$ be an enumeration. Let $h : \mathbb{Z}^k \to (a,b)$ with $0 < a \le b < \infty$ and $g(k) = \langle k \rangle^{-N}$ for some $N \ge 0$. Then the weight*

$$w(k,A) := \begin{cases} \frac{h(k)}{\varphi_k(A)} + \frac{g(k)}{\varphi(A)} & (k,A) \in Q \\ 1 & (k,A) \in Q^c \end{cases} \tag{27}$$

satisfies the properties (i), (ii) and (iii).

Proof. Using the definition (27) and positivity of the involved functions, we have that

$$W_k \ge h^2(k) \sum_{A\in\Omega_k} \varphi_k(A)^{-2} = h^2(k) \sum_{i=1}^{|\Omega_k|} i^{-2} \ge a^2. \tag{28}$$

This shows (iii).

Suppose that $(k,A) \in Q$. We use

$$\frac{1}{2} w(k,A)^2 \le \frac{h^2(k)}{\varphi_k(A)^2} + \frac{g^2(k)}{\varphi(A)^2} \tag{29}$$

to estimate W_k from above. The formula (29) gives

$$\frac{1}{2} W_k \le \sum_{A\in\Omega_k} \left(\frac{h^2(k)}{\varphi_k(A)^2} + \frac{g^2(k)}{\varphi(A)^2} \right) \le h^2(k) \sum_{i=1}^{|\Omega_k|} i^{-2} + \langle k \rangle^{-2N} \sum_{i=1}^{|\Omega_k|} i^{-2}. \tag{30}$$

Since $\langle k \rangle^{-2N} \le 1$ and $h(k) \le b$ for any $k \in \mathbb{Z}^n$, we obtain that $W_k \le 2C(1+b^2)$ where $C = \sum_{i=1}^{\infty} i^{-2} < \infty$. This shows (ii).

Using the definition (27) and positivity of the involved functions, we can directly estimate that

$$|w(k,A)| \ge \min\{1, \frac{1}{\varphi(A)} \langle k \rangle^{-N}\} = \frac{1}{\varphi(A)} \langle k \rangle^{-N}. \tag{31}$$

This shows that $w(\cdot, A)$ is at most of polynomial decay (i). □

Remark 2.3. Proposition 2.2 generalizes for $w(k, A)|_Q = h(k)\psi(k, A) + g(k)\omega(A)$ with the conditions that $h(k)$ is bounded from above and below, $g(k)$ has at most of polynomial decay and is bounded above, the sums of $\omega(A)^2$ over Ω_k are uniformly bounded from above, and the sums of $\psi(k, A)^2$ over Ω_k are uniformly bounded from below and above.

Remark 2.4. If a weight w satisfies the conditions (i) and (ii), then it can be normalized as $\tilde{w}(k, A) := \frac{w(k,A)}{\sqrt{W_k}}$. The normalized weight $\tilde{w}$ has the property that $\tilde{W}_k = 1$ for any $k \in \mathbb{Z}^n$. Moreover, since $w(k, A)$ is at most of polynomial decay and $\sqrt{W_k} \leq C$ for some $C > 0$, it follows that $\tilde{w}$ is at most of polynomial decay.

We can construct weights that satisfty the assumptions of theorem 1.3 by defining $w(k, A) = 2^{-\varphi_k(A)}$ for any $(k, A) \in Q$ and $w(k, A) = 1$ if $(k, A) \notin Q$. If $d < n - 1$, then $\sum_{A\in\Omega_k} w(k, A) = 1$ for any $k \in \mathbb{Z}^n$, and the series $\sum_{A\in\Omega_k} w(k, A)$ are absolutely convergent.

2.4. **Basic properties of periodic Radon transforms.** In this section, we state and prove some basic properties of R_d. Some of these properties were used earlier in the special cases in [7, 10]. We have chosen to include most of the proofs here for completeness.

2.4.1. *Periodic Radon transforms for integrable functions.* Let $T = (t_1, \ldots, t_d) \in \mathbb{R}^d$ and $A = \{v_1, \ldots, v_d\} \in G^n_d$. We can define $R_d f(\cdot, A)$ for $L^1(\mathbb{T}^n)$ functions simply as

$$R_{d,A}f(x) := \int_{[0,1]^d} f(x + t_1v_1 + \cdots + t_dv_d)dt_1 \cdots dt_d \tag{32}$$

where the formula is defined for a.e. $x \in \mathbb{T}^n$. We lighten our notation by denoting the corresponding linear combinations by $T \cdot A = t_1v_1 + \cdots + t_dv_d$ with respect to the enumeration of A. The following basic properties are valid.

Lemma 2.3. *Suppose that $f \in L^1(\mathbb{T}^n)$ and $A \in G^n_d$. Then $R_{d,A}f$ can be defined by the formula (32) for a.e. $x \in \mathbb{T}^n$. Moreover,*

- *(i) this definition coincides with the distributional definition: for every $f \in L^1(\mathbb{T}^n)$ and $g \in L^\infty(\mathbb{T}^n)$ it holds that $(R_{d,A}f, g) = (f, R_{d,A}g)$;*
- *(ii) $R_{d,A} : L^p(\mathbb{T}^n) \to L^p(\mathbb{T}^n)$ is 1-Lipschitz for any $p \in [1, \infty]$.*
- *(iii) Suppose that $f \in \mathcal{T}'$, $A \in G^n_d$ and $R_d f(\cdot, A) \in L^1(\mathbb{T}^n)$. Then $R_{d,A}f(x + S \cdot A) = R_{d,A}f(x)$ for a.e. $x \in \mathbb{T}^n$ and every $S \in \mathbb{R}^d$.*

We postpone the proof of lemma 2.3 for a while. We remark that lemma 2.3 is a simple generalization of [10, Lemma 7], which was stated in [10] without a proof. We need to first introduce some useful notations.

Let $q = n-d$ and V be the linear subspace of $\mathbb{R}^n$ spanned by A. Now there exist distinct unit vectors $e_{1_A}, \ldots, e_{q_A} \in \mathbb{R}^n$ along the positive coordinate axes, $\{e_1, \ldots, e_n\}$, such that $e_{i_A} \notin V$ and $E_A := \{v_1, \ldots, v_d, e_{1_A}, \ldots, e_{q_A}\}$ spans $\mathbb{R}^n$. We define $\varphi_A : [0, 1]^n \to \mathbb{R}^n$ by the formula

$$\varphi_A(t_1, \ldots, t_q, s_1, \ldots, s_d) = t_1e_{1_A} + \cdots + t_qe_{q_A} + s_1v_1 + \cdots + s_dv_d. \tag{33}$$

We may write $T = (t_1, \dots, t_q)$, $S = (s_1, \dots, s_d)$ and $dx = dSdT = dTdS$ to shorten notation.

Remark 2.5. These coordinates are not unique, but we suppose that we have fixed some $e_{1_A}, \dots, e_{q_A}$ for every $A \in G^n_d$. The specific choice is not important in our method.

Next we discuss some elementary properties of the coordinates φ_A. The image of φ_A is an n-parallelepiped when interpreted in $\mathbb{R}^n$. A simple calculation shows that $|\det(D\varphi_A)| = \left|\det(v_1, \dots, v_n, e_{1_A}, \dots, e_{e_{q_A}})\right| \in \mathbb{Z}_+$, which is also equal to the volume of the n-parallelepiped spanned by E_A. The corners of the parallelepiped, $\varphi_A(T,S)$ with $T \in \{0,1\}^q, S \in \{0,1\}^d$, have integer coordinates as well. It can be argued that the coordinates (33) wrap around the torus $|\det(D\varphi_A)|$ times when projected into $\mathbb{T}^n$, i.e. $|\det(D\varphi_A)| = \left|\varphi_A^{-1}(x)\right|$ for any $x \in \mathbb{T}^n$.

Let us denote the Lebesgue measure on $\mathbb{T}^n$ by dm and on $[0,1]^n$ by dx. We thus have the change of coordinates formula for integrals of measurable functions in the form of

$$\begin{aligned} \int_{\mathbb{T}^n} f dm &= \frac{1}{|\det(D\varphi_A)|} \int_{[0,1]^n} f \circ \varphi_A \, |\det(D\varphi_A)| \, dx \\ &= \int_{[0,1]^n} f \circ \varphi_A dx. \end{aligned} \tag{34}$$

The formula (34), in a slightly different form, was used in the proofs given in [10]. The connection to [10] is explained with more details in remark 2.6.

Remark 2.6. Let $n = 2, d = 1$, $v = (v^1, v^2) \in \mathbb{Z}^2 \setminus \{0\}$ and $A = \{v\}$. Suppose that v is not parallel to e_1, which in turn implies that $v^2 \neq 0$. If we choose $E_A = \{e_1\}$, then the formula $|\det(D\varphi_A)| = \left|v^2\right|$ holds and it is easy to check that the coordinates wrap $\left|v^2\right|$ times around $\mathbb{T}^2$. If v is parallel to e_1, then one chooses $E_A = \{e_2\}$ instead of e_1. This is in-line with the formulas derived in [10] but there the coordinates were scaled so that they wrap around $\mathbb{T}^2$ exactly once.

Now we are ready to prove lemma 2.3.

Proof of lemma 2.3. The properties (i) and (iii) follow easily from the definitions, and the proofs are thus omitted.

We show first that the mapping $R_{d,A}$ is well defined by the formula (32). Let $\tilde{0} = (0, \dots, 0) \in \mathbb{R}^d$. We get from Fubini's theorem and the formula (34) that

$$\int_{\mathbb{T}^n} f dm = \int_{[0,1]^q} R_{d,A} f(\varphi_A(T, \tilde{0})) dT \tag{35}$$

and $R_{d,A} f(\varphi_A(T, \tilde{0})) \in L^1([0,1]^q)$. It follows from the definition (32) of $R_{d,A} f$ that

$$R_{d,A} f(\varphi_A(T, \tilde{0})) = R_{d,A} f(\varphi_A(T, S)) \tag{36}$$

for all $S \in \mathbb{R}^d$.

We show that $R_{d,A}f$ is a measurable function. Suppose for simplicity that f is real valued. Let $\alpha > 0$ and define the sets

$$X_\alpha = \{ T \in [0,1]^q \,;\, R_{d,A}f(\varphi_A(T,\tilde{0})) > \alpha \}. \tag{37}$$

We have already proved that the set X_α is measurable for any $\alpha > 0$. Now we get from the formula (36) that

$$\{ p \in [0,1]^n \,;\, R_{d,A}f(\varphi_A(p)) > \alpha \} = X_\alpha \times [0,1]^d. \tag{38}$$

The set $X_\alpha \times [0,1]^d$ is measurable as a product of measurable sets. Since φ_A is a smooth change of coordinates, we first find that $\varphi_A(X_\alpha \times [0,1]^d)$ is measurable, and thus $R_{d,A}f$ is measurable. If f is complex valued, then the above argument can be done separately for the real and imaginary parts as $R_{d,A}$ is linear.

Now we are ready to prove the property (ii). Suppose that $f \in L^p(\mathbb{T}^n)$ and $p \in [1,\infty)$. The formulas (34) and (36), and Hölder's inequality give

$$\begin{aligned}
\int_{\mathbb{T}^n} |R_{d,A}f|^p \, dm &= \int_{[0,1]^q} \int_{[0,1]^d} |R_{d,A}f \circ \varphi_A|^p \, dx \\
&= \int_{[0,1]^q} \left|(R_{d,A}f)(\varphi_A(T,\tilde{0}))\right|^p dT \\
&\leq \int_{[0,1]^q} (R_{d,A}\,|f|^p)(\varphi_A(T,\tilde{0}))dT \\
&= \|f\|^p_{L^p(\mathbb{T}^n)} < \infty.
\end{aligned} \tag{39}$$

Hence Tonelli's theorem implies that $R_{d,A}f \in L^p(\mathbb{T}^n)$. If $p = \infty$, then trivially $\|R_{d,A}f\|_{L^\infty(\mathbb{T}^n)} \leq \|f\|_{L^\infty(\mathbb{T}^n)}$. □

2.4.2. *Mapping properties of periodic Radon transforms.* We first recall the inversion formula in [7].

Theorem 2.4 (Eq. (2) in [7])**.** *Let $f \in \mathcal{T}'$, $k \in \mathbb{Z}^n$ and $A \in \mathbf{Gr}(d,n)$. Then $\widehat{R_d f}(k,A) = \hat{f}(k)\delta_{k\perp A}$, where*

$$\delta_{k\perp A} = \begin{cases} 1 & \text{if } k\perp A \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

It is evident that for every $k \in \mathbb{Z}^n$ there exists $A \in \mathbf{Gr}(d,n)$ such that $k \perp A$, see [1, p. 11] and [7, Lemma 9]. This directly gives a reconstructive inversion procedure for R_d. In section 3, we derive new inversion formulas which might provide computational advantage in practice (cf. [10] when $n = 2$ and $d = 1$).

Lemma 2.5. *Let $A \in \mathbf{Gr}(d,n)$.*

(i) If $P : \mathcal{T}' \to \mathcal{T}'$ acts as a Fourier multiplier $(p_k)_{k\in\mathbb{Z}^n}$, then $[P, R_{d,A}] = 0$.

(ii) $R_{d,A} : L^p_s(\mathbb{T}^n) \to L^p_s(\mathbb{T}^n)$ is 1-Lipschitz for any $p \in [1,\infty]$.

Proof. (i) This is a simple application of theorem 2.4. We calculate that

$$\widehat{R_d(Pf)}(k,A) = \widehat{Pf}(k)\delta_{k\perp A} = p_k\hat{f}(k)\delta_{k\perp A} = \widehat{P(R_d f)}(k,A). \tag{41}$$

(ii) Suppose that $f \in L^p_s(\mathbb{T}^n)$. Now $h := (1-\Delta)^{s/2} f \in L^p(\mathbb{T}^n)$. Notice that $R_{d,A}h \in L^p(\mathbb{T}^n)$ by lemma 2.3. We have by the property (i) that $(1-\Delta)^{s/2} R_{d,A} f = R_{d,A}h \in L^p(\mathbb{T}^n)$. Hence $R_{d,A}f \in L^p_s(\mathbb{T}^n)$. We can conclude that

$$\|R_{d,A}f\|_{L^p_s(\mathbb{T}^n)} = \|R_{d,A}h\|_{L^p(\mathbb{T}^n)} \leq \|h\|_{L^p(\mathbb{T}^n)} = \|f\|_{L^p_s(\mathbb{T}^n)} \tag{42}$$

by lemma 2.3. □

The next lemma generalizes [10, Proposition 11] to many different directions.

Lemma 2.6. *(i) Let $l \in [1,\infty)$. Suppose that for any $A \in \mathbf{Gr}(d,n)$ there exists $C_A > 0$ such that $w(k,A) = C_A$ for every $k \perp A$. Moreover, suppose that*

$$C^l_w := \sum_{A \in \mathbf{Gr}(d,n)} C^l_A < \infty. \tag{43}$$

Then the Radon transform $R_d : L^p_s(\mathbb{T}^n) \to L^{p,l}_s(X_{d,n}; w)$ is C_w-Lipschitz.

(ii) Suppose that for any $A \in \mathbf{Gr}(d,n)$ there exists $C_A > 0$ such that $w(k,A) = C_A$ for every $k \perp A$. Moreover, suppose that

$$C_w = \sup_{A \in \mathbf{Gr}(d,n)} C_A < \infty. \tag{44}$$

Then the Radon transform $R_d : L^p_s(\mathbb{T}^n) \to L^{p,\infty}_s(X_{d,n}; w)$ is C_w-Lipschitz.

(iii) Suppose that there exists $C_w > 0$ such that

$$\sum_{A \in \Omega_k} w(k,A)^2 \leq C^2_w, \quad \Omega_k := \{\, A \in \mathbf{Gr}(d,n)\,;\, k \perp A \,\} \tag{45}$$

for any $k \in \mathbb{Z}^n$. Then the Radon transform $R_d : H^s(\mathbb{T}^n) \to L^{2,2}_s(X_{d,n}; w)$ is C_w-Lipschitz.

Proof. (i) We have that

$$\|R_{d,A}f\|_{L^p_s(\mathbb{T}^n)} \leq \|f\|_{L^p_s(\mathbb{T}^n)} \tag{46}$$

for any $A \in \mathbf{Gr}(d,n)$ by lemma 2.5. Theorem 2.4 implies that

$$F_{w(\cdot,A)} R_{d,A} f(x) = \sum_{k \perp A} w(k,A) \hat{f}(k) e^{2\pi i k \cdot x}. \tag{47}$$

Now it follows from the triangle inequality and (47) that

$$\begin{aligned} \|R_d f\|^l_{L^{p,l}_s(X_{d,n};w)} &= \sum_{A \in \mathbf{Gr}(d,n)} C^l_A \|\sum_{k \perp A} \langle k \rangle^s \hat{f}(k) e^{2\pi i k \cdot x}\|^l_{L^p(\mathbb{T}^n)} \\ &\leq C^l_w \|f\|^l_{L^p_s(\mathbb{T}^n)}. \end{aligned} \tag{48}$$

(ii) A calculation similar to the proof of (i) shows that

$$\|R_d f\|_{L^{p,\infty}_s(X_{d,n};w)} \leq \|f\|_{L^p_s(\mathbb{T}^n)} \sup_{A \in \mathbf{Gr}(d,n)} C_A. \tag{49}$$

(iii) We have that

$$
\begin{aligned}
\|R_d f\|^2_{L^{2,2}_s(X_{d,n};w)} &= \sum_{A\in\mathbf{Gr}(d,n)} \|\sum_{k\perp A} w(k,A)\,\langle k\rangle^s \hat{f}(k) e^{2\pi i k\cdot x}\|^2_{L^2(\mathbb{T}^n)} \\
&= \sum_{A\in\mathbf{Gr}(d,n)} \sum_{k\perp A} w(k,A)^2 \left|\langle k\rangle^s \hat{f}(k)\right|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \sum_{A\in\Omega_k} w(k,A)^2 \langle k\rangle^{2s} \left|\hat{f}(k)\right|^2 \\
&\le C_w^2 \|f\|^2_{L^2_s(\mathbb{T}^n)}
\end{aligned}
\tag{50}
$$

where the order of summation can be interchanged by non-negativity of the terms. □

Remark 2.7. If $d = n-1$, then the only restriction on w in the case of (iii) is $\sum_{A\in\mathbf{Gr}(n-1,n)} w(0,A)^2 < \infty$. This follows since each $A \in \mathbf{Gr}(n-1,n)$ has a unique normal direction.

2.4.3. *Adjoint and normal operators.* Next, we study the adjoint and normal operators of R_d when the image side is equipped with the Hilbert space $L^{2,2}_s(X_{d,n};w)$ satisfying the assumptions (iii) of lemma 2.6. This generalizes the considerations in [10, Section 2.4] into higher dimensions and for any $1 \le d \le n-1$.

Proof of theorem 1.1. Let $f \in H^s(\mathbb{T}^n)$ and $g \in L^{2;2}_s(X_{d,n};w)$. Using the definition of the inner product (20), we get

$$
\begin{aligned}
(R_d f, g)_{L^{2,2}_s(X_{d,n};w)} &= \sum_{A\in\mathbf{Gr}(d,n)} (F_{w(\cdot,A)} R_d f, F_{w(\cdot,A)} g)_{H^s(\mathbb{T}^n)} \\
&= \sum_{A\in\mathbf{Gr}(d,n)} \sum_{k\perp A} w(k,A)^2 \langle k\rangle^{2s} \hat{f}(k)\hat{g}(k,A)^* \\
&= \sum_{k\in\mathbb{Z}^n} \sum_{A\in\Omega_k} w(k,A)^2 \langle k\rangle^{2s} \hat{f}(k)\hat{g}(k,A)^* \\
&= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} \hat{f}(k) \left(\sum_{A\in\Omega_k} w(k,A)^2 \hat{g}(k,A) \right)^* \\
&=: (f, R_d^* g)_{H^s(\mathbb{T}^n)}
\end{aligned}
\tag{51}
$$

where we can interchange the order of the summation by the Cauchy–Schwarz inequality as it implies that the series is absolutely convergent.

We have that

$$
\begin{aligned}
\widehat{R_d^* R_d f}(k) &= \sum_{A\in\Omega_k} w(k,A)^2 \widehat{R_d f}(k,A) \\
&= \sum_{A\in\Omega_k} w(k,A)^2 \hat{f}(k)\delta_{k\perp A} \\
&= \hat{f}(k) \sum_{A\in\Omega_k} w(k,A)^2
\end{aligned}
\tag{52}
$$

by the formula for the adjoint and theorem 2.4. □

We prove corollary 1.2 on inversion formulas and stability estimates next.

Proof of corollary 1.2. (i) We first calculate that

$$\|F_{W_k^{-1}} R_d^* g\|_{H^s(\mathbb{T}^n)}^2 = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} \frac{1}{W_k^2} \left| \sum_{A\in\Omega_k} w(k,A)^2 \hat{g}(k,A) \right|^2 \tag{53}$$

for any $g \in L_s^{2,2}(X_{d,n}; w)$. The triangle inequality and Hölder's inequality for the sequences $w(k,A)$ and $w(k,A)\,|\hat{g}(k,A)|$ over $A \in \Omega_k$ gives that

$$\left| \sum_{A\in\Omega_k} w(k,A)^2 \hat{g}(k,A) \right|^2 \leq W_k \left(\sum_{A\in\Omega_k} w(k,A)^2 \,|\hat{g}(k,A)|^2 \right). \tag{54}$$

Recall that

$$\|g\|_{L_s^{2,2}(X_{d,n};w)}^2 = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} \sum_{A\in\mathbf{Gr}(d,n)} w(k,A)^2 \,|\hat{g}(k,A)|^2 \tag{55}$$

after a rearrangement of the series. We can conclude from the formulas (53), (54) and (55) that $\|F_{W_k^{-1}} R_d^* g\|_{H^s(\mathbb{T}^n)} \leq \frac{1}{c_w}\|g\|_{L_s^{2,2}(X_{d,n};w)}$.

(ii) This is a simple calculation using the formula for the normal operator:

$$(R_d f, R_d f)_{L_s^{2,2}(X_{d,n};w)} = (f, F_{W_k} f)_{H^s(\mathbb{T}^n)} \geq \inf_{k\in\mathbb{Z}^n} W_k \|f\|_{H^s(\mathbb{T}^n)}^2 \tag{56}$$

if $f \in H^s(\mathbb{T}^n)$.

(iii) We have by remark 2.4 that $\tilde{w}$ is a weight that satisfies the assumptions of theorem 1.1 and $\tilde{W}_k = 1$ for any $k \in \mathbb{Z}^n$. Therefore, the corresponding adjoint $R_d^{*,\tilde{w}}$ is well-defined, and $R_d^{*,\tilde{w}} R_d f = f$ for any $f \in \mathcal{T}'$ by theorem 1.1. □

3. Inversion formulas

We have already proved one new inversion formula in corollary 1.2 for $H^s(\mathbb{T}^n)$ functions. In this section, we prove three other inversion formulas. One of the formulas generalizes the inversion formula for R_1 on $L^1(\mathbb{T}^2)$ proved in [10, Theorem 1 and Theorem 8]. The second inversion formula is a corollary of the first one and remains valid for any distribution. The third inversion formula takes a slightly different approach and shows that a distribution $f \in \mathcal{T}'$ is a weighted sum of the data $R_{d,A}f$ over the set $\mathbf{Gr}(d,n)$. These formulas might have practical value.

Proposition 3.1 (The first inversion formula)**.** *Let $A \in \mathbf{Gr}(d,n)$ and $k \in \mathbb{Z}^n$. Suppose that $f \in \mathcal{T}'$ and $R_{d,A}f \in L^1(\mathbb{T}^2)$. If $k \perp A$, then*

$$\hat{f}(k) = \int_{[0,1]^q} R_{d,A} f(\varphi_A(T,0)) \exp(-2\pi i (k_{1_A} t_{1_A} + \cdots + k_{q_A} t_{q_A})) dT. \tag{57}$$

Proof. Fubini's theorem, theorem 2.4 and the formula (34) implies that

$$\begin{aligned} &\widehat{R_{d,A} f}(k) \\ &= \int_{[0,1]^q} \int_{[0,1]^d} R_{d,A} f(\varphi_A(T,S)) \exp(-2\pi i k \cdot \varphi_A(T,S)) dS dT. \end{aligned} \tag{58}$$

Since $k \perp A$, a simple calculation shows that

$$k \cdot \varphi_A(T, S) = k_{1_A} t_{1_A} + \cdots + k_{q_A} t_{q_A}, \tag{59}$$

and lemma 2.3 implies that

$$R_{d,A} f(\varphi_A(T, S)) = R_{d,A} f(\varphi_A(T, 0)) \tag{60}$$

for a.e. $T \in [0, 1]^q$.

Hence, using the formulas (59) and (60), we may simplify the formula (58) into the form

$$\begin{aligned} &\widehat{R_{d,A} f}(k) \\ &= \int_{[0,1]^q} R_{d,A} f(\varphi_A(T, 0)) \exp(-2\pi i (k_{1_A} t_{1_A} + \cdots + k_{q_A} t_{q_A})) dT. \end{aligned} \tag{61}$$

□

Remark 3.1. The proof shows that instead of choosing $S = 0$, we may choose any other values for the S-coordinates as well.

We immediately get the following corollary from proposition 3.1 and lemma 2.3.

Corollary 3.2. *Suppose that $f \in L^1(\mathbb{T}^n)$. Then the inversion formula (57) is valid.*

Remark 3.2. One could prove corollary 3.2 directly without using lemma 2.3 and theorem 2.4 (or proposition 3.1). This proof is given for the geodesic X-ray transform in [10] and it could be adapted to this setting as well.

Recall that the structure theorem of periodic distributions [16, Theorem 2.4.5] states that for any $f \in \mathcal{T}'$ there exist $h \in C(\mathbb{T}^n)$ and $s \geq 0$ such that

$$f = (1 - \Delta)^s h. \tag{62}$$

We get another corollary of proposition 3.1 and lemma 2.5.

Corollary 3.3 (The second inversion formula)**.** *Let $A \in \mathbf{Gr}(d, n)$ and $k \in \mathbb{Z}^n$. Suppose that $f \in \mathcal{T}'$ and $f = (1 - \Delta)^s h$, $h \in C(\mathbb{T}^n)$. If $k \perp A$, then*

$$\hat{f}(k) = \langle k \rangle^{2s} \widehat{R_{d,A} h}(k) = \widehat{R_{d,A} f}(k) \tag{63}$$

where $\widehat{R_{d,A} h}(k)$ can be calculated by the formula (57).

We now prove our third inversion formula stated in the introduction.

Proof of theorem 1.3. Using theorem 2.4, we calculate that

$$\mathcal{F}(F_{w(\cdot,A)} R_{d,A} f)(k) = w(k, A) \hat{f}(k) \delta_{k \perp A}. \tag{64}$$

Hence, we get

$$\begin{aligned} \mathcal{F}\left(\sum_{A \in \mathbf{Gr}(d,n)} F_{w(\cdot,A)} R_{d,A} f \right)(k) &= \sum_{A \in \mathbf{Gr}(d,n)} w(k, A) \hat{f}(k) \delta_{k \perp A} \\ &= \hat{f}(k) \sum_{A \in \Omega_k} w(k, A) \\ &= \hat{f}(k) \end{aligned} \tag{65}$$

Suppose now that $d = n-1$ and $\hat{f}(0) = 0$. Notice that $|\Omega_k| = 1$ if $k \neq 0$ and $\Omega_0 = \mathbf{Gr}(n-1, n)$. Hence, the formula (7) follows by choosing any weight w such that

$$\sum_{A \in \mathbf{Gr}(n-1,d)} w(0, A) = 1, w(0, A) \geq 0, \tag{66}$$

and $w(k, A) = 1$ for any $A \in \mathbf{Gr}(n-1, n)$ and $k \neq 0$. □

4. Stability estimates and regularization methods

In this section, we look at stability estimates for functions in the Bessel potential spaces when $p \neq \infty$. We also generalize the Tikhonov regularization methods developed in [10]. In the Tikhonov regularization part, we restrict our study to the functions in $H^s(\mathbb{T}^n)$, as done in [10]. Our results on regularization are new for any $1 \leq d \leq n-1$ when $n \geq 3$, and the stability estimates are new in any dimension.

4.1. Stability estimates and the Sobolev inequality.

Recall that in corollary 1.2 we obtained the estimate

$$\|f\|^2_{H^s(\mathbb{T}^n)} \leq \frac{1}{c_w^2} \|R_d f\|^2_{L^{2,2}_s(X_{d,n};w)} \tag{67}$$

if the weight w is such that the normal operator $R_d^* R_d$ has a uniform lower bound $\frac{1}{c_w^2}$ as a Fourier multiplier. The condition on the weight w is that $c_w^2 \leq W_k = \sum_{A \in \Omega_k} w(k, A)^2 \leq C_w^2$ for some uniform $c_w, C_w > 0$. This implies stability on $L^p_s(\mathbb{T}^n)$ if $p \leq 2$, as we will show later. We can reach stability estimates for $p > 2$ using the Sobolev inequality on $\mathbb{T}^n$.

Theorem 4.1 (Sobolev inequality [20]). *Let $f \in \mathcal{T}'$. Suppose that $s > 0$ and $1 < q < p < \infty$ satisfy $s/n \geq q^{-1} - p^{-1}$. Then*

$$\|f\|_{L^p(\mathbb{T}^n)} \leq C \|f\|_{L^q_s(\mathbb{T}^n)} \tag{68}$$

for some $C > 0$ that does not depend on f.

A proof of the Sobolev inequality on $\mathbb{T}^n$ is given in [3, Corollary 1.2].

Lemma 4.2. *Let $l \in [1, \infty]$ and $g : \mathbf{Gr}(d, n) \to \mathcal{T}'$.*

(i) If $t \in \mathbb{R}$, $s > 0$, and $1 < q < p < \infty$ satisfy $s/n \geq q^{-1} - p^{-1}$, then

$$\|g\|_{L^{p,l}_t(X_{d,n};w)} \leq C \|g\|_{L^{q,l}_{t+s}(X_{d,n};w)} \tag{69}$$

for some $C > 0$ that does not depend on g.

(ii) If $1 \leq p < q \leq \infty$, then for any $s \in \mathbb{R}$ holds

$$\|g\|_{L^{p,l}_s(X_{d,n};w)} \leq \|g\|_{L^{q,l}_s(X_{d,n};w)}. \tag{70}$$

Proof. (i) We have

$$\|g(\cdot, A)\|_{L^p(\mathbb{T}^n; w(\cdot, A))} \leq C \|g(\cdot, A)\|_{L^q_s(\mathbb{T}^n; w(\cdot, A))} \tag{71}$$

for any $A \in \mathbf{Gr}(d, n)$ by the Sobolev inequality where $C > 0$ does not depend on f, A and w. Now (69) with $t = 0$ follows from the definition of the norms $\|\cdot\|_{L^{q,l}_s(X_{d,n};w)}$ and the inequality (71).

Fix any $z \in \mathbb{R}$. Define then the function $\tilde{g} : \mathbf{Gr}(d,n) \to \mathcal{T}'$ by the formula $\tilde{g}(\cdot, A) = (1-\Delta)^{z/2} g(\cdot, A)$. Now (69) with $t = 0$ implies

$$\|g\|_{L^{p,l}_z(X_{d,n};w)} = \|\tilde{g}\|_{L^{p,l}_0(X_{d,n};w)} \leq C\|\tilde{g}\|_{L^{q,l}_s(X_{d,n};w)} = C\|g\|_{L^{q,l}_{z+s}(X_{d,n};w)}. \tag{72}$$

(ii) The inequality (70) can be proved similarly. Now the Sobolev inequality is replaced by the inequality $\|f\|_{L^p_s(\mathbb{T}^n)} \leq \|f\|_{L^q_s(\mathbb{T}^n)}$, which holds since $m(\mathbb{T}^n) = 1$ and $p \leq q$. □

Theorem 1.1 and lemma 4.2 imply the following, slightly more general, shifted stability estimates.

Proposition 4.3 (Shifted stability estimates)**.** *Let w be a weight such that $c_w^2 \leq W_k \leq C_w^2$ for some uniform constants $c_w, C_w > 0$. Let $f \in \mathcal{T}'$, $s \in \mathbb{R}$, and $s(p,n) := n\left|\frac{p-2}{2p}\right|$.*

(i) If $1 < p \leq 2$, then

$$\|f\|_{L^p_s(\mathbb{T}^n)} \leq C_1\|R_d f\|_{L^{2,2}_s(X_{d,n};w)} \leq C_2\|R_d f\|_{L^{p,2}_{s+s(p,n)}(X_{d,n};w)}, \tag{73}$$

where $C_1, C_2 > 0$ do not depend on f. If $p = 1$, then the first inequality of (73) holds.

(ii) If $2 \leq p < \infty$, then

$$\|f\|_{L^p_s(\mathbb{T}^n)} \leq C_1\|R_d f\|_{L^{2,2}_{s+s(p,n)}(X_{d,n};w)} \leq C_2\|R_d f\|_{L^{p,2}_{s+s(p,n)}(X_{d,n};w)}, \tag{74}$$

where $C_1, C_2 > 0$ do not depend on f.

Proof. (i) Suppose that $f \in \mathcal{T}'$ and $1 \leq p \leq 2$. Let $h = (1-\Delta)^{s/2} f$. We have that $\|h\|_{L^p(\mathbb{T}^n)} \leq \|h\|_{L^2(\mathbb{T}^n)}$ since $p \leq 2$ and $m(\mathbb{T}^n) = 1$. This implies that $\|f\|_{L^p_s(\mathbb{T}^n)} \leq \|f\|_{L^2_s(\mathbb{T}^n)}$. Now the first inequality follows from corollary 1.2.

Suppose additionally that $1 < p < 2$. Choose $s^p = n\frac{2-p}{2p} > 0$ in the part (i) of lemma 4.2. Now it holds that

$$\|R_d f\|_{L^{2,2}_s(X_{d,n};w)} \leq \|R_d f\|_{L^{p,2}_{s+s^p}(X_{d,n};w)} \tag{75}$$

for any $s \in \mathbb{R}$.

(ii) Suppose that $f \in \mathcal{T}'$ and $p > 2$. Choose in the Sobolev inequality (68) that $q = 2$. Now we can calculate that the Sobolev inequality is valid if $s \geq n\frac{p-2}{2p}$. Let us define that $s_p = n\frac{p-2}{2p} > 0$. Hence, $\|f\|_{L^p(\mathbb{T}^n)} \leq C\|f\|_{H^{s_p}(\mathbb{T}^n)}$.

Let now $s \in \mathbb{R}$ and $f \in L^p_s(\mathbb{T}^n)$. We then have that

$$\begin{aligned}\|f\|_{L^p_s(\mathbb{T}^n)} &= \|(1-\Delta)^{s/2} f\|_{L^p(\mathbb{T}^n)} \\ &\leq C\|(1-\Delta)^{s/2} f\|_{H^{s_p}(\mathbb{T}^n)} = C\|f\|_{H^{s+s_p}(\mathbb{T}^n)}.\end{aligned} \tag{76}$$

Now the first inequality follows from the part (i) of the theorem. The second inequality follows from the part (ii) of lemma 4.2 since $p > 2$. □

Remark 4.1. For any $f \in \mathcal{T}'$ there exists $s \geq 0$ such that $f \in L^p_{-s}(\mathbb{T}^n)$ for any $p \in [1, \infty]$ by the structure theorem of periodic distributions.

4.2. **Tikhonov minimization problem.** We will show that $P^\alpha_{w,s-r}R^*_d g$ is the unique minimizer of (8) when $l = 2$. We first analyze the regularity properties of $P^\alpha_{w,z}$ and $P^\alpha_{w,s-r}R^*_d$. Then we understand which space the regularized reconstruction $P^\alpha_{w,s-r}R^*_d g$ lives in when $g \in L^{2,2}_r(X_{d,n}; w)$. First of all, $R^*_d : L^{2,2}_r(X_{d,n}; w) \to H^r(\mathbb{T}^n)$. On the other hand, $P^\alpha_{w,z} : H^r(\mathbb{T}^n) \to H^{r+2z}(\mathbb{T}^n)$ for any $r, z \in \mathbb{R}$ since W_k is uniformly bounded from below. We conclude that $P^\alpha_{w,s-r}R^*_d : L^{2,2}_r(X_{d,n}; w) \to H^{2s-r}(\mathbb{T}^n)$.

We are not ready to prove theorem 1.4. The proof uses the same ideas as the proof of [10, Theorem 2]. The proof presented here also explains some missing details about the splitting of the minimization problem into the real and imaginary parts in (81), (82) and (83). This is one of the crucial parts of the proof of [10, Theorem 2] though it is not mentioned at all in [10].

Proof of theorem 1.4. We have that

$$\begin{aligned}
&\|R_d f - g\|^2_{L^{2,2}_r(X_{d,n};w)} \\
&= \sum_{A\in\mathbf{Gr}(d,n)} \sum_{k\perp A} \langle k\rangle^{2r} w(k,A)^2 \left|\hat f(k) - \hat g(k,A)\right|^2 \\
&+ \sum_{A\in\mathbf{Gr}(d,n)} \sum_{k\not\perp A} \langle k\rangle^{2r} w(k,A)^2 \left|\hat g(k,A)\right|^2 .
\end{aligned} \tag{77}$$

Since the second term of (77) is independent of f, it can be neglected in the minimization problem (8). On the other hand,

$$\begin{aligned}
&\sum_{A\in\mathbf{Gr}(d,n)} \sum_{k\perp A} \langle k\rangle^{2r} w(k,A)^2 \left|\hat f(k) - \hat g(k,A)\right|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2r} \sum_{A\in\Omega_k} w(k,A)^2 \left|\hat f(k) - \hat g(k,A)\right|^2 .
\end{aligned} \tag{78}$$

We next expand the term

$$\alpha\|f\|^2_{H^s(\mathbb{T}^n)} = \alpha \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} \left|\hat f(k)\right|^2 . \tag{79}$$

We can conclude that a solution to the minimization problem (8) is a minimizer of

$$\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2r} \left(\alpha \langle k\rangle^{2s-2r} \left|\hat f(k)\right|^2 + \sum_{A\in\Omega_k} w(k,A)^2 \left|\hat f(k) - \hat g(k,A)\right|^2 \right) . \tag{80}$$

Hence, a minimizer of (80) must minimize

$$H(f) := \alpha \langle k\rangle^{2s-2r} \left|\hat f(k)\right|^2 + \sum_{A\in\Omega_k} w(k,A)^2 \left|\hat f(k) - \hat g(k,A)\right|^2 \tag{81}$$

for each $k \in \mathbb{Z}^n$.

To proceed, we need to minimize the real part and the imaginary part of (81) separately. Let us write the real and imaginary parts of the involved terms simply as $f_r(k) := \Re(\hat f(k))$, $f_i(k) := \Im(\hat f(k))$, $g_r(k,A) := \Re(\hat g(k,A))$

and $g_i(k,A) := \Im(\hat{g}(k,A))$ to keep our notation shorter. Now, we define the operators

$$R(f) := \alpha \langle k\rangle^{2s-2r} f_r(k)^2 + \sum_{A\in\Omega_k} w(k,A)^2(f_r(k) - g_r(k,A))^2 \tag{82}$$

and

$$I(f) := \alpha \langle k\rangle^{2s-2r} f_i(k)^2 + \sum_{A\in\Omega_k} w(k,A)^2(f_i(k) - g_i(k,A))^2. \tag{83}$$

These functions have the property that $R(f) + I(f) = H(f)$. Moreover, if H is minimized, then R and I are minimized, and vice versa.

We show how the minimization is done for the real part. As the minimization for the imaginary part is similar, we do not repeat the calculations twice. We expand the second term of (82), and get

$$\begin{aligned}&\sum_{A\in\Omega_k} w(k,A)^2(f_r(k) - g_r(k,A))^2\\ &\quad= W_k f_r(k)^2 - 2f_r(k)\sum_{A\in\Omega_k} w(k,A)^2 g_r(k,A) + \sum_{A\in\Omega_k} w(k,A)^2 g_r(k,A)^2.\end{aligned} \tag{84}$$

The last term of (84) does not depend on f, so it can be neglected in the minimization. Thus, we have arrived to the minimization problem

$$-2f_r(k)\sum_{A\in\Omega_k} w(k,A)^2 g_r(k,A) + (W_k + \alpha\langle k\rangle^{2s-2r}) f_r(k)^2. \tag{85}$$

Simple calculus shows that the minimizer of (85) is

$$f_r(k) = \frac{\sum_{A\in\Omega_k} w(k,A)^2 g_r(k,A)}{W_k + \alpha\langle k\rangle^{2s-2r}} = \Re(\mathcal{F}(P^\alpha_{w,s-r}R_d^* g)(k)). \tag{86}$$

We can similarly calculate that the unique minimizer of the minimization problem associated to the imaginary part (83) is $f_i(k) = \Im(\mathcal{F}(P^\alpha_{w,s-r}R_d^* g)(k))$. This shows that the unique minimizer of (81) satisfies $\hat{f}(k) = \mathcal{F}(P^\alpha_{w,s-r}R_d^* g)(k)$.

Hence, the unique minimizer of (8) is $f = P^\alpha_{w,s-r}R_d^* g$. The claimed regularity of f follows from the discussion preceding the proof. □

Remark 4.2. If $l \neq 2$, the analysis of the Tikhonov minimization problem becomes more difficult but it might still be possible to adapt the method also in that case (when $p = 2$).

4.3. **Regularization strategies.** Let X and Y be subsets of Banach spaces and $F : X \to Y$ a continuous mapping. A family of continuous maps $\mathcal{R}_\alpha : Y \to X$ with $\alpha \in (0, \alpha_0]$, $\alpha_0 > 0$, is called a *regularization strategy* if $\lim_{\alpha\to 0} \mathcal{R}_\alpha(F(x)) = x$ for any $x \in X$. A choice of regularization parameter $\alpha(\epsilon)$ with $\lim_{\epsilon\to 0}\alpha(\epsilon) = 0$ is called *admissible* if

$$\lim_{\epsilon\to 0}\sup_{y\in Y}\left\{\|\mathcal{R}_{\alpha(\epsilon)}y - x\|_X\,;\,\|y - F(x)\|_Y \le \epsilon\right\} = 0 \tag{87}$$

holds for any $x \in X$ [4, 12].

We will show that the solution found in theorem 1.4 to the Tikhonov minimization problem (8) is an admissible regularization strategy with a quantitative stability estimate. Our proof follows that of [10, Theorem 3].

Proof of theorem 1.5. Let $\alpha > 0$. Theorem 1.1 implies that

$$P^\alpha_{w,s} R^*_d (R_d f + g) - f = (P^\alpha_{w,s} F_{W_k} - \mathrm{Id}) f + P^\alpha_{w,s} R^*_d g. \tag{88}$$

To estimate the first term on the right hand side of (88), we calculate that

$$P^\alpha_{w,s} F_{W_k} - \mathrm{Id} = -\frac{\alpha W_k^{-1} \langle k \rangle^{2s}}{1 + \alpha W_k^{-1} \langle k \rangle^{2s}} \tag{89}$$

as a Fourier multiplier. This shows that $\|P^\alpha_{w,s} F_{W_k} - \mathrm{Id}\|_{H^r(\mathbb{T}^n) \to H^r(\mathbb{T}^n)} = 1$ as W_k is bounded from below and above. It follows from the dominated convergence theorem that $\|(P^\alpha_{w,s} F_{W_k} - \mathrm{Id}) f\|_r^2 \to 0$ as $\alpha \to 0$ if $f \in H^r(\mathbb{T}^n)$.

Suppose that $\|g\|_{L^{2,2}_t(X_{d,n};w)} \leq \epsilon$. We have that $\|R^*_d\| = \|R_d\| = C_w$ by lemma 2.6. Hence $\|R^*_d g\|^2_{H^t(\mathbb{T}^n)} \leq C_w^2 \epsilon^2$. This implies that

$$\begin{aligned} \|P^\alpha_{w,s} R^*_d g\|^2_{H^r(\mathbb{T}^n)} &\leq C_w^2 \epsilon^2 \sup_{k \in \mathbb{Z}^n} \left(\frac{W_k^{-1}}{1 + \alpha W_k^{-1} \langle k \rangle^{2s}} \right)^2 \langle k \rangle^{2r-2t} \\ &\leq C_w^2 \epsilon^2 c_w^{-4} \sup_{k \in \mathbb{Z}^n} \left(\frac{1}{1 + \alpha C_w^{-2} \langle k \rangle^{2s}} \right)^2 \langle k \rangle^{2r-2t} \\ &\leq C_w^6 c_w^{-4} \alpha^{-2} \epsilon^2 \end{aligned} \tag{90}$$

where the last inequality follows using $-4s + 2r - 2t \leq 0$. We can conclude that

$$\|P^\alpha_{w,s} R^*_d g\|_{H^r(\mathbb{T}^n)} \leq C_w^3 c_w^{-2} \frac{\epsilon}{\alpha}. \tag{91}$$

This shows that choosing $\alpha = \sqrt{\epsilon}$ gives a regularization strategy.

Suppose now that $\delta > 0$. The proof of the estimate (11) is similar to that of [10]. Using the formula (89), we get that

$$\|P^\alpha_{w,s} F_{W_k} - \mathrm{Id}\|_{H^{r+\delta}(\mathbb{T}^n) \to H^r(\mathbb{T}^n)} = \sup_{k \in \mathbb{Z}^n} \frac{\alpha W_k^{-1} \langle k \rangle^{2s-\delta}}{1 + \alpha W_k^{-1} \langle k \rangle^{2s}}. \tag{92}$$

We can estimate the norm by defining the functions

$$F_k(x) := \frac{\alpha W_k^{-1} x^{2s-\delta}}{1 + \alpha W_k^{-1} x^{2s}}. \tag{93}$$

The formula [10, Eq. (38)] implies that the maximum value of F_k is $(W_k^{-1}\alpha)^{\delta/2s} C(\delta/2s)$ if $\alpha \leq W_k(2s/\delta - 1)$. We see that $\alpha \leq W_k(2s/\delta - 1)$ holds as we assumed that $\alpha \leq c_w^2(2s/\delta - 1)$.

We obtain that

$$\begin{aligned} &\|(P^\alpha_{w,s} F_{W_k} - \mathrm{Id})\|_{H^{r+\delta}(\mathbb{T}^n) \to H^r(\mathbb{T}^n)} \\ &\leq \sup_{k \in \mathbb{Z}^n, x \in \mathbb{R}} F_k(x) \leq (c_w^{-2} \alpha)^{\delta/2s} C(\delta/2s). \end{aligned} \tag{94}$$

Hence

$$\|(P^\alpha_{w,s} F_{W_k} - \mathrm{Id}) f\|_{H^r(\mathbb{T}^n)} \leq (c_w^{-2} \alpha)^{\delta/2s} C(\delta/2s) \|f\|_{H^{r+\delta}(\mathbb{T}^n)}. \tag{95}$$

Now the formulas (91) and (95) imply the quantitative estimate (11). □

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