JYU DISSERTATIONS 126

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Cardioid-Type Domains and Regularity of Homeomorphic Extensions

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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List of included articles

This dissertation consists of an introductory part and the following two publications:

[**A**] H. Xu, *Weighted estimates for diffeomorphic extensions of homeomorphisms*, To appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.

[**B**] H. Xu, *Optimal extensions of conformal mappings from the unit disk to cardioid-type domains*, Preprint arXiv:1905.09351, 2019.

INTRODUCTION

1. PRELIMINARIES

Let $\Omega \subset \mathbb{R}$ be a bounded domain. Then Ω is an open interval. Let $I = (-1, 1)$. If $\varphi : \partial I \to \partial\Omega$ is a homeomorphism, there is a harmonic diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h = \varphi$ on ∂I. Actually we may define h as a linear mapping. Especially, for any bounded domain $\Omega \subset \mathbb{R}$ there is a harmonic diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h(I) = \Omega$ and $h \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$.

Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. If $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a homeomorphism, then by the Schoenflies theorem there is a homeomorphic extension h of φ to the entire plane. If Ω is additionally convex, then by the Radó-Kneser-Choquet Theorem, h can be chosen to be a harmonic diffeomorphism on $\mathbb{D}$;

$$h(z) = P[\varphi](z) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{1 - |z|^2}{|z - \xi|^2} \varphi(\xi) \, |d\xi| : \ \mathbb{D} \to \Omega, \tag{1.1}$$

see [5, 6, 13, 19, 22]. However, h cannot, in general, be required to be diffeomorphic on entire $\mathbb{R}^2$. A natural question arises:

Question 1.1. *Let h be as in* (1.1). *What is the degree of integrability of the derivatives of h?*

Let us review results related to Question 1.1. Let Ω be a bounded convex domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. Let $h : \mathbb{D} \to \Omega$ be the harmonic extension of φ as in (1.1). In 2007, G. C. Verchota [25] proved that $h \in W^{1,p}(\mathbb{D})$ for all $p < 2$ but not necessarily for $p = 2$. In 2009, T. Iwaniec, G. J. Martin and C. Sbordone improved on [11] by showing that the derivatives of h belong to weak-L^2 with sharp estimates. Actually

$$\int_{\mathbb{D}} |Dh(z)|^2 \, dz \approx \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|\varphi(\xi) - \varphi(\eta)|^2}{|\xi - \eta|^2} \, |d\xi| \, |d\eta|, \tag{1.2}$$

since harmonic functions minimize the L^2-energy and the right-hand side of (1.2) is the trace norm of $\dot{W}^{1,2}(\mathbb{D})$. It was further shown in [3] that if additionally $\partial\Omega$ is a C^1-regular Jordan curve then

$$\int_{\mathbb{D}} |Dh(z)|^2 \, dz < \infty \Leftrightarrow \int_{\partial\Omega} \int_{\partial\Omega} |\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|| \, |d\xi| \, |d\eta| < \infty.$$

If $\Omega = \mathbb{D}$, it was proved recently in [14] that for any $\lambda \in (-1, +\infty)$ the following are equivalent:

(i) $\int_{\mathbb{D}} |Dh(z)|^2 \log^\lambda(e + |Dh(z)|) \, dz < +\infty$;
(ii) $\int_{\mathbb{D}} |Dh(z)|^2 \log^\lambda(2(1 - |z|)^{-1}) \, dz < +\infty$;
(iii) $\int_{\mathbb{S}^1} \int_{\mathbb{S}^1} |\log|\varphi^{-1}(\eta) - \varphi^{-1}(\xi)||^{\lambda+1} \, |d\eta| \, |d\xi| < +\infty$.

When $\Omega \subset \mathbb{R}^2$ is a non-convex Jordan domain, there exists a homeomorphism $\varphi : \mathbb{S}^1 \to \partial\Omega$ for which the harmonic extension fails to map $\mathbb{D}$ homeomorphically onto Ω, see [5, 13]. Hence we cannot use the harmonic extension to produce a diffeomorphic extension. However there is a diffeomorphic extension from $\mathbb{D}$ onto Ω of φ. Indeed, since a Jordan domain Ω is simply connected, by the Riemann mapping theorem there is a conformal mapping

$$g : \mathbb{D} \to \Omega. \tag{1.3}$$

By the Osgood-Carathéodory theorem, g can be extended to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$, still denoted g. Then $g \circ P[g^{-1} \circ \varphi] : \mathbb{D} \to \Omega$ is a diffeomorphic extension of φ. A natural question arises:

Question 1.2. *Given a bounded non-convex Jordan domain Ω and a homeomorphism $\varphi : \mathbb{S}^1 \to \partial\Omega$, how good a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ can we find?*

Let g be as above. Then $g : \overline{\mathbb{D}} \to \overline{\Omega}$ is harmonic and maps $\mathbb{D}$ diffeomorphically onto Ω. This g belongs to $W^{1,2}(\mathbb{D}, \Omega)$ by the area formula and the Cauchy-Riemann equations for conformal mappings. By the Schoenflies theorem, we may extend g to a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ but one cannot in general find such an extension with $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, see [26]. However a nice extension can be found when $\partial\Omega$ is sufficiently regular, e.g. satisfies the three-point condition.

Let us recall that a Jordan domain $\Omega \subset \mathbb{R}^2$ satisfies the **three-point condition** if there is a constant $C \geq 1$ such that for each pair of points $z_1, z_2 \in \partial\Omega \setminus \{\infty\}$,

$$\min_{j=1,2} \operatorname{diam}(\gamma_j) \leq C|z_1 - z_2|$$

where γ_1, γ_2 are the components of $\partial\Omega \setminus \{z_1, z_2\}$.

We continue by recalling the definitions of two classes of homeomorphisms. Let $\Omega \subset \mathbb{R}^2$ and $\Omega' \subset \mathbb{R}^2$ be domains. A homeomorphism $f : \Omega \to \Omega'$ is called **K-quasiconformal** if $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ and if there is a constant $K \geq 1$ such that

$$|Df(z)|^2 \leq K J_f(z)$$

holds for $\mathcal{L}^2$-a.e. $z \in \Omega$. Note that 1-quasiconformal mappings are conformal. More generally, we say that a homeomorphism $f : \Omega \to \Omega'$ has **finite distortion** if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ and

$$|Df(z)|^2 \leq K_f(z) J_f(z) \qquad \mathcal{L}^2\text{-a.e. } z \in \Omega,$$

where

$$K_f(z) = \begin{cases} \frac{|Df(z)|^2}{J_f(z)} & \text{for all } z \in \{J_f > 0\}, \\ 1 & \text{for all } z \in \{J_f = 0\}. \end{cases}$$

Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and g be a conformal mapping as in (1.3). If additionally $\partial\Omega$ satisfies the three-point condition, by [18, Theorem 8.3], g can be extended to a quasiconformal mapping from $\overline{\mathbb{D}}^c$ onto $\overline{\Omega}^c$. If $\partial\Omega$ does not satisfy the three-point condition, no such quasiconformal extension of g exists. This motivates the following question.

Question 1.3. *Let Ω be a Jordan domain and let $g : \mathbb{D} \to \Omega$ be a conformal mapping. Under which conditions on $\partial\Omega$, can we have a homeomorphic extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion of g? If we have, how good an extension can we obtain?*

More generally, we may ask:

Question 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. By the Schoenflies theorem, there is a homeomorphic extension of φ to the entire plane. How good an extension can we find?*

Let us give partial answers to Question 1.4. First of all, if $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a diffeomorphism, by results from differential topology (see [10]) there exists a diffeomorphic extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ of φ. Secondly, if $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a bi-Lipschitz mapping, there exists a bi-Lipschitz extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ of φ, see [12, 23]. Generally, if $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a Lipschitz homeomorphism, L. V. Kovalev showed in [16,17] that there exists a Lipschitz homeomorphic extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ of φ. If $\varphi : \mathbb{S}^1 \to \partial\Omega$ is quasisymmetric, then by [24] there exists a quasisymmetric extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ of φ.

In this thesis, we give partial answers to Question 1.2 and to Question 1.3.

2. QUESTION 1.2

A. Koski and J. Onninen studied a general case of Question 1.2 in [15]: Sobolev homeomorphic extensions in the case when $\partial\Omega$ is rectifiable. We next study Question 1.2 in the special case when Ω is an internal chord-arc domain.

Recall that a Jordan domain $\Omega \subset \mathbb{R}^2$ is an **internal chord-arc Jordan domain** if $\partial\Omega$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \partial\Omega$,

$$\ell(w_1, w_2) \leq C\lambda_\Omega(w_1, w_2), \tag{2.1}$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial\Omega$ joining w_1 to w_2, and $\lambda_\Omega(w_1, w_2)$ is the **internal distance** between w_1, w_2, which is defined as

$$\lambda_\Omega(w_1, w_2) = \inf_\alpha \ell(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha \subset \Omega$ joining w_1 and w_2; if there is no rectifiable curve joining w_1 and w_2, we set $\lambda_\Omega(w_1, w_2) = \infty$; cf. [21, Section 3.1] or [4, Section 2]. Note that every bounded convex domain is an internal chord-arc domain, and the boundary of an internal chord-arc domain is rectifiable. If (2.1) holds for the Euclidean distance instead of the internal distance, we call Ω a **chord-arc domain**. Naturally, every chord-arc Jordan domain is an internal chord-arc domain, but there are internal chord-arc domains that fail to be chord-arc; e.g. the standard cardioid domain

$$\Delta = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 < 0\}. \tag{2.2}$$

This is a prime example of internal chord-arc domains; the boundary of such a domain can only contain internal cusps.

Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc domain with the internal distance λ_Ω. Assume that $h : \mathbb{D} \to \Omega$ is a diffeomorphism and $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a homeomorphism. Set $\delta(z) = 1 - |z|$. Given $p > 1, \alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define

$$I_1(p, \alpha, \lambda, h) = \int_{\mathbb{D}} |Dh(z)|^p \delta^\alpha(z) \log^\lambda(2\delta^{-1}(z))\, dz,$$

$$I_2(p, \alpha, \lambda, h) = \int_{\mathbb{D}} |Dh(z)|^p \log^\lambda(e + |Dh(z)|)\delta^\alpha(z)\, dz,$$

$$\mathcal{U}(p, \alpha, \lambda, \varphi) = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{\lambda_\Omega^p(\varphi(\xi), \varphi(\eta))}{|\xi - \eta|^{p-\alpha}} \log^\lambda \big(e + \frac{\lambda_\Omega(\varphi(\xi), \varphi(\eta))}{|\xi - \eta|}\big)\, |d\eta|\, |d\xi|,$$

$$\mathcal{A}_{p,\alpha,\lambda}(t) = \int_1^t -x^{1+\alpha-p} \log_2^\lambda(x^{-1})\, dx \qquad \forall t \geq 0,$$

$$\mathcal{V}(p, \alpha, \lambda, \varphi) = \int_{\partial\Omega} \big(\int_{\partial\Omega} \mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|)\, |d\eta| \big)^{p-1} |d\xi|.$$

The following theorem from [A] gives an answer to Question 1.2 when Ω is an internal chord-arc domain.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. There is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which, for any $p > 1$, we have that*

(1) *if either $\alpha \in (p-2, +\infty)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in (-\infty, -1)$,*
$$\text{both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h) \text{ are finite.}$$
(2) *if either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$,*
$$\text{both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h) \text{ are comparable to } \mathcal{U}(p,\alpha,\lambda,\varphi).$$
Moreover for any $p \in (1, 2]$
$$\text{both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h) \text{ dominate } \mathcal{V}(p,\alpha,\lambda,\varphi),$$
while
$$\mathcal{V}(p,\alpha,\lambda,\varphi) \text{ controls both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h)$$
for all $p \in [2, +\infty)$. Furthermore both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are in general comparable to $\mathcal{V}(p,\alpha,\lambda,\varphi)$ only for $p = 2$.

Given $p > 1$ and $\alpha \in (-\infty, -1)$, $\lambda \in \mathbb{R}$, or $\alpha = -1$, $\lambda \in [-1, \infty)$, we have that $I_1(p,\alpha,\lambda,h) = \infty$ for each homeomorphic extension $h : \mathbb{D} \to \Omega$ of φ. This also holds for $I_2(p,\alpha,\lambda,h)$ when $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.

When $p = 2$, $\alpha = 0$ and $\lambda > -1$, Theorem 2.1 was proved in [14]. We next sketch the proof of Theorem 2.1. One begins by proving the following lemma.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. Given $p > 1$ and $\alpha \in (-\infty, -1)$, $\lambda \in \mathbb{R}$, or $\alpha = -1$, $\lambda \in [-1, \infty)$, we have that $I_1(p,\alpha,\lambda,h) = \infty$ for each homeomorphic extension $h : \mathbb{D} \to \Omega$ of φ. This also holds for $I_2(p,\alpha,\lambda,h)$ when $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.*

By Lemma 2.2, it suffices to prove Theorem 2.1 (1) and (2). One first proves a special case of Theorem 2.1 (1) and (2):
$$\Omega = \mathbb{D} \text{ and } h = P[\varphi] \text{ is the harmonic extension of } \varphi. \tag{2.3}$$
Given $j \in \mathbb{N}$ and $k = 1, ..., 2^j$, let
$$I_{j,k} = [2\pi(k-1)2^{-j}, 2\pi k 2^{-j}] \text{ and } \Gamma_{j,k} = \{e^{i\theta} : \theta \in I_{j,k}\}.$$
Set $\ell(\Gamma_{j,k})$ be the length of $\Gamma_{j,k}$. Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define
$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda$$
and
$$\mathcal{E}_2(p,\alpha,\lambda,\varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \left(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right)^p \log^\lambda \left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \ell(\Gamma_{j,k})^{2+\alpha}.$$
The idea of proof of Theorem 2.1 (1) and (2) for the case (2.3) is to connect $I_1(p,\alpha,\lambda,h)$, $I_2(p,\alpha,\lambda,h)$, $\mathcal{U}(p,\alpha,\lambda,\varphi)$ and $\mathcal{V}(p,\alpha,\lambda,\varphi)$ with either $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ or $\mathcal{E}_2(p,\alpha,\lambda,\varphi)$. Secondly,

one proves Theorem 2.1 (1) and (2) via this special case. Since Ω is an internal chord-arc domain, there exists a bi-Lipschitz mapping $g : (\mathbb{S}^1, |\cdot|) \to (\partial\Omega, \lambda_\Omega)$. Here $|\cdot|$ is the Euclidean distance and λ_Ω is the internal distance. Moreover by [4, Theorem 4.7] we have a diffeomorphic bi-Lipschitz extension $\tilde{g} : (\mathbb{D}, |\cdot|) \to (\Omega, \lambda_\Omega)$ of g. Let $h = \tilde{g} \circ P[g^{-1} \circ \varphi]$. We have that $h : \mathbb{D} \to \Omega$ is a diffeomorphic extension of φ. Moreover

$$I_1(p, \alpha, \lambda, h) \approx I_1(p, \alpha, \lambda, P[g^{-1} \circ \varphi]),\ I_2(p, \alpha, \lambda, h) \approx I_2(p, \alpha, \lambda, P[g^{-1} \circ \varphi]),$$

$$\mathcal{U}(p, \alpha, \lambda, \varphi) \approx \mathcal{U}(p, \alpha, \lambda, g^{-1} \circ \varphi),\ \mathcal{V}(p, \alpha, \lambda, \varphi) \approx \mathcal{V}(p, \alpha, \lambda, g^{-1} \circ \varphi).$$

Hence Theorem 2.1 (1) and (2) follow from Theorem 2.1 (1) and (2) for the case (2.3).

3. QUESTION 1.3

C.-Y. Guo et al. [7, 8] have studied Question 1.3 in general settings. Our following results are more explicit.

Let Δ be the standard cardioid domain as in (2.2). The cardioid curve $\partial\Delta$ contains an inner-cusp point of asymptotic polynomial degree 3/2. We next introduce a class of cardioid-type domains Δ_s, whose boundaries contain internal polynomial cusps of order s with $s > 1$, see FIGURE 1. We study Question 1.3 for $\Omega = \Delta_s$ and for $\Omega = \Delta$. For technical reasons we

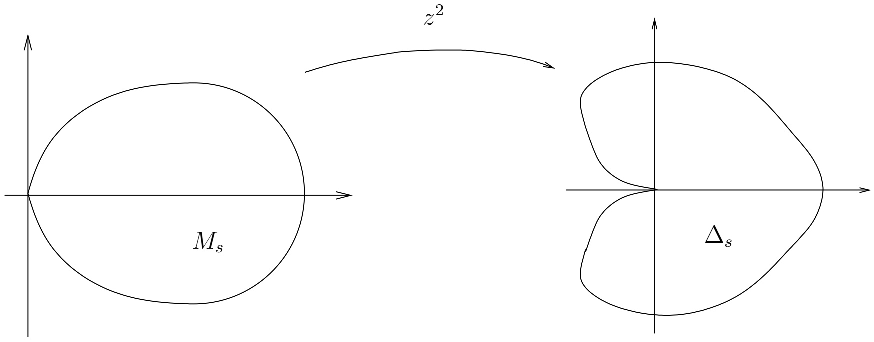

FIGURE 1. M_s and Δ_s

do this in the following manner. Denote

$$\ell_1(s) = \{(u, v) \in \mathbb{R}^2 : u \in [-1, 0],\ v = (-u)^s\}$$

and

$$\ell_2(s) = \{(u, v) \in \mathbb{R}^2 : u \in [-1, 0],\ v = -(-u)^s\}.$$

Write $\ell_1(s)$ and $\ell_2(s)$ in the polar coordinate system as

$$\ell_1(s) = \{Re^{i\Theta} : R = (-u)(1 + (-u)^{2(s-1)})^{\frac{1}{2}} \text{ and } \Theta = \pi - \arctan((-u)^{s-1}) \text{ for } u \in [-1, 0]\}$$

and

$$\ell_2(s) = \{Re^{i\Theta} : R = (-u)(1 + (-u)^{2(s-1)})^{\frac{1}{2}} \text{ and } \Theta = -\pi + \arctan((-u)^{s-1}) \text{ for } u \in [-1, 0]\}.$$

Take the branch of complex-valued function $z = w^{1/2}$ with $1^{1/2} = 1$. Denote by $\ell_1^m(s)$ and $\ell_2^m(s)$ the images of $\ell_1(s)$ and $\ell_2(s)$ under the preceding $z = w^{1/2}$, respectively. Then we can write $\ell_1^m(s)$ and $\ell_2^m(s)$ in the polar coordinate system as

$$\ell_1^m(s) = \{re^{i\theta} : r = \sqrt{-u}(1 + (-u)^{2(s-1)})^{\frac{1}{4}} \\ \text{and } \theta = \frac{\pi - \arctan((-u)^{s-1})}{2} \text{ for } u \in [-1, 0]\}$$

and

$$\ell_2^m(s) = \{re^{i\theta} : r = \sqrt{-u}(1 + (-u)^{2(s-1)})^{\frac{1}{4}} \\ \text{and } \theta = \frac{-\pi + \arctan((-u)^{s-1})}{2} \text{ for } u \in [-1, 0]\}.$$

Denote by z_1 and z_2 the end points of $\ell_1^m(s) \cup \ell_2^m(s)$. Notice that there is a unique circle sharing both the tangent of $\ell_1^m(s)$ at z_1 and the one of $\ell_2^m(s)$ at z_2. This circle is divided into two arcs by z_1 and z_2. Concatenating $\ell_1^m(s) \cup \ell_2^m(s)$ with the arc located on the right-hand side of the line through z_1 and z_2, we then obtain a Jordan curve $\ell^m(s)$. Denote by $\ell(s)$ the image of $\ell^m(s)$ under z^2. Let

$$M_s \text{ and } \Delta_s \text{ be the interior domains of } \ell^m(s) \text{ and } \ell(s), \text{ respectively.} \tag{3.1}$$

Then Δ_s is the desired cardioid-type domain with degree s. Moreover z^2 maps M_s conformally onto Δ_s.

The following theorem from [B] gives an answer to Question 1.3 when $\Omega = \Delta_s$.

Theorem 3.1. *Let g be a conformal map from $\mathbb{D}$ onto Δ_s, where Δ_s is defined in* (3.1) *with $s > 1$. Suppose that $\mathcal{F}_s(g)$ is the collection of homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion such that $f|_{\mathbb{D}} = g$. Then $\mathcal{F}_s(g) \neq \emptyset$. Moreover*

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F}_s(g) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)\} = +\infty, \tag{3.2}$$

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}_s(g),\ K_f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)\} = \max\left\{1, \frac{1}{s-1}\right\}, \tag{3.3}$$

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}_s(g) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \textit{ for a fixed } p > 1 \textit{ and } K_f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)\} \\ = \max\left\{\frac{1}{s-1}, \frac{3p}{(2s-1)p + 4 - 2s}\right\}, \tag{3.4}$$

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F}_s(g),\ f^{-1} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)\} = \frac{2(s+1)}{2s-1} \tag{3.5}$$

and

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F}_s(g),\ K_{f^{-1}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)\} = \frac{s+1}{s-1}. \tag{3.6}$$

One first proves $\mathcal{F}_s(g) \neq \emptyset$. Let M_s be as in (3.1). We use M_s as an intermediate domain between $\mathbb{D}$ and Δ_s. By the Riemann mapping theorem, there is a conformal mapping from $\mathbb{D} \cap \mathbb{R}^2_+$ onto $M_s \cap \mathbb{R}^2_+$ such that $\mathbb{D} \cap \mathbb{R}$ is mapped onto $M_s \cap \mathbb{R}$. It follows from the Schwarz reflection principle that there is a conformal mapping

$$g_s : \mathbb{D} \to M_s. \tag{3.7}$$

such that $g_s(\bar{z}) = \overline{g_s(z)}$ for all $z \in \mathbb{D}$. Moreover by the Osgood-Carathéodory theorem g_s has a homeomorphic extension from $\overline{\mathbb{D}}$ onto $\overline{M_s}$, still denoted g_s. We prove in [B] that g_s is a bi-Lipschitz mapping on $\overline{\mathbb{D}}$. By [23, Theorem A], there is a bi-Lipschitz mapping $g_s^c : \mathbb{D}^c \to M_s^c$ such that $g_s^c|_{\mathbb{S}^1} = g_s$. Let

$$G_s(z) = \begin{cases} g_s(z) & \forall z \in \overline{\mathbb{D}}, \\ g_s^c(z) & \forall z \in \mathbb{D}^c. \end{cases} \tag{3.8}$$

Then G_s is an orientation-preserving bi-Lipschitz mapping. Let g be as in Theorem 3.1, g_s be as in (3.7) and $h_s = z^2 \circ g_s$. Since $h_s : \mathbb{D} \to \Delta_s$ is conformal, there is a Möbius transformation

$$m_s(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z} \qquad \text{where } \theta \in [0, 2\pi] \text{ and } |a| < 1$$

such that $g(z) = h_s \circ m_s(z)$ for all $z \in \mathbb{D}$. Since $m_s : \mathbb{S}^1 \to \mathbb{S}^1$ is a bi-Lipschitz mapping, by [23, Theorem A] there is a bi-Lipschitz mapping $m_s^c : \mathbb{D}^c \to \Delta_s^c$ such that $m_s^c|_{\mathbb{S}^1} = m_s$. Define

$$\mathfrak{M}_s(z) = \begin{cases} m_s(z) & \forall z \in \overline{\mathbb{D}}, \\ m_s^c(z) & \forall z \in \mathbb{D}^c. \end{cases} \tag{3.9}$$

Then $\mathfrak{M}_s : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving bi-Lipschitz mapping. Define

$$\begin{aligned} \mathcal{E}_s = \{f : f : \mathbb{R}^2 \to \mathbb{R}^2 &\text{ is a homeomorphism of finite distortion} \\ &\text{and } f(z) = z^2 \text{ for all } z \in \overline{M_s}\}. \end{aligned}$$

If $E \in \mathcal{E}_s$, we can obtain

$$E \circ G_s \circ \mathfrak{M}_s \in \mathcal{F}_s(g). \tag{3.10}$$

We now divide the construction of E into two steps: Step 1 deals with the construction in a neighborhood of the cusp point, see FIGURE 2; Step 2 gives the construction on the domain away from the cusp point. Fix $s > 1$, and define

$$\eta(x) = \sqrt{x}(1 + x^{2(s-1)})^{\frac{1}{4}} \qquad \text{for all } x > 0.$$

For a given $t \ll 1$, let

$$L_t^1 = \eta((t/2)^2),\ L_t^2 = \eta(t^2),\ \sigma_t = L_t^2 - L_t^1,$$
$$Q_t = \overline{B(0, L_t^2)} \setminus (B(0, L_t^1) \cup M_s),$$
$$\tilde{Q}_t = \{(x, y) \in \mathbb{R}^2 : x \in [-t^2, -(t/2)^2],\ |y| \le |x|^s\}.$$

Let $f_1(x, y) = xe^{iy}$. By stretching in one direction and keeping the other direction invariant, we define f_3 such that $\tilde{R}_t = f_3(\tilde{Q}_t)$ is a rectangle. Analogously, we define f_2 such that $R_t = f_2 \circ f_1^{-1}(Q_t)$ is a square. Denote by P_1, P_2, P_3, P_4 and $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ the four vertices of $\tilde{R}_t$ and R_t, respectively. Then

$$P_1 = (L_t^1, \frac{\sigma_t}{2}),\ P_2 = (L_t^2, \frac{\sigma_t}{2}),\ P_3 = (L_t^2, -\frac{\sigma_t}{2}),\ P_4 = (L_t^1, -\frac{\sigma_t}{2})$$

and

$$\tilde{P}_1 = ((t/2)^2, t^{2s}),\ \tilde{P}_2 = (t^2, t^{2s}),\ \tilde{P}_3 = (t^2, -t^{2s}),\ \tilde{P}_4 = ((t/2)^2, -t^{2s}).$$

Since ∂M_s is mapped onto $\partial \Delta_s$ by z^2, the line segment $\tilde{P}_1\tilde{P}_2$ is mapped onto P_1P_2 by

$$(u, t^{2s}) \mapsto \Big(\eta(u), \frac{\sigma_t}{2}\Big) \qquad \forall u \in [(t/2)^2, t^2], \tag{3.11}$$

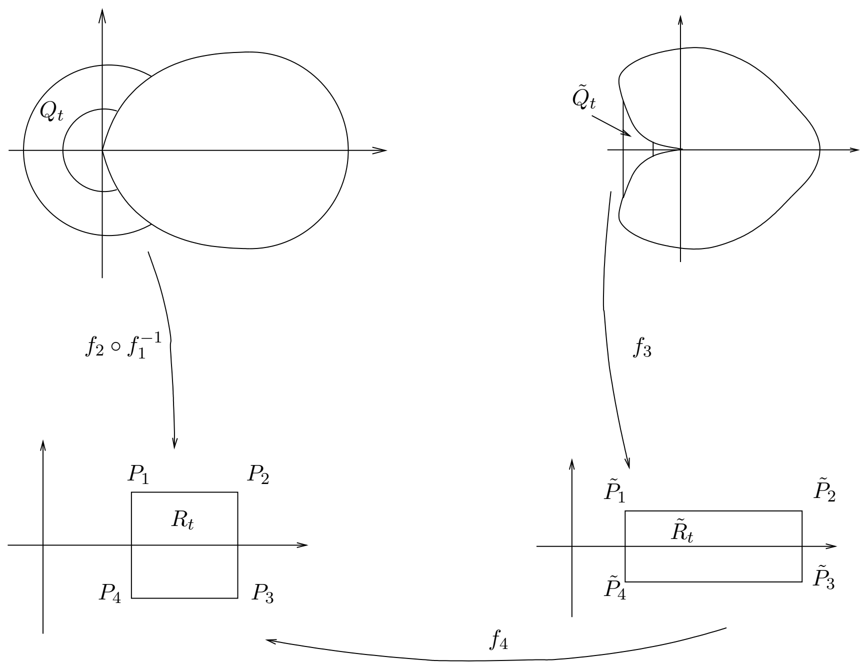

FIGURE 2. The construction $f_3^{-1} \circ f_4^{-1} \circ f_2 \circ f_1^{-1} : Q_t \to \tilde{Q}_t$

and the line segment $\tilde{P}_4\tilde{P}_3$ is mapped onto P_4P_3 by

$$(u, -t^{2s}) \mapsto \left(\eta(u), -\frac{\sigma_t}{2}\right) \qquad \forall u \in [(t/2)^2, t^2].$$

Define

$$f_4(u, v) = \left(\eta(u), \frac{\sigma_t}{2t^{2s}} v\right) \qquad \forall (u, v) \in \tilde{R}_t. \tag{3.12}$$

We have that

$$F_t = f_3^{-1} \circ f_4^{-1} \circ f_2 \circ f_1^{-1}.$$

is a diffeomorphism from Q_t onto $\tilde{Q}_t$.

For a fixed large j_0, we now consider the set Q_t with $t = 2^{-j}$ for all $j \geq j_0$. Define

$$E_1 = \sum_{j=j_0}^{+\infty} F_{2^{-j}} \chi_{Q_{2^{-j}}}.$$

Let $\Omega_1 = \cup_{j=j_0}^{+\infty} Q_{2^{-j}}$ and $\tilde{\Omega}_1 = \cup_{j=j_0}^{+\infty} \tilde{Q}_{2^{-j}}$. We can prove that $E_1 : \Omega_1 \to \tilde{\Omega}_1$ is a homeomorphism of finite distortion. Set

$$\Omega_2 = M_s^c \setminus \Omega_1 \text{ and } \tilde{\Omega}_2 = \Delta_s^c \setminus \tilde{\Omega}_1.$$

We have that both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$ are chord-arc curves. Define

$$h(z) = \begin{cases} E_1(z) & \forall z \in \partial\Omega_2 \cap \partial\Omega_1, \\ z^2 & \forall z \in \partial\Omega_2 \cap \partial M_s. \end{cases}$$

We have that $h : \partial\Omega_2 \to \partial\tilde{\Omega}_2$ is a bi-Lipschitz mapping. By [12] there is a bi-Lipschitz extension $E_2 : \Omega_2 \to \tilde{\Omega}_2$ of h. Define

$$E(x,y) = \begin{cases} E_1(x,y) & \text{for all } (x,y) \in \Omega_1, \\ E_2(x,y) & \text{for all } (x,y) \in \Omega_2, \\ (x^2-y^2, 2xy) & \text{for all } (x,y) \in \overline{M_s}. \end{cases} \tag{3.13}$$

We can prove that $E \in \mathcal{E}_s$.

The following lemma shows the integrability degrees of potential extensions.

Lemma 3.2. *Let Δ_s be as in (3.1) with $s > 1$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism of finite distortion such that f maps $\mathbb{D}$ conformally onto Δ_s. We have that*

1. *if $f^{-1} \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p \geq 1$ then $p < 2(s+1)/(2s-1)$,*
2. *if $K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$ then $q < s + 1/(s-1)$,*
3. *if $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$ then $q < \max\{1, 1/(s-1)\}$,*
4. *if $s > 2$, $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_f \in L^q_{\text{loc}}$ for some $q \in (0,1)$, then $q < 3p/((2s-1)p + 4 - 2s)$.*

We next sketch the proof of (3.3). Analogously, we can prove (3.2), (3.4), (3.5) and (3.6). By Lemma 3.2 (1), it suffices to construct $f \in \mathcal{F}_s(g)$ such that $K_f \in L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q < \max\{1, 1/(s-1)\}$. Let G_s be as in (3.8) and $\mathfrak{M}_s$ be as in (3.9). If $E \in \mathcal{E}_s$, we can prove that

$$\int_A K^q_{E\circ G_s\circ\mathfrak{M}_s}(z)\,dz \approx \int_{G_s\circ\mathfrak{M}_s(A)} K^q_E(w)\,dw \tag{3.14}$$

for any $q \geq 0$ and any compact set $A \subset \mathbb{R}^2$. If $E \in \mathcal{E}_s$ with $K_E \in L^q_{\text{loc}}$ for all $q < \max\{1, 1/(s-1)\}$, then by (3.10) and (3.14) we can define $f = E \circ G_s \circ \mathfrak{M}_s$. Let E be as in (3.13). We can prove that $K_E \in L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q < 1/(s-1)$. Therefore (3.3) holds whenever $s \in (1,2)$.

We next consider the case $s \in [2,\infty)$. It suffices to construct a mapping $E \in \mathcal{E}_s$ such that $K_E \in L^q_{\text{loc}}$ for all $q < 1$. We now redefine $f_4^{-1} : R_t \to \tilde{R}_t$ as in (3.12), see FIGURE 3. Let α_t

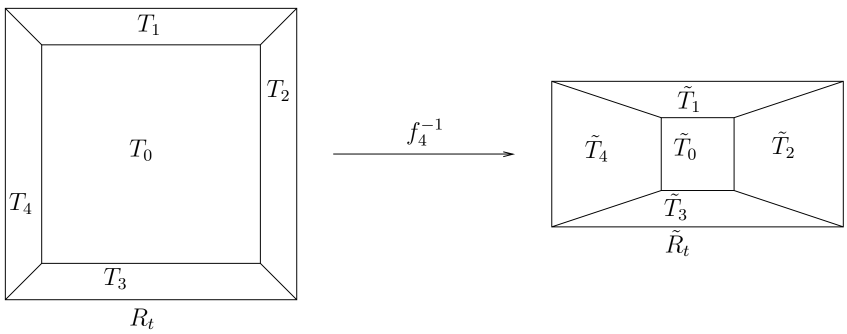

FIGURE 3. The redefined $f_4^{-1} : R_t \to \tilde{R}_t$

and β_t be the length of sides of $\tilde{R}_t$, and γ_t be the length of a side of R_t. Let $\tilde{T}_0 = \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3\tilde{Q}_4$ be the square concentric with $\tilde{R}_t$ and with side length $\beta_t/2$. Set

$$\delta_t = \exp(-t^{-1}) \qquad \text{for } t > 0$$

and let $T_0 = Q_1Q_2Q_3Q_4$ be the square concentric with R_t and with side length $\gamma_t(1-2\delta_t)$. We divide $R_t \setminus T_0$ into four isosceles trapezoids $T_1,\ T_2,\ T_3$ and T_4. Similarly, we obtain isosceles trapezoids $\tilde{T}_1,\ \tilde{T}_2,\ \tilde{T}_3,\ \tilde{T}_4$ from $\tilde{R}_t \setminus \tilde{T}_0$. We first define a diffeomorphism $A = (A_1, A_2)$ from T_1 onto $\tilde{T}_1$. We define a unique linear mapping A_2. Based on A_2, by (3.11) we define A_1^{-1}. Actually the process of defining A can be expressed as filling T_1 by copies of the stretched line segment P_1P_2. By symmetry, we can obtain a diffeomorphism from T_3 onto $\tilde{T}_3$. We next define a diffeomorphism $B = (B_1, B_2)$ from T_2 onto $\tilde{T}_2$. We define a unique linear mapping B_1. Based on the aim that B match A well on $T_1 \cap T_2$, we uniquely define B_2. By symmetry, we can obtain a diffeomorphism from T_4 onto $\tilde{T}_4$. Now we have the exact formulas for the mappings from Q_1Q_2 onto $\tilde{Q}_1\tilde{Q}_2$, Q_1Q_4 onto $\tilde{Q}_1\tilde{Q}_4$, from Q_4Q_3 onto $\tilde{Q}_4\tilde{Q}_3$ and from Q_2Q_3 onto $\tilde{Q}_2\tilde{Q}_3$. We take the natural extension $C : T_0 \to \tilde{T}_0$ of our boundary map. By the above construction, we have redefined f_4^{-1}. Following all processes for (3.13), we then define a new E. We can prove that $\int_X K_E^q < +\infty$ for all compact set $X \subset \mathbb{R}^2$, $q \in (0,1)$ and each $s > 1$. Hence we finish (3.3) for $s \in [2,\infty)$.

The following theorem from [B] gives an answer to Question 1.3 when $\Omega = \Delta$.

Theorem 3.3. *Let $\mathcal{F}$ be the collection of homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion such that $f(z) = (z+1)^2$ for all $z \in \mathbb{D}$. Then $\mathcal{F} \neq \emptyset$. Moreover*

$$\sup\{p \in [1,+\infty) : f \in \mathcal{F} \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)\} = +\infty, \tag{3.15}$$

$$\sup\{q \in (0,+\infty) : f \in \mathcal{F},\ K_f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)\} = 2, \tag{3.16}$$

$$\begin{aligned}&\sup\{q \in (0,+\infty) : f \in \mathcal{F} \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2) \textit{ for some } p > 1 \textit{ and } K_f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)\}\\ &= 1,\end{aligned} \tag{3.17}$$

$$\sup\{p \in [1,+\infty) : f \in \mathcal{F},\ f^{-1} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)\} = \frac{5}{2} \tag{3.18}$$

and

$$\sup\{q \in (0,+\infty) : f \in \mathcal{F},\ K_{f^{-1}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)\} = 5. \tag{3.19}$$

We next briefly explain the main ideas for the proof of Theorem 3.3. Let Δ be as in (2.2). The representation of $\partial\Delta$ in Cartesian coordinates is

$$(x^2+y^2)^2 - 4x(x^2+y^2) - 4y^2 = 0.$$

Hence we can parametrize $\partial\Delta$ in a neighborhood of the origin as

$$\tilde{\Gamma}_0 = \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = d(x)\},$$

where $j_0 \gg 1$ and $d(x) = \frac{-x^3(4-x)}{2-x^2+2x+\sqrt{1+2x}}$. Since $d(x) \approx |x|^3$ for all $|x| \ll 1$, there are $c_1 > 0,\ c_2 > 0$ such that

$$-c_1x^3 \le d(x) \le -c_2x^3 \qquad \forall x \in [-2^{-j_0}, 0].$$

Denote

$$\begin{aligned}\tilde{\Gamma}_1 &= \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = -c_1x^3\},\\ \tilde{\Gamma}_2 &= \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = -c_2x^3\},\\ \tilde{\Gamma}_3 &= \{(x,y) \in \mathbb{R}^2 : x = -2^{-j_0}, y^2 \in [c_1(2^{-j_0})^3, d(-2^{-j_0})\},\\ \tilde{\Gamma}_4 &= \{(x,y) \in \mathbb{R}^2 : x = -2^{-j_0}, y^2 \in [d(-2^{-j_0}), c_2(2^{-j_0})^3]\}.\end{aligned}$$

Let $\tilde{\Omega}_u$ and $\tilde{\Omega}_d$ be the domains bounded by $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_4$ and $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_3$, respectively. Denote by Ω_u, Ω_d and Γ_k for $k = 0, ..., 4$ the images of $\tilde{\Omega}_u, \tilde{\Omega}_d$ and $\tilde{\Gamma}_k$ under the branch of complex-valued function $z^{1/2}$ with $1^{1/2} = 1$, respectively.

We first prove the existence of an extension in Theorem 3.3, see FIGURE 4. Let $r =$

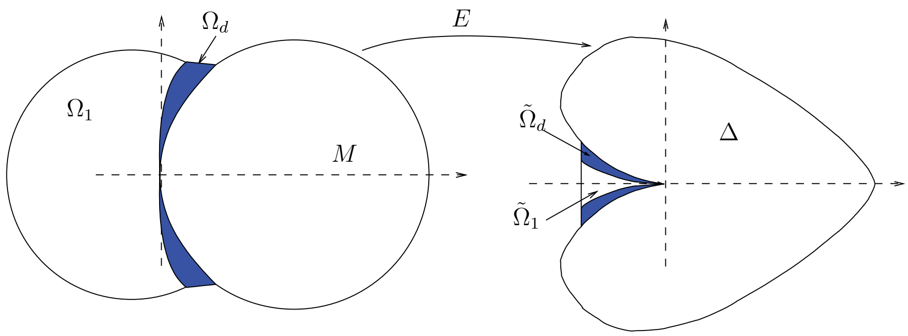

FIGURE 4. The existence of an extension

$(2^{-2j_0} + c_1 2^{-3j_0})^{1/4}$. Denote

$$M = \{(x+1, y) \in \mathbb{R}^2 : (x, y) \in \mathbb{D}\},$$

$$\Omega_1 = \overline{B(0,r)} \setminus (M \cup \Omega_d), \ \ \Omega_2 = \mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_d \cup M),$$

$$\tilde{\Omega}_1 = \{(x, y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 \leq c_1|x|^3\} \text{ and } \tilde{\Omega}_2 = \mathbb{R}^2 \setminus (\tilde{\Omega}_1 \cup \tilde{\Omega}_d \cup \Delta).$$

Analogously to the arguments for $\mathcal{F}_s(g)$ in Theorem 3.1, we define $E_1 : \Omega_1 \to \tilde{\Omega}_1$ and $E_2 : \Omega_2 \to \tilde{\Omega}_2$. Here $\eta(x) = \sqrt{x}(1 + c_1 x)^{1/4}$ and $s = 3/2$. Define

$$E(x, y) = \begin{cases} E_1(x, y) & \forall\ (x, y) \in \Omega_1, \\ E_2(x, y) & \forall\ (x, y) \in \Omega_2, \\ (x^2 - y^2, 2xy) & \forall\ (x, y) \in M \cup \Omega_d, \end{cases} \tag{3.20}$$

and $f_0(x, y) = E(x+1, y)$. By analogous arguments as for $\mathcal{F}_s(g)$ in Theorem 3.1, we have that $f_0 \in \mathcal{F}$.

We next prove (3.16). Suppose $f \in \mathcal{F}$. Then $\hat{f}(u, v) = f(u-1, v)$ is a homeomorphism of finite distortion on $\mathbb{R}^2$ and $\hat{f}(M \setminus \Omega_u) = \Delta \setminus \tilde{\Omega}_u$. A result analogous to Lemma 3.2 shows that if $K_{\hat{f}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ then $q < 2$. Therefore if $K_f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ then $q < 2$. In order to prove (3.16), it then suffices to construct a mapping $f_0 \in \mathcal{F}$ such that $K_{f_0} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 2$. Let E be as in (3.20) and $f_0(x, y) = E(x+1, y)$. Then we can prove that $f_0 \in \mathcal{F}$ and $K_{f_0} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 2$.

The strategies to prove (3.15), (3.17), (3.18) and (3.19) are same as the one to prove (3.16).

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Included articles

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Weighted estimates for diffeomorphic extensions of homeomorphisms

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Weighted estimates for diffeomorphic extensions of homeomorphisms

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. Then there is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ. We study the relationship between weighted integrability of the derivatives of h and double integrals of φ and of φ^{-1}.

Keywords: Poisson extension, diffeomorphism, internal chord-arc domain.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Suppose that φ is a homeomorphism from the unit circle $\mathbb{S}^1$ onto $\partial\Omega$. Then, by Radó [13], Kneser [7], Choquet [3] and Lewy [10], the complex-valued Poisson extension h of φ is a diffeomorphism from $\mathbb{D}$ onto Ω. We are interested in the integrability degrees of the derivatives of h. In 2007, G. C. Verchota [14] proved that the derivatives of h may fail to be square integrable but that they are necessarily p-integrable over $\mathbb{D}$ for all $p < 2$. In 2009, T. Iwaniec, G. J. Martin and C. Sbordone improved on [5] by showing that the derivatives belong to weak-L^2 with sharp estimates. Actually

$$\int_{\mathbb{D}} |Dh(z)|^2 \, dz \approx \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|\varphi(\xi) - \varphi(\eta)|^2}{|\xi - \eta|^2} \, |d\xi| \, |d\eta|, \tag{1.1}$$

since harmonic functions minimize the L^2-energy and the right-hand side of (1.1) is the trace norm of $\dot{W}^{1,2}(\mathbb{D})$. In [1], it was further shown that if additionally $\partial\Omega$ is a C^1-regular Jordan curve then

$$\int_{\mathbb{D}} |Dh(z)|^2\, dz < \infty \Leftrightarrow \int_{\partial\Omega}\int_{\partial\Omega} |\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||\,|d\xi|\,|d\eta| < \infty. \tag{1.2}$$

All the above results require the target domain to be convex.

If Ω is a bounded non-convex Jordan domain, then there exists a homeomorphism $\varphi : \mathbb{S}^1 \to \partial\Omega$ for which the harmonic extension fails to map $\mathbb{D}$ homeomorphically onto Ω, see [3, 7]. Hence we cannot use the harmonic extension to produce a diffeomorphic extension. Nevertheless, (weighted) analogs of the results as (1.2) for diffeomorphic extensions in the case of an internal chord-arc Jordan domain exist, see [9]. For the definition of (internal) chord-arc domains, we refer to Definition 2.1. Notice that each bounded convex Jordan domain is a chord-arc domain. In this paper, we generalize the results in [9] to the weighted L^p-setting.

Let Ω be an internal chord-arc Jordan domain with the internal distance λ_Ω. Assume that $h : \mathbb{D} \to \Omega$ is a diffeomorphism and $\varphi : \mathbb{S}^1 \to \partial\Omega$ is a homeomorphism. Set $\delta(z) = 1 - |z|$. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define

$$I_1(p,\alpha,\lambda,h) = \int_{\mathbb{D}} |Dh(z)|^p \delta^\alpha(z) \log^\lambda(2\delta^{-1}(z))\, dz,$$

$$I_2(p,\alpha,\lambda,h) = \int_{\mathbb{D}} |Dh(z)|^p \log^\lambda(e + |Dh(z)|)\delta^\alpha(z)\, dz,$$

$$\mathcal{U}(p,\alpha,\lambda,\varphi) = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{\lambda_\Omega^p(\varphi(\xi),\varphi(\eta))}{|\xi-\eta|^{p-\alpha}} \log^\lambda\big(e + \frac{\lambda_\Omega(\varphi(\xi),\varphi(\eta))}{|\xi-\eta|}\big)\,|d\eta|\,|d\xi|,$$

$$\mathcal{A}_{p,\alpha,\lambda}(t) = \int_1^t -x^{1+\alpha-p}\log_2^\lambda(x^{-1})\, dx \qquad \forall t \geq 0,$$

$$\mathcal{V}(p,\alpha,\lambda,\varphi) = \int_{\partial\Omega}\Big(\int_{\partial\Omega} \mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|)\,|d\eta|\Big)^{p-1}|d\xi|.$$

Our main result is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an internal chord-arc Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. There is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which, for any $p > 1$, we have that*

(1) if either $\alpha \in (p-2, +\infty)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in (-\infty, -1)$,

both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are finite.

(2) if either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$,

$$\text{both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h) \text{ are comparable to } \mathcal{U}(p,\alpha,\lambda,\varphi).$$

Moreover whenever $p \in (1,2]$

$$\text{both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h) \text{ dominate } \mathcal{V}(p,\alpha,\lambda,\varphi),$$

while

$$\mathcal{V}(p,\alpha,\lambda,\varphi) \text{ controls both } I_1(p,\alpha,\lambda,h) \text{ and } I_2(p,\alpha,\lambda,h)$$

for all $p \in [2, +\infty)$. Furthermore both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are in general comparable to $\mathcal{V}(p,\alpha,\lambda,\varphi)$ only for $p=2$.

For any $p > 1$, there is no diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_1(p,\alpha,\lambda,h) < +\infty$ for either $\alpha \in (-\infty,-1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1,+\infty)$; or for which $I_2(p,\alpha,\lambda,h) < +\infty$ for some $\alpha \in (-\infty,-1]$ and $\lambda \in \mathbb{R}$.

Motivated by (1.2), one could hope to use $\mathcal{V}(p,\alpha,\lambda,\varphi)$ to control both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$. Example 4.2 together with Example 4.3 shows that $\mathcal{V}(p,\alpha,\lambda,\varphi)$ is comparable to $I_1(p,\alpha,\lambda,h)$ or $I_2(p,\alpha,\lambda,h)$ only when $p=2$. Theorem 1.1 does not cover the case where $p > 1$, $\alpha = -1$ and $\lambda \in (-\infty,-1)$. We will return to this case in a future paper.

The structure of this paper is the following. In the next section, we give some preliminaries. Section 3 is the proof of Theorem 1.1. The final section contains several examples related to Theorem 1.1 (2).

2 Preliminaries

By $s \gg 1$ and $t \ll 1$ we mean that s is sufficiently large and t is sufficiently small, respectively. By $f \lesssim g$ we mean that there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for every x. If $f \lesssim g$ and $g \lesssim f$ we may denote $f \approx g$. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of all positive integers and the set of all real numbers. Let $\mathcal{L}^2$ (respectively $\mathcal{L}^1$) be the 2-dimensional (1-dimensional) Lebesgue measure. For sets $E \in \mathbb{R}^2$ and $F \in \mathbb{R}^2$, let $\operatorname{diam}(E)$ be the diameter of E, and $\operatorname{dist}(E,F)$ be the Euclidean distance between E and F. Let $B(p,r)$ be the disk with center P and radius r.

Definition 2.1. A Jordan domain $\Omega \subset \mathbb{R}^2$ is an **internal chord-arc Jordan domain** if $\partial\Omega$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \partial\Omega$,

$$\ell(w_1,w_2) \leq C\lambda_\Omega(w_1,w_2), \tag{2.1}$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial\Omega$ joining w_1 to w_2, and $\lambda_\Omega(w_1, w_2)$ is the **internal distance** between w_1, w_2, which is defined as

$$\lambda_\Omega(w_1, w_2) = \inf_\alpha \ell(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha \subset \Omega$ joining w_1 and w_2; if there is no rectifiable curve joining w_1 and w_2, we set $\lambda_\Omega(w_1, w_2) = \infty$; cf. [12, Section 3.1] or [2, Section 2].

If (2.1) holds for the Euclidean distance instead of the internal distance, we call Ω be a **chord-arc domain**. Naturally, every chord-arc Jordan domain is an internal chord-arc domain, but there are internal chord-arc domains that fail to be chord-arc; e.g. the standard cardioid domain

$$\Delta = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 < 0\}.$$

2.1 Dyadic decomposition

Given $j \in \mathbb{N}$ and $k = 1, ..., 2^j$, let

$$I_{j,k} = [2\pi(k-1)2^{-j}, 2\pi k 2^{-j}],\ \Gamma_{j,k} = \{e^{i\theta} : \theta \in I_{j,k}\}. \tag{2.2}$$

Then $\{I_{j,k}\}$ is a dyadic decomposition of $[0, 2\pi]$ and $\{\Gamma_{j,k}\}$ is a dyadic decomposition of $\mathbb{S}^1$. We call $\Gamma_{j,k}$ a j-level dyadic arc. Moreover we have that

$$\ell(\Gamma_{j,k}) \approx 2^{-j} \qquad \forall j \in \mathbb{N} \text{ and } k = 1, ..., 2^j. \tag{2.3}$$

Based on (2.2), there is a decomposition of the unit disk $\mathbb{D}$ given by $\{Q_{j,k} : j \in \mathbb{N} \text{ and } k = 1, ..., 2^j\}$, where

$$Q_{j,k} = \left\{re^{i\theta} : 1 - 2^{1-j} \le r \le 1 - 2^{-j} \text{ and } \theta \in I_{j,k}\right\}. \tag{2.4}$$

By (2.3) it follows that

$$\mathcal{L}^2(Q_{j,k}) \approx 2^{-2j} \approx \ell(\Gamma_{j,k})^2 \qquad \forall j \in \mathbb{N} \text{ and } k = 1, ..., 2^j. \tag{2.5}$$

Moreover there is a uniform constant $C > 0$ such that for any $Q_{j,k}$ there is a disk $B_{j,k}$ satisfying

$$B_{j,k} \subset Q_{j,k} \subset CB_{j,k}. \tag{2.6}$$

2.2 A_p weights

Definition 2.2. For a given $p \in (1, +\infty)$, a locally integrable function $w : \mathbb{R}^2 \to [0, +\infty)$ is an A_p weight if there is a constant $C > 0$ such that for any disk $B \subset \mathbb{R}^2$ we have that

$$\frac{1}{\mathcal{L}^2(B)} \int_B w(x)\, dx \leq C\Big(\frac{1}{\mathcal{L}^2(B)} \int_B w(x)^{\frac{1}{1-p}}\, dx\Big)^{1-p}.$$

Next, w is an A_1 weight if there is a constant $C > 0$ such that

$$\frac{1}{\mathcal{L}^2(B)} \int_B w(z)\, dz \leq C w(x)$$

for each disk $B \subset \mathbb{R}^2$ and all $x \in B$.

For more information on A_p weights, we recommend [4, 6, 11]. Let $\delta(x) = \operatorname{dist}(\mathbb{S}^1, x)$. Given $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, we define

$$w_{\alpha,\lambda}(x) = \begin{cases} \delta(x)^\alpha \log^\lambda(2\delta^{-1}(x)) & 0 \leq |x| \leq 2, \\ \log^\lambda(2) & |x| \geq 2. \end{cases} \tag{2.7}$$

It is well known that $w_{\alpha,0}$ belongs to A_p. We now generalize this to all $\lambda \in \mathbb{R}$.

Proposition 2.3. *Let $p \geq 1$ and $w_{\alpha,\lambda}$ be as in (2.7). Then $w_{\alpha,\lambda}$ is an A_p weight for all $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. The idea of proof is to use the Jones factorization of A_p weights (see [6]), i.e. we should prove $w_{\alpha,\lambda} = w_1 w_2^{1-p}$ for two A_1 weights w_1 and w_2.

We first consider the case $\lambda \geq 0$. For a given $\alpha \in (-1, p-1)$, there uniquely exist $a_1 \in (0, 1)$ and $a_2 \in (0, 1)$ such that $\alpha = a_1(-1) + a_2(p-1)$. Set $\alpha_1 = -a_1$, $\alpha_2 = -a_2$, $\lambda_1 = p\lambda$ and $\lambda_2 = \lambda$. We define

$$w_1(x) = \begin{cases} \delta(x)^{\alpha_1} \log^{\lambda_1}(2\delta^{-1}(x)) & 0 \leq |x| \leq 2, \\ \log^{\lambda_1}(2) & |x| \geq 2, \end{cases} \tag{2.8}$$

and

$$w_2(x) = \begin{cases} \delta(x)^{\alpha_2} \log^{\lambda_2}(2\delta^{-1}(x)) & 0 \leq |x| \leq 2, \\ \log^{\lambda_2}(2) & |x| \geq 2. \end{cases} \tag{2.9}$$

We next prove that w_1 is an A_1 weight, i.e.

$$⨍_B w_1(x)dx \lesssim \inf_{x \in B} w_1(x) \tag{2.10}$$

for every disk $B \subset \mathbb{R}^2$. Let $d_B = \mathrm{dist}(B, \mathbb{S}^1)$.

Case 1: $d_B \geq \mathrm{diam}(B)/2$. We have that

$$d_B \leq \delta(x) \leq 3d_B \qquad \forall x \in B. \tag{2.11}$$

If $1 \leq d_B$, then $\delta(x) \geq 1$ for all $x \in B$. Therefore $w_1(x) = \log^{\lambda_1}(2)$ whenever $x \in B$. Of course (2.10) holds now. If $3d_B \leq 1$, then $w_1(x) = \delta(x)^{\alpha_1} \log^{\lambda_1}(2\delta^{-1}(x))$ for all $x \in B$. By (2.11) it hence follows that $w_1(x) \approx d_B^{\alpha_1} \log^{\lambda_1}\left(2d_B^{-1}\right)$ whenever $x \in B$. Therefore (2.10) holds. If $d_B < 1 < 3d_B$, let $B_1 = \{x \in B : d_B < \delta(x) < 1\}$ and $B_2 = \{x \in B : 1 \leq \delta(x) < 3d_B\}$. Then $B = B_1 \cup B_2$ and

$$w_1(x) = \log^{\lambda_1}(2) \qquad \text{whenever } x \in B_2. \tag{2.12}$$

Since

$$\left[t^{\alpha_1} \log^{\lambda_1}\left(2t^{-1}\right)\right]' = t^{\alpha_1 - 1} \log^{\lambda_1}(2t^{-1}) \left(\alpha_1 - \frac{\lambda_1}{\log(2t^{-1})}\right) < 0, \tag{2.13}$$

for all $t \in (0, 1]$, we have that

$$w_1(x) \leq d_B^{\alpha_1} \log^{\lambda_1}(2d_B^{-1}) \leq \frac{\log^{\lambda_1}(6)}{3^{\alpha_1}} \qquad \forall x \in B_1. \tag{2.14}$$

Combining (2.12) and (2.14) implies that

$$\begin{aligned}
\fint_B w_1(x)dx &= \frac{1}{\mathcal{L}^2(B)} \left(\int_{B_1} w_1 + \int_{B_2} w_1\right) \\
&\leq \frac{1}{\mathcal{L}^2(B)} \left(\mathcal{L}^2(B_1) \frac{\log^{\lambda_1}(6)}{3^{\alpha_1}} + \mathcal{L}^2(B_2) \log^{\lambda_1}(2)\right) \\
&\lesssim \log^{\lambda_1}(2) = \inf_{x \in B} w_1(x).
\end{aligned}$$

Case 2: $d_B < \mathrm{diam}(B)/2$ and $\mathrm{diam}(B) \leq 2/3$. Pick $x' \in \partial B$ and $x_0 \in \mathbb{S}^1$ such that $\mathrm{dist}(B, \mathbb{S}^1) = |x' - x_0|$. Let $r_B = 3\mathrm{diam}(B)/2$. Since

$$|x - x_0| \leq |x - x'| + |x' - x_0| \leq r_B$$

for all $x \in B$, we have $B \subset B(x_0, r_B)$. Let $E = \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \mathbb{S}^1) < r_B\}$. Then $B(x_0, r_B) \subset E$. Since $\mathcal{L}^2(B(x_0, r_B)) = \pi r_B^2$ and $\mathcal{L}^2(E) = 4\pi r_B$, the maximal number of pairwise disjoint open disks $B(x, r_B)$ with $x \in \mathbb{S}^1$ is less than $4r_B^{-1}$. We have that

$$\begin{aligned}
\frac{1}{\mathcal{L}^2(B)} \int_B w_1(x)\, dx &\leq \frac{1}{\mathcal{L}^2(B)} \int_{B(x_0, r_B)} w_1(x)\, dx \\
&\lesssim \frac{r_B}{\mathcal{L}^2(B)} \int_E w_1(x)\, dx \approx \frac{1}{r_B} \int_0^{r_B} t^{\alpha_1} \log^{\lambda_1}\left(2t^{-1}\right) dt.
\end{aligned} \tag{2.15}$$

Notice that

$$\left[t^{\alpha_1+1}\log^{\lambda_1}\left(2t^{-1}\right)\right]' = t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\left(\alpha_1+1-\frac{\lambda_1}{\log(2t^{-1})}\right) \qquad t>0. \tag{2.16}$$

Since $\lim_{t\to 0^+}\alpha_1+1-\frac{\lambda_1}{\log(2t^{-1})} = \alpha_1+1$ and $\alpha_1+1-\frac{\lambda_1}{\log(2t^{-1})}$ is decreasing with respect to $t>0$, there exists $\epsilon\in(0,1)$ determined by α_1 and λ_1 such that $\alpha_1+1-\frac{\lambda_1}{\log(2\epsilon^{-1})}\ge(\alpha_1+1)/2$. We then obtain from (2.16) that

$$\left[t^{\alpha_1+1}\log^{\lambda_1}\left(2t^{-1}\right)\right]' \ge \frac{\alpha_1+1}{2}t^{\alpha_1}\log^{\lambda_1}(2t^{-1})$$

for all $t\in[0,\epsilon r_B]$. Therefore

$$\int_0^{\epsilon r_B} t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\,dt = \frac{2(\epsilon r_B)^{\alpha_1+1}}{\alpha_1+1}\log^{\lambda_1}(2(\epsilon r_B)^{-1}) \lesssim r_B^{\alpha_1+1}\log^{\lambda_1}\left(2r_B^{-1}\right) \tag{2.17}$$

Moreover by (2.13) we have that

$$\int_{\epsilon r_B}^{r_B} t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\,dt \le (r_B-\epsilon r_B)(\epsilon r_B)^{\alpha_1}\log^{\lambda_1}(2(\epsilon r_B)^{-1}) \lesssim r_B^{\alpha_1+1}\log^{\lambda_1}(2r_B^{-1}). \tag{2.18}$$

Combining (2.15), (2.17) with (2.18) implies that

$$\frac{1}{|B|}\int_B w_1(x)\,dx \lesssim r_B^{\alpha_1}\log^{\lambda_1}\left(2r_B^{-1}\right).$$

Together with

$$r_B^{\alpha_1}\log^{\lambda_1}\left(2r_B^{-1}\right) = \inf_{t\in[0,r_B]} t^{\alpha_1}\log^{\lambda_1}\left(2t^{-1}\right) = \inf_{x\in E} w_1(x) \le \inf_{x\in B} w_1(x),$$

we hence obtain (2.10).

Case 3: $d_B<\mathrm{diam}(B)/2$ and $\mathrm{diam}(B)>2/3$. Let x' and x_0 be as in Case 2. Then $|x'| = 1+\mathrm{dist}(x',\mathbb{S}^1)\le 1+\mathrm{diam}(B)2^{-1}$. Together with the fact that $|x-x'|\le\mathrm{diam}(B)$ for all $x\in B$, we have $B\subset B(0,1+r_B)$. Moreover by (2.17) and (2.18), we obtain that

$$\int_{B(0,2)} w_1(x)\,dx = \int_{B(0,1)} + \int_{B(0,2)\setminus B(0,1)} = 4\pi\int_0^1 t^{\alpha_1}\log^{\lambda_1}(2t^{-1})\,dt \approx 1.$$

Therefore

$$\begin{aligned}
\frac{1}{\mathcal{L}^2(B)}\int_B w_1(x)\,dx \lesssim& \frac{1}{r_B^2}\left(\int_{B(0,2)} + \int_{B(0,1+r_B)\setminus B(0,2)}\right)\\
\lesssim& \frac{1}{r_B^2}\left(\mathcal{L}^2(B(0,2)) + \log^{\lambda_1}(2)\mathcal{L}^2(B(0,1+r_B)\setminus B(0,2))\right)\\
\lesssim& 1.
\end{aligned} \tag{2.19}$$

Moreover by the monotonicity of $t^{\alpha_1}\log^{\lambda_1}(2t^{-1})$ on $(0,+\infty)$, we have that

$$\log^{\lambda_1}(2) = \inf_{t\in[0,1+r_B]} t^{\alpha_1}\log^{\lambda_1}\left(2t^{-1}\right) = \inf_{x\in B(0,1+r_B)} w_1(x) \le \inf_{x\in B} w_1(x). \tag{2.20}$$

By combining (2.19) with (2.20), we obtain (2.10).

By the analogous arguments as for (2.10), we obtain $w_2 \in A_1$. Therefore the Jones factorization theorem implies that $w_{\alpha,\lambda} \in A_p$ for all $\alpha \in (-1, p-1)$ and $\lambda \ge 0$.

When $\lambda < 0$, define w_1 and w_2 as in (2.8) and (2.9) with $\lambda_1 = -\lambda$, $\lambda_2 = 2\lambda(1-p)^{-1}$ and both α_1 and α_2 invariant. By the same arguments as for the case $\lambda \ge 0$, we obtain that $w_{\alpha,\lambda} \in A_p$ whenever $\alpha \in (-1, p-1)$ and $\lambda < 0$. □

2.3 A class of functions

We define the Hardy-Littlewood maximal function for a Lebesgue measurable function f in $\mathbb{R}^2$ as

$$M_f(x) = \sup_{x\in B} \fint_B |f(z)|dz = \sup_{x\in B} \frac{1}{|B|}\int_B |f(z)|\,dz$$

where the supremum is taken over all disks $B \subset \mathbb{R}^2$ containing x. Let $p \in (1,\infty)$ and w be a weight. It is well-known that

$$\int_{\mathbb{R}^2} |M_f(x)|^p w(x)\,dx \lesssim \int_{\mathbb{R}^2} |f(x)|^p w(x)\,dx$$

if and only if w is an A_p weight. We generalize this to weighted Orlicz spaces. We begin with some definitions.

Definition 2.4. Let $\mathcal{F}$ be the collection of $\Phi : [0,\infty) \to [0,\infty)$, which is increasing and satisfies $\lim_{t\to0}\Phi(t) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. We say that $\Phi \in \mathcal{F}$ is the Young function, if Φ is convex on $[0,\infty)$ and $\lim_{t\to0}\Phi(t)/t = \lim_{t\to\infty} t/\Phi(t) = 0$.

Definition 2.5. We say that a function $\Phi : [0,+\infty) \to [0,+\infty)$ satisfies the Δ_2-condition if there is a constant $C > 0$ such that

$$\Phi(2t) \le C\Phi(t)$$

for all $t \in [0,+\infty)$.

Let $\Phi \in \mathcal{F}$ satisfying the Δ_2-condition. Put

$$h_\Phi(s) = \sup_{t>0} \frac{\Phi(st)}{\Phi(t)} \qquad s > 0.$$

We define the lower index of Φ by

$$i(\Phi) = \lim_{s\to 0} \frac{\log h_\Phi(s)}{\log s} = \sup_{0<s<1} \frac{\log h_\Phi(s)}{\log s}.$$

The quantity $i(\Phi)$ is well defined, see [8]. The following lemma is from [8, Theorem 2.1.1].

Lemma 2.6. *Let Φ be a Young function satisfying the Δ_2-condition and w be a weight on $\mathbb{R}^2$. Then the following conditions are equivalent:*

1. $\int_{\mathbb{R}^2} \Phi(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Phi(|f(x)|)w(x)\,dx$,
2. $w \in A_{i(\Phi)}$.

We next consider a special class of Young functions. Given $p > 1$ and $\lambda \in \mathbb{R}$, we set

$$\Phi_{p,\lambda}(t) = t^p \log^\lambda(e+t) \qquad \text{for } t \in [0, +\infty). \tag{2.21}$$

Proposition 2.7. *Let $\Phi_{p,\lambda}$ be as in (2.21) with $p > 1$ and $\lambda \geq 0$. Then $\Phi_{p,\lambda}$ is a Young function and satisfies the Δ_2-condition on $[0,\infty)$. Moreover $i(\Phi_{p,\lambda}) = p$.*

Proof. Simple calculations show that

$$\Phi'_{p,\lambda}(t) = \left(p\log(e+t) + \lambda\frac{t}{e+t}\right) t^{p-1}\log^{\lambda-1}(e+t) \tag{2.22}$$

and

$$\begin{aligned}\Phi''_{p,\lambda}(t) = &\left(p(p-1)\log^2(e+t) + \lambda e\frac{t}{(e+t)^2}\log(e+t) + R_{p,\lambda}(t)\right)\\ &\times t^{p-2}\log^{\lambda-2}(e+t)\end{aligned} \tag{2.23}$$

where $R_{p,\lambda}(t) = \lambda(\lambda-1)(t(e+t)^{-1})^2 + \lambda(2p-1)t(e+t)^{-1}\log(e+t)$. Since $p\log(e+t) + \lambda t(e+t)^{-1} > 0$ for all $t \in (0,+\infty)$, it follows from (2.22) that $\Phi'_{p,\lambda}(t) > 0$ for all $t > 0$. Therefore $\Phi_{p,\lambda}$ is strictly increasing on $[0,\infty)$. If $\lambda \geq 1$, we have that

$$R_{p,\lambda}(t) \geq 0 \qquad \text{for all } t \geq 0. \tag{2.24}$$

Whenever $0 \leq \lambda < 1$, since $t/(e+t) < 1$ and $\log(e+t) \geq 1$ for all $t \geq 0$ we have that

$$\begin{aligned}R_{p,\lambda}(t) =& \frac{t}{e+t}\left(\lambda(\lambda-1)\frac{t}{e+t} + \lambda(2p-1)\log(e+t)\right)\\ \geq& \frac{t}{e+t}\big(\lambda(\lambda-1) + \lambda(2p-1)\big) = \frac{t}{e+t}\left(\lambda^2 + 2\lambda(p-1)\right) \geq 0\end{aligned} \tag{2.25}$$

for all $t \geq 0$. By (2.23), (2.24) and (2.25), we have that $\Phi''_{p,\lambda}(t) \geq 0$ for all $t \geq 0$. Therefore $\Phi_{p,\lambda}$ is convex on $[0, +\infty)$. Hence $\Phi_{p,\lambda}$ is a Young function whenever $p > 1$ and $\lambda \geq 0$.

Since both t^p and $\log^\lambda(e+t)$ satisfy the Δ_2-condition on $[0, +\infty)$, $\Phi_{p,\lambda}$ satisfies Δ_2-condition also. Since $h_{\Phi_{p,\lambda}}(s) = s^p$ whenever $s \in (0, 1)$, we have $i(\Phi_{p,\lambda}) = p$. □

Remark 2.8. For $p > 1$, let $\Phi_{p,\lambda}$ be as in (2.21) with $\lambda \geq 0$. Assume that w is an A_p weight. Given a Lebesgue measurable function f, by Lemma 2.6 and Proposition 2.7 we have that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(|f(x)|)w(x)\,dx.$$

Let $\Phi_{p,\lambda}$ be as in (2.21) with $p > 1$ and $\lambda < 0$. By (2.22) and (2.23), we have that both monotonicity and convexity of $\Phi_{p,\lambda}$ may fail whenever $t \ll 1$, but still hold for all $t \gg 1$. We modify $\Phi_{p,\lambda}$ in a neighborhood of the origin such that it satisfies all conclusions in Proposition 2.7.

Since $2^{-1}(p+1)\log(e+t) \leq p\log(e+t) + \lambda t(e+t)^{-1}$ whenever $t \gg 1$, by (2.22) there is a constant $t_2 \gg 1$ such that

$$\frac{(p+1)\Phi_{p,\lambda}(t)}{2t} = \frac{p+1}{2} t^{p-1}\log^\lambda(e+t) \leq \Phi'_{p,\lambda}(t) \tag{2.26}$$

for all $t \geq t_2$. Without loss of generality, we assume that $\Phi_{p,\lambda}$ is strictly increasing and convex on $[t_2, \infty)$. Since $pt_2t^{p-1} + t^p \leq (p+1)t_2t^{p-1} \leq t_2^p \log^\lambda(e+t_2)$ for any $t \ll 1$, we have that

$$pt^{p-1}(t_2 - t) \leq \Phi_{p,\lambda}(t_2) - t^p \tag{2.27}$$

for all $t \ll 1$. Moreover when $t \leq t_2(p-1)/(p+1)$, we have that

$$\frac{\Phi_{p,\lambda}(t_2) - t^p}{t_2 - t} \leq \frac{\Phi_{p,\lambda}(t_2)}{t_2 - t} \leq \frac{(p+1)\Phi_{p,\lambda}(t_2)}{2t_2}. \tag{2.28}$$

Therefore by (2.26), (2.27) and (2.28), there exists a constant $t_1 \ll 1$ such that

$$pt_1^{p-1} \leq \frac{\Phi_{p,\lambda}(t_2) - t_1^p}{t_2 - t_1} \leq \frac{(p+1)\Phi_{p,\lambda}(t_2)}{2t_2} \leq \Phi'_{p,\lambda}(t_2).$$

Let $k = (\Phi_{p,\lambda}(t_2) - t_1^p)/(t_2 - t_1)$. Given $p > 1$ and $\lambda < 0$, we define

$$\Psi_{p,\lambda}(t) = \begin{cases} t^p & 0 \leq t < t_1, \\ k(t - t_1) + t_1^p & t_1 \leq t < t_2, \\ \Phi_{p,\lambda}(t) & t_2 \leq t. \end{cases} \tag{2.29}$$

Proposition 2.9. *The function $\Psi_{p,\lambda}$ is a Young function and satisfies the Δ_2-condition on $[0, \infty)$. Moreover $i(\Psi_{p,\lambda}) = p$.*

Proof. It is easy to see that $\Psi_{p,\lambda}$ is strictly increasing, continuous and convex on $[0, +\infty)$. Hence $\Psi_{p,\lambda}$ is a Young function. To prove the Δ_2-condition, it suffices to check that

$$\Psi_{p,\lambda}(2t) \le C\Psi_{p,\lambda}(t) \tag{2.30}$$

for all $t \in [0, +\infty)$. In fact, (2.30) is trivial if either $t \ge t_2$ or $2t < t_1$. Whenever $t \in [t_1/2, t_2]$, by the monotonicity of $\Psi_{p,\lambda}$ we have that

$$\frac{\Psi_{p,\lambda}(2t)}{\Psi_{p,\lambda}(2t_2)} \le 1 \le \frac{\Psi_{p,\lambda}(t)}{\Psi_{p,\lambda}(t_1/2)}.$$

Let $s \ll 1$. Without loss of generality, we assume $s \le t_1/t_2$. In order to prove $i(\Psi_{p,\lambda}) = p$, we first estimate $h_{\Psi_{p,\lambda}}(s)$. By (2.29), we have that

$$\frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \begin{cases} s^p & \forall t \in (0, t_1), \\ \frac{(st)^p}{k(t-t_1)+t_1^p} \approx s^p & \forall t \in [t_1, t_2). \end{cases} \tag{2.31}$$

Moreover we obtain that

$$\frac{s^p}{\log^\lambda(e + t_2)} \le \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{(st)^p}{\Phi_{p,\lambda}(t)} \le \frac{s^p}{\log^\lambda(e + t_1 s^{-1})} \tag{2.32}$$

for all $t \in [t_2, t_1/s)$ and

$$\frac{t_1^p s^p}{t_2^p \log^\lambda(e + t_2 s^{-1})} \le \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{k(st - t_1) + t_1^p}{\Phi_{p,\lambda}(t)} \le \frac{\Phi_{p,\lambda}(t_2)s^p}{t_1^p \log^\lambda(e + t_1 s^{-1})} \tag{2.33}$$

for all $t \in [t_1/s, t_2/s)$. Assume $t \in [t_2/s, +\infty)$. It follows that

$$\frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} = \frac{\Phi_{p,\lambda}(st)}{\Phi_{p,\lambda}(t)} = s^p \left(\frac{\log(e + st)}{\log(e + t)}\right)^\lambda. \tag{2.34}$$

By the monotonicity of function $(\log(s) + \log(e + \cdot)) \log^{-1}(e + \cdot)$, we have that

$$\frac{\log(e + st)}{\log(e + t)} \ge \frac{\log(s) + \log(e + t)}{\log(e + t)} \ge \frac{\log(s) + \log(e + t_2 s^{-1})}{\log(e + t_2 s^{-1})} \ge \frac{\log(t_2)}{\log(e + t_2 s^{-1})}$$

for all $t \ge t_2/s$. Hence we derive from (2.34) that

$$s^p \le \frac{\Psi_{p,\lambda}(st)}{\Psi_{p,\lambda}(t)} \le \log^\lambda(t_2) \frac{s^p}{\log^\lambda(e + t_2 s^{-1})}. \tag{2.35}$$

Combining (2.31), (2.32), (2.33) with (2.35) implies that

$$s^p \lesssim h_{\Psi_{p,\lambda}}(s) \lesssim s^p \log^{-\lambda}(s^{-1}) \tag{2.36}$$

whenever $s \ll 1$. By (2.36), we therefore have that $i(\Psi_{p,\lambda}) = p$. □

Remark 2.10. For $p > 1$ and $\lambda < 0$, let $\Psi_{p,\lambda}$ be as in (2.29) and $\Phi_{p,\lambda}$ be as in (2.21). Analogously to Remark 2.8, we have that

$$\int_{\mathbb{R}^2} \Psi_{p,\lambda}(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Psi_{p,\lambda}(|f(x)|)w(x)\,dx. \tag{2.37}$$

Since $\lim_{t\to 0+} \Psi_{p,\lambda}(t)/\Phi_{p,\lambda}(t) = 1$, it follows that

$$\Psi_{p,\lambda}(t) \approx \Phi_{p,\lambda}(t) \tag{2.38}$$

whenever $t \in [0, +\infty)$. Hence we derive from (2.37) that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_f(x))w(x)\,dx \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(|f(x)|)w(x)\,dx.$$

3 Proof of Theorem 1.1

We begin by proving the following special case of Theorem 1.1.

Theorem 3.1. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the harmonic extension of φ. For any $p > 1$, we have that*

(1) if either $\alpha \in (p-2, +\infty)$ and $\lambda \in \mathbb{R}$, or $\alpha = p-2$ and $\lambda \in (-\infty, -1)$,

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are finite.

(2) if either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$, or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$, then

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are comparable to $\mathcal{U}(p, \alpha, \lambda, \varphi)$.

Moreover whenever $p \in (1, 2]$

both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ dominate $\mathcal{V}(p, \alpha, \lambda, \varphi)$,

while

$\mathcal{V}(p, \alpha, \lambda, \varphi)$ controls both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$

for all $p \in [2, +\infty)$. Furthermore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are in general comparable to $\mathcal{V}(p, \alpha, \lambda, \varphi)$ only for $p = 2$.

(3) if either $\alpha \in (-\infty, -1)$ and $\lambda \in \mathbb{R}$, or $\alpha = -1$ and $\lambda \in [-1, +\infty)$, we have that $I_1(p, \alpha, \lambda, h) = \infty$. While $I_2(p, \alpha, \lambda, h) = \infty$ for all $\alpha \in (-\infty, -1]$ and $\lambda \in \mathbb{R}$.

Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. Given $p > 1, \alpha \in \mathbb{R}, \lambda \in \mathbb{R}$, we define

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda$$

and

$$\mathcal{E}_2(p, \alpha, \lambda, \varphi) = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \Phi_{p,\lambda}\Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big) \ell(\Gamma_{j,k})^{2+\alpha}$$

where $\Phi_{p,\lambda}(t)$ is from (2.21).

Lemma 3.2. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. For any $p > 1, \alpha \in (-1, +\infty)$ and every $\lambda \in \mathbb{R}$, the dyadic energies $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ and $\mathcal{E}_2(p, \alpha, \lambda, \varphi)$ are equivalent.*

Proof. We first consider the case $\lambda \geq 0$. Let $\Phi_{p,\lambda}$ be as in (2.21). Since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ and $\ell(\Gamma_{j,k}) \approx 2^{-j}$ for all $j \in \mathbb{N}$ and $k \in \{1, ..., 2^j\}$, by the monotonicity and Δ_2-property of the standard logarithm we have that

$$\Phi_{p,\lambda}\Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big) \lesssim \Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big)^p \log^\lambda \big(e + 2\pi \cdot 2^j\big) \lesssim \Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big)^p j^\lambda.$$

Hence

$$\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim \mathcal{E}_1(p, \alpha, \lambda, \varphi). \tag{3.1}$$

Given $p > 1$ and $\alpha \in (-1, +\infty)$, there is $\beta \in (0, 1)$ such that $\alpha > (1 - \beta)p - 1 > -1$. Define

$$\chi_{j,k} = \begin{cases} 1 & \text{if } \ell(\varphi(\Gamma_{j,k})) \geq 2^{-j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

We decompose $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ as

$$\begin{aligned} \mathcal{E}_1(p, \alpha, \lambda, \varphi) = &\sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \chi_{j,k} \\ &+ \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda (1 - \chi_{j,k}) \\ =: &\mathcal{E}_1'(p, \alpha, \lambda, \varphi) + \mathcal{E}_1''(p, \alpha, \lambda, \varphi). \end{aligned} \tag{3.2}$$

Whenever $\ell(\varphi(\Gamma_{j,k})) \geq 2^{-j\beta}$, by (2.3) we have $j^\lambda \lesssim \log^\lambda \left(e + \ell(\varphi(\Gamma_{j,k}))\ell(\Gamma_{j,k})^{-1}\right)$. Therefore

$$\begin{aligned} \mathcal{E}_1'(p,\alpha,\lambda,\varphi) \lesssim & \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\right) \\ = & \mathcal{E}_2(p,\alpha,\lambda,\varphi). \end{aligned} \tag{3.3}$$

Moreover, by (2.3) we have that

$$\mathcal{E}_1''(p,\alpha,\lambda,\varphi) \leq \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{-\beta j p} 2^{j(p-2-\alpha)} j^\lambda = \sum_{j=1}^{+\infty} 2^{-j((\beta-1)p+1+\alpha)} j^\lambda < \infty. \tag{3.4}$$

We conclude from (3.2), (3.3) and (3.4) that there is a constant $C > 0$ such that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) \lesssim C + \mathcal{E}_2(p,\alpha,\lambda,\varphi). \tag{3.5}$$

From (3.1) and (3.5) it follows that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) \text{ and } \mathcal{E}_2(p,\alpha,\lambda,\varphi) \text{ are comparable whenever } \lambda \geq 0. \tag{3.6}$$

Analogously to (3.6), we have that $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ and $\mathcal{E}_2(p,\alpha,\lambda,\varphi)$ are comparable whenever $\lambda < 0$. □

Lemma 3.3. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism, and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the Poisson homeomorphic extension of φ. For any $p > 1$, we have that $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\alpha \in (-1,+\infty)$ and $\lambda \in \mathbb{R}$, while $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ is controlled by $I_1(p,\alpha,\lambda,h)$ for all $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$.*

Proof. We first prove that $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ for all $\alpha > -1$ and all $\lambda \in \mathbb{R}$. Let $w_{\alpha,\lambda}$ be as in (2.7). For any $j \in \mathbb{N}$ and $1 \leq k \leq 2^j$, by (2.4) and (2.3) we have that

$$w_{\alpha,\lambda}(z) \approx 2^{-j\alpha} j^\lambda \approx \ell(\Gamma_{j,k})^\alpha j^\lambda \tag{3.7}$$

for all $z \in Q_{j,k}$. Hence

$$I_1(p,\alpha,\lambda,h) \approx \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{-j\alpha} j^\lambda \int_{Q_{j,k}} |Dh(z)|^p \, dz. \tag{3.8}$$

Let $\mathcal{P}(\Gamma_{j,k})$ be the technical decomposition of $\mathbb{S}^1$ based on $\Gamma_{j,k}$ in [9, Section 2.1]. As shown in [9, Proof (iv) $\Rightarrow$ (i)], for any $j \in \mathbb{N}$ and $k = 1, ..., 2^j$ we have that

$$|Dh(z)| \lesssim \sum_{n \leq j} \sum_{m \in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}. \tag{3.9}$$

for all $z \in Q_{j,k}$. Here $\Gamma_{n,m} \in \mathcal{P}(\Gamma_{j,k})$ and $\sharp i_n \leq 3$ for all $n \leq j$, see [9, Section 2.1]. Let $\alpha > -1$. There is $q_0 > 1$ such that $p/q_0 - 1 - \alpha < 0$. Denote by p_0 the exponent conjugate to q_0. Via Hölder's inequality we derive from (3.9) that

$$|Dh(z)|^p \lesssim \left(\sum_{n\leq j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n(\frac{1}{q_0}+\frac{1}{p_0})}}\right)^p \leq \left(\sum_{n\leq j}\sum_{m\in i_n} 2^{\frac{nq}{q_0}}\right)^{\frac{p}{q}} \sum_{n\leq j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}}$$

$$\approx 2^{\frac{jp}{q_0}} \sum_{n\leq j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \tag{3.10}$$

for all $z \in Q_{j,k}$. By (2.5), (3.8) and (3.10) we have that

$$I_1(p,\alpha,\lambda,h) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda \sum_{n\leq j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{q_0}}}. \tag{3.11}$$

Moreover given a dyadic arc $\Gamma_{n,m}$, for any $j \geq n$ it is shown in [9, Section 2.1] that

$$\sharp\{\Gamma : \Gamma \text{ is a } j\text{-level dyadic arc and } \Gamma_{n,m} \in \mathcal{P}(\Gamma)\} \leq 3 \cdot 2^{j-n}. \tag{3.12}$$

From Fubini's theorem and (3.12) we obtain that

$$\begin{aligned}
&\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda \sum_{n\leq j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \\
=&\sum_{n=1}^{\infty}\sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \sum_{n\leq j}\sum_{k} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda \\
\lesssim&\sum_{n=1}^{\infty}\sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^p}{2^{-\frac{np}{p_0}}} \sum_{n\leq j} 2^{j(\frac{p}{q_0}-2-\alpha)} j^\lambda 2^{j-n} \\
=&\sum_{n=1}^{\infty}\sum_{m=1}^{2^n} \ell(\varphi(\Gamma_{n,m}))^p 2^{n(\frac{p}{p_0}-1)} \sum_{n\leq j} 2^{j(\frac{p}{q_0}-1-\alpha)} j^\lambda.
\end{aligned} \tag{3.13}$$

Moreover when $p/q_0 - 1 - \alpha < 0$ we have that

$$\sum_{n\leq j} 2^{j(\frac{p}{q_0}-1-\alpha)} j^\lambda \approx 2^{n(\frac{p}{q_0}-1-\alpha)} n^\lambda. \tag{3.14}$$

By (3.11), (3.13), (3.14) and (2.3), we conclude that

$$I_1(p,\alpha,\lambda,h) \lesssim \sum_{n=1}^{\infty}\sum_{m=1}^{2^n} \ell(\varphi(\Gamma_{n,m}))^p 2^{n(p-2-\alpha)} n^\lambda \approx \mathcal{E}_1(p,\alpha,\lambda,\varphi).$$

We next prove that $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ is controlled by $I_1(p,\alpha,\lambda,h)$ for all $\alpha\in(-1,p-1)$ and $\lambda\in\mathbb{R}$. By [9, (3.17)], there is $j_0>1$ such that

$$\ell(\varphi(\Gamma_{j,k}))\lesssim\frac{1}{\ell(\Gamma_{j,k})}\int_{CQ_{j,k}\cap\mathbb{D}}|Dh(z)|\,dz \tag{3.15}$$

for all $j\geq j_0$ and $k\in\{1,...,2^j\}$. Set $H(z)=|Dh(z)|\chi_{\mathbb{D}}(z)$. By (2.6) we have that

$$\fint_{CQ_{j,k}\cap\mathbb{D}}|Dh(z)|dz\leq\fint_{CC'B_{j,k}}H(z)dz\leq\fint_{Q_{j,k}}M_H(w)dw, \tag{3.16}$$

where the last inequality comes from the fact that $\fint_{CC'B_{j,k}}H(z)dz\leq M_H(w)$ for all $w\in Q_{j,k}$. Combining (3.15) with (3.16) implies that

$$\ell(\varphi(\Gamma_{j,k}))\lesssim\ell(\Gamma_{j,k})\fint_{Q_{j,k}}M_H(z)dz \tag{3.17}$$

for all $j\geq j_0$ and $k=1,...,2^j$. By Jensen's inequality and (2.5), we deduce from (3.17) that

$$\ell(\varphi(\Gamma_{j,k}))^p\lesssim\ell(\Gamma_{j,k})^{p-2}\int_{Q_{j,k}}M_H^p(z)\,dz. \tag{3.18}$$

By (3.7) and (3.18), there is then a constant $C>0$ such that
(3.19)

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi)\lesssim C+\sum_{j=j_0}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}}M_H^p(z)w_{\alpha,\lambda}(z)\,dz\leq C+\int_{\mathbb{R}^2}M_H^p(z)w_{\alpha,\lambda}(z)\,dz.$$

Moreover, for any $\alpha\in(-1,p-1)$ and $\lambda\in\mathbb{R}$, from Proposition 2.3 and Remark 2.8 it follows that

$$\int_{\mathbb{R}^2}M_H^p(z)w_{\alpha,\lambda}(z)\,dz\lesssim\int_{\mathbb{R}^2}H^p(z)w_{\alpha,\lambda}(z)\,dz=I_1(p,\alpha,\lambda,h). \tag{3.20}$$

By (3.19) and (3.20) we conclude that $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ is controlled by $I_1(p,\alpha,\lambda,h)$ for all $\alpha\in(-1,p-1)$ and $\lambda\in\mathbb{R}$. $\square$

Lemma 3.4. *Let $\varphi:\mathbb{S}^1\to\mathbb{S}^1$ be a homeomorphism, and $h=P[\varphi]:\mathbb{D}\to\mathbb{D}$ be the Poisson homeomorphic extension of φ. For any $p>1$, we have that $I_2(p,\alpha,\lambda,h)\lesssim\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\alpha\in(-1,+\infty)$ and $\lambda\in\mathbb{R}$, while $\mathcal{E}_2(p,\alpha,\lambda,\varphi)$ is controlled by $I_2(p,\alpha,\lambda,h)$ for all $\alpha\in(-1,p-1)$ and $\lambda\in\mathbb{R}$.*

Proof. We first consider that case $\lambda \geq 0$. Let $\Phi_{p,\lambda}$ be as in (2.21). Proposition 2.7 shows that $\Phi_{p,\lambda}(t)$ is increasing and satisfies Δ_2-property on $[0, +\infty)$. From (3.7) and (3.9) we have that

$$\begin{aligned} I_2(p,\alpha,\lambda,h) &= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi_{p,\lambda}(|Dh(z)|)w_{\alpha,0}(z)\,dz \\ &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Phi_{p,\lambda}\big(\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big). \end{aligned} \tag{3.21}$$

Moreover since $\ell(\varphi(\Gamma_{n,m})) \leq 2\pi$ for all $n \in \mathbb{N}$ and $m = 1, ..., 2^n$, it follows that

$$\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \lesssim \sum_{n\leq j}\frac{1}{2^{-n}} \lesssim 2^j.$$

for any $j \geq 1$. Therefore

$$\log^\lambda\big(e + \sum_{n\leq j}\sum_{m}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big) \lesssim \log^\lambda(e+2^j) \lesssim j^\lambda \tag{3.22}$$

for all $j \geq 1$. By (3.21) and (3.22) we obtain that

$$I_2(p,\alpha,\lambda,h) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^\lambda \big(\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big)^p. \tag{3.23}$$

The analogous arguments as for $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ in Lemma 3.3 imply that

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^\lambda \big(\sum_{n\leq j}\sum_{m\in i_n}\frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big)^p \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi). \tag{3.24}$$

We conclude from (3.23) and (3.24) that $I_2(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$.

Applying $\Phi_{p,\lambda}$ to the both sides of (3.17), via Proposition 2.7 and Jensen's inequality we have that

$$\Phi_{p,\lambda}\big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\big) \lesssim \Phi_{p,\lambda}\big(\fint_{Q_{j,k}} M_H(z)dz\big) \leq \fint_{Q_{j,k}} \Phi_{p,\lambda}(M_H(z))dz \tag{3.25}$$

for all $j \geq j_0$ and $k \in \{1, ..., 2^j\}$. By (2.5), (3.7) and (3.25), we then obtain that

$$\begin{aligned} \mathcal{E}_2(p,\alpha,\lambda,\varphi) &\lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\int_{Q_{j,k}} \Phi_{p,\lambda}(M_H(z))w_{\alpha,0}(z)\,dz \\ &\leq \int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_H(z))w_{\alpha,0}(z)\,dz. \end{aligned} \tag{3.26}$$

Moreover, for any $\alpha \in (-1, p-1)$ it follows from Remark 2.8 that

$$\int_{\mathbb{R}^2} \Phi_{p,\lambda}(M_H(z))w_{\alpha,0}(z)\,dz \lesssim \int_{\mathbb{R}^2} \Phi_{p,\lambda}(H(z))w_{\alpha,0}(z)\,dz = I_2(p,\alpha,\lambda,h). \tag{3.27}$$

By (3.26) and (3.27) we conclude that $\mathcal{E}_2(p,\alpha,\lambda,\varphi) \lesssim I_2(p,\alpha,\lambda,h)$.

We next consider the case $\lambda < 0$. Let $\Psi_{p,\lambda}$ be as in (2.29). By the analogous arguments as for (3.21), we have that

$$\int_{\mathbb{D}} \Psi_{p,\lambda}(|Dh(z)|)w_{\alpha,0}(z)\,dz \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Psi_{p,\lambda}\big(\sum_{n\le j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}\big). \tag{3.28}$$

Set $S_{j,k} = \sum_{n\le j}\sum_{m\in i_n} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}}$. It follows from (2.38) and (3.28) that

$$I_2(p,\alpha,\lambda,h) \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Phi_{p,\lambda}(S_{j,k}). \tag{3.29}$$

Since $\alpha > -1$, there is $\beta > 0$ such that $\beta p \le 1+\alpha$. Define

$$\chi(j,k) = \begin{cases} 1 & \text{if } S_{j,k} < 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Phi_{p,\lambda}(S_{j,k}) &= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Phi_{p,\lambda}(\chi(j,k)S_{j,k}) \\ &\quad + \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2}\Phi_{p,\lambda}((1-\chi(j,k))S_{j,k}) =: \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned} \tag{3.30}$$

Since $\log^\lambda(e + S_{j,k}) \le \log^\lambda(e) = 1$, we have that

$$\sum\nolimits_1 \le \sum\nolimits_1 2^{-j(\alpha+2)}(S_{j,k})^p \le \sum_{j=1}^{\infty} 2^{j(p\beta-\alpha-1)} < \infty. \tag{3.31}$$

Whenever $S_{j,k} \ge 2^{j\beta}$, it follows that $\log^\lambda(e + S_{j,k}) \lesssim j^\lambda$. Via the analogous arguments as for $I_1(p,\alpha,\lambda,h) \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi)$ in Lemma 3.3, it then follows that

$$\sum\nolimits_2 \le \sum_{j=1}^{\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} j^\lambda S_{j,k}^p \lesssim \mathcal{E}_1(p,\alpha,\lambda,\varphi). \tag{3.32}$$

From (3.29), (3.30), (3.31) and (3.32), we conclude that there is a constant $C > 0$ such that $I_2(p, \alpha, \lambda, h) \lesssim C + \mathcal{E}_1(p, \alpha, \lambda, \varphi)$.

By the analogous arguments as for $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$ whenever $\lambda \geq 0$, we have that

$$\sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{\alpha+2} \Psi_{p,\lambda}\Big(\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})}\Big) \lesssim \int_{\mathbb{R}^2} \Psi_{p,\lambda}(|Dh|(z)) w_{\alpha,0}(z)\, dz. \tag{3.33}$$

It follows from (2.38) that $\mathcal{E}_2(p, \alpha, \lambda, \varphi) \lesssim I_2(p, \alpha, \lambda, h)$. □

Proof of Theorem 3.1 (1). From Lemma 3.3 and Lemma 3.4, we have that both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are dominated by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ for all $p > 1, \alpha \in (-1, +\infty)$ and each $\lambda \in \mathbb{R}$. Moreover since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ for all $j \geq 1$ and $1 \leq k \leq 2^j$, we have that

$$\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \leq (2\pi)^{p-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) = (2\pi)^p.$$

Therefore both $I_1(p, \alpha, \lambda, h)$ and $I_2(p, \alpha, \lambda, h)$ are controlled by $\sum_{j=1}^{\infty} 2^{j(p-2-\alpha)} j^\lambda$ whenever $\alpha \in (-1, +\infty)$ and $\lambda \in \mathbb{R}$. Notice that $\sum_{j=1}^{\infty} 2^{j(p-2-\alpha)} j^\lambda < \infty$ whenever either $p - 2 < \alpha$ and $\lambda \in \mathbb{R}$, or $p - 2 = \alpha$ and $\lambda < -1$. We hence complete Theorem 3.1 (1). □

By Example 4.4, there are homeomorphisms $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that, for their harmonic extensions $P[\varphi]$, both $I_1(p, \alpha, \lambda, P[\varphi])$ and $I_2(p, \alpha, \lambda, P[\varphi])$ may be finite or infinite for either some $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or some $\alpha = p - 2$ and $\lambda \in [-1, +\infty)$. How can we characterize both $I_1(p, \alpha, \lambda, P[\varphi]) < \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) < \infty$? As shown in [9], double integrals of the inverse mapping over the boundary are potential choices.

Lemma 3.5. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. For any $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, $\mathcal{V}(p, \alpha, \lambda, \varphi)$ is dominated by $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ whenever $p \in (1, 2]$; while $\mathcal{E}_1(p, \alpha, \lambda, \varphi)$ is controlled by $\mathcal{V}(p, \alpha, \lambda, \varphi)$ if $p \in [2, +\infty)$.*

Proof. We first consider the case $p \in (1, 2]$. Given $\xi \in \mathbb{S}^1$ and $t \geq 0$, set

$$E_t(\xi) = \{\eta \in \mathbb{S}^1 : |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| < t\}.$$

By Fubini's theorem we have that

$$\begin{aligned}\int_{\mathbb{S}^1}\mathcal{A}_{p,\alpha,\lambda}(|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|)\,|d\eta| &= \int_{\mathbb{S}^1}\int_{|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|}^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt\,|d\eta| \\ &= \int_0^1\int_{\mathbb{S}^1} -\mathcal{A}'_{p,\alpha,\lambda}(t)\chi_{E_t(\xi)}\,|d\eta|\,dt \\ &= \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\,dt. \qquad (3.34)\end{aligned}$$

Moreover, from Jensen's inequality and Minkowski's inequality it follows that

$$\begin{aligned}&\Big(\int_{\mathbb{S}^1}\big(\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\,dt\big)^{p-1}\,|d\xi|\Big)^{\frac{1}{p-1}} \\ \lesssim &\Big(\int_{\mathbb{S}^1}\big(\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\,dt\big)^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1} \\ \leq &\int_0^1\Big(\int_{\mathbb{S}^1}\big(-\mathcal{A}'_{p,\alpha,\lambda}(t)\mathcal{L}^1(E_t(\xi))\big)^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1}\,dt \\ = &\int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_t(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1}\,dt. \qquad (3.35)\end{aligned}$$

Combining (3.34) with (3.35) implies that

$$\begin{aligned}\mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi) &\lesssim \int_0^1 -\mathcal{A}'_{p,\alpha,\lambda}(t)\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_t(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1}\,dt \\ &\leq \sum_{j=1}^{+\infty}\int_{2^{-j}}^{2^{1-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt\,\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1}. \qquad (3.36)\end{aligned}$$

Since $E_{2^{1-j}}(\xi)\subset\cup_{i=k-1}^{k+1}\varphi(\Gamma_{j,i})$ for all $j\in\mathbb{N},\ k=1,...,2^j$ and all $\xi\in\varphi(\Gamma_{j,k})$, we have that

$$\begin{aligned}\Big(\int_{\mathbb{S}^1}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1} &= \Big(\sum_{k=1}^{2^j}\int_{\varphi(\Gamma_{j,k})}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1} \\ &\leq \Big(\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))\Big(\sum_{i=k-1}^{k+1}\ell(\varphi(\Gamma_{j,i}))\Big)^{\frac{1}{p-1}}\Big)^{p-1} \\ &\leq \sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^{p-1}\sum_{i=k-1}^{k+1}\ell(\varphi(\Gamma_{j,i})). \qquad (3.37)\end{aligned}$$

Moreover by Young's inequality we have that

$$\begin{aligned}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^{p-1}\ell(\varphi(\Gamma_{j,k-1})) &\leq \sum_{k=1}^{2^j}\frac{1}{p}\ell(\varphi(\Gamma_{j,k}))^p+\frac{p}{p-1}\ell(\varphi(\Gamma_{j,k-1}))^p\\ &\lesssim \sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^p \end{aligned}\tag{3.38}$$

and

$$\begin{aligned}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^{p-1}\ell(\varphi(\Gamma_{j,k+1})) &\leq \sum_{k=1}^{2^j}\frac{1}{p}\ell(\varphi(\Gamma_{j,k}))^p+\frac{p}{p-1}\ell(\varphi(\Gamma_{j,k+1}))^p\\ &\lesssim \sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^p. \end{aligned}\tag{3.39}$$

Combining (3.37), (3.38) with (3.39) implies that

$$\left(\int_{\mathbb{S}^1}\mathcal{L}^1(E_{2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\right)^{p-1}\lesssim\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^p\tag{3.40}$$

for all $j\in\mathbb{N}$. Let

$$\Lambda_\lambda(t)=\begin{cases}t^{\lambda+1} & \lambda\neq -1,\\ \log t & \lambda=-1.\end{cases}$$

For any $j\in\mathbb{N}$, we have that

$$\int_{2^{-j}}^{2^{1-j}} t^{-1}\log_2^{\lambda}(t^{-1})\;dt\approx-\int_{2^{-j}}^{2^{1-j}}\;d\Lambda_\lambda(\log_2(t^{-1}))=\Lambda_\lambda(j)-\Lambda_\lambda(j-1)\approx j^\lambda.\tag{3.41}$$

It follows (3.41) and (2.3) that

$$\int_{2^{-j}}^{2^{1-j}}-\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt\approx 2^{j(p-2-\alpha)}\int_{2^{-j}}^{2^{1-j}}\frac{1}{t}\log_2^{\lambda}(t^{-1})\,dt\approx\ell(\Gamma_{j,k})^{2+\alpha-p}j^\lambda.\tag{3.42}$$

By combining (3.36), (3.40) with (3.42), we conclude that

$$\mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi)\lesssim\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j}\ell(\varphi(\Gamma_{j,k}))^p\ell(\Gamma_{j,k})^{2+\alpha-p}j^\lambda=\mathcal{E}_1(p,\alpha,\lambda,\varphi).$$

We next consider the case $p \in [2, +\infty)$. By the analogous arguments as for (3.36) we have that

(3.43)
$$\mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi) \gtrsim \sum_{j=5}^{+\infty} \int_{\pi 2^{1-j}}^{\pi 2^{2-j}} -\mathcal{A}'_{p,\alpha,\lambda}(t)\,dt \Big(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{\pi 2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1}.$$

Since $\varphi(\Gamma_{j,k}) \subset E_{\pi 2^{1-j}}(\xi)$ for all $j \geq 5$, $k \in \{1, ..., 2^j\}$ and all $\xi \in \varphi(\Gamma_{j,k})$, we have that

$$\begin{aligned}
\Big(\int_{\mathbb{S}^1} \mathcal{L}^1(E_{\pi 2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1} &= \Big(\sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \mathcal{L}^1(E_{\pi 2^{1-j}}(\xi))^{\frac{1}{p-1}}\,|d\xi|\Big)^{p-1} \\
&\geq \Big(\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))\ell(\varphi(\Gamma_{j,k}))^{\frac{1}{p-1}}\Big)^{p-1} \\
&\geq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p. \qquad (3.44)
\end{aligned}$$

By (3.42), (3.43) and (3.44), there is a constant $C > 0$ such that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) = \sum_{j=1}^{4}\sum_{k=1}^{2^j} + \sum_{j=5}^{+\infty}\sum_{k=1}^{2^j} \lesssim C + \mathcal{V}^{\frac{1}{p-1}}(p,\alpha,\lambda,\varphi).$$

□

We next prove Theorem 3.1 (2).

Lemma 3.6. *Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. For any $p \in (1, +\infty)$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$, we have that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ and $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ are comparable.*

Proof. We first prove that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ is controlled by $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$. Given $\xi \in \mathbb{S}^1$ and $\eta \in \mathbb{S}^1$, let $\ell(\xi\eta)$ be the arc length of the shorter arc in $\mathbb{S}^1$ connecting ξ and η. Given $j \geq 1$ and $\xi \in \mathbb{S}^1$, set

$$A_j = \{(\xi,\eta) \in \mathbb{S}^1 \times \mathbb{S}^1 : \pi 2^{-j} < \ell(\xi\eta) \leq \pi 2^{1-j}\}$$

and $A_j(\xi) = \{\eta \in \mathbb{S}^1 : (\xi,\eta) \in A_j\}$. Notice that $\lambda_{\mathbb{D}}$ is the Euclidean distance. We have that

$$\begin{aligned}
\mathcal{U}(p,\alpha,\lambda,\varphi) &= \sum_{j=1}^{+\infty} \int_{A_j} \Phi_{p,\lambda}\Big(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\Big)|\xi-\eta|^{\alpha}\,|d\eta|\,|d\xi| \\
&= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \int_{\Gamma_{j,k}} \int_{A_j(\xi)} \Phi_{p,\lambda}\Big(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\Big)|\xi-\eta|^{\alpha}\,|d\eta|\,|d\xi|. \qquad (3.45)
\end{aligned}$$

Notice that

$$|\xi-\eta| \approx \ell(\Gamma_{j,k}) \text{ and } |\varphi(\xi)-\varphi(\eta)| \leq \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i})) \leq 2\pi \tag{3.46}$$

for all $j \in \mathbb{N}$, $k \in \{1, ..., 2^j\}$, $\xi \in \Gamma_{j,k}$ and $\eta \in A_j(\xi)$. It then follows that

$$\begin{aligned}\Phi_{p,\lambda}\left(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\right)|\xi-\eta|^{\alpha} &\lesssim \left(\sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))\right)^{p} \ell(\Gamma_{j,k})^{\alpha-p} \log^{\lambda}(e+2\pi\cdot 2^{j}) \\ &\lesssim \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))^{p} \ell(\Gamma_{j,k})^{\alpha-p} j^{\lambda} \qquad (3.47)\end{aligned}$$

for all $\lambda \geq 0$, $\xi \in \Gamma_{j,k}$ and $\eta \in A_j(\xi)$. Since

$$\mathcal{L}^1(A_j(\xi)) \approx \ell(\Gamma_{j,k}) \tag{3.48}$$

for all $j \in \mathbb{N}$, $k = 1, ..., 2^j$ and $\xi \in \Gamma_{j,k}$, we derive from (3.45) and (3.47) that

$$\begin{aligned}\mathcal{U}(p,\alpha,\lambda,\varphi) &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \sum_{i=k-1}^{k+1} \ell(\varphi(\Gamma_{j,i}))^{p} \ell(\Gamma_{j,k})^{\alpha-p} j^{\lambda} \int_{\Gamma_{j,k}} \int_{A_j(\xi)} |d\eta|\,|d\xi| \\ &\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^{p} \ell(\Gamma_{j,k})^{2+\alpha-p} j^{\lambda} = \mathcal{E}_1(p,\alpha,\lambda,\varphi)\end{aligned}$$

whenever $\lambda \geq 0$.

Since φ is homeomorphic, for any $j \in \mathbb{N}$ and $k \in \{1, ..., 2^j\}$ there are $\xi'_{j,k} \in \Gamma_{j,k}$ and $\eta'_{j,k} \in A_j(\xi'_{j,k})$ such that

$$\begin{aligned}&\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)|\xi'_{j,k}-\eta'_{j,k}|^{\alpha} \\ &= \max\Big\{\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\Big)|\xi-\eta|^{\alpha} : \xi \in \Gamma_{j,k} \text{ and } \eta \in A_j(\xi)\Big\}. \qquad (3.49)\end{aligned}$$

Since $0 < \alpha+1 < p$, there is $\beta \in (-1,0)$ such that $0 < (1+\beta)p < \alpha+1$. Define

$$\chi(j,k) = \begin{cases} 1 & \text{if } |\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})| \leq 2^{j\beta}, \\ 0 & \text{otherwise.}\end{cases}$$

From (3.45), (3.49), (3.48), (2.3) and (3.46), we obtain that

$$\mathcal{U}(p,\alpha,\lambda,\varphi) \le \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)$$

$$= \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)\chi(j,k)$$

$$(3.50)\qquad + \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})|}{|\xi'_{j,k}-\eta'_{j,k}|}\Big)(1-\chi(j,k)) =: \sum\nolimits_1 + \sum\nolimits_2 .$$

Since $\log^\lambda(e+|\varphi(\xi'_{j,k})-\varphi(\eta'_{j,k})||\xi'_{j,k}-\eta'_{j,k}|^{-1}) \le 1$ for all $\lambda<0$, $j\in\mathbb{N}$ and $1\le k\le 2^j$, by (3.46) and (2.3) we have that

$$(3.51)\qquad \sum\nolimits_1 \lesssim \sum_{j=1}^{+\infty} 2^{-2j}\sum_{k=1}^{2^j} 2^{j((1+\beta)p-\alpha)} = \sum_{j=1}^{+\infty} 2^{j((1+\beta)p-\alpha-1)} < +\infty.$$

Moreover we derive from (3.46) that

$$\sum\nolimits_2 \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha-p}\Big(\sum_{i=k-1}^{k+1}\ell(\varphi(\Gamma_{j,k}))\Big)^p \log^\lambda(2^{j(1+\beta)})$$

$$(3.52)\qquad \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda = \mathcal{E}_1(p,\alpha,\lambda,\varphi).$$

for all $\lambda<0$. Combining (3.50), (3.51) with (3.52) implies that there is a constant $C>0$ such that $\mathcal{U}(p,\alpha,\lambda,\varphi)\lesssim C+\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ for all $\lambda<0$.

We next prove that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ dominates $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$. Given $j\ge 3$ and $\xi\in\mathbb{S}^1$, set

$$B_j = \{(\xi,\eta)\in\mathbb{S}^1\times\mathbb{S}^1 : \pi 2^{2-j} < \ell(\xi\eta) \le \pi 2^{3-j} \text{ with } \arg\eta > \arg\xi\}$$

and $B_j(\xi)=\{\eta\in\mathbb{S}^1 : (\xi,\eta)\in B_j\}$. We have that

$$(3.53)\qquad \sum_{j=3}^{+\infty}\sum_{k=1}^{2^j}\int_{\Gamma_{j,k-1}}\int_{B_j(\xi)} \Phi_{p,\lambda}\Big(\frac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|}\Big)|\xi-\eta|^\alpha\,|d\eta|\,|d\xi| = \mathcal{U}(p,\alpha,\lambda,\varphi).$$

Since φ is homeomorphic, for any $j \ge 3$ and $1 \le k \le 2^j$ there are $\xi''_{j,k} \in \Gamma_{j,k-1}$ and $\eta''_{j,k} \in B_j(\xi''_{j,k})$ such that

$$\begin{aligned}
&\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|}\Big)|\xi''_{j,k} - \eta''_{j,k}|^{\alpha} \\
&= \min\Big\{\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi) - \varphi(\eta)|}{|\xi - \eta|}\Big)|\xi - \eta|^{\alpha} : \xi \in \Gamma_{j,k-1} \text{ and } \eta \in B_j(\xi)\Big\}.
\end{aligned} \tag{3.54}$$

Notice that

$$|\xi''_{j,k} - \eta''_{j,k}| \approx \ell(\Gamma_{j,k}) \text{ and } 2 \ge |\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})| \gtrsim \ell(\varphi(\Gamma_{j,k})) \tag{3.55}$$

whenever $j \ge 3$ and $k \in \{1, ..., 2^j\}$. Since $\mathcal{L}^1(B_j(\xi)) \approx \ell(\Gamma_{j,k})$ for all $j \ge 3,\ k = 1, ..., 2^j$ and $\xi \in \mathbb{S}^1$, it follows from (3.53), (3.54) and (3.55) that

$$\sum_{j=3}^{+\infty}\sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^{2+\alpha}\Phi_{p,\lambda}\Big(\frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|}\Big)| \lesssim \mathcal{U}(p, \alpha, \lambda, \varphi). \tag{3.56}$$

Moreover, for any $\lambda \le 0$ we obtain from (3.55) that

$$j^{\lambda} \lesssim \log^{\lambda}(e + 2^{1+j}) \lesssim \log^{\lambda}\Big(e + \frac{|\varphi(\xi''_{j,k}) - \varphi(\eta''_{j,k})|}{|\xi''_{j,k} - \eta''_{j,k}|}\Big) \tag{3.57}$$

for all $j \in \mathbb{N}$ and all $k = 1, ..., 2^j$. From (3.55), (3.56) and (3.57), there is a constant $C > 0$ such that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) = C + \sum_{j=3}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^{\lambda} \lesssim C + \mathcal{U}(p, \alpha, \lambda, \varphi)$$

for all $\lambda \le 0$. For any $\lambda > 0$, by (3.55) and (3.56) there is a constant $C > 0$ such that

$$\sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^{\lambda}\big(2^j \ell(\varphi(\Gamma_{j,k}))\big) \lesssim C + \mathcal{U}(p, \alpha, \lambda, \varphi). \tag{3.58}$$

Let β be same as in (3). Set

$$\chi_{j,k} = \begin{cases} 1 & \text{if } \ell(\varphi(\Gamma_{j,k})) \le 2^{j\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\mathcal{E}_1(p,\alpha,\lambda,\varphi) = \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda \chi_{j,k}$$

$$+ \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} j^\lambda (1-\chi_{j,k}) =: \sum{}^1 + \sum{}^2 . \tag{3.59}$$

Moreover

$$\sum{}^1 \leq \sum_{j=1}^{+\infty} 2^{j((1+\beta)p-\alpha-1)} j^\lambda < \infty \tag{3.60}$$

and

$$\sum{}^2 \lesssim \sum_{j=1}^{+\infty}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \ell(\Gamma_{j,k})^{2+\alpha-p} \log^\lambda \left(2^j \ell(\varphi(\Gamma_{j,k}))\right). \tag{3.61}$$

From (3.59), (3.60), (3.61) and (3.58), we have that $\mathcal{U}(p,\alpha,\lambda,\varphi)$ controls $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\lambda > 0$. $\square$

Proof of Theorem 3.1 (2). By Lemma 3.2, Lemma 3.3 and Lemma 3.4, for any $p > 1$ we have that both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are comparable to $\mathcal{E}_1(p,\alpha,\lambda,\varphi)$ whenever $\alpha \in (-1,p-1)$ and $\lambda \in \mathbb{R}$. By Lemma 3.6, we hence conclude comparability of both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ with $\mathcal{U}(p,\alpha,\lambda,\varphi)$ for all $p > 1, \alpha \in (-1,p-1)$ and every $\lambda \in \mathbb{R}$. By Lemma 3.5, we can dominate $\mathcal{V}(p,\alpha,\lambda,\varphi)$ by either $I_1(p,\alpha,\lambda,h)$ or $I_2(p,\alpha,\lambda,h)$ whenever $p \in (1,2]$, while both $I_1(p,\alpha,\lambda,h)$ and $I_2(p,\alpha,\lambda,h)$ are controlled by $\mathcal{V}(p,\alpha,\lambda,\varphi)$ for all $p \in [2,+\infty)$. Moreover from Example 4.2 and Example 4.3, we have that $\mathcal{V}(p,\alpha,\lambda,\varphi)$ is comparable to either $I_1(p,\alpha,\lambda,h)$ or $I_2(p,\alpha,\lambda,h)$ only when $p = 2$. $\square$

Towards the proof of Theorem 3.1 (3), we have the following general result.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and $\varphi : \mathbb{S}^1 \to \partial\Omega$ be a homeomorphism. For any $p > 1$, there is no diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_1(p,\alpha,\lambda,h) < +\infty$ for either $\alpha \in (-\infty,-1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1,+\infty)$; or for which $I_2(p,\alpha,\lambda,h) < +\infty$ for some $\alpha \in (-\infty,-1]$ and $\lambda \in \mathbb{R}$.*

Proof. Assume that there is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_1(p,\alpha,\lambda,h) < +\infty$ for either $\alpha \in (-\infty,-1)$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \in [-1,+\infty)$. Then $h \in W^{1,p}(\mathbb{D},\Omega)$. Let

$$\mathbb{S}_r = \{\xi \in \mathbb{R}^2 : |\xi| = r\} \text{ and } \operatorname{osc}_{\mathbb{S}_r} h = \sup\{|h(\xi_1) - h(\xi_2)| : \xi_1, \xi_2 \in \mathbb{S}_r\}.$$

By the ACL-property of Sobolev mappings, we have that

$$\mathrm{osc}_{\mathbb{S}_r} h \le \int_{\mathbb{S}_r} |Dh(\xi)|\,|d\xi| \tag{3.62}$$

for $\mathcal{L}^1$-a.e. $r \in [0,1)$. By Jensen's inequality we derive from (3.62) that

$$\begin{aligned}(\mathrm{osc}_{\mathbb{S}_r} h)^p \le &(\mathrm{osc}_{\mathbb{S}_r} h)^p r^{1-p} \lesssim \int_{\mathbb{S}_r} |Dh(\xi)|^p\,|d\xi| \\ = &w_{\alpha,\lambda}^{-1}(1-r)\int_{\mathbb{S}_r} |Dh(\xi)|^p w_{\alpha,\lambda}(1-r)\,|d\xi|. \end{aligned}\tag{3.63}$$

Let $\mathbb{D}_r = \{z \in \mathbb{R}^2 : |z| < r\}$. Since h is a homeomorphism, we have $\mathrm{osc}_{\mathbb{D}_r} h = \mathrm{osc}_{\mathbb{S}_r} h$. Hence

$$\mathrm{osc}_{\mathbb{S}_r} h \text{ is increasing with respect to } r \in [0,1). \tag{3.64}$$

Moreover $w_{\alpha,\lambda}(1-r) \approx 2^{-\alpha j} j^\lambda$ for all $j \ge 0$ and $r \in (1-2^{-j}, 1-2^{-j-1}]$. By (3.63), (3.64) and Fubini's theorem, we obtain that

$$\begin{aligned}\sum_{j=1}^{+\infty} (\mathrm{osc}_{\mathbb{S}_{1-2^{-j}}} h)^p 2^{-(\alpha+1)j} j^\lambda \le &\sum_{j=1}^{+\infty} \int_{1-2^{-j}}^{1-2^{-j-1}} (\mathrm{osc}_{\mathbb{S}_r} h)^p w_{\alpha,\lambda}(1-r)\,dr \\ \lesssim &\sum_{j=1}^{+\infty} \int_{1-2^{-j}}^{1-2^{-j-1}} \int_{\mathbb{S}_r} |Dh(\xi)|^p w_{\alpha,\lambda}(1-r)\,|d\xi|\,dr \\ = &I_1(p,\alpha,\lambda,h). \end{aligned}\tag{3.65}$$

By the assumption at the beginning, we derive from (3.65) that

$$\sum_{j=1}^{+\infty} (\mathrm{osc}_{\mathbb{S}_{1-2^{-j}}} h)^p 2^{-(\alpha+1)j} j^\lambda < +\infty \tag{3.66}$$

for either $\alpha < -1$ and $\lambda \in \mathbb{R}$ or $\alpha = -1$ and $\lambda \ge -1$. Hence by (3.64) we have that $\mathrm{osc}_{\mathbb{S}_{1-2^{-j}}} h = 0$ for all $j \ge 1$. Therefore there is a constant C such that $h(z) = C$ for all $z \in \mathbb{D}$. This contradicts the homeomorphicity of h. We conclude that the assumption at the beginning cannot hold.

We next assume that there is a diffeomorphic extension $h : \mathbb{D} \to \Omega$ of φ for which $I_2(p,\alpha,\lambda,h) < +\infty$ for some $\alpha \in (-\infty,-1]$ and $\lambda \in \mathbb{R}$. It is not difficult to see that $h \in W^{1,1}(\mathbb{D},\Omega)$. We first let $\lambda \ge 0$. Proposition 2.7 shows that $\Phi_{p,\lambda}$ is convex. Analogously to (3.65), we have

$$\sum_{j=1}^{+\infty} \Phi_{p,\lambda}\left(\frac{\mathrm{osc}_{\mathbb{S}_{1-2^{-j}}} \mathrm{Re} h}{2\pi}\right) 2^{-(\alpha+1)j} \lesssim I_2(p,\alpha,\lambda,h). \tag{3.67}$$

Analogous arguments as below (3.66) imply that there is a contradiction under the above assumption. We next let $\lambda < 0$. Proposition 2.9 shows that $\Psi_{p,\lambda}$ is convex. Analogously to (3.67), we obtain from (2.38) that

$$\begin{aligned}\sum_{j=1}^{+\infty} \Psi_{p,\lambda}\left(\frac{\operatorname{osc}_{\mathbb{S}_{1-2^{-j}}} \operatorname{Re} h}{2\pi}\right) 2^{-(\alpha+1)j} \lesssim & \int_{\mathbb{D}} \Psi_{p,\lambda}(|Dh(z)|) w_{\alpha,0}(z)\, dz \\ \approx & I_2(p,\alpha,\lambda,h).\end{aligned}$$

Analogous arguments as below (3.66) imply that there is a contradiction under the above assumption. □

Proof of Theorem 1.1. Let λ_Ω be the internal distance and $|\cdot|$ be the Euclidean distance. As the proof of [9, Theorem 1] shows that there exist a bi-Lipschitz mapping $g : (\mathbb{S}^1, |\cdot|) \to (\partial\Omega, \lambda_\Omega)$ and a diffeomorphic bi-Lipschitz extension $\tilde{g} : (\mathbb{D}, |\cdot|) \to (\Omega, \lambda_\Omega)$ of g. Let $h = \tilde{g} \circ P[g^{-1} \circ \varphi]$. Then $h : \mathbb{D} \to \Omega$ is a diffeomorphic extension of φ. Moreover

$$I_1(p,\alpha,\lambda,h) \approx I_1(p,\alpha,\lambda,P[g^{-1}\circ\varphi]),\ \ I_2(p,\alpha,\lambda,h) \approx I_2(p,\alpha,\lambda,P[g^{-1}\circ\varphi]),$$

$$\mathcal{U}(p,\alpha,\lambda,\varphi) \approx \mathcal{U}(p,\alpha,\lambda,g^{-1}\circ\varphi),\ \ \mathcal{V}(p,\alpha,\lambda,\varphi) \approx \mathcal{V}(p,\alpha,\lambda,g^{-1}\circ\varphi).$$

Hence Theorem 1.1 (1) and (2) follow from Theorem 3.1. By Lemma 3.7, we complete the proof of Theorem 1.1. □

4 Examples

In this section, we give examples related to Theorem 3.1 (2). We first decompose $[0,1]$. For a given $s \in (0,+\infty)$, let

$$j_n^s = [2^{\frac{n}{s}}] \tag{4.1}$$

be the largest integer less than $2^{n/s}$. There is $n_0^s \geq 1$ such that

$$2^{-2-j_n^s} \geq 2^{-j_{n+1}^s} \text{ and } 2^{-j_n^s} \leq 4^{-n} \qquad \forall n \geq n_0^s - 1. \tag{4.2}$$

Step 1. Let

$$I_1 = I_{1,1} = (a_{1,1}, a_{1,2}) \qquad \text{where } a_{1,1} = 4^{-1} \text{ and } a_{1,2} = 1 - 4^{-1}.$$

Renumber the elements in $T_1 = \{0,1\} \cup \partial I_1$ as $\{b_{1,i_1} : i_1 = 1, ..., 4\}$ such that $b_{1,i_1'} < b_{1,i_1''}$ if $i_1' < i_1''$.

Step 2. Let

$$I_{2,1} = (b_{1,1} + 4^{-2}, b_{1,2} - 4^{-2}) \text{ and } I_{2,2} = (b_{1,3} + 4^{-2}, b_{1,4} - 4^{-2}).$$

Set $I_2 = \cup_{i=1}^{2} I_{2,i}$, and renumber the elements in $T_2 = T_1 \cup \partial I_2$ as $\{b_{2,i_2} : i_2 = 1, ..., 8\}$ such that $b_{2,i'_2} < b_{2,i''_2}$ if $i'_2 < i''_2$.

After Step (n-1), we have $\{I_{n-1,k_{n-1}} : k_{n-1} = 1, ..., 2^{n-2}\}$, $I_{n-1} = \cup_{k_{n-1}=1}^{2^{n-2}} I_{n-1,k_{n-1}}$ and $T_{n-1} := T_{n-2} \cup \partial I_{n-1} = \{b_{n-1,i_{n-1}} : i_{n-1} = 1, ..., 2^n\}$ where $b_{n-1,i'_{n-1}} < b_{n-1,i''_{n-1}}$ if $i'_{n-1} < i''_{n-1}$. In the following Step n, set

$$I_{n,k_n} := (b_{n-1,2k_n-1} + 4^{-n}, b_{n-1,2k_n} - 4^{-n}) \qquad \text{for } k_n = 1, ..., 2^{n-1}. \tag{4.3}$$

and $I_n = \cup_{k_n=1}^{2^{n-1}} I_{n,k_n}$. After renumbering the elements in $T_n = T_{n-1} \cup \partial I_n$ as above, we can proceed to Step (n+1). Moreover we must replace I_{n,k_n} in (4.3) by

$$I_{n,k_n} = (b_{n-1,2k_n-1} + 2^{-j_n^s}, b_{n-1,2k_n} - 2^{-j_n^s}) \tag{4.4}$$

whenever $n \geq n_0^s$. Let $I = \cup_{n=1}^{\infty} I_n$ and $R = [0,1] \setminus I$. Then $R \neq \emptyset$. We finally decompose $[0,1]$ as

$$R \cup I. \tag{4.5}$$

We next give an estimate on the length of I_{n,k_n}. Since $\mathcal{L}^1(I_{n,k_n}) = 2^{-j_{n-1}} - 2^{1-j_n}$ for all $n \geq n_0 + 1$ and $k_n \in \{1, ..., 2^{n-1}\}$, by the first inequality in (4.2) we have that

$$\mathcal{L}^1(I_{n,k_n}) \geq 2^{-1-j_{n-1}}. \tag{4.6}$$

for all $n \geq n_0 + 1$ and $k_n \in \{1, ..., 2^{n-1}\}$. When $n = n_0$, from (4.4) and the second estimate in (4.2) we have that

$$\mathcal{L}^1(I_{n,k_n}) = 4^{1-n_0} - 2^{1-j_{n_0}} \geq 4^{-n_0+1/2} > 4^{-n_0} \tag{4.7}$$

for all $k_n = 1, ..., 2^{n-1}$. Whenever $1 \leq n \leq n_0 - 1$ and $k_n \in \{1, ..., 2^{n-1}\}$, we have $\mathcal{L}^1(I_{n,k_n}) = 4^{-n}$. Let $C_1(s) = \min\{2^{j_{n-1}-2n} : 1 \leq n \leq n_0\}$. Then

$$\mathcal{L}^1(I_{n,k_n}) \geq C_1(s) 2^{-j_{n-1}} \tag{4.8}$$

for all $1 \leq n \leq n_0$ and $k_n \in \{1, ..., 2^{n-1}\}$. By (4.6), (4.7) and (4.8), we obtain that there is a constant $C(s) > 0$ such that

$$\mathcal{L}^1(I_{n,k_n}) \geq C(s) 2^{-j_{n-1}} \tag{4.9}$$

for all $n \in \mathbb{N}$ and $k_n \in \{1, ..., 2^{n-1}\}$.

Define

$$f_{n,s}^1(x) = \sum_{k_n=1}^{2^{n-1}} \frac{2k_n - 1}{2^n} \chi_{\overline{I_{n,k_n}}}(x) \text{ and } f_s^1(x) = \sum_{n=1}^{+\infty} f_{n,s}^1(x). \tag{4.10}$$

For any $x \in R$ and any $n \geq n_0^s$, there is $b_n \in \partial I_n$ such that $|b_n - x| = \inf_{b \in \partial I_n} |b - x|$. By (4.4) and (4.10), we have that

$$|b_n - x| \leq 2^{-j_n} \text{ and } |f_s^1(b_{n+1}) - f_s^1(b_n)| < 2^{-n-1}.$$

It follows that $\lim_{n\to+\infty} b_n = x$ and $\{f^1(b_n)\}$ is a Cauchy sequence. Therefore

$$f_s(x) = \begin{cases} f_s^1(x) & \text{if } x \in I, \\ \lim_{n\to+\infty} f_s^1(b_n) & \text{if } x \in R. \end{cases} \tag{4.11}$$

is a well-defined function on $[0, 1]$.

Proposition 4.1. *Let f_s be as in (4.11) with $s \in (0, +\infty)$. Then $f_s(0) = 0$, $f_s(1) = 1$ and f_s is increasing on $[0, 1]$. Moreover there is a constant $C(s) > 0$ such that*

$$|f(x) - f(y)| \log^s(|x - y|^{-1}) \leq C(s) \tag{4.12}$$

for all $x, y \in [0, 1]$ with $x \neq y$.

Proof. By (4.11), we have that $f_s(0) = \lim_{n\to\infty} f_s^1(2^{-j_n}) = \lim_{n\to\infty} 2^{-n} = 0$. Analogously $f_s(1) = 1$.

We next prove the monotonicity of f_s. Let $x_1 \in [0, 1]$, $x_2 \in [0, 1]$ with $x_1 \leq x_2$. If $x_1 \in I_{n,k'_n}$ and $x_2 \in I_{n,k''_n}$ with $k'_n \leq k''_n$, from (4.11) we have that

$$f_s(x_1) \leq f_s(x_2). \tag{4.13}$$

Assume $x_1 \in I_{n_1,k_{n_1}}$ and $x_2 \in I_{n_2,k_{n_2}}$ with $n_1 \neq n_2$. Let $q = |n_2 - n_1|$. If $n_1 < n_2$, from the construction of $\{I_{n,k_n}\}$ we have that $k_{n_2} \geq 2^q(k_{n_1} - 1) + 2^{q-1} + 1$. It then follows from (4.10) that

$$f_s(x_2) \geq \frac{2\big(2^q(k_{n_1} - 1) + 2^{q-1} + 1\big) - 1}{2^{n_1} 2^q} > f_s(x_1). \tag{4.14}$$

If $n_2 < n_1$, from the construction of $\{I_{n,k_n}\}$ we have that

$$k_{n_2} \geq \begin{cases} \left[\frac{k_{n_1}}{2^q}\right] + 1 & \text{if } 0 \leq \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] \leq 1/2, \\ \left[\frac{k_{n_1}}{2^q}\right] + 2 & \text{if } 1/2 < \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] < 1. \end{cases}$$

It follows that

$$2k_{n_2} - 1 \geq 2(\frac{k_{n_1}}{2^q} + 1/2) - 1 = 2\frac{k_{n_1}}{2^q} \qquad \text{if } 0 \leq \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] \leq 1/2 \tag{4.15}$$

and

$$2k_{n_2} - 1 \geq 2(\frac{k_{n_1}}{2^q} + 1) - 1 = 2\frac{k_{n_1}}{2^q} + 1 \qquad \text{if } 1/2 < \frac{k_{n_1}}{2^q} - \left[\frac{k_{n_1}}{2^q}\right] < 1. \tag{4.16}$$

By combining (4.15) with (4.16), we deduce from (4.10) that

$$f_s(x_2) > f_s(x_1). \tag{4.17}$$

Assume $x_1 \in R$ and $x_2 \in I$. By (4.11), there is $\{b_n\} \subset \partial I$ such that $\lim\limits_{n\to\infty} b_n = x_1$. Together with $x_1 < x_2$, it follows that $b_n < x_2$ whenever $n \gg 1$. Via the arguments for (4.13), (4.14) and (4.17), we have that

$$f_s^1(b_n) \leq f_s(x_2) \qquad \forall n \gg 1. \tag{4.18}$$

By taking limit for (4.18), we have that

$$f_s(x_1) \leq f_s(x_2). \tag{4.19}$$

Assume either $x_1 \in I$ and $x_2 \in R$, or $x_1 \in R$ and $x_2 \in R$. Via analogous arguments as for (4.19), we can also prove $f_s(x_1) \leq f_s(x_2)$ at these two cases. By preceding arguments, we conclude that f_s is increasing on $[0, 1]$.

We next prove (4.12). Let $T_n = \{b_{n,i_n} : i_n = 1, ..., 2^{n+1}\}$ with $n \in \mathbb{N}$ and $f_{i,s}^1$ be as in (4.10). For a given $n \in \mathbb{N}$, define

$$f_{n,s}^2(x) = \sum_{i=1}^{2^n} \left(\frac{2^{j_n}}{2^n}(x - b_{n,2i-1}) + \frac{i-1}{2^n} \right) \chi_{[b_{n,2i-1}, b_{n,2i}]}(x),$$

$$f_{n,s}(x) = f_{n,s}^2(x) + \sum_{i=1}^{n} f_{i,s}^1(x). \tag{4.20}$$

Then $f_{n,s}$ is piecewise affine, increasing and continuous on $[0, 1]$. Furthermore we claim:

(i) $\lim_{n\to\infty} f_{n,s}(x_0) = f_s(x_0)$ for all $x_0 \in [0, 1]$,

(ii) there are constant $C(s) > 0$ and $N(s) > 0$ such that

$$\sup\left\{|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) : x, y \in [0,1] \text{ and } x \neq y\right\} \leq C(s)$$

for all $n \geq N(s)$.

If both (i) and (ii) hold, we can prove (4.12).

We first prove (i). Let $x_0 \in [0,1]$. If $x_0 \in I$, without loss of generality we assume that $x_0 \in I_{n_0,k_{n_0}}$. From (4.11) and (4.20), we have that $f_n(x_0) = f(x_0)$ for all $n \geq n_0$. Therefore (i) holds. If $x_0 \in R$, from (4.11) there is $\{b_n\} \subset \partial I$ such that $\lim_{n\to\infty} b_n = x_0$ and $\lim_{n\to\infty} f_s^1(b_n) = f_s(x_0)$. Moreover by (4.20), we have that

$$|f_{n,s}(x_0) - f_s^1(b_n)| = |f_{n,s}(x_0) - f_{n,s}(b_n)| \leq 2^{-n}.$$

Together with $|f_{n,s}(x_0) - f_s(x_0)| \leq |f_{n,s}(x_0) - f_s^1(b_n)| + |f_s^1(b_n) - f_s(x_0)|$, we have that (i) also holds at this case.

We next prove (ii). Given $n \geq 1,\ x \in [0,1]$ and $y \in [0,1]$ with $x < y$, set

$$k_n(x,y) = \#\{I_{m,k_m} :\ I_{m,k_m} \subset [x,y] \text{ for } m = 1, ..., n \text{ and } k_m = 1, ..., 2^{m-1}\}.$$

Then $0 \leq k_n(x,y) \leq 2^n - 1$.

Assume $k_n(x,y) = 0$. If $x \in \cup_{m=1}^n I_m$, there are $m \in \{1, ..., n\}$ and $k_m \in \{1, ..., 2^{m-1}\}$ such that $x \in I_{m,k_m}$. For the location of y, possibly we have that

$$y \in I_{m,m_k},\ y \in I_{m,m_k+1},\ \text{or } y \in [0,1] \setminus (\cup_{m=1}^n I_m). \tag{4.21}$$

If $y \in I_{m,m_k}$, by (4.20) we have that

$$f_{n,s}(x) = f_{n,s}(y) \qquad \forall n \geq m.$$

If $y \in I_{m,m_k+1}$, then $|x - y| \geq 2^{-j_n}$. It follows from (4.20) that

$$|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \leq 2^{-n} \log^s(2^{j_n}) < 1. \tag{4.22}$$

If $y \in [0,1] \setminus (\cup_{m=1}^n I_m)$, there is $x_0 \in [x,y) \cap T_n$ such that

$$0 < y - x_0 < 2^{-j_n} \text{ and } f_{n,s}(x) = f_{n,s}(x_0). \tag{4.23}$$

Since there is $n_1^s > 0$ such that $\log(2^{j^s_{n_1^s}}) - s > 0$, we have that

$$t \log^s(t^{-1}) \leq 2^{-j_n^s} \log^s(2^{j_n^s}) < 2^{n-j_n} \tag{4.24}$$

for all $n \geq n_1^s$ and every $t \in (0, 2^{-j_n^s}]$. By (4.20), (4.23) and (4.24), we then have that

$$
\begin{aligned}
|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \leq & \frac{|f_{n,s}(x_0) - f_{n,s}(y)|}{|x_0 - y|} |x_0 - y| \log^s(|x_0 - y|^{-1}) \\
< & \frac{2^{j_n}}{2^n} \frac{2^n}{2^{j_n}} = 1
\end{aligned} \tag{4.25}
$$

whenever $n \geq N(s) := \max\{n_0^s, n_1^s\}$. If $x \in [0,1] \setminus (\cup_{m=1}^n I_m)$, for the location of y we possibly have that

$$
y \in [0,1] \setminus (\cup_{m=1}^n I_m), \;\; y \in \cup_{m=1}^n I_m.
$$

If $y \in [0,1] \setminus (\cup_{m=1}^n I_m)$, then $0 < y - x < 2^{-j_n}$. By (4.20) and (4.24) we have that

$$
|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) = \frac{2^{j_n}}{2^n} |x-y| \log^s(|x-y|^{-1}) < 1 \tag{4.26}
$$

for all $n \geq N(s)$. If $y \in \cup_{m=1}^n I_m$, by analogous arguments as for (4.25) we have that

$$
|f_{n,s}(y) - f_{n,s}(x)| \log^s(|x-y|^{-1}) < 1 \tag{4.27}
$$

for all $n \geq N(s)$. By (4), (4.22), (4.25), (4.26) and (4.27), we conclude that

$$
|f_{n,s}(y) - f_{n,s}(x)| \log^s(|x-y|^{-1}) < 1 \tag{4.28}
$$

for all $n \geq N(s)$ and $k_n(x,y) = 0$.

Assume $k_n(x,y) \in \{1, ..., 2^n - 1\}$. Define

$$
x' = \inf\{e \in I_{m,k_m} : I_{m,k_m} \subset [x,y] \text{ for } m = 1, ..., n \text{ and } k_m = 1, ..., 2^{m-1}\}
$$

and

$$
y' = \sup\{e \in I_{m,k_m} : I_{m,k_m} \subset [x,y] \text{ for } m = 1, ..., n \text{ and } k_m = 1, ..., 2^{m-1}\}.
$$

If $k_n(x,y) = 1$, by (4.20) we have that

$$
f_{n,s}(x') = f_{n,s}(y'). \tag{4.29}
$$

If $2^m \leq k_n(x,y) \leq 2^{m+1} - 1$ for $m = 1, ..., n-1$, by (4.5), (4.9) and (4.20) we have that

$$
|x - y| \geq \mathcal{L}^1(I_{n-m,k_{n-m}}) \geq C(s) 2^{-j_{n-m-1}^s}
$$

and

$$|f_{n,s}(x') - f_{n,s}(y')| = \frac{2 + ... + 2^m}{2^n} < 2^{m+1-n}.$$

Whenever $n \geq n_0^s + 1$, it follows from (4.1) that

$$\begin{aligned} |f_{n,s}(x') - f_{n,s}(y')| \log^s(|x-y|^{-1}) \leq & 2^{m+1-n} \log^s \left(C^{-1} 2^{j^s_{n-m-1}}\right) \\ \leq & C(s) 2^{m+1-n} j^s_{n-m-1} < C(s). \end{aligned} \tag{4.30}$$

Notice that there are two cases for the location of x

$$x \in (x' - 2^{-j_n}, x'],\ \ x \in \cup_{m=1}^n I_m.$$

If $x \in (x' - 2^{-j_n}, x']$, by analogous arguments as for (4.26) we have that

$$|f_{n,s}(x) - f_{n,s}(x')| \log^s(|x-y|^{-1}) < 1 \qquad \text{whenever } n \geq N(s). \tag{4.31}$$

If $x \in \cup_{m=1}^n I_m$, same arguments as (4.25) imply (4.31). Analogously, we have that

$$|f_{n,s}(y') - f_{n,s}(y)| \log^s(|x-y|^{-1}) < 1 \qquad \text{whenever } n \geq N(s). \tag{4.32}$$

Since

$$\begin{aligned} & |f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \\ = & \big(|f_{n,s}(x) - f_{n,s}(x')| + |f_{n,s}(x') - f_{n,s}(y')| + |f_{n,s}(y') - f_{n,s}(y)|\big) \log^s(|x-y|^{-1}), \end{aligned}$$

by (4.29), (4.30), (4.31) and (4.32) there is a constant $C(s) > 0$ such that

$$|f_{n,s}(x) - f_{n,s}(y)| \log^s(|x-y|^{-1}) \leq C(s) \tag{4.33}$$

whenever $n \geq N(s)$ and $k_n(x, y) \in \{1, ..., 2^n - 1\}$. By (4.28) and (4.33), we finish the proof of (ii). $\square$

Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism. In the following we denote by $P[\varphi] : \mathbb{D} \to \mathbb{D}$ the harmonic extension of φ.

Example 4.2. *For a given $p \in (1, 2)$, there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\mathcal{V}(p, p-2, 0, \varphi) < \infty$, $I_1(p, p-2, 0, P[\varphi]) = \infty$ and $I_2(p, p-2, 0, P[\varphi]) = \infty$.*

Proof. We first introduce a class of self-homeomorphisms on $\mathbb{S}^1$ and their properties. Let f_s be as in (4.11) with $s \in (0, +\infty)$. Define

$$g_s(x) = \frac{f_s(x) + x}{2} \qquad x \in [0, 1]. \tag{4.34}$$

Then $g_s : [0,1] \to [0,1]$ is strictly increasing and continuous, i.e. g_s is homeomorphic. Moreover by (4.1), there is a constant $C(s) > 0$ such that

$$|g_s(x) - g_s(y)| \leq C(s) \log^{-s}(|x-y|^{-1}) \tag{4.35}$$

for all $x, y \in [0,1]$ with $x \neq y$. Let arg$z \in (-\pi, \pi]$ be the principal value of the argument z. Define

$$\varphi_s(z) = \exp\left(i2\pi \left[g_s\left(\frac{\arg z}{2\pi}\right) - g_s(\frac{1}{2}) + \frac{1}{2}\right]\right) \qquad z \in \mathbb{S}^1. \tag{4.36}$$

Then $\varphi_s : \mathbb{S}^1 \to \mathbb{S}^1$ is homeomorphic and $\varphi(e^{i\pi}) = e^{i\pi}$. Next we prove that

$$|\varphi_s(z_1) - \varphi_s(z_2)| \lesssim \log^{-s}(|z_1 - z_2|^{-1}) \tag{4.37}$$

for all $z_1, z_2 \in \mathbb{S}^1$ with $z_1 \neq z_2$. Let $\Gamma(z_1, z_2)$ be the arc in $\mathbb{S}^1$ joining z_1 to z_2 with smaller length. Denote by $\ell(\Gamma(z_1, z_2))$ the length of $\Gamma(z_1, z_2)$. In order to prove (4.37), it is enough to consider the case $\ell(\Gamma(z_1, z_2)) \ll 1$. If $e^{i\pi} \notin \Gamma(z_1, z_2)$, we have that

$$|\arg z_1 - \arg z_2| \approx |z_1 - z_2| \text{ and } \left|g_s\left(\frac{\arg z_1}{2\pi}\right) - g_s\left(\frac{\arg z_2}{2\pi}\right)\right| \approx |\varphi_s(z_1) - \varphi_s(z_2)|$$

whenever $\ell(\Gamma(z_1, z_2)) \ll 1$. Together with (4.35), we then have that

$$\begin{aligned} |\varphi_s(z_1) - \varphi_s(z_2)| \approx & \left|g_s\left(\frac{\arg z_1}{2\pi}\right) - g_s\left(\frac{\arg z_2}{2\pi}\right)\right| \\ \lesssim & \log^{-s}(|\arg z_1 - \arg z_2|^{-1}) \approx \log^{-s}(|z_1 - z_2|^{-1}). \end{aligned} \tag{4.38}$$

If $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi(z_1), \varphi(e^{i\pi}))) > \ell(\Gamma(\varphi(e^{i\pi}), \varphi(z_2)))$, there is $z_0 \in \Gamma(z_1, e^{i\pi})$ such that

$$|\varphi_s(z_1) - \varphi_s(z_2)| \lesssim |\varphi_s(z_1) - \varphi_s(z_0)|. \tag{4.39}$$

Same arguments as for (4.38) imply that

$$|\varphi_s(z_1) - \varphi_s(z_0)| \lesssim \log^{-s}(|z_1 - z_0|^{-1}) \lesssim \log^{-s}(|z_1 - z_2|^{-1}). \tag{4.40}$$

Combining (4.39) with (4.40) therefore implies that (4.37) holds when $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi_s(z_1), \varphi_s(e^{i\pi}))) > \ell(\Gamma(\varphi(e^{i\pi})$. Analogously, we can prove that (4.37) holds when $e^{i\pi} \in \Gamma(z_1, z_2)$ and $\ell(\Gamma(\varphi_s(z_1), \varphi_s(e^{i\pi}))) \leq \ell(\Gamma(\varphi_s(e^{i\pi}), \varphi_s(z_2)))$.

Let $p \in (1, 2)$. There is $s \in (1, +\infty)$ such that $p - 1 < 1/s < 1$. Based on this s, we obtain a homeomorphism $\varphi = \varphi_s : \mathbb{S}^1 \to \mathbb{S}^1$, where φ_s is from (4.36). By

Jensen's inequality and (4.37), we have that

$$\begin{aligned}\mathcal{V}(p,p-2,0,\varphi) =& \int_{\mathbb{S}^1} \Big(\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\Big)^{p-1}|d\xi| \\ \le& \Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi|\Big)^{p-1} \\ \lesssim& \Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} |\xi-\eta|^{-\frac{1}{s}}\,|d\eta|\,|d\xi|\Big)^{p-1} < +\infty.\end{aligned}$$

Let n_0^s be as in (4.2) with s chosen above. For any $n \ge n_0^s$ and any $j_n < j \le j_{n+1}$, by (4.34) and (4.11) we have that

$$\begin{aligned}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p =& 2\pi \sum_{k=1}^{2^j} \mathcal{L}^1(g_s([(k-1)2^{-j}, k2^{-j}]))^p \\ \gtrsim& \sum_{k=1}^{2^j} (f_s(k2^{-j}) - f_s((k-1)2^{-j}))^p = 2^{(n+1)(1-p)}.\end{aligned} \tag{4.41}$$

Notice that $j_{n+1} - j_n \approx 2^{n/s}$ whenever $n \ge n_0$. We then derive from (4.41) that

$$\mathcal{E}_1(p,p-2,0,\varphi) \ge \sum_{n=n_0}^{+\infty} \sum_{j_n<j\le j_{n+1}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \gtrsim \sum_{n=n_0}^{+\infty} 2^{n(1-p+\frac{1}{s})} = +\infty.$$

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, it follows that $I_1(p,p-2,0,P[\varphi]) = \infty$ and $I_2(p,p-2,0,P[\varphi]) = \infty$. □

Example 4.3. *For a given $p \in (2,+\infty)$, there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\mathcal{V}(p,p-2,0,\varphi) = \infty$, $I_1(p,p-2,0,P[\varphi]) < \infty$ and $I_2(p,p-2,0,P[\varphi]) < \infty$.*

Proof. Since $p \in (2,+\infty)$, there is $s \in (0,1)$ such that $p-1 > 1/s > 1$. Based on this chosen s, we obtain a homeomorphism $\varphi = \varphi_s : \mathbb{S}^1 \to \mathbb{S}^1$, where φ_s is from (4.36). In order to prove $\mathcal{V}(p,p-2,0,\varphi) = \infty$, by Jensen's inequality it suffices to prove that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi| = +\infty. \tag{4.42}$$

For any $\sigma \in \mathbb{S}^1$ and $\tau \in \mathbb{S}^1$, let $\ell(\sigma,\tau)$ be the arc length of the shorter arc in $\mathbb{S}^1$ joining σ and τ. Let n_0^s be from (4.2) with s chosen above. For any $n \ge n_0^s$, set

$$\Gamma_n = \{(\xi,\eta) \in \mathbb{S}^1 \times \mathbb{S}^1 : \pi 2^{1-j_{n+1}} < \ell(\varphi^{-1}(\xi), \varphi^{-1}(\eta)) \le \pi 2^{1-j_n}\}.$$

We have that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi| \geq \sum_{n=n_0}^{+\infty}\int_{\Gamma_n} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi|$$

$$\gtrsim \sum_{n=n_0}^{+\infty} j_n \int_{\Gamma_n} |d\eta|\,|d\xi|. \tag{4.43}$$

Given $n \geq n_0^s$ and $k = 1, ..., 2^n$, let

$$\Gamma'_{n,k} = \exp(i2\pi[b_{n,2k-1}, 2^{-j_{n+1}} + b_{n,2k-1}]),$$

$$\Gamma''_{n,k} = \exp(i2\pi[2^{-j_n} - 2^{-j_{n+1}} + b_{n,2k-1}, 2^{-j_n} + b_{n,2k-1}]).$$

For any $\xi \in \varphi(\Gamma'_{n,k})$ and $\eta \in \varphi(\Gamma''_{n,k})$, we have that

$$2\pi(2^{-j_n} - 2^{1-j_{n+1}}) \leq \ell(\varphi^{-1}(\xi), \varphi^{-1}(\eta)) \leq \pi \cdot 2^{1-j_n}. \tag{4.44}$$

Notice that by (4.2) we have that $2^{-j_{n+1}} < 2^{-j_n} - 2^{1-j_{n+1}}$ whenever $n \geq n_0^s$. It then follows from (4.44) that

$$\varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k}) \subset \Gamma_n \tag{4.45}$$

for all $n \geq n_0^s$ and all $k = 1, ..., 2^n$. Moreover from (4.36), (4.34) and (4.11), it follows that

$$\ell(\varphi(\Gamma'_{n,k})) = 2\pi\mathcal{L}^1(g([b_{n,2k-1}, 2^{-j_{n+1}} + b_{n,2k-1}]))$$

$$\geq \pi(f_s(2^{-j_{n+1}}) - f_s(0)) = \pi 2^{-n-1}. \tag{4.46}$$

for all $n \geq n_0^s$ and all $k = 1, ..., 2^n$. Similarly

$$\ell(\varphi(\Gamma''_{n,k})) \geq \pi 2^{-n-1}. \tag{4.47}$$

Since $(\varphi(\Gamma'_{n,k}) \times \varphi(\Gamma''_{n,k})) \cap (\varphi(\Gamma'_{n,j}) \times \varphi(\Gamma''_{n,j})) = \emptyset$ for all $n \geq n_0^s$ and $k, j \in \{1, ..., 2^n\}$ with $k \neq j$, it follows (4.45), (4.46) and (4.47) that

$$\int_{\Gamma_n} |d\eta|\,|d\xi| \geq \sum_{k=1}^{2^n} \int_{\varphi(\Gamma'_{n,k})\times\varphi(\Gamma''_{n,k})} |d\xi|\,|d\eta| \geq \pi^2 2^{-n-2} \tag{4.48}$$

for all $n \geq n_0^s$. Combining (4.43) with (4.48) hence implies that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|^{-1}\,|d\eta|\,|d\xi| \gtrsim \sum_{n=n_0}^{+\infty} \frac{j_n}{2^n} \approx \sum_{n=n_0}^{+\infty} \frac{2^{\frac{n}{s}}}{2^n} = +\infty.$$

Therefore (4.42) is complete.

For any $n \geq n_0$ and $j_n < j \leq j_{n+1}$, by (4.36), (4.34), (4.11) and Jensen's inequality we have that

$$\begin{aligned}\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p =& 2\pi \sum_{k=1}^{2^j} \mathcal{L}^1(g_s([(k-1)2^{-j}, k2^{-j}]))^p \\ \lesssim& \sum_{k=1}^{2^j} (f_s(k2^{-j}) - f_s((k-1)2^{-j}))^p + \sum_{k=1}^{2^j} 2^{-pj} \\ =& 2^{(1-p)(n+1)} + 2^{(1-p)j} \end{aligned} \tag{4.49}$$

Notice $j_{n+1} - j_n \approx 2^{n/s}$ whenever $n \geq n_0^s$. We then derive from (4.49) that

$$\begin{aligned}\sum_{j=j_{n_0}+1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \lesssim& \sum_{n=n_0}^{+\infty} \sum_{j_n < j \leq j_{n+1}} 2^{(1-p)(n+1)} + \sum_{j=j_{n_0}+1}^{+\infty} 2^{(1-p)j} \\ \approx& \sum_{n=n_0}^{+\infty} 2^{n(1-p+\frac{1}{s})} + \sum_{j=j_{n_0}+1}^{+\infty} 2^{(1-p)j} < +\infty. \end{aligned} \tag{4.50}$$

Since $\sum_{j=1}^{j_{n_0}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p$ is finite, it follows from (4.50) that $\mathcal{E}_1(p, p-2, 0, \varphi) < +\infty$. Moreover by Lemma 3.3 and Lemma 3.4 we have that $I_1(p, p-2, 0, P[\varphi]) < +\infty$ and $I_2(p, p-2, 0, P[\varphi]) < +\infty$. □

Example 4.4. *There is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that both $I_1(p, \alpha, \lambda, P[\varphi]) < +\infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) < +\infty$ hold for all $p > 1$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. Moreover for any $p > 1$, there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $I_1(p, \alpha, \lambda, p[\varphi]) = \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) = \infty$ whenever either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$.*

Proof. Take $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ as the identity mapping. We have that

$$\mathcal{E}_1(p, \alpha, \lambda, \varphi) \approx \sum_{j=1}^{+\infty} 2^{j(p-2-\alpha)} j^\lambda 2^j (2^{1-j}\pi)^p \approx \sum_{j=1}^{+\infty} 2^{-j(1+\alpha)} j^\lambda < +\infty$$

whenever $p > 1$, $\alpha \in (-1, p-1)$ and $\lambda \in \mathbb{R}$. Therefore by Lemma 3.3 and Lemma 3.4 both $I_1(p, \alpha, \lambda, P[\varphi])$ and $I_2(p, \alpha, \lambda, P[\varphi])$ are finite now.

For a given $p > 1$, set j_n in (4.1) as $[e^{2^{n(p-1)}}]$. There is $n_0 \geq 1$ such that (4.2) holds for all $n \geq n_0 - 1$. By following the arguments for (4.5), we have f as in (4.11). Moreover by same arguments as in the proof of Proposition 4.1, there is a constant $C > 0$ depending only on p such that

$$|f(x) - f(y)| \log^{\frac{1}{p-1}} \log(|x-y|^{-1}) \leq C$$

for all $x, y \in [0,1]$ with $x \neq y$. As in (4.36), we finally obtain a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. For any $n \geq n_0$ and $j_n < j \leq j_{n+1}$, by analogous arguments for (4.41) we have that

$$\sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \gtrsim 2^{n(1-p)}. \tag{4.51}$$

Notice that $\sum_{j_n<j\leq j_{n+1}} j^{-1} \approx \log j_{n+1} - \log j_n \gtrsim 2^{n(p-1)}$ for all $n \geq n_0$. For any $\lambda \in [-1, +\infty)$ it then follows from (4.51) that

$$\begin{aligned}\mathcal{E}_1(p, p-2, \lambda, \varphi) &\geq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p j^{-1} \geq \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} j^{-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p \\ &\gtrsim \sum_{n=n_0}^{+\infty} 2^{n(p-1)} \cdot 2^{n(1-p)} = +\infty. \end{aligned} \tag{4.52}$$

For any $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$, we have that $2^{j(p-2-\alpha)} j^\lambda \gtrsim j^{-1}$ whenever $j \gg 1$. Without loss of generality, we assume that $2^{j(p-2-\alpha)} j^\lambda \gtrsim j^{-1}$ for all $n \geq n_0$ and $j_n < j \leq j_{n+1}$. Hence from (4.52) we have that

$$\begin{aligned}\mathcal{E}_1(p, \alpha, \lambda, \varphi) &\geq \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p 2^{j(p-2-\alpha)} j^\lambda \\ &\gtrsim \sum_{n=n_0}^{+\infty} \sum_{j_n<j\leq j_{n+1}} \frac{1}{j} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^p = +\infty \end{aligned} \tag{4.53}$$

for all $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we conclude from (4.52) and (4.53) that for any $p > 1$ there is a homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $I_1(p, \alpha, \lambda, P[\varphi]) = \infty$ and $I_2(p, \alpha, \lambda, P[\varphi]) = \infty$ whenever either $\alpha \in (-1, p-2)$ and $\lambda \in \mathbb{R}$ or $\alpha = p-2$ and $\lambda \in [-1, +\infty)$. $\square$

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[**B**]

Optimal extensions of conformal mappings from the unit disk to cardioid-type domains

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OPTIMAL EXTENSIONS OF CONFORMAL MAPPINGS FROM THE UNIT DISK TO CARDIOID-TYPE DOMAINS

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Abstract. The conformal mapping $f(z) = (z+1)^2$ from $\mathbb{D}$ onto the standard cardioid has a homeomorphic extension of finite distortion to entire $\mathbb{R}^2$. We study the optimal regularity of such extensions, in terms of the integrability degree of the distortion and of the derivatives, and these for the inverse. We generalize all outcomes to the case of conformal mappings from $\mathbb{D}$ onto cardioid-type domains.

1. Introduction

The standard cardioid domain

$$\Delta = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 - 4x(x^2 + y^2) - 4y^2 < 0\} \tag{1.0.1}$$

is the image of the unit disk $\mathbb{D}$ under the conformal mapping $g(z) = (z + 1)^2$. Since the origin is an inner-cusp point of $\partial\Delta$, the Ahlfors' three-point property fails, and hence $\partial\Delta$ is not a quasicircle. Therefore the preceding conformal mapping does not possess a quasiconformal extension to the entire plane. However, there is a homeomorphic extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ by the Schoenflies theorem, see [10, Theorem 10.4]. Recall that homeomorphisms of finite distortion form a much larger class of homeomorphisms than quasiconformal mappings. A natural question arises: can we extend g as a homeomorphism of finite distortion? If we can, how good an extension can we find? Our first result gives a rather complete answer.

Theorem 1.1. *Let $\mathcal{F}$ be the collection of homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion such that $f(z) = (z + 1)^2$ for all $z \in \mathbb{D}$. Then $\mathcal{F} \neq \emptyset$. Moreover*

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F} \cap W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)\} = +\infty, \tag{1.0.2}$$

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F},\ K_f \in L^q_{loc}(\mathbb{R}^2)\} = 2, \tag{1.0.3}$$

$$\begin{aligned}&\sup\{q \in (0, +\infty) : f \in \mathcal{F} \cap W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2) \text{ for some } p > 1 \text{ and } K_f \in L^q_{loc}(\mathbb{R}^2)\}\\ &= 1,\end{aligned} \tag{1.0.4}$$

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F},\ f^{-1} \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)\} = \frac{5}{2} \tag{1.0.5}$$

and

$$\sup\{q \in (0, +\infty) : f \in \mathcal{F},\ K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)\} = 5. \tag{1.0.6}$$

The cardioid curve $\partial\Delta$ contains an inner-cusp point of asymptotic polynomial degree $3/2$. Motivated by this, we introduce a family of cardioid-type domains Δ_s with degree $s > 1$, see (2.3.2). Our second result is an analog of Theorem 1.1.

Theorem 1.2. *Let g be a conformal map from $\mathbb{D}$ onto Δ_s, where Δ_s is defined in* (2.3.2) *and $s > 1$. Suppose that $\mathcal{F}_s(g)$ is the collection of homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion such that $f|_{\mathbb{D}} = g$. Then $\mathcal{F}_s(g) \neq \emptyset$. Moreover*

$$\sup\{p \in [1, +\infty) : f \in \mathcal{F}_s(g) \cap W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)\} = +\infty, \tag{1.0.7}$$

Key words and phrases. Extensions; Homeomorphisms of finite distortion; Inner cusp.

$$\sup\{q \in (0,+\infty) : f \in \mathcal{F}_s(g),\ K_f \in L^q_{loc}(\mathbb{R}^2)\} = \max\left\{\frac{1}{s-1}, 1\right\}, \tag{1.0.8}$$

$$\begin{aligned} &\sup\{q \in (0,+\infty) : f \in \mathcal{F}_s(g) \cap W^{1,p}_{loc}(\mathbb{R}^2,\mathbb{R}^2) \textit{ for some } p>1 \textit{ and } K_f \in L^q_{loc}(\mathbb{R}^2)\} \\ &= \max\left\{\frac{1}{s-1}, \frac{3p}{(2s-1)p+4-2s}\right\}, \end{aligned} \tag{1.0.9}$$

$$\sup\{p \in [1,+\infty) : f \in \mathcal{F}_s(g),\ f^{-1} \in W^{1,p}_{loc}(\mathbb{R}^2,\mathbb{R}^2)\} = \frac{2(s+1)}{2s-1} \tag{1.0.10}$$

and

$$\sup\{q \in (0,+\infty) : f \in \mathcal{F}_s(g),\ K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)\} = \frac{s+1}{s-1}. \tag{1.0.11}$$

Extendability questions similar to Theorem 1.2 have also been studied in [3,4,8].

In Section 2, we recall some basic definitions and facts. We also introduce auxiliary mappings and domains. In Section 3, we give upper bounds for integrability degrees of potential extensions. Section 4 is devoted to the proof of Theorem 1.2. In Section 5, we prove Theorem 1.1.

2. Preliminaries

2.1. **Notation.** By $s \gg 1$ and $t \ll 1$ we mean that s is sufficiently large and t is sufficiently small, respectively. By $f \lesssim g$ we mean that there exists a constant $M > 0$ such that $f(x) \le Mg(x)$ for every x. We write $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$ hold. By $\mathcal{L}^2$ (respectively $\mathcal{L}^1$) we mean the 2-dimensional (1-dimensional) Lebesgue measure. Furthermore we refer to the disk with center P and radius r by $B(P,r)$, and $S(P,r) = \partial B(P,r)$. For a set $E \subset \mathbb{R}^2$ we denote by $\overline{E}$ the closure of E. If $A \in \mathbb{R}^{2\times 2}$ is a matrix, $adjA$ is the adjoint matrix of A.

2.2. **Basic definitions and facts.**

Definition 2.1. Let $\Omega \subset \mathbb{R}^2$ and $\Omega' \subset \mathbb{R}^2$ be domains. A homeomorphism $f : \Omega \to \Omega'$ is called K-quasiconformal if $f \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ and if there is a constant $K \ge 1$ such that

$$|Df(z)|^2 \le KJ_f(z)$$

holds for $\mathcal{L}^2$-a.e. $z \in \Omega$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^2$ be a domain. We say that a mapping $f : \Omega \to \mathbb{R}^2$ has finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$, $J_f \in L^1_{\mathrm{loc}}(\Omega)$ and

$$|Df(z)|^2 \le K_f(z)J_f(z) \qquad \mathcal{L}^2\text{-a.e. } z \in \Omega, \tag{2.2.1}$$

where

$$K_f(z) = \begin{cases} \frac{|Df(z)|^2}{J_f(z)} & \text{for all } z \in \{J_f > 0\}, \\ 1 & \text{for all } z \in \{J_f = 0\}. \end{cases}$$

Definition 2.3. Given $A \subset \mathbb{R}^2$, a map $f : A \to \mathbb{R}^2$ is called an (l,L)-bi-Lipschitz mapping if $0 < l \le L < \infty$ and

$$l|x-y| \le |f(x)-f(y)| \le L|x-y|$$

for all $x, y \in A$.

If $\Omega \subset \mathbb{R}^2$ is a domain and $f : \Omega \to \mathbb{R}^2$ is an orientation-preserving bi-Lipschiz mapping, then f is quasiconformal.

Definition 2.4. Given a function φ defined on set $A \subset \mathbb{R}^2$, its modulus of continuity is defined as

$$\omega(\delta) \equiv \omega(\delta, \varphi, A) = \sup\{|\varphi(z_1) - \varphi(z_2)| : z_1, z_2 \in A,\ |z_1 - z_2| \le \delta\}$$

for $\delta \ge 0$. Then φ is called Dini-continuous if

$$\int_0^\pi \frac{\omega(t)}{t}\, dt < \infty,$$

where the integration bound π can be replaced by any positive constant.

We say that a curve C is *Dini-smooth* if it has a parametrization $\alpha(t)$ for $t \in [0, 2\pi]$ so that $\alpha'(t) \neq 0$ for all $t \in [0, 2\pi]$ and α' is Dini-continuous.

Definition 2.5. Let $\Omega \subset \mathbb{R}^2$ be open and $f : \Omega \to \mathbb{R}^2$ be a mapping. We say that f satisfies the Lusin (N) condition if $\mathcal{L}^2(f(E)) = 0$ for any $E \subset \Omega$ with $\mathcal{L}^2(E) = 0$. Similarly, f satisfies the Lusin (N^{-1}) condition if $\mathcal{L}^2(f^{-1}(E)) = 0$ for any $E \subset \Omega$ with $\mathcal{L}^2(E) = 0$.

Lemma 2.1. ([6, Theorem A.35]) *Let $\Omega \subset \mathbb{R}^2$ be open and $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^2)$. Suppose that η is a nonnegative Borel measurable function on $\mathbb{R}^2$. Then*

$$\int_\Omega \eta(f(x))|J_f(x)|\, dx \le \int_{f(\Omega)} \eta(y) N(f, \Omega, y)\, dy, \tag{2.2.2}$$

where the multiplicity function $N(f, \Omega, y)$ of f is defined as the number of preimages of y under f in Ω. Moreover (2.2.2) *is an equality if we assume in addition that f satisfies the Lusin (N) condition.*

Lemma 2.2. ([6, Lemma A.28]) *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism which belongs to $W^{1,1}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. Then f is differentiable $\mathcal{L}^2$-a.e. on $\mathbb{R}^2$.*

Lemma 2.2 and a simple computation show that

$$\max_{\theta \in [0,2\pi]} |\partial_\theta f(z)| = K_f(z) \min_{\theta \in [0,2\pi]} |\partial_\theta f(z)| \qquad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2 \tag{2.2.3}$$

when $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism of finite distortion. Here $\partial_\theta f(z) = \cos(\theta) f_x(z) + \sin(\theta) f_y(z)$ for $\theta \in [0, 2\pi]$.

Lemma 2.3. ([5, Theorem 1.2], [6, Theorem 1.6]) *Let $\Omega \subset \mathbb{R}^2$ be a domain and $f : \Omega \to \mathbb{R}^2$ be a homeomorphism of finite distortion. Then $f^{-1} : f(\Omega) \to \Omega$ is also a homeomorphism of finite distortion. Moreover*

$$|Df^{-1}(y)|^2 \le K_{f^{-1}}(y) J_{f^{-1}}(y) \qquad \mathcal{L}^2\text{-a.e. } y \in f(\Omega). \tag{2.2.4}$$

Lemma 2.4. ([14, Theorem 2.1.11]) *Let all $\Omega \subset \mathbb{R}^2$, $\Omega_1 \subset \mathbb{R}^2$ and $\Omega_2 \subset \mathbb{R}^2$ be open, and $T \in Lip(\Omega_1, \Omega_2)$. Suppose that both $f \in W^{1,p}_{loc}(\Omega, \Omega_1)$ and $T \circ f \in L^p_{loc}(\Omega, \Omega_2)$ hold for some p with $1 \le p \le \infty$. Then $T \circ f \in W^{1,p}_{loc}(\Omega, \Omega_2)$ and*

$$D(T \circ f)(z) = DT(f(z)) Df(z) \qquad \mathcal{L}^2\text{-a.e. } z \in \Omega.$$

Definition 2.6. A rectifiable Jordan curve Γ in the plane is a chord-arc curve if there is a constant $C > 0$ such that

$$\ell_\Gamma(z_1, z_2) \le C|z_1 - z_2|$$

for all $z_1, z_2 \in \Gamma$, where $\ell_\Gamma(z_1, z_2)$ is the length of the shorter arc of Γ joining z_1 and z_2.

It is a well-known fact that a chord-arc curve is the image of the unit circle under a bi-Lipschitz mappings of the plane, see [7]. Thus chord-arc curves form a special class of quasicircles. The connections between chord-arc curves and quasiconformal theory can be found in [1, 12].

2.3. **Definition of cardioid-type domains.** Let $s>1$. We introduce a class of cardioid-type domains Δ_s whose boundaries contain internal polynomial cusps of order s, see FIGURE 1. For technical reasons we do this in the following manner. Denote

$$\ell_1(s)=\{(u,v)\in\mathbb{R}^2 : u\in[-1,0],\ v=(-u)^s\}$$

and

$$\ell_2(s)=\{(u,v)\in\mathbb{R}^2 : u\in[-1,0],\ v=-(-u)^s\}.$$

Write $\ell_1(s)$ and $\ell_2(s)$ in the polar coordinate system as

$$\ell_1(s)=\{Re^{i\Theta} : R=(-u)(1+(-u)^{2(s-1)})^{\frac{1}{2}} \\ \text{and } \Theta=\pi-\arctan((-u)^{s-1}) \text{ for } u\in[-1,0]\}$$

and

$$\ell_2(s)=\{Re^{i\Theta} : R=(-u)(1+(-u)^{2(s-1)})^{\frac{1}{2}} \\ \text{and } \Theta=-\pi+\arctan((-u)^{s-1}) \text{ for } u\in[-1,0]\}.$$

Take the branch of complex-valued function $z=w^{1/2}$ with $1^{1/2}=1$. Denote by $\ell_1^m(s)$ and $\ell_2^m(s)$ the images of $\ell_1(s)$ and $\ell_2(s)$ under the preceding $z=w^{1/2}$, respectively. Then we can write $\ell_1^m(s)$ and $\ell_2^m(s)$ in the polar coordinate system as

$$\ell_1^m(s)=\{re^{i\theta} : r=\sqrt{-u}(1+(-u)^{2(s-1)})^{\frac{1}{4}} \\ \text{and } \theta=\frac{\pi-\arctan((-u)^{s-1})}{2} \text{ for } u\in[-1,0]\} \tag{2.3.1}$$

and

$$\ell_2^m(s)=\{re^{i\theta} : r=\sqrt{-u}(1+(-u)^{2(s-1)})^{\frac{1}{4}} \\ \text{and } \theta=\frac{-\pi+\arctan((-u)^{s-1})}{2} \text{ for } u\in[-1,0]\}.$$

Denote by z_1 and z_2 the end points of $\ell_1^m(s)\cup\ell_2^m(s)$. Notice that there is a unique circle sharing both the tangent of $\ell_1^m(s)$ at z_1 and the one of $\ell_2^m(s)$ at z_2. This circle is divided into two arcs by z_1 and z_2. Concatenating $\ell_1^m(s)\cup\ell_2^m(s)$ with the arc located on the right-hand side of the line through z_1 and z_2, we then obtain a Jordan curve $\ell^m(s)$. Denote by $\ell(s)$ the image of $\ell^m(s)$ under z^2. Let

$$M_s \text{ and } \Delta_s \text{ be the interior domains of } \ell^m(s) \text{ and } \ell(s), \text{ respectively.} \tag{2.3.2}$$

Then Δ_s is the desired cardioid-type domain with degree s. Moreover $\ell^m(s)$, $\ell(s)$, M_s and Δ_s are symmetric with respect to the real axis.

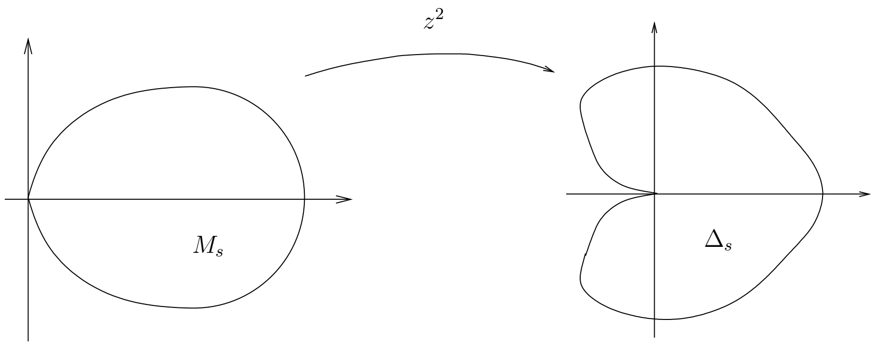

FIGURE 1. M_s and Δ_s

By the Riemann mapping theorem, there is a conformal mapping from $\mathbb{D}\cap\mathbb{R}^2_+$ onto $M_s\cap\mathbb{R}^2_+$ such that $\mathbb{D}\cap\mathbb{R}$ is mapped onto $M_s\cap\mathbb{R}$. It follows from the Schwarz reflection principle that there is a conformal mapping

$$g_s:\mathbb{D}\to M_s. \tag{2.3.3}$$

such that $g_s(\bar{z})=\overline{g_s(z)}$ for all $z\in\mathbb{D}$. Moreover by the Osgood-Carathéodory theorem g_s has a homeomorphic extension from $\overline{\mathbb{D}}$ onto $\overline{M_s}$, still denoted g_s.

Lemma 2.5. *Let M_s and g_s be as in* (2.3.2) *and* (2.3.3) *with $s>1$. Then g_s is a bi-Lipschitz mapping on $\overline{\mathbb{D}}$.*

Proof. If ∂M_s were a Dini-smooth Jordan curve, from [11, Theorem 3.3.5] it would follow that g_s' is continuous on $\overline{\mathbb{D}}$ and $g_s'(z)\neq 0$ for all $z\in\overline{\mathbb{D}}$. Since M_s is convex, the mean value theorem would then yield that g_s is a bi-Lipschitz map from $\overline{\mathbb{D}}$ onto $\overline{M_s}$.

In order to prove that ∂M_s is a Dini-smooth Jordan curve, we first analyze ∂M_s in a neighborhood of the origin. For any point in ℓ_1^m with Euclidean coordinate (x,y), we have

$$x=r\cos\theta \text{ and } y=r\sin\theta. \tag{2.3.4}$$

where both r and θ share the expression in (2.3.1). We then obtain that

$$r\approx\sqrt{-u},\ \theta\approx\frac{\pi}{2},\ \frac{\partial r}{\partial u}\approx\frac{-1}{\sqrt{-u}} \text{ and } \frac{\partial\theta}{\partial u}\approx(-u)^{s-2} \tag{2.3.5}$$

whenever $|u|\ll 1$. Therefore from (2.3.4) and (2.3.5), it follows that

$$x\approx(-u)^{s-\frac{1}{2}},\ y\approx(-u)^{\frac{1}{2}},\ \frac{\partial x}{\partial u}\approx-(-u)^{s-\frac{3}{2}} \text{ and } \frac{\partial y}{\partial u}\approx-(-u)^{-\frac{1}{2}}.$$

Together with symmetry of ∂M_s, we conclude that $\frac{\partial x}{\partial y}\approx|y|^{2(s-1)}$ whenever $|y|\ll 1$. Next, notice that the part of ∂M_s away from the origin is piecewise smooth. By parametrizing ∂M_s as $\alpha(y)=(x(y),y)$, we then obtain that the modulus of continuity of α' satisfies

$$\omega(\delta,\alpha',\partial M_s)\leq\max\{\delta^{2(s-1)},\delta\}\qquad\forall\delta\ll 1.$$

Consequently α' is Dini-continuous. Therefore ∂M_s is a Dini-smooth Jordan curve. $\square$

Remark 2.1. Since $g_s:\mathbb{S}^1\to\partial M_s$ is a bi-Lipschitz map by Lemma 2.5, via [13, Theorem A] there is a bi-Lipschitz mapping $g_s^c:\mathbb{D}^c\to M_s^c$ such that $g_s^c|_{\mathbb{S}^1}=g_s$. Let

$$G_s(z)=\begin{cases} g_s(z) & \forall z\in\overline{\mathbb{D}},\\ g_s^c(z) & \forall z\in\mathbb{D}^c.\end{cases} \tag{2.3.6}$$

Then G_s is an orientation-preserving bi-Lipschitz mapping.

Lemma 2.6. *Let $h_1:\mathbb{R}^2\to\mathbb{R}^2$ be a homeomorphism of finite distortion, and $h_2:\mathbb{R}^2\to\mathbb{R}^2$ be an (l,L)-bi-Lipschitz, orientation-preserving mapping. Then $h_1\circ h_2$ is a homeomorphism of finite distortion.*

Proof. Since h_2 is an orientation-preserving bi-Lipschitz mapping, we have that h_2 is quasiconformal. From [2, Corollary 3.7.6] it then follows that

$$h_2 \text{ satisfies Lusin } (N) \text{ and } (N^{-1}) \text{ condition,} \tag{2.3.7}$$

$$J_{h_2}>0\quad\mathcal{L}^2\text{-a.e. on }\mathbb{R}^2. \tag{2.3.8}$$

By Lemma 2.2 we have

$$\text{both } h_1 \text{ and } h_2 \text{ are differentiable } \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{2.3.9}$$

From (2.3.9) and (2.3.7) it therefore follows that $h_1\circ h_2$ is differentiable $\mathcal{L}^2$-a.e. on $\mathbb{R}^2$, and

$$D(h_1\circ h_2)(z)=Dh_1(h_2(z))Dh_2(z)\qquad\mathcal{L}^2\text{-a.e. } z\in\mathbb{R}^2. \tag{2.3.10}$$

By (2.3.10), Lemma 2.1 and (2.3.7), we then have that

$$\int_M |J_{h_1\circ h_2}(z)|\,dz = \int_M |J_{h_1}(h_2(z))||J_{h_2}(z)|\,dz = \int_{h_2(M)} |J_{h_1}(w)|\,dw < \infty \tag{2.3.11}$$

for any compact set $M \subset \mathbb{R}^2$, where the last inequality is from $J_{h_1} \in L^1_{\text{loc}}$. Moreover, from (2.3.10) and the distortion inequalities for h_1 and h_2 it follows that

$$\begin{aligned} |D(h_1\circ h_2)(z)|^2 \le& |Dh_1(h_2(z))|^2|Dh_2(z)|^2 \le K_{h_1}(h_2(z))K_{h_2}(z)J_{h_1}(h_2(z))J_{h_2}(z)\\ =& K_{h_1}(h_2(z))K_{h_2}(z)J_{h_1\circ h_2}(z) \end{aligned} \tag{2.3.12}$$

for $\mathcal{L}^2$-a.e. $z \in \mathbb{R}^2$.

To prove that $h_1 \circ h_2$ is a homeomorphism of finite distortion, via (2.3.11) and (2.3.12) it is sufficient to prove that $h_1 \circ h_2 \in W^{1,1}_{\text{loc}}$. Since h_2 is an (l, L)-bi-Lipschitz orientation-preserving mapping, by (2.3.9) and (2.2.3) we then have that

$$l \le |Dh_2(z)| \le L \text{ and } 1 \le K_{h_2}(z) \le \frac{L}{l} \qquad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{2.3.13}$$

From(2.3.8), (2.3.13) and (2.2.1) it then follows that

$$\frac{l^3}{L} \le J_{h_2}(z) \le L^2 \qquad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{2.3.14}$$

By (2.3.10), (2.3.13), (2.3.14) and Lemma 2.1, we therefore have

$$\begin{aligned} \int_M |D(h_1\circ h_2)(z)|\,dz \le& \int_M |Dh_1(h_2(z))|\frac{|Dh_2(z)|}{J_{h_2}(z)}J_{h_2}(z)\,dz\\ \approx& \int_M |Dh_1(h_2(z))|J_{h_2}(z)\,dz\\ =& \int_{h_2(M)} |Dh_1(w)|\,dw < \infty \end{aligned}$$

for any compact set $M \subset \mathbb{R}^2$, where the last inequality is from $h_1 \in W^{1,1}_{\text{loc}}$. □

3. Bounds for integrability degrees

For a given $s > 1$, let M_s as in (2.3.2). Define

$$\begin{aligned} \mathcal{E}_s = \{f : f :& \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a homeomorphism of finite distortion}\\ &\text{and } f(z) = z^2 \text{ for all } z \in \overline{M_s}\}. \end{aligned} \tag{3.0.1}$$

Lemma 3.1. *Let $\mathcal{E}_s$ be as in (3.0.1) with $s > 1$, and $f \in \mathcal{E}_s$. Suppose that $f^{-1} \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p \ge 1$. Then necessarily $p < 2(s + 1)/(2s - 1)$.*

Proof. Given $x \in (-1, 0)$, denote by I_x the line segment connecting the points $(x, |x|^s)$ and $(x, -|x|^s)$. Since $f^{-1} \in W^{1,p}_{\text{loc}}$ for some $p \ge 1$, by the ACL-property of Sobolev functions it follows that

$$\operatorname{osc}_{I_x} f^{-1} \le \int_{I_x} |Df^{-1}(x, y)|\,dy \tag{3.0.2}$$

holds for $\mathcal{L}^1$-a.e. $x \in (-1, 0)$. Applying Jensen's inequality to (3.0.2), we have

$$\frac{(\operatorname{osc}_{I_x} f^{-1})^p}{(-x)^{s(p-1)}} \le \int_{I_x} |Df^{-1}(x, y)|^p\,dy. \tag{3.0.3}$$

Since $f(z) = z^2$ for all $z \in \partial M_s$, we have

$$(-x)^{1/2} \lesssim \operatorname{osc}_{I_x} f^{-1} \qquad \forall x \in (-1, 0). \tag{3.0.4}$$

Combining (3.0.3) with (3.0.4), we hence obtain

$$(-x)^{\frac{p}{2}-s(p-1)} \lesssim \int_{I_x} |Df^{-1}(x,y)|^p\,dy \qquad \mathcal{L}^1\text{-a.e. } x \in (-1,0). \tag{3.0.5}$$

Integrating (3.0.5) with respect to $x \in (-1,0)$ therefore implies

$$\int_{-1}^{0} (-x)^{\frac{p}{2}-s(p-1)}\,dx \lesssim \int_{B(0,\sqrt{2})} |Df^{-1}(x,y)|^p\,dx\,dy. \tag{3.0.6}$$

Since $f^{-1} \in W^{1,p}_{\mathrm{loc}}$, from (3.0.6) we necessarily obtain $\frac{p}{2} - s(p-1) > -1$, which is equivalent to $p < 2(s+1)/(2s-1)$. $\square$

Our next proof borrows some ideas from [9, Theorem 1].

Lemma 3.2. *Let $\mathcal{E}_s$ be as in* (3.0.1) *with $s > 1$. Let $f \in \mathcal{E}_s$ and suppose that $K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)$ for a given $q \geq 1$. Then $q < (s+1)/(s-1)$.*

Proof. For a given $t \ll 1$, we denote

$$E_t = \{(x,y) \in \mathbb{R}^2 : x \in (-t^2, -(\frac{t}{2})^2) \text{ and } y = -|x|^s\}$$

and

$$F_t = \{(x,y) \in \mathbb{R}^2 : x \in (-t^2, -(\frac{t}{2})^2) \text{ and } y = |x|^s\}.$$

Let $\tilde{E}_t = f^{-1}(E_t)$ and $\tilde{F}_t = f^{-1}(F_t)$. Set

$$L_t^1 = \min\{|z| : z \in \tilde{F}_t\},\ L_t^2 = \max\{|z| : z \in \tilde{F}_t\},$$

$$L_t = \operatorname{dist}(\tilde{E}_t, \tilde{F}_t),\ L_0 = \max\{|f^{-1}(z)| : \mathrm{Re}z = -1, \mathrm{Im}z \in [-1,1]\}.$$

Since $f(z) = z^2$ for all $z \in \partial M_s$, we have $L_t^1 \approx t/2$, $L_t^2 \approx t$ and $L_t \approx t$ whenever $t \ll 1$. Given $w \in A_t := \{w \in \mathbb{R}^2 : L_t^1 \leq |w| \leq L_t^2\}$, set $\rho(w) = L_t^2/(L_t|w|)$. Define

$$v(z) = \begin{cases} 1 & \text{for all } z \in B(0,L_0) \setminus A_t, \\ \inf_{\gamma_z} \int_{\gamma_z} \rho\,ds & \text{for all } z \in A_t, \end{cases} \tag{3.0.7}$$

where the infimum is taken over all curves $\gamma_z \subset A_t$ joining z and $\tilde{E}_t$. From (3.0.7) it follows that for any $z_1,\ z_2 \in A_t$ and any curve $\gamma_{z_1 z_2} \subset A_t$ connecting z_1 and z_2 we have

$$|v(z_1) - v(z_2)| \leq \int_{\gamma_{z_1 z_2}} \rho\,ds. \tag{3.0.8}$$

Therefore v is a Lipschitz function on A_t. By Rademacher's theorem, v is differentiable $\mathcal{L}^2$-a.e. on A_t. Hence (3.0.8) together with the continuity of ρ gives

$$|Dv(z)| \leq \rho(z) \qquad \mathcal{L}^2\text{-a.e. } z \in A_t. \tag{3.0.9}$$

Integrating (3.0.9) over $\tilde{Q}_t = A_t \setminus M_s$ then yields

$$\int_{\tilde{Q}_t} |Dv|^2 \leq \int_{\tilde{Q}_t} \rho^2 \approx \int_{L_t^1}^{L_t^2} \frac{1}{r}\,dr \approx \log 2. \tag{3.0.10}$$

By Lemma 2.3 we have $f^{-1} \in W^{1,1}_{\mathrm{loc}}$. Let $u = v \circ f^{-1}$. From Lemma 2.4 we then have $u \in W^{1,1}_{\mathrm{loc}}(f(B(0,L_0)))$ and

$$|Du(z)| \leq |Dv(f^{-1}(z))||Df^{-1}(z)| \qquad \mathcal{L}^2\text{-a.e. in } f(A_t). \tag{3.0.11}$$

By (3.0.7), $v(z)=0$ for all $z\in\tilde{E}_t$. Hence $u(z)=0$ for all $z\in E_t$. Whenever $z\in\tilde{F}_t$, we have $\mathcal{L}^1(\gamma_z)\geq L_t$ for any curve $\gamma_z\subset A_t$ joining z and $\tilde{E}_t$. Therefore $v(z)\geq 1$ for all $z\in\tilde{F}_t$. Hence $u(z)\geq 1$ for all $z\in F_t$. By the ACL-property of Sobolev functions and Hölder's inequality, we therefore have that

$$1\leq\int_{-x^s}^{x^s}|Du(x,y)|\,dy\leq\left(\int_{-x^s}^{x^s}|Du(x,y)|^p\,dy\right)^{\frac{1}{p}}(2x^s)^{\frac{p-1}{p}}\tag{3.0.12}$$

for any $p>1$ and $\mathcal{L}^1$-a.e. $x\in[-t^2,-(t/2)^2]$. Define

$$R_t=\{(x,y)\in\mathbb{R}^2: x\in(-t^2,-(t/2)^2),\ y\in(-|x|^s,|x|^s)\}.$$

Fubini's theorem and (3.0.12) then give

$$\begin{aligned}\int_{R_t}|Du(x,y)|^p\,dx\,dy&=\int_{-t^2}^{-(t/2)^2}\int_{-x^s}^{x^s}|Du(x,y)|^p\,dy\,dx\\&\gtrsim\int_{-t^2}^{-(t/2)^2}x^{s(1-p)}\,dx\approx t^{2(1+s(1-p))}.\end{aligned}\tag{3.0.13}$$

Set $Q_t=f(\tilde{Q}_t)$. Then for any $z\in R_t\setminus Q_t$ there is an open disk $B_z\subset R_t\setminus Q_t$ such that $z\in B_z$ and $u|_{B_z}\equiv 1$. Therefore

$$\int_{Q_t}|Du|^p\geq\int_{Q_t\cap R_t}|Du|^p=\int_{R_t}|Du|^p.\tag{3.0.14}$$

Combining (3.0.13) with (3.0.14) gives that

$$t^{2(1+s(1-p))}\lesssim\int_{Q_t}|Du|^p\tag{3.0.15}$$

for all $p\geq 1$.

For any $p\in(0,2)$, by (3.0.11), (2.2.4) and Hölder's inequality we have

$$\begin{aligned}\int_{Q_t}|Du|^p&\leq\int_{Q_t}|Dv\circ f^{-1}|^p|Df^{-1}|^p\\&\leq\int_{Q_t}|Dv\circ f^{-1}|^pJ_{f^{-1}}^{\frac{p}{2}}K_{f^{-1}}^{\frac{p}{2}}\\&\leq\left(\int_{Q_t}|Dv\circ f^{-1}|^2J_{f^{-1}}\right)^{\frac{p}{2}}\left(\int_{Q_t}K_{f^{-1}}^{\frac{p}{2-p}}\right)^{\frac{2-p}{2}}\\&\leq\left(\int_{\tilde{Q}_t}|Dv|^2\right)^{\frac{p}{2}}\left(\int_{Q_t}K_{f^{-1}}^{\frac{p}{2-p}}\right)^{\frac{2-p}{2}}\end{aligned}\tag{3.0.16}$$

where the last inequality comes from Lemma 2.1. Let $q=p/(2-p)$. Via (3.0.10) and (3.0.15), we conclude from (3.0.16) that

$$t^{2(1+q+s(1-q))}\lesssim\int_{Q_t}K_{f^{-1}}^q\tag{3.0.17}$$

for all $q\geq 1$. We now consider the set Q_t for $t=2^{-j}$ with $j\geq j_0$ for a fixed large j_0. Since

$$\sum_{j=j_0}^{\infty}\chi_{Q_{2^{-j}}}(x)\leq 2\chi_{\mathbb{D}}(x)\qquad\forall x\in\mathbb{R}^2,$$

by (3.0.17) we have that

$$\sum_{j=j_0}^{+\infty}2^{j2(s(q-1)-q-1)}\lesssim\sum_{j=j_0}^{+\infty}\int_{Q_{2^{-j}}}K_{f^{-1}}^q\leq 2\int_{\mathbb{D}}K_{f^{-1}}^q.\tag{3.0.18}$$

The series in (3.0.18) diverges when $q \geq \frac{s+1}{s-1}$ and hence $K_{f^{-1}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ can only hold when $q < (s+1)/(s-1)$. $\square$

We continue with properties of our homeomorphism f. The following lemma is a version of [3, Theorem 4.4].

Lemma 3.3. *Let $\mathcal{E}_s$ be as in* (3.0.1) *with $s > 1$. If $f \in \mathcal{E}_s$ and $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$, then $q < \max\{1, 1/(s-1)\}$.*

Proof. Denote

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-1, 0),\ x_2 \in (-|x_1|^s, |x_1|^s)\}.$$

For a given $t \ll 1$, set

$$\Omega^1_t = \{(x_1, x_2) \in \Omega : x_1 \in (-1, -t^2)\},$$

$$\tilde{Q}_t = \{(x_1, x_2) \in \Omega : x_1 \in [-t^2, -(\frac{t}{2})^2]\} \text{ and } \Omega^2_t = \Omega \setminus (\Omega^1_t \cup \tilde{Q}_t).$$

Define

$$v(x_1, x_2) = \begin{cases} 1 & \forall (x_1, x_2) \in \Omega^1_t, \\ 1 - \left(\int_{-t^2}^{-(t/2)^2} \frac{dx}{(-x)^s} \right)^{-1} \int_{-t^2}^{x_1} \frac{dx}{(-x)^s} & \forall (x_1, x_2) \in \tilde{Q}_t, \\ 0 & \forall (x_1, x_2) \in \Omega^2_t. \end{cases} \tag{3.0.19}$$

Then v is a Lipschitz function on Ω. Let $u = v \circ f$. By Lemma 2.4, we have $u \in W^{1,1}_{\mathrm{loc}}(f^{-1}(\Omega))$ and

$$Du(z) = Dv(f(z))Df(z) \qquad \mathcal{L}^2\text{-a.e. } z \in f^{-1}(\Omega). \tag{3.0.20}$$

Let $P_1 = f^{-1}((-t^2, t^{2s}))$, $P_2 = f^{-1}((-(t/2)^2, (t/2)^{2s}))$ and O be the origin. Denote by L^1_t and L^2_t the length of line segment P_1P_2 and of P_1O, respectively. Then $L^1_t < L^2_t$. Since $f(z) = z^2$ for all $z \in \partial M_s$, we have

$$L^1_t \approx \frac{t}{2} \text{ and } L^2_t \approx t \qquad \text{whenever } t \ll 1. \tag{3.0.21}$$

Let $\hat{S}(P_1, r) = S(P_1, r) \cap f^{-1}(\Omega)$. From the ACL-property of Sobolev functions and Hölder's inequality, we have that

$$\mathrm{osc}_{\hat{S}(P_1,r)} u \leq \int_{\hat{S}(P_1,r)} |Du|\, ds \leq (2\pi r)^{\frac{p-1}{p}} \left(\int_{\hat{S}(P_1,r)} |Du|^p\, ds \right)^{\frac{1}{p}} \tag{3.0.22}$$

for any $p > 1$ and $\mathcal{L}^1$-a.e. $r \in (L^1_t, L^2_t)$. Since $\mathrm{osc}_{\hat{S}(P_1,r)} u = 1$ for all $r \in (L^1_t, L^2_t)$, we conclude from (3.0.22) that

$$\int_{\hat{S}(P_1,r)} |Du|^p\, ds \gtrsim r^{1-p} \qquad \mathcal{L}^1\text{-a.e. } r \in (L^1_t, L^2_t). \tag{3.0.23}$$

Let $A_t = f^{-1}(\Omega) \cap B(P_1, L^2_t) \setminus \overline{B(P_1, L^1_t)}$. By Fubini's theorem and (3.0.21), we deduce from (3.0.23) that

$$\int_{A_t} |Du|^p = \int_{L^1_t}^{L^2_t} \int_{\hat{S}(P_1,r)} |Du|^p\, ds\, dr \gtrsim \int_{L^1_t}^{L^2_t} r^{1-p}\, dr \approx t^{2-p}. \tag{3.0.24}$$

Let $Q_t = f^{-1}(\tilde{Q}_t)$. From (3.0.19), we have $|Du(z)| = 0$ for all $z \in A_t \setminus Q_t$. We hence conclude from (3.0.24) that

$$\int_{Q_t} |Du|^p \geq \int_{Q_t \cap A_t} |Du|^p = \int_{A_t} |Du|^p \gtrsim t^{2-p} \tag{3.0.25}$$

for any $p \geq 1$.

From (3.0.20), (2.2.1) and Hölder's inequality, it follows that for any $p \in (0,2)$

$$\int_{Q_t} |Du|^p \le \int_{Q_t} |Dv \circ f|^p |Df|^p \le \int_{Q_t} |Dv \circ f|^p J_f^{\frac{p}{2}} K_f^{\frac{p}{2}}$$

$$\le \left(\int_{Q_t} |Dv \circ f|^2 J_f \right)^{\frac{p}{2}} \left(\int_{Q_t} K_f^{\frac{p}{2-p}} \right)^{\frac{2-p}{2}}$$

$$\le \left(\int_{\tilde{Q}_t} |Dv|^2 \right)^{\frac{p}{2}} \left(\int_{Q_t} K_f^{\frac{p}{2-p}} \right)^{\frac{2-p}{2}}, \tag{3.0.26}$$

where the last inequality is from Lemma 2.1. From (3.0.19), we have that

$$\int_{\tilde{Q}_t} |Dv(x_1,x_2)|^2 \, dx_1 \, dx_2 = \left(\int_{-t^2}^{-(t/2)^2} \frac{dx}{(-x)^s} \right)^{-2} \int_{-t^2}^{-(t/2)^2} \int_{-|x_1|^s}^{|x_1|^s} \frac{1}{(-x_1)^{2s}} \, dx_2 \, dx_1$$

$$\approx \left(\int_{-t^2}^{-(t/2)^2} \frac{dx}{(-x)^s} \right)^{-1} \approx t^{2(s-1)}. \tag{3.0.27}$$

Let $q = p/(2-p)$. Then $q \in [1, +\infty)$ whenever $p \in [1,2)$. Combining (3.0.27), (3.0.25) with (3.0.26) yields

$$t^{2+2(1-s)q} \lesssim \int_{Q_t} K_f^q \tag{3.0.28}$$

for all $q \ge 1$. We now consider the set Q_t for $t = 2^{-j}$ with $j \ge j_0$ for a fixed large j_0. Analogously to (3.0.18), it follows from (3.0.28) that

$$\sum_{j=j_0}^{+\infty} 2^{2j((s-1)q-1)} \lesssim \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K_f^q \le 2 \int_{B(0,1)} K_f^q. \tag{3.0.29}$$

Whenever $s \ge 2$, the sum in (3.0.29) diverges if $q \ge 1$. Whenever $s \in (1,2)$, the sum in (3.0.29) also diverges if $q \ge 1/(s-1)$. Hence $K_f \in L^q_{loc}(\mathbb{R}^2)$ is possible only when $q < \max\{1, 1/(s-1)\}$. □

In Lemma 3.3, we obtained an estimate for those q for which $K_f \in L^q_{\text{loc}}$. We continue with the additional assumption that $f \in W^{1,p}_{\text{loc}}$ for some $p > 1$.

Lemma 3.4. *Let $\mathcal{E}_s$ be as in* (3.0.1) *with $s > 2$. If $f \in \mathcal{E}_s$, $f \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \in (0,1)$, then $q < 3p/((2s-1)p + 4 - 2s)$.*

Proof. Let f be a homeomorphism with the above properties. By [5, Theorem 4.1] we have $f^{-1} \in W^{1,r}_{\text{loc}}(\mathbb{R}^2)$ where

$$r = \frac{(q+1)p - 2q}{p-q}.$$

Moreover

$$r < \frac{2(s+1)}{2s-1} \Leftrightarrow q < \frac{3p}{(2s-1)p + 4 - 2s}.$$

Hence the claim follows from Lemma 3.1. □

Remark 3.1. Notice that in the proof of Lemma 3.3 we only care about the property of f in a small neighborhood of the origin. Let $t \ll 1$. By modifying $\partial M_s \cap B(0,t)$, we may generalize Lemma 3.3. For example, we modify $\partial M_{3/2} \cap B(0,t)$ such that its image under $f(z) = z^2$ is

$$\{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0],\ y^2 = c|x|^3\}$$

where c is a positive constant. If $K_f \in L^q_{\text{loc}}(\mathbb{R}^2)$ for some $q \ge 1$, by the analogous arguments as for Lemma 3.3 we have $q < 2$. Similarly, one may extend Lemma 3.1, Lemma 3.2 and Lemma 3.4 to the above setting.

Lemma 3.5. *Let Δ_s be as in (2.3.2) with $s > 1$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism of finite distortion such that f maps $\mathbb{D}$ conformally onto Δ_s. We have that*

(1) *if $f^{-1} \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p \geq 1$ then $p < 2(s+1)/(2s-1)$,*
(2) *if $K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$ then $q < (s+1)/(s-1)$,*
(3) *if $K_f \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$ then $q < \max\{1, 1/(s-1)\}$,*
(4) *if $s > 2$, $f \in W^{1,p}_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_f \in L^q_{loc}$ for some $q \in (0,1)$, then $q < 3p/((2s-1)p + 4 - 2s)$.*

Proof. Let g_s be as in (2.3.3), and $h_s = z^2 \circ g_s$. Since $h_s : \mathbb{D} \to \Delta_s$ is conformal, there is a Möbius transformation

$$m_s(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z} \qquad \text{where } \theta \in [0, 2\pi] \text{ and } |a| < 1$$

such that $f(z) = h_s \circ m_s(z)$ for all $z \in \mathbb{D}$. Since $m_s : \mathbb{S}^1 \to \mathbb{S}^1$ is a bi-Lipschitz mapping, by [13, Theorem A] there is a bi-Lipschitz mapping $m_s^c : \mathbb{D}^c \to \Delta_s^c$ such that $m_s^c|_{\mathbb{S}^1} = m_s$. Define

$$\mathfrak{M}_s(z) = \begin{cases} m_s(z) & z \in \overline{\mathbb{D}}, \\ m_s^c(z) & z \in \mathbb{D}^c. \end{cases} \tag{3.0.30}$$

Then $\mathfrak{M}_s : \mathbb{R}^2 \to \mathbb{R}^2$ is a bi-Lipschitz, orientation-preserving mapping. Let G_s be as in (2.3.6). Define

$$F = f \circ \mathfrak{M}_s^{-1} \circ G_s^{-1} : \mathbb{R}^2 \to \mathbb{R}^2.$$

Lemma 2.6 implies that $F \in \mathcal{E}_s$, where $\mathcal{E}_s$ is from (3.0.1). From Lemma 2.3 and Lemma 2.2, it follows that

$$\text{both } f^{-1} \text{ and } F^{-1} \text{ are differentiable } \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{3.0.31}$$

Since

$$\begin{aligned} \frac{|f^{-1}(z_1) - f^{-1}(z_2)|}{|z_1 - z_2|} = & \frac{|F^{-1}(z_1) - F^{-1}(z_2)|}{|z_1 - z_2|} \frac{|(G_s^{-1}(F^{-1}(z_1)) - (G_s^{-1}(F^{-1}(z_2))|}{|F^{-1}(z_1) - F^{-1}(z_2)|} \times \\ & \times \frac{|\mathfrak{M}_s^{-1}(G_s^{-1} \circ F^{-1}(z_1)) - \mathfrak{M}_s^{-1}(G_s^{-1} \circ F^{-1}(z_2))|}{|G_s^{-1} \circ F^{-1}(z_1) - G_s^{-1} \circ F^{-1}(z_2)|} \end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}^2$ with $z_1 \neq z_2$, by (3.0.31) and the bi-Lipschitz properties of G_s^{-1} and $\mathfrak{M}_s^{-1}$ we have that

$$|Df^{-1}(z)| \approx |DF^{-1}(z)|, \tag{3.0.32}$$

$$\max_{\theta \in [0,2\pi]} |\partial_\theta f^{-1}(z)| \approx \max_{\theta \in [0,2\pi]} |\partial_\theta F^{-1}(z)|, \quad \min_{\theta \in [0,2\pi]} |\partial_\theta f^{-1}(z)| \approx \min_{\theta \in [0,2\pi]} |\partial_\theta F^{-1}(z)| \tag{3.0.33}$$

for $\mathcal{L}^2$-a.e. $z \in \mathbb{R}^2$. If $f^{-1} \in W^{1,p}_{\text{loc}}$ for some $p \geq 1$, Lemma 3.2 together with (3.0.34) gives $p < 2(s+1)/(2s-1)$. By (3.0.33) and (2.2.3) we have that

$$K_{f^{-1}}(z) \approx K_{F^{-1}}(z) \qquad \mathcal{L}^2\text{-a.e. } z \in \mathbb{R}^2. \tag{3.0.34}$$

If $K_{f^{-1}} \in L^q_{loc}(\mathbb{R}^2)$ for some $q \geq 1$, combining (3.0.32) and Lemma 3.1 then yields $q < (s+1)/(s-1)$.

By Lemma 2.6 and and Lemma 2.2, we have that

$$\text{both } f \text{ and } F \text{ are differentiable } \mathcal{L}^2\text{-a.e. on } \mathbb{R}^2. \tag{3.0.35}$$

From [2, Corollary 3.7.6], $G_s \circ \mathfrak{M}_s$ satisfies Lusin (N) and (N^{-1}) conditions. Since

$$\begin{aligned} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} = & \frac{|F(G_s \circ \mathfrak{M}_s(z_1)) - F(G_s \circ \mathfrak{M}_s(z_2))|}{|G_s \circ \mathfrak{M}_s(z_1) - G_s \circ \mathfrak{M}_s(z_2)|} \frac{|G_s(\mathfrak{M}_s(z_1)) - G_s(\mathfrak{M}_s(z_2))|}{|\mathfrak{M}_s(z_1) - \mathfrak{M}_s(z_2)|} \times \\ & \times \frac{|\mathfrak{M}_s(z_1) - \mathfrak{M}_s(z_2)|}{|z_1 - z_2|} \end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}^2$ with $z_1 \neq z_2$, from (3.0.35) and the bi-Lipschitz properties of G_s and $\mathfrak{M}_s$ we have that

$$|Df(z)| \approx |DF(G_s \circ \mathfrak{M}_s(z))|, \tag{3.0.36}$$

$$\max_{\theta\in[0,2\pi]}|\partial_\theta f(z)| \approx \max_{\theta\in[0,2\pi]}|\partial_\theta F(G_s\circ\mathfrak{M}_s(z))|, \tag{3.0.37}$$

$$\min_{\theta\in[0,2\pi]}|\partial_\theta f(z)| \approx \min_{\theta\in[0,2\pi]}|\partial_\theta F(G_s\circ\mathfrak{M}_s(z))| \tag{3.0.38}$$

for $\mathcal{L}^2$-a.e. $z\in\mathbb{R}^2$. By (2.2.3), (3.0.37) and (3.0.38) we have that

$$K_f(z)\approx K_F(G_s\circ\mathfrak{M}_s(z)) \qquad \mathcal{L}^2\text{-a.e. } z\in\mathbb{R}^2. \tag{3.0.39}$$

Via the same reasons as for (2.3.14), we have that

$$J_{G_s\circ\mathfrak{M}_s}(z)\approx 1 \qquad \mathcal{L}^2\text{-a.e. } z\in\mathbb{R}^2. \tag{3.0.40}$$

By (3.0.40) and Lemma 2.1, we derive from (3.0.39) that

$$\begin{aligned}\int_A K_f^q(z)\,dz &= \int_A K_F^q(G_s\circ\mathfrak{M}_s(z))\frac{J_{G_s\circ\mathfrak{M}_s}(z)}{J_{G_s\circ\mathfrak{M}_s}(z)}\,dz\\ &\approx \int_A K_F^q(G_s\circ\mathfrak{M}_s(z))J_{G_s\circ\mathfrak{M}_s}(z)\,dz = \int_{G_s\circ\mathfrak{M}_s(A)} K_F^q(w)\,dw\end{aligned} \tag{3.0.41}$$

for any $q\ge 0$ and any compact set $A\subset\mathbb{R}^2$. By (3.0.36) and Lemma 2.1, we obtain that

$$\begin{aligned}\int_A |Df(z)|^p &= \int_A |DF(G_s\circ\mathfrak{M}_s(z))|^p\frac{J_{G_s\circ\mathfrak{M}_s}(z)}{J_{G_s\circ\mathfrak{M}_s}(z)}\,dz\\ &\approx \int_A |DF(G_s\circ\mathfrak{M}_s(z))|^p J_{G_s\circ\mathfrak{M}_s}(z)\,dz = \int_{G_s\circ\mathfrak{M}_s(A)} |DF|^p(w)\,dw\end{aligned} \tag{3.0.42}$$

for any $p\ge 0$. If $K_f\in L^q_{loc}(\mathbb{R}^2)$ for some $q\ge 1$, Lemma 3.3 together with (3.0.41) gives that $q<\max\{1,1/(s-1)\}$. If $f\in W^{1,p}_{\mathrm{loc}}$ and $K_f\in L^q_{\mathrm{loc}}$ for some $p>1$ and some $q\in(0,1)$, combining Lemma 3.4 with (3.0.42) then implies $q<3p/((2s-1)p+4-2s)$. □

A result related to Lemma 3.5 (3) appeared in [3, Theorem 4.4].

4. Proof of Theorem 1.2

4.1. $\mathcal{F}_s(f)\neq\emptyset$.

Proof. Let $g:\mathbb{D}\to\Delta_s$ be a conformal mapping with $s>1$. Analogously to (3.0.30), there is a bi-Lipschitz mapping $\mathfrak{M}_s:\mathbb{R}^2\to\mathbb{R}^2$. Let G_s be as in (2.3.6) and $\mathcal{E}_s$ be defined in (3.0.1). If $E\in\mathcal{E}_s$, by Lemma 2.6 we have $E\circ G_s\circ\mathfrak{M}_s\in\mathcal{F}_s(g)$. We now divide the construction of E into two steps: Step 1 deals with the construction in a neighborhood of the cusp point, see FIGURE 2; Step 2 gives the construction on the domain away from the cusp point.

Step 1: Fix $s>1$, and define

$$\eta(x)=\sqrt{x}(1+x^{2(s-1)})^{\frac14} \qquad \text{for all } x>0. \tag{4.1.1}$$

Then

$$\eta'(x)=\frac{(1+x^{2(s-1)})^{\frac14}}{2\sqrt{x}}\left(1+\frac{(s-1)x^{2s-2}}{1+x^{2(s-1)}}\right). \tag{4.1.2}$$

For a given $t\ll 1$, let

$$L_t^1=\eta((t/2)^2),\ \ L_t^2=\eta(t^2) \text{ and } \sigma_t=L_t^2-L_t^1. \tag{4.1.3}$$

Then $L_t^1\approx t/2$, $L_t^2\approx t$ and $\sigma_t\approx t/2$ whenever $t\ll 1$. Set

$$Q_t=\overline{B(0,L_t^2)}\setminus(B(0,L_t^1)\cup M_s), \text{ and } f_1(x,y)=xe^{iy} \quad \forall x\ge 0 \text{ and } y\in[0,2\pi]. \tag{4.1.4}$$

Let $\ell(r)$ be the length of $f_1^{-1}(Q_t)\cap\{(x,y)\in\mathbb{R}^2 : x=r\}$. Define

$$f_2(r,\theta)=\left(r,\frac{\sigma_t}{\ell(r)}(\pi-\theta)\right)\qquad\forall(r,\theta)\in f_1^{-1}(Q_t).\tag{4.1.5}$$

Since ∂M_s is mapped onto $\partial\Delta_s$ by z^2, we have that

$$\ell(r)=\pi+\arctan\tau^{2(s-1)}\text{ and }r=\eta(\tau^2)\tag{4.1.6}$$

for all $\tau\in(t/2,t)$. Then $\ell(r)\approx\pi$ and $r\approx\tau$ whenever $\tau\ll1$. From (4.1.2), it follows that $\frac{\partial r}{\partial\tau}\approx1$. Together with $\frac{\partial\ell}{\partial\tau}\approx\tau^{2s-3}$, we have that

$$\frac{\partial\ell(r)}{\partial r}\approx r^{2s-3}\qquad\text{for all } r\ll1.\tag{4.1.7}$$

Denote $R_t=f_2\circ f_1^{-1}(Q_t)$. Then $R_t=[L_t^1,L_t^2]\times[-\sigma_t/2,\sigma_t/2]$. Combining (4.1.4) with (4.1.5) implies

$$f_1\circ f_2^{-1}(x,y)=\left(-x\cos\frac{\ell(x)y}{\sigma_t},x\sin\frac{\ell(x)y}{\sigma_t}\right)\quad\forall(x,y)\in R_t.$$

Therefore

$$Df_1\circ f_2^{-1}(x,y)=\begin{bmatrix}-\cos\frac{\ell(x)y}{\sigma_t}+\frac{xy\ell'(x)}{\sigma_t}\sin\frac{\ell(x)y}{\sigma_t} & \frac{x\ell(x)}{\sigma_t}\sin\frac{\ell(x)y}{\sigma_t}\\ \sin\frac{\ell(x)y}{\sigma_t}+\frac{xy\ell'(x)}{\sigma_t}\cos\frac{\ell(x)y}{\sigma_t} & \frac{x\ell(x)}{\sigma_t}\cos\frac{\ell(x)y}{\sigma_t}\end{bmatrix}.\tag{4.1.8}$$

By (4.1.3), (4.1.6) and (4.1.7), we deduce from (4.1.8) that

$$|Df_1\circ f_2^{-1}(x,y)|\lesssim1\text{ and }J_{f_1\circ f_2^{-1}}(x,y)=-\frac{x\ell(x)}{\sigma}\approx-1\tag{4.1.9}$$

for all $t\ll1$ and each $(x,y)\in R_t$. Since $K_{f_1\circ f_2^{-1}}\ge1$, from (4.1.9) we have

$$K_{f_1\circ f_2^{-1}}\approx1.\tag{4.1.10}$$

By (4.1.9) again we have that

$$|Df_2\circ f_1^{-1}|=\frac{|adjDf_1\circ f_2^{-1}|}{|J_{f_1\circ f_2^{-1}}|}\approx|Df_1\circ f_2^{-1}|\lesssim1\text{ and }J_{f_2\circ f_1^{-1}}=\frac{1}{J_{f_1\circ f_2^{-1}}}\approx-1.\tag{4.1.11}$$

Analogously to (4.1.10), we have that

$$K_{f_2\circ f_1^{-1}}(x,y)\approx1\qquad\forall t\ll1\text{ and }\forall(x,y)\in Q_t.\tag{4.1.12}$$

Let

$$\tilde{Q}_t=\{(x,y)\in\mathbb{R}^2 : x\in[-t^2,-(t/2)^2],\ |y|\le|x|^s\}.$$

Define

$$f_3(u,v)=\left(-u,\frac{t^{2s}}{(-u)^s}v\right)\qquad\forall(u,v)\in\tilde{Q}_t.$$

Then f_3 is diffeomorphic and

$$Df_3(u,v)=\begin{bmatrix}-1 & 0\\ \frac{st^{2s}}{(-u)^{s+1}}v & \frac{t^{2s}}{(-u)^s}\end{bmatrix}.\tag{4.1.13}$$

From (4.1.13) we have that

$$|Df_3|\lesssim1\text{ and }J_{f_3}\approx-1\qquad\forall(u,v)\in\tilde{Q}_t.\tag{4.1.14}$$

Analogously to (4.1.10), we have that

$$K_{f_3}(u,v)\approx1\qquad\forall t\ll1\text{ and }\forall(u,v)\in\tilde{Q}_t.\tag{4.1.15}$$

Let $\tilde{R}_t=f_3(\tilde{Q}_t)$. Then $\tilde{R}_t=[(t/2)^2,t^2]\times[-t^{2s},t^{2s}]$. The same reasons as for (4.1.11) and (4.1.12) imply that

$$|Df_3^{-1}(x,y)|\lesssim1,\ J_{f_3^{-1}}(x,y)\approx-1\text{ and }K_{f_3^{-1}}(x,y)\approx1\tag{4.1.16}$$

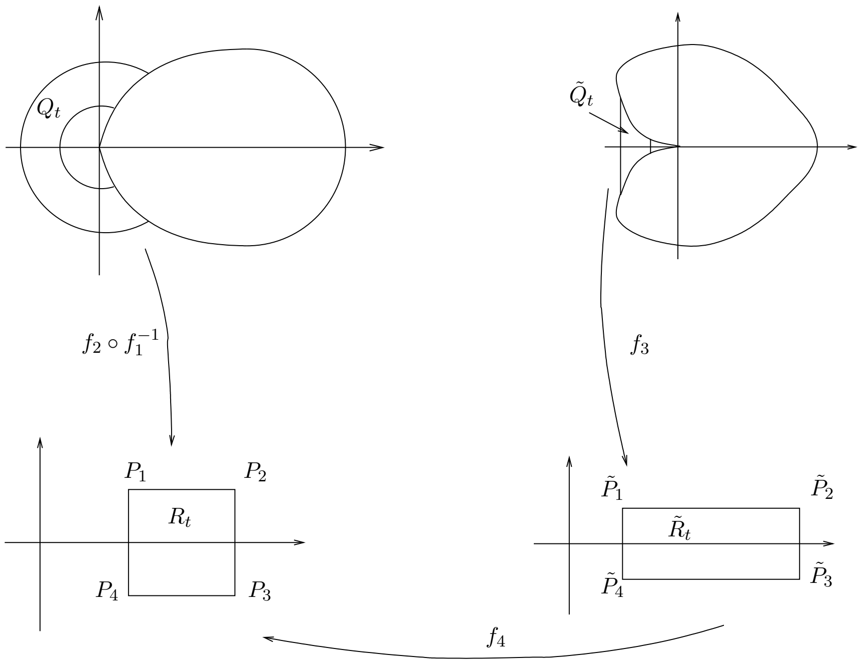

FIGURE 2. The construction $f_3^{-1} \circ f_4^{-1} \circ f_2 \circ f_1^{-1} : Q_t \to \tilde{Q}_t$

for all $t \ll 1$ and $(x, y) \in \tilde{R}_t$.

Denote by P_1, P_2, P_3, P_4 and $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ the four vertices of $\tilde{R}_t$ and R_t, respectively. Then

$$P_1 = (L_t^1, \frac{\sigma_t}{2}),\ P_2 = (L_t^2, \frac{\sigma_t}{2}),\ P_3 = (L_t^2, -\frac{\sigma_t}{2}),\ P_4 = (L_t^1, -\frac{\sigma_t}{2})$$

and

$$\tilde{P}_1 = ((t/2)^2, t^{2s}),\ \tilde{P}_2 = (t^2, t^{2s}),\ \tilde{P}_3 = (t^2, -t^{2s}),\ \tilde{P}_4 = ((t/2)^2, -t^{2s}).$$

Since ∂M_s is mapped onto $\partial \Delta_s$ by z^2, the line segment $\tilde{P}_1\tilde{P}_2$ is mapped onto P_1P_2 by

$$(u, t^{2s}) \mapsto \Big(\eta(u), \frac{\sigma_t}{2}\Big) \qquad \forall u \in [(t/2)^2, t^2],$$

and the line segment $\tilde{P}_4\tilde{P}_3$ is mapped onto P_4P_3 by

$$(u, -t^{2s}) \mapsto \Big(\eta(u), -\frac{\sigma_t}{2}\Big) \qquad \forall u \in [(t/2)^2, t^2].$$

Define

$$f_4(u, v) = \Big(\eta(u), \frac{\sigma_t}{2t^{2s}} v\Big) \qquad \forall (u, v) \in \tilde{R}_t. \tag{4.1.17}$$

Then f_4 is a diffeomorphism from $\tilde{R}_t$ onto R_t and

$$Df_4(u, v) = \begin{bmatrix} \eta'(u) & 0 \\ 0 & \frac{\sigma_t}{2t^{2s}} \end{bmatrix}. \tag{4.1.18}$$

By (4.1.2) and (4.1.3) we have that $\eta'(u) \approx t^{-1}$ and $\frac{\sigma_t}{2t^{2s}} \approx t^{1-2s}$ whenever $t \ll 1$ and $(u, v) \in \tilde{R}_t$. It follows from (4.1.18) that

$$|Df_4(u, v)| \approx t^{1-2s} \text{ and } J_{f_4}(u, v) \approx t^{-2s} \tag{4.1.19}$$

for all $t \ll 1$ and all $(u,v) \in \tilde{R}_t$. Then

$$K_{f_4}(u,v) = \frac{|Df_4(u,v)|^2}{J_{f_4}(u,v)} \approx t^{2-2s} \qquad \forall t \ll 1 \text{ and } (u,v) \in \tilde{R}_t. \tag{4.1.20}$$

The same reasons as for (4.1.11) and (4.1.12) imply that

$$|Df_4^{-1}(x,y)| \approx t,\ J_{f_4^{-1}}(x,y) \approx t^{2s} \text{ and } K_{f_4^{-1}}(x,y) \approx t^{2-2s} \tag{4.1.21}$$

for all $t \ll 1$ and all $(x,y) \in R_t$.

Define

$$F_t = f_3^{-1} \circ f_4^{-1} \circ f_2 \circ f_1^{-1}.$$

Then F_t is a diffeomorphism from Q_t onto $\tilde{Q}_t$. Therefore

$$DF_t(z) = Df_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z))Df_4^{-1}(f_2 \circ f_1^{-1}(z))D(f_2 \circ f_1^{-1})(z)$$

for all $z \in Q_t$. From (4.1.16), (4.1.21) and (4.1.11) it then follows that

$$\begin{aligned}\int_{Q_t} |DF_t|^p\, dz \le& \int_{Q_t} |Df_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1})|^p |Df_4^{-1}(f_2 \circ f_1^{-1})|^p |Df_2 \circ f_1^{-1}|^p\, dz \\ \lesssim& t^p \mathcal{L}^2(Q_t) \approx t^{2+p}\end{aligned} \tag{4.1.22}$$

for any $p \ge 0$. By Lemma 2.1 we have that

$$\begin{aligned}\int_{Q_t} |J_{F_t}(z)|\, dz =& \int_{Q_t} |J_{f_3^{-1}}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z))||J_{f_4^{-1}}(f_2 \circ f_1^{-1}(z))||J_{f_2\circ f_1^{-1}}(z)|\, dz \\ \le& \int_{f_2\circ f_1^{-1}(Q_t)} |J_{f_3^{-1}}(f_4^{-1})||J_{f_4^{-1}}| \\ \le& \int_{f_4^{-1}\circ f_2 \circ f_1^{-1}(Q_t)} |J_{f_3^{-1}}| \le \mathcal{L}^2(\tilde{Q}_t).\end{aligned} \tag{4.1.23}$$

For a fixed large j_0, we now consider the set Q_t with $t = 2^{-j}$ for all $j \ge j_0$. Define

$$E_1 = \sum_{j=j_0}^{+\infty} F_{2^{-j}} \chi_{Q_{2^{-j}}}. \tag{4.1.24}$$

Denote $\Omega_1 = \cup_{j=j_0}^{+\infty} Q_{2^{-j}}$ and $\tilde{\Omega}_1 = \cup_{j=j_0}^{+\infty} \tilde{Q}_{2^{-j}}$. Then E_1 is a homeomorphism from Ω_1 onto $\tilde{\Omega}_1$, and satisfies (2.2.1) for E_1 on $\mathcal{L}^2$-a.e. Ω_1. In order to prove that E_1 has finite distortion on Ω_1, it thus suffices to prove that $E_1 \in W^{1,1}_{\mathrm{loc}}(\Omega_1)$ and $J_{E_1} \in L^1_{\mathrm{loc}}(\Omega_1)$. Actually, from (4.1.22) and (4.1.23) we have that

$$\int_{\Omega_1} |DE_1|^p = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} |DF_{2^{-j}}(z)|^p\, dz \lesssim \sum_{j=j_0}^{+\infty} 2^{-j(2+p)} < \infty \tag{4.1.25}$$

and

$$\int_{\Omega_1} |J_{E_1}| = \sum_{j=j_0}^{\infty} \int_{Q_{2^{-j}}} |J_{F_{2^{-j}}}| \le \sum_{j=j_0}^{\infty} \mathcal{L}^2(\tilde{Q}_{2^{-j}}) = \mathcal{L}^2(\tilde{\Omega}_1) < \infty \tag{4.1.26}$$

for all $p \ge 1$.

Step 2: Denote

$$\Omega_2 = M_s^c \setminus \Omega_1 \text{ and } \tilde{\Omega}_2 = \Delta_s^c \setminus \tilde{\Omega}_1.$$

Notice that both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$ are piecewise smooth Jordan curves with non-zero angles at the two corners. Therefore both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$ are chord-arc curves. By [7] there are bi-Lipschitz mappings

$$H_1 : \mathbb{R}^2 \to \mathbb{R}^2 \text{ and } H_2 : \mathbb{R}^2 \to \mathbb{R}^2 \tag{4.1.27}$$

such that $H_1(\mathbb{S}^1) = \partial\Omega_2$ and $H_2(\mathbb{S}^1) = \partial\tilde{\Omega}_2$. Define

$$h(z) = \begin{cases} E_1(z) & \forall z \in \partial\Omega_2 \cap \partial\Omega_1, \\ z^2 & \forall z \in \partial\Omega_2 \cap \partial M_s. \end{cases}$$

Then h is a bi-Lipschitz mapping in terms of the arc lengths. By the chord-arc properties of both $\partial\Omega_2$ and $\partial\tilde{\Omega}_2$, we have that h is also a bi-Lipschitz mapping with respect to the Euclidean distances. Taking (4.1.27) into account, we conclude that $H_2^{-1} \circ h \circ H_1 : \mathbb{S}^1 \to \mathbb{S}^1$ is a bi-Lipschitz mapping. By [13, Theorem A] there is then a bi-Lipschitz mapping

$$H : \mathbb{R}^2 \to \mathbb{R}^2 \tag{4.1.28}$$

such that $H|_{\mathbb{S}^1} = H_2^{-1} \circ h \circ H_1$. Define

$$E_2 = H_2 \circ H \circ H_1^{-1}. \tag{4.1.29}$$

By (4.1.27) and (4.1.28), we have that E_2 is a bi-Lipschitz extension of h. Furthermore since $\deg_{M_s}(h, w) = 1$, we obtain that E_2 is orientation-preserving. Hence E_2 is a quasiconformal mapping. The same reasons as for (2.3.13) and (2.3.14) imply

$$|DE_2(z)|,\ K_{E_2}(z) \text{ and } J_{E_2}(z) \text{ are bounded from both above and below} \tag{4.1.30}$$

for $\mathcal{L}^2$-a.e. $z \in \mathbb{R}^2$, and

$$|DE_2^{-1}(w)|,\ K_{E_2}^{-1}(w) \text{ and } J_{E_2}^{-1}(w) \text{ are bounded from both above and below} \tag{4.1.31}$$

for $\mathcal{L}^2$-a.e. $w \in \mathbb{R}^2$.

Via (4.1.24) and (4.1.29), we define

$$E(x, y) = \begin{cases} E_1(x, y) & \text{for all } (x, y) \in \Omega_1, \\ E_2(x, y) & \text{for all } (x, y) \in \Omega_2, \\ (x^2 - y^2, 2xy) & \text{for all } (x, y) \in \overline{M_s}. \end{cases} \tag{4.1.32}$$

By the properties of E_1 and E_2, we conclude that $E \in \mathcal{E}_s$. □

4.2. (1.0.7), (1.0.10) and (1.0.11).

Proof of (1.0.7). Let $g : \mathbb{D} \to \Delta_s$ be conformal, where Δ_s is defined in (2.3.2) with $s > 1$. In order to prove (1.0.7), it is enough to construct $f \in \mathcal{F}_s(g)$ such that $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. let E be as in (4.1.32). Then $E \in \mathcal{E}_s$. By (4.1.25), (4.1.30) and the fact that $E(z) = z^2$ for all $z \in M_s$, we obtain that $E \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. Let G_s be as in (2.3.6) and $\mathfrak{M}_s$ be as in (3.0.30). By Lemma 2.6 and the analogous arguments as for (3.0.42), we can define $f = E \circ G_s \circ \mathfrak{M}_s$. □

Proof of (1.0.10). Let $g : \mathbb{D} \to \Delta_s$ be conformal, where Δ_s is defined in (2.3.2) with $s > 1$. In order to prove (1.0.10), by Lemma 3.5 (1) it is enough to construct a mapping $f \in \mathcal{F}_s(g)$ such that $f^{-1} \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < 2(s + 1)/(2s - 1)$. Let G_s be as in (2.3.6) and $\mathfrak{M}_s$ be defined in (3.0.30). If there is a mapping $E \in \mathcal{E}_s$ such that $E^{-1} \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < 2(s + 1)/(2s - 1)$, by Lemma 2.6 and analogous arguments as for (3.0.32) we can define $f = E \circ G_s \circ \mathfrak{M}_s$.

Let E be as in (4.1.32). Then $E \in \mathcal{E}_s$. By (4.1.14), (4.1.19) and (4.1.9) we have that

$$|DF_{2^{-j}}^{-1}(w)| \leq |Df_1 \circ f_2^{-1}(f_4 \circ f_3(w))||Df_4(f_3(w))||Df_3(w)| \lesssim 2^{j(2s-1)}$$

for all $j \geq j_0$ and $\mathcal{L}^2$-a.e. $w \in \tilde{Q}_{2^{-j}}$. Together with $\mathcal{L}^2(\tilde{Q}_{2^{-j}}) \approx 2^{-2j(s+1)}$, we hence obtain that

$$\int_{\tilde{\Omega}_1} |DE_1^{-1}|^p = \sum_{j=j_0}^{+\infty} \int_{\tilde{Q}_{2^{-j}}} |DF_{2^{-j}}^{-1}|^p \lesssim \sum_{j=j_0}^{+\infty} 2^{-j(2(s+1)+p(1-2s))} < \infty \tag{4.2.1}$$

for all $p < 2(s + 1)/(2s - 1)$. Since

$$|DE^{-1}(u, v)| \lesssim (u^2 + v^2)^{-1/4} \qquad \forall (u, v) \in \Delta_s, \tag{4.2.2}$$

by a change of variables we have that

$$\int_{\Delta_s} |DE^{-1}(w)|^p\,dw \lesssim \int_0^{2\pi}\int_0^1 r^{1-\frac{p}{2}}\,dr\,d\theta \approx \int_0^1 r^{1-\frac{p}{2}}\,dr < \infty \tag{4.2.3}$$

for all $p < 2(s+1)/(2s-1)$. By (4.1.31), (4.2.1) and (4.2.3), we conclude that $E^{-1} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ for all $p < 2(s+1)/(2s-1)$. □

Proof of (1.0.11). Let $g : \mathbb{D} \to \Delta_s$ be conformal, where Δ_s is defined in (2.3.2) with $s > 1$. In order to prove (1.0.11), by Lemma 3.5 (2) it is enough to construct a mapping $f \in \mathcal{F}_s(g)$ such that $K_{f^{-1}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < (s+1)/(s-1)$. Let G_s be as in (2.3.6) and $\mathfrak{M}_s$ be as in (3.0.30). If there is a mapping $E \in \mathcal{E}_s$ such that $K_{E^{-1}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < (s+1)/(s-1)$, by Lemma 2.6 and analogous argument as for (3.0.34) we can define $f = E \circ G_s \circ \mathfrak{M}_s$.

Let E be as in (4.1.32). Then $E \in \mathcal{E}_s$. From (4.1.10), (4.1.20) and (4.1.15), we have that

$$K_{F^{-1}_{2^{-j}}}(w) = K_{f_1\circ f_2^{-1}}(f_4 \circ f_3(w))K_{f_4}(f_3(w))K_{f_3}(w) \approx 2^{j(2s-2)}$$

for all $j \geq j_0$ and $\mathcal{L}^2$-a.e. $w \in \tilde{Q}_{2^{-j}}$. Together with $\mathcal{L}^2(\tilde{Q}_{2^{-j}}) \approx 2^{-j2(s+1)}$, we then obtain that

$$\int_{\tilde{\Omega}_1} K^q_{E^{-1}} = \sum_{j=j_0}^{+\infty} \int_{\tilde{Q}_{2^{-j}}} K^q_{F^{-1}_{2^{-j}}} \lesssim \sum_{j=j_0}^{+\infty} 2^{2j[(s-1)q-(s+1)]} < \infty \tag{4.2.4}$$

for all $q < (s+1)/(s-1)$. By (4.1.31), (4.2.4) and the fact that E is conformal on M_s, we conclude that $K_{E^{-1}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < (s+1)/(s-1)$.

□

4.3. (1.0.8).

Proof. Let $g : \mathbb{D} \to \Delta_s$ be conformal, where Δ_s is defined as (2.3.2) with $s > 1$. In order to prove (1.0.8), via Lemma 3.5 (3) it is enough to construct a mapping $f \in \mathcal{F}_s(g)$ such that $K_f \in L^q_{\mathrm{loc}}$ for all $q < \max\{1, 1/(s-1)\}$. Let G_s be as in (2.3.6) and $\mathfrak{M}_s$ be as in (3.0.30). If $E \in \mathcal{E}_s$ such that $K_E \in L^q_{\mathrm{loc}}$ for all $q < \max\{1, 1/(s-1)\}$, by Lemma 2.6 and analogous arguments as for (3.0.41) we can define $f = E \circ G_s \circ \mathfrak{M}_s$.

Let E be as in (4.1.32). Then $E \in \mathcal{E}_s$. From (4.1.16), (4.1.21) and (4.1.12), it follows that

$$K_{F_{2^{-j}}}(z) = K_{f_3^{-1}}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z))K_{f_4^{-1}}(f_2 \circ f_1^{-1}(z))K_{f_2\circ f_1^{-1}}(z) \approx 2^{2j(s-1)}$$

for all $j \geq j_0$ and $\mathcal{L}^2$-a.e. $z \in Q_{2^{-j}}$. Together with $\mathcal{L}^2(Q_{2^{-j}}) \approx 2^{-2j}$ we then have that

$$\int_{\Omega_1} K^q_E = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K^q_{F_{2^{-j}}} \approx \sum_{j=j_0}^{+\infty} 2^{2j(q(s-1)-1)} < \infty \tag{4.3.1}$$

for all $q < 1/(s-1)$. By (4.3.1), (4.1.30) and the fact that E is conformal on M_s, we conclude that $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 1/(s-1)$. Therefore we have proved (1.0.8) whenever $s \in (1, 2)$.

We next consider the case $s \in [2, \infty)$. It is enough to construct a mapping $E \in \mathcal{E}_s$ such that $K_E \in L^q_{\mathrm{loc}}$ for all $q < 1$. Except for redefining $f_4^{-1} : R_t \to \tilde{R}_t$ as in (4.1.17), we follow all processes in Section 4.1 to define a new E, see FIGURE 3. Let α_t and β_t be the length of sides of $\tilde{R}_t$, and γ_t be the length of a side of R_t. Whenever $t \ll 1$, we have that

$$\alpha_t = t^2 - (t/2)^2 \approx t^2,\ \ \beta_t = 2t^{2s} \text{ and } \gamma_t = \eta(t^2) - \eta((t/2)^2) \approx t. \tag{4.3.2}$$

Let $\tilde{T}_0 = \tilde{Q}_1\tilde{Q}_2\tilde{Q}_3\tilde{Q}_4$ be the concentric square of $\tilde{R}_t$ with side length $\beta_t/2$. Set

$$\delta_t = \exp(-t^{-1}) \qquad \text{for } t > 0 \tag{4.3.3}$$

and let $T_0 = Q_1Q_2Q_3Q_4$ be the concentric square of R_t with side length $\gamma_t(1 - 2\delta_t)$. We divide $R_t \setminus T_0$ into four isosceles trapezoids $T_1,\ T_2,\ T_3$ and T_4. Similarly, we obtain isosceles trapezoids $\tilde{T}_1,\ \tilde{T}_2,\ \tilde{T}_3,\ \tilde{T}_4$ from $\tilde{R}_t \setminus \tilde{T}_0$, see FIGURE 3.

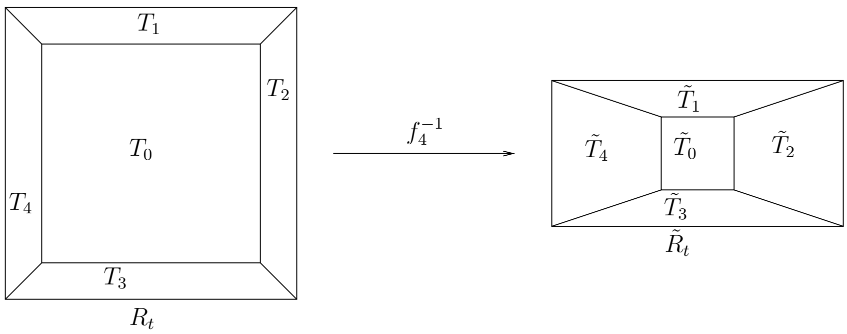

FIGURE 3. The redefined $f_4^{-1} : R_t \to \tilde{R}_t$

We first define a diffeomorphism from T_1 onto $\tilde{T}_1$. Define

$$A_2(x,y) = \frac{\beta_t}{4\delta_t\gamma_t}\left(y - \gamma_t(\frac{1}{2} - \delta_t)\right) + \frac{\beta_t}{4} \qquad \forall (x,y) \in T_1. \tag{4.3.4}$$

For a given $(x,y) \in T_1$, let $(x_p, y) = P_1Q_1 \cap \{(X,Y) \in \mathbb{R}^2 : Y = y\}$, $(\tilde{x}_p, A_2) = \tilde{P}_1\tilde{Q}_1 \cap \{(X,Y) \in \mathbb{R}^2 : Y = A_2(x,y)\}$, $\ell(y)$ be the length of $T_1 \cap \{(X,Y) : Y = y\}$, and $\tilde{\ell}(y)$ be the length of $\tilde{T}_1 \cap \{(X,Y) : Y = A_2\}$. Denote $(P_1)_1$ by the first coordinate of P_1. Then

$$x_p = -y + \frac{\gamma_t}{2} + (P_1)_1 \text{ and } \tilde{x}_p = \frac{2\alpha_t - \beta_t}{\beta_t}\left(\frac{\beta_t}{2} - A_2\right) + (\tilde{P}_1)_1, \tag{4.3.5}$$

$$\ell(y) = 2y \approx \gamma_t \text{ and } \tilde{\ell}(y) = \frac{4\alpha_t - 2\beta_t}{\beta_t}A_2(x,y) + \beta_t - \alpha_t \geq \frac{\beta_t}{2}. \tag{4.3.6}$$

Let $u = \frac{\gamma_t}{\ell(y)}(x - x_p) + (P_1)_1$ for $(x,y) \in T_1$, and η be as in (4.1.1). Define

$$A_1(x,y) = \frac{\tilde{\ell}(y)}{\alpha_t}\left(\eta^{-1}(u) - (\tilde{P}_1)_1\right) + \tilde{x}_p \qquad \forall (x,y) \in T_1. \tag{4.3.7}$$

By (4.3.7) and (4.3.4), we have that

$$A = (A_1, A_2) \tag{4.3.8}$$

is a diffeomorphism from T_1 onto $\tilde{T}_1$. We next give some estimates for A. By (4.3.2) we have that

$$\frac{\partial A_2(x,y)}{\partial y} = \frac{\beta_t}{4\delta_t\gamma_t} \approx \frac{t^{2s-1}}{\delta_t} \qquad \forall (x,y) \in T_1. \tag{4.3.9}$$

From (4.1.2), (4.3.6) and (4.3.2) it follows that

$$\frac{\partial A_1(x,y)}{\partial x} = \frac{\tilde{\ell}(y)}{\alpha_t}(\eta^{-1})'(u)\frac{\partial u}{\partial x} \approx \frac{\tilde{\ell}(y)}{t} \qquad \forall (x,y) \in T_1. \tag{4.3.10}$$

Moreover, by (4.3.5) and (4.3.6) we have that

$$\frac{\partial x_p}{\partial y} = -1,\ \frac{\partial \tilde{x}_p}{\partial y} = \frac{\beta_t - 2\alpha_t}{\beta_t}\frac{\partial A_2}{\partial y},\ \frac{\partial \ell(y)}{\partial y} = 2 \text{ and } \frac{\partial \tilde{\ell}(y)}{\partial y} = \frac{4\alpha_t - 2\beta_t}{\beta_t}\frac{\partial A_2}{\partial y}. \tag{4.3.11}$$

It follows from (4.3.11) that

$$
\begin{aligned}
\frac{\partial A_1}{\partial y} =& \frac{\partial \tilde{x}_p}{\partial y} + \frac{\partial \tilde{\ell}(y)}{\alpha_t \partial y}\left(\eta^{-1}(u) - (\tilde{P}_1)_1\right) + \frac{\tilde{\ell}(y)}{\alpha_t}(\eta^{-1})'(u)\frac{\partial u}{\partial y} \\
=& \frac{2\alpha_t - \beta_t}{\beta_t}\frac{\partial A_2}{\partial y}\left[-1 + \frac{2}{\alpha_t}(\eta^{-1}(u) - (\tilde{P}_1)_1)\right] \\
& + \frac{\gamma_t \tilde{\ell}(y)}{\alpha_t \ell(y)}(\eta^{-1})'(u)\left[1 - \frac{2}{\ell(y)}(x - x_p)\right]. \qquad (4.3.12)
\end{aligned}
$$

Notice that $0 \le \eta^{-1}(u) - (\tilde{P}_1)_1 \le \alpha_t$ and $0 \le x - x_p \le \ell(y)$ for all $(x, y) \in T_1$. Therefore (4.3.12) together with (4.3.2) and (4.3.9) implies

$$
\left|\frac{\partial A_1(x,y)}{\partial y}\right| \lesssim \frac{2\alpha_t - \beta_t}{\beta_t}\frac{\partial A_2(x,y)}{\partial y} \approx \frac{t}{\delta_t} \qquad \forall (x,y) \in T_1. \qquad (4.3.13)
$$

We conclude from (4.3.9), (4.3.10) and (4.3.13) that

$$
|DA(x,y)| \lesssim \max\left\{\left|\frac{\partial A_1}{\partial x}\right|,\ \left|\frac{\partial A_1}{\partial y}\right|,\ \left|\frac{\partial A_2}{\partial x}\right|,\ \left|\frac{\partial A_2}{\partial y}\right|\right\} \lesssim \frac{t}{\delta_t} \qquad (4.3.14)
$$

and

$$
J_A(x,y) = \frac{\partial A_1}{\partial x}\frac{\partial A_2}{\partial y} \approx \frac{t^{2s-2}\tilde{\ell}(y)}{\delta_t} \qquad (4.3.15)
$$

for all $t \ll 1$ and all $(x, y) \in T_1$. Moreover by (4.3.14), (4.3.15) and (4.3.6) we have that

$$
K_A(x,y) = \frac{|DA(x,y)|^2}{J_A(x,y)} \lesssim \frac{t^{4-2s}}{\delta_t \tilde{\ell}(y)} \lesssim \frac{t^{4(1-s)}}{\delta_t} \qquad (4.3.16)
$$

holds for all $t \ll 1$ and all $(x, y) \in T_1$.

We next define a diffeomorphism from T_2 onto $\tilde{T}_2$. Denote by P_c and $\tilde{P}_c$ be the center of R_t and $\tilde{R}_t$, respectively. Given $(x, y) \in T_2$, we define

$$
B_1(x,y) = \frac{2\alpha_t - \beta_t}{4\delta_t\gamma_t}\left(x - (P_c)_1 - \frac{\gamma_t}{2}\right) + (\tilde{P}_c)_1 + \frac{\alpha_t}{2},\ B_2(x,y) = y\frac{a(x - (P_c)_1) + b}{c(x - (P_c)_1) + d},
$$

where $a,\ b,\ c,\ d$ satisfy

$$
a\gamma_t(\frac{1}{2} - \delta_t) + b = \frac{\beta_t}{4},\ a\frac{\gamma_t}{2} + b = \frac{\beta_t}{2},\ c\gamma_t(\frac{1}{2} - \delta_t) + d = \gamma_t(\frac{1}{2} - \delta_t),\ c\frac{\gamma_t}{2} + d = \frac{\gamma_t}{2}. \qquad (4.3.17)
$$

Then

$$
B = (B_1, B_2) \qquad (4.3.18)
$$

is a diffeomorphism from T_2 onto $\tilde{T}_2$. By (4.3.2) we have that

$$
\frac{\partial B_1(x,y)}{\partial x} = \frac{2\alpha_t - \beta_t}{4\delta_t\gamma_t} \approx \frac{t}{\delta_t} \qquad \forall (x,y) \in T_2. \qquad (4.3.19)
$$

Moreover, from (4.3.17) and (4.3.2) we have that

$$
\frac{\partial B_2(x,y)}{\partial y} = \frac{a(x - (P_c)_1) + b}{c(x - (P_c)_1) + d} \approx \frac{\beta_t}{\gamma_t} \approx t^{2s-1} \qquad (4.3.20)
$$

and

$$
\left|\frac{\partial B_2(x,y)}{\partial x}\right| = \frac{|y(ad - bc)|}{[c(x - (P_c)_1) + d]^2} \lesssim \frac{\gamma_t b}{\gamma_t^2} \approx t^{2s-1} \qquad (4.3.21)
$$

for all $(x, y) \in T_2$. We then conclude from (4.3.19), (4.3.20) and (4.3.21) that

$$
|DB(x,y)| \lesssim \max\left\{\left|\frac{\partial B_1}{\partial x}\right|,\ \left|\frac{\partial B_1}{\partial y}\right|,\ \left|\frac{\partial B_2}{\partial x}\right|,\ \left|\frac{\partial B_2}{\partial y}\right|\right\} \lesssim \frac{t}{\delta_t} \qquad (4.3.22)
$$

and

$$J_B(x,y) = \frac{\partial B_1}{\partial x}\frac{\partial B_2}{\partial y} \approx \frac{t^{2s}}{\delta_t}. \tag{4.3.23}$$

for all $t \ll 1$ and all $(x,y) \in T_2$. Moreover by (4.3.22) and (4.3.23) we have that

$$K_B(x,y) = \frac{|DB(x,y)|^2}{J_B(x,y)} \lesssim \frac{t^{2(1-s)}}{\delta_t} \tag{4.3.24}$$

for all $t \ll 1$ and all $(x,y) \in T_2$.

We next construct a diffeomorphism $C : T_0 \to \tilde{T}_0$. By (4.3.8) and (4.3.18) we have that Q_1Q_2 is mapped onto $\tilde{Q}_1\tilde{Q}_2$ by $A_1(\cdot, \gamma_t(1/2 - \delta_t)$, and Q_2Q_3 is mapped onto $\tilde{Q}_2\tilde{Q}_3$ by $B_2((P_c)_1 + \gamma_t(1/2 - \delta_t), \cdot)$. For a given $(x,y) \in T_0$, define

$$C(x,y) = \left(A_1\big(x, \gamma_t(\frac{1}{2} - \delta_t)\big), B_2\big((P_c)_1 + \gamma_t(\frac{1}{2} - \delta_t), y\big)\right). \tag{4.3.25}$$

Then $C : T_0 \to \tilde{T}_0$ is diffeomorphic. By (4.3.10) and (4.3.20), we have that

$$\frac{\partial}{\partial x}A_1(x, \gamma_t(1/2 - \delta_t) \approx t^{2s-1}, \ \frac{\partial}{y}B_2((P_c)_1 + \gamma_t(1/2 - \delta_t), y) \approx t^{2s-1}$$

for all $(x,y) \in T_0$. Therefore

$$|DC(x,y)| \lesssim t^{2s-1} \text{ and } K_C(x,y) \approx 1 \tag{4.3.26}$$

for all $t \ll 1$ and all $(x,y) \in T_0$.

Via (4.3.8), (4.3.18) and (4.3.25), we redefine $f_4^{-1} : R_t \to \tilde{R}_t$ in (4.1.17) as

$$f_4^{-1}(x,y) = \begin{cases} A(x,y) & \forall (x,y) \in T_1, \\ B(x,y) & \forall (x,y) \in T_2, \\ (A_1(x,-y), -A_2(x,-y)), & \forall (x,y) \in T_3, \\ (2(\tilde{P}_c)_1 - B_1(2(P_c)_1 - x, y), B_2(2(P_c)_1 - x, y)) & \forall (x,y) \in T_4, \\ C(x,y) & \forall (x,y) \in T_0. \end{cases} \tag{4.3.27}$$

Like in Section 4.1, by taking a fixed $j_0 \gg 1$ we then define $F_{2^{-j}} : Q_{2^{-j}} \to \tilde{Q}_{2^{-j}}$ for all $j \geq j_0$, $E_1 : \Omega_1 \to \tilde{\Omega}_1$, $E_2 : \Omega_2 \to \tilde{\Omega}_2$ and $E : \mathbb{R}^2 \to \mathbb{R}^2$. It is not difficult to see that the new-defined E is a homeomorphism such that $E(z) = z^2$ for all $z \in \overline{M_s}$ and satisfies (2.2.1) for E on $\mathcal{L}^2$-a.e. $\mathbb{R}^2$. To show $E \in \mathcal{E}_s$, it is then enough to prove that $E \in W^{1,1}_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ and $J_E \in L^1_{\rm loc}(\mathbb{R}^2)$. By (4.1.11), (4.1.16), (4.3.14), (4.3.22) and (4.3.26), we have that

$$\begin{aligned} DF_{2^{-j}}(z) =& Df_3^{-1}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z)) Df_4^{-1}(f_2 \circ f_1^{-1}(z)) D(f_2 \circ f_1^{-1})(z) \\ \lesssim& \begin{cases} \frac{2^{-j}}{\delta_{2^{-j}}} & \mathcal{L}^2\text{-a.e. } z \in f_1 \circ f_2^{-1}(\cup_{k=1}^4 T_k), \\ 2^{j(1-2s)} & \mathcal{L}^2\text{-a.e. } z \in f_1 \circ f_2^{-1}(T_0), \end{cases} \end{aligned} \tag{4.3.28}$$

for all $j \geq j_0$. Notice that

$$\mathcal{L}^2(T_0) = (\gamma_{2^{-j}}(1 - 2\delta_{2^{-j}}))^2 \approx 2^{-2j}, \ \mathcal{L}^2(T_k) = \delta_{2^{-j}}\gamma_{2^{-j}}^2(1 - \delta_{2^{-j}}) \approx \delta_{2^{-j}}2^{-2j}$$

for all $k = 1, 2, 3, 4$ and all $j \geq j_0$. It hence follows from (4.1.9) that

$$\mathcal{L}^2(f_1 \circ f_2^{-1}(T_0)) \approx 2^{-2j}, \ \mathcal{L}^2(f_1 \circ f_2^{-1}(T_k)) \approx \delta_{2^{-j}}2^{-2j} \quad \text{for all } k = 1, 2, 3, 4. \tag{4.3.29}$$

By (4.3.28) and (4.3.29) we then have that

$$\int_{Q_{2^{-j}}} |DF_{2^{-j}}| = \sum_{k=0}^4 \int_{f_1 \circ f_2^{-1}(T_k)} |DF_{2^{-j}}| \lesssim 2^{-3j} + 2^{-j(2s+1)} \lesssim 2^{-3j} \qquad \forall j \geq j_0.$$

Therefore

$$\int_{\Omega_1} |DE_1| = \sum_{j=j_0}^{\infty} \int_{Q_{2^{-j}}} |DF_{2^{-j}}| \lesssim \sum_{j=j_0}^{\infty} 2^{-3j} < \infty. \tag{4.3.30}$$

By (4.1.30), (4.3.30) and the fact that $E(z) = z^2$ for all $z \in M_s$, we have that $E \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. Analogously to (4.1.26), we have that

$$\int_{\Omega_1} |J_{E_1}| \leq \mathcal{L}^2(\tilde{\Omega}_1) < \infty. \tag{4.3.31}$$

From (4.1.30), (4.3.31) and the fact that $E(z) = z^2$ for all $z \in M_s$, we have that $J_E \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

We next show $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 1$. By (4.1.12), (4.1.16), (4.3.16), (4.3.24) and (4.3.26), we have that

$$K_{F_{2^{-j}}}(z) \lesssim \begin{cases} \frac{2^{4j(s-1)}}{\delta_{2^{-j}}} & \forall\ z \in f_1 \circ f_2^{-1}(T_1 \cup T_3), \\ \frac{2^{2j(s-1)}}{\delta_{2^{-j}}} & \forall\ z \in f_1 \circ f_2^{-1}(T_2 \cup T_4), \\ 1 & \forall\ z \in f_1 \circ f_2^{-1}(T_0). \end{cases} \tag{4.3.32}$$

for all $j \geq j_0$. For any $q \geq 0$, via (4.3.29) and (4.3.32) we obtain that

$$\int_{Q_{2^{-j}}} K^q_{F_{2^{-j}}} = \sum_{k=0}^{4} \int_{f_1 \circ f_2^{-1}(T_k)} K^q_{F_{2^{-j}}} \lesssim \delta_{2^{-j}}^{1-q} 2^{j(4q(s-1)-2)}(1 + 2^{2qj(1-s)}) + 2^{-2j}$$

for all $j \geq j_0$. Therefore

$$\begin{aligned} \int_{\Omega_1} K^q_E &= \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K^q_{F_{2^{-j}}} \\ &\lesssim \sum_{j=j_0}^{+\infty} \exp((q-1)2^j) 2^{j(4q(s-1)-2)}(1 + 2^{j2q(1-s)}) + \sum_{j=j_0}^{+\infty} 2^{-2j} < +\infty \end{aligned} \tag{4.3.33}$$

for all $q \in (0,1)$ and each $s > 1$. By (4.1.30), (4.3.33) and the fact that E is conformal on M_s, we conclude that $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q \in (0,1)$. □

4.4. (1.0.9).

Proof of (1.0.9). Let $g : \mathbb{D} \to \Delta_s$ be conformal, where Δ_s is defined in (2.3.2) with $s > 1$. In order to prove (1.0.9), via Lemma 3.5 (4) it is enough to construct $f \in \mathcal{F}_s(g)$ such that $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_f \in L^q_{\mathrm{loc}}$ for all $q < \max\{1/(s-1), 3p/((2s-1)p + 4 - 2s)\}$.

We consider the case $s \in (1,2]$ first. Let G_s be as in (2.3.6) and $\mathfrak{M}_s$ be as in (3.0.30). If $E \in \mathcal{E}_s$ satisfying that $E \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 1$ and $K_E \in L^q_{\mathrm{loc}}$ for all $q < 1/(s-1)$, by Lemma 2.6 and the analogous arguments as for (3.0.41) and (3.0.42), we can define $f = E \circ G_s \circ \mathfrak{M}_s$. We now let E be as in (4.1.32). Then $E \in \mathcal{E}_s$. By (4.1.25), (4.1.30) and the fact that $E(z) = z^2$ for all $z \in M_s$, we obtain that $E \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p \geq 1$. From (4.1.16), (4.1.21) and (4.1.12), it follows that

$$K_{F_{2^{-j}}}(z) = K_{f_3^{-1}}(f_4^{-1} \circ f_2 \circ f_1^{-1}(z)) K_{f_4^{-1}}(f_2 \circ f_1^{-1}(z)) K_{f_2 \circ f_1^{-1}}(z) \approx 2^{(2s-2)j}$$

for all $j \geq j_0$ and $\mathcal{L}^2$-a.e. $z \in Q_{2^{-j}}$. Together with $\mathcal{L}^2(Q_{2^{-j}}) \approx 2^{-2j}$, we then obtain

$$\int_{\Omega_1} K^q_E = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K^q_{F_{2^{-j}}} \approx \sum_{j=j_0}^{+\infty} 2^{-j2(1+q(1-s))} < \infty \tag{4.4.1}$$

for all $q < 1/(s-1)$. By (4.4.1), (4.1.30) and the fact that E is conformal on M_s, we have that $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 1/(s-1)$.

We turn to the case $s > 2$. Let $M(p,s) = 3p/((2s-1)p + 4 - 2s)$ with $p > 1$. Analogously to the case $s \in (1,2]$, it is enough to construct $E \in \mathcal{E}_s$ such that $E \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all

$q \in (0, M(p,s))$. Redefining δ_t in (4.3.3) as $\delta_t = t^{\frac{p+2}{p-1}} \log^{\frac{p}{p-1}}(t^{-1})$. We follow the methods in Section 4.3 to define a new f_4^{-1}. Set $j_0 \gg 1$. There are then new $F_{2^{-j}} : Q_{2^{-j}} \to \tilde{Q}_{2^{-j}}$ for all $j \geq j_0$, $E_1 : \Omega_1 \to \tilde{\Omega}_1$, $E_2 : \Omega_2 \to \tilde{\Omega}_2$ and $E : \mathbb{R}^2 \to \mathbb{R}^2$. It is not difficult to see that the new E is homeomorphic, satisfies (2.2.1) for E on $\mathcal{L}^2$-a.e. $\mathbb{R}^2$ and $J_E \in L^1_{\text{loc}}(\mathbb{R}^2)$. To show that E satisfies all requirements, it is enough to check that $E \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $K_E \in L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q \in (0, M(p,s))$.

From (4.1.11), (4.1.16), (4.3.14), (4.3.22) and (4.3.26) we have that

$$|DF_{2^{-j}}(z)| \lesssim \begin{cases} \frac{2^{-j}}{\delta_{2^{-j}}} & \forall z \in f_1 \circ f_2^{-1}(\cup_{k=1}^4 T_k), \\ 2^{j(1-2s)} & \forall z \in f_1 \circ f_2^{-1}(T_0), \end{cases} \tag{4.4.2}$$

for all $j \geq j_0$. It follows from (4.4.2) and (4.3.29) that

$$\int_{Q_{2^{-j}}} |DF_{2^{-j}}|^p = \sum_{k=0}^{4} \int_{f_1 \circ f_2^{-1}(T_k)} |DF_{2^{-j}}|^p \lesssim \delta_{2^{-j}}^{1-p} 2^{-j(2+p)} + 2^{j(p(1-2s)-2)}.$$

Therefore

$$\int_{\Omega_1} |DE|^p = \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} |DF_{2^{-j}}|^p \lesssim \sum_{j=j_0}^{+\infty} \frac{1}{j^p} + \sum_{j=j_0}^{+\infty} 2^{-j(p(2s-1)+2)} < \infty. \tag{4.4.3}$$

By (4.4.3), (4.1.30) and the fact that $E(z) = z^2$ for all $z \in M_s$, we conclude that $E \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. By (4.1.11), (4.1.12), Lemma 2.1 and (4.1.16), we have

$$\begin{aligned} \int_{f_1 \circ f_2^{-1}(T_1)} K^q_{F_{2^{-j}}} &\approx \int_{f_1 \circ f_2^{-1}(T_1)} K^q_{f_3^{-1}}(f_4^{-1} \circ f_2 \circ f_1^{-1}) K^q_{f_4^{-1}}(f_2 \circ f_1^{-1}) K^q_{f_2 \circ f_1^{-1}} \left| J_{f_2 \circ f_1^{-1}} \right| \\ &\leq \int_{T_1} K^q_{f_3^{-1}}(f_4^{-1}) K^q_{f_4^{-1}} \\ &\lesssim \int_{T_1} K^q_{f_4^{-1}} \end{aligned} \tag{4.4.4}$$

for all $q \geq 0$ and all $j \geq j_0$. Notice $\tilde{\ell}(\gamma_{2^{-j}}/2) = \alpha_{2^{-j}}$ and $\tilde{\ell}(\gamma_{2^{-j}}(\frac{1}{2} - \delta_{2^{-j}})) = \beta_{2^{-j}}/2$ for all $j \geq 1$. By Fubini's theorem, (4.3.16), (4.3.6) and (4.3.2) we then have

$$\begin{aligned} \int_{T_1} K^q_{f_4^{-1}} &\lesssim \int_{\gamma_{2^{-j}}(\frac{1}{2}-\delta_{2^{-j}})}^{\frac{\gamma_{2^{-j}}}{2}} \int_{x_p}^{x_p+\ell(y)} \left(\frac{2^{j(2s-4)}}{\delta_{2^{-j}} \tilde{\ell}(y)} \right)^q dx\, dy \\ &\approx \frac{2^{jq(2s-4)} \gamma_{2^{-j}}}{\delta^q_{2^{-j}}} \int_{\gamma_{2^{-j}}(\frac{1}{2}-\delta_{2^{-j}})}^{\frac{\gamma_{2^{-j}}}{2}} \frac{1}{\tilde{\ell}^q(y)}\, dy \\ &= \frac{2^{jq(2s-4)} \gamma_{2^{-j}}}{(1-q)\delta^q_{2^{-j}}} \frac{2\delta_{2^{-j}} \gamma_{2^{-j}}}{2\alpha_{2^{-j}} - \beta_{2^{-j}}} \left(\tilde{\ell}^{1-q}(\frac{\gamma_{2^{-j}}}{2}) - \tilde{\ell}^{1-q}(\gamma_{2^{-j}}(\frac{1}{2} - \delta_{2^{-j}})) \right) \\ &\lesssim \frac{\delta^{1-q}_{2^{-j}} 2^{-2j[1+q(1-s)]}}{1 - M(p,s)} \end{aligned} \tag{4.4.5}$$

for any fixed $q \in (0, M(p,s))$. Combining (4.4.4) with (4.4.5) implies that

$$\int_{f_1 \circ f_2^{-1}(T_1)} K^q_{F_{2^{-j}}} \lesssim \delta^{1-q}_{2^{-j}} 2^{-2j[1+q(1-s)]} \qquad \forall j \geq j_0. \tag{4.4.6}$$

By symmetry of f_4^{-1} between T_1 and T_3, it follows from (4.4.6) that

$$\int_{f_1 \circ f_2^{-1}(T_3)} K^q_{F_{2^{-j}}} = \int_{f_1 \circ f_2^{-1}(T_1)} K^q_{F_{2^{-j}}} \lesssim \delta^{1-q}_{2^{-j}} 2^{-2j[1+q(1-s)]} \tag{4.4.7}$$

for all $j \geq j_0$. By (4.3.32) and (4.3.29), we have that

$$\int_{f_1\circ f_2^{-1}(T_0)} K^q_{F_{2^{-j}}} \lesssim 2^{-2j} \tag{4.4.8}$$

and

$$\int_{f_1\circ f_2^{-1}(T_2\cup T_4)} K^q_{F_{2^{-j}}} \lesssim \delta_{2^{-j}} 2^{-2j} \left(\frac{2^{2j(s-1)}}{\delta_{2^{-j}}}\right)^q = \delta_{2^{-j}}^{1-q} 2^{2j[q(s-1)-1]} \tag{4.4.9}$$

for all $j \geq j_0$. From (4.4.6), (4.4.7), (4.4.8) and (4.4.9), we conclude that

$$\begin{aligned} \int_{\Omega_1} K^q_E &= \sum_{j=j_0}^{+\infty} \int_{Q_{2^{-j}}} K^q_{F_{2^{-j}}} = \sum_{j=j_0}^{+\infty} \sum_{k=0}^{4} \int_{f_1\circ f_2^{-1}(T_k)} K^q_{F_{2^{-j}}} \\ &\lesssim \sum_{j=j_0}^{+\infty} 2^{-2j} + 2^{-j\left(\frac{(p+2)(1-q)}{p-1}+2[1+q(1-s)]\right)} \log^{\frac{p(1-q)}{p-1}}\left(2^j\right). \end{aligned} \tag{4.4.10}$$

Note that

$$\frac{(p+2)(1-q)}{p-1} + 2[1+q(1-s)] > 0 \Leftrightarrow q < M(p,s).$$

It from (4.4.10) follows that $\int_{\Omega_1} K^q_E < \infty$ for all $q \in (0, M(p,s))$. Together with (4.1.30) and the fact that E is conformal on M_s, we conclude that $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q \in (0, M(p,s))$. $\square$

5. Proof of Theorem 1.1

Proof. Let Δ be as in (1.0.1). The representation of $\partial\Delta$ in Cartesian coordinates is

$$(x^2+y^2)^2 - 4x(x^2+y^2) - 4y^2 = 0.$$

Hence we can parametrize $\partial\Delta$ in a neighborhood of the origin as

$$\tilde{\Gamma}_0 = \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = d(x)\},$$

where $j_0 \gg 1$ and $d(x) = \frac{-x^3(4-x)}{2-x^2+2x+\sqrt{1+2x}}$. Since $d(x) \approx |x|^3$ for all $|x| \ll 1$, there are $c_1 > 0,\ c_2 > 0$ such that

$$-c_1 x^3 \leq d(x) \leq -c_2 x^3 \qquad \forall x \in [-2^{-j_0}, 0].$$

Denote

$$\begin{aligned} \tilde{\Gamma}_1 &= \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = -c_1 x^3\}, \\ \tilde{\Gamma}_2 &= \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 = -c_2 x^3\}, \\ \tilde{\Gamma}_3 &= \{(x,y) \in \mathbb{R}^2 : x = -2^{-j_0}, y^2 \in [c_1(2^{-j_0})^3, d(-2^{-j_0})\}, \\ \tilde{\Gamma}_4 &= \{(x,y) \in \mathbb{R}^2 : x = -2^{-j_0}, y^2 \in [d(-2^{-j_0}), c_2(2^{-j_0})^3]\}. \end{aligned}$$

Let $\tilde{\Omega}_u$ and $\tilde{\Omega}_d$ be the domains bounded by $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_4$ and $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_3$, respectively. Denote by Ω_u, Ω_d and Γ_k for $k = 0, ..., 4$ the images of $\tilde{\Omega}_u, \tilde{\Omega}_d$ and $\tilde{\Gamma}_k$ under the branch of complex-valued function $z^{1/2}$ with $1^{1/2} = 1$, respectively.

We first prove the existence of an extension, see FIGURE 4. Let $r = (2^{-2j_0} + c_1 2^{-3j_0})^{1/4}$. Denote

$$M = \{(x+1, y) \in \mathbb{R}^2 : (x,y) \in \mathbb{D}\},$$

$$\Omega_1 = \overline{B(0,r)} \setminus (M \cup \Omega_d),\ \ \Omega_2 = \mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_d \cup M),$$

$$\tilde{\Omega}_1 = \{(x,y) \in \mathbb{R}^2 : x \in [-2^{-j_0}, 0], y^2 \leq c_1|x|^3\} \text{ and } \tilde{\Omega}_2 = \mathbb{R}^2 \setminus (\tilde{\Omega}_1 \cup \tilde{\Omega}_d \cup \Delta).$$

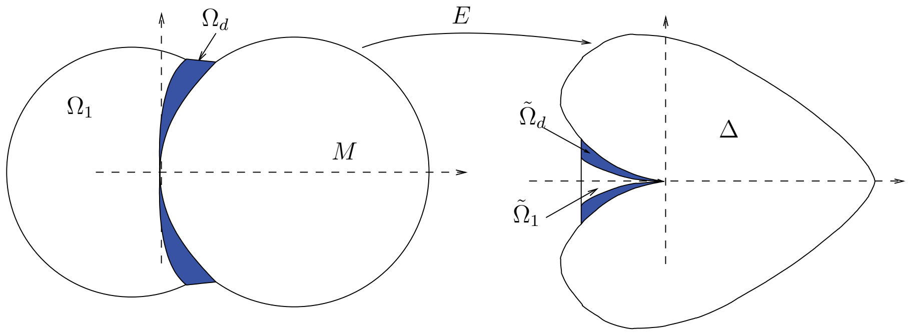

FIGURE 4. The existence of an extension

Analogously to the arguments in Section 4.1, we define $E_1 : \Omega_1 \to \tilde{\Omega}_1$ and $E_2 : \Omega_2 \to \tilde{\Omega}_2$. Here $\eta(x) = \sqrt{x}(1 + c_1 x)^{1/4}$ and $s = 3/2$. Define

$$E(x, y) = \begin{cases} E_1(x, y) & \forall\ (x, y) \in \Omega_1, \\ E_2(x, y) & \forall\ (x, y) \in \Omega_2, \\ (x^2 - y^2, 2xy) & \forall\ (x, y) \in M \cup \Omega_d, \end{cases} \tag{5.0.1}$$

and $f_0(x, y) = E(x + 1, y)$. By the analogous arguments as in Section 4.1, we have that $f_0 \in \mathcal{F}$.

We next prove (1.0.3). Suppose $f \in \mathcal{F}$. Then $\hat{f}(u, v) = f(u - 1, v)$ is a homeomorphism of finite distortion on $\mathbb{R}^2$ and $\hat{f}(M \setminus \Omega_u) = \Delta \setminus \tilde{\Omega}_u$. By Remark 3.1, we have that if $K_{\hat{f}} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ then $q < 2$. Therefore if $K_f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ then $q < 2$. In order to prove (1.0.3), it then suffices to construct a mapping $f_0 \in \mathcal{F}$ such that $K_{f_0} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 2$. Let E be as in (5.0.1) and $f_0(x, y) = E(x + 1, y)$. Then $f_0 \in \mathcal{F}$. The same arguments as for the case $s \in (1, 2)$ in Section 4.3 show that $K_E \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 2$. Therefore $K_{f_0} \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for all $q < 2$.

The strategies to prove (1.0.2), (1.0.4), (1.0.5) and (1.0.6) are same as the one to prove (1.0.3). We leave the details to the interested reader. □

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