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MONOTONICITY AND LOCAL UNIQUENESS FOR THE HELMHOLTZ EQUATION

BASTIAN HARRACH, VALTER POHJOLA AND MIKKO SALO

This work extends monotonicity-based methods in inverse problems to the case of the Helmholtz (or stationary Schrödinger) equation $(\Delta + k^2 q)u = 0$ in a bounded domain for fixed nonresonance frequency $k > 0$ and real-valued scattering coefficient function q. We show a monotonicity relation between the scattering coefficient q and the local Neumann-to-Dirichlet operator that holds up to finitely many eigenvalues. Combining this with the method of localized potentials, or Runge approximation, adapted to the case where finitely many constraints are present, we derive a constructive monotonicity-based characterization of scatterers from partial boundary data. We also obtain the local uniqueness result that two coefficient functions q_1 and q_2 can be distinguished by partial boundary data if there is a neighborhood of the boundary part where $q_1 \geq q_2$ and $q_1 \not\equiv q_2$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with unit outer normal ν. For a fixed nonresonance frequency $k > 0$, we study the relation between a real-valued scattering coefficient function $q \in L^\infty(\Omega)$ in the Helmholtz equation (or time-independent Schrödinger equation)

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega \tag{1}$$

and the local (or partial) Neumann-to-Dirichlet (NtD) operator

$$\Lambda(q) : L^2(\Sigma) \to L^2(\Sigma), \quad g \mapsto u|_\Sigma,$$

where $u \in H^1(\Omega)$ solves (1) with Neumann data

$$\partial_\nu u|_{\partial\Omega} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

Here $\Sigma \subseteq \partial\Omega$ is assumed to be an arbitrary nonempty relatively open subset of $\partial\Omega$. Since k is a nonresonance frequency, $\Lambda(q)$ is well-defined and is easily shown to be a self-adjoint compact operator.

We will show that

$$q_1 \leq q_2 \quad \text{implies} \quad \Lambda(q_1) \leq_{\text{fin}} \Lambda(q_2),$$

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where the inequality on the left-hand side is to be understood pointwise almost everywhere, and the right-hand side denotes that $\Lambda(q_2) - \Lambda(q_1)$ possesses only finitely many negative eigenvalues. Based on a slightly stronger quantitative version of this monotonicity relation, and an extension of the technique of localized potentials [Gebauer 2008] to spaces with finite codimension, we deduce the following local uniqueness result for determining q from $\Lambda(q)$.

Theorem 1.1. *Let $O \subseteq \overline{\Omega}$ be a connected relatively open set with $O \cap \Sigma \neq \varnothing$ and $q_1 \leq q_2$ on O. Then*

$$\Lambda(q_1) = \Lambda(q_2) \qquad \textit{implies} \qquad q_1 = q_2 \quad \textit{in } O.$$

Moreover, if $q_1|_O \not\equiv q_2|_O$, then $\Lambda(q_2) - \Lambda(q_1)$ has infinitely many positive eigenvalues.

Theorem 1.1 will be proven in Section 5. Note that this result removes the assumption $q_1, q_2 \in L^\infty_+(\Omega)$ from the local uniqueness result in [Harrach and Ullrich 2017], and that it implies global uniqueness if $q_1 - q_2$ is piecewise-analytic; see Corollary 5.2. Note also that in dimension $n = 2$, Imanuvilov, Uhlmann and Yamamoto [Imanuvilov et al. 2015] have proven global uniqueness with partial boundary data for potentials $q \in W^{1,p}(\Omega)$, $p > 2$. Compared to the result in [Imanuvilov et al. 2015], Theorem 1.1 is both less restrictive, as it holds for L^∞-potentials and any dimension $n \geq 2$, and more restrictive, as it relies on a local definiteness condition that is not required in [Imanuvilov et al. 2015].

Additionally to Theorem 1.1, we will also derive a constructive monotonicity-based method to detect a scatterer in an otherwise homogeneous domain. Let the scatterer $D \subseteq \Omega$ be an open set such that $\overline{D} \subseteq \Omega$ and the complement $\Omega \setminus \overline{D}$ is connected, and let

$$\begin{aligned} q(x) &= 1 \quad \text{for } x \in \Omega \setminus D \text{ (a.e.), and} \\ 1 < q_{\min} \leq q(x) &\leq q_{\max} \quad \text{for } x \in D \text{ (a.e.),} \end{aligned}$$

with constants $q_{\min}, q_{\max} > 0$. For an open set $B \subseteq \Omega$, we define the self-adjoint compact operator

$$T_B : L^2(\Sigma) \to L^2(\Sigma), \qquad \int_\Sigma g T_B h \, \mathrm{d}s := \int_B k^2 u_1^{(g)} u_1^{(h)} \, \mathrm{d}x,$$

where $u_1^{(g)}, u_1^{(h)} \in H^1(\Omega)$ solve (1) with $q \equiv 1$ and Neumann data g, h respectively.

Theorem 1.2. *For all $0 < \alpha \leq q_{\min} - 1$,*

$$B \subseteq D \quad \textit{if and only if} \quad \alpha T_B \leq_{\text{fin}} \Lambda(q) - \Lambda(1).$$

We will also give a bound on the number of negative eigenvalues in the case $B \subseteq D$, and prove a similar result for scatterers with negative contrast in Section 6.

Let us give some references on related works and comment on the origins and relevance of our result. The inverse problem considered in this work is closely related to the inverse conductivity problem of determining the positive conductivity function γ in the equation $\nabla \cdot (\gamma \nabla u) = 0$ in a bounded domain in $\mathbb{R}^n$ from knowledge of the associated Neumann-to-Dirichlet operator. This is also known as the problem of electrical impedance tomography or the Calderón problem [1980; 2006]. For a short list of seminal contributions for full boundary data let us refer to [Kohn and Vogelius 1984; 1985; Druskin 1998; Sylvester and Uhlmann 1987; Nachman 1996; Astala and Päivärinta 2006; Haberman and Tataru

2013; Caro and Rogers 2016]. For the uniqueness problem with partial boundary data there are rather precise results if $n = 2$ (see [Imanuvilov et al. 2010; 2015] and the survey [Guillarmou and Tzou 2013]), but in dimensions $n \geq 3$ it is an open question whether measurements on an arbitrary open set $\Sigma \subseteq \partial\Omega$ suffice to determine the unknown coefficient. We refer to [Kenig et al. 2007; Isakov 2007; Kenig and Salo 2013; Krupchyk and Uhlmann 2016] and the overview article [Kenig and Salo 2014] for known results, which either impose strong geometric restrictions on the inaccessible part of the boundary or require measurements of Dirichlet and Neumann data on sets that cover a neighborhood of the so-called front face

$$F(x_0) = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) \leq 0\}$$

for a point x_0 outside the closed convex hull of Ω. Also note that partial boundary data determines full boundary data by unique continuation if there exists a connected neighborhood of the full boundary on which the coefficient is known, so that uniqueness also holds in this case; see [Ammari and Uhlmann 2004].

Theorem 1.1, as well as the previous work [Harrach and Ullrich 2017], give uniqueness results where the measurements are made on an arbitrary open set $\Sigma \subseteq \partial\Omega$. Our result shows that a coefficient change in the positive or negative direction in a neighborhood of Σ (or an open subset of Σ) always leads to a change in the Neumann–Dirichlet-operator irrespectively of what happens outside this neighborhood, or the geometry or topology of the domain. Note however that our uniqueness result requires that there is a neighborhood of the boundary part on which the coefficient change is of definite sign. Our uniqueness result does not cover coefficient changes that are infinitely oscillating between positive and negative values when approaching the boundary.

Our result is based on combining monotonicity estimates (similar to those originally derived in [Kang et al. 1997; Ikehata 1998]) with localized potentials. Other theoretical uniqueness results have been obtained by this approach in [Arnold and Harrach 2013; Gebauer 2008; Harrach 2009; 2012; Harrach and Seo 2010; Harrach and Ullrich 2017]. Also note that monotonicity relations have been used in various ways in the study of inverse problems; see, e.g., [Kohn and Vogelius 1984; 1985; Isakov 1988; Alessandrini 1990; Ikehata 1999], where uniqueness results are established by methods that involve monotonicity conditions and blow-up arguments.

Monotonicity-based methods for detecting regions (or inclusions) where a coefficient function differs from a known background have been introduced by Tamburrino and Rubinacci [2002] for the inverse conductivity problem. In that paper they propose to simulate boundary measurements for a number of test regions and then use the fact that a monotonicity relation between the simulated and the true measurements will hold, if the test region lies inside the true inclusion. The work [Harrach and Ullrich 2013] used the technique of localized potentials [Gebauer 2008] to prove that this is really an if-and-only-if relation for the case of continuous measurements modeled by the NtD operator. Moreover, [Harrach and Ullrich 2013] also showed that this if-and-only-if relation still holds when the simulated measurements are replaced by linearized approximations so that the monotonicity method can be implemented without solving any forward problems other than that for the known background medium. For a list of recent works on monotonicity-based methods, let us refer to [Harrach et al. 2015; 2019; Harrach and Ullrich

2015; Harrach and Minh 2016; 2018; Maffucci et al. 2016; Tamburrino et al. 2016; Barth et al. 2017; Garde 2018; Garde and Staboulis 2017; 2019; Harrach and Lin 2017; Su et al. 2017; Ventre et al. 2017; Brander et al. 2018; Griesmaier and Harrach 2018; Zhou et al. 2018; Harrach and Meftahi 2019; Harrach 2019; Harrach and Lin 2019].

Previous monotonicity-based results often considered second-order equations with positive bilinear forms, such as the conductivity equation. So far, this positivity has been the key to proving monotonicity inequalities between the coefficient and the Neumann-to-Dirichlet operator, and previous results fail to hold in general for equations involving a positive frequency $k > 0$ (or a negative potential for the Schrödinger equation). In this article, we remove this limitation and introduce methods for more general elliptic models. We will focus on the Helmholtz equation in a bounded domain as a model case, but the ideas might be applicable to inverse boundary value and scattering problems for, e.g., Helmholtz, Maxwell, and elasticity equations. The main technical novelty of this work is that we treat compact perturbations of positive bilinear forms by extending the monotonicity relations to only hold true up to finitely many eigenvalues, and extend the localized potentials arguments to hold on spaces of finite codimension.

It should also be noted that the localized potentials arguments in [Gebauer 2008] stem from the ideas of the factorization method that was originally developed for scattering problems involving far-field measurements of the Helmholtz equation by Kirsch [1998], see also [Kirsch and Grinberg 2008], and then extended to the inverse conductivity problem in [Brühl and Hanke 2000; Brühl 2001]; see also the overview article [Harrach 2013]. For the inverse conductivity problem, the monotonicity method has the advantage over the factorization method that it allows a convergent regularized numerical implementation, see [Harrach and Ullrich 2013, Remark 3.5; Garde and Staboulis 2019], and that it can also be used for the indefinite case where anomalies of larger and smaller conductivity are present. The localized potentials approach in [Gebauer 2008] has recently been extended to show the possibility of localizing and concentrating electromagnetic fields in [Harrach et al. 2018].

The paper is structured as follows. In Section 2 we discuss the well-posedness of the Helmholtz equation outside resonance frequencies, introduce the Neumann-to-Dirichlet-operators, and give a unique continuation result from sets of positive measure. Sections 3 and 4 contain the main theoretical tools for this work. In Section 3, we introduce a Loewner order of compact self-adjoint operators that holds up to finitely many negative eigenvalues, and show that increasing the scattering index monotonically increases the Neumann-to-Dirichlet-operator in the sense of this new order. We also characterize the connection between the finite number of negative eigenvalues that have to be excluded in the Loewner ordering and the Neumann eigenvalues for the Laplacian. Section 4 extends the localized potentials result from [Gebauer 2008] to the Helmholtz equation and shows that the energy terms appearing in the monotonicity relation can be controlled in spaces of finite codimension. We give two independent proofs of this result, one using a functional analytic relation between operator norms and the ranges of their adjoints, and an alternative proof that is based on a Runge approximation argument. Sections 5 and 6 then contain the main results of this work on local uniqueness for the bounded Helmholtz equation and the detection of scatterers by monotonicity comparisons; see Theorem 1.1 and 1.2 above.

A preliminary version of these results has been published as the extended abstract [Harrach et al. 2017]. The bound on the number of negative eigenvalues in the monotonicity inequalities derived in this work has recently been improved in [Harrach et al. 2019].

2. The Helmholtz equation in a bounded domain

We start by summarizing some properties of the Neumann-to-Dirichlet-operators, discuss well-posedness and the role of resonance frequencies, and state a unique continuation result for the Helmholtz equation in a bounded domain.

2A. ***Neumann-to-Dirichlet operators.*** Throughout this work, let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, denote a bounded domain with Lipschitz boundary and outer unit normal ν, and let $\Sigma \subseteq \partial\Omega$ be an open subset of $\partial\Omega$. For a frequency $k \geq 0$ and a real-valued scattering coefficient function $q \in L^\infty(\Omega)$, we consider the Helmholtz equation with (partial) Neumann boundary data $g \in L^2(\Sigma)$, i.e., finding $u \in H^1(\Omega)$ with

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases} \tag{2}$$

We also denote the solution by $u_q^{(g)}$ instead of u if the choice of g and q is not clear from the context.

The Neumann problem (2) is equivalent to the variational formulation of finding $u \in H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla v - k^2 q u v)\,\mathrm{d}x = \int_{\partial\Omega} g v|_{\partial\Omega}\,\mathrm{d}s \quad \text{for all } v \in H^1(\Omega). \tag{3}$$

We introduce the bounded linear operators

$$\begin{aligned} I &: H^1(\Omega) \to H^1(\Omega), \\ j &: H^1(\Omega) \to L^2(\Omega), \\ M_q &: L^2(\Omega) \to L^2(\Omega), \end{aligned}$$

where I denotes the identity operator, j is the compact embedding from H^1 to L^2, and M_q is the multiplication operator by q. We furthermore use $\langle \cdot, \cdot \rangle$ to denote the $H^1(\Omega)$ inner product and define the operators

$$K := j^* j \quad \text{and} \quad K_q := j^* M_q j,$$

which are compact self-adjoint linear operators from $H^1(\Omega)$ to $H^1(\Omega)$. By

$$\gamma_\Sigma : H^1(\Omega) \to L^2(\Sigma), \quad v \mapsto v|_\Sigma,$$

we denote the compact trace operator.

With this notation (3) can be written as

$$\langle (I - K - k^2 K_q)u, v \rangle = \int_{\partial\Omega} g(\gamma_\Sigma v)\,\mathrm{d}s \quad \text{for all } v \in H^1(\Omega),$$

so that the Neumann problem for the Helmholtz equation (2) is equivalent to the equation

$$(I - K - k^2 K_q)u = \gamma_\Sigma^* g. \tag{4}$$

Our results on identifying the scattering coefficient q will require that $I - K - k^2 K_q$ is continuously invertible, which is equivalent to the fact that k *is not a resonance frequency*, or, equivalently; that 0 *is not a Neumann eigenvalue*, see Lemmas 2.2 and 3.10. Note that this implies, in particular, that $k > 0$ and $q \not\equiv 0$. We can then define the Neumann-to-Dirichlet operator (with Neumann data prescribed and Dirichlet data measured on the same open subset $\Sigma \subseteq \partial\Omega$)

$$\Lambda(q) : L^2(\Sigma) \to L^2(\Sigma), \quad g \mapsto u|_\Sigma, \quad \text{where } u \in H^1(\Omega) \text{ solves (2).}$$

The Neumann-to-Dirichlet operator satisfies

$$\Lambda(q) = \gamma_\Sigma (I - K - k^2 K_q)^{-1} \gamma_\Sigma^*, \tag{5}$$

which shows that $\Lambda(q)$ is a compact self-adjoint linear operator.

We will show in Section 3 that there is a monotonicity relation between the scattering coefficient q and the Neumann-to-Dirichlet-operator $\Lambda(q)$. Increasing q will increase $\Lambda(q)$ in the sense of operator definiteness up to finitely many eigenvalues. The number of eigenvalues that do not follow the increase will be bounded by the number defined in the following lemma. Note that here, and throughout the paper, we always count the number of eigenvalues of a compact self-adjoint operator with multiplicity according to the dimension of the associated eigenspaces.

Lemma 2.1. *Given $k > 0$ and $q \in L^\infty(\Omega)$, let $d(q)$ be the number of eigenvalues of $K + k^2 K_q$ that are larger than* 1, *and let $V(q)$ be the sum of the associated eigenspaces. Then $d(q) = \dim(V(q)) \in \mathbb{N}_0$ is finite, and*

$$\int_\Omega (|\nabla v|^2 - k^2 q |v|^2)\, \mathrm{d}x \geq 0 \quad \textit{for all } v \in V(q)^\perp,$$

where $V(q)^\perp$ denotes the orthocomplement of $V(q)$ in $H^1(\Omega)$.

Proof. Since

$$\langle (I - K - k^2 K_q) v, v \rangle = \int_\Omega (|\nabla v|^2 - k^2 q |v|^2)\, \mathrm{d}x,$$

the assertion follows from the spectral theorem for compact self-adjoint operators. □

We will show in Lemma 3.10 that $d(q)$ agrees with the number of positive Neumann eigenvalues of $\Delta + k^2 q$. If $q(x) \leq q_{\max} \in \mathbb{R}$ for all $x \in \Omega$ (a.e.) then $d(q) \leq d(q_{\max})$, and $d(q_{\max})$ is the number of Neumann eigenvalues of the Laplacian Δ that are larger than $-k^2 q_{\max}$; see Corollary 3.11.

2B. *Resonance frequencies.* We now summarize some results on the solvability of the Helmholtz equation (2) outside of resonance frequencies.

Lemma 2.2. *Let $q \in L^\infty(\Omega)$.*

(a) *For each $k \geq 0$, the following properties are equivalent*:

(i) *For each $F \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of*

$$(\Delta + k^2 q) u = F \quad \textit{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = g, \tag{6}$$

and the solution depends linearly and continuously on F and g.

(ii) *The homogeneous Neumann problem*

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = 0, \tag{7}$$

admits only the trivial solution $u \equiv 0$.

(iii) *The operator* $I - K - k^2 K_q : H^1(\Omega) \to H^1(\Omega)$ *is continuously invertible.*

We call k *a* ***resonance frequency*** *if the properties* (i)–(iii) *do not hold.*

(b) *If* $q \not\equiv 0$, *then the set of resonance frequencies is countable and discrete.*

Proof. (a) Clearly, (i) implies (ii), and, using the equivalence of (2) and (4), (ii) implies that $I - K - k^2 K_q$ is injective. Since K and K_q are compact, the operator $I - K - k^2 K_q$ is Fredholm of index 0. Hence, injectivity of $I - K - k^2 K_q$ already implies that $I - K - k^2 K_q$ is continuously invertible, so that (ii) implies (iii). Finally, $u \in H^1(\Omega)$ solves (6) if and only if

$$\int_\Omega (\nabla u \cdot \nabla v - k^2 quv)\,\mathrm{d}x = -\int_\Omega Fv\,\mathrm{d}x + \int_{\partial\Omega} gv|_{\partial\Omega}\,\mathrm{d}S \quad \text{for all } v \in H^1(\Omega).$$

This is equivalent to

$$\langle (I - K - k^2 K_q)u, v\rangle = -\int_\Omega Fj(v)\,\mathrm{d}x + \int_{\partial\Omega} g\gamma_{\partial\Omega}(v)\,\mathrm{d}s \quad \text{for all } v \in H^1(\Omega),$$

and thus equivalent to

$$(I - K - k^2 K_q)u = -j^* F + \gamma_{\partial\Omega}^* g,$$

so that (iii) implies (i).

(b) We extend I, K, and K_q to the Sobolev space of complex-valued functions

$$I, K, K_q : H^1(\Omega; \mathbb{C}) \to H^1(\Omega; \mathbb{C}).$$

For $k \in \mathbb{C}$ we then define

$$R(k) := K + k^2 K_q : H^1(\Omega; \mathbb{C}) \to H^1(\Omega; \mathbb{C}).$$

$R(k)$ is a family of compact operators depending analytically on $k \in \mathbb{C}$. The analytic Fredholm theorem, see, e.g., [Reed and Simon 1972, Theorem VI.14], now implies that either $I - R(k)$ is not invertible for all $k \in \mathbb{C}$, or that there is a countable discrete set $Z \subseteq \mathbb{C}$ such that $I - R(k)$ is continuously invertible when $k \in \mathbb{C} \setminus Z$. Hence, to prove (b), it suffices to show that there exists $k \in \mathbb{C}$ for which $I - R(k)$ is invertible.

We will show that this is the case for any $0 \neq k \in \mathbb{C}$ with $\mathrm{Re}(k^2) = 0$. In fact, $(I - R(k))u = 0$ implies

$$0 = \int_\Omega (\nabla u \cdot \nabla v - k^2 quv)\,\mathrm{d}x \quad \text{for all } v \in H^1(\Omega; \mathbb{C}).$$

Using $v := \bar{u}$ and taking the real part yields that $0 = \int_\Omega |\nabla u|^2\,\mathrm{d}x$, which shows that u must be constant, and that

$$\int_\Omega k^2 quv\,\mathrm{d}x = 0 \quad \text{for all } v \in H^1(\Omega; \mathbb{C}).$$

Together with $k^2 \neq 0$, and $q \not\equiv 0$, this shows that $u \equiv 0$. Hence, $I - R(k)$ is injective and thus invertible for all $0 \neq k \in \mathbb{C}$ with $\mathrm{Re}(k^2) = 0$. □

2C. *Unique continuation.* We will make use of a unique continuation property for the Helmholtz equation from sets of positive measure. In two dimensions, this follows from a standard reduction to quasiconformal mappings. However, since we could not find a proof in the literature, we will first give the argument following [Alessandrini 2012] and references therein (in fact [Alessandrini 2012] proves strong unique continuation for more general equations). See also [Astala et al. 2009] for basic facts on quasiconformal mappings in the plane.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a connected open set, and suppose that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution of*

$$-\operatorname{div}(A\nabla u) + du = 0 \quad \text{in } \Omega,$$

where $A \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ is symmetric and satisfies $A(x)\xi \cdot \xi \geq c_0|\xi|^2$ for some $c_0 > 0$, and $d \in L^{q/2}(\Omega)$ for some $q > 2$. If u vanishes in a set E of positive measure, then $u \equiv 0$ in Ω.

Proof. It is enough to show that u vanishes in some ball, since then weak (or strong) unique continuation [Alessandrini 2012] implies that $u \equiv 0$. Let x_0 be a point of density 1 in E and let $U_r := B_r(x_0)$ and $E_r := E \cap U_r$. There is $r_0 > 0$ so that if $r < r_0$, then $U_r \subset \Omega$ and E_r has positive measure.

We will now work in U_r. Observe first that there is $p > 2$ so that $u \in W^{1,p}(U_r)$ [Astala et al. 2009, Theorem 16.1.4]. In particular u is Hölder continuous and we may assume (after removing a set of measure zero from E) that $u(x) = 0$ for all $x \in E_r$. The first step is to show that $\nabla u = 0$ a.e. on E_r. Let N_1 be the set of points in E_r where u is not differentiable, and let N_2 be the set of points of density < 1 in E_r. Then N_1 and N_2 have zero measure. Fix a point $x \in E_r \setminus (N_1 \cup N_2)$ and a unit direction e. There is a sequence (x_j) with $x_j \in B(x, 1/j) \cap E_r$ so that $\left|(x_j - x)/|x_j - x| - e\right| \leq 1/j$ for j large (for if not, then all points in E_r near x would be outside a fixed sector in direction e, which contradicts the fact that x has density 1). Since u is differentiable at x,

$$u(x_j) - u(x) = \nabla u(x) \cdot (x_j - x) + o(|x - x_j|).$$

Dividing by $|x - x_j|$ and using that $u(x_j) = u(x) = 0$ implies that $\nabla u(x) \cdot e = 0$. It follows that ∇u vanishes in $E_r \setminus (N_1 \cup N_2)$, so indeed

$$u = 0 \quad \text{in } E_r, \qquad \nabla u = 0 \quad \text{a.e. in } E_r. \tag{8}$$

The next step is to reduce to the case where $d = 0$. As in [Alessandrini 2012, Proposition 2.4], we choose r small enough so that there is a nonvanishing $w \in W^{1,p}(U_r)$ satisfying

$$-\operatorname{div}(A\nabla w) + dw = 0 \quad \text{in } U_r,$$
$$\tfrac{1}{2} \leq w \leq 2 \quad \text{in } U_r, \qquad \|\nabla w\|_{L^p(U_r)} \leq 1.$$

We write $v = u/w$. It follows that $v \in W^{1,p}(U_r)$ is a weak solution of

$$-\operatorname{div}(\tilde{A}\nabla v) = 0 \quad \text{in } U_r,$$

where $\tilde{A} = w^2 A$ is L^∞ and uniformly elliptic. Moreover, (8) implies that

$$v = 0 \quad \text{in } E_r, \qquad \nabla v = 0 \quad \text{a.e. in } E_r. \tag{9}$$

To prove the lemma, we will show that $v \equiv 0$ in some ball.

Let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $\tilde{A}\nabla v$ is divergence-free, there is a real-valued function $\tilde{v} \in H^1(U_r)$ satisfying

$$\nabla \tilde{v} = J(\tilde{A}\nabla v). \tag{10}$$

Such a function $\tilde{v}$ is unique up to an additive constant. Define

$$f = v + i\tilde{v}.$$

As in [Alessandrini 2012], $f \in H^1(U_r)$ solves an equation of the form

$$\partial_{\bar{z}} f = \mu \partial_z f + \nu \bar{\partial}_z f \quad \text{in } U_r,$$

where $\|\mu\|_{L^\infty(U_r)} + \|\nu\|_{L^\infty(U_r)} < 1$. It follows that f is a quasiregular map and by the Stoilow factorization [Astala et al. 2009, Theorem 5.5.1] it has the representation

$$f(z) = F(\chi(z)), \quad z \in U,$$

where χ is a quasiconformal map $\mathbb{C} \to \mathbb{C}$ and F is a holomorphic function on $\chi(U)$.

Finally, the Jacobian determinant J_f of f is given by

$$J_f(z) = F'(\chi(z)) J_\chi(z).$$

Using (9) and (10), we see that $J_f = 0$ a.e. in E_r. Moreover, since χ is quasiconformal, J_χ can only vanish in a set of measure zero [Astala et al. 2009, Corollary 3.7.6]. It follows that $F'(\chi(z)) = 0$ for a.e. $z \in E_r$. Then the Taylor series of the analytic function F' at $\chi(x_0)$ must vanish (otherwise one would have $F'(\chi(z)) = (\chi(z) - \chi(x_0))^N g(\chi(z))$, where $g(\chi(x_0)) \neq 0$ and the only zero near x_0 would be $z = x_0$). Thus $F' = 0$ near x_0, so F is constant, f is also constant, and $v = 0$ near x_0. □

We can now state the unique continuation property for any dimension $n \geq 2$ in the form that we will utilize in the later sections. As in [Harrach and Ullrich 2013, Definition 2.2] we say that a relatively open subset $O \subseteq \overline{\Omega}$ is *connected to* Σ if O is connected and $\Sigma \cap O \neq \varnothing$.

Theorem 2.4. (a) *Let $u \in H^1(\Omega)$ solve*

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega. \tag{11}$$

If $u|_E = 0$ for a subset $E \subseteq \Omega$ with positive measure then $u(x) = 0$ for all $x \in \Omega$ (a.e.)

(b) *Let $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$, and*

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega \setminus C$$

for a closed set C for which $\overline{\Omega} \setminus C$ is connected to Σ. If $u|_\Sigma = 0$ and $\partial_\nu u|_\Sigma = 0$, then $u(x) = 0$ for all $x \in \Omega \setminus C$ (a.e.)

Proof. For $n = 2$, (a) follows from Lemma 2.3. For $n \geq 3$, (a) is shown in [Harrach and Ullrich 2017, Theorem 4.2] (see also [Regbaoui 2001, proof of Theorem 2.1]) by combining the following two results:

(i) If $u \in H^1(\Omega)$ solves (11) and vanishes on a measurable set of positive measure then u has a zero of infinite order; see, e.g., [de Figueiredo and Gossez 1992, Proposition 3; Hadi and Tsouli 2001, Theorem 2.1].

(ii) The trivial solution $u \equiv 0$ is the only $H^1(\Omega)$-solution of (11) that has a zero of infinite order; see, e.g, [Hörmander 1985, Theorem 17.2.6].

Part (b) follows from (a) by extending u by zero on $B \setminus \Omega$, where B is a small ball with $B \cap \partial\Omega \subseteq \Sigma$; see the proof of Lemma 4.4(c) in [Harrach and Ullrich 2017]. □

3. Monotonicity and localized potentials for the Helmholtz equation

In this section we show that increasing the scattering coefficient leads to a larger Neumann-to-Dirichlet operator in a certain sense. For this result, the Neumann-to-Dirichlet operators are ordered by an extension of the Loewner order of compact self-adjoint operators that holds up to finitely many negative eigenvalues.

3A. ***A Loewner order up to finitely many eigenvalues.*** We start by giving a rigorous definition and characterization of this ordering.

Definition 3.1. Let $A, B : X \to X$ be two self-adjoint compact linear operators on a Hilbert space X. For a number $d \in \mathbb{N}_0$, we write

$$A \leq_d B \quad \text{or} \quad \langle Ax, x\rangle \leq_d \langle Bx, x\rangle$$

if $B - A$ has at most d negative eigenvalues. We also write $A \leq_{\text{fin}} B$ if $A \leq_d B$ holds for some $d \in \mathbb{N}_0$, and we write $A \leq B$ if $A \leq_d B$ holds for $d = 0$.

Note that for $d = 0$ this is the standard partial ordering of compact self-adjoint operators in the sense of operator definiteness (also called Loewner order). Also note that "$\leq_{\text{fin}}$" and "$\leq_d$" (for $d \neq 0$) are not partial orders since they are clearly not antisymmetric. Obviously, "$\leq_{\text{fin}}$" and "$\leq_d$" are reflexive, and "$\leq_{\text{fin}}$" is also transitive (see Lemma 3.4 below) and thus a so-called preorder.

To characterize this new ordering, we will make use of the following lemma.

Lemma 3.2. *Let $A : X \to X$ be a self-adjoint compact linear operator on a Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ inducing the norm $\|\cdot\|$. Let $d \in \mathbb{N}_0$ and $r \in \mathbb{R}$, $r \geq 0$.*

(a) *The following statements are equivalent*:

(i) *A has at most d eigenvalues larger than r.*

(ii) *There exists a compact self-adjoint operator $F : X \to X$ with*

$$\dim(\mathcal{R}(F)) \leq d \qquad \textit{and} \qquad \langle (A - F)x, x\rangle \leq r\|x\|^2 \quad \textit{for all } x \in X,$$

where $\mathcal{R}(F)$ stands for the range of F.

(iii) *There exists a subspace* $W \subset X$ *with* $\operatorname{codim}(W) \le d$ *such that*

$$\langle Aw, w\rangle \le r\|w\|^2 \quad \textit{for all } w \in W.$$

(iv) *There exists a subspace* $V \subset X$ *with* $\dim(V) \le d$ *such that*

$$\langle Av, v\rangle \le r\|v\|^2 \quad \textit{for all } v \in V^\perp.$$

(b) *The following statements are equivalent*:

(i) A *has* (*at least*) d *eigenvalues larger than* r.

(ii) *There exists a subspace* $V \subset X$ *with* $\dim(V) \ge d$ *such that*

$$\langle Av, v\rangle > r\|v\|^2 \quad \textit{for all } v \in V.$$

Proof. (a) We start by showing that (i) implies (ii). Let A have at most d eigenvalues larger than $r \ge 0$. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the nonzero eigenvalues of A, ordered in such a way that $\lambda_k \le r$ for $k > d$. Let $\mathcal{N}(A)$ denote the kernel of A and let $(v_k)_{k\in\mathbb{N}} \in X$ be a sequence of corresponding eigenvectors forming an orthonormal basis of $\mathcal{N}(A)^\perp$. Then

$$Ax = \sum_{k=1}^{\infty} \lambda_k v_k \langle v_k, x\rangle \quad \text{for all } x \in X,$$

and (ii) follows with $F : X \to X$ defined by

$$F : x \mapsto \sum_{k=1}^{d} \lambda_k v_k \langle v_k, x\rangle \quad \text{for all } x \in X.$$

The implication from (ii) to (iii) follows by setting $W := \mathcal{N}(F)$ since

$$\operatorname{codim}(W) = \dim(W^\perp) = \dim(\mathcal{R}(F)) \le d$$

and

$$\langle Aw, w\rangle = \langle (A - F)w, w\rangle \ge 0.$$

Part (iii) implies (iv) by setting $V := W^\perp$.

To show that (iv) implies (i), we assume that (i) is not true, so that A has at least $d + 1$ eigenvalues larger than $r \ge 0$. We sort the positive eigenvalues of A in decreasing order to obtain

$$\lambda_1 \ge \cdots \ge \lambda_d \ge \lambda_{d+1} > r.$$

Then, by the Courant–Fischer–Weyl min-max principle, see, e.g., [Lax 2002, p. 318], we have that the minimum over all d-dimensional subspaces $V \subset X$ must satisfy

$$\min_{\substack{V\subset X\\ \dim(V)=d}} \max_{\substack{v\in V^\perp\\ \|v\|=1}} \langle Av, v\rangle = \lambda_{d+1} > r,$$

which shows that (iv) cannot be true. Hence, (iv) implies (i).

(b) This can be shown analogously to (a). Part (ii) follows from (i) by choosing V as the sum of eigenspaces for eigenvalues larger than r, and (ii) implies (i) by using the Courant–Fischer–Weyl min-max principle. □

Corollary 3.3. *Let $A, B : X \to X$ be two self-adjoint compact linear operators on a Hilbert space X with inner product $\langle \cdot, \cdot \rangle$. For any number $d \in \mathbb{N}_0$, the following statements are equivalent*:

(a) $A \leq_d B$.

(b) *There exists a compact self-adjoint operator $F : X \to X$ with*

$$\dim(\mathcal{R}(F)) \leq d \qquad \text{and} \qquad \langle (B - A + F)x, x \rangle \geq 0 \quad \text{for all } x \in X.$$

(c) *There exists a subspace $W \subset X$ with* $\operatorname{codim}(W) \leq d$ *such that*

$$\langle (B - A)w, w \rangle \geq 0 \quad \text{for all } w \in W.$$

(d) *There exists a subspace $V \subset X$ with* $\dim(V) \leq d$ *such that*

$$\langle (B - A)v, v \rangle \geq 0 \quad \text{for all } v \in V^{\perp}.$$

Proof. This follows from Lemma 3.2(a) with $r = 0$ and A replaced by $A - B$. □

Lemma 3.4. *Let $A, B, C : X \to X$ be self-adjoint compact linear operators on a Hilbert space X. For $d_1, d_2 \in \mathbb{N}_0$*

$$A \leq_{d_1} B \quad \text{and} \quad B \leq_{d_2} C \qquad \text{implies} \qquad A \leq_{d_1+d_2} C,$$
$$A \leq_{\text{fin}} B \quad \text{and} \quad B \leq_{\text{fin}} C \qquad \text{implies} \qquad A \leq_{\text{fin}} C.$$

Proof. This follows from the characterization in Corollary 3.3(b). □

3B. *A monotonicity relation for the Helmholtz equation.* With this new ordering, we can show a monotonicity relation between the scattering index and the Neumann-to-Dirichlet-operators. Note that the dimension bound in the last line of the following theorem has recently been improved to $d(q_2) - d(q_1)$ in [Harrach et al. 2019].

Theorem 3.5. *Let $q_1, q_2 \in L^\infty(\Omega) \setminus \{0\}$. Assume that $k > 0$ is not a resonance for q_1 or q_2, and let $d(q_2) \in \mathbb{N}_0$ be defined as in Lemma 2.1.*

Then there exists a subspace $V \subset L^2(\Sigma)$ with $\dim(V) \leq d(q_2)$ *such that*

$$\int_\Sigma g(\Lambda(q_2) - \Lambda(q_1))g \, \mathrm{d}s \geq \int_\Omega k^2(q_2 - q_1)|u_1^{(g)}|^2 \, \mathrm{d}x \quad \text{for all } g \in V^{\perp}.$$

In particular

$$q_1 \leq q_2 \quad \text{implies} \quad \Lambda(q_1) \leq_{d(q_2)} \Lambda(q_2).$$

Remark 3.6. Note that by interchanging q_1 and q_2, Theorem 3.5 also yields that there exists a subspace $V \subset L^2(\Sigma)$ with $\dim(V) \leq d(q_1)$ such that

$$\int_\Sigma g(\Lambda(q_2) - \Lambda(q_1))g \, \mathrm{d}s \leq \int_\Omega k^2(q_2 - q_1)|u_2^{(g)}|^2 \, \mathrm{d}x \quad \text{for all } g \in V^{\perp}.$$

To prove Theorem 3.5 we will use the following lemmas.

Lemma 3.7. *Let $q_1, q_2 \in L^\infty(\Omega) \setminus \{0\}$. Assume that $k > 0$ is not a resonance for q_1 or q_2. Then, for all $g \in L^2(\Sigma)$,*

$$\int_\Sigma g(\Lambda(q_2) - \Lambda(q_1))g \,\mathrm{d}s + \int_\Omega k^2(q_1 - q_2)|u_1^{(g)}|^2 \,\mathrm{d}x = \int_\Omega (|\nabla(u_2^{(g)} - u_1^{(g)})|^2 - k^2 q_2 |u_2^{(g)} - u_1^{(g)}|^2) \,\mathrm{d}x,$$

where $u_1^{(g)}, u_2^{(g)}$ solve the Helmholtz equation (2) with Neumann boundary data g and $q = q_1,\ q = q_2$ respectively.

Proof. Define the bilinear form

$$B_q(u, v) = \int_\Omega (\nabla u \cdot \nabla v - k^2 q u v) \,\mathrm{d}x, \quad u, v \in H^1(\Omega).$$

Writing $u_1 = u_1^{(g)}$ and $u_2 = u_2^{(g)}$, from the definition of the NtD map and from (3) we have

$$\int_\Sigma g\Lambda(q_1)g \,\mathrm{d}s = \int_\Sigma (\partial_\nu u_1)u_1 \,\mathrm{d}s = 2\int_\Sigma (\partial_\nu u_2)u_1 \,\mathrm{d}s - \int_\Sigma (\partial_\nu u_1)u_1 \,\mathrm{d}s = 2B_{q_2}(u_2, u_1) - B_{q_1}(u_1, u_1)$$

and

$$\int_\Sigma g\Lambda(q_2)g \,\mathrm{d}s = \int_\Sigma (\partial_\nu u_2)u_2 \,\mathrm{d}s = B_{q_2}(u_2, u_2).$$

We thus obtain that

$$\begin{aligned}\int_\Sigma g(\Lambda(q_2) - \Lambda(q_1))g \,\mathrm{d}s &= B_{q_2}(u_2, u_2) - 2B_{q_2}(u_2, u_1) + B_{q_1}(u_1, u_1)\\ &= B_{q_2}(u_2 - u_1, u_2 - u_1) - B_{q_2}(u_1, u_1) + B_{q_1}(u_1, u_1).\end{aligned}$$

□

We will show that the bilinear forms in the right-hand sides in Lemma 3.7 are positive up to a finite-dimensional subspace.

Lemma 3.8. *Let $q_1, q_2 \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. There exists a subspace $V \subset L^2(\Sigma)$ with $\dim(V) \le d(q_2)$ such that for all $g \in V^\perp$*

$$\int_\Omega \left(|\nabla(u_2^{(g)} - u_1^{(g)})|^2 - k^2 q_2 |u_2^{(g)} - u_1^{(g)}|^2\right) \mathrm{d}x \ge 0.$$

Proof. Using Lemma 2.1, we have

$$\int_\Omega \left(|\nabla(u_2^{(g)} - u_1^{(g)})|^2 - k^2 q_2 |u_2^{(g)} - u_1^{(g)}|^2\right) \mathrm{d}x \ge 0$$

for all $g \in L^2(\Sigma)$ with $u_2^{(g)} - u_1^{(g)} \in V(q_2)^\perp$. The solution operators

$$S_j : L^2(\Sigma) \to H^1(\Omega), \quad g \mapsto u_j^{(g)}, \quad \text{where } u_j^{(g)} \in H^1(\Omega) \text{ solves (2)}, \quad j \in \{1, 2\},$$

are linear and bounded, and

$$(S_2 - S_1)g = u_2^{(g)} - u_1^{(g)} \in V(q_2)^\perp \quad \text{if and only if} \quad g \in ((S_2 - S_1)^* V(q_2))^\perp.$$

Since $\dim(S_2 - S_1)^* V(q_2) \le \dim V(q_2) = d(q_2)$, the assertion follows with $V := (S_2 - S_1)^* V(q_2)$. □

Proof of Theorem 3.5. This now immediately follows from combining Lemmas 3.7 and 3.8. □

3C. *The number of negative eigenvalues.* We will now further investigate the number $d(q) \in \mathbb{N}_0$ (defined in Lemma 2.1) that bounds the number of negative eigenvalues in the monotonicity relations derived in Section 3B. We will show that $d(q)$ depends monotonously on the scattering index q and show that $d(q)$ is less than or equal to the number of Neumann eigenvalues for the Laplacian which are larger than $-k^2 q_{\max}$, where $q_{\max} \geq q(x)$ for all $x \in \Omega$ (a.e.).

Lemma 3.9. *Let $q_1, q_2 \in L^\infty(\Omega)$. Then $q_1 \leq q_2$ implies $d(q_1) \leq d(q_2)$.*

Proof. The inequality $q_1 \leq q_2$ implies that $K_{q_1} \leq K_{q_2}$. Hence, the assertion follows from the equivalence of (a) and (c) in Corollary 3.3. □

Lemma 3.10. *Let $q \in L^\infty(\Omega)$, and $k \in \mathbb{R}$.*

(*a*) *There is a countable and discrete set of real values*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \to -\infty$$

(*called* ***Neumann eigenvalues***) *such that*

$$(\Delta + k^2 q)u = \lambda u \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = 0, \tag{12}$$

admits a nontrivial solution (*called a* ***Neumann eigenfunction***) $0 \not\equiv u \in H^1(\Omega)$ *if and only if* $\lambda \in \{\lambda_1, \lambda_2, \ldots\}$, *and there is an orthonormal basis* $(u_1, u_2, \ldots)$ *of* $L^2(\Omega)$ *such that* $u_j \in H^1(\Omega)$ *is a Neumann eigenfunction for* λ_j.

(b) *If* λ *is not a Neumann eigenvalue, then the problem*

$$(\Delta + k^2 q)u = \lambda u + F \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = g, \tag{13}$$

has a unique solution $u \in H^1(\Omega)$ *for any* $F \in L^2(\Omega)$ *and* $g \in L^2(\partial\Omega)$, *and the solution operator is linear and bounded.*

(c) *Let* $N_+ := \operatorname{span}\{u_j : \lambda_j > 0\}$. *Then* $\dim(N_+) < \infty$,

$$N_- := \overline{\operatorname{span}\{u_j : \lambda_j \leq 0\}} = \{v \in H^1(\Omega) : v \perp_{L^2} N_+\} \tag{14}$$

is a complement of N_+ (*in* $H^1(\Omega)$), *and*

$$\int_\Omega |\nabla v|^2 - k^2 q v^2 \, \mathrm{d}x < 0 \quad \text{for all } v \in N_+, \tag{15}$$

$$\int_\Omega |\nabla v|^2 - k^2 q v^2 \, \mathrm{d}x \geq 0 \quad \text{for all } v \in N_-, \tag{16}$$

where the closure in (14) is taken with respect to the $H^1(\Omega)$*-norm, and* $\perp_{L^2}$ *denotes orthogonality with respect to the* L^2 *inner product.*

(d) $d(q)$ *is the number of positive Neumann eigenvalues of* $\Delta + k^2 q$; *i.e.*, $d(q) = \dim(N_+)$.

(e) 0 *is a Neumann eigenvalue if and only if* $k > 0$ *is a resonance frequency.*

Proof. (a) Define $c := k^2\|q\|_{L^\infty(\Omega)} + 1 > 0$ and $R := I - K - k^2K_q + cK$. Then R is coercive and thus continuously invertible. Using the equivalent variational formulation of (12), we have that $\lambda \in \mathbb{R}$ is a Neumann eigenvalue with Neumann eigenfunction $u \not\equiv 0$ if and only if

$$\int_\Omega (-\nabla u \cdot \nabla v + k^2 quv)\,\mathrm{d}x = \lambda \int_\Omega uv\,\mathrm{d}x \quad \text{for all } v \in H^1(\Omega),$$

which is equivalent to

$$(I - K - k^2K_q)u = -\lambda Ku$$

and thus to

$$Ru = (I - K - k^2K_q + cK)u = (c - \lambda)Ku. \tag{17}$$

This shows that c cannot be a Neumann eigenvalue since $Ru \not\equiv 0$ for $u \not\equiv 0$. Moreover, using $K = j^*j$, the invertibility of R, and the injectivity of j, we have that (17) is equivalent to

$$\frac{1}{c-\lambda}(ju) = jR^{-1}j^*(ju).$$

This shows that $\lambda \in \mathbb{R}$ is a Neumann eigenvalue with Neumann eigenfunction $u \in H^1(\Omega)$ if and only if $ju \in L^2(\Omega)$ is an eigenfunction of $jR^{-1}j^* : L^2(\Omega) \to L^2(\Omega)$ with eigenvalue $1/(c-\lambda)$. Since j is injective, and every eigenfunction of $jR^{-1}j^*$ lies in the range of j, this is a one-to-one correspondence, and the dimension of the corresponding eigenspaces is the same. Since $jR^{-1}j^*$ is a compact, self-adjoint, positive operator, the assertions in (a) follow from the spectral theorem on self-adjoint compact operators.

(b) This follows from the fact that $I - K - k^2K_q - \lambda K$ is Fredholm of index 0 and thus continuously invertible if it is injective.

(c) $\dim(N_+) < \infty$ follows from (a). We define

$$N_- := \overline{\operatorname{span}\{u_j : \lambda_j \le 0\}} \quad \text{and} \quad \widetilde{N}_- := \{v \in H^1(\Omega) : v \perp_{L^2} N_+\}.$$

$\widetilde{N}_-$ is closed with respect to the H^1-norm and contains all u_j with $\lambda_j \le 0$, so that $N_- \subseteq \widetilde{N}_-$. To show $N_- = \widetilde{N}_-$, we argue by contradiction. If $N_- \subsetneq \widetilde{N}_-$, then there would exist a $0 \ne v \in \widetilde{N}_-$ with $\langle u_j, v\rangle = 0$ for all u_j with $\lambda_j \le 0$. Using

$$\begin{aligned} 0 = \langle u_j, v\rangle &= \int_\Omega (\nabla u_j \cdot \nabla v + u_j v)\,\mathrm{d}x \\ &= \int_\Omega (\nabla u_j \cdot \nabla v - k^2 q u_j v)\,\mathrm{d}x + \int_\Omega (1 + k^2 q) u_j v\,\mathrm{d}x \\ &= \int_\Omega (1 + k^2 q - \lambda_j) u_j v\,\mathrm{d}x, \end{aligned}$$

and the fact that $\lambda_j \to -\infty$, it would follow that $v \perp_{L^2} u_j$ for all but finitely many u_j. Since $v \perp_{L^2} N_+$, and $(u_1, u_2, \ldots)$ is an orthonormal basis of $L^2(\Omega)$, v must then be a finite combination of u_j with $\lambda_j \le 0$, which would imply that $v = 0$. Hence, $N_- = \widetilde{N}_-$, so that the equality in (14) is proven.

Obviously, $N_+ \cap N_- = 0$ and every $v \in H^1(\Omega)$ can be written as

$$v = \sum_{\lambda_j > 0} \left(\int_\Omega v u_j \, \mathrm{d}x \right) u_j + \left(v - \sum_{\lambda_j > 0} \left(\int_\Omega v u_j \, \mathrm{d}x \right) u_j \right) \in N_+ + N_-,$$

which shows that N_- is a complement of N_+.

To show (15), we use the L^2-orthogonality of the u_j to obtain for all $v = \sum_{\lambda_j > 0} \alpha_j u_j \in N_+$

$$\begin{aligned} \int_\Omega (|\nabla v|^2 - k^2 q v v) \, \mathrm{d}x &= \sum_{\lambda_j > 0} \alpha_j \int_\Omega (\nabla u_j \cdot \nabla v - k^2 q u_j v) \, \mathrm{d}x \\ &= - \sum_{\lambda_j > 0} \alpha_j \lambda_j \int_\Omega u_j v \, \mathrm{d}x = - \sum_{\lambda_j > 0} \alpha_j^2 \lambda_j \int_\Omega u_j^2 \, \mathrm{d}x < 0. \end{aligned}$$

Since every $v \in N_-$ is an $H^1(\Omega)$-limit of finite linear combinations of u_j with $\lambda_j \le 0$, (16) follows with the same argument.

(d) Inequality (15) can be written as

$$\langle (K + k^2 K_q) v, v \rangle > \|v\|^2 \quad \text{for all } v \in N_+.$$

Lemma 3.2(b) implies that the number $d(q)$ of eigenvalues of $K + k^2 K_q$ larger than 1 must be at least $\dim(N_+)$. Likewise, (16) can be written as

$$\langle (K + k^2 K_q) v, v \rangle \le \|v\|^2 \quad \text{for all } v \in N_-.$$

Hence, Lemma 3.2(a) shows that $d(q)$ is at most $\operatorname{codim}(N_-) = \dim(N_+)$.

(e) This is trivial. □

Corollary 3.11. *If $q \in L^\infty(\Omega)$ and $q(x) \le q_{\max} \in \mathbb{R}$ for all $x \in \Omega$ (a.e.), then $d(q) \le d(q_{\max})$, and $d(q_{\max})$ is the number of Neumann eigenvalues of the Laplacian Δ that are larger than $-k^2 q_{\max}$.*

Proof. Obviously, the number of positive Neumann eigenvalues of $\Delta + k^2 q_{\max}$ agrees with the number of Neumann eigenvalues of the Laplacian Δ that are greater than $-k^2 q_{\max}$. Hence, the assertion follows from Lemmas 3.9 and 3.10(d). □

Remark 3.12. One can show, by using constant potentials, that for the Helmholtz equation $\Lambda_{q_2} - \Lambda_{q_1}$ can actually have negative eigenvalues when $q_1 \le q_2$. This shows that in Theorem 3.5 it is indeed necessary to work modulo a finite-dimensional subspace. The details will appear in a subsequent work.

4. Localized potentials for the Helmholtz equation

We now extend the result in [Gebauer 2008] to the Helmholtz equation and prove that we can control the energy terms appearing in the monotonicity relation in spaces of finite codimension. We will first state the result and prove it using a functional analytic relation between operator norms and the ranges of their adjoints in Section 4A. Section 4B then gives an alternative proof that is based on a Runge approximation argument.

4A. *Localized potentials.* Our main result on controlling the solutions of the Helmholtz equation in spaces of finite codimension is the following theorem.

Theorem 4.1. *Let $q \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. Let $B, D \subseteq \overline{\Omega}$ be measurable, $B \setminus \overline{D}$ possess positive measure, and $\overline{\Omega} \setminus \overline{D}$ be connected to Σ.*

Then for any subspace $V \subset L^2(\Sigma)$ with $\dim V < \infty$, there exists a sequence $(g_j)_{j\in\mathbb{N}} \subset V^\perp$ such that

$$\int_B |u_q^{(g_j)}|^2 \,\mathrm{d}x \to \infty \quad \text{and} \quad \int_D |u_q^{(g_j)}|^2 \,\mathrm{d}x \to 0,$$

where $u_q^{(g_j)} \in H^1(\Omega)$ solves the Helmholtz equation (2) with Neumann boundary data g_j.

The arguments that we will use to prove Theorem 4.1 in this subsection also yield a simple proof for the following elementary result. We formulate it as a theorem since we will utilize it in the next section to control energy terms in monotonicity inequalities for different scattering coefficients.

Theorem 4.2. *Let $q_1, q_2 \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. If $q_1(x) = q_2(x)$ for all x (a.e.) outside a measurable set $D \subset \Omega$, then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \int_D |u_1^{(g)}|^2 \,\mathrm{d}x \le \int_D |u_2^{(g)}|^2 \,\mathrm{d}x \le c_2 \int_D |u_1^{(g)}|^2 \,\mathrm{d}x \quad \text{for all } g \in L^2(\Sigma),$$

where $u_1^{(g)}, u_2^{(g)} \in H^1(\Omega)$ solve the Helmholtz equation (2) with Neumann boundary data g and $q = q_1$, $q = q_2$ respectively.

To prove Theorems 4.1 and 4.2 we will formulate and prove several lemmas. Let us first note that the assertion of Theorem 4.1 already holds if we can prove it for a subset of B with positive measure. We will use the subset $B \cap C$, where C is a small closed ball constructed in the next lemma.

Lemma 4.3. *Let $B, D \subseteq \overline{\Omega}$ be measurable, $B \setminus \overline{D}$ possess positive measure, and $\overline{\Omega} \setminus \overline{D}$ be connected to Σ. Then there exists a closed ball C such that $B \cap C$ has positive measure, $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (\overline{D} \cup C)$ is connected to Σ.*

Proof. Let x be a point of Lebesgue density 1 in $B \setminus \overline{D}$. Then the closure C of a sufficiently small ball centered in x will satisfy $B \cap C$ has positive measure, $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (\overline{D} \cup C)$ is connected to Σ. □

Now we follow the general approach in [Gebauer 2008]. We formulate the energy terms in Theorem 4.1 as norms of operator evaluations and characterize their adjoints. Then we characterize the ranges of the adjoints using the unique continuation property, and prove Theorem 4.1 using a functional-analytic relation between norms of operator evaluations and ranges of their adjoints.

Lemma 4.4. *Let $q \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. For a measurable set $D \subset \Omega$ we define*

$$L_D : L^2(\Sigma) \to L^2(D), \quad g \mapsto u|_D,$$

where $u \in H^1(\Omega)$ solves (2). Then L_D is a compact linear operator, and its adjoint satisfies

$$L_D^* : L^2(D) \to L^2(\Sigma), \quad f \mapsto v|_\Sigma,$$

where v solves

$$\Delta v + k^2 q v = f\chi_D, \quad \partial_\nu v|_{\partial\Omega} = 0. \tag{18}$$

Proof. With the operators I, j, and K_q defined as in Section 2A and (4) we have

$$L_D = R_D j (I - K - k^2 K_q)^{-1} \gamma_\Sigma^*,$$

where $R_D : L^2(\Omega) \to L^2(D)$ is the restriction operator $v \to v|_D$. Hence, L_D is a linear compact operator, and its adjoint is

$$L_D^* = \gamma_\Sigma (I - K - k^2 K_q)^{-1} j^* R_D^*.$$

Thus $L_D^* f = v|_\Sigma$, where $v \in H^1(\Omega)$ solves

$$(I - K - k^2 K_q) v = j^* R_D^* f;$$

i.e., for all $w \in H^1(\Omega)$,

$$\int_\Omega (\nabla v \cdot \nabla w - k^2 q v w)\,\mathrm{d}x = \langle (I - K - k^2 K_q) v, w\rangle = \langle j^* R_D^* f, w\rangle = \int_D f w\,\mathrm{d}x,$$

which is the variational formulation equivalent to (18). □

Lemma 4.5. *Let $q \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. Let $B, D \subseteq \overline{\Omega}$ be measurable and $C \subseteq \overline{\Omega}$ be a closed set such that $B \cap C$ has positive measure, $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (\overline{D} \cup C)$ is connected to Σ. Then,*

$$\mathcal{R}(L_{B\cap C}^*) \cap \mathcal{R}(L_D^*) = \{0\}, \tag{19}$$

and $\mathcal{R}(L_{B\cap C}^*), \mathcal{R}(L_D^*) \subset L^2(\Sigma)$ are both dense (and thus in particular infinite-dimensional).

Proof. It follows from the unique continuation property in Theorem 2.4(a) that $L_{B\cap C}$ and L_D are injective. Hence $\mathcal{R}(L_{B\cap C}^*)$ and $\mathcal{R}(L_D^*)$ are dense subspaces of $L^2(\Sigma)$.

The characterization of the adjoint operators in Lemma 4.4 shows that

$$B \cap C \subseteq C \quad \text{implies} \quad \mathcal{R}(L_{B\cap C}^*) \subseteq \mathcal{R}(L_C^*).$$

Hence, (19) follows a fortiori if we can show that

$$\mathcal{R}(L_C^*) \cap \mathcal{R}(L_D^*) = \{0\}.$$

To show this let $h \in \mathcal{R}(L_C^*) \cap \mathcal{R}(L_D^*)$. Then there exist $f_C \in L^2(C)$, $f_D \in L^2(D)$, and $v_C, v_D \in H^1(\Omega)$ such that

$$\Delta v_C + k^2 q v_C = f_C \chi_C, \quad \partial_\nu v|_{\partial\Omega} = 0,$$
$$\Delta v_D + k^2 q v_D = f_D \chi_D, \quad \partial_\nu v|_{\partial\Omega} = 0,$$

and $v_C|_\Sigma = h = v_D|_\Sigma$.

It follows from the unique continuation property in Theorem 2.4(b) that $v_C = v_D$ on the connected set $\Omega \setminus (C \cup \overline{D})$. Hence,

$$v := \begin{cases} v_C = v_D & \text{on } \Omega \setminus (C \cup \overline{D}), \\ v_C & \text{on } \overline{D}, \\ v_D & \text{on } C \end{cases}$$

defines an $H^1(\Omega)$-function solving

$$\Delta v + k^2 q v = 0, \quad \partial_\nu v|_{\partial\Omega} = 0,$$

so that $v = 0$ and thus $h = v_C|_\Sigma = v_D|_\Sigma = v|_\Sigma = 0$. □

Lemma 4.6. *Let X, Y and Z be Hilbert spaces, and $A_1 : X \to Y$ and $A_2 : X \to Z$ be linear bounded operators. Then*

$$\exists c > 0 : \|A_1 x\| \le c\|A_2 x\| \quad \forall x \in X \qquad \textit{if and only if} \qquad \mathcal{R}(A_1^*) \subseteq \mathcal{R}(A_2^*).$$

Proof. This is proven for reflexive Banach spaces in [Gebauer 2008, Lemma 2.5]. Note that one direction of the implication also holds in nonreflexive Banach spaces; see [Gebauer 2008, Lemma 2.4]. □

Lemma 4.7. *Let $V, X, Y \subset Z$ be subspaces of a real vector space Z. If*

$$X \cap Y = \{0\} \quad \textit{and} \quad X \subseteq Y + V,$$

then $\dim(X) \le \dim(V)$.

Proof. Let $(x_j)_{j=1}^m \subset X$ be a linearly independent sequence of m vectors. Then there exist $(y_j)_{j=1}^m \subset Y$ and $(v_j)_{j=1}^m \subset V$ such that $x_j = y_j + v_j$ for all $j = 1, \dots, m$. We will prove the assertion by showing that the sequence $(v_j)_{j=1}^m$ is linearly independent. To this end let $\sum_{j=1}^m a_j v_j = 0$ with $a_j \in \mathbb{R}$, $j = 1, \dots, m$. Then

$$\sum_{j=1}^m a_j x_j = \sum_{j=1}^m a_j (y_j + v_j) = \sum_{j=1}^m a_j y_j \in Y,$$

so that $\sum_{j=1}^m a_j x_j = 0$. Since $(x_j)_{j=1}^m \subset X$ is linearly independent, it follows that $a_j = 0$ for all $j = 1, \dots, m$. This shows that $(v_j)_{j=1}^m$ is linearly independent. □

Proof of Theorem 4.1. Let $q \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. Let $B, D \subseteq \overline{\Omega}$ be measurable, $B \setminus \overline{D}$ possess positive measure, and $\overline{\Omega} \setminus \overline{D}$ be connected to Σ. Using Lemma 4.3 we obtain a closed set $C \subseteq \overline{\Omega}$ such that $B \cap C$ has positive measure, $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (\overline{D} \cup C)$ is connected to Σ.

Let $V \subset L^2(\Sigma)$ be a subspace with $d := \dim(V) < \infty$. Since V is finite-dimensional and thus closed, there exists an orthogonal projection operator $P_V : L^2(\Sigma) \to L^2(\Sigma)$ with

$$\mathcal{R}(P_V) = V, \quad P_V^2 = P_V, \quad \text{and} \quad P_V = P_V^*.$$

From Lemma 4.5, we have that $\mathcal{R}(L^*_{B\cap C}) \cap \mathcal{R}(L^*_D) = 0$ and that $\mathcal{R}(L^*_{B\cap C})$ is infinite-dimensional. So it follows from Lemma 4.7 that

$$\mathcal{R}(L^*_{B\cap C}) \not\subseteq \mathcal{R}(L^*_D) + V = \mathcal{R}(L^*_D) + \mathcal{R}(P_V^*).$$

Since $B \cap C \subseteq B$ implies that $\mathcal{R}(L^*_{B\cap C}) \subseteq \mathcal{R}(L^*_B)$, and since (using block operator matrix notation)

$$\mathcal{R}((L_D^* \;\; P_V^*)) \subseteq \mathcal{R}(L_D^*) + \mathcal{R}(P_V^*),$$

we obtain that

$$\mathcal{R}(L_B^*) \not\subseteq \mathcal{R}((L_D^* \;\; P_V^*)) = \mathcal{R}\left(\begin{pmatrix} L_D \\ P_V \end{pmatrix}^*\right).$$

It then follows from Lemma 4.6 that there cannot exist a constant $C > 0$ with

$$\|L_B g\|^2 \le C^2 \left\| \begin{pmatrix} L_D \\ P_V \end{pmatrix} g \right\|^2 = C^2 \|L_D g\|^2 + C^2 \|P_V g\|^2 \quad \text{for all } g \in L^2(\Sigma).$$

Hence, there must exist a sequence $(\tilde{g}_k)_{k\in\mathbb{N}} \subseteq L^2(\Sigma)$ with

$$\|L_B \tilde{g}_k\| \to \infty \quad \text{and} \quad \|L_D \tilde{g}_k\|, \|P_V \tilde{g}_k\| \to 0.$$

Thus, $g_k := \tilde{g}_k - P_V \tilde{g}_k \in V^\perp \subseteq L^2(\Sigma)$ and

$$\|L_B g_k\| \ge \|L_B \tilde{g}_k\| - \|L_B\| \, \|P_V \tilde{g}_k\| \to \infty \quad \text{and} \quad \|L_D g_k\| \to 0,$$

which shows the assertion. □

Proof of Theorem 4.2. Let $q_1, q_2 \in L^\infty(\Omega)$ for which $k > 0$ is not a resonance, and let $q_1(x) = q_2(x)$ for all x (a.e.) outside a measurable set $D \subset \Omega$. We denote by $L_{q_1,D}$ and $L_{q_2,D}$ the operators from Lemma 4.4 for $q = q_1$ and $q = q_2$. For $f \in L^2(D)$, we then have

$$L^*_{q_1,D} f = v_1|_\Sigma \quad \text{and} \quad L^*_{q_2,D} f = v_2|_\Sigma,$$

where $v_1, v_2 \in H^1(\Omega)$ solve

$$\Delta v_1 + k^2 q_1 v_1 = f\chi_D, \quad \partial_\nu v_1|_{\partial\Omega} = 0,$$
$$\Delta v_2 + k^2 q_2 v_2 = f\chi_D, \quad \partial_\nu v_2|_{\partial\Omega} = 0.$$

Since this also implies

$$\Delta v_1 + k^2 q_2 v_1 = f\chi_D + k^2(q_2 - q_1)v_1, \quad \partial_\nu v_1|_{\partial\Omega} = 0,$$
$$\Delta v_2 + k^2 q_1 v_2 = f\chi_D + k^2(q_1 - q_2)v_2, \quad \partial_\nu v_2|_{\partial\Omega} = 0,$$

and $q_1 - q_2$ vanishes (a.e.) outside D, it follows that

$$v_1|_\Sigma = L^*_{q_2,D}(f + k^2(q_2 - q_1)v_1) \quad \text{and} \quad v_2|_\Sigma = L^*_{q_1,D}(f + k^2(q_1 - q_2)v_2).$$

Hence, $\mathcal{R}(L^*_{q_1,D}) = \mathcal{R}(L^*_{q_2,D})$, so that the assertion follows from Lemma 4.6. □

4B. *Localized potentials and Runge approximation.* In this subsection we give an alternative proof of Theorem 4.1 that is based on a Runge approximation argument that characterizes whether a given function $\varphi \in L^2(O)$ on a measurable subset $O \subseteq \Omega$ can be approximated by functions in a subspace of solutions of the Helmholtz equation in Ω. Throughout this subsection let $q \in L^\infty(\Omega) \setminus \{0\}$ for which $k > 0$ is not a resonance. We will prove the following theorem.

Theorem 4.8. *Let $D \subseteq \Omega$ be a measurable set and $C \subset \Omega$ be a closed ball for which $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (C \cup \overline{D})$ is connected to Σ.*

Then for any subspace $V \subset L^2(\Sigma)$ with $\dim V < \infty$, there exists a function $\varphi \in L^2(C \cup \overline{D})$ that can be approximated (in the $L^2(C \cup \overline{D})$-norm) by solutions $u \in H^1(\Omega)$ of

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega \qquad \text{with} \qquad \partial_\nu u|_{\partial\Omega\setminus\Sigma} = 0, \quad \partial_\nu u|_\Sigma \in V^\perp,$$

and satisfies

$$\varphi|_{\overline{D}} \equiv 0 \quad \textit{and} \quad \varphi|_B \not\equiv 0$$

for all subsets $B \subseteq C$ with positive measure.

Before we prove Theorem 4.8, let us first show that it implies Theorem 4.1.

Corollary 4.9. *Let B, $D \subseteq \overline{\Omega}$ be measurable, $B \setminus \overline{D}$ possess positive measure, and $\overline{\Omega} \setminus \overline{D}$ be connected to Σ. Then for any subspace $V \subset L^2(\Sigma)$ with $\dim V < \infty$, there exists a sequence $(g_j)_{j\in\mathbb{N}} \subset V^\perp$ such that*

$$\int_B |u_q^{(g_j)}|^2 \,\mathrm{d}x \to \infty \quad \textit{and} \quad \int_D |u_q^{(g_j)}|^2 \,\mathrm{d}x \to 0,$$

where $u_q^{(g_j)} \in H^1(\Omega)$ solves the Helmholtz equation (2) with Neumann boundary data g_j.

Proof. As in Lemma 4.3, we can find a closed ball $C \subset \Omega$ so that $B \cap C$ has positive measure, $C \cap \overline{D} = \varnothing$, and $\overline{\Omega} \setminus (\overline{D} \cup C)$ is connected to Σ. Using Theorem 4.8, there exists $\varphi \in L^2(C \cup \overline{D})$ and a sequence of solutions $(\tilde{u}^{(j)})_{j\in\mathbb{N}} \subset H^1(\Omega)$ of $(\Delta + k^2 q)\tilde{u}^{(j)} = 0$ in Ω with $\partial_\nu \tilde{u}^{(j)}|_{\partial\Omega\setminus\Sigma} = 0$, $\partial_\nu \tilde{u}^{(j)}|_\Sigma \in V^\perp$,

$$\|\tilde{u}^{(j)}|_{B\cap C}\|_{L^2(B\cap C)} \to \|\varphi\|_{L^2(B\cap C)} > 0 \quad \text{and} \quad \|\tilde{u}^{(j)}|_{\overline{D}}\|_{L^2(\overline{D})} \to 0.$$

Obviously, the scaled sequence

$$g^{(j)} := \frac{\partial_\nu \tilde{u}^{(j)}}{\sqrt{\|\tilde{u}^{(j)}|_{\overline{D}}\|_{L^2(\overline{D})}}} \in V^\perp$$

satisfies the assertion. □

To prove Theorem 4.8, we start with an abstract characterization showing whether a given function $\varphi \in L^2(O)$ on a measurable set $O \subseteq \Omega$ is a limit of functions from a subspace of solutions of the Helmholtz equation in Ω. For the sake of readability, we write $v\chi_O \in L^2(\Omega)$ for the zero extension of a function $v \in L^2(O)$, and we write the dual pairing on $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ as an integral over $\partial\Omega$.

Lemma 4.10. *Let $O \subseteq \Omega$ be measurable. Let $H \subseteq H^1(\Omega)$ be a (not necessarily closed) subspace of solutions of $(\Delta + k^2 q)u = 0$ in Ω.*

A function $\varphi \in L^2(O)$ can be approximated on O by solutions $u \in H$ in the sense that

$$\inf_{u\in H} \|\varphi - u\|_{L^2(O)} = 0$$

if and only if $\int_O \varphi v \,\mathrm{d}x = 0$ for all $v \in L^2(O)$ for which the solution $w \in H^1(\Omega)$ of

$$(\Delta + k^2 q)w = v\chi_O \quad \textit{and} \quad \partial_\nu w|_{\partial\Omega} = 0 \tag{20}$$

satisfies $\int_{\partial\Omega} \partial_\nu u|_{\partial\Omega} w|_{\partial\Omega} \,\mathrm{d}s = 0$ for all $u \in H$.

Proof. Let

$$\mathcal{R} := \{u|_O : u \in H\} \subseteq L^2(O).$$

Let $v \in L^2(O)$ and $w \in H^1(\Omega)$ solve (20). Then $v \in \mathcal{R}^\perp$ if and only if, for all $u \in H$,

$$0 = \int_O uv \,\mathrm{d}x = \int_\Omega u(\Delta + k^2 q)w \,\mathrm{d}x = \int_\Omega w(\Delta + k^2 q)u \,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu u|_{\partial\Omega} w|_{\partial\Omega} \,\mathrm{d}s = -\int_{\partial\Omega} \partial_\nu u|_{\partial\Omega} w|_{\partial\Omega} \,\mathrm{d}s.$$

Hence, the assertion follows from $\overline{\mathcal{R}} = (\mathcal{R}^\perp)^\perp$ (where orthogonality and closures are taken with respect to the $L^2(O)$ inner product). □

Now we characterize the functions w appearing in Lemma 4.10 for a setting that will be considered in the proof of Theorem 4.8.

Lemma 4.11. *Let V be a finite-dimensional subspace of $L^2(\Sigma)$, and $O \subset \Omega$ be a closed set for which the complement $\overline{\Omega} \setminus O$ is connected to Σ.*

We define the spaces

$$W := \{w \in H^1(\Omega) : \exists v \in L^2(O) \text{ s.t. } (\Delta + k^2 q)w = v\chi_O,\ \partial_\nu w|_{\partial\Omega} = 0,\ w|_\Sigma \in V\},$$
$$W_0 := \{w \in H^1(\Omega) : \exists v \in L^2(O) \text{ s.t. } (\Delta + k^2 q)w = v\chi_O,\ \partial_\nu w|_{\partial\Omega} = 0,\ w|_\Sigma = 0\}.$$

Then the codimension $d := \dim(W/W_0)$ of W_0 in W is at most $\dim(V)$; i.e., there exist functions $w_1, \dots, w_d \in W$ such that every $w \in W$ can be written as

$$w = w_0 + \sum_{j=1}^{d} a_j w_j,$$

with (w-dependent) $w_0 \in W_0$ and $a_1, \dots, a_d \in \mathbb{R}$.

Proof. W_0 is the kernel of the restricted trace operator

$$\gamma_\Sigma|_W : W \to V, \quad w \mapsto w|_\Sigma.$$

Hence, the codimension of W_0 as a subspace of W is

$$\dim(W/W_0) = \dim(\mathcal{R}(\gamma_\Sigma|_W)) \le \dim(V),$$

which proves the assertion. □

Proof of Theorem 4.8. Let $D \subseteq \Omega$ be a measurable set and $C \subset \Omega$ be a closed ball for which $C \cap \overline{D} = \varnothing$ and $\overline{\Omega} \setminus (C \cup \overline{D})$ is connected to Σ. Let V be a finite-dimensional subspace of $L^2(\Sigma)$.

To apply Lemma 4.10, we set $O := C \cup \overline{D}$ and

$$H := \{u \in H^1(\Omega) : (\Delta + k^2 q)u = 0 \text{ in } \Omega,\ \partial_\nu u|_{\partial\Omega\setminus\Sigma} = 0,\ \partial_\nu u|_\Sigma \in V^\perp\}.$$

Then $w \in H^1(\Omega)$ satisfies (20) and

$$\int_{\partial\Omega} \partial_\nu u|_{\partial\Omega} w|_{\partial\Omega}\, \mathrm{d}s = 0$$

for all $u \in H$ if and only if $w \in W$, with W defined in Lemma 4.11. Hence, by Lemma 4.10, a function $\varphi \in L^2(C \cup \overline{D})$ can be approximated by solutions $u \in H$ if and only if

$$\int_{C\cup\overline{D}} \varphi(\Delta + k^2 q)w\, \mathrm{d}x = 0 \quad \text{for all } w \in W. \tag{21}$$

Thus, the assertion of Theorem 4.8 follows if we can show that there exists $\varphi \in L^2(C \cup \overline{D})$ that satisfies (21) and vanishes on D but not on any subset of C having positive measure.

To construct such a φ, we first note that the Helmholtz equation (2) on Ω is uniquely solvable for all Neumann data $g \in L^2(\Sigma)$, and by unique continuation, linearly independent Neumann data yield solutions whose restrictions to the open ball C° are linearly independent. Hence, there exists an infinite number of linearly independent solutions

$$\varphi_j \in H^1(C^\circ) \qquad \text{with} \qquad (\Delta + k^2 q)\varphi_j = 0 \quad \text{in } C^\circ, \quad j \in \mathbb{N}. \tag{22}$$

We extend φ_j by zero on $\overline{D} \cup \partial C$ to $\varphi_j \in L^2(O)$.

Every $w_0 \in W_0$, with W_0 from Lemma 4.11, must possess zero Cauchy data $w_0|_{\partial C} = 0$ and $\partial_\nu w_0|_{\partial C} = 0$ by unique continuation. Hence, for all $w_0 \in W_0$, and $j \in \mathbb{N}$,

$$\int_O \varphi_j(\Delta + k^2 q) w_0 \,\mathrm{d}x = \int_C \varphi_j(\Delta + k^2 q) w_0 \,\mathrm{d}x = \int_{\partial C} (\varphi_j|_{\partial C} \partial_\nu w_0|_{\partial C} - \partial_\nu \varphi_j|_{\partial C} w_0|_{\partial C}) \,\mathrm{d}s = 0.$$

Moreover, by a dimensionality argument, there must exist a nontrivial finite linear combination φ of the infinitely many linearly independent φ_j such that

$$\int_O \varphi(\Delta + k^2 q) w_k \,\mathrm{d}x = 0$$

for the finitely many functions $w_1, \dots, w_d \in W$ from Lemma 4.11. Thus, using Lemma 4.11, we have constructed a function $\varphi \in L^2(O)$ with $\varphi|_{\overline{D}} \equiv 0$, $\varphi|_{C^\circ} \not\equiv 0$, and

$$\int_O \varphi(\Delta + k^2 q) w \,\mathrm{d}x = 0 \quad \text{for all } w \in W = W_0 + \operatorname{span}\{w_1, \dots, w_d\}.$$

Moreover, φ solves (22), so that the unique continuation result from measurable sets in Theorem 2.4 also yields that $\varphi|_B \not\equiv 0$ for all $B \subseteq C^\circ$ with positive measure. Since ∂C is a null set, the latter also holds for all $B \subseteq C$ with positive measure. As explained above, the assertion of Theorem 4.8 now follows from Lemma 4.10. □

5. Local uniqueness for the Helmholtz equation

We are now able to prove the first main result in this work, announced as Theorem 1.1 in the Introduction, and extend the local uniqueness result in [Harrach and Ullrich 2017] to the case of negative potentials and $n \geq 2$.

As in Section 2A, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denote a bounded Lipschitz domain, and let $\Sigma \subseteq \partial\Omega$ be an arbitrarily small, relatively open part of the boundary $\partial\Omega$. For $q_1, q_2 \in L^\infty(\Omega)$ let

$$\Lambda(q_1), \Lambda(q_2) : L^2(\Sigma) \to L^2(\Sigma), \quad \Lambda(q_1) : g \mapsto u_1|_\Sigma, \quad \Lambda(q_2) : g \mapsto u_2|_\Sigma,$$

be the Neumann-to-Dirichlet operators for the Helmholtz equation

$$(\Delta + k^2 q)u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else}, \end{cases} \tag{23}$$

with $q = q_1$ and $q = q_2$ respectively, and let $k > 0$ be such that it is not a resonance for q_1 or q_2.

Theorem 5.1. *Let $q_1 \le q_2$ in a relatively open set $O \subseteq \overline{\Omega}$ that is connected to Σ. Then*

$$q_1|_O \not\equiv q_2|_O \quad \textit{implies} \quad \Lambda(q_1) \neq \Lambda(q_2).$$

Moreover, in that case, $\Lambda(q_2) - \Lambda(q_1)$ has infinitely many positive eigenvalues.

Proof. If $q_1|_O \not\equiv q_2|_O$ then there exists a subset $B \subseteq O$ with positive measure, and a constant $c > 0$ such that $q_2(x) - q_1(x) \ge c$ for all $x \in B$ (a.e.). From the monotonicity inequality in Theorem 3.5 we have that $\Lambda(q_2) - \Lambda(q_1) \ge_{\text{fin}} A$, where

$$A : L^2(\Sigma) \to L^2(\Sigma), \qquad \int_\Sigma hAg \,\mathrm{d}s = \int_\Omega k^2(q_2 - q_1) u_1^{(g)} u_1^{(h)} \,\mathrm{d}x.$$

Note that $A = S_1^* j^* k^2 M_{q_1 - q_2} j S_1$, where $S_1 : g \mapsto u_1^{(g)}$ is the solution operator and $j : H^1(\Omega) \to L^2(\Omega)$ is the compact inclusion, so A is indeed a compact, self-adjoint linear operator on $L^2(\Sigma)$.

We will now prove the assertion by contradiction and assume that $\Lambda(q_2) - \Lambda(q_1) \le_{\text{fin}} 0$. Then, the transitivity result in Lemma 3.4 gives that $A \le_{\text{fin}} 0$. By the characterization in Corollary 3.3, there would exist a finite-dimensional subspace $V \subseteq L^2(\partial\Omega)$, with

$$\begin{aligned} 0 &\ge \int_\Omega k^2(q_2 - q_1)|u_1^{(g)}|^2 \,\mathrm{d}x \\ &= \int_O k^2(q_2 - q_1)|u_1^{(g)}|^2 \,\mathrm{d}x + \int_{\Omega\setminus O} k^2(q_2 - q_1)|u_1^{(g)}|^2 \,\mathrm{d}x \\ &\ge c \int_B k^2 |u_1^{(g)}|^2 \,\mathrm{d}x - C \int_{\Omega\setminus O} k^2 |u_1^{(g)}|^2 \,\mathrm{d}x \end{aligned}$$

for all $g \in V^\perp$, where $C := (\|q_1\|_{L^\infty(\Omega)} + \|q_2\|_{L^\infty(\Omega)})$ and $u_1^{(g)}$ solves (23) with $q = q_1$.

However, using the localized potentials from Theorem 4.1 with $D := \overline{\Omega} \setminus O$, there must exist a Neumann datum $g \in V^\perp$ with

$$c \int_B k^2 |u_1^{(g)}|^2 \,\mathrm{d}x > C \int_{\Omega\setminus O} k^2 |u_1^{(g)}|^2 \,\mathrm{d}x,$$

which contradicts the above inequality. Hence, $\Lambda(q_2) - \Lambda(q_1)$ must have infinitely many negative eigenvalues, and in particular $\Lambda(q_2) \neq \Lambda(q_1)$. □

Proof of Theorem 1.1. The result is an immediate consequence of Theorem 5.1. □

Theorem 5.1 shows that two scattering coefficient functions can be distinguished from knowledge of the partial boundary measurements if their difference is of definite sign in a neighborhood of Σ (or any open subset of Σ since $\Lambda(\Sigma)$ determines the boundary measurements on all smaller parts). This definite sign condition is satisfied for piecewise-analytic functions, see, e.g., [Harrach and Ullrich 2013, Theorem A.1], but the authors are not aware of other named function spaces, with less regularity, where infinite oscillations between positive and negative values when approaching the boundary can be ruled out. In the following corollary the term piecewise-analytic is understood with respect to a partition in finitely many subdomains with piecewise C^∞-boundaries; see [Harrach and Ullrich 2013] for a precise definition.

Corollary 5.2. *If $q_1 - q_2$ is piecewise-analytic on Ω then*

$$\Lambda(q_1) = \Lambda(q_2) \quad \textit{if and only if} \quad q_1 = q_2.$$

Proof. This follows from Theorem 1.1 and [Harrach and Ullrich 2013, Theorem A.1]. □

6. Detecting the support of a scatterer

We will now show that an unknown scatterer, where the refraction index is either higher or lower than an otherwise homogeneous background value, can be reconstructed by simple monotonicity comparisons.

6A. ***Scatterer detection by monotonicity tests.*** As before, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with Lipschitz boundary. The domain is assumed to contain an open set (the scatterer) $D \subseteq \Omega$ with $\overline{D} \subset \Omega$ and connected complement $\Omega \setminus \overline{D}$. We assume that the scattering index satisfies $q(x) = 1$ in $\Omega \setminus D$ (a.e.) and that there exist constants $q_{\min}, q_{\max} \in \mathbb{R}$ so that either

$$1 < q_{\min} \leq q(x) \leq q_{\max} \quad \text{for all } x \in D \text{ (a.e.)}$$

or

$$q_{\min} \leq q(x) \leq q_{\max} < 1 \quad \text{for all } x \in D \text{ (a.e.)}.$$

$\Lambda(q)$ denotes the Neumann-to-Dirichlet operator for the domain containing the scatterer, and $\Lambda(1)$ is the Neumann-to-Dirichlet operator for a homogeneous domain with $q \equiv 1$. For both cases, we assume that $k > 0$ is not a resonance.

For an open set $B \subseteq \Omega$ (e.g., a small ball), we define the operator

$$T_B : L^2(\Sigma) \to L^2(\Sigma), \qquad \int_\Sigma g T_B h \, \mathrm{d}s := \int_B k^2 u_1^{(g)} u_1^{(h)} \, \mathrm{d}x,$$

where $u_1^{(g)}, u_1^{(h)} \in H^1(\Omega)$ solve (2) with $q \equiv 1$ and Neumann boundary data g and h respectively. Obviously, T_B is a compact self-adjoint linear operator.

The following two theorems show that D can be reconstructed by comparing $\Lambda(q) - \Lambda(1)$ with T_B in the sense of the Loewner order up to finitely many eigenvalues introduced in Section 3A.

Theorem 6.1. *Let*

$$1 < q_{\min} \leq q(x) \leq q_{\max} \quad \textit{for all } x \in D \textit{ (a.e.)},$$

and let $d(q_{\max})$ be defined as in Lemma 2.1 (which also equals the number of Neumann eigenvalues of the Laplacian Δ that are larger than $-k^2 q_{\max}$; see Corollary 3.11).

(a) *If $B \subseteq D$ then*

$$\alpha T_B \leq_{d(q_{\max})} \Lambda(q) - \Lambda(1) \quad \textit{for all } \alpha \leq q_{\min} - 1.$$

(b) *If $B \not\subseteq D$ then, for all $\alpha > 0$, $\Lambda(q) - \Lambda(1) - \alpha T_B$ has infinitely many negative eigenvalues.*

Theorem 6.2. *Let*

$$q_{\min} \leq q(x) \leq q_{\max} < 1 \quad \textit{for all } x \in D \textit{ (a.e.)},$$

and let $d(1)$ be defined as in Lemma 2.1 (which also equals the number of Neumann eigenvalues of the Laplacian Δ that are larger than $-k^2$; see Corollary 3.11).

(a) *If $B \subseteq D$ then there exists $\alpha_{\max} > 0$ such that*

$$\alpha T_B \leq_{d(1)} \Lambda(1) - \Lambda(q) \quad \textit{for all } \alpha \leq \alpha_{\max}.$$

(b) *If $B \not\subseteq D$ then, for all $\alpha > 0$, $\Lambda(1) - \Lambda(q) - \alpha T_B$ has infinitely many negative eigenvalues.*

6B. *Proof of Theorems 6.1 and 6.2.* We prove both results by combining the monotonicity relations and localized potentials results from the last subsections.

Proof of Theorem 6.1. By the monotonicity relation in Theorem 3.5 there exists a subspace $V \subset L^2(\Sigma)$ with $\dim(V) \leq d(q) \leq d(q_{\max})$ (see Corollary 3.11) and

$$\int_\Sigma g(\Lambda(q) - \Lambda(1))g \, \mathrm{d}s \geq \int_\Omega k^2(q-1)|u_1^{(g)}|^2 \, \mathrm{d}x \quad \text{for all } g \in V^\perp.$$

If $B \subseteq D$ and $\alpha \leq q_{\min} - 1$, then $q - 1 \geq \alpha \chi_B$, so that for all $g \in L^2(\Sigma)$

$$\int_\Omega k^2(q-1)|u_1^{(g)}|^2 \, \mathrm{d}x \geq \int_B k^2 \alpha |u_1^{(g)}|^2 \, \mathrm{d}x = \alpha \int_\Sigma g T_B g \, \mathrm{d}s.$$

Hence, if $B \subseteq D$ and $\alpha \leq q_{\min} - 1$, then

$$\int_\Sigma g(\Lambda(q) - \Lambda(1))g \, \mathrm{d}s \geq \alpha \int_\Sigma g T_B g \, \mathrm{d}s \quad \text{for all } g \in V^\perp,$$

which proves (a).

To prove (b) by contradiction, let $B \not\subseteq D$, $\alpha > 0$, and assume that

$$\Lambda(q) - \Lambda(1) \geq_{\text{fin}} \alpha T_B. \tag{24}$$

Using the monotonicity relation in Remark 3.6 together with Theorem 4.2, there exists a finite-dimensional subspace $V \subset L^2(\Sigma)$ and a constant $C > 0$, so that for all $g \in V^\perp$

$$\int_\Sigma g(\Lambda(q) - \Lambda(1))g \, \mathrm{d}s \leq \int_D k^2(q-1)|u_q^{(g)}|^2 \, \mathrm{d}x \leq C \int_D k^2(q-1)|u_1^{(g)}|^2 \, \mathrm{d}x. \tag{25}$$

Combining (24) and (25) using the transitivity result from Lemma 3.4, there exists a finite-dimensional subspace $\widetilde{V} \subset L^2(\Sigma)$ with

$$\alpha \int_B k^2 |u_1^{(g)}|^2 \, \mathrm{d}x \leq C \int_D k^2(q-1)|u_1^{(g)}|^2 \, \mathrm{d}x \quad \text{for all } g \in \widetilde{V}^\perp.$$

However, this is contradicted by the localized potentials result in Theorem 4.1, which guarantees the existence of a sequence $(g_j)_{j \in \mathbb{N}} \subset \widetilde{V}^\perp$ with

$$\int_B |u_1^{(g_j)}|^2 \, \mathrm{d}x \to \infty \quad \text{and} \quad \int_D |u_1^{(g_j)}|^2 \, \mathrm{d}x \to 0.$$

Hence, $\Lambda(q) - \Lambda(1) - \alpha T_B$ cannot have only finitely many negative eigenvalues. □

Proof of Theorem 6.2. The proof is analogous to that of Theorem 6.1. We state it for the sake of completeness. Let

$$q_{\min} \le q(x) \le q_{\max} < 1 \quad \text{for all } x \in D \text{ (a.e.)}.$$

If $B \subseteq D$, then by the monotonicity relation in Remark 3.6, together with Theorem 4.2, we have

$$\begin{aligned}\int_\Sigma g(\Lambda(q)-\Lambda(1))g\,\mathrm{d}s &\le_{d(1)} \int_\Omega k^2(q-1)|u_q^{(g)}|^2\,\mathrm{d}x \le -\int_D k^2(1-q_{\max})|u_q^{(g)}|^2\,\mathrm{d}x \\ &\le -c(1-q_{\max})\int_D k^2|u_1^{(g)}|^2\,\mathrm{d}x \le -c(1-q_{\max})\int_B k^2|u_1^{(g)}|^2\,\mathrm{d}x \\ &= -c(1-q_{\max})\int_\Sigma gT_B g\,\mathrm{d}s,\end{aligned}$$

with a constant $c > 0$ from Theorem 4.2. This shows that $B \subseteq D$ implies

$$\alpha T_B \le_{d(1)} \Lambda(1)-\Lambda(q) \quad \text{for all } \alpha \le c(1-q_{\max}) =: \alpha_{\max},$$

so that (a) is proven.

To prove (b) by contradiction, let $B \not\subseteq D$, $\alpha > 0$, and assume that

$$\Lambda(1)-\Lambda(q) \ge_{\text{fin}} \alpha T_B. \tag{26}$$

By the monotonicity relation in Theorem 3.5, we have

$$\int_\Sigma g(\Lambda(1)-\Lambda(q))g\,\mathrm{d}s \le_{\text{fin}} \int_D k^2(1-q)|u_1^{(g)}|^2\,\mathrm{d}x. \tag{27}$$

Combining (26) and (27) using the transitivity result from Lemma 3.4, we have

$$\alpha\int_B k^2|u_1^{(g)}|^2\,\mathrm{d}x \le_{\text{fin}} \int_D k^2(1-q)|u_1^{(g)}|^2\,\mathrm{d}x.$$

However, this is contradicted by Theorem 4.1, which guarantees (for each finite-dimensional space $V \subset L^2(\Sigma)$) the existence of a sequence $(g_j)_{j\in\mathbb{N}} \subset V^\perp$ with

$$\int_B |u_0^{(g_j)}|^2\,\mathrm{d}x \to \infty \quad \text{and} \quad \int_D |u_0^{(g_j)}|^2\,\mathrm{d}x \to 0.$$

Hence, $\Lambda(1)-\Lambda(q)-\alpha T_B$ cannot have only finitely many negative eigenvalues, which shows (b). □

6C. ***Remarks and extensions.*** We finish this section with some remarks on possible extensions of our results. Theorems 6.1 and 6.2 hold with analogous proofs also for the case that the homogeneous background scattering index is replaced by a known inhomogeneous function $q_0 \in L^\infty(\Omega)$. Using the concept of the inner and outer support from [Harrach and Ullrich 2013] (see also [Kusiak and Sylvester 2003; Gebauer and Hyvönen 2008; Harrach and Seo 2010] for the origins of this concept), we can also treat the case where $\Omega \setminus \overline{D}$ is not connected or where there is no clear jump of the scattering index. The monotonicity tests will then determine D up to the difference of the inner and outer support. Moreover, the so-called indefinite case that the domain contains scatterers with higher and lower refractive indices can be treated by shrinking a large test region analogously to [Harrach and Ullrich 2013]; see also [Garde and Staboulis 2019].

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