Noname manuscript No.
(will be inserted by the editor)

Resolvent estimates for the magnetic Schrödinger operator in dimensions ≥ 2

Cristóbal J. Meroño · Leyter Potenciano-Machado · Mikko Salo

Received: date / Accepted: date

Abstract It is well known that the resolvent of the free Schrödinger operator on weighted L^2 spaces has norm decaying like $\lambda^{-\frac{1}{2}}$ at energy λ. There are several works proving analogous high frequency estimates for magnetic Schrödinger operators, with large long or short range potentials, in dimensions $n \geq 3$. We prove that the same estimates remain valid in all dimensions $n \geq 2$.

Keywords Resolvent estimates · Schrödinger equation · magnetic potentials · limiting absorption principle

1 Introduction

Resolvent estimates for Schrödinger operators play a fundamental role in stationary scattering theory [12,11] and in inverse scattering [8]. They are also useful when proving Strichartz, smoothing, and dispersive estimates, eigenvalue estimates, as well as local energy decay for wave and Schrödinger equations (see e.g. [6,3,14,2,10]). In many of these applications it is important to understand the high frequency behavior of the resolvent, i.e. how the norm bounds depend on the frequency (or energy).

C.M. was supported by Spanish government predoctoral grant BES-2015-074055 and projects MTM2014-57769-C3-1-P and MTM2017-85934-C3-3-P. L.P. and M.S. were supported by the Academy of Finland (Centre of Excellence in Inverse Modeling and Imaging, grant numbers 284715 and 309963) and by the European Research Council under Horizon 2020 (ERC CoG 770924).

C. J. Meroño
Universidad Politécnica de Madrid, ETSI Caminos, Departmento de Matemática e Informática, Campus Ciudad Universitaria, 28040 Madrid
E-mail: cj.merono@upm.es

L. Potenciano-Machado
University of Jyväskylä, Department of Mathematics and Statistics, PO Box 35, 40014, Finland
E-mail: leyter.m.potenciano@jyu.fi

M. Salo
University of Jyväskylä, Department of Mathematics and Statistics, PO Box 35, 40014, Finland
E-mail: mikko.j.salo@jyu.fi

A standard high frequency resolvent estimate for the free Schrödinger operator in $\mathbb{R}^n$ (see e.g. [1], [16, Section 7.1]) states that

$$\|\partial^\alpha((-\Delta-\lambda\pm i\varepsilon)^{-1}f)\|_{L^2_{-\delta}}\le C\lambda^{\frac{|\alpha|-1}{2}}\|f\|_{L^2_\delta} \tag{1.1}$$

where $\lambda\ge 1$, $0<\varepsilon\le 1$, $\delta>1/2$, $|\alpha|\le 1$ and C is independent of λ and f. The spaces $L^2_\delta=L^2_\delta(\mathbb{R}^n)$, $\delta\in\mathbb{R}$, are the weighted Agmon spaces, and their norm is given by

$$\|f\|_{L^2_\delta}=\|\langle x\rangle^\delta f\|$$

where $\langle x\rangle=(1+|x|^2)^{1/2}$ and $\|\cdot\|$ denotes the $L^2(\mathbb{R}^n)$ norm.

In this work, we consider a first order perturbation of the Laplacian, the magnetic Schrödinger operator in $\mathbb{R}^n$, $n\ge 2$, given by

$$H=(D+W)^2+V \tag{1.2}$$

where $D=-i\nabla$, the magnetic potential $W:\mathbb{R}^n\to\mathbb{R}^n$ is a vector field, and the electrostatic potential $V:\mathbb{R}^n\to\mathbb{R}$ is a scalar function. To start, we assume that $W\in L^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and $V\in L^\infty(\mathbb{R}^n,\mathbb{R})$. Under these assumptions, H can be considered as a self-adjoint operator in $L^2(\mathbb{R}^n)$, in the sense of Proposition A.1.

A direct perturbation argument shows that (1.1) remains valid when $-\Delta$ is replaced by the magnetic Schrödinger operator H, provided that for some $\sigma>0$,

$$\|\langle x\rangle^{1+\sigma}W\|_{L^\infty} \text{ is sufficiently small,} \qquad \|\langle x\rangle^{1+\sigma}V\|_{L^\infty}<\infty,$$

and provided that λ is sufficiently large (depending on V).

If the magnetic potential W is large, the perturbation argument fails, and several works have been devoted to understanding high frequency resolvent estimates. The articles [6,7,9] employ harmonic analysis methods and prove high frequency resolvent estimates assuming that W is continuous and the potentials are of short range type. Analogous estimates were proved earlier in [13, Théorème (5.1)] for smooth long range potentials satisfying symbol type bounds, also when the Euclidean metric is replaced by an asymptotically Euclidean metric with no trapped geodesics. The proof was based on a microlocal version of the Mourre commutator method, which in turn is an instance of a positive commutator method.

In the Euclidean case, the works [3,4,15] prove high frequency resolvent estimates for long and short range potentials having low regularity, with the most general results given in [15]. Their proofs involve positive commutator arguments combined with ODE techniques, including Carleman estimates, that are valid under low regularity assumptions. We mention also the works [17,18], in which the Morawetz multiplier method, also related to positive commutator arguments, is used to allow magnetic potentials with singularities.

Many of the previously mentioned works explicitly assume that the dimension is $n\ge 3$. It is the purpose of this article to show that high frequency resolvent estimates for the magnetic Schrödinger operator, under low regularity assumptions like in [15], remain valid in all dimensions $n\ge 2$.

In this work we assume that the potentials V and W in (1.2) have both long range and short range parts $V^L,V^S\in L^\infty(\mathbb{R}^n,\mathbb{R})$ and $W^L,W^S\in L^\infty(\mathbb{R}^n,\mathbb{R}^n)$ satisfying the following conditions:

$$V=V^L+V^S,\quad W=W^L+W^S, \tag{1.3}$$

$$|\nabla V^L| \le C\langle x\rangle^{-1-\sigma}, \quad |W^L| \le C\langle x\rangle^{-\sigma}, \quad |\nabla W^L| \le C\langle x\rangle^{-1-\sigma}. \tag{1.4}$$

$$|V^S| \le C\langle x\rangle^{-1-\sigma}, \quad |W^S| \le C\langle x\rangle^{-1-\sigma}, \tag{1.5}$$

for some $\sigma > 0$. In some results we will use also the stronger condition

$$|\nabla \cdot W^S| \le C\langle x\rangle^{-1-\sigma}. \tag{1.6}$$

We can state now the main result in this work.

Theorem 1.1. *Let $n \ge 2$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) = \lambda$, $|\mathrm{Im}(z)| \le 1$ and $\mathrm{Im}(z) \ne 0$. Assume that $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ and $W \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfy (1.3) – (1.5). Then, for any $\delta > 1/2$, there exist positive constants $C = C(n, \delta)$ and $\lambda_0 = \lambda_0(n, \delta, V, W)$ such that for every $\lambda \ge \lambda_0$, the resolvent $R(z) = (H - z)^{-1}$ satisfies the estimate*

$$\|\partial^{\alpha_1} R(z)\partial^{\alpha_2} f\|_{L^2_{-\delta}} \le C\lambda^{\frac{|\alpha_1|+|\alpha_2|-1}{2}} \|f\|_{L^2_\delta}, \tag{1.7}$$

whenever $|\alpha_1|, |\alpha_2| \le 1$ and $f \in L^2_\delta$. Moreover, if one also assumes the condition (1.6) on the short range magnetic potential, then the estimate

$$\|\partial^{\alpha} R(z) f\|_{L^2_{-\delta}} \le C\lambda^{\frac{|\alpha|-1}{2}} \|f\|_{L^2_\delta}, \tag{1.8}$$

holds for every $|\alpha| \le 2$.

Estimate (1.7) is analogous to the results in [15, Theorem 1.1] but it holds also for $n = 2$. Condition (1.6) already appears in [4]. However, the results in both the above mentioned papers hold under the following slightly weaker conditions on the long range potentials:

$$\partial_r V^L \le C\langle x\rangle^{-1-\sigma}, \quad |W^L| \le C\langle x\rangle^{-\sigma}, \quad |\partial_r W^L| \le C\langle x\rangle^{-1-\sigma}. \tag{1.9}$$

Here $\partial_r := \frac{x}{|x|} \cdot \nabla$ denotes the radial derivative. In our case, the main a priori estimates in this work (see Lemma 2.4 and Proposition 5.1 below) are also obtained under essentially the weaker long range conditions (1.9). The stronger conditions (1.4) are only needed for the final density argument used to prove Theorem 1.1.

In the proof of Theorem 1.1 the self-adjointness of H is essential so that the resolvent $R(z)$ can be defined as a bounded operator in L^2 for all $z \in \mathbb{C}$ with $\mathrm{Im}(z) \ne 0$. But, in general, for $\lambda > 0$ the operator $(H - \lambda)^{-1}$ cannot be expected to be a bounded operator in L^2. Nonetheless, it is well known that the limiting absorption principle (see e.g. [12,11]) provides a way to define the resolvent operators

$$R(\lambda \pm i0)f := \lim_{\mathrm{Im}(z)\to 0^\pm} R(z)f, \tag{1.10}$$

as bounded operators from L^2_δ to $L^2_{-\delta}$ for $\delta > 1/2$.

Under certain restrictions, a limiting absorption principle is proved in [11, Theorem 30.2.10] in the presence of long range and short range magnetic potentials. Then, it follows from this result that the resolvent $R(\lambda \pm i0)$ will satisfy the same bounds as $R(z)$ in Theorem 1.1. We state this with more precision in the following theorem.

Theorem 1.2. *Assume that the hypotheses from Theorem 1.1 hold, together with (1.6). Additionally, assume that W^S is continuous. Then there is a discrete set $\Lambda \subset \mathbb{R}_+$ (which is empty if $W^L = V^L = 0$) such that the resolvent for H at energy $\lambda \in \mathbb{R}_+ \setminus \Lambda$ with $\lambda \geq \lambda_0$ satisfies*

$$\|\partial^\alpha R(\lambda \pm i0) f\|_{L^2_{-\delta}} \leq C\lambda^{\frac{|\alpha|-1}{2}} \|f\|_{L^2_\delta}, \tag{1.11}$$

for any $|\alpha| \leq 2$ and $f \in L^2_\delta$.

Our proof of Theorem 1.1 employs analogous methods to the ones used in [14, 4, 15]. As in [15], we begin by proving a global Carleman type estimate for the case $W^S = 0$, $V^S = 0$. Nonetheless, the specific Carleman estimate in this work and the approach used to prove it, differ from [15]. To obtain this estimate we use a positive commutator argument based on the construction of a suitable conjugate operator as in [14, Section 6.1], and integration by parts. The conjugate operator is chosen carefully in order to have an estimate that is valid in any dimension $n \geq 2$.

The commutator argument is explained in Section 2 and the proof of the Carleman estimate is given in Section 3. This estimate, stated in Lemma 2.4, would already imply (1.8) for $|\alpha| \leq 1$ assuming stronger conditions on W^S and $\nabla \cdot W^S$. Then, following [15], in Section 4 we shift the previous estimate to lower index Sobolev spaces to prepare for the inclusion of the low regularity term $\nabla \cdot W^S$. To conclude the proof of Theorem 1.1, in Section 5 we include the short range perturbation and we extend the final a priori estimate, which holds for C_c^∞ functions, to the appropriate spaces using the Friedrichs lemma. To finish, Theorem 1.2 is proved in the last section.

2 The commutator method

We will first describe a positive commutator argument for the free resolvent following the presentation in [14, Section 6]. Define

$$P := -\Delta.$$

We will construct a first order differential operator A ("conjugate operator") such that $i[P, A]$ is positive. If this is true, then for any $\lambda \in \mathbb{R}$ and for any test function $u \in C_c^\infty(\mathbb{R}^n)$ we have

$$(i[P, A]u, u) = (i[P - \lambda, A]u, u) = i(Au, (P - \lambda)u) - i((P - \lambda)u, A^*u).$$

Here and below, we write $(\,\cdot\,,\,\cdot\,)$ and $\|\cdot\|$ for the inner product and norm on $L^2(\mathbb{R}^n)$. If $(i[P, A]u, u)$ is sufficiently positive so that it controls weighted versions of Au and A^*u, we can use Young's inequality with $\varepsilon > 0$ in the form $2|(Au, (P - \lambda)u)| \leq \varepsilon \|Au\|^2 + \varepsilon^{-1}\|(P - \lambda)u\|^2$ (also with suitable weights) to obtain a resolvent type estimate with $(P - \lambda)u$ on the right.

To motivate the choice of the conjugate operator $A = a_k(x)D_k + b(x)$, we note that P and A have principal symbols $p(x, \xi) = |\xi|^2$ and $a(x, \xi) = a_k(x)\xi_k$. Notice that we are omitting summation symbols over repeated indices (we will continue to use this convention in the rest of the paper). Then the commutator $i[P,A]$ has principal symbol given by the Poisson bracket

$$\{p,a\} = \nabla_\xi p \cdot \nabla_x a - \nabla_x p \cdot \nabla_\xi a = 2\partial_j a_k(x)\xi_j\xi_k.$$

We want the last quantity to be suitably positive. If we choose $a_k = 2\partial_k\varphi$ for some function φ, which happens for instance if $A = i[P,\varphi]$, then $\{p,a\} = 4\varphi''(x)\xi\cdot\xi$ where φ'' is the Hessian matrix of φ. Thus, if φ is a suitable convex function, we expect that $A = i[P,\varphi]$ could have the required properties.

We now consider the operator $P+L+V^L$, where L is the long range magnetic perturbation given by

$$Lu := W^L \cdot Du + D\cdot(W^L u). \tag{2.1}$$

From now on, we assume that W^L and V^L satisfy the conditions (1.4), or conditions (1.9) together with the assumption $\operatorname{div}(W^L) = \nabla\cdot W^L \in L^2(\mathbb{R}^n)$, so that the term $D\cdot(W^L u) = W^L\cdot Du - i\operatorname{div}(W^L)u$ is always well defined as an L^2 function for $u \in C_c^\infty(\mathbb{R}^n)$.

Lemma 2.1. *Let $\varphi \in C^4(\mathbb{R}^n)$ be real valued, and let*

$$A := i[P,\varphi] = 2\partial_j\varphi D_j - i(\Delta\varphi).$$

Let also $z\in\mathbb{C}$. Then, for any $u \in C_c^\infty(\mathbb{R}^n)$ one has

$$\begin{aligned}&4(\varphi'' Du, Du) - ((\Delta^2\varphi)u,u)\\ &= -2\operatorname{Im}(Au,(P+L+V^L-z)u) + 4\operatorname{Im}(z)(Au,u) - (i[L+V^L,A]u,u).\end{aligned} \tag{2.2}$$

Proof. With the given choice of A, we compute

$$\begin{aligned} i[P,A]u &= -i\Delta(2\partial_j\varphi D_j u - i(\Delta\varphi)u) + i(2\partial_j\varphi D_j - i(\Delta\varphi))(\Delta u)\\ &= -2i\Delta\partial_j\varphi D_j u - 4i\partial_{jk}\varphi D_j\partial_k u - (\Delta^2\varphi)u - 2\partial_j\Delta\varphi\partial_j u\\ &= 4D_k(\partial_{jk}\varphi D_j u) - (\Delta^2\varphi)u.\end{aligned}$$

Note that $(Au,v) = (u,Av)$ for $u,v\in C_c^\infty(\mathbb{R}^n)$. The result follows since

$$\begin{aligned}&4(\varphi'' Du,Du) - ((\Delta^2\varphi)u,u) = (i[P,A]u,u)\\ &= (i[P+L+V^L-z,A]u,u) - (i[L+V^L,A]u,u),\end{aligned}$$

and

$$\begin{aligned}(i[P+L+V^L-z,A]u,u) = i(Au,(P+L+V^L-\bar z)u)&\\ - i((P+L+V^L-z)u,Au).&\quad\square\end{aligned}$$

Note that the left hand side of (2.2) is independent of z. To describe the dependence on $\operatorname{Re}(z)$, we need the following lemma.

Lemma 2.2. *Let $\eta\in C^2(\mathbb{R}^n)$ be real valued, and let $z\in\mathbb{C}$. Then for any $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned}2\operatorname{Re}(z)\int\eta|u|^2 = -\int(\Delta\eta)|u|^2 + 2\int\eta|\nabla u|^2&\\ -\int\eta(((P-z)u)\bar u + u(P-\bar z)\bar u).&\end{aligned} \tag{2.3}$$

Proof. This is a direct integration by parts:

$$\begin{aligned}-\int (\Delta\eta)|u|^2 &= -\int \eta\Delta(u\bar{u})\\ &= -2\int \eta|\nabla u|^2 - \int \eta((\Delta u)\bar{u} + u\overline{\Delta u}).\end{aligned}$$

The result follows by writing $\Delta u = (\Delta + z)u - zu$ in the last terms on the right. □

We can combine the previous lemmas with suitable choices of φ and η to obtain an a priori estimate for the long range magnetic resolvent. We need a weight φ with a large positive Hessian, so that we can later absorb certain terms on the left hand side of (2.2). See Remark 2.5 below for motivation for the choice of φ.

Let $\tau \geq 1$ and write $r = |x|$. We consider a radial weight

$$\varphi := \tilde{\varphi}e^{\tau\psi}$$

with $\psi(x) = \psi_0(r)$ and $\tilde{\varphi}(x) = \tilde{\varphi}_0(r)$, for some $\tilde{\varphi}_0, \psi_0 : (0,\infty) \to \mathbb{R}$. By direct computation one can show that the Hessian of φ satisfies

$$\varphi'' = e^{\tau\psi}\left(\tilde{\varphi}'' + \tau\nabla\tilde{\varphi}\otimes\nabla\psi + \tau\nabla\psi\otimes\nabla\tilde{\varphi} + \tau^2\tilde{\varphi}\nabla\psi\otimes\nabla\psi + \tau\tilde{\varphi}\psi''\right).$$

First we choose $\tilde{\varphi}(x) = \tilde{\varphi}_0(r) = \langle r\rangle$. With this choice, the Hessian of $\tilde{\varphi}$ is positive semidefinite, i.e. $\tilde{\varphi}''(x) \geq 0$. Also, let us take $\psi(x) = \psi_0(r) = 1 - \langle r\rangle^{1-2\delta}$. Writing explicitly the Hessian of ψ in terms of the derivatives of $\psi_0(r)$, yields

$$(\psi''(x)\nabla u, \nabla u) = (\psi_0''(r)\partial_r u, \partial_r u) + (\frac{\psi_0'(r)}{r}\nabla^{\perp}u, \nabla^{\perp}u),$$

where $\nabla^{\perp}u := \nabla u - (\nabla u\cdot\hat{x})\hat{x}$ with $\hat{x} = x/|x|$, so that $|\nabla u|^2 = (\partial_r u)^2 + |\nabla^{\perp}u|^2$ (notice that ψ'' denotes the Hessian matrix of ψ and $\psi_0'' = \frac{d^2\psi_0}{d\,r^2}$). Thus, using the condition $\tilde{\varphi}'' \geq 0$, we get

$$\begin{aligned}(\varphi''(x)\nabla u, \nabla u) \geq{}& 2\tau(e^{\tau\psi}\tilde{\varphi}_0'\psi_0'\partial_r u, \partial_r u) + \tau^2(e^{\tau\psi}\tilde{\varphi}_0(\psi_0')^2\partial_r u, \partial_r u)\\ &+ \tau(e^{\tau\psi}\tilde{\varphi}_0\psi_0''\partial_r u, \partial_r u) + \tau(e^{\tau\psi}\tilde{\varphi}_0\frac{\psi_0'}{r}\nabla^{\perp}u, \nabla^{\perp}u).\end{aligned} \tag{2.4}$$

We remark that, since φ and ψ are increasing functions, all the terms in the right hand side of (2.4) are positive, except for the third one.

Lemma 2.3. *Let $1/2 < \delta < 1$, and let $\varphi(x) = \langle x\rangle e^{\tau\psi}$, with $\psi(x) = \psi_0(r) = 1 - \langle r\rangle^{1-2\delta}$ and $\tau \geq 1$. Then, if $\beta = 2(1-\delta)(2\delta - 1)$ we have that*

$$(\varphi''(x)\nabla u, \nabla u) \geq \beta\tau\|e^{\frac{1}{2}\tau\psi}w_1\nabla u\|^2 + \tau^2\|e^{\frac{1}{2}\tau\psi}w_2\partial_r u\|^2, \tag{2.5}$$

where $w_1^2(x) = \langle x\rangle^{-2\delta}$, and $w_2^2(x) = \langle r\rangle(\psi_0'(r))^2$. Moreover, if α is a multi-index such that $|\alpha| = N$, then

$$|\partial^{\alpha}\varphi(x)| \leq C_N\tau^N\langle x\rangle^{1-N}e^{\tau\psi}. \tag{2.6}$$

Since the proof is a straightforward computation, we leave it to Appendix A. As a consequence (2.6) we get that

$$|\Delta\varphi(x)| \leq C\tau^2\langle x\rangle^{-1}e^{\tau\psi}, \qquad |\Delta^2\varphi(x)| \leq C\tau^4\langle x\rangle^{-3}e^{\tau\psi}, \tag{2.7}$$

which will be used in the proof of the following lemma. We state now the main Carleman type estimate for the long range magnetic resolvent.

Lemma 2.4. *Let $1/2 < \delta < \infty$ and $\tau > 6\beta^{-1}$. Consider $W^L \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\nabla \cdot W^L \in L^2_{loc}(\mathbb{R}^n)$ and $V^L \in L^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying (1.9) for some $\sigma > 0$. Then for any $v \in C_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned}\mathrm{Re}(z)\|v\|^2_{L^2_{-\delta}} + \|\nabla v\|^2_{L^2_{-\delta}} \leq 64\beta^{-1}\tau^{-1}\|e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)e^{-\frac{1}{2}\tau\psi}v\|^2_{L^2_\delta} \\ + C(\beta,\tau)|\mathrm{Im}(z)|\mathrm{Re}(z)^{1/2}\|v\|^2,\end{aligned} \tag{2.8}$$

whenever $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| \leq 1$ and $\mathrm{Re}(z) \geq C(n, \delta, \tau, W^L, V^L) \geq 1$.

This lemma will be proved in the next section. We mention that the condition $\nabla \cdot W^L \in L^2_{loc}(\mathbb{R}^n)$ is only necessary so that the right hand side of (2.8) is well defined.

As explained in Remark 2.5 below, the factor τ^{-1} in the right hand side of (2.8) is the key to absorb a general short range magnetic perturbation. On the other hand, the dependence on τ of the last term on the right hand side is not a source of concern. Once the short range potentials will be introduced, we will fix the value of τ, and then the factor $\mathrm{Im}(z)$ will lead to an estimate for the error term involving the whole magnetic operator (see Lemma 5.2).

Remark 2.5. In [14, Section 6], a resolvent estimate with $W = 0$ is proved using a commutator argument with the weight $|x| - \langle x\rangle^{2-2\delta}$. To avoid problems at the origin one can use the analogous smooth weight $\varphi = \langle x\rangle - \langle x\rangle^{2-2\delta}$. This weight has a positive Hessian growing as $\langle x\rangle^{2\delta}$ and satisfies $\Delta^2\varphi > 0$ for $n \geq 3$, and this, thanks to (2.2), leads after some work to a resolvent estimate of the kind

$$\mathrm{Re}(z)\|u\|^2_{L^2_{-\delta}} + \|\nabla u\|^2_{L^2_{-\delta}} \leq C\|(P + L + V^L - z)u\|^2_{L^2_\delta}. \tag{2.9}$$

The same estimate can be obtained in dimension $n = 2$ using that for this choice of weight one has $|\Delta^2\varphi| \leq C\langle x\rangle^{-3}$, so that the bilaplacian term in (2.2) can be controlled appropriately.

Unfortunately, it is not possible to absorb in (2.9) a general short range magnetic perturbation, unless W^S is assumed to be small enough. To deal with the general case we want the Hessian of φ to be as large as needed. The exponential weight $\varphi = \langle x\rangle e^{\tau\psi}$ works satisfactorily since it has a large positive Hessian that grows with the parameter τ, as shown in Lemma 2.3. The main difference with [14, Section 6] is that now this choice leads naturally to the Carleman type estimate (2.8). This kind of estimate is in line with the results in [15]. We have chosen ψ to be bounded above and below so that the exponentials can be removed later from the estimates.

3 Proof of Lemma 2.4

We now prove Lemma 2.4 using the commutator method introduced in the previous section. We start from the identity (2.2). One hopes to be able to bound the terms on the right with an appropriate norm of $(P + L + V^L - z)u$, or to absorb them in the left hand side using the term $(\varphi'' Du, Du)$. In this sense, it will be useful to state the following lemma that we prove at the end of this section.

Lemma 3.1. *Let* $M := \tau^3 C(n, \delta)(\|\langle x\rangle^{1+\sigma}\partial_r W^L\|_{L^\infty} + \|\langle x\rangle^{\sigma} W^L\|_{L^\infty})$. *Then for any* $u \in C_c^\infty(\mathbb{R}^n)$ *we have*

$$|i([L, A]u, u)| \leq \|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + \left(M^2 + 2\right) \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2. \tag{3.1}$$

In the proof of Lemma 2.4 we are also helped by Lemma 2.2 which gives to (2.8) the appropriate dependence with $\mathrm{Re}(z)$ by introducing in the left the "large" term $\mathrm{Re}(z)\int \eta |u|^2$, for suitable η.

Proof of Lemma 2.4. Notice that if (2.8) holds for one δ, then it also hold with the same constant for every $\delta' > \delta$. Therefore, without loss of generality, we consider $1/2 < \delta < \min\{(\sigma+1)/2, 1\}$. Then, Lemmas 2.1 and 2.3 give that

$$\begin{aligned} 4\beta\tau\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 4\tau^2\|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2 &\leq ((\Delta^2\varphi)u, u) \\ - 2\mathrm{Im}(Au, (P + L + V^L - z)u) &+ 4\mathrm{Im}(z)(Au, u) - (i[L + V^L, A]u, u). \end{aligned}$$

Now, since φ is a radial function and $\partial_r\varphi \geq 0$,

$$-(i[V^L, A]u, u) = 2(\partial_r\varphi \partial_r V^L u, u) \leq 2(\partial_r\varphi(\partial_r V^L)_+ u, u),$$

where $(f)_+$ is the nonnegative part of f. Hence, using that $w_1^{-2} \leq \langle x\rangle^{1+\sigma}$ and that $\partial_r\varphi \leq 2\tau e^{\tau\psi}$, we have

$$\begin{aligned} 4\beta\tau\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 4\tau^2\|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2 &\leq ((\Delta^2\varphi)u, u) \\ + |2\mathrm{Im}(Au, (P + L + V^L - z)u)| + 4|\mathrm{Im}(z)(Au, u)| &+ |(i[L, A]u, u)| \\ + 4\tau\|\langle x\rangle^{1+\sigma}(\partial_r V^L)_+\|_{L^\infty} &\|e^{\frac{1}{2}\tau\psi} w_1 u\|^2. \end{aligned} \tag{3.2}$$

We now apply Lemma 2.2 with $\eta = w_1^2 e^{\tau\psi}$. Using that $|\nabla w_1| \leq w_1$, $|\Delta w_1^2| \leq 2n w_1^2$, and that the derivatives of ψ are bounded, one can see that $|\Delta(w_1^2 e^{\tau\psi})| \leq C\tau^2 w_1^2 e^{\tau\psi}$ for suitable C depending on the dimension. We use this fact in (2.3) and multiply the resulting inequality by $\beta\tau$. This yields

$$\begin{aligned} (2\beta\tau\mathrm{Re}(z) - C\tau^3)\|e^{\frac{1}{2}\tau\psi} w_1 u\|^2 \leq 2\beta\tau&\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 \\ &+ 2\beta\tau|\mathrm{Re}((P - z)u, e^{\tau\psi} w_1^2 u)|. \end{aligned} \tag{3.3}$$

Adding estimates (3.2) and (3.3) and moving two terms to the left hand side, we get

$$\begin{aligned} 2\beta\tau\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 4\tau^2\|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2 + (2\beta\tau\mathrm{Re}(z) - K_1)\|e^{\frac{1}{2}\tau\psi} w_1 u\|^2 \\ \leq |((\Delta^2\varphi)u, u)| + |(i[L, A]u, u)| + 2|\mathrm{Im}(Au, (P + L + V^L - z)u)| \\ + 2|\mathrm{Im}(z)(Au, u)| + 2\beta\tau|((P - z)u, e^{\tau\psi} w_1^2 u)|, \end{aligned}$$

where $K_1 = C\tau^3 + 4\tau \|\langle x\rangle^{1+\sigma}(\partial_r V^L)_+\|_{L^\infty}$. Then, Lemma 3.1 and estimate (2.7) allow us to absorb the first two terms on the right into the left hand side. Thus

$$
\begin{aligned}
&(2\beta\tau - 1)\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 4\tau^2 \|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2 + (2\beta\tau \mathrm{Re}(z) - K_2)\, \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2 \\
&\quad \leq 2|(Au, (P + L + V^L - z)u)| + 2|\mathrm{Im}(z)(Au, u)| + 2\beta\tau |((P-z)u, e^{\tau\psi} w_1^2 u)|\,.
\end{aligned}
$$

where $K_2 = K_1 + C\tau^4 + M^2 + 2$.

Since $w_1 \leq 1 \leq w_1^{-1}$, the last term on the right hand side satisfies

$$
\begin{aligned}
&2\beta\tau |((P-z)u, e^{\tau\psi} w_1^2 u)| \leq 2\beta\tau \|w_1^{-1} e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)u\| \|e^{\frac{1}{2}\tau\psi} w_1 u\| \\
&+ \|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 2\beta\tau(2\beta\tau \|W^L\|_{L^\infty}^2 + (2+\tau)\|W^L\|_{L^\infty} + \|V^L\|_{L^\infty}) \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2,
\end{aligned}
$$

where we have used Young's inequality, and the fact that by (2.1) and integration by parts one has

$$
\begin{aligned}
|(Lu, e^{\tau\psi} w_1^2 u)| &\leq |(W^L u, \nabla(e^{\tau\psi} w_1^2 u))| + |(W^L \cdot \nabla u, e^{\tau\psi} w_1^2 u)| \\
&\leq 2\|W^L\|_{L^\infty} \|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\| \|e^{\frac{1}{2}\tau\psi} w_1 u\| + (2+\tau)\|W^L\|_{L^\infty} \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2,
\end{aligned}
$$

since $|\nabla(e^{\tau\psi} w_1^2)| \leq (2+\tau) e^{\tau\psi} w_1^2$. This and Young's inequality lead to the estimate

$$
\begin{aligned}
&(2\beta\tau - 2)\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 4\tau^2 \|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2 + (2\beta\tau \mathrm{Re}(z) - K_3)\, \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2 \\
&\quad \leq \|w_1^{-1} e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)u\|^2 + 2|\mathrm{Im}(z)(Au, u)| \\
&\qquad + 2|(Au, (P + L + V^L - z)u)|, \qquad (3.4)
\end{aligned}
$$

where now $K_3 = K_2 + 2\beta\tau(2\beta\tau\|W^L\|_{L^\infty}^2 + (2+\tau)\|W^L\|_{L^\infty} + \|V^L\|_{L^\infty}) + \beta^2\tau^2$.

Again, we estimate the last term on the right of (3.4). We have that

$$
\begin{aligned}
|w_1 Au| &\leq 2 w_1 e^{\tau\psi}(r\langle r\rangle^{-1} + \tau\langle r\rangle \psi_0'(r))|\partial_r u| + w_1 |\Delta\varphi||u| \\
&\leq 2e^{\tau\psi}(w_1 + \tau w_2)|\partial_r u| + C\tau^2 e^{\tau\psi}|w_1 u|
\end{aligned}
$$

where we have used (2.7) and that $w_1^2\langle r\rangle = \langle r\rangle^{1-2\delta} \leq 1$ (recall that $w_2^2 = \langle r\rangle \psi_0'(r)^2$). Then, we can apply several times Young's inequality with suitable values of ε, to obtain

$$
\begin{aligned}
2|(w_1 Au, w_1^{-1}(P + L + V^L - z)u)| &\leq 4\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + 4\tau^2 \|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2 \\
&\quad + C^2\tau^4 \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2 + 8\|w_1^{-1} e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)u\|^2.
\end{aligned}
$$

With these choices, returning to the estimate (3.4), the $4\tau^2\|e^{\frac{1}{2}\tau\psi} w_2 \partial_r u\|^2$ terms can be cancelled out. This yields

$$
\begin{aligned}
&(2\beta\tau - 6)\|e^{\frac{1}{2}\tau\psi} w_1 \nabla u\|^2 + \Big(2\beta\tau \mathrm{Re}(z) - K_3 - C^2\tau^4\Big)\, \|e^{\frac{1}{2}\tau\psi} w_1 u\|^2 \\
&\quad \leq 16\|w_1^{-1} e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)u\|^2 + 2|\mathrm{Im}(z)(Au, u)|. \qquad (3.5)
\end{aligned}
$$

We still need to control the last term on the right, that is $|\mathrm{Im}(z)(Au, u)|$. Unfortunately, it cannot be absorbed directly in the left hand side, since $|\nabla\varphi|$

does not have any decay at infinity. We proceed as follows. Using that $|\nabla\varphi| \le (1+\tau)e^{\tau} := a(\tau)$, Young's inequality yields

$$\begin{aligned}|\mathrm{Im}(z)(Au,u)| &= |\mathrm{Im}(z)((\nabla\varphi\cdot Du,u)+(u,\nabla\varphi\cdot Du))|\\ &\le 2a(\tau)|\mathrm{Im}(z)|\|\nabla u\|\|u\| \le |\mathrm{Im}(z)|\left(\frac{1}{\mathrm{Re}(z)^{1/2}}\|\nabla u\|^2 + a(\tau)^2\mathrm{Re}(z)^{1/2}\|u\|^2\right).\end{aligned} \tag{3.6}$$

We now estimate $\|\nabla u\|$ as follows:

$$\|\nabla u\|^2 = (Pu,u) = ((P-z)u,u) + \mathrm{Re}(z)\|u\|^2 + i\,\mathrm{Im}(z)\|u\|^2.$$

Taking the real part and adding and subtracting the long range potentials leads to

$$\begin{aligned}\|\nabla u\|^2 &= \mathrm{Re}((P-z)u,u) + \mathrm{Re}(z)\|u\|^2\\ &= \mathrm{Re}((P+L+V^L-z)u,u) - ((L+V^L)u,u) + \mathrm{Re}(z)\|u\|^2.\end{aligned}$$

Integrating by parts, as we did previously, gives

$$|((L+V^L)u,u)| \le |(W^L u,\nabla u)| + |(W^L\cdot\nabla u,u)| + \|V^L\|_{L^\infty}\|u\|^2,$$

and hence, using Young's inequality and taking $\mathrm{Re}(z) > 2\|W^L\|_{L^\infty}^2 + \|V^L\|_{L^\infty}$, yields

$$\begin{aligned}\frac{1}{2}\|\nabla u\|^2 &\le |((P+L+V^L-z)u,u)|\\ &\qquad + (2\|W^L\|_{L^\infty}^2 + \|V^L\|_{L^\infty} + \mathrm{Re}(z))\|u\|^2\\ &\qquad\qquad \le |((P+L+V^L-z)u,u)| + 2\mathrm{Re}(z)\|u\|^2.\end{aligned}$$

Inserting this in (3.6) and using that $|\mathrm{Im}(z)| \le 1 \le \mathrm{Re}(z)$ we get that

$$|\mathrm{Im}(z)(Au,u)| \le 2|((P+L+V^L-z)u,u)| + C(\tau)|\mathrm{Im}(z)|\mathrm{Re}(z)^{1/2}\|u\|^2.$$

Returning to (3.5) and using that $e^{\frac{1}{2}\tau\psi} \ge 1$ together with the previous estimate, yields

$$\begin{aligned}&(2\beta\tau-6)\|e^{\frac{1}{2}\tau\psi}w_1\nabla u\|^2 + (2\beta\tau\mathrm{Re}(z)-K_4)\,\|e^{\frac{1}{2}\tau\psi}w_1 u\|^2\\ &\qquad \le 32\|w_1^{-1}e^{\frac{1}{2}\tau\psi}(P+L+V^L-z)u\|^2 + C(\tau)|\mathrm{Im}(z)|\mathrm{Re}(z)^{1/2}\|u\|^2,\end{aligned}$$

where $K_4 = K_3 + C^2\tau^4 + 1$. To finish, we use the fact that

$$\nabla(e^{\frac{1}{2}\tau\psi}u) = e^{\frac{1}{2}\tau\psi}\nabla u + \frac{1}{2}\tau e^{\frac{1}{2}\tau\psi}(\nabla\psi)u,$$

in the first term on the left, and we fix $v = e^{\frac{1}{2}\tau\psi}u$. Then, since $|\nabla\psi| \le 1$, we obtain that

$$\|w_1\nabla v\|^2 \le 2\|e^{\frac{1}{2}\tau\psi}w_1\nabla u\|^2 + \tau^2\|e^{\frac{1}{2}\tau\psi}w_1 u\|^2,$$

and consequently

$$\begin{aligned}&(\beta\tau-3)\|w_1\nabla v\|^2+(2\beta\tau\mathrm{Re}(z)-K_4-(\beta\tau-3)\tau^2)\|w_1 v\|^2\\&\qquad\leq 32\|w_1^{-1}e^{\frac12\tau\psi}(P+L+V^L-z)e^{-\frac12\tau\psi}v\|^2+C(\tau)|\mathrm{Im}(z)|\mathrm{Re}(z)^{1/2}\|v\|^2,\end{aligned}$$

where in the last term we have used again that $e^{\frac12\tau\psi}\geq 1$. We choose now $\tau>6\beta^{-1}$, so that $\beta\tau-3\geq\frac12\beta\tau$, and $\mathrm{Re}(z)>\tau^{-1}\beta^{-1}(K_4+(\beta\tau-3)\tau^2)$. This gives

$$\begin{aligned}&\|w_1\nabla v\|^2+\mathrm{Re}(z)\|w_1 v\|^2\\&\quad\leq\frac{64}{\beta\tau}\|w_1^{-1}e^{\frac12\tau\psi}(P+L+V^L-z)e^{-\frac12\tau\psi}v\|^2+C(\tau,\beta)|\mathrm{Im}(z)|\mathrm{Re}(z)^{1/2}\|v\|^2,\end{aligned}$$

which proves the claim. □

We now give the proof of Lemma 3.1. Here the long range conditions on the potentials play an essential role.

Proof of Lemma 3.1. A direct computation shows that

$$i[L,A]u=4(\varphi''W^L-(\nabla W^L)\nabla\varphi)\cdot Du+2i(\nabla\varphi\cdot\nabla(\nabla\cdot W^L)-W^L\cdot(\nabla\Delta\varphi))u$$

where ∇W^L is the Jacobian matrix of W^L. We are going to use (2.6) several times. We begin by studying the first and last terms. Since $2\delta-1<\sigma$, we get

$$\begin{aligned}&|4(\varphi''W^L\cdot Du,u)-2i(W^L\cdot(\nabla\Delta\varphi)u,u)|\\&\qquad\leq C\tau^3\|\langle x\rangle^\sigma W^L\|_{L^\infty}(\|e^{\frac12\tau\psi}w_1\nabla u\|\|e^{\frac12\tau\psi}w_1u\|+\|e^{\frac12\tau\psi}w_1u\|^2).\end{aligned}\tag{3.7}$$

Also, since φ is radial, $\nabla W^L(\nabla\varphi)=(\partial_r\varphi)\partial_r W^L$, which implies

$$|((\nabla W^L)\nabla\varphi\cdot\nabla u,u)|\leq C\tau\|\langle x\rangle^{1+\sigma}\partial_r W^L\|_{L^\infty}\|e^{\frac12\tau\psi}w_1\nabla u\|\|e^{\frac12\tau\psi}w_1u\|.\tag{3.8}$$

Let us estimate the remaining term. By the Leibniz rule we have that

$$\begin{aligned}(\nabla\varphi\cdot\nabla(\nabla\cdot W^L))u&=\nabla\cdot((\nabla W^L)(\nabla\varphi)u)\\&\qquad-\nabla W^L(\nabla\varphi)\cdot\nabla u-(\partial_j\partial_k\varphi)\partial_k W^L_j u.\end{aligned}$$

We expand again the last term, so that we only have terms with radial derivatives of the magnetic potential,

$$(\partial_j\partial_k\varphi)\partial_k W^L_j u=\nabla\cdot(\varphi''(W^L)u)-(\nabla\Delta\varphi)W^L u-\varphi''(W^L)\cdot\nabla u,$$

where recall that we denote the Hessian of φ by φ''. Hence, integrating by parts the first term on the right hand side of the previous two expressions yields

$$\begin{aligned}\Big|\Big((\nabla\varphi\cdot\nabla(\nabla\cdot W^L))u,u\Big)\Big|&\leq C\tau^3(\|\langle x\rangle^{1+\sigma}\partial_r W^L\|_{L^\infty}+\|\langle x\rangle^\sigma W^L\|_{L^\infty})\times\dots\\&\qquad(\|e^{\frac12\tau\psi}w_1\nabla u\|\|e^{\frac12\tau\psi}w_1u\|+\|e^{\frac12\tau\psi}w_1u\|^2).\end{aligned}$$

Then, writing $M:=C\tau^3(\|\langle x\rangle^{1+\sigma}\partial_r W^L\|_{L^\infty}+\|\langle x\rangle^\sigma W^L\|_{L^\infty})$ for an appropriate constant $C=C(n,\delta)$, one has

$$\begin{aligned}|i([L,A]u,u)|&\leq 2M(\|e^{\frac12\tau\psi}w_1\nabla u\|\|e^{\frac12\tau\psi}w_1u\|+\|e^{\frac12\tau\psi}w_1u\|^2)\\&\leq\|e^{\frac12\tau\psi}w_1\nabla u\|^2+\Big(M^2+2\Big)\|e^{\frac12\tau\psi}w_1u\|^2.\end{aligned}$$

□

4 Shifting the long range estimate to H^{-1}

Even with the help of the large parameters τ and $\mathrm{Re}(z)$, we cannot introduce directly the short range magnetic perturbation in the right hand side of (2.8). This is due to the fact that $\nabla \cdot W^S$ is not necessarily an $L^\infty(\mathbb{R}^n)$ function under the assumed conditions (just (1.5)), which implies that the short range magnetic perturbation is not in general bounded as an operator from H^1 to L^2. However, as pointed out in [15], it is bounded from H^1 to H^{-1}. Motivated by this fact, we are going to derive a better version of estimate (2.8) now from H^1 to H^{-1}. Of course, this is not necessary when assuming the extra condition (1.6). In this case, we will just show by analogous methods that (2.8) can be improved to an estimate from H^2 to L^2.

From now on, just to simplify notation, we switch to the conventions of semi-classical analysis.

Definition 4.1. Let k be a nonnegative integer. We define the space $H^k_{scl}(\mathbb{R}^n) := H^k_{scl}$ as the $H^k(\mathbb{R}^n)$-Sobolev space with semiclassical parameter $h > 0$, equipped with the norm

$$\|u\|^2_{H^k_{scl}} = \sum_{|\alpha|\le k} \|h^{|\alpha|}\partial^\alpha u\|^2.$$

We also consider its dual space $H^{-k}_{scl}(\mathbb{R}^n) := H^{-k}_{scl}$ with norm given by

$$\|u\|_{H^{-k}_{scl}} = \sup_{\vartheta \in C^\infty_0(\mathbb{R}^n)\setminus\{0\}} \frac{\left|\langle u, \vartheta\rangle_{\mathbb{R}^n}\right|}{\|\vartheta\|_{H^k_{scl}}},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the distribution duality in $\mathbb{R}^n$.

In these estimates, the semiclassical structure emerges naturally by taking $h = \mathrm{Re}(z)^{-1/2}$, so that we now have $z = h^{-2} + i\mathrm{Im}(z)$. Also, let us write $w(x) = w_1(x) = \langle x\rangle^{-\delta}$. Under this framework, (2.8) can be written as follows:

$$\|wv\|^2_{H^1_{scl}} \le 64\beta^{-1}\tau^{-1}h^2\|w^{-1}e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)e^{-\frac{1}{2}\tau\psi}v\|^2 + bh\|v\|^2, \quad (4.1)$$

where $b = C(\beta, \tau)|\mathrm{Im}(z)|$. We want to prove the following proposition.

Proposition 4.2. *Assume that all the conditions in the statement of Lemma 2.4 hold, and that $\nabla \cdot W^L \in L^\infty(\mathbb{R}^n)$. Then, for any $v \in C^\infty_c(\mathbb{R}^n)$,*

$$\|wv\|^2_{H^{1+a}_{scl}} \le Cbh\|v\|^2_{H^{-1+a}_{scl}} + C\beta^{-1}\tau^{-1}h^2\|w^{-1}e^{\frac{1}{2}\tau\psi}(P + L + V^L - z)e^{-\frac{1}{2}\tau\psi}v\|^2_{H^{-1+a}_{scl}}, \quad (4.2)$$

whenever $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| \le 1$, $h < h_0(n, \delta, \tau, W^L, V^L) < 1$ and $a = 0, 1$. Here C is an absolute constant.

In the remaining part of this section, to simplify further the notation, we put

$$G_{h,\tau} := h^2 e^{\frac{1}{2}\tau\psi}(P + L + V^L)e^{-\frac{1}{2}\tau\psi}. \quad (4.3)$$

Recall that L was defined in (2.1). To prove the estimate (4.2) in the case $a = 0$, instead of commuting with the operator $\langle hD\rangle^{-1}$ to shift estimate (4.1) one derivative down, we follow [15] and commute with a resolvent operator (in the case $a = 1$ we only need to get an extra one derivative gain in (4.1)). In both cases, we need the following result.

Lemma 4.3. *Let $h > 0$, $\tau \geq 1$, $k \in \mathbb{R}$, and $a = 0, 1$. Consider in (4.3) $V^L \in L^\infty(\mathbb{R}^n, \mathbb{R})$, $W^L \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot W^L \in L^\infty(\mathbb{R}^n)$, and $\psi \in C^2(\mathbb{R}^n)$ independent of h and τ such that $|\nabla\psi|, |\Delta\psi| < \infty$. Then, for all $u \in H_{scl}^{-1+a}$ we have that*

$$\|w^k (G_{h,\tau} - i)^{-1} w^{-k} u\|_{H_{scl}^{1+a}} \leq 4\|u\|_{H_{scl}^{-1+a}}, \tag{4.4}$$

if $a = 0, 1$ and $h < c\tau^{-1}$ for a constant $c = c(V^L, W^L, \psi, k)$.

This lemma is essentially [15, Lemma 3.2]. Nonetheless, for the interested reader we give a proof in the appendix. We now prove estimate (4.2).

Proof of Proposition 4.2. To simplify the computations, throughout this proof we denote $G := G_{h,\tau}$. We can combine (4.1) and Lemma 4.3 to get (4.2), using that the identity

$$v = (h^2 z - i)(G - i)^{-1} v + (G - i)^{-1}(G - h^2 z)v, \tag{4.5}$$

holds for any $z \in \mathbb{C}$. Multiplying (4.5) by the weight w and taking the H_{scl}^{1+a} norm squared we have

$$\|wv\|^2_{H^{1+a}_{scl}} \leq 8\|w(G - i)^{-1} v\|^2_{H^{1+a}_{scl}} + 2\|w(G - i)^{-1}(G - h^2 z)v\|^2_{H^{1+a}_{scl}}, \tag{4.6}$$

since $|i - h^2 z| \leq 2$. From here we split the proof into two cases.
Case $a = 0$. We have shown that (4.1) holds for $v \in C_c^\infty$. We can easily extend this estimate for $v \in H^2_\delta$. Indeed, take a sequence of functions $v_j \in C_c^\infty$ such that $v_j \to v$ in H^2_δ. Applying (4.1) to the v_j, we can pass to the limit using that G is bounded from H^2_δ to L^2_δ. By Lemma 4.3 with $k = -1$, $(G - i)^{-1} v \in H^2_\delta$ so by the previous density argument, we can apply (4.1) to the first term of (4.6) with $a = 0$. Then

$$\|wv\|^2_{H^1_{scl}} \leq C\beta^{-1}\tau^{-1}h^{-2}\|w^{-1}(G - h^2 z)(G - i)^{-1} v\|^2 + Cbh\|(G - i)^{-1} v\|^2 + 2\|w(G - i)^{-1}(G - h^2 z)v\|^2_{H^1_{scl}}.$$

Using that the operators $(G - h^2 z)$ and $(G - i)^{-1}$ commute in the first term on the right hand side and taking h small enough such that $\tau^{-1}h^{-2}\beta^{-1} > 1$, yields

$$\|wv\|^2_{H^1_{scl}} \leq C\beta^{-1}\tau^{-1}h^{-2}\|w^{-1}(G - i)^{-1}(G - h^2 z)v\|^2 + Cbh\|(G - i)^{-1} v\|^2 + 2\beta^{-1}\tau^{-1}h^{-2}\|w(G - i)^{-1}(G - h^2 z)v\|^2_{H^1_{scl}}.$$

Hence, applying Lemma 4.3 with $k = -1$, $k = 0$, and $k = 1$ to each term on the right and using that $w \leq w^{-1}$ in the last term, we finally obtain

$$\|wv\|^2_{H^1_{scl}} \leq C\beta^{-1}\tau^{-1}h^{-2}\|w^{-1}(G - h^2 z)v\|^2_{H^{-1}_{scl}} + Cbh\|v\|^2_{H^{-1}_{scl}},$$

which combined with (4.3) yields the desired estimate.
Case $a = 1$. By a straightforward computation and applying Lemma 4.3 with $k, a = 1$ to each term on the right of (4.6) we get

$$\|wv\|^2_{H^2_{scl}} \leq 32(4\|wv\|^2 + \|w(G - h^2 z)v\|^2).$$

Hence, using (4.1) in the first term, and taking $\tau^{-1}h^{-2} > 1$ in the second gives the desired estimate, as in the previous case. □

5 Absorbing the short range potentials

In this section we finally prove Theorem 1.1. The first step is to introduce the short range perturbation in (4.2). Once we have an estimate for the full operator, we can fix an appropriate value of τ and remove the exponential conjugation. The final step is to extend by density the resulting estimate with the help of Friedrichs lemma. In this step we need to slightly strengthen the assumptions assuming (1.4) instead of just (1.9).

First, recall that $H = (D+W)^2 + V$. For any $u \in C_c^\infty(\mathbb{R}^n)$ we can write that

$$\begin{aligned} Hu &= Pu + W \cdot Du + D \cdot (Wu) + (V + |W|^2)u \\ &= Pu + 2W \cdot Du - i\mathrm{div}(W)u + (V + |W|^2)u, \end{aligned} \tag{5.1}$$

where one can verify that the term $\mathrm{div}(W)u$ is well defined as a distribution in H^{-1}, since $W \in L^\infty(\mathbb{R}^n)$ (we use the notation $-i\mathrm{div}(W)u$ instead of $(D \cdot W)u$ to avoid possible confusions). To simplify the main computations, in the rest of this section we are going to reduce to the case of the Hamiltonian $Hu = Pu + 2W \cdot Du - i\mathrm{div}(W)u + Vu$, omitting the term $|W|^2$. This is harmless, since $|W^L|^2$ behaves as a long range scalar potential and $|W^S|^2 + 2W^L \cdot W^S$ as a short range one, so essentially they can be included in V^L and V^S.

Proposition 5.1. *Assume that $W \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\nabla \cdot W^L \in L^\infty(\mathbb{R}^n)$, and $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ satisfy (1.3), (1.5), and (1.9) for some $\sigma > 0$. Let $1/2 < \delta < \infty$ and $\tau > \tau_0(\delta, W^S, V^S) \geq 1$. Then for any $v \in C_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned} \|wv\|^2_{H^{1+a}_{scl}} \leq\ & C(\beta,\tau)|\mathrm{Im}(z)|h\|v\|^2 \\ & + C\beta^{-1}\tau^{-1}h^2\|w^{-1}e^{\frac{1}{2}\tau\psi}(H-z)e^{-\frac{1}{2}\tau\psi}v\|^2_{H^{-1+a}_{scl}}. \end{aligned} \tag{5.2}$$

whenever $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| \leq 1$, $a = 0$, and $h \leq h_0(n, \delta, \tau, W, V) < 1$. Moreover, under the extra assumption (1.6) the previous estimate also holds for $a = 1$.

Proof. Again, we assume $1/2 < \delta < \min\{(\sigma+1)/2, 1\}$ without loss of generality. We first consider the case $a = 0$. Adding and subtracting the terms with the short range perturbation in the right hand side of (4.2), we have

$$\begin{aligned} &\|wv\|^2_{H^1_{scl}} \leq C\beta^{-1}\tau^{-1}h^2\|w^{-1}e^{\frac{1}{2}\tau\psi}(H-z)e^{-\frac{1}{2}\tau\psi}v\|^2_{H^{-1}_{scl}} \\ &+C\beta^{-1}\tau^{-1}\|w^{-1}h(2W^S\cdot D + i\tau W^S\cdot\nabla\psi - i\mathrm{div}(W^S) + V^S)v\|^2_{H^{-1}_{scl}} + Cbh\|v\|^2_{H^{-1}_{scl}}. \end{aligned}$$

As we mentioned previously, we can estimate the term $\mathrm{div}(W^S)v$ in the H^{-1}_{scl} norm, in fact, we have

$$w^{-1}\mathrm{div}(W^S)v = \nabla \cdot (w^{-1}W^S v) - W^S \cdot \nabla(w^{-1}v),$$

in the sense of distributions. Thus,

$$\begin{aligned} \|w^{-1}h\,\mathrm{div}(W^S)v\|_{H^{-1}_{scl}} &\leq \|h\nabla\cdot(w^{-1}W^S v)\|_{H^{-1}_{scl}} + \|W^S \cdot h\nabla(w^{-1}v)\|_{H^{-1}_{scl}} \\ &\leq \|w^{-1}W^S v\| + \|w^{-2}W^S\|_{L^\infty}\|w^2 h\nabla(w^{-1}v)\|_{H^{-1}_{scl}} \leq C\|w^{-2}W^S\|_{L^\infty}\|wv\|, \end{aligned} \tag{5.3}$$

since $|\nabla w^{-1}| \leq \delta w^{-1}$. Therefore, using that $|\nabla\psi| \leq 1$ and $w^{-1} \geq 1$, we obtain

$$\|wv\|^2_{H^1_{scl}} \leq C\beta^{-1}\tau^{-1}h^2\|w^{-1}e^{\frac{1}{2}\tau\psi}(H-z)e^{-\frac{1}{2}\tau\psi}v\|^2_{H^{-1}_{scl}}$$
$$+ C(\beta)\|wv\|^2_{H^1_{scl}}\Big((\tau^{-1}+\tau h^2)\|w^{-2}W^S\|^2_{L^\infty} + \tau^{-1}h^2\|w^{-2}V^S\|^2_{L^\infty}\Big) + Cbh\|v\|^2$$

for an appropriate constant $C(\beta) > 0$. Since $w^{-2} \leq \langle x\rangle^{1+\sigma}$, the short range conditions on the potentials guarantee that the L^∞ norms appearing in the previous estimate are finite. Hence taking $\tau > 4C(\beta)\|\langle x\rangle^{1+\sigma}W^S\|^2_{L^\infty}$ and $h^2 < \big(4C(\beta)\tau(\|\langle x\rangle^{1+\sigma}W^S\|^2_{L^\infty} + \|\langle x\rangle^{1+\sigma}V^S\|^2_{L^\infty})\big)^{-1}$ to absorb the middle term on the right in the left hand side, and using that $b = C(\delta,\tau)|\mathrm{Im}(z)|$ yields the desired result.

The case $a = 1$ is even simpler since we do not need the integration by parts in (5.3) thanks to the fact that the norm $\|\langle x\rangle^{1+\sigma}\nabla\cdot W^S\|_{L^\infty}$ is finite by (1.6). □

We are going to use the previous proposition to prove Theorem 1.1, but first we need a couple of lemmas. The first one is necessary to control the term with the factor $|\mathrm{Im}(z)|$ in (5.2).

Lemma 5.2. *Let $W \in L^\infty(\mathbb{R}^n,\mathbb{R}^n)$, $V \in L^\infty(\mathbb{R}^n,\mathbb{R})$, and $u \in C_c^\infty(\mathbb{R}^n)$. Then*

$$|\mathrm{Im}(z)|\|u\|^2 \leq |((H-z)u,u)|.$$

Proof. This follows by the symmetry of the operator H: in fact, by integration by parts, $\mathrm{Im}(Hu,u) = 0$, and therefore

$$\mathrm{Im}(z)\|u\|^2 = \mathrm{Im}(zu,u) = -\mathrm{Im}((H-z)u,u). \qquad \square$$

We now state the Friedrichs lemma as in Lemma 17.1.5 of [11] (see also Lemma 1.5.2 of [5]).

Lemma 5.3. *Let $v \in L^2$ and let $|a(x)-a(y)| \leq M|x-y|$ if $x,y \in \mathbb{R}^n$. If $\Phi \in C_c^\infty$ and $\Phi_\varepsilon(x) = \Phi(x/\varepsilon)\varepsilon^{-n}$, then*

$$\|(aD_jv)*\Phi_\varepsilon - a(D_jv*\Phi_\varepsilon)\|_{L^2} = o(1) \text{ as } \varepsilon \to 0.$$

We can now prove the main result of this paper.

Proof of Theorem 1.1. Let $u \in C_c^\infty(\mathbb{R}^n)$. We fix a sufficiently large τ so that (5.2) holds, and choose $v = e^{\frac{1}{2}\tau\psi}u$. Next, we remove the exponentials by using that $1 \leq e^{\frac{1}{2}\tau\psi} \leq e^{\frac{1}{2}\tau}$ (there are some extra terms appearing in the left hand side due to the H^{1+a}_{scl} norm, but they can be absorbed easily for $h < c\tau^{-1}$). Hence the estimate

$$\|wu\|^2_{H^{1+a}_{scl}} \leq C|\mathrm{Im}(z)|h\|u\|^2 + Ch^2\|w^{-1}(H-z)u\|^2_{H^{-1+a}_{scl}}$$

holds for $a = 0,1$, depending on the conditions assumed on W, and for some $C = C(\delta,V,W) > 0$ (also depending on the fixed τ). Then, we can apply Lemma 5.2 and Young's inequality to the first term on the right. Thus

$$C|\mathrm{Im}(z)|h\|u\|^2 \leq \frac{1}{4}\|wu\|^2_{H^1_{scl}} + C^2h^2\|w^{-1}(H-z)u\|^2_{H^{-1}_{scl}}.$$

This yields the estimate

$$\|wu\|_{H^{1+a}_{scl}} \le C(\delta, V, W) h \|w^{-1}(H-z)u\|_{H^{-1+a}_{scl}}. \tag{5.4}$$

This estimate holds under the assumption that $u \in C_c^\infty$. We are going to extend it for any $u \in H^1_{scl}$ such that $w^{-1}(H-z)u \in H^{-1+a}_{scl}$. We restrict ourselves to the case of $a = 0$, since $a = 1$ follows from the same arguments with minor modifications (the condition (1.6) is again essential in the case $a = 1$ so that the short range terms are bounded in L^2 instead of just in H^{-1}). Also, we now drop temporarily the semiclassical spaces since all the convergence arguments we need work independently of h.

Let $\Phi_\varepsilon(x) := \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$, where Φ is a standard smooth mollifier, and $\chi_\varepsilon := \chi(\varepsilon x)$ where $0 \le \chi(x) \le 1$ is a smooth cut-off function such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ if $|x| \ge 2$. Let also $u \in H^1$, and $u_\varepsilon = \chi_\varepsilon(u * \Phi_\varepsilon)$. Then $u_\varepsilon \in C_c^\infty$ and we have that $u_\varepsilon \to u$ in H^1 as $\varepsilon \to 0$. We would like to show that $w^{-1}(H-z)u_\varepsilon \to w^{-1}(H-z)u$ in H^{-1} when $\varepsilon \to 0$. Notice that since the potentials are bounded, for any $u \in H^1$ we have $(H-z)u \in H^{-1}$.

We decompose H in a long range Hamiltonian and a short range perturbation $Hu = (H^L + S)u$ where

$$\begin{aligned} H^L u &= (-\Delta + 2W^L \cdot D + V^L)u, \\ Su &= (2W^S \cdot D - i\operatorname{div}(W^S) - i\operatorname{div}(W^L) + V^S)u. \end{aligned}$$

The perturbation S is bounded from H^1 to H^{-1}, in fact a better estimate holds:

$$\|w^{-1}Su\|_{H^{-1}} \le C\|wu\|. \tag{5.5}$$

This follows directly from the short range conditions (1.5) on W^S, V^S and the long range conditions (1.4) on W^L (we have already controlled the term $\nabla \cdot W^S$ in (5.3)). Therefore, it is enough to show that $w^{-1}(H^L - z)u_\varepsilon \to w^{-1}(H^L - z)u$ in H^{-1} when $\varepsilon \to 0$.

Now, let $v_\varepsilon = u * \Phi_\varepsilon$ so that $u_\varepsilon = \chi_\varepsilon v_\varepsilon$. Then

$$\begin{aligned} \|\chi_\varepsilon w^{-1}(H^L - z)v_\varepsilon - w^{-1}(H^L - z)u_\varepsilon\|_{H^{-1}} &= \|w^{-1}[\chi_\varepsilon, H^L - z]v_\varepsilon\|_{H^{-1}} \\ = \|w^{-1}(\Delta\chi_\varepsilon + 2\nabla\chi_\varepsilon \cdot \nabla + 2iW^L \cdot \nabla\chi_\varepsilon)v_\varepsilon\|_{H^{-1}} &\le C\varepsilon^{1-\delta}\|v_\varepsilon\|, \end{aligned} \tag{5.6}$$

where we have used that $|w^{-1}\nabla\chi_\varepsilon|, |w^{-1}\Delta\chi_\varepsilon| \le C\varepsilon^{1-\delta}$ to get the last inequality.

By the Friedrichs lemma, commuting the convolution with the long range Hamiltonian H^L, one gets an error term which is small in the L^2 norm as $\varepsilon \to 0$. Since

$$\begin{aligned} &\|\Phi_\varepsilon * [w^{-1}(H^L - z)u] - w^{-1}(H^L - z)v_\varepsilon\| \le \\ &\quad \|\Phi_\varepsilon * (w^{-1}\Delta u) - w^{-1}(\Delta v_\varepsilon)\| + 2\|\Phi_\varepsilon * (w^{-1}W^L \cdot \nabla u) - w^{-1}(W^L \cdot \nabla v_\varepsilon)\| \\ &\qquad + \|\Phi_\varepsilon * [w^{-1}(V^L - z)u] - w^{-1}(V^L - z)v_\varepsilon\|, \end{aligned} \tag{5.7}$$

we can verify this term by term. First, if $1/2 < \delta < 1$ (which can be assumed without loss of generality), w^{-1} and all its derivatives are Lipschitz functions in $\mathbb{R}^n$. As a consequence, as $\varepsilon \to 0$,

$$\|\Phi_\varepsilon * (w^{-1}\Delta u) - w^{-1}(\Delta v_\varepsilon)\| = o(1),$$

applying Lemma 5.3. To control the last two terms in the same way, we need to impose the long range conditions $w^{-1}|\nabla V^L| \leq C$ and $w^{-1}|\nabla W^L| \leq C$ on the potentials so that both $w^{-1}V^L$ and $w^{-1}W^L$ have bounded gradients in $\mathbb{R}^n$. Then, using this in (5.7) yields

$$\|\Phi_\varepsilon * [w^{-1}(H^L - z)u] - w^{-1}(H^L - z)v_\varepsilon\|_{H^{-1}} = o(1). \tag{5.8}$$

As a consequence, using (5.6) and (5.8) and using the fact that one has $w^{-1}(H^L - z)u \in H^{-1}$, we get that

$$\begin{aligned}
&\|w^{-1}(H^L - z)u_\varepsilon - w^{-1}(H^L - z)u\|_{H^{-1}} \\
&\quad \leq \|w^{-1}(H^L - z)u_\varepsilon - \chi_\varepsilon w^{-1}(H^L - z)v_\varepsilon\|_{H^{-1}} \\
&\qquad + \|\chi_\varepsilon w^{-1}(H^L - z)v_\varepsilon - \chi_\varepsilon \Phi_\varepsilon * [w^{-1}(H^L - z)u]\|_{H^{-1}} \\
&\qquad + \|\chi_\varepsilon \Phi_\varepsilon * [w^{-1}(H^L - z)u] - w^{-1}(H^L - z)u\|_{H^{-1}} = o(1),
\end{aligned}$$

and hence $w^{-1}(H^L - z)u_\varepsilon \to w^{-1}(H^L - z)u$ in H^{-1}. We can use now (5.5) to conclude that $w^{-1}(H - z)u_\varepsilon \to w^{-1}(H - z)u$ when $\varepsilon \to 0$. This shows that (5.4) holds (with $a = 0$) for any $u \in H^1$.

Let us now introduce the resolvent operator $R(z) = (H - z)^{-1}$. Under the conditions assumed on the potentials, H is self-adjoint (see Proposition A.1 in the Appendix). As a consequence $R(z)$ is well defined for every $z \in \mathbb{C}$ such that $\mathrm{Im}(z) \neq 0$ and it satisfies the estimate

$$\|R(z)f\| \leq \frac{1}{|\mathrm{Im}(z)|}\|f\|,$$

for every $f \in L^2$. This means that $R(z)g$, $\mathrm{Im}(z) \neq 0$, is well defined for $g \in L^2_\delta \subset L^2$. Also, if $g \in C_c^\infty$ the previous estimates imply that $u = R(z)g \in H^1$ (or H^2, if (1.6) holds). To see this, notice we have that $Hu = g + zu$, and since $u \in L^2$, all the terms in Hu must have at least H^{-1} regularity, except perhaps for the term Δu (for the short range perturbation see (5.5)). But since g is smooth, we must also have $\Delta u \in H^{-1}$ which shows that $u \in H^1$.

Therefore we can apply (5.4) with $a = 0$ to the function $u = R(z)g$, taking $g = h\partial^{\alpha_2} f$, for $f \in C_c^\infty$ and $|\alpha_2| \leq 1$. With this choice of g we can finally get rid of the semiclassical H^{-1}_{scl} norm in the right hand side of (5.4), and using that $h^{-2} = \mathrm{Re}(z) = \lambda$, this yields

$$\lambda \|R(z)\partial^{\alpha_2} f\|^2_{L^2_{-\delta}} + \|\partial^{\alpha_1} R(z)\partial^{\alpha_2} f\|^2_{L^2_{-\delta}} \leq C(\delta, V, W)\lambda^{|\alpha_2|}\|f\|^2_{L^2_\delta},$$

for every $f \in C_c^\infty$ and $|\alpha_1| = 1$. This estimate is the same as (1.7), and since C_c^∞ is dense in L^2_δ it can be extended for every $f \in L^2_\delta$. This is enough to finish the proof of (1.7). As mentioned previously, the proof of (1.8) from (5.4) with $a = 1$ is completely analogous to the case $a = 0$. This concludes the proof of the main theorem. □

6 The limiting absorption principle

In this section we prove Theorem 1.2 from Theorem 1.1. The fact that one can define the resolvent $R(\lambda \pm i0)$ as a bounded operator between the L^2_δ and $L^2_{-\delta}$ spaces is known as the limiting absorption principle.

To define the resolvent when $\mathrm{Im}(z) = 0$ one needs show that the limit $R(\lambda \pm i0)f = \lim_{\varepsilon\to 0} R(\lambda \pm i\varepsilon)f$ exists in $L^2_{-\delta}$. This follows from (1.8) if one can show that

$$\|\partial_r u - i\lambda u\|_{L^2_{\delta-1}} \le C\|(H-z)u\|_{L^2_\delta}, \tag{6.1}$$

holds for $1/2 < \delta < 1$ and $u \in H^1$, or other analogous condition. The previous estimate is known as a Sommerfeld radiation condition, see [14,17] for more details. In our case we do not prove a Sommerfeld radiation condition like (6.1), we use instead the limiting absorption principle already proved in [11, Theorem 30.2.10]. This holds assuming that W^S is continuous in addition to the conditions assumed in Theorem 1.1. To state Hörmander's result we need to introduce the Agmon-Hörmander space B and its dual B^*.

$$\|v\|_B = \sum_{j=1}^{\infty}\left(R_j \int_{X_j} |v|^2\, dx\right)^{1/2} < \infty,$$

$$\|v\|_{B^*} = \sup_{j>0}\left(R_j^{-1} \int_{X_j} |v|^2\, dx\right)^{1/2} < \infty,$$

where $R_0 = 0$, $R_j = 2^{j-1}$ for $j > 1$ and $X_j = \{x \in \mathbb{R}^n : R_{j-1} \le |x| \le R_j\}$.

Theorem 6.1 (Theorem 30.2.10 of [11])**.** *Assume that W and V satisfy (1.3)–(1.6). Also, assume that W^S is continuous. Then the eigenvalues $\lambda > 0$ of H are of finite multiplicity, and form a set Λ which is discrete in $\mathbb{R}_+$. Moreover, if $\lambda \in \mathbb{R}_+ \setminus \Lambda$ and $\mathrm{Re}(z) = \lambda$, then $\partial^\alpha R(z)f \to \partial^\alpha R(\lambda \pm i0)f$ in the weak* topology of B^* for every $f \in B$ and $|\alpha| \le 2$, as $z \to \lambda$ in the respective complex half planes.*

With this theorem we can finally define the resolvent operator $R(\lambda \pm i0)$ in order to prove Theorem 1.2, but we would like to have convergence in the $L^2_{-\delta}$ spaces. This follows from the next brief lemma.

Lemma 6.2. *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence in B^* such that $u_j \to u$ in the weak* topology of B^*. Then $u_j \to u$ converges weakly in $L^2_{-\delta}$.*

Proof. It follows directly from the fact that $\|v\|_B \le C\|v\|_{L^2_\delta}$ (and hence that $L^2_\delta \subset B$ and $B^* \subset L^2_{-\delta}$ continuously). □

Proof of Theorem 1.2. By the previous lemma and Theorem 6.1 we have that, for every $f \in L^2_\delta$ and $\lambda \in \mathbb{R}_+ \setminus \Lambda$, $\partial^\alpha R(z)f \to \partial^\alpha R(\lambda \pm i0)f$ converges weakly in $L^2_{-\delta}$ as $\mathrm{Im}(z) \to 0$. Now, let $f \in L^2_\delta$. By Theorem 1.1, we have that if $\mathrm{Im}(z) \ne 0$, $\partial^\alpha R(z)f$ is bounded in $L^2_{-\delta}$, and the right hand side of (1.8) is independent of $\mathrm{Im}(z)$. Since bounded sets are precompact in the weak topology, this implies that there is a positive sequence $(\varepsilon_j)_{j\ge 1}$, $\varepsilon_j \to 0$, such that

$$\partial^\alpha R(\lambda \pm i\varepsilon_j)f \rightharpoonup \partial^\alpha R(\lambda \pm i0)f \quad \text{weakly in } L^2_{-\delta}.$$

As a consequence, $\|\partial^\alpha R(\lambda \pm i0)f\|_{L^2_{-\delta}} \le \liminf_{j\to\infty} \|\partial^\alpha R(\lambda \pm i\varepsilon_j)f\|_{L^2_{-\delta}}$. Theorem 1.1 yields directly the estimate (1.11). □

A Appendix

We now show that H is self-adjoint with form domain H^1. We define the sesquilinear form $q_H(u,v) := (u, Hv)$ for $u \in H^1$ and $v \in C_c^\infty$. Under these assumptions, by integration by parts one can show that

$$q_H(u,v) = (Du, Dv) + (Du, Wv) + (Wu, Dv) + (u, (V + |W|^2)v). \tag{A.1}$$

Then, since $W \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $q_H(u,v)$ makes sense for all $u, v \in H^1$.

Proposition A.1. *Let $W \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$. Then there is a unique self-adjoint operator H with form domain H^1, such that (A.1) holds for all $u, v \in H^1$.*

Proof. The proof follows from [12, Theorem X.17]. Thanks to this theorem, it is enough to show that the form q_H is relatively bounded with respect to the form associated to the negative Laplacian, that is $q_{-\Delta}(u,v) = (Du, Dv)$. This is immediate by Young's inequality. If $u \in H^1$, for every $\varepsilon > 0$ one has

$$|(Du, Wu) + (Wu, Du) + (u, (V + |W|^2)u)| \leq \varepsilon \|\nabla u\|^2 + ((\varepsilon^{-1} + 1)\|W\|_{L^\infty}^2 + \|V\|_{L^\infty})\|u\|^2,$$

so actually the relative bound is zero. □

We now give the proof of a couple of auxiliary results used in the paper.

Proof of Lemma 2.3. We have that

$$\begin{aligned} \tilde{\varphi}_0(r) &= \langle r \rangle, & \tilde{\varphi}_0'(r) &= r\langle r \rangle^{-1}, \\ \psi_0'(r) &= (2\delta - 1)r\langle r \rangle^{-2\delta-1}, & \psi_0''(r) &= (2\delta - 1)\langle r \rangle^{-2\delta-3}\left(1 - 2\delta r^2\right). \end{aligned}$$

First, we combine the first and third terms on the right hand side of (2.4) and show that

$$\left(e^{\tau\psi}(\tilde{\varphi}_0\psi_0''(r) + 2\tilde{\varphi}_0'(r)\psi_0'(r))\partial_r u, \partial_r u\right) > \alpha(e^{\tau\psi}\langle r \rangle^{-2\delta}\partial_r u, \partial_r u), \tag{A.2}$$

for $\alpha = (2 - 2\delta)(2\delta - 1)$. This follows from the fact that

$$\begin{aligned} \tilde{\varphi}_0(r)\psi_0''(r) + 2\tilde{\varphi}_0'(r)\psi_0'(r) &= (2\delta - 1)\langle r \rangle^{-2\delta-2}(1 + (2 - 2\delta)r^2) \\ &> (2 - 2\delta)(2\delta - 1)\langle r \rangle^{-2\delta}, \end{aligned}$$

since $0 < 2 - 2\delta < 1$. Then, using (A.2) in (2.4) we obtain that

$$\begin{aligned} (\varphi''(x)\nabla u, \nabla u) &\geq \alpha\tau(e^{\tau\psi}\langle r \rangle^{-2\delta}\partial_r u, \partial_r u) \\ &\quad + \tau^2(e^{\tau\psi}\tilde{\varphi}_0(\psi_0'(r))^2\partial_r u, \partial_r u) + \tau(e^{\tau\psi}\tilde{\varphi}_0\frac{\psi_0'}{r}\nabla^\perp u, \nabla^\perp u). \end{aligned} \tag{A.3}$$

Therefore, using that $\tilde{\varphi}_0\frac{\psi_0'}{r} = (2\delta - 1)\langle r \rangle^{-2\delta}$, and that $\alpha < (2\delta - 1)$ we get

$$\begin{aligned} (\varphi''(x)\nabla u, \nabla u) &\geq \alpha\tau(e^{\tau\psi}\langle r \rangle^{-2\delta}\nabla u, \nabla u) \\ &\quad + \tau^2(e^{\tau\psi}\tilde{\varphi}_0(\psi_0'(r))^2\partial_r u, \partial_r u). \end{aligned} \tag{A.4}$$

This yields (2.5). (2.6) follows by direct computation. □

Proof of Lemma 4.3. The proof is similar to [15, Lemma 3.2]. First we define $\widetilde{\psi} := \tau^{-1}k\log w + \frac{1}{2}\psi$, so that we have $w^k e^{\frac{1}{2}\tau\psi} = e^{\tau\widetilde{\psi}}$. By the conditions assumed on φ and since $\tau \geq 1$, we have that $|\nabla\widetilde{\psi}|, |\Delta\widetilde{\psi}| \leq C(k)$, where we remark that $C(k) \geq 1$ can be chosen to be independent of h and τ. By direct computation we get

$$\begin{aligned} w^k(G_{h,\tau} - i)w^{-k}v &= (e^{\tau\widetilde{\psi}}h^2\left(P + 2W^L \cdot D - i\mathrm{div}(W^L) + V^L\right)e^{-\tau\widetilde{\psi}} - i)\,v \\ &= (h^2P - i)v + Q_{h,\tau}v, \end{aligned}$$

where $Q_{h,\tau}$ is a first order operator defined by

$$\begin{aligned} Q_{h,\tau} v = h^2 \left(-\tau^2 |\nabla \widetilde{\psi}|^2 + \tau \Delta \widetilde{\psi} + 2i\tau W^L \cdot \nabla \widetilde{\psi} - i \mathrm{div}(W^L) + V^L \right) v \\ + 2\, h (\tau \nabla \widetilde{\psi} - i W^L) \cdot h \nabla v. \end{aligned} \tag{A.5}$$

Using the Fourier transform, one can easily check that

$$\|(h^2 P - i)^{-1} v\|_{H^{1+a}_{scl}} \le 2 \|v\|_{H^{-1+a}_{scl}}.$$

We also consider the resolvent identity

$$(h^2 P - i + Q_{h,\tau})^{-1} = (h^2 P - i)^{-1} + (h^2 P - i)^{-1} Q_{h,\tau} (h^2 P - i + Q_{h,\tau})^{-1},$$

which allows us to show that

$$\begin{aligned} \|(h^2 P - i + Q_{h,\tau})^{-1} v\|_{H^{1+a}_{scl}} \le 2 \|v\|_{H^{-1+a}_{scl}} + 2 \|Q_{h,\tau}\|_{\mathcal{L}(H^{1+a}_{scl},\, H^{-1+a}_{scl})} \\ \times \|(h^2 P - i + Q_{h,\tau})^{-1} v\|_{H^{1+a}_{scl}}. \end{aligned} \tag{A.6}$$

Then, since $a = 0, 1$, taking the L^2 norm of (A.5) we obtain

$$\|Q_{h,\tau}\|_{\mathcal{L}(H^{1+a}_{scl},\, H^{-1+a}_{scl})} \le \|Q_{h,\tau}\|_{\mathcal{L}(H^{1+a}_{scl},\, L^2)} \le \frac{1}{4},$$

whenever

$$h < \tau^{-1} 18 C(k)^2 (1 + \|W^L\|_{L^\infty} + \|V^L\|_{L^\infty} + \|\nabla \cdot W^L\|_{L^\infty})^{-1}.$$

This implies the desired result by absorbing the second term on the right hand side of (A.6) in the left. □

References

1. S. Agmon, Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** no. 2, 151–218, (1975).
2. N. Burq, D. Dos Santos Ferreira, K. Krupchyk, From semiclassical Strichartz estimates to uniform L^p resolvent estimates on compact manifolds, Int. Math. Res. Not. **16**, 5178–5218, (2018).
3. F. Cardoso, C. Cuevas, G. Vodev, High frequency resolvent estimates for perturbations by large long-range magnetic potentials and applications to dispersive estimates, Annales Henri Poincaré. A Journal of Theoretical and Mathematical Physics **14**, no. 1, 95–117, (2013).
4. F. Cardoso, C. Cuevas, G. Vodev, Resolvent estimates for perturbations by large magnetic potentials, J. Math. Phys. **55** no. 2, 023502, 8, (2014).
5. P. Cherrier, A. Milani, *Linear and quasi-linear evolution equations in Hilbert spaces*, Grad. Stud. Math. **125**, American Mathematical Society, Providence, RI, 2012.
6. M.B. Erdogan, M. Goldberg, W. Schlag, Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in $\mathbb{R}^3$, Journal of EMS **10** no. 2, 507–531, (2008).
7. M.B. Erdogan, M. Goldberg, W. Schlag, Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions, Forum Math. **21**, no. 4, 687–722, (2009).
8. G. Eskin, *The analysis of linear partial differential operators*. Grad. Stud. Math. **123**, American Mathematical Society, Providence, RI, (2011).
9. M. Goldberg, Strichartz estimates for Schrödinger operators with a non-smooth magnetic potential, Discrete Contin. Dyn. Syst. **31** no. 1, 109–118, (2011).
10. C. Guillarmou, A. Hassell, K. Krupchyk, Eigenvalue bounds for non-self-adjoint Schrödinger operators with non-trapping metrics, arXiv:1709.09759 [math.AP] (2017).
11. L. Hörmander, *The analysis of linear partial differential operators*, vol. I-IV, Springer-Verlag, Berlin, (1983–1985).
12. M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. I-IV, Academic Press, (1975–1980).

13. D. Robert, Asymptotique de la phase de diffusion à haute énergie pour des perturbations du second ordre du laplacien, Annales de l'ENS **25**, 107–134 (1992).
14. I. Rodnianski, T. Tao, Effective limiting absorption principles, and applications, Comm. Math. Phys. **333**, no. 1, 1–95, (2015).
15. G. Vodev, Resolvent estimates for the magnetic Schrödinger operator, Anal. PDE. **7** no. 7, 1639–1648, (2014).
16. D.R. Yafaev, Mathematical scattering theory. Analytic theory. AMS (2010).
17. M. Zubeldia, The forward problem for the electromagnetic Helmholtz equation, PhD Thesis (2012).
18. M. Zubeldia, Limiting absorption principle for the electromagnetic Helmholtz equation with singular potentials, Proc. Roy. Soc. Edinburgh Sect. A **144**, no. 4, 857–890 (2014).