APPROXIMATION OF $W^{1,p}$ SOBOLEV HOMEOMORPHISM BY DIFFEOMORPHISMS AND THE SIGNS OF THE JACOBIAN

DANIEL CAMPBELL, STANISLAV HENCL, AND VILLE TENGVALL

Dedicated to Professor Jan Malý on his 60th birthday

Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$, be a domain and $1 \leq p < [n/2]$, where $[a]$ stands for the integer part of a. We construct a homeomorphism $f \in W^{1,p}((-1,1)^n, \mathbb{R}^n)$ such that $J_f = \det Df > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. It follows that there are no diffeomorphisms (or piecewise affine homeomorphisms) f_k such that $f_k \to f$ in $W^{1,p}$.

1. Introduction

The problem of approximating homeomorphisms $f : \mathbb{R}^n \supseteq \Omega \longrightarrow f(\Omega) \subseteq \mathbb{R}^n$ with either diffeomorphisms or piecewise-affine homeomorphisms has proven to be both very challenging and of great interest in a variety of contexts. As far as we know, in the simplest non-trivial setting (i.e. $n = 2$, approximations in the L^∞-norm) the problem was solved by Radó [38]. Due to its fundamental importance in geometric topology, the problem of finding piecewise affine homeomorphic approximations in the L^∞-norm and dimensions $n > 2$ was deeply investigated in the '50s and '60s. In particular, it was solved by Moise [33] and Bing [8] in the case $n = 3$ (see also the survey book [34]), while for contractible spaces of dimension $n \geq 5$ the result follows from theorems of Connell [13], Bing [9], Kirby [29] and Kirby, Siebenmann and Wall [30] (for a proof see, e.g., Rushing [40, Theorem 4.11.1.] or Luukkainen [31]). Finally, twenty years later, while studying the class of quasi-conformal manifolds, Donaldson and Sullivan [16] proved that the result is false in dimension 4.

After the L^∞-approximation problem had been completely solved, the question of approximating homeomorphisms revived again in the altogether different context for variational models in nonlinear elasticity. Let us briefly explain this. Let $\Omega \subset \mathbb{R}^n$ be a domain which models a body made out of homogeneous elastic material, and suppose that a mapping $f : \Omega \to \mathbb{R}^n$ is modeling the deformation of this body with prescribed boundary values. If we want to study the properties of the deformation in the setting of nonlinear elasticity theory of Antman, Ball and Ciarlet, see e.g. [2, 3, 4, 12], we are led to study the existence and regularity properties of minimizers of the energy functionals of the form

$$I(f) = \int_\Omega W(Df)\, \mathrm{d}x,$$

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where $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ is so-called stored-energy functional, and Df is the differential matrix of a deformation f. In order for this model to be physically relevant we have to require this model to satisfy the following conditions:

(W1) $W(A) \to +\infty$ as $\det A \to 0$, which prevents too high compression of the elastic body.

(W2) $W(A) = +\infty$ if $\det A \leq 0$, which guarantees that the orientation is preserved.

In particular, it follows that if f is an admissible deformation with finite energy, then we have

$$J_f(x) := \det Df(x) > 0 \quad \text{for a.e. } x \in \Omega.$$

Using other assumptions one can prove that the mapping with finite energy is continuous and one-to-one, which corresponds to the non-impenetrability of the matter. Therefore the natural candidate for a minimizer is in fact a homeomorphism. Hence, when we study this model it is natural to restrict our attention only on Sobolev homeomorphisms where the Jacobian does not change sign.

As pointed out by Ball in [5, 6] (who ascribes the question to Evans [18]), an important issue toward understanding the regularity of the minimizers in this setting would be to show the existence of minimizing sequences given by piecewise affine homeomorphisms or by diffeomorphisms. In particular, a first step would be to prove that any homeomorphism $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, $p \in [1,+\infty)$, can be approximated in $W^{1,p}$ by piecewise affine ones or smooth ones. One very significant reason why this would be desirable, is that regularity is typically often proven by testing the weak equation or the variation formulation by the solution itself; but unless one has some a priori regularity of the solution, such a test may not make sense. In order to solve this problem it would be possible to test the equation with a smooth test mapping which is close to the given homeomorphism instead. Here we see the necessity for the approximations to be homeomorphisms whose image is the same as that of the approximated map, otherwise this sequence would have nothing in common with our original problem. Besides non-linear elasticity, an approximation result of homeomorphisms with diffeomorphisms would be a very useful tool in and of itself as it would allow a number of proofs to be significantly simplified and lead to some stronger results. Let us note that finding diffeomorphisms near a given homeomorphism is not an easy task, as the usual approximation techniques like mollification or Lipschitz extension using the maximal operator destroy, in general, injectivity.

Let us describe the results in this direction. The first positive results were achieved by Mora-Corral [35] on planar homeomorphisms smooth outside a point and by Bellido and Mora-Corral [7] on approximation in Hölder continuous maps. Let us also note that the problem of approximation by smooth or piecewise affine planar homeomorphisms are in fact equivalent by the result of Mora-Corral and Pratelli [36]. The celebrated breakthrough result in the area which stimulated much interest in the subject was given by Iwaniec, Kovalev and Onninen in [27], [28], where they found diffeomorphic approximations to any homeomorphism $f \in W^{1,p}(\Omega,\mathbb{R}^2)$, for any $1 < p < \infty$ in the $W^{1,p}$ norm. The remaining missing case $p = 1$ in the plane has been solved by Hencl and Pratelli in [25] by a different method. This method was extended by Campbell [10] to give a different proof of the $W^{1,p}$, $p > 1$, case and to prove the result also for Orlicz-Sobolev spaces. The problem of approximating homeomorphisms with diffeomorphisms cannot be considered entirely closed even in the planar case. Another problem mentioned in [27] is to approximate both a map and its inverse simultaneously in $W^{1,p}$. The first results in this area was given by Daneri and Pratelli in [15] for all $1 \leq p < \infty$ under the additional assumption that the mapping is bi-Lipschitz. Recently Pratelli [37] has answered this question for $p = 1$ (without any additional assumptions) using the technique of [25]. The cases $p > 1$ (especially $p = 2$) which are even more important in terms of their application are still open.

And even more interesting open problem is the approximation of Sobolev homeomorphism in dimension $n = 3$ as there are no results in this direction so far. The only breakthrough result in higher dimension is the result of Hencl and Vejnar in [26] that there is a homeomorphism in $W^{1,1}$ for $n \geq 4$ which cannot be approximated by diffeomorphisms. The main result of this paper is the following extension, which shows that the problem is not in the special choice of nonreflexive space $W^{1,1}$.

Theorem 1.1. *Let $n \geq 4$ and $1 \leq p < [n/2]$. Then there exists a homeomorphism $f \in W^{1,p}((-1,1)^n, \mathbb{R}^n)$ such that there are no diffeomorphisms (or piecewise affine homeomorphisms) $f_k : (-1,1)^n \to \mathbb{R}^n$ such that $f_k \to f$ in $W^{1,p}_{\mathrm{loc}}((-1,1)^n, \mathbb{R}^n)$.*

Here $[n/2]$ denotes the integer part of $n/2$, i.e. $1 \leq p < 2$ for $n = 4, 5$, $1 \leq p < 3$ for $n = 6, 7$ and so on. This result is deeply connected with the sign of the Jacobian of a homeomorphism. As we mentioned before in models of nonlinear elasticity one usually assumes that $J_f > 0$ a.e. (or at least $J_f \geq 0$ a.e.). It is therefore natural to ask if this condition is automatically satisfied (up to a reflection) in the reasonable class of mappings. This problem was promoted by Hajlasz, see e.g. Goldstein and Hajlasz [21]. As each homeomorphism on a domain is either sense-preserving or sense-reversing (see Preliminaries) we can equivalently ask if the topological (sense-preserving) and analytical ($J_f \geq 0$) notion of orientation are the same.

Another reason to study nonnegativity of the Jacobian comes from the well-known area formula which is one of the most fundamental tools in the area. For a Sobolev homeomorphism $f : \Omega \to \mathbb{R}^n$ for which the Lusin's condition (N) (i.e. sets of null measure are always mapped to sets of null measure) holds we have

$$\int_{\Omega} \eta(f(x))\, |J_f(x)|\, \mathrm{d}x = \int_{f(\Omega)} \eta(y)\, \mathrm{d}y \tag{1.1}$$

for every nonnegative Borel function $\eta : f(\Omega) \to [0, \infty]$ (see Federer [19]). If we knew that $J_f \geq 0$ a.e. we could write the formula (1.1) without absolute values.

It is relatively easy to show that every topologically sense-preserving Sobolev homeomorphism which is differentiable almost everywhere has nonnegative Jacobian almost everywhere, see [32, Lemma 2.14]. Therefore every sense-preserving planar homeomorphism in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, and more generally every sense-preserving homeomorphism in $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $p > n - 1$, satisfies $J_f \geq 0$ a.e. (see [23, Corollary 2.25 and Theorem 5.22.]). However, when we study homeomorphisms in $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $n \geq 3$ and $1 \leq p \leq n - 1$ it might happen that the mapping is nowhere differentiable even under some additional assumptions, see e.g. [14]. Thus the previous argument which heavily uses differentiability of the mapping cannot be used anymore when $f \in W^{1,p}$, $p \in [1, n - 1]$.

In [24] Hencl and Malý were able to overcome the difficulties caused by the lack of differentiability by giving the first nontrivial positive answer to the question about the nonnegativity of the Jacobian of Sobolev homeomorphisms. More precisely, they showed that every sense-preserving Sobolev homeomorphism $f \in W^{1,p}_{\rm loc}(\Omega, \mathbb{R}^n)$ with $p > [n/2]$ has nonnegative Jacobian at almost every point. The proof was based on the approximative differentiability of Sobolev mappings and on the topological invariance of the linking number under homeomorphisms. The restriction $p > [n/2]$ in their proof comes from the linking number argument where one has to require the mapping to behave geometrically nicely on both "links". Here we show that somewhat surprisingly the strange exponent $[n/2]$ is indeed the borderline exponent for this question.

Theorem 1.2. *Let $n \geq 4$ and $1 \leq p < [n/2]$. Then there is a homeomorphism $f \in W^{1,p}((-1,1)^n, \mathbb{R}^n)$ such that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure.*

This result for $p = 1$ was shown by Hencl and Vejnar in [26] and as in their paper Theorem 1.1 now follows easily. Indeed, assume on the contrary that f from the statement can be approximated by diffeomorphisms (or piecewise affine homeomorphisms) $\{f_k\}_{k=1}^\infty$, then the pointwise limit of a subsequence (which we denote the same) satisfies

$$Df_k(x) \to Df(x) \quad \text{and} \quad J_{f_k}(x) \to J_f(x)$$

for almost every $x \in (-1,1)^n$. As f_k are locally Lipschitz we know that $J_{f_k} \geq 0$ a.e. in $(-1,1)^n$ or $J_{f_k} \leq 0$ a.e. in $(-1,1)^n$, see e.g. [24] and [23, Theorem 5.22]. The pointwise limit of nonnegative (or nonpositive) functions J_{f_k} cannot change sign which gives us contradiction.

Let us also recall that the Jacobian of a $W^{1,p}$, $1 \leq p < n$, Sobolev homeomorphism may behave strangely as it may vanish a.e. (see [22], [11] and [17]). As mentioned before the Jacobian of a homeomorphism cannot change sign if $p > [n/2]$ by [24] and therefore the method of sign-changing Jacobian for providing a counterexample in Theorem 1.1 cannot be improved to $p > [n/2]$. On the other hand, there might be a different way of producing a counterexample to the Ball-Evans approximation problem or there might be even a positive result in $\mathbb{R}^n$, $n \geq 4$, for $W^{1,p}$ if p is large enough (but definitely we must have $p \geq [n/2]$). Also the question whether the Jacobian can have both positive and negative Jacobian in a sets of positive measure in the borderline case $p = [n/2]$ remains open.

Now we outline the rough idea of our construction. We fix a Cantor type set $\mathcal{C}_A \subset (-1,1)$ of positive measure and we set

$$\begin{aligned}\mathcal{K}_A :=& \big(\mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times [-1,1]\big) \cup \big(\mathcal{C}_A \times \mathcal{C}_A \times [-1,1] \times \mathcal{C}_A\big) \cup \\ &\cup \big(\mathcal{C}_A \times [-1,1] \times \mathcal{C}_A \times \mathcal{C}_A\big) \cup \big([-1,1] \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A\big).\end{aligned} \tag{1.2}$$

We also fix a Cantor type set $\mathcal{C}_B \subset (-1,1)$ of zero measure (in fact its Hausdorff dimension δ is small) and define the set $\mathcal{K}_B$ similarly as above. Our first mapping $S_q : \mathbb{R}^n \to \mathbb{R}^n$ squeezes $\mathcal{K}_A$ onto $\mathcal{K}_B$ homeomorphically in a natural way. Then we find a bi-Lipschitz sense-preserving homeomorphism F such that

$$F(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \text{ for every } x \in \mathcal{K}_B. \tag{1.3}$$

Indeed, we can find a direction in $\mathbb{R}^4$ such that the projection of $\mathcal{K}_B$ to the corresponding hyperplane is one-to-one. The rough reason for that is that the set of directions where the projection is not one-to-one has Hausdorff dimension at most

$$\dim \mathcal{K}_B + \dim \mathcal{K}_B = 2 + 6\delta \tag{1.4}$$

(starting+ending point of the direction) and this is smaller than 3-the dimension of all directions. This projection of $\mathcal{K}_B$ can be extended to the homeomorphism $g : \mathbb{R}^4 \to \mathbb{R}^4$ which is bi-Lipschitz. By the turnover of the 3-dimensional hyperplane with respect to x_1 direction (which can be done by a sense-preserving homeomorphism of $\mathbb{R}^4$) and the composition with g^{-1} we obtain our mapping F. In view of the turnover of the hyperplane we obtain the key property (1.3). At last we find a mapping $S_t : \mathbb{R}^n \to \mathbb{R}^n$ which stretches $\mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$ back to $\mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A$ such that lines in $\mathcal{K}_B$ through the Cantor set are not prolonged too much and that S_t is locally Lipschitz outside of $\mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$.

We verify that $f = S_t \circ F \circ S_q$ belongs to $W^{1,p}$ by using the ACL property. It is thus crucial for us that lines parallel to coordinate axes that intersect $\mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A$ are mapped to lines by S_q, then to the same lines (with possibly reverse orientation in x_4-direction) by F (see (1.3)) and to something of reasonable length by S_t. To control the derivative on the lines parallel to coordinate axes that do not intersect $\mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A$ we use explicit form of mappings S_q and S_t and it is essential for us that F is Lipschitz everywhere and that S_t is locally Lipschitz far away from $\mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$.

Let us compare this result to the methods in [26]. In [26] the authors only showed that the length of the images of line segments are finite (which is enough for $\int |Df| < \infty$) but here we need to write explicit formulas for the mappings and to differentiate them, which requires much more details, precision and a delicate case study. More importantly there are three new main essential ingredients here. First there is a gap in the argument of [26] in the construction of the last mapping. During our detailed estimates we have found this gap and we have repaired it by giving a different last mapping S_t such that lines in $\mathcal{K}_B$ through the Cantor set are not prolonged too much. Secondly in [26] it was enough to find any bi-Lipschitz extension of the projection to construct a mapping F. Here we need to know that line segments close to $\mathcal{K}_B$ but far away from $\mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$ are mapped to line segments (see Section 3) so that the partial derivatives corresponding to different directions do not mix (and the big derivative in one direction is not multiplied by a big derivative in other direction). This requires a novel construction of the mapping F in Section 3. The third main ingredient is the extension to higher dimension as in $W^{1,1}$ it was enough to extend simply as $\tilde{f}(x) = (f(x_1, x_2, x_3, x_4), x_5, \dots, x_n)$. Here it requires much more work and it is essential for us to consider not only line segments through $\mathcal{C}_A \times \mathcal{C}_A \times \ldots \times \mathcal{C}_A$ as in (1.2) but $[n/2] - 1$ dimensional planes through the Cantor set (i.e. 2 dimensional planes for $n = 6, 7$ and so on). Then F reflects not only line segments through $\mathcal{C}_A \times \mathcal{C}_A \times \ldots \times \mathcal{C}_A$ (see (1.3)) but it reflects $[n/2] - 1$ dimensional planes through the Cantor set as the analogy of (1.4) is now

$$\dim \mathcal{K}_B + \dim \mathcal{K}_B = 2([n/2] - 1) + 2(n - [n/2] + 1)\delta < n - 1.$$

This allows us to control the derivative only on lines that do not belong to this $[n/2]-1$ dimensional planes and the measure of this set is very small close to the Cantor set.

2. Preliminaries

2.1. **Notation.** A point $x \in \mathbb{R}^n$ in coordinates is denoted as $(x_1, x_2, \dots, x_n)$. We denote by $|x| := \sqrt{\sum_{i=1}^n x_i}$ the Euclidean norm of a point $x \in \mathbb{R}^n$, and $\|x\| := \sup_i |x_i|$ will denote the supremum norm of x. We also define the distance of two sets $A, B \subset \mathbb{R}^n$ as

$$\operatorname{dist}(A, B) := \inf\{|x - y| : x \in A \text{ and } y \in B\}.$$

We will denote by

$$Q(a, r) := (a_1 - r, a_1 + r) \times \dots \times (a_n - r, a_n + r)$$

the open cube centered at $a \in \mathbb{R}^n$ with sidelength $2r > 0$. The interior of a set $A \subset \mathbb{R}^n$ is sometimes denoted also by A°.

We will denote by $C := C(p_1, \dots, p_k)$ a positive constant which depends only on the given parameters $p_1, \dots p_k$. The constant C might change from line to line. Furthermore, for given functions f and g we denote $f \lesssim g$ if there exists a positive constant $C > 0$ such that $f(x) \leq Cg(x)$ for all points x. If both conditions $f \lesssim g$ and $g \lesssim f$ are satisfied we denote $f \sim g$.

2.2. **Sobolev spaces and the ACL condition.** Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $f : \Omega \to \mathbb{R}^m$ belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$, $1 \leq p < \infty$, if f is p-integrable and if the coordinate functions of f have p-integrable distributional derivatives. We say that f belongs to the space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ if $f \in W^{1,p}(\Omega', \mathbb{R}^m)$ for every subdomain $\Omega' \subset\subset \Omega$.

Let $i \in \{1, 2, \dots, n\}$ and denote by π_i the projection on the given hyperplane $H_i = \{x \in \mathbb{R}^m : x_i = 0\}$ perpendicular to the x_i-axis. We say that a mapping $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is absolutely continuous on lines (abbr. $f \in \mathrm{ACL}(\Omega, \mathbb{R}^m)$) if the following ACL conditions holds:

(ACL) For every cube $Q(a, r) = (a_1 - r, a_1 + r) \times \dots \times (a_n - r, a_n + r) \subset\subset \Omega$ and for every $i \in \{1, 2, \dots, n\}$ the coordinate functions of the mapping

$$f^i(t; x) := f(x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n)$$

are absolutely continuous on $(a_i - r, a_i + r)$ for $\mathcal{L}^{n-1}$-almost every $x \in \pi_i(Q(a, r))$.

The following characterization of Sobolev spaces is classical and can be found e.g. in [1, Section 3.11] and [23, Theorem A.15].

Proposition 2.1. *Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be an open set and $f \in L^p(\Omega, \mathbb{R}^m)$. Then $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ if and only if there is a representative of f which is a $\mathrm{ACL}(\Omega, \mathbb{R}^m)$ mapping with locally L^p-integrable partial derivatives on Ω.*

2.3. **Topological degree.** For a given smooth map f from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ we can define the *topological degree* as

$$\deg(f, \Omega, y_0) = \sum_{\{x \in \Omega : f(x) = y_0\}} \operatorname{sgn}(J_f(x))$$

if $J_f(x) \neq 0$ for each $x \in f^{-1}(y_0)$. This definition can be extended to arbitrary continuous mappings and each point, see e.g. [20].

A continuous mapping $f : \Omega \to \mathbb{R}^n$ is called *sense-preserving* if

$$\deg(f, \Omega', y_0) > 0$$

for all subdomains $\Omega' \subset\subset \Omega$ and for all $y_0 \in f(\Omega') \setminus f(\partial\Omega')$. Similarly we call f *sense-reversing* if $\deg(f, \Omega', y_0) < 0$ for all Ω' and $y_0 \in f(\Omega') \setminus f(\partial\Omega')$. Let us recall that each homeomorphism on a domain is either sense-preserving or sense-reversing, see e.g. [39, II.2.4., Theorem 3].

2.4. **Hausdorff dimension.** Let $\alpha > 0$. We define α-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^n$ by

$$\mathcal{H}^\alpha(E) = \lim_{\varepsilon \to 0^+} \mathcal{H}^\alpha_\varepsilon(E),$$

where for a given $\varepsilon > 0$ we define

$$\mathcal{H}^\alpha_\varepsilon(E) = \inf\left\{ \sum_{i=1}^\infty (\operatorname{diam} A_i)^\alpha : E \subset \bigcup_{i=1}^\infty A_i,\ \operatorname{diam} A_i < \varepsilon \right\}.$$

We define the Hausdorff dimension of a set E as

$$\dim_{\mathcal{H}}(E) = \sup\{\alpha > 0 : \mathcal{H}^\alpha(E) = \infty\} = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\}.$$

We point out that Lipschitz mappings do not raise the Hausdorff dimension of a set and furthermore if $E = \bigcup_{i=1}^\infty E_i$ then

$$\dim_{\mathcal{H}}(E) = \sup_i \dim_{\mathcal{H}}(E_i).$$

2.5. **Construction of the Cantor set C_A and the set $\mathcal{K}_A$.** Denote by $\mathbb{V}$ the set of 2^4 vertices of the cube $[-1, 1]^4$. The sets

$$\mathbb{V}^k = \mathbb{V} \times \cdots \times \mathbb{V}, \quad k \in \mathbb{N},$$

will serve as the set of indices for our construction of Cantor sets.

We will define next the Cantor set C_A with positive measure for our construction. For this fix $\alpha > 0$. Let us define the sequence $\{a_k\}_{k=0}^\infty$ by setting

$$a_k = \frac{1}{2}\left(1 + \frac{1}{(k+1)^\alpha}\right).$$

Set $z_0 = 0$ and let us define

$$r_k = 2^{-k} a_k.$$

It follows that $Q(z_0, r_0) = (-1, 1)^4$ and further we proceed by induction. For $\boldsymbol{v}(k) = (v_1, \ldots, v_k) \in \mathbb{V}^k$ we denote $\boldsymbol{w}(k) = (v_1, \ldots, v_{k-1})$ and we define

$$z_{\boldsymbol{v}(k)} = z_{\boldsymbol{w}(k)} + \frac{1}{2} r_{k-1} v_k = z_0 + \frac{1}{2} \sum_{j=1}^k r_{j-1} v_j,$$

$$Q'_{\boldsymbol{v}(k)} = Q\big(z_{\boldsymbol{v}(k)}, 2^{-k} a_{k-1}\big) \text{ and } Q_{\boldsymbol{v}(k)} = Q\big(z_{\boldsymbol{v}(k)}, 2^{-k} a_k\big).$$

Formally we should write $\boldsymbol{w}(\boldsymbol{v}(k))$ instead of $\boldsymbol{w}(k)$ but for the simplification of the notation we will avoid this. Sometimes we may even denote $\boldsymbol{v}$ and $\boldsymbol{w}$ instead of $\boldsymbol{v}(k)$ and $\boldsymbol{w}(k)$.

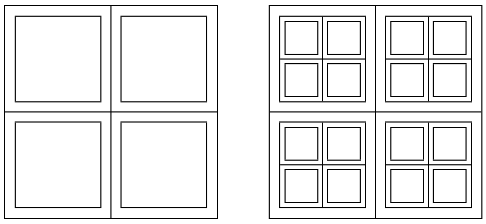

FIGURE 1. Two-dimensional projection of the cubes $Q_{\boldsymbol{v}(k)}$ and $Q'_{\boldsymbol{v}(k)}$ for $k=1,2$.

Then for the measure of the k-th frame $\mathbb{A}_{\boldsymbol{v}(k)} := Q'_{\boldsymbol{v}(k)} \setminus Q_{\boldsymbol{v}(k)}$, $k \in \mathbb{N}$, we have

$$\mathcal{L}^4(\mathbb{A}_{\boldsymbol{v}(k)}) = 2^{-4k+4}(a_{k-1}^4 - a_k^4) = 2^{-4k}\Bigg[\Big(1+\frac{1}{k^\alpha}\Big)^4 - \Big(1+\frac{1}{(k+1)^\alpha}\Big)^4\Bigg]. \tag{2.1}$$

The number of the cubes in $\{Q_{\boldsymbol{v}(k)} : \boldsymbol{v}(k) \in \mathbb{V}^k\}$ is 2^{4k}. It is not difficult to find out that the resulting Cantor set

$$\bigcap_{k=1}^{\infty} \bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k} Q_{\boldsymbol{v}(k)} =: C_A[\{a_k\}_{k=0}^\infty] = \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A$$

is a product of 4 Cantor sets $\mathcal{C}_A$ in $\mathbb{R}$. Moreover, the measure of the set C_A can be calculated as

$$\mathcal{L}^4(C_A) = \lim_{k\to\infty} 2^{4k}(2a_k 2^{-k})^4 = \lim_{k\to\infty}\Big(1+\frac{1}{(k+1)^\alpha}\Big)^4 = 1. \tag{2.2}$$

Furthermore, we may write the 1-dimensional Cantor set $\mathcal{C}_A$ as

$$\mathcal{C}_A = \bigcap_{k=1}^{\infty}\bigcup_{i=1}^{2^k} I_{i,k}$$

where $I_{i,k}$ are closed intervals of length $2^{-k}\big(1+\frac{1}{(k+1)^\alpha}\big)$, $I_{i,k}\cap I_{j,k}=\emptyset$ for $i\neq j$, and $I_{2i-1,k}\cup I_{2i,k}\subset I_{i,k-1}$. Throughout this paper we will also denote

$$U_k := \bigcup_{i=1}^{2^k} I_{i,k}, \quad \mathcal{M}_k := U_k\times U_k\times U_k\times U_k, \quad P_k := U_k\times U_k\times U_k,$$

and in view (2.2) it is easy to see that

$$\mathcal{H}^1\big(U_k\setminus\mathcal{C}_A\big) \le 2^k 2^{-k}\Big(1+\frac{1}{(k+1)^\alpha}\Big) - 1 \le \frac{C}{k^\alpha}. \tag{2.3}$$

Further we denote

$$\begin{aligned}\mathcal{A}_k :=& \big(U_k\times U_k\times U_k\times\mathbb{R}\big)\cup\big(U_k\times U_k\times\mathbb{R}\times U_k\big)\\ &\cup\big(U_k\times\mathbb{R}\times U_k\times U_k\big)\cup\big(\mathbb{R}\times U_k\times U_k\times U_k\big).\end{aligned}$$

It is easy to see that

$$C_A = \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A = \bigcap_{k=1}^{\infty} \mathcal{M}_k.$$

Furthermore, we also denote

$$\begin{aligned}\mathcal{K}_A :=&(\mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times [-1,1]) \cup (\mathcal{C}_A \times \mathcal{C}_A \times [-1,1] \times \mathcal{C}_A)\\ &\cup (\mathcal{C}_A \times [-1,1] \times \mathcal{C}_A \times \mathcal{C}_A) \cup ([-1,1] \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A),\end{aligned}$$

and then we have

$$\mathcal{K}_A = [-1,1]^4 \cap \bigcap_{k=1}^{\infty} \mathcal{A}_k.$$

It is easy to see that $\mathcal{L}^4(\mathcal{K}_A) > 0$. Analogously to (2.3) we can estimate
(2.4)

$$\mathcal{H}^3(P_k \setminus P_{k+1}) \leq \Bigl(2^k 2^{-k}\bigl(1 + \frac{1}{(k+1)^{\alpha}}\bigr)\Bigr)^3 - \Bigl(2^{k+1} 2^{-(k+1)}\bigl(1 + \frac{1}{(k+2)^{\alpha}}\bigr)\Bigr)^3 \leq \frac{C}{k^{\alpha+1}}.$$

2.6. **Construction of the Cantor set C_B and the set $\mathcal{K}_B$.** Next, we will define the Cantor set C_B of zero measure for our construction. The definition of the index set $\mathbb{V}^k$ remains the same as in the subsection 2.5.

To define C_B we fix $0 < \delta < 1/7$. Let us define the sequence $\{b_k\}_{k=0}^{\infty}$ by setting

$$b_k = 2^{-k\beta},$$

where $\beta = \frac{1-\delta}{\delta}$. Analogously to the previous section we set $\hat{z}_0 = 0$ and define

$$\hat{r}_k = 2^{-k} b_k.$$

Then it follows that $Q(\hat{z}_0, \hat{r}_0) = (-1,1)^4$ and further we proceed by induction. For $\boldsymbol{v}(k) = (v_1, \dots, v_k) \in \mathbb{V}^k$ we denote $\boldsymbol{w}(k) = (v_1, \dots, v_{k-1})$ and we define

$$\hat{z}_{\boldsymbol{v}(k)} = \hat{z}_{\boldsymbol{w}(k)} + \frac{1}{2}\hat{r}_{k-1} v_k = \hat{z}_0 + \frac{1}{2}\sum_{j=1}^{k} \hat{r}_{j-1} v_j,$$

$$\hat{Q}'_{\boldsymbol{v}(k)} = Q\bigl(\hat{z}_{\boldsymbol{v}(k)}, 2^{-k} b_{k-1}\bigr) \text{ and } \hat{Q}_{\boldsymbol{v}(k)} = Q\bigl(\hat{z}_{\boldsymbol{v}(k)}, 2^{-k} b_k\bigr).$$

Index $\boldsymbol{w}(k) = (v_1, \dots, v_{k-1})$ is called as the *parent* of the index $\boldsymbol{v}(k) = (v_1, \dots, v_k)$. For the measure of the k-th frame $\mathbb{B}_{\boldsymbol{v}(k)} := \hat{Q}'_{\boldsymbol{v}(k)} \setminus \hat{Q}_{\boldsymbol{v}(k)}$, $k \in \mathbb{N}$, we have

(2.5) $$\mathcal{L}^4(\mathbb{B}_{\boldsymbol{v}(k)}) = 2^{-4k+4}(b_{k-1}^4 - b_k^4) = 2^{-4k+4-4\beta k}(2^{4\beta} - 1).$$

Analogously to the previous section, it is not difficult to find out that the resulting Cantor set

$$\bigcap_{k=1}^{\infty} \bigcup_{\boldsymbol{v}(k) \in \mathbb{V}^k} \hat{Q}_{\boldsymbol{v}(k)} =: C_B[\{b_k\}_{k=0}^{\infty}] = \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$$

is a product of n Cantor sets $\mathcal{C}_B$ in $\mathbb{R}$. Moreover, the measure of the set C_B can be calculated as

(2.6) $$\mathcal{L}^4(C_B) = \lim_{k\to\infty} 2^{4k}(2b_k 2^{-k})^n = \lim_{k\to\infty} 2^{4-4\beta k} = 0.$$

Furthermore, we may write the 1-dimensional Cantor set $\mathcal{C}_B$ as

$$\mathcal{C}_B = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} \hat{I}_{i,k}$$

where $\hat{I}_{i,k}$ are closed intervals of length $2b_k 2^{-k}$, $\hat{I}_{i,k} \cap \hat{I}_{j,k} = \emptyset$ for $i \neq j$, and $\hat{I}_{2i-1,k} \cup \hat{I}_{2i,k} \subset \hat{I}_{i,k-1}$. Throughout this paper we denote

$$\hat{U}_k := \bigcup_{i=1}^{2^k} \hat{I}_{i,k}\,, \quad \hat{\mathcal{M}}_k := \hat{U}_k \times \hat{U}_k \times \hat{U}_k \times \hat{U}_k\,, \quad \hat{P}_k = \hat{U}_k \times \hat{U}_k \times \hat{U}_k. \tag{2.7}$$

Furthermore, we also denote

$$\begin{aligned} \hat{\mathcal{A}}_k :=& \big(\hat{U}_k \times \hat{U}_k \times \hat{U}_k \times \mathbb{R}\big) \cup \big(\hat{U}_k \times \hat{U}_k \times \mathbb{R} \times \hat{U}_k\big) \\ &\cup \big(\hat{U}_k \times \mathbb{R} \times \hat{U}_k \times \hat{U}_k\big) \cup \big(\mathbb{R} \times \hat{U}_k \times \hat{U}_k \times \hat{U}_k\big), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \mathcal{K}_B :=& (\mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times [-1,1]) \cup (\mathcal{C}_B \times \mathcal{C}_B \times [-1,1] \times \mathcal{C}_B) \\ &\cup (\mathcal{C}_B \times [-1,1] \times \mathcal{C}_B \times \mathcal{C}_B) \cup ([-1,1] \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B). \end{aligned}$$

It is easy to see that $\mathcal{L}^4(\mathcal{K}_B) = 0$. Furthermore, we may find out that $\dim_{\mathcal{H}} \mathcal{C}_B = \delta$ as in the k-th step of construction we have 2^k intervals of length $2b_k 2^{-k} = 2 \cdot 2^{-k-k\beta} = 2 \cdot 2^{-\frac{k}{\delta}}$. Therefore, as $0 < \delta < 1/7$, we conclude that

$$\dim_{\mathcal{H}} \mathcal{K}_B \leq 1 + 3\delta < \frac{3}{2}.$$

2.7. **The mapping S_q.** Suppose that C_A and C_B are the Cantor sets in subsections 2.5 and 2.6. Let $q : \mathbb{R} \to \mathbb{R}$ be the natural piecewise linear homeomorphism which takes each interval in the set $U_k \setminus U_{k+1}$, $k \in \mathbb{N}$, onto corresponding interval in $\hat{U}_k \setminus \hat{U}_{k+1}$ linearly. Then it is easy to see that q is an odd function, i.e. $q(-s) = -q(s)$ for every $s \in \mathbb{R}$. We define the homeomorphism $S_q : (-1,1)^n \to (-1,1)^n$ by setting

$$S_q(x_1, \ldots, x_n) = (q(x_1), \ldots, q(x_n)).$$

It is easy to see that S_q maps $\mathcal{K}_A$ onto $\mathcal{K}_B$. Moreover, we may notice that S_q is a Lipschitz mapping which takes each line segment parallel to x_i-axis to a line segment parallel to x_i-axis for every $i = 1, 2, 3, 4$. Furthermore, we also have that:

(1) For each $x \in (-1,1)^4$ such that $x_i \in U_k \setminus U_{k+1}$, $i = 1, 2, 3, 4$, we have

$$|D_i S_q(x)| = \frac{b_k - b_{k+1}}{a_k - a_{k+1}} \leq C k^{\alpha+1} 2^{-\beta k}, \tag{2.9}$$

where the constant $C = C(\alpha, \beta) > 0$ depends only on parameters α and β.

(2) For each $x \in (-1,1)^4$ such that $x_i \in \mathcal{C}_A$, $i = 1, 2, 3, 4$, we have

$$|D_i S_q(x)| = 0.$$

Here and in what follows $D_i g$ denotes the derivative of a mapping g along the x_i-direction for $i \in \{1, 2, 3, 4\}$.

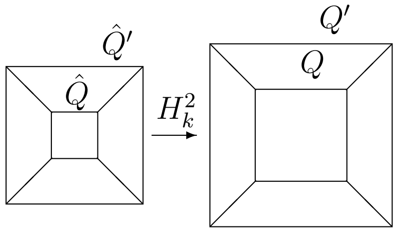

FIGURE 2. The transformation of $\hat{Q}' \setminus \hat{Q}^\circ$ onto $Q' \setminus Q^\circ$ in two dimensions.

2.8. **Frames to frames mapping of** $(n-1)$**-dimensional Cantor sets.** Suppose that $n \geq 3$. Analogously to the constructions of C_A and C_B we can define the $(n-1)$-dimensional Cantor type sets

$$\underbrace{\mathcal{C}_B \times \cdots \times \mathcal{C}_B}_{n-1 \text{ times}} \text{ and } \underbrace{\mathcal{C}_A \times \cdots \times \mathcal{C}_A}_{n-1 \text{ times}}.$$

We will need to find a mapping which maps the first set onto the second and the corresponding frames around it to corresponding frames around the second set. Instead of the index set $\mathbb{V}^k$ we use now the set $\mathbb{W}^k$ where $\mathbb{W}$ denotes the vertices of the cube $[-1,1]^{n-1}$. Analogously to previous notation we denote $\boldsymbol{w} \in \mathbb{W}^k$ instead of $\boldsymbol{v} \in \mathbb{V}^k$ and we work with cubes

$$Q'_{\boldsymbol{w}(k)}, Q_{\boldsymbol{w}(k)}, \hat{Q}'_{\boldsymbol{w}(k)} \text{ and } \hat{Q}_{\boldsymbol{w}(k)}$$

defined analogously to subsections 2.3 and 2.4 but now in $n-1$ dimensions.

We will find a sequence of homeomorphisms $H_k^{n-1} : (-1,1)^{n-1} \to (-1,1)^{n-1}$. We set $H_0^{n-1}(x) = x$ and we proceed by induction. We will give a mapping F_1 which stretches each cube $\hat{Q}_{\boldsymbol{w}}$, $\boldsymbol{w} \in \mathbb{W}^1$, homogeneously so that $H_1^{n-1}(\hat{Q}_{\boldsymbol{w}})$ equals $Q_{\boldsymbol{w}}$. On the annulus $\hat{Q}'_{\boldsymbol{w}} \setminus \hat{Q}_{\boldsymbol{w}}$, H_1^{n-1} is defined to be an appropriate radial map with respect to $\hat{z}_{\boldsymbol{w}}$ and $z_{\boldsymbol{w}}$ in the image in order to make H_1^{n-1} a homeomorphism. The general step is the following: If $k > 1$, H_k^{n-1} is defined as H_{k-1}^{n-1} outside the union of all cubes $\hat{Q}'_{\boldsymbol{w}}$, $\boldsymbol{w} \in \mathbb{W}^k$. Further, H_k^{n-1} remains equal to H_{k-1}^{n-1} at the centers of cubes $\hat{Q}_{\boldsymbol{w}}$, $\boldsymbol{w} \in \mathbb{W}^k$. Then H_k^{n-1} stretches each cube $\hat{Q}_{\boldsymbol{w}}$, $\boldsymbol{w} \in \mathbb{W}^k$, homogeneously so that $H_k^{n-1}(\hat{Q}_{\boldsymbol{w}})$ equals $Q_{\boldsymbol{w}}$. On the annulus $\hat{Q}'_{\boldsymbol{w}} \setminus \hat{Q}_{\boldsymbol{w}}$, H_k^{n-1} is defined to be an appropriate radial map with respect to $\hat{z}_{\boldsymbol{w}}$ in preimage and $z_{\boldsymbol{w}}$ in image to make H_k^{n-1} a homeomorphism (see Fig. 2). Notice that the Jacobian determinant $J_{H_k^{n-1}}(x)$ will be strictly positive almost everywhere in $(-1,1)^{n-1}$.

In the following definition of H_k^{n-1} we use the notation $\|x\|$ for the supremum norm of $x \in \mathbb{R}^{n-1}$. The mappings H_k^{n-1}, $k \in \mathbb{N}$, are formally defined as
(2.10)

$$H_k^{n-1}(x) = \begin{cases} H_{k-1}^{n-1}(x) & \text{for } x \notin \bigcup_{\boldsymbol{w} \in \mathbb{W}^k} \hat{Q}'_{\boldsymbol{w}} \\ H_{k-1}^{n-1}(\hat{z}_{\boldsymbol{w}}) + (\alpha_k \|x - \hat{z}_{\boldsymbol{w}}\| + \beta_k) \frac{x - \hat{z}_{\boldsymbol{w}}}{\|x - \hat{z}_{\boldsymbol{w}}\|} & \text{for } x \in \hat{Q}'_{\boldsymbol{w}} \setminus \hat{Q}_{\boldsymbol{w}},\ \boldsymbol{w} \in \mathbb{W}^k \\ H_{k-1}^{n-1}(\hat{z}_{\boldsymbol{w}}) + \frac{r_k}{\tilde{r}_k}(x - \hat{z}_{\boldsymbol{w}}) & \text{for } x \in \hat{Q}_{\boldsymbol{w}},\ \boldsymbol{w} \in \mathbb{W}^k \end{cases}$$

where the constants α_k and β_k are given by

$$\alpha_k \hat{r}_k + \beta_k = r_k \text{ and } \alpha_k \tfrac{\hat{r}_{k-1}}{2} + \beta_k = \tfrac{r_{k-1}}{2}. \tag{2.11}$$

It is not difficult to find out that each H_k^{n-1} is a homeomorphism and maps

$$\bigcup_{\boldsymbol{w}\in\mathbb{W}^k} \hat{Q}_{\boldsymbol{w}} \text{ onto } \bigcup_{\boldsymbol{w}\in\mathbb{W}^k} Q_{\boldsymbol{w}}.$$

The limit $H^{n-1}(x) = \lim_{k\to\infty} H_k^{n-1}(x)$ is clearly one-to-one and continuous and therefore a homeomorphism. Moreover, it is easy to see that H^{n-1} is differentiable almost everywhere (as $\mathcal{L}^{n-1}(\mathcal{C}_B^{n-1}) = 0$) and maps $\mathcal{C}_B^{n-1}$ onto $\mathcal{C}_A^{n-1}$.

Fix $j \in \mathbb{N}$. We claim that the mapping H^{n-1} is Lipschitz on $(\hat{U}_j)^{n-1} \setminus (\hat{U}_{j+1})^{n-1}$ where the sets $\hat{U}_j$ are defined analogously to subsection 2.6 (the Lipschitz constant of course depends on the fixed j). This is in fact easy to see as the mapping is given by simple formula (2.10) on each $\hat{Q}'_{\boldsymbol{w}} \setminus \hat{Q}_{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \mathbb{W}^j$. Analogously to [23, Lemma 2.1 and proof of Theorem 4.10] we can estimate

$$|DH_j^{n-1}(x)| = |DH^{n-1}(x)| \sim \max\Big\{\frac{r_j}{\hat{r}_j}, \alpha_j\Big\} \leq C\max\{2^{\beta j}, 2^{\beta j} j^{-(\alpha+1)}\} \leq C2^{\beta j} \tag{2.12}$$

for every $x \in \hat{Q}'_{\boldsymbol{w}} \setminus \hat{Q}_{\boldsymbol{w}}$ and $\boldsymbol{w} \in \mathbb{W}^j$. This is because

$$\begin{aligned} |D_l H_j^{n-1}(x)| &\leq \frac{r_j}{\hat{r}_j} \leq C2^{\beta j} \text{ for } l \neq i \text{ and} \\ |D_i H_j^{n-1}(x)| &\leq \alpha_j \leq C2^{\beta j} j^{-(\alpha+1)} \end{aligned} \tag{2.13}$$

if x_i is the direction which realizes the supremum norm distance from the center of the cube $\hat{z}_{\boldsymbol{w}}$. From (2.10) it is also easy to see that

$$|DH_j^{n-1}(x)| \sim \frac{r_j}{\hat{r}_j} \leq C2^{\beta j} \text{ for } x \in \hat{Q}_{\boldsymbol{w}} \text{ and } \boldsymbol{w} \in \mathbb{W}^j. \tag{2.14}$$

In our construction we will need to know that for each $\alpha \in (0,1)$ and $k \in \mathbb{N}$ the mapping

$$\alpha H_{3k-3}^{n-1}(x) + (1-\alpha) H_{3k}^{n-1}(x) \tag{2.15}$$

is a homeomorphisms. Outside of $\bigcup_{\boldsymbol{w}\in\mathbb{W}^{3k-3}} \hat{Q}_{\boldsymbol{w}}$ both mapping are equal and hence the mapping is a homeomorphism there. Let us fix $\hat{Q}_{\boldsymbol{w}}$ for some $\boldsymbol{w} \in \mathbb{W}^{3k-3}$. We know by (2.10) that H_{3k}^{n-1} is a frame to frame mapping on $\hat{Q}_{\boldsymbol{w}}$ which maps corresponding squares with sizes $\tilde{r}_{3k-2}$ (resp. $\tilde{r}_{3k-1}$ and $\tilde{r}_{3k}$) to squares with sizes r_{3k-2} (resp. r_{3k-1} and r_{3k}). We also know by (2.10) that H_{3k-3}^{n-1} is a linear mapping

$$\frac{r_{3k-3}}{\tilde{r}_{3k-3}}(x - \tilde{z}_{\boldsymbol{w}}) \text{ on } \hat{Q}_{\boldsymbol{w}}$$

but this can be also viewed as a frame to frame mapping on $\hat{Q}_{\boldsymbol{w}}$ which maps corresponding squares with sizes $\tilde{r}_{3k-2}$ (resp. $\tilde{r}_{3k-1}$ and $\tilde{r}_{3k}$) to squares with sizes $\frac{r_{3k-3}}{\tilde{r}_{3k-3}}\tilde{r}_{3k-2}$ (resp. $\frac{r_{3k-3}}{\tilde{r}_{3k-3}}\tilde{r}_{3k-1}$ and $\frac{r_{3k-3}}{\tilde{r}_{3k-3}}\tilde{r}_{3k}$). Thus it is not difficult to see that the mapping (2.15) on $\hat{Q}_{\boldsymbol{w}}$ is a frame to frame mapping which maps corresponding squares with sizes $\tilde{r}_{3k-2}$ (resp. $\tilde{r}_{3k-1}$ and $\tilde{r}_{3k}$) to squares with sizes

$\alpha r_{3k-2}+(1-\alpha)\frac{r_{3k-3}}{\tilde{r}_{3k-3}}\tilde{r}_{3k-2}$ (resp. $\alpha r_{3k-1}+(1-\alpha)\frac{r_{3k-3}}{\tilde{r}_{3k-3}}\tilde{r}_{3k-1}$ and $\alpha r_{3k}+(1-\alpha)\frac{r_{3k-3}}{\tilde{r}_{3k-3}}\tilde{r}_{3k}$).

Analogously to the fact that each H_k defined by (2.10) is a homeomorphism we can conclude that the mapping (2.15) given by formula analogous to (2.10) is also a homeomorphism.

3. A sense-preserving bi-Lipschitz mapping F equal to a reflection in the last variable on $\mathcal{K}_B$

This section is dedicated to constructing a bi-Lipschitz mapping which equals the reflection in the last variable on $\mathcal{K}_B$. Especially, this means that the mapping will map lines in $\mathcal{K}_B$ to lines in $\mathcal{K}_B$. In fact even more than this the mapping will map certain line segments close to $\mathcal{K}_B$ to line segments (recall that $\mathcal{K}_B$ and $\hat{U}_k$ are defined in subsection 2.5). Also see Fig. 3.

Theorem 3.1. *If $\beta > 0$ is sufficiently large in the definition of the Cantor set C_B in subsection 2.6 then there exists a mapping $F : (-1,1)^4 \to (-1,1)^4$, which is a sense-preserving bi-Lipschitz extension of the map*

$$F(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4) \quad x \in \mathcal{K}_B, \tag{3.1}$$

and a constant $N_F \in \mathbb{N}$ such that for each $j, k \in \mathbb{N}$ satisfying $N_F < j \leq k$ the image of the intersection of a line parallel to e_i with the set

$$A_{i,j-N_F-1,k+N_F} := \{x \in \mathbb{R}^4 : x_i \in [-1,1] \setminus \hat{U}_{j-N_F-1}, x_l \in \hat{U}_{k+N_F}, l \neq i\}$$

in the map F is a line segment parallel to e_i which lies in the set

$$A_{i,j-1,k} = \{x \in \mathbb{R}^4 : x_i \in [-1,1] \setminus \hat{U}_{j-1}, x_l \in \hat{U}_k, l \neq i\}.$$

Moreover, the derivative along this segment satisfies

$$D_i F(x) = \begin{cases} e_i & \text{if } i = 1,2,3 \\ -e_i & \text{if } i = 4 \end{cases}$$

for every $x \in A_{i,j-N_F-1,k+N_F}$.

The concept of the following type of mapping is key to our proof. We will show the obvious fact that they are bi-Lipschitz maps.

Definition 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $v \in \mathbb{R}^n$ be a vector such that $v_n \neq 0$. Denote $X := \mathbb{R}^{n-1} \times \{0\}$. Let $g : X \to \mathbb{R}$ be a Lipschitz function and define a projection P_v of $\mathbb{R}^n$ onto X as follows*

$$P_v(x) = x - \frac{x_n}{v_n} v. \tag{3.2}$$

Then we define the spaghetti strand map $F_{g,v}$ as follows

$$F_{g,v}(x) = x + v g(P_v(x)).$$

Lemma 3.3. *Spaghetti strand maps from Definition 3.2 are bi-Lipschitz maps.*

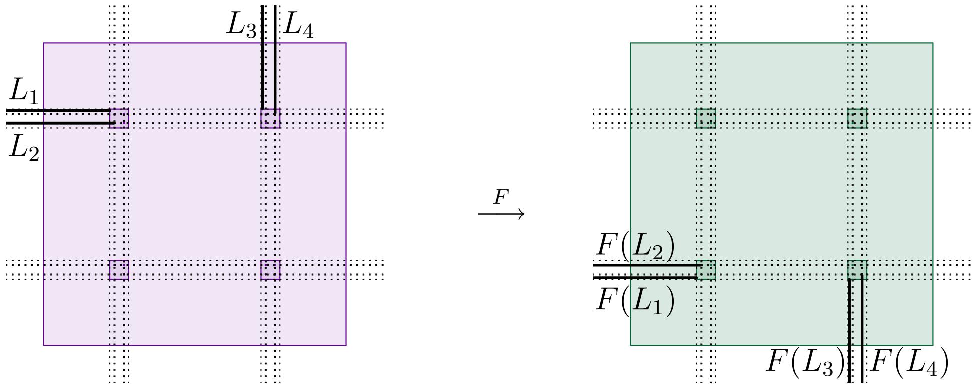

FIGURE 3. A sense preserving bi-Lipschitz map that reflects in e_4 and maps certain lines to lines

Proof. It is easy to see that every spaghetti strand map is Lipschitz as a composition of Lipschitz maps. Moreover $P_v(\alpha v) = 0$ for each $\alpha \in \mathbb{R}$ which implies that

$$x + vg(P_v(x)) - vg\big(P_v(x + vg(P_v(x)))\big) = x + vg(P_v(x)) - vg(P_v(x)) = x$$

and hence the inverse of a spaghetti strand map is the spaghetti strand map corresponding to $-g$. This inverse is also Lipschitz and therefore we see that these maps are bi-Lipschitz. $\square$

Firstly, let us outline our strategy for the rest of the section. We construct F from the composition of two spaghetti strand maps. Firstly we must choose a vector v and prove that the projection P_v is one-to-one on the set $\mathcal{K}_B$ and further there exists a Lipschitz function g so that $F_{g,v}(x) = P_v(x)$ for all $x \in \mathcal{K}_B$. This step is contained in Lemma 3.4. If we take $u = (-v_1, -v_2, \dots, -v_{n-1}, v_n)$ then we can define $F = F_{g,u} \circ F_{g,v}$ and it is not difficult to deduce that (3.1) holds (this is done in (3.37) below).

Lemma 3.4. *Let* $v = \big(\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, 1\big)$, $u = \big(-\frac{1}{16}, -\frac{1}{8}, -\frac{1}{4}, 1\big)$. *Then there is* $\beta \geq 6$ *and a corresponding set* $\mathcal{K}_B$ *given by the subsection 2.6 such that* P_v *is one-to-one on* $\mathcal{K}_B$, *and the function* g *defined on* $P_v(\mathcal{K}_B)$ *as* $g(P_v(x)) = -x_4$ *can be extended onto* X *as a Lipschitz function. Furthermore, it is possible to find a Lipschitz extension of the function* g *which guarantees that*

$$D_i\big(F_{g,u} \circ F_{g,v}\big)(x) = \begin{cases} e_i & \text{if } i = 1,2,3 \\ -e_i & \text{if } i = 4 \end{cases} \tag{3.3}$$

whenever $k \in \mathbb{N}$, $x_i \in [-1,1] \setminus \hat{U}_k$ *and* $x_j \in \hat{U}_{k+2}$ *for all* $j \neq i$.

Proof. Let us start by defining some notation we will use throughout the proof. We will denote $\tilde{v} := \big(\frac{1}{16}, \frac{1}{8}, \frac{1}{4}\big)$. Furthermore, if $\hat{Q}_{\boldsymbol{v}(k)} := Q(\hat{z}_{\boldsymbol{v}(k)}, \hat{r}_k)$, $\boldsymbol{v}(k) \in \mathbb{V}^k$, are the cubes used in the definition of the Cantor set C_B in subsection 2.6, then we define

$$\hat{G}^i_{\boldsymbol{v}(k)} := \hat{Q}_{\boldsymbol{v}(k)} + \mathbb{R}e_i. \tag{3.4}$$

These sets are called *k-bars*.

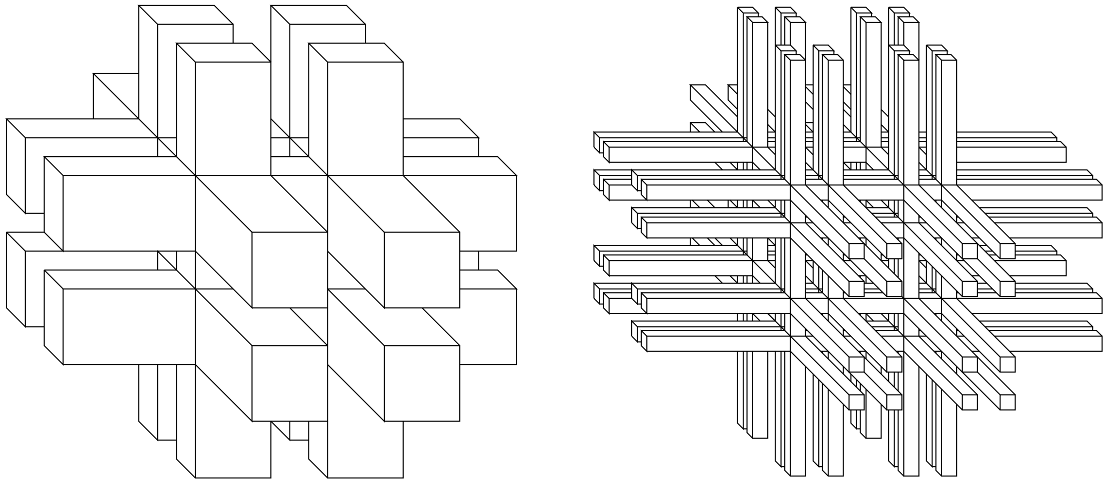

FIGURE 4. All 1-bars and 2-bars in three dimensions.

By the construction of the Cantor set we have $Q(\hat{z}_{\boldsymbol{v}(k-2)}, \hat{r}_{k-2}) \cap Q(\hat{z}_{\hat{\boldsymbol{v}}(k-2)}, \hat{r}_{k-2}) = \emptyset$, whenever $\boldsymbol{v}(k) \neq \hat{\boldsymbol{v}}(k)$. Therefore we have the equality for the so-called "sliced" bar

$$\hat{S}^i_{\boldsymbol{v}(k)} := \hat{G}^i_{\boldsymbol{v}(k)} \setminus \Big(\bigcup_{\boldsymbol{w} \in \mathbb{V}^{k-2}_{\boldsymbol{v}(k)}(i)} Q\big(\hat{z}_{\boldsymbol{w}}, \hat{r}_{k-2}\big)\Big) = \hat{G}^i_{\boldsymbol{v}(k)} \setminus \Big(\bigcup_{\boldsymbol{w} \in \mathbb{V}^{k-2}} Q\big(\hat{z}_{\boldsymbol{w}}, \hat{r}_{k-2}\big)\Big), \tag{3.5}$$

where

$$\mathbb{V}^{k-2}_{\boldsymbol{v}(k)}(i) := \big\{ \boldsymbol{w} \in \mathbb{V}^{k-2} : (\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_i) \cap Q(\hat{z}_{\boldsymbol{w}}, \hat{r}_{k-2}) \neq \emptyset \big\}.$$

It is easy to see that there is $\beta_1 > 0$ such that for $\beta \geq \beta_1$ (in the definition of C_B) we can replace the index set $\mathbb{V}^{k-2}_{\boldsymbol{v}(k)}(i)$ by much nicer set $\mathbb{V}^{k-2}$ in the definition of $\hat{S}^i_{\boldsymbol{v}(k)}$.

More precisely, a sliced k-bar $\hat{S}^i_{\boldsymbol{v}(k)}$ can be considered as a k-bar where we have removed all the cubes around the Cantor set from the $(k-2)$-nd generation of the construction.

In similar fashion we also define

$$\hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)} := \Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(\mathbb{R}e_i)\Big) \setminus \Big(\bigcup_{\boldsymbol{w} \in \mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)} Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_{k-1}\big)\Big) \subset X, \tag{3.6}$$

where $Q^3(z, r)$ denotes the 3 dimensional cube in $X := \mathbb{R}^3 \times \{0\}$ with radius $r > 0$ and centered at $z \in X$ and $q \geq \frac{5}{4}$ is a constant we will determine later. We also denote "sliced k-pipes" as follows

$$\hat{H}^i_{\boldsymbol{v}(k)} := \partial_X \Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(\mathbb{R}e_i)\Big) \setminus \Big(\bigcup_{\boldsymbol{w} \in \mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)} Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_{k-1}\big)\Big) \subset X,$$

where $\partial_X A$ denotes the relative boundary of a set A in X. We will later see that also in the definition of the sets $\hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ and $\hat{H}^i_{\boldsymbol{v}(k)}$ the index sets $\mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)$ can be replaced by $\mathbb{V}^{k-1}$ when $\beta > 0$ in the definition of C_B is just large enough.

Now let us briefly outline the rest of the proof. We prove that our choice of a vector v gives that P_v is one-to-one on $\mathcal{K}_B$. Then we prove that each $\hat{S}^i_{\boldsymbol{v}(k)}$ is projected into

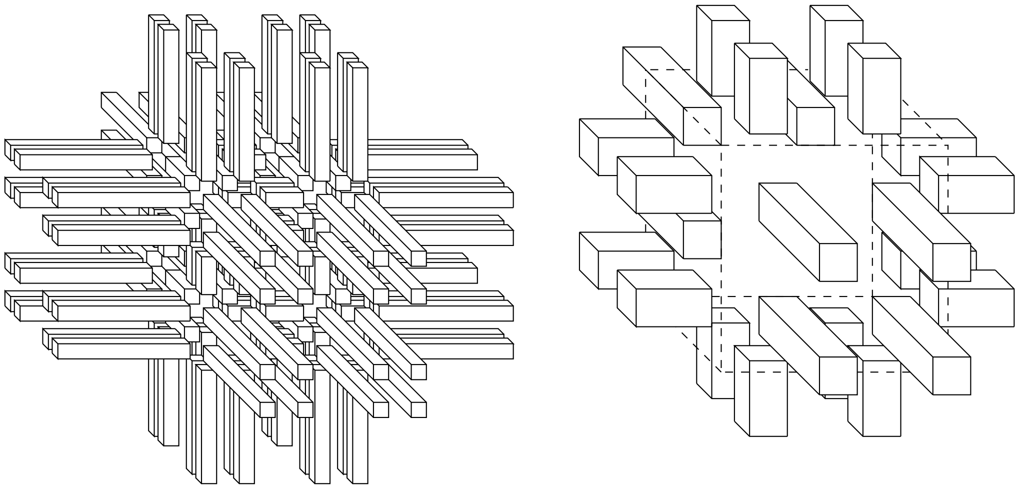

FIGURE 5. In the picture on the left we have all sliced 2-bars by 1-st generation cubes in three dimensions. In the later we slice k generational bars with $k-2$ generation cubes. A choice of β guarantees that in comparison the bars are as thin as required in comparison to the cube. In the picture on the right we have zoomed in one of the removed cubes (drawn with dashed line) from the picture on the left.

$\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$ which, for fixed k, are pairwise disjoint. This allows us to define a Lipschitz function g on $\mathbb{R}^3 \times \{0\}$ such that $F_{g,v} = P_v$ on $\mathcal{K}_B$. A careful extension of g onto $\mathbb{R}^3 \times \{0\}$ guarantees (3.3). We divide the proof into several steps.

Step 1: The projection is one-to-one on C_B. Our first step is simple, we want to show that the projection is one-to-one on the set $C_B = \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$. Consider the first stage of our Cantor construction, i.e. we have the cube $\hat{Q}_0 = Q(0,1)$ and the set of cubes $\hat{Q}_{\boldsymbol{v}(1)} := Q(\hat{z}_{\boldsymbol{v}(1)}, \hat{r}_1)$, $\boldsymbol{v}(1) \in \mathbb{V}$. We will show that the images of these 2^4 cubes in P_v are pairwise disjoint. Then we can use the same calculations to show that the projections of the next generation of cubes in our construction are also pairwise disjoint because the construction is self-similar. We can repeat this argument inductively to get that P_v is one-to-one on C_B. Therefore it suffices to show that the images of $\hat{Q}_{\boldsymbol{v}(1)}$ are pairwise disjoint. Although this step is slightly redundant it aids the understanding of the reader and so we include it here.

We will deal with two separate cases. The first case is where we are considering the projections of a pair of boxes $\hat{Q}_{\boldsymbol{v}(1)}$ and $\hat{Q}_{\hat{\boldsymbol{v}}(1)}$, whose centers have the same 4-th coordinate. The second case is where $(\hat{z}_{\boldsymbol{v}(1)})_4 \neq (\hat{z}_{\hat{\boldsymbol{v}}(1)})_4$. For any of the first generation cubes $\hat{Q}_{\boldsymbol{v}(1)}$ we can calculate its image in P_v using (3.2) and $v = \big(\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, 1\big)$ as

$$P_v(\hat{Q}_{\boldsymbol{v}(1)}) = Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(1)}), \hat{r}_1\big) + (-\hat{r}_1, \hat{r}_1)\big(\tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}\big) \subset Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(1)}), \hat{r}_1(1 + \tfrac{1}{4})\big), \tag{3.7}$$

where

$$P_v(\hat{z}_{\boldsymbol{v}(1)}) = \hat{z}_{\boldsymbol{v}(1)} - (\hat{z}_{\boldsymbol{v}(1)})_4(\tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}, 1) = \big((\hat{z}_{\boldsymbol{v}(1)})_1, (\hat{z}_{\boldsymbol{v}(1)})_2, (\hat{z}_{\boldsymbol{v}(1)})_3, 0\big) \mp \tfrac{1}{2}(\tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}, 0). \tag{3.8}$$

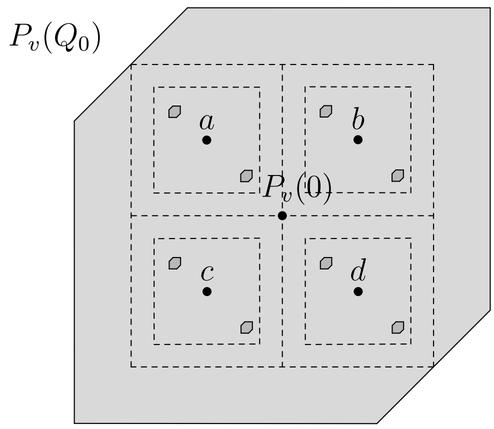

FIGURE 6. An illustration of the image of two generations of cubes in the projection P_v from three dimensions to the plane. For printing reasons we have now increased significantly $\hat{r}_1$ and changed somewhat v. The shaded regions (one big and eight smaller ones) describe the images of the cubes in P_v. The black dot in the middle describes the point $P_v(0) = 0$ the center of the large cube. The other four black dots a, b, c and d describe the centers of the small dashed cubes of radius $\frac{1}{4} + \frac{5}{4}\hat{r}_1$ which always contain the image of a pair of cubes symmetrical about the hyperplane, see Case 2B. In one case we consider a pair of cubes symmetrical about the hyperplane and the images of their centers are separated by $\tilde{v}$. In the other cases the images of cubes are disjoint because they lie in different dotted cubes, which are disjoint.

Case 1 (Step 1): In the first case we have a distinct pair of centers $\hat{z}_{\boldsymbol{v}(1)}$ and $\hat{z}_{\hat{\boldsymbol{v}}(1)}$ such that $(\hat{z}_{\boldsymbol{v}(1)})_4 = (\hat{z}_{\hat{\boldsymbol{v}}(1)})_4$. Since the pair is distinct we can find at least one $i \in \{1, 2, 3\}$ such that

$$|(\hat{z}_{\boldsymbol{v}(1)})_i - (\hat{z}_{\hat{\boldsymbol{v}}(1)})_i| = 1.$$

This means that $|\hat{z}_{\boldsymbol{v}(1)} - \hat{z}_{\hat{\boldsymbol{v}}(1)}| \geq 1$. But since $(\hat{z}_{\boldsymbol{v}(1)})_4 = (\hat{z}_{\hat{\boldsymbol{v}}(1)})_4$ we have

$$P_v(\hat{z}_{\boldsymbol{v}(1)}) - \hat{z}_{\boldsymbol{v}(1)} = P_v(\hat{z}_{\hat{\boldsymbol{v}}(1)}) - \hat{z}_{\hat{\boldsymbol{v}}(1)},$$

and therefore

$$|P_v(\hat{z}_{\boldsymbol{v}(1)}) - P_v(\hat{z}_{\hat{\boldsymbol{v}}(1)})| = |\hat{z}_{\boldsymbol{v}(1)} - \hat{z}_{\hat{\boldsymbol{v}}(1)}| \geq 1.$$

This together with (3.7) and the fact that $2\hat{r}_1(1 + \frac{1}{4}) < 1$ (recall that $\hat{r}_1 = 2^{-1}2^{-\beta}$ with $\beta \geq 6$) implies that

$$P_v(\hat{Q}_{\boldsymbol{v}(1)}) \cap P_v(\hat{Q}_{\hat{\boldsymbol{v}}(1)}) = \emptyset.$$

Case 2A (Step 1): Suppose now that $(\hat{z}_{\boldsymbol{v}(1)})_4 \neq (\hat{z}_{\hat{\boldsymbol{v}}(1)})_4$. We shall consider first a pair of boxes, whose centers $\hat{z}_{\boldsymbol{v}(1)}$ and $\hat{z}_{\hat{\boldsymbol{v}}(1)}$ are on a line parallel to e_4. To see that the images of these boxes are disjoint we observe that

$$P_v(\hat{z}_{\boldsymbol{v}(1)}) - P_v(\hat{z}_{\hat{\boldsymbol{v}}(1)}) = P_v(\hat{z}_{\boldsymbol{v}(1)} - \hat{z}_{\hat{\boldsymbol{v}}(1)}) = P_v(\pm e_4) = \mp\left(\tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}\right). \tag{3.9}$$

Furthermore, as

$$P_v(\hat{Q}_{\boldsymbol{v}(1)}) \subset Q(P_v(\hat{z}_{\boldsymbol{v}(1)}), \tfrac{5}{4}\hat{r}_1), \tag{3.10}$$

and since $2\hat{r}_1\frac{5}{4} < \frac{1}{16} < \|P_v(\hat{z}_{\boldsymbol{v}(1)}) - P_v(\hat{z}_{\hat{\boldsymbol{v}}(1)})\|$ then the projection of $\hat{Q}_{\boldsymbol{v}(1)}$ and $\hat{Q}_{\hat{\boldsymbol{v}}(1)}$ must be disjoint. Here $\|x\| := \sup_i |x_i|$ denotes the supremum norm.

Case 2B (Step 1): We still need to consider the pairs of cubes with centers that vary from each other in the 4-th variable and in another variable. In other words, let us suppose that it holds for $\hat{z}_{\boldsymbol{v}(1)}$ and $\hat{z}_{\hat{\boldsymbol{v}}(1)}$ that

$$(\hat{z}_{\boldsymbol{v}(1)})_4 \neq (\hat{z}_{\hat{\boldsymbol{v}}(1)})_4 \quad \text{and} \quad (\hat{z}_{\boldsymbol{v}(1)})_i \neq (\hat{z}_{\hat{\boldsymbol{v}}(1)})_i \text{ for some } i \in \{1,2,3\},$$

and let us denote

$$a := ((\hat{z}_{\boldsymbol{v}(1)})_1, (\hat{z}_{\boldsymbol{v}(1)})_2, (\hat{z}_{\boldsymbol{v}(1)})_3, 0) \quad \text{and} \quad b = ((\hat{z}_{\hat{\boldsymbol{v}}(1)})_1, ((\hat{z}_{\hat{\boldsymbol{v}}(1)})_2, ((\hat{z}_{\hat{\boldsymbol{v}}(1)})_3, 0).$$

By applying (3.7) and (3.8) we get

$$P_v(Q_{\boldsymbol{v}(1)}) \subset Q\big(a, \tfrac{1}{4} + \tfrac{5}{4}\hat{r}_1\big) \quad \text{and} \quad P_v(Q_{\hat{\boldsymbol{v}}(1)}) \subset Q\big(b, \tfrac{1}{4} + \tfrac{5}{4}\hat{r}_1\big),$$

where $\frac{1}{4} + \frac{5}{4}\hat{r}_1 < \frac{9}{64} < \frac{1}{2}$. Thus, it follows from the fact $|a-b| \geq 1$ that

$$\begin{aligned}\operatorname{dist}(P_v(\hat{Q}_{\boldsymbol{v}(1)}), P_v(\hat{Q}_{\hat{\boldsymbol{v}}(1)})) &\geq \operatorname{dist}\big(Q\big(a, \tfrac{1}{4} + \tfrac{5}{4}\hat{r}_1\big), Q\big(b, \tfrac{1}{4} + \tfrac{5}{4}\hat{r}_1\big)\big) \\ &\geq |a-b| - 2\big(\tfrac{1}{4} + \tfrac{5}{4}\hat{r}_1\big) > 0,\end{aligned}$$

which gives us that the sets $P_v(\hat{Q}_{\boldsymbol{v}(1)})$ and $P_v(\hat{Q}_{\hat{\boldsymbol{v}}(1)})$ are disjoint. This implies that the remaining pairs of cubes to consider (i.e. the pairs of cubes with centers that vary from each other in the 4-th variable and in another variable) are also disjoint.

It follows now from Cases 1, 2A and 2B that images of the first generation cubes $\hat{Q}_{\boldsymbol{v}(1)}$ in the projection P_v are pairwise disjoint. The self similarity argument mentioned above implies that P_v is one-to-one on C_B. The reason why the self similarity argument works here is because the ratio $\hat{r}_{k-1}/\hat{r}_k = 2^{\beta+1}$ is not depending on k. Geometrically this means that if we rescale a cube $\hat{Q}_{\boldsymbol{v}(k-1)}$ and the smaller cubes $\hat{Q}_{\boldsymbol{v}(k)}$ which lies inside this cube by factor $2^{k-1}2^{\beta(k-1)}$ we see that there will be as much space to project the cubes of the k-th step as there was in the first step (see Fig. 7).

Step 2: The projection is one-to-one on $\mathcal{K}_B$. We will start this step by showing that if a and b are any two vertices of $Q(0, \frac{1}{2})$ and $e_i, e_j \in \mathbb{R}^4$ are two (possibly identical) canonical basis vectors of $\mathbb{R}^4$, then

$$P_v(a + \mathbb{R}e_i) \cap P_v(b + \mathbb{R}e_j) = P_v\big((a + \mathbb{R}e_i) \cap (b + \mathbb{R}e_j)\big). \tag{3.11}$$

This gives us that if ℓ and $\hat{\ell}$ are two distinct lines parallel to coordinate axes through some vertices a and b of $Q(0, \frac{1}{2})$ then their projections $P_v(\ell)$ and $P_v(\hat{\ell})$ meet at most at one point which is the image $P_v(z)$ of the intersection point z of ℓ and $\hat{\ell}$. We use this to show that images of sliced k-bars are disjoint and finally by this observation and by Step 1 we will conclude that P_v is one-to-one on $\mathcal{K}_B$. It is good to remark that the argument bellow does not work if the dimension of the space is three or smaller.

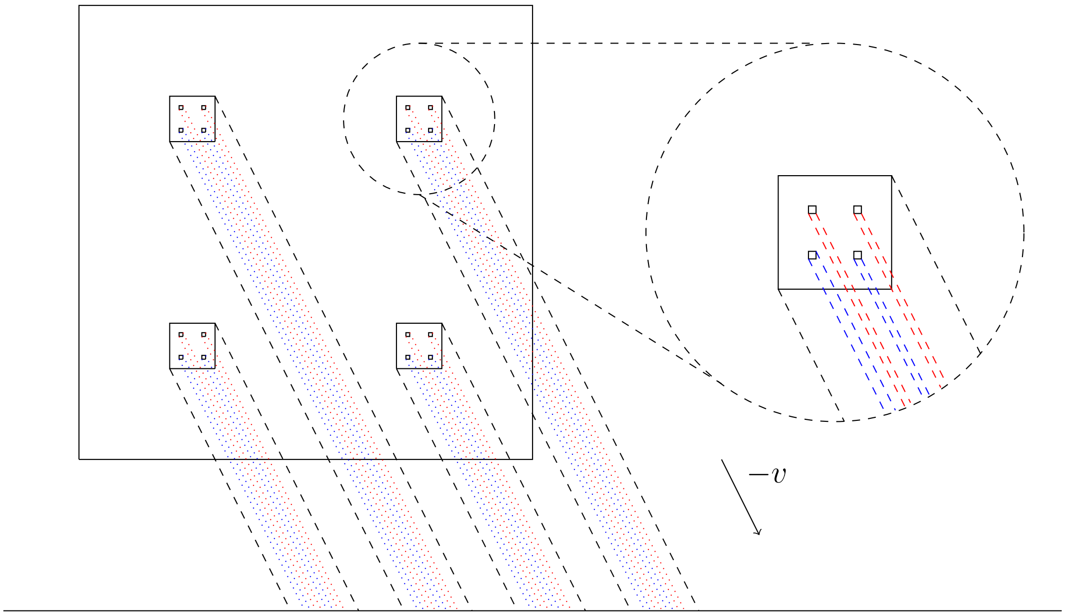

FIGURE 7. An illustration of the idea behind the self similarity argument.

Step 2A: Proving the equation (3.11). To prove (3.11) it suffices to show that

$$P_v(a+\mathbb{R}e_i)\cap P_v(b+\mathbb{R}e_j)\subset P_v\big((a+\mathbb{R}e_i)\cap(b+\mathbb{R}e_j)\big) \tag{3.12}$$

as the opposite inclusion is obvious. To prove (3.12) we recall the following elementary dimension formula for the linear map $P_v:\mathbb{R}^4\to X$:

$$\dim\big(\ker P_v\big)+\dim\big(\operatorname{im} P_v\big)=4,$$

where $\ker P_v$ stands for the kernel of the linear map P_v, and $\operatorname{im} P_v$ equals the image $P_v(\mathbb{R}^4)$. It is easy to see that $\dim\big(\operatorname{im} P_v\big)=3$ from the definition of P_v and from the observation that

$$P_v(e_l)=\begin{cases} e_l & \text{if } l=1,2,3\\ e_4-v & \text{if } l=4.\end{cases} \tag{3.13}$$

Thus, when $v=\big(\frac{1}{16},\frac{1}{8},\frac{1}{4},1\big)$ we conclude that

$$\ker P_v=\langle v\rangle,$$

where $\langle v\rangle$ stands for the linear span of the vector v. This follows from the fact that $\dim\ker P_v=1$ and $v\in\ker P_v$.

Next, we may assume that $P_v(a+\mathbb{R}e_i)\cap P_v(b+\mathbb{R}e_j)\neq\emptyset$ as otherwise the inclusion in (3.12) is obvious. Then there exists $t\in\mathbb{R}$ and $s\in\mathbb{R}$ such that

$$P_v(a+te_i-b-se_j)=0,$$

or equivalently, there exists $t,s\in\mathbb{R}$ and $r\in\mathbb{R}$ such that

$$(a-b)+te_i-se_j=rv. \tag{3.14}$$

To prove (3.11) we need to show that the equation (3.14) can have only trivial solutions (i.e. solutions for which $r=0$). In other words, we need to show that if the intersection $\big((a-b)+\mathbb{R}e_i\oplus\mathbb{R}e_j\big)\cap\langle v\rangle$ is nonempty, then

$$\big((a-b)+\mathbb{R}e_i\oplus\mathbb{R}e_j\big)\cap\langle v\rangle=\{0\}.$$

Because $(a+b)+\mathbb{R}e_i\oplus\mathbb{R}e_j$ is an affine vector space which is parallel to coordinate axes, and

$$\dim\big((a+b)+\mathbb{R}e_i\oplus\mathbb{R}e_j\big)\leq 2,$$

it is easy to see that for the vector $v=\big(\frac{1}{16},\frac{1}{8},\frac{1}{4},1\big)$ the intersection $\big((a-b)+\mathbb{R}e_i\oplus\mathbb{R}e_j\big)\cap\langle v\rangle$ can contain at most one point z. Then there are two possible cases we need to consider:

Case 1: Suppose first that $(a+\mathbb{R}e_i)\cap(b+\mathbb{R}e_j)\neq\emptyset$. In this case it follows that $z=0$ and the claim will follow because all the solutions to (3.14) are then trivial.

Case 2: Let us next assume that $(a+\mathbb{R}e_i)\cap(b+\mathbb{R}e_j)=\emptyset$. Then, because a and b were assumed to be vertices of $Q(0,\frac{1}{2})$ it follows that there is an index $i_1\notin\{i,j\}$ such that

$$(a-b)_{i_1}\in\{1,-1\}.$$

Moreover, because $\dim\langle e_i,e_j,e_{i_1}\rangle<4$ it will follow that there is also an index $i_2\notin\{i,j,i_1\}$ such that

$$(a-b)_{i_2}\in\{1,0,-1\}.$$

However, this is a contradiction with the fact that the equation (3.14) was assumed to have a solution. Indeed, otherwise it would follow that there is $r\in\mathbb{R}$ such that

$$|rv_{i_1}|=1\quad\text{and}\quad|rv_{i_2}|\in\{1,0\}$$

which is not the case when $v=\big(\frac{1}{16},\frac{1}{8},\frac{1}{4},1\big)$.

Step 2B: Proving that the sets $\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$ are disjoint. Recall now that

$$\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}:=\Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}),q\hat{r}_k\big)+P_v(\mathbb{R}e_i)\Big)\setminus\Big(\bigcup_{\boldsymbol{w}\in\mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)}Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}),\tfrac{7}{8}\hat{r}_{k-1}\big)\Big),$$

where $\mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i):=\big\{\boldsymbol{w}\in\mathbb{V}^{k-1}:(\hat{z}_{\boldsymbol{v}(k)}+\mathbb{R}e_i)\cap Q(\hat{z}_{\boldsymbol{w}},\hat{r}_{k-1})\neq\emptyset\big\}$. We claim that if we choose $\beta>0$ sufficiently large in the definition of the Cantor set C_B then:

(1) We may replace the index set $\mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)$ in the definition of $\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$ by the index set $\mathbb{V}^{k-1}$. This will be only a technical detail which helps us to work with sets $\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$ more easily.
(2) The sets $\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$ are pairwise disjoint for each fixed $k\in\mathbb{N}$ (recall that $\hat{r}_k=2^{-k}2^{-\beta k}$).

Proof of (1): We need to show that for every fixed q there exists $\beta_2 := \beta_2(q) > 0$ such that if we choose $\beta \geq \beta_2$ in the definition of the Cantor set C_B then for each fixed $\boldsymbol{v}(k) \in \mathbb{V}^k$ we have

$$\Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(\mathbb{R}e_i)\Big) \cap Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_{k-1}\big) = \emptyset,$$

whenever $\boldsymbol{w} \in \mathbb{V}^{k-1} \setminus \mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)$. It suffices to prove this for $k = 2$ because after this the general case follows from the self similarity of the construction.

First, we may find $\beta_2^1(q) > 0$ such that if $\beta \geq \beta_2^1$ in the definition of C_B then we have

$$Q^3(P_v(\hat{z}_{\boldsymbol{v}(2)}), q\hat{r}_2) + P_v(\mathbb{R}e_i) \subset\subset Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_1\big) + P_v(\mathbb{R}e_i),$$

whenever $\boldsymbol{w} \in \mathbb{V}$ is the parent of a given index $\boldsymbol{v}(2) \in \mathbb{V}^2$.

Next, by applying (3.11) and continuity of P_v we may find $\beta_2^2 > 0$ such that if $\beta \geq \beta_2^2$ in the definition of the Cantor set C_B then

$$\Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_1\big) + P_v(\mathbb{R}e_i)\Big) \cap \Big(Q^3\big(P_v(\hat{z}_{\hat{\boldsymbol{w}}}), \tfrac{7}{8}\hat{r}_1\big) + P_v(\mathbb{R}e_j)\Big) = \emptyset,$$

whenever $\boldsymbol{w}, \hat{\boldsymbol{w}} \in \mathbb{V}$ are indices for which the intersection of the lines $l_{\boldsymbol{w}} = \hat{z}_{\boldsymbol{w}} + \mathbb{R}e_i$ and $l_{\hat{\boldsymbol{w}}} = \hat{z}_{\hat{\boldsymbol{w}}} + \mathbb{R}e_j$ is empty, i.e. if the intersection of lines is empty then the intersection of small neighborhoods is also empty.

Suppose now that $\beta \geq \beta_2 := \max\{\beta_2^1, \beta_2^2\}$. Let us fix $\boldsymbol{v}(2) \in \mathbb{V}^2$ and suppose that $\boldsymbol{w} \in \mathbb{V}$ is the parent of $\boldsymbol{v}(2)$. Let us also assume that $\hat{\boldsymbol{w}} \in \mathbb{V} \setminus \mathbb{V}^1_{\boldsymbol{v}(2)}(i)$. Then it follows that the lines $l_{\boldsymbol{w}} = \hat{z}_{\boldsymbol{w}} + \mathbb{R}e_i$ and $l_{\hat{\boldsymbol{w}}} = \hat{z}_{\hat{\boldsymbol{w}}} + \mathbb{R}e_i$ do not intersect each other, and therefore

$$\begin{aligned}&\Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(2)}), q\hat{r}_2\big) + P_v(\mathbb{R}e_i)\Big) \cap Q^3\big(P_v(\hat{z}_{\hat{\boldsymbol{w}}}), \tfrac{7}{8}\hat{r}_1\big)\\ &\qquad \subset \Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_1\big) + P_v(\mathbb{R}e_i)\Big) \cap \Big(Q^3\big(P_v(\hat{z}_{\hat{\boldsymbol{w}}}), \tfrac{7}{8}\hat{r}_1\big) + P_v(\mathbb{R}e_i)\Big) = \emptyset,\end{aligned}$$

and (1) follows and we may write from now on

$$\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)} = \Big(Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(\mathbb{R}e_i)\Big) \setminus \Big(\bigcup_{\boldsymbol{w} \in \mathbb{V}^{k-1}} Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_{k-1}\big)\Big).$$

Proof of (2): Again, by the self similarity of the construction it is enough to prove (2) in the case $k = 1$. Let us first assume that z is one of the vertices of the cube $Q(0, \frac{1}{2})$. Then, recalling that the center of the cube $\hat{Q}_0$ is $\tilde{z}_0 = 0$, we have that

$$\|P_v(z) - P_v(\hat{z}_0)\| = \|P_v(z) - P_v(0)\| = \|z - z_4 v\| < \tfrac{7}{8}. \tag{3.15}$$

This gives us that $P_v(z) \in Q^3(P_v(\hat{z}_0), \frac{7}{8}\hat{r}_0)$ for each vertex z of the cube $Q(0, \frac{1}{2})$. Suppose next that $\hat{z}_{\boldsymbol{v}(1)}$ and $\hat{z}_{\hat{\boldsymbol{v}}(1)}$ are two (possibly identical) vertices of $Q(0, \frac{1}{2})$, and consider two (nonidentical) lines $M_1 = \hat{z}_{\boldsymbol{v}(1)} + \mathbb{R}e_i$ and $M_2 = \hat{z}_{\hat{\boldsymbol{v}}(1)} + \mathbb{R}e_j$ through the points $\hat{z}_{\boldsymbol{v}(1)}$ and $\hat{z}_{\hat{\boldsymbol{v}}(1)}$. Then by applying (3.15) to points $\hat{z}_{\boldsymbol{v}(1)}$ and $\hat{z}_{\hat{\boldsymbol{v}}(1)}$, and using (3.11) we get

$$\big(P_v(M_1) \setminus Q^3\big(P_v(\hat{z}_0), \tfrac{7}{8}\hat{r}_0\big)\big) \cap \big(P_v(M_2) \setminus Q^3\big(P_v(\hat{z}_0), \tfrac{7}{8}\hat{r}_0\big)\big) = \emptyset.$$

Therefore, by linearity and Lipschitz continuity of P_v and by the fact that the lines $P_v(M_1)$ and $P_v(M_2)$ intersect at most at one point it is easy to see that there exists

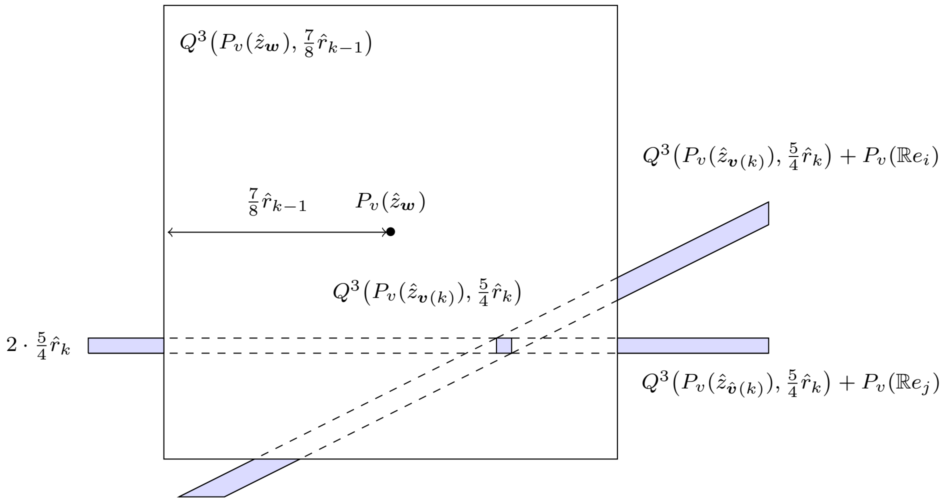

FIGURE 8. When we choose $\beta > 0$ large enough the ratio $\hat{r}_{k-1}/\hat{r}_k = 2^{1+\beta}$ will be very large. This gives enough space for the lines $P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_i)$ and $P_v(\hat{z}_{\hat{\boldsymbol{v}}(k)} + \mathbb{R}e_j)$ to recede from each other before they reach the boundary of the big cube. Especially, it follows from this and the linearity of the mapping P_v that the intersection of the sets $Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), \frac{5}{4}\hat{r}_k\big) + P_v(\mathbb{R}e_i)$ and $Q^3\big(P_v(\hat{z}_{\hat{\boldsymbol{v}}(k)}), \frac{5}{4}\hat{r}_k\big) + P_v(\mathbb{R}e_j)$ is empty outside the cubes $Q^3\big(P_v(\hat{z}_{\boldsymbol{w}}), \frac{7}{8}\hat{r}_{k-1}\big)$, $\boldsymbol{w} \in \mathbb{V}^{k-1}$.

$\beta_3 > 0$ such that if we choose $\beta \geq \beta_3$ in the definition of the Cantor set C_B then we get

$$\begin{aligned}\hat{\mathcal{S}}^i_{\boldsymbol{v}(1)} \cap \hat{\mathcal{S}}^j_{\hat{\boldsymbol{v}}(1)} =&\Big(\big(Q^3\big((P_v(\hat{z}_{\boldsymbol{v}(1)}), q\hat{r}_1\big) + P_v(\mathbb{R}e_i)\big)\cap\big(Q^3\big((P_v(\hat{z}_{\hat{\boldsymbol{v}}(1)}), q\hat{r}_1\big) + P_v(\mathbb{R}e_j)\big)\Big)\\ &\setminus Q^3\big(P_v(\hat{z}_0), \tfrac{7}{8}\hat{r}_0\big) = \emptyset,\end{aligned}$$

see also Fig. 8. By working through all the combinations $\boldsymbol{v}(1), \hat{\boldsymbol{v}}(1) \in \mathbb{V}$ we may also assume that $\beta_3 > 0$ is independent on the pair $(\boldsymbol{v}(1), \hat{\boldsymbol{v}}(1)) \in \mathbb{V} \times \mathbb{V}$. This gives us that the sets $\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$ are pairwise disjoint for $k = 1$.

To see that $\hat{\mathcal{S}}^i_{\boldsymbol{v}(k)} \cap \hat{\mathcal{S}}^j_{\hat{\boldsymbol{v}}(k)} = \emptyset$ for $k \geq 2$ one may apply self similarity of the construction together with the previous argument where $k = 1$. Self similarity argument applies to this situation as the ratio $\hat{r}_{k-1}/\hat{r}_k = 2^{1+\beta}$ stays the same for every $k \in \mathbb{N}$ (see also Fig. 8).

Step 2C: Proving the inclusion $P_v(\hat{S}^i_{\boldsymbol{v}(k)}) \subset \hat{\mathcal{S}}^i_{\boldsymbol{v}(k)}$. Let us next recall the definition of k-bars

$$\hat{G}^i_{\boldsymbol{v}(k)} := \hat{Q}_{\boldsymbol{v}(k)} + \mathbb{R}e_i.$$

We also recall (see the paragraph after (3.5)) that there exists $\beta_1 > 0$ such that if we choose $\beta \geq \beta_1$ in the definition of the Cantor set C_B, then we may define the corresponding sliced k-bars for k-bars as

$$\hat{S}^i_{\boldsymbol{v}(k)} := \hat{G}^i_{\boldsymbol{v}(k)} \setminus \Big(\bigcup_{\boldsymbol{w}\in\mathbb{V}^{k-2}} Q(\hat{z}_{\boldsymbol{w}}, \hat{r}_{k-2})\Big).$$

We want to show that $P_v(\hat{S}^i_{\boldsymbol{v}(k)}) \subset \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$. For this we first observe that for every $x, y \in \mathbb{R}^4$ such that $\|x - y\| < \hat{r}_k$, where $\|.\|$ denotes the maximum norm, we have by (3.2)

$$\|P_v(x) - P_v(y)\| = \big(x_1 - y_1 - (x_4 - y_4)v_1, x_2 - y_2 - (x_4 - y_4)v_2,\\ x_3 - y_3 - (x_4 - y_4)v_3, 0\big) \leq \tfrac{5}{4}\|x - y\| < \tfrac{5}{4}\hat{r}_k.$$

Thus, it follows that

$$P_v(\hat{S}^i_{\boldsymbol{v}(k)}) \subset Q^3\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), \tfrac{5}{4}\hat{r}_k\big) + P_v(\mathbb{R}e_i), \tag{3.16}$$

and hence the requirement that $q \geq \frac{5}{4}$ in the definition (3.6). Therefore it suffices now to show that for every $x \in \hat{S}^i_{\boldsymbol{v}(k)}$ we have

$$\|P_v(x) - P_v(\hat{z}_{\boldsymbol{w}})\| > \tfrac{7}{8}\hat{r}_{k-1}, \tag{3.17}$$

whenever $\boldsymbol{w} \in \mathbb{V}^{k-1}$. Actually, by assuming that $\beta \geq \beta_2$ we need to verify (3.17) only for every $\boldsymbol{w} \in \mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)$. For this, let us assume that $x \in \hat{S}^i_{\boldsymbol{v}(k)}$ and $\boldsymbol{w} \in \mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)$. Then we have to consider two different cases:

(i) Suppose first that $i \neq 4$. If we denote $y := \hat{z}_{\boldsymbol{w}}$ we get $|x_4 - y_4| < \hat{r}_{k-1}$ and $|x_i - y_i| \geq \hat{r}_{k-2}$. Thus, it follows that

$$\|P_v(x) - P_v(y)\| = \|x - y - (x_4 - y_4)v\| > \hat{r}_{k-2} - \tfrac{1}{4}\hat{r}_{k-1} > \tfrac{7}{8}\hat{r}_{k-1},$$

simply because we know that $\beta > 2$ and (3.17) follows.

(ii) Next we assume that $i = 4$. If we write $y := \hat{z}_{\boldsymbol{v}(k)} - x$ we get that

$$|y_4| \geq \hat{r}_{k-2} \text{ and } |y_i| < \hat{r}_{k-1} \text{ for all } i = 1, 2, 3.$$

These estimates give us

$$\|P_v(\hat{z}_{\boldsymbol{v}(k)}) - P_v(x)\| = \|P_v(y)\| = \|y - y_4 v\| \geq |\tfrac{1}{4}y_4 - y_3| \geq \tfrac{1}{4}\hat{r}_{k-2} - \hat{r}_{k-1} > \tfrac{7}{8}\hat{r}_{k-1},$$

which implies (3.17) by having $\beta > 3$.

Therefore, by combining (3.16) and (3.17) together we conclude that $P_v(\hat{S}^i_{\boldsymbol{v}(k)}) \subset \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ as we wanted.

Step 2D: Conclusion of Step 2. In Step 1 we have already showed that P_v is one-to-one on C_B. Thus, it suffices to show that P_v is one-to-one also on $\mathcal{K}_B \setminus C_B$ and then we can easily see, for example by the linearity of P_v, that P_v is in fact one-to-one on $\mathcal{K}_B$.

To see that P_v is one-to-one on $\mathcal{K}_B \setminus \mathcal{C}_B$ suppose that ℓ and $\hat{\ell}$ are two distinct lines in $\mathcal{K}_B$. It is easy to see that P_v is one-to-one along these lines and thus it suffices to show that

$$P_v\big(\ell \setminus C_B\big) \cap P_v\big(\hat{\ell} \setminus C_B\big) = \emptyset.$$

For this we observe that the intersection of ℓ and $\hat{\ell}$ is either an empty set or one point which lies in the set C_B. Therefore we may find an index $N \in \mathbb{N}$, and sequences $\{\hat{S}^i_{\boldsymbol{v}(k)}\}_{k=N}^{\infty}$ and $\{\hat{S}^j_{\hat{\boldsymbol{v}}(k)}\}_{k=N}^{\infty}$ of sliced k-bars such that $\{\ell \cap \hat{S}^i_{\boldsymbol{v}(k)}\}_{k=N}^{\infty}$ and $\{\hat{\ell} \cap \hat{S}^j_{\hat{\boldsymbol{v}}(k)}\}_{k=N}^{\infty}$ are two sequences of sets, and it holds that

$$\lim_{k \to \infty} \ell \cap \hat{S}^i_{\boldsymbol{v}(k)} = \ell \setminus C_B, \quad \lim_{k \to \infty} \hat{\ell} \cap \hat{S}^j_{\hat{\boldsymbol{v}}(k)} = \hat{\ell} \setminus C_B, \text{ and } \quad \hat{S}^i_{\boldsymbol{v}(k)} \cap \hat{S}^j_{\hat{\boldsymbol{v}}(k)} = \emptyset$$

for every $k \geq N$. Furthermore, by step 2B and step 2C we have $P_v(\hat{S}^i_{\boldsymbol{v}(k)}) \subset \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ and $P_v(\hat{S}^j_{\hat{\boldsymbol{v}}(k)}) \subset \hat{\boldsymbol{\mathcal{S}}}^j_{\hat{\boldsymbol{v}}(k)}$ where $\hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)} \cap \hat{\boldsymbol{\mathcal{S}}}^j_{\hat{\boldsymbol{v}}(k)} = \emptyset$, and therefore

$$P_v\big(\ell \setminus C_B\big) \cap P_v\big(\hat{\ell} \setminus C_B\big) \subset \lim_{k\to\infty} P_v\big(\hat{S}^i_{\boldsymbol{v}(k)}\big) \cap P_v\big(\hat{S}^j_{\hat{\boldsymbol{v}}(k)}\big) \subset \lim_{k\to\infty} \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)} \cap \hat{\boldsymbol{\mathcal{S}}}^j_{\hat{\boldsymbol{v}}(k)} = \emptyset,$$

which ends this step.

Step 3: Defining the function g on X. We have that the sets $\hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ (recall that their definition is dependent on a positive parameter q) are disjoint if distinct. Therefore also the sets, which one could call (punctured) pipes,

$$\begin{aligned}\hat{H}^i_{\boldsymbol{v}(k)} := \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}} &:= \partial_X\Big(Q^3(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k) + P_v(\mathbb{R}e_i)\Big) \setminus \Big(\bigcup_{\boldsymbol{w}\in\mathbb{V}^{k-1}_{\boldsymbol{v}(k)}(i)} Q^3(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_{k-1})\Big)\\ &= \partial_X\Big(Q^3(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k) + P_v(\mathbb{R}e_i)\Big) \setminus \Big(\bigcup_{\boldsymbol{w}\in\mathbb{V}^{k-1}} Q^3(P_v(\hat{z}_{\boldsymbol{w}}), \tfrac{7}{8}\hat{r}_{k-1})\Big),\end{aligned} \tag{3.18}$$

are pairwise disjoint sets for distinct bars. Here $\partial_X A$ denotes the relative boundary of a set A in $X = \mathbb{R}^3 \times \{0\}$.

It is worth noticing that lines in $\mathcal{K}_B$ parallel to e_i are contained in the interior of $\hat{G}^i_{\boldsymbol{v}(k)}$-type bars and therefore also the projection of the line is contained in the 3-dimensional interior of the projection of the bar. Especially the projection of a line in $\mathcal{K}_B$ never intersects a punctured pipe. When we say that the projection of a line in $\mathcal{K}_B$ is inside a pipe $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$ we mean that the line in $\mathcal{K}_B$ lies in the bar $\hat{G}^i_{\boldsymbol{v}(k)}$ from which we derived the pipe. In fact we can claim not only that the projection of lines in $\mathcal{K}_B$ do not intersect pipes, but further we know that the projection of a line in $\mathcal{K}_B$ lies inside the projection of some $(k+1)$-bar and that means that there are no lines in $\mathcal{K}_B$ whose projection intersects the set

$$L^i_{\boldsymbol{v}(k)} := \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)} \setminus \Big(\bigcup_{\boldsymbol{v}(k+1)\in\mathbb{V}^{k+1}} \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k+1)}\Big). \tag{3.19}$$

With respect to this fact we will extend our Lipschitz function g in the following way. We will define g on the projection of lines in $\mathcal{K}_B$ and on punctured pipes. We will show that our definition is Lipschitz and then extend it in a Lipschitz way inside $L^i_{\boldsymbol{v}(k)}$. We will take care during the extension to guarantee that (3.3) holds, which is not difficult. Then there will be some remaining part of X where we can define g practically arbitrarily as long as we maintain the Lipschitz property.

As mentioned in our outline, we will define

$$g(P_v(x)) = -x_n \text{ for } x \in \mathcal{K}_B.$$

We now wish to show that this can be extended in a Lipschitz way onto X. Our argument will make use of pipes, but for pipes of type $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$, with $i = 1, 2, 3$ it is slightly more simple than for $i = 4$. We will deal with the simpler case first then note the difference for the case $i = 4$.

Step 3A. First we take a pipe $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$, with $i = 1, 2, 3$ and $k \geq 2$. Then we define

$$g(x) = -(\hat{z}_{\boldsymbol{v}(k)})_4 \quad \text{for all } x \in \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}.$$

Let us note that this definition is well-defined, if $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}} = \hat{H}^i_{\hat{z}_{\tilde{\boldsymbol{v}}(k)}}$ then $(\hat{z}_{\boldsymbol{v}(k)})_4 = (\hat{z}_{\tilde{\boldsymbol{v}}(k)})_4$. Now it is very easy to notice that if we have two pipes, one inside another (that is $\hat{z}_{\boldsymbol{v}(k+1)} + \mathbb{R}e_i$ intersects $\hat{Q}_{\boldsymbol{v}(k)}$), then

$$\text{dist}(\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}, \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k+1)}}) \geq C\hat{r}_k \text{ for a suitable } C > 0 \text{ independent of } k. \tag{3.20}$$

Further considering $x \in \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$ and $y \in \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k+1)}}$ we have

$$|g(x) - g(y)| = |-(\hat{z}_{\boldsymbol{v}(k)})_4 + (\hat{z}_{\boldsymbol{v}(k+1)})_4| = \tfrac{1}{2}\hat{r}_k.$$

Considering two distinct pipes of the same generation, both inside $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$ we see that

$$\text{dist}(\hat{H}^i_{\hat{z}_{\hat{\boldsymbol{v}}(k+1)}}, \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k+1)}}) \geq C\hat{r}_k \text{ for a suitable } C > 0 \text{ independent of } k.$$

Furthermore, for $x \in \hat{H}^i_{\hat{z}_{\hat{\boldsymbol{v}}(k+1)}}$ and $y \in \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k+1)}}$ we have

$$|g(x) - g(y)| = |-(z_{\hat{\boldsymbol{v}}(k+1)})_4 + (z_{\boldsymbol{v}(k+1)})_4| \leq \hat{r}_k.$$

This proves that g, thus defined, on the pipes $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$, $i = 1, 2, 3$, is Lipschitz with respect to parallel pipes.

Step 3B. Now consider a line l through the Cantor set C_B parallel to e_i, $i \in \{1, 2, 3\}$, whose projection lies inside the pipe $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$. For each such line l we define

$$g(P_v(l)) = -x_4 \quad \text{where } x \in l \cap C_B. \tag{3.21}$$

Next, we calculate that

$$\text{dist}(P_v(l), \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}) \geq C\hat{r}_k \text{ for a suitable } C \text{ independent of } k.$$

On the other hand we may observe that g is constant on each line $P_v(l)$ described above, and thus by taking $z \in P_v(l), y \in \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$ we observe

$$|g(z) - g(y)| = |-x_4 + (\hat{z}_{\boldsymbol{v}(k+1)})_4| \leq 2\hat{r}_k.$$

But this shows that we have defined g Lipschitz on the set of pipes and projection of lines through the Cantor set for those pipes and lines parallel to e_i, $i = 1, 2, 3$.

Strictly speaking we should check that our definition of g is Lipschitz, when we compare $x \in \hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$ and $y \in \hat{H}^j_{\hat{z}_{\boldsymbol{v}(k)}}$ for $i, j \in \{1, 2, 3\}$ also for $i \neq j$ but the considerations and calculations from step 1 and step 2 show that the distance between these pipes is at least $C\hat{r}_k$ and $|g(x) - g(y)| \leq 2\hat{r}_k$ and so this part of the argument is easy.

Step 3C. Now we define g on $L^i_{\boldsymbol{v}(k)}$, $i = 1, 2, 3$, as follows (recall that $L^i_{\boldsymbol{v}(k)}$ are defined in (3.19)). Choose $i \in \{1, 2, 3\}$ and fix a 2-dimensional hyperplane $Y_i \subset\subset X$ perpendicular to e_i, such that Y_i intersects all of the pipes $\hat{H}^i_{\hat{z}_{\boldsymbol{v}(k)}}$, $k \geq 2$. We may write

$$Y_i = \Big\{ y_i e_i + \sum_{j \neq i} t_j e_j : t_j \in \mathbb{R} \text{ for every } j \neq i \Big\},$$

where $y_i \in \mathbb{R}$ is fixed. We define a projection π_{Y_i} from X onto Y_i by

$$(\pi_{Y_i}(x))_j = \begin{cases} x_j & j \neq i \\ y_i & j = i. \end{cases}$$

We use the McShane extension theorem on the hyperplane Y_i to extend g on those parts of the set $Y_i \cap \big(\bigcup_{k=2}^\infty \hat{\mathcal{S}}^i_{\hat{z}_{\boldsymbol{v}(k)}}\big)$ where we did not define g during the previous steps (step 3A and 3B) and then we define g at other points x in $\hat{\mathcal{S}}^i := \bigcup_{k=2}^\infty \hat{\mathcal{S}}^i_{\hat{z}_{\boldsymbol{v}(k)}}$ by simply projecting x onto Y_i and then using g. In other words

$$g(x) = g(\pi_{Y_i}(x)) \quad \text{for all } x \in \hat{\mathcal{S}}^i. \tag{3.22}$$

Thus defined the function g is constant on the intersection of lines parallel to e_i with the set $\hat{\mathcal{S}}^i$.

Step 3D. Our argument in the projection of bars parallel to e_4 is identical to the previous, up to the fact that we do not define g as constant equal to $-(\hat{z}_{\boldsymbol{v}(k)})_4$ on pipes generated by $\hat{G}^4_{\boldsymbol{v}(k)}$-type bars but by using an appropriate affine function. For this we recall that $\tilde{v} = \big(\frac{1}{16}, \frac{1}{8}, \frac{1}{4}\big)$ and we denote

$$Y_4 := \{w \in \mathbb{R}^3 : \langle w, \tilde{v}\rangle = 0\}.$$

Then we may separate $\mathbb{R}^3$ into the direct sum $\mathbb{R}\tilde{v} \oplus Y_4$. Now, suppose that $\lambda_0 \in \mathbb{R}$ and $w_0 \in Y_4$ are such that

$$P_v(\hat{z}_{\boldsymbol{v}(k)}) = w_0 + \lambda_0 \tilde{v}. \tag{3.23}$$

Then, if $\tilde{x} \in \hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ we may find $\lambda \in \mathbb{R}$ and $w \in Y_4$ such that $\tilde{x} = w + \lambda\tilde{v}$ which leads us to define

$$g(\tilde{x}) = \lambda - \lambda_0 - (\hat{z}_{\boldsymbol{v}(k)})_4 \quad \text{for every } \tilde{x} \in \hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4). \tag{3.24}$$

We proceed to prove that by defining

$$\begin{aligned} g(\tilde{x}) &= \lambda - \lambda_0 - (\hat{z}_{\boldsymbol{v}(k)})_4 && \text{for every } \tilde{x} \in \hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4) \\ g(P_v(x)) &= -x_4 && \text{for every } x \in \mathcal{K}_B \end{aligned}$$

we get a Lipschitz function on $\hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4) \cup P_v(\mathcal{K}_B)$. A first observation is that for every $x \in \hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4$ we find α such that $x = \hat{z}_{\boldsymbol{v}(k)} + \alpha e_4$ and then by (3.13) and (3.23) we get

$$P_v(x) = P_v(\hat{z}_{\boldsymbol{v}(k)}) + \alpha P_v(e_4) = \big(w_0 + \lambda_0\tilde{v} - \alpha\tilde{v}, 0\big).$$

Now we will apply (3.24) with $\tilde{x} = P_v(x) = w_0 + \lambda_0\tilde{v} - \alpha\tilde{v}$ to get

$$g(P_v(x)) = \lambda_0 - \alpha - \lambda_0 - (\hat{z}_{\boldsymbol{v}(k)})_4 = -x_4 \text{ for all } x \in \hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4.$$

The rest of the argument will be a case of proving that g has similar values on $\hat{H}^4_{\boldsymbol{v}(k)}$ (up to an error of $C\hat{r}_k$) and the distance between $P_v(C_B + \mathbb{R}e_4)$ and $\hat{H}^4_{\boldsymbol{v}(k)}$ is $C\hat{r}_k$. Let us continue to expound.

Our choice of $\beta > 1$ guarantees that the Cantor set C_B is at a distance of at least $\frac{1}{4}\hat{r}_k$ from the boundary of the cubes $\hat{Q}_{\hat{z}_{\boldsymbol{v}(k)}}$. Now we will take any $c \in C_B \cap \hat{Q}_{\boldsymbol{v}(k)}$ and (recall the definition of k-bars from (3.4)) we will see that

$$c + \mathbb{R}e_4 + Q(0, \tfrac{1}{4}\hat{r}_k) \subset\subset \hat{G}^4_{\hat{z}_{\boldsymbol{v}(k)}}.$$

Our projection P_v is continuous onto X and therefore there is a $C > 0$ such that

$$P_v(c + \mathbb{R}e_4) + Q^3(0, C\hat{r}_k) \subset P_v(\hat{G}^4_{\hat{z}_{\boldsymbol{v}(k)}})$$

implying that there is $C_1 > 0$ such that

$$\operatorname{dist}(\hat{H}^4_{\boldsymbol{v}(k)}, P_v(c + \mathbb{R}e_4)) \geq C_1\hat{r}_k \tag{3.25}$$

for all $\boldsymbol{v}(k)$, all c and all k. Exactly the same argument gives that

$$\operatorname{dist}(\hat{H}^4_{\boldsymbol{v}(k)}, P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)) \geq C_1\hat{r}_k. \tag{3.26}$$

Furthermore we can make the opposite estimates since for some $C > 0$

$$c + \mathbb{R}e_4 + Q(0, C\hat{r}_k) \supset \hat{G}^4_{\hat{z}_{\boldsymbol{v}(k)}}$$

and the continuity of our projection gives

$$\operatorname{dist}(x, P_v(c + \mathbb{R}e_4)) \leq C_2\hat{r}_k \tag{3.27}$$

for any $x \in \hat{H}^4_{\boldsymbol{v}(k)}$ and similarly

$$\operatorname{dist}(x, P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)) \leq C_2\hat{r}_k, \tag{3.28}$$

for any $x \in \hat{H}^4_{\boldsymbol{v}(k)}$.

Now we will be able to show that g is a Lipschitz function when restricted to $\hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$. For every point $x \in \hat{H}^4_{\boldsymbol{v}(k)}$ we find a point $w \in P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ such that $g(x) = g(w)$ and $|x - w|$ is bounded by a constant multiple of $\hat{r}_k$. Finally, since g is linear on the line $P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ we will be able to prove the desired Lipschitz quality of g by (3.26), when y is close to x and w, and by $|w - y| \approx |g(w) - g(y)|$, when y is far from x and w.

Now recall that $P_v(\mathbb{R}e_4) = \mathbb{R}\tilde{v}$ (see (3.13)) and Y_4 is the linear space perpendicular to $\tilde{v}$. We take $x \in \hat{H}^4_{\boldsymbol{v}(k)}$ and claim that there exists a unique $w \in P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4) \cap (x + Y_4)$, which is obvious because $P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ and $x + Y_4$ are a pair of perpendicular affine spaces in a 3 dimensional space and the sum of their dimensions is 3. Quite simply because $x - w \in Y_4$ and $P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ is perpendicular to Y_4 we see that w is the closest point to x in $P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$. Using (3.26) and (3.28) we may estimate

$$C_1\hat{r}_k \leq |w - x| \leq C_2\hat{r}_k.$$

Also, since (3.24) gives that g is constant on the intersection of any affine plane parallel to Y_4 with the set $\hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$, we have that

$$g(x) = g(w).$$

Now we may take any $x \in \hat{H}^4_{\boldsymbol{v}(k)}$, its corresponding $w \in (x + Y_4) \cap P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ and any $y \in P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ and calculate

$$|g(y) - g(x)| = |g(y) - g(w)| = |\langle (y-w), \tfrac{\tilde{v}}{|\tilde{v}|^2} \rangle| = |\tilde{v}|^{-1}|y-w|$$
$$|y-x| \geq |y-w| - |w-x| \geq |y-w| - C_2\hat{r}_k.$$

When $|y-w| > 2C_2\hat{r}_k$ then $|y-w| > 2|x-w|$ and therefore $|x-y| > \frac{1}{2}|w-y|$ and we may estimate

$$\frac{|g(x)-g(y)|}{|x-y|} \leq \frac{2|g(w)-g(y)|}{|w-y|} \leq \frac{2|w-y|}{|\tilde{v}||w-y|}.$$

When $|y-w| \leq 2C_2\hat{r}_k$ we use (3.26)

$$\frac{|g(x)-g(y)|}{|x-y|} \leq \frac{|w-y|}{|\tilde{v}|C_1\hat{r}_k} \leq \frac{2C_2}{C_1|\tilde{v}|}.$$

By a very similar argument we will proceed to prove that g is C_3-Lipschitz when restricted to $P_v(\mathcal{K}_B) \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$. Take a point $c \in C_B$ and the unique $\hat{z}_{\boldsymbol{v}(k)}$ such that $c \in Q_{\boldsymbol{v}(k)}$. First we observe that

$$g(P_v(\hat{z}_{\boldsymbol{v}(k)})) = -(\hat{z}_{\boldsymbol{v}(k)})_4 \text{ and } g(P_v(c)) = -c_4$$

and

$$|c_4 - (\hat{z}_{\boldsymbol{v}(k)})_4| < C\hat{r}_k.$$

If $x \in P_v(c + \mathbb{R}e_4)$ and $y \in P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ then

$$|g(x) - g(y)| \leq \langle (x-y), \tfrac{\tilde{v}}{|\tilde{v}|^2} \rangle + C\hat{r}_k.$$

Further, distance estimates similar to (3.25)–(3.28) hold also for $P_v(c + \mathbb{R}e_4)$ and $P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$, and therefore

$$|x-y| \geq C\langle (x-y), \frac{\tilde{v}}{|\tilde{v}|^2} \rangle + C\hat{r}_k.$$

Hence

$$\frac{|g(x)-g(y)|}{|x-y|} \leq \frac{\langle (x-y), \frac{\tilde{v}}{|\tilde{v}|^2} \rangle + C\hat{r}_k}{C\langle (x-y), \frac{\tilde{v}}{|\tilde{v}|^2} \rangle + C\hat{r}_k} < C_3.$$

This means, that g is C_3-Lipschitz when restricted to

$$P_v(\mathcal{K}_B) \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4).$$

Now we can show that the restriction of g to $P_v(\mathcal{K}_B) \cup \hat{H}^4_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ is Lipschitz. Assume that we have $x \in \hat{H}^4_{\boldsymbol{v}(k)}$ and $y \in P_v(\mathcal{K}_B)$. If

$$|g(x) - g(y)| \leq 2C_2C_3\hat{r}_k$$

then (3.25) says that g has been defined Lipschitz. Therefore we consider the case

$$|g(x) - g(y)| > 2C_2C_3\hat{r}_k.$$

We have a $w \in (x + Y_4) \cap P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ and $|x-w| < C_2\hat{r}_k$. Since g is C_3-Lipschitz when restricted to the set $P_v(\mathcal{K}_B) \cup \bigcup_{\boldsymbol{v}(k) \in \mathbb{V}^k} P_v(\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4)$ we have that

$$|w-y| \geq \frac{|g(w)-g(y)|}{C_3} = \frac{|g(x)-g(y)|}{C_3}$$

and therefore

$$|x-y| \geq |w-y| - |x-w| \geq \frac{|g(x)-g(y)|}{C_3} - C_2\hat{r}_k.$$

We get

$$\frac{|g(x)-g(y)|}{|x-y|} \leq \frac{C_3|g(x)-g(y)|}{|g(x)-g(y)| - C_3C_2\hat{r}_k} \leq \frac{C_3}{1-\frac{1}{2}}.$$

So we prove that g is $2C_3C_2C_1^{-1}$-Lipschitz when restricted to the set

$$P_v(\mathcal{K}_B) \cup \bigcup P_v(z_{\boldsymbol{v}(k)} + \mathbb{R}e_4) \cup \bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k} \hat{H}^4_{\hat{z}_{\boldsymbol{v}(k)}}.$$

Of course self-similarity means that $2C_3C_2C_1^{-1}$ is independent of k.

It is not hard to estimate that

$$\operatorname{dist}(\hat{H}^4_{\hat{z}_{\boldsymbol{v}(k)}}, \hat{H}^4_{\hat{z}_{\tilde{\boldsymbol{v}}(\tilde{k})}}) \approx \hat{r}_{\min\{k,\tilde{k}\}}.$$

Therefore we see that the definition (3.24) is Lipschitz on the collection of all e_4 pipes. We use the construction described before (3.22) to get a Lipschitz extension which guarantees that

$$g(x+t\tilde{v}) = g(x) + t \tag{3.29}$$

everywhere in $\hat{\mathcal{S}}^4_{\boldsymbol{v}(k)}$, this time by projecting onto Y_4.

Where not yet defined we may extend g Lipschitz arbitrarily, for example by the McShane extension theorem.

Step 3E: verifying the condition (3.3)**.** Now it is quite simple to notice that we have

$$D_iF_{g,u} \circ F_{g,v}(x) = e_i,\ i = 1,2,3 \text{ and } D_4F_{g,u} \circ F_{g,v}(x) = -e_4$$

whenever $x_i \in [-1,1] \setminus \hat{U}_k$ and $x_j \in \hat{U}_{k+2}$ for all $j \neq i$. This can be seen from the following arguments. Firstly, it follows from (3.13) that $P_v(x+te_i) = P_v(x) + te_i$ for $i = 1,2,3$. On the other hand, one can see from (3.22) that if $P_v(x)$ and $P_v(x) + te_i$ lies in $\hat{\mathcal{S}}^i_{\hat{z}_{\boldsymbol{v}(k)}}$, then $g(P_v(x)) = g(P_v(x+te_i))$. Thus, we have

$$F_{g,v}(x+te_i) = x + te_i + vg(P_v(x+te_i)) = F_{g,v}(x) + te_i$$

and the similar identity holds for $F_{g,u}$. It follows that for each $i = 1,2,3$ it holds

$$\lim_{t\to 0} \frac{F_{g,u}(F_{g,v}(x+te_i)) - F_{g,u}(F_{g,v}(x))}{t} = e_i \quad \text{whenever } P_v(x) \in \hat{\mathcal{S}}^i_{\hat{z}_{\boldsymbol{v}(k)}}. \tag{3.30}$$

The argument for D_4 is similar. We know (see (3.29)) that g has the following property on a line segment parallel to $\tilde{v}$ e.g. $\{P_v(x) + t\tilde{v}, t \in I\}$ which happens to lie in $\hat{\mathcal{S}}^4_{\hat{z}_{\boldsymbol{v}(k)}}$,

$$g(P_v(x) + t\tilde{v}) = g(P_v(x)) + t.$$

Now take a line segment in $\hat{S}^4_{\hat{z}_{\boldsymbol{v}(k)}}$ parallel to e_4. From (3.13) we know that

$$P_v(x+te_4) = P_v(x) - t\tilde{v}$$

and therefore

$$g(P_v(x+te_4)) = g(P_v(x)) - t.$$

Recalling that $v = (\tilde{v}, 1)$ we get

$$\begin{aligned} F_{g,v}(x + te_4) &= x + te_4 + vg(P_v(x + te_4)) \\ &= x + te_4 - tv + vg(P_v(x)) \\ &= F_{g,v}(x) - t\tilde{v}. \end{aligned} \tag{3.31}$$

At this point we need to finally choose q in the definition of $\hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ (see (3.6)). For every $i \in \{1, 2, 3\}$ we have defined g in (3.21) on projection of line segments through $\hat{z}_{\boldsymbol{v}(k)}$ so that $g = -(\hat{z}_{\boldsymbol{v}(k)})_4$ and hence for every $x \in (\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_i) \cap \hat{S}^i_{\boldsymbol{v}(k)}$ we have

$$(F_{g,v}(x))_4 = (x + vg(P_v(x)))_4 = x_4 + v_4(-(\hat{z}_{\boldsymbol{v}(k)})_4) = x_4 - (\hat{z}_{\boldsymbol{v}(k)})_4 = 0$$

and hence

$$\big(F_{g,v}\big((\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_i) \cap \hat{S}^i_{\boldsymbol{v}(k)}\big)\big)_4 = 0.$$

Analogously for $i = 4$ we defined g in (3.24) so that

$$\big(F_{g,v}\big((\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_4) \cap \hat{S}^4_{\boldsymbol{v}(k)}\big)\big)_4 = 0$$

for all $\hat{z}_{\boldsymbol{v}(k)}$. Since for all $x \in \hat{S}^i_{\boldsymbol{v}(k)}$ we find a $y \in (\hat{z}_{\boldsymbol{v}(k)} + \mathbb{R}e_i) \cap \hat{S}^i_{\boldsymbol{v}(k)}$ such that $\|x - y\| \le \hat{r}_k$ we see that

$$|F_{g,v}(y) - F_{g,v}(x)| < C\hat{r}_k$$

and so by Lipschitz continuity of P_u,

$$|P_u(F_{g,v}(y)) - P_u(F_{g,v}(x))| < C\hat{r}_k.$$

Therefore we find a $q \ge \frac{5}{4}$ which will now ensure that (note that there is q in the definition of $\hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ but not in the definition of $\hat{S}^i_{\boldsymbol{v}(k)}$)

$$P_u(F_{g,v}(x)) \in \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)} \text{ for every } x \in \hat{S}^i_{\boldsymbol{v}(k)}. \tag{3.32}$$

Now using (3.31), applying $F_{g,u}$ and using (3.29) with (3.32) we get

$$\begin{aligned} F_{g,u} \circ F_{g,v}(x + te_4) &= F_{g,v}(x) - t\tilde{v} + ug(P_u(F_{g,v}(x)) - t\tilde{v}) \\ &= F_{g,v}(x) - t\tilde{v} + ug(P_u(F_{g,v}(x)) - tu \\ &= F_{g,v}(x) + ug(P_u(F_{g,v}(x)) - te_4 \\ &= F_{g,u} \circ F_{g,v}(x) - te_4, \end{aligned} \tag{3.33}$$

where we used $u = (-\tilde{v}, 1)$. Now (3.33) easily gives us what we wanted to prove, i.e. $D_4 F_{g,u} \circ F_{g,v} = -e_4$.

Given this, it suffices to realize that

$$\bigcup_{\boldsymbol{v}(k) \in \mathbb{V}^k} \hat{S}^i_{\boldsymbol{v}(k)} = \{x \in \mathbb{R}^n : x_i \in [-1, 1] \setminus \hat{U}_{k-2},\ x_j \in \hat{U}_k \text{ for all } j \neq i\}$$

and that $P_v(\hat{S}^i_{\boldsymbol{v}(k)}) \subset \hat{\boldsymbol{\mathcal{S}}}^i_{\boldsymbol{v}(k)}$ to see that (3.3) is satisfied. This ends the proof of the lemma. □

Lemma 3.5. *Let F be a C-bi-Lipschitz map defined on $Q(0, 1)$ that maps $\mathcal{K}_B$ onto $\mathcal{K}_B$ and C_B onto C_B. Then there exists a constant $\tilde{C} > 0$ such that for every $x \in Q(0, 1)$ we have*

$$\tilde{C}^{-1} \operatorname{dist}(x, \mathcal{K}_B) < \operatorname{dist}(F(x), \mathcal{K}_B) < \tilde{C} \operatorname{dist}(x, \mathcal{K}_B) \tag{3.34}$$

and

$$\tilde{C}^{-1}\operatorname{dist}(x, C_B) < \operatorname{dist}(F(x), C_B) < \tilde{C}\operatorname{dist}(x, C_B). \tag{3.35}$$

Proof. We prove the first inequality in (3.34) by contradiction. Assume that we have a sequence $\{x_k\}_{k=1}^{\infty}$ of points with the following property,

$$\operatorname{dist}(F(x_k), \mathcal{K}_B) < \tfrac{1}{k}\operatorname{dist}(x_k, \mathcal{K}_B).$$

Then applying F^{-1} to the points $F(x_k)$ and using the fact that F^{-1} is Lipschitz map which maps $\mathcal{K}_B$ onto $\mathcal{K}_B$ we get that

$$\operatorname{dist}(x_k, \mathcal{K}_B) = \operatorname{dist}\big(F^{-1}(F(x_k)), F^{-1}(\mathcal{K}_B)\big) \le C\operatorname{dist}(F(x_k), \mathcal{K}_B) < C\frac{1}{k}\operatorname{dist}(x_k, \mathcal{K}_B)$$

for all k, which is a contradiction. Therefore we see that there exists some constant $\tilde{C}_1 > 0$ such that

$$\tilde{C}_1^{-1}\operatorname{dist}(x, \mathcal{K}_B) \le \operatorname{dist}(F(x), \mathcal{K}_B) \quad \text{for all } x.$$

The second inequality in (3.34) is implied by the first and the fact that F^{-1} is a bi-Lipschitz mapping, which maps $\mathcal{K}_B$ onto $\mathcal{K}_B$. Thus we my find also a constant $\tilde{C}_2 > 0$ such that

$$\operatorname{dist}(F(x), \mathcal{K}_B) < \tilde{C}_2\operatorname{dist}(x, \mathcal{K}_B) \quad \text{for all } x.$$

The proof of the two inequalities in (3.35) goes similarly and thus we may find constants $\tilde{C}_3, \tilde{C}_4 > 0$ such that

$$\tilde{C}_3^{-1}\operatorname{dist}(x, C_B) < \operatorname{dist}(F(x), C_B) < \tilde{C}_4\operatorname{dist}(x, C_B) \quad \text{for all } x.$$

The claim follows now by taking $\tilde{C} = \max\{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\}$. □

Proof of Theorem 3.1. First we need to find a suitable Cantor set C_B. For this we need to assume that $\beta \ge \max\{6, \beta_1, \beta_2, \beta_3\}$ in the definition of C_B in subsection 2.6 where $\beta_1, \beta_2, \beta_3$ are described in the proof of Lemma 3.4. Taking this Cantor set C_B we may apply Lemma 3.4. From Lemma 3.4 we get a vector v, such that P_v is one-to-one on the set $\mathcal{K}_B$ and further the function $g(P_v(x)) = -x_4$ on $P_v(\mathcal{K}_B)$ has a Lipschitz extension on $X = \mathbb{R}^3 \times \{0\}$, which we have defined in the end of the the proof of Lemma 3.4. Define the vector $u = (-v_1, -v_2, -v_3, v_4)$ and recall that we have defined $F_{g,v} : \mathbb{R}^4 \to \mathbb{R}^4$ as

$$F_{g,v}(x) = x + vg(P_v(x)). \tag{3.36}$$

Then consider the image of a point $x \in \mathcal{K}_B$ for the map $F := F_{g,u} \circ F_{g,v}$. First we observe that

$$F_{g,v}(x) = x + vg(P_v(x)) = x - \frac{x_4}{v_4}v = P_v(x) \quad \text{for every } x \in \mathcal{K}_B.$$

Furthermore, it is easy to see that the projections P_v and P_u are identities when restricted to $X = P_v(\mathbb{R}^4) = P_u(\mathbb{R}^4)$, which gives us

$$P_u(F_{g,v}(x)) = P_u(P_v(x)) = P_v(x) \quad \text{for all } x \in \mathcal{K}_B.$$

Therefore, for each $x \in \mathcal{K}_B$ we can calculate

$$\begin{aligned} F_{g,u} \circ F_{g,v}(x) &= x + vg(P_v(x)) + ug(P_u(F_{g,v}(x))) \\ &= x + vg(P_v(x)) + ug(P_v(x)) \\ &= x - v\frac{x_4}{v_4} - u\frac{x_4}{v_4} \\ &= (x_1, x_2, x_3, -x_4). \end{aligned} \tag{3.37}$$

This means that $F_{g,u} \circ F_{g,v}$ is exactly the reflection in the last coordinate on $\mathcal{K}_B$ as in (3.1).

If we redefine g so that it is constant on a small ball B in X then we can find a point $x \in \mathbb{R}^4$, which is mapped to the center of B by $F_{g,v}$. The projection P_v is continuous and so $V = P_v^{-1}(\frac{1}{2}B)$ is open. Now we call

$$U := \{y \in V; P_u(y) \in \tfrac{1}{2}B\}$$

which is also an open neighbourhood of $P_v(x) = F_{g,v}(x)$. Then $W = F_{-g,v}(U)$ is an open neighbourhood of x mapped by $F_{g,v}$ onto U. Then $P_u(F_{g,v}(w)) \in B$ for all $w \in W$. Let us denote by λ the constant value of g on B, then we have

$$g(P_v(w)) = g(P_u(F_{g,v}(w))) = \lambda.$$

Hereby we see using (3.36) that

$$F_{g,u} \circ F_{g,v}(w) = w + vg(P_v(w)) + ug(P_u(F_{g,v}(w))) = w + \lambda v + \lambda u = w + 2\lambda e_4$$

for all $w \in W$ which is an open set containing x. Our mapping $f = F_{g,u} \circ F_{g,v}$ is a translation on V and the translation is obviously sense preserving. Now $F_{g,u} \circ F_{g,v}$ is a bi-Lipschitz map that can equal a translation everywhere on a ball and therefore must be sense-preserving. This ends the first part of the proof.

Next, it follows from Lemma 3.4 that if $N_F \in \mathbb{N}$ is arbitrary and $N_F < j \leq k$ then F maps each line segment I_i parallel to e_i which lies in

$$A_{i,j-N_F-1,k+N_F} = \{x \in \mathbb{R}^4 : x_i \in [-1,1] \setminus \hat{U}_{j-N_F-1},\ x_l \in \hat{U}_{k+N_F} \text{ for } l \neq i\}$$

to a line segments parallel to e_i as the derivative along the segment satisfies

$$D_iF(x) = \begin{cases} e_i & \text{if } i = 1,2,3 \\ -e_i & \text{if } i = 4 \end{cases}$$

for every $x \in A_{i,j-N_F-1,k+N_F}$. Therefore it suffices to show that there exists $N_F \in \mathbb{N}$ such that the image $F(I_i)$ of such a line segment I_i lies always in the set $A_{i,j-1,k}$.

Let us start by recalling from (2.7) that $\hat{U}_k = \bigcup_i \hat{I}_{i,k}$, where by choosing the center points of the intervals to be $\hat{z}_{i,k}$ we can write $\hat{I}_{i,k} = [\hat{z}_{i,k} - \hat{r}_k, \hat{z}_{i,k} + \hat{r}_k]$. Thus

$$\hat{U}_k \subset \bigcup_i [\hat{z}_{i,k} - 2\hat{r}_k, \hat{z}_{i,k} + 2\hat{r}_k].$$

This immediately gives that

$$\{y \in \mathbb{R} : \operatorname{dist}(y, \mathcal{C}_B) < \hat{r}_{k+1}\} \subset \hat{U}_k \subset \{y \in \mathbb{R} : \operatorname{dist}(y, \mathcal{C}_B) < 2\hat{r}_k\}, \tag{3.38}$$

and it follows that

$$A_{i,j-1,k} \supset \{x \in \mathbb{R}^4 : \operatorname{dist}(x_i, \mathcal{C}_B) > 2\hat{r}_{j-1},\ \operatorname{dist}(x_l, \mathcal{C}_B) < \hat{r}_{k+1} \text{ for } l \neq i\} =: \hat{A}_{i,j-1,k}.$$

Therefore, it is enough to show that there is $N_F \in \mathbb{N}$ such that $F(x) \in \hat{A}_{i,j-1,k}$ for every $x \in A_{i,j-N_F-1,k+N_F}$ whenever $N_F < j \le k$.

Suppose that $N_F \in \mathbb{N}$ and assume that $x \in A_{i,j-N_F-1,k+N_F}$ where $N_F < j \le k$. Then it follows from (3.38) that

$$\text{(3.39)} \qquad \operatorname{dist}(x, C_B + \mathbb{R}e_i) < c_1 \hat{r}_{k+N_F+1} \quad \text{and} \quad \operatorname{dist}(x, C_B) \ge c_2^{-1} \hat{r}_{j-N_F-1},$$

where the constants $c_1 > 0$ and $c_2 > 0$ depend only on the dimension $n = 4$ and on β. If we apply the bi-Lipschitz property of F to (3.39), and the fact that $F(C_B + \mathbb{R}e_i) = C_B + \mathbb{R}e_i$ and $F(C_B) = C_B$ we get

$$\operatorname{dist}(F(x), C_B + \mathbb{R}e_i) \le C \operatorname{dist}(x, C_B + \mathbb{R}e_i) < C c_1 \hat{r}_{k+N_F+1}$$

and with the help of Lemma 3.5

$$\operatorname{dist}(F(x), C_B) \ge C^{-1} \operatorname{dist}(x, C_B) \ge (C c_2)^{-1} \hat{r}_{j-N_F-1},$$

where $C \ge 1$ stands for the bi-Lipschitz constant of F. Thus, if we choose $N_F \in \mathbb{N}$ such that $\hat{r}_{N_F} < \min\{(Cc_1)^{-1}, \frac{1}{3}(Cc_2)^{-1}\}$ and use the fact that $\hat{r}_k = 2^{-k-\beta k} = \hat{r}_1^k$ we have that

$$\text{(3.40)} \qquad \operatorname{dist}(F(x), C_B + \mathbb{R}e_i) < \hat{r}_{k+1} \quad \text{and} \quad \operatorname{dist}(F(x), C_B) > 3\hat{r}_{j-1}.$$

On the other hand, if we apply the triangle inequality to the point $y = F(x)$ we get

$$3\hat{r}_{j-1} < \operatorname{dist}(y, C_B) \le \operatorname{dist}(y_i, \mathcal{C}_B) + \operatorname{dist}(y, C_B + \mathbb{R}e_i) < \operatorname{dist}(y_i, \mathcal{C}_B) + \hat{r}_{k+1},$$

where y_i is the i-th coordinate of y. Thus, it follows that $\operatorname{dist}(y_i, \mathcal{C}_B) > 2\hat{r}_{j-1}$. Furthermore, as it follows from (3.40) that

$$\operatorname{dist}(y_l, \mathcal{C}_B) \le \operatorname{dist}(F(x), C_B + \mathbb{R}e_i) < \hat{r}_{k+1} \quad \text{for each } l \ne i,$$

we get that $F(x) \in \hat{A}_{i,j-1,k}$ and the claim follows. □

4. The mapping S_t

The purpose of this section is to define a mapping which stretches C_B back onto C_A and has the properties listed in Lemma 4.1. We use the notation $\hat{U}_k$, $\hat{\mathcal{M}}_k$ and $\hat{\mathcal{A}}_k$ introduced in (2.7) and (2.8) and we recall that

$$C_B = \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B = \bigcap_{k=1}^{\infty} \hat{\mathcal{M}}_k\,.$$

Lemma 4.1. *There exists a sense-preserving homeomorphisms $S_t : (-1,1)^4 \to (-1,1)^4$ such that:*

(i) *S_t maps C_B onto C_A and $S_t = S_q^{-1}$ on C_B.*
(ii) *Mapping S_t is locally Lipschitz on $(-1,1)^4 \setminus C_B$.*
(iii) *If L_i is a line parallel to x_i-axis with $L_i \cap (\hat{U}_k)^4 \ne \emptyset$ then*

$$|D_i S_t(x)| \le C \frac{2^{\beta k}}{k^{\alpha+1}}$$

for every $x \in L_i \cap \big((\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4\big)$.

(iv) *If* $k \le j \le 3k+2$ *and* $x \in (\hat{U}_j \setminus \hat{U}_{j+1}) \times \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big)$ *then*

$$|DS_t(x)| \le C2^{\beta j}.$$

The same holds for $x \in \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big) \times (\hat{U}_j \setminus \hat{U}_{j+1})$ *and also for two other permutations of coordinates.*

(v) *If* $x \in \hat{U}_{3k+3} \times \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big)$ *then*

$$|DS_t(x)| \le C2^{\beta(3k+3)}.$$

The same holds for $x \in \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big) \times (\hat{U}_{3k+3})^3$ *and also for two other permutations of coordinates.*

Proof. In order to aid our construction, let us first recall and define some notation we will use. We recall that if $C_A = \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A$ and $C_B = \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B \times \mathcal{C}_B$ are the Cantor sets defined in subsection 2.5 and 2.6 then we may write

$$\mathcal{C}_A = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} I_{i,k} \quad \text{and} \quad \mathcal{C}_B = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} \hat{I}_{i,k}, \tag{4.1}$$

where the closed intervals $I_{i,k}$ and $\hat{I}_{i,k}$ have the lengths

$$\ell_k = \mathcal{L}^1(I_{i,k}) = 2^{-k}\Big(1 + \frac{1}{(k+1)^\alpha}\Big) \quad \text{and} \quad \hat{\ell}_k = \mathcal{L}^1(\hat{I}_{i,k}) = 2^{-k\beta-k+1}. \tag{4.2}$$

Moreover, we have $I_{i,k} \cap I_{j,k} = \emptyset$ for $i \neq j$, $I_{2i-1,k} \cup I_{2i,k} \subset I_{i,k-1}$ and $I_{i,k}$ lies more to the left than $I_{i+1,k}$ (similar properties holds also for the intervals $\hat{I}_{i,k}$).

Then there exists a natural function $t : \mathbb{R} \to \mathbb{R}$ which maps $\mathcal{C}_B$ onto $\mathcal{C}_A$. In fact t is a uniform limit of functions $t_k : \mathbb{R} \to \mathbb{R}$, $k = 0, 1, 2, \ldots$, such that

(1) $t_0(x) = x$,
(2) t_k maps each $\hat{I}_{i,k}$ onto $I_{i,k}$ linearly,
(3) t_k maps each of the three parts of $\hat{I}_{i,k-1} \setminus (\hat{I}_{2i-1,k} \cup \hat{I}_{2i,k})$ onto the corresponding parts of $I_{i,k-1} \setminus (I_{2i-1,k} \cup I_{2i,k})$ linearly, and
(4) $t_k = t_{k-1}$ outside $\bigcup_{i=1}^{2^k} \hat{I}_{i,k-1}$.

Note that then we have $t = q^{-1}$ where q is the function defined in subsection 2.7. It follows that $(t(x_1), t(x_2), t(x_3), t(x_4)) = S_q^{-1}(x)$.

The definition of the mapping S_t will make use of the standard frame-to-frame maps H^3_k, H^3 and H^4_k, H^4 described in Section 2.8. In a rough, intuitive sense we want a map that behaves very much like H^4_k on parts of the frame "far away" from $\mathcal{K}_B$, (i.e. in $\big((\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4\big) \setminus \hat{\mathcal{A}}_k$) and on hyperplanes in $\hat{\mathcal{A}}_k$ perpendicular to lines in $\mathcal{K}_B$ acts like the higher iterations of the frame-to-frame map H^3_{3k}. Our strategy is to define a map which equals (up to some isometric rotation) H^3_{3k} on each face of each cube in $(\hat{U}_k)^4 = \bigcup_{\boldsymbol{v}(k) \in \mathbb{V}^k} Q_{\boldsymbol{v}(k)}$. We extend this mapping into $(\hat{U}_k)^4 \setminus (\hat{U}_{k+1})^4$ simply as the frame to frame mapping H^4_k on $\big((\hat{U}_k)^4 \setminus (\hat{U}_{k+1})^4\big) \setminus \hat{\mathcal{A}}_k$. Inside the "tubes" of type $\big((\hat{U}_k)^4 \setminus (\hat{U}_{k+1})^4\big) \cap \hat{\mathcal{A}}_k$ we use a suitable convex combination of the maps defined on the faces to extend the map inside the frame.

We refer to the i-th canonical projection π_i as the linear map

$$\pi_i(x) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_4).$$

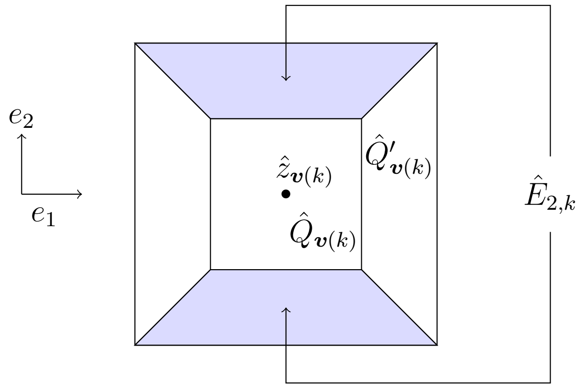

FIGURE 9. The part of the set $\hat{E}_{2,k}$ which lies inside the frame $\hat{Q}'_{\boldsymbol{v}(k)} \setminus \hat{Q}_{\boldsymbol{v}(k)}$ in two dimensions.

Take $i \in \{1,2,3,4\}$. We will also denote $\tilde{x}^i = (x_1, \dots x_{i-1}, x_{i+1}, \dots x_4) \in \mathbb{R}^3$. Using this notation we define the linear isomorphic isometry $L_i : \mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{4-i} \to \mathbb{R}^3$ defined as $L_i(\pi_i(x)) = \tilde{x}^i$. Furthermore, we define

$$H_k^{3,i}(x) = L_i^{-1} \circ H_k^3 \circ L_i \circ \pi_i(x),$$
$$H^{3,i}(x) = L_i^{-1} \circ H^3 \circ L_i \circ \pi_i(x).$$

For a point $x \in (\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4$ we will define the functions

$$d_{i,k}(x) = \frac{\min\{|x_i - (\hat{z}_{\boldsymbol{v}(k)})_i| : \boldsymbol{v}(k) \in \mathbb{V}^k\} - \hat{r}_k}{\frac{1}{2}\hat{r}_{k-1} - \hat{r}_k}.$$

The set $\hat{I}_{i,k-1} \setminus (\hat{I}_{2i-1,k} \cup \hat{I}_{2i,k})$ of intervals, whose union is $\hat{U}_{k-1} \setminus \hat{U}_k$, can be decomposed to four closed (maximal) intervals with disjoint interiors so that each function $d_{i,k}$ is linear in x_i on each of these four subintervals. Further, if $x \in (\hat{U}_{k-1})^4$ and $x_i \in \hat{U}_{k-1} \setminus \hat{U}_k$ then we have

$$d_{i,k}(x) = 4\frac{\operatorname{dist}(x_i, \hat{U}_k)}{\hat{\ell}_{k-1} - 2\hat{\ell}_k} \tag{4.3}$$

which takes values between 0 and 1. Using these functions we can divide the frame into the parts where we are furthest from its center in the direction e_i, which are the sets (see Fig. 9.)

$$\begin{aligned}\hat{E}_{i,k} &= \big\{x \in (\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4 : \min_{\boldsymbol{v}(k)\in\mathbb{V}^k} |x_i - (\hat{z}_{\boldsymbol{v}(k)})_i| \ge \min_{\boldsymbol{v}(k)\in\mathbb{V}^k} |x_j - (\hat{z}_{\boldsymbol{v}(k)})_j| \text{ for all } j \ne i\big\} \\ &= \{x \in (\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4 : d_{i,k}(x) \ge d_{j,k}(x) \text{ for all } j \ne i\}.\end{aligned}$$

For technical reasons it is also convenient to define the corresponding sets $E_{i,k}$ in the target, i.e.,

$$E_{i,k} = \big\{y \in (U_{k-1})^4 \setminus (U_k)^4 : \min_{\boldsymbol{v}(k)\in\mathbb{V}^k} |y_i - (z_{\boldsymbol{v}(k)})_i| \ge \min_{\boldsymbol{v}(k)\in\mathbb{V}^k} |y_j - (z_{\boldsymbol{v}(k)})_j| \text{ for all } j \ne i\big\}.$$

We will next use the convex combinations of the maps $H^{3,i}_{3k-3}$ and $H^{3,i}_{3k}$ in the sets $\hat{E}_{i,k}$ together with some correction mapping to define the mapping S_t.

We cut the set $\hat{E}_{i,k}$ into hyperplane slices with hyperplanes perpendicular to e_i, $\hat{E}_{i,k}\cap\{x_i=c\}$. On these planes we apply $L_i\circ\pi_i$, which shifts it onto a corresponding hyperplane $\{x_i=0\}$ and then rotates it onto $\mathbb{R}^3$. Then we can apply a convex combination of the 3-dimensional frame-to-frame maps

$$A_{i,k}(x):=d_{i,k}(x)H^3_{3k-3}(x)+(1-d_{i,k}(x))H^3_{3k}(x)$$

and then reverse the rotation using L_i^{-1}. Now we shift the hyperplane into the right position by adjusting the i-coordinate so that it corresponds to the i-th coordinate of the frame-to-frame map, i.e. $((H^4_k)(x))_i=t(x_i)$. In summary, we define

$$S_t(x)=\underbrace{d_{i,k}(x)H^{3,i}_{3k-3}(x)+(1-d_{i,k}(x))H^{3,i}_{3k}(x)+t(x_i)e_i}_{=A_{i,k}(x)+t(x_i)e_i}\text{ for }x\in\hat{E}_{i,k}. \tag{4.4}$$

We need to show that S_t defines a homeomorphism which satisfies all the conditions (i)-(v) in Lemma 4.1.

Step 1: Proving that S_t is a homeomorphism. First we show that (4.4) yields a homeomorphism. The first observation in this direction is that S_t maps $(\hat{U}_{k-1})^4\setminus(\hat{U}_k)^4$ onto $(U_k)^4\setminus(U_{k+1})^4$ for each $k\in\mathbb{N}$. To see this we observe that in the expression

$$S_t(x)=A_{i,k}(x)+t(x_i)e_i\text{ for every }x\in\hat{E}_{i,k}$$

the mapping $A_{i,k}$ maps each hyperplane in $\hat{E}_{i,k}$ perpendicular to e_i homeomorphically to a hyperplane in $E_{i,k}$ perpendicular to e_i. Moreover, at the end of subsection 2.8 we have shown that for each fixed $\alpha\in(0,1)$ the mapping

$$\alpha H^3_{3k-3}(x)+(1-\alpha)H^3_{3k}(x)$$

is a homeomorphism in $\mathbb{R}^3$ and thus $A_{i,k}(x)$ on the hyperplane is a homeomorphism. Furthermore, we have that $t(\hat{U}_k)=U_k$ for every k. Thus, it is quite easy to see that S_t actually maps each set $\hat{E}_{i,k}$ onto $E_{i,k}$ homeomorphically.

Next we will show that S_t defines a homeomorphism from $(\hat{U}_{k-1})^4\setminus(\hat{U}_k)^4$ onto $(U_{k-1})^4\setminus(U_k)^4$. For this it suffices to show that in the critical set

$$\hat{E}_{i,k}\cap\hat{E}_{j,k}=\bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k}\left\{x\in\hat{Q}'_{\boldsymbol{v}(k)}\setminus\hat{Q}_{\boldsymbol{v}(k)}:|x-\hat{z}_{\boldsymbol{v}(k)}|=|x_i-(\hat{z}_{\boldsymbol{v}(k)})_i|=|x_j-(\hat{z}_{\boldsymbol{v}(k)})_j|\right\}$$

the expressions in (4.4) coincide. This gives us that S_t is continuous in $(\hat{U}_{k-1})^4\setminus(\hat{U}_k)^4$. For this, let us assume that $x\in\hat{E}_{i,k}\cap\hat{E}_{j,k}$. Then one can show that

$$(H^{3,i}_{3k-3}(x))_j=(H^{3,i}_{3k}(x))_j=t(x_j)\quad\text{and}\quad(H^{3,i}_{3k-3}(x))_i=(H^{3,i}_{3k}(x))_i=t(x_i).$$

Thus, we get that

$$\begin{aligned}H^{3,i}_{3k-3}(x)&=H^{3,j}_{3k-3}(x)+(H^{3,i}_{3k-3}(x))_je_j-(H^{3,j}_{3k-3}(x))_ie_i\\&=H^{3,j}_{3k-3}(x)+t(x_j)e_j-t(x_i)e_i.\end{aligned}$$

and

$$\begin{aligned}H^{3,i}_{3k}(x)&=H^{3,j}_{3k}(x)+(H^{3,i}_{3k}(x))_je_j-(H^{3,j}_{3k}(x))_ie_i\\&=H^{3,j}_{3k}(x)+t(x_j)e_j-t(x_i)e_i.\end{aligned}$$

Thus, it follows that

$$\begin{aligned} A_{i,k}(x) + t(x_i)e_i &= d_{i,k}(x)H^{3,i}_{3k-3}(x) + (1 - d_{i,k}(x))H^{3,i}_{3k}(x) + t(x_i)e_i \\ &= d_{j,k}(x)H^{3,i}_{3k-3}(x) + (1 - d_{j,k}(x))H^{3,i}_{3k}(x) + t(x_i)e_i \\ &= d_{j,k}(x)H^{3,j}_{3k-3}(x) + (1 - d_{j,k}(x))H^{3,j}_{3k}(x) + t(x_j)e_j \\ &= A_{j,k}(x) + t(x_j)e_j, \end{aligned}$$

as we wanted.

We have now shown that S_t defines a homeomorphism on each set $(\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4$. Next, we will show that S_t defines a homeomorphism on $(-1,1)^4 \setminus C_B$. Because S_t is a homeomorphisms on each set $(\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4$ and these sets are pairwise disjoint it suffices to show that in the critical set

$$C_k := \overline{(\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4} \cap \overline{(\hat{U}_k)^4 \setminus (\hat{U}_{k+1})^4}$$

the expressions in (4.4) coincide. For this, it suffices to prove that for every $i = 1, 2, 3, 4$ these expressions coincide along the lines

$$l^i_{\boldsymbol{v}(k)} = \hat{z}_{\boldsymbol{v}(k)} + \hat{x}_i + se_i, \quad s \in \mathbb{R},$$

where $\hat{x}_i = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_4)$ with $|x_j| < \hat{r}_k$ for every $j \neq i$. However, this is clear as

$$\begin{aligned} \lim_{s\to\hat{r}_k^+} S_t(\hat{z}_{\boldsymbol{v}(k)} + \hat{x}_i + se_i) &= \lim_{s\to\hat{r}_k^+} \Big(\overbrace{\big(d_{i,k}H^{3,i}_{3k-3} + (1 - d_{i,k})H^{3,i}_{3k}\big)(\hat{z}_{\boldsymbol{v}(k)} + \hat{x}_i + se_i)}^{=A_{i,k}(\hat{z}_{\boldsymbol{v}(k)}+\hat{x}_i+se_i)} + t(s)e_i \Big) \\ &= H^{3,i}_{3k}\big(\hat{z}_{\boldsymbol{v}(k)} + \hat{x}_i + \hat{r}_k e_i\big) + t(\hat{r}_k)e_i \\ &= \lim_{s\to\hat{r}_k^-} \Big(\big(d_{i,k+1}H^{3,i}_{3k} + (1 - d_{i,k+1})H^{3,i}_{3k+1}\big)(\hat{z}_{\boldsymbol{v}(k)} + \hat{x}_i + se_i) + t(s)e_i \Big) \\ &= \lim_{s\to\hat{r}_k^-} S_t(\hat{z}_{\boldsymbol{v}(k)} + \hat{x}_i + se_i), \end{aligned}$$

thus we have shown that S_t defines a homeomorphism on $(-1,1)^4 \setminus C_B$.

Finally, since S_t is a homeomorphism on all frames that sends frames to frames, S_t is extended homeomorphically as $S_t(x) = S_q^{-1}(x) = (t(x_1), t(x_2), t(x_3), t(x_4))$ to C_B. Especially, S_t will then take C_B onto C_A, and thus (i) follows. It is also easy to see that for a fixed k the mapping H^3_{3k} is Lipschitz and hence S_t defined by (4.4) is a locally Lipschitz mappings on on $(-1,1)^4 \setminus C_B$ which implies (ii).

Step 2: Calculating the derivatives of S_t. We now calculate the derivative of the mapping S_t on $\hat{U}^4_{k-1} \setminus \hat{U}^4_k$. More precisely, we want to verify the conditions (iii)-(v). In the following calculations we will rely on (2.13) to calculate the derivative.

Step 2A: Proving the condition (iii). Suppose that L_i is a line parallel to x_i-axis with $L_i \cap (\hat{U}_k)^4 \neq \emptyset$. Then it follows that $L_i \cap \big((\hat{U}_{k-1})^4 \setminus (\hat{U}_k)^4\big) \subset \hat{E}_{i,k}$.

(1) Let us first assume that $x \in L_i \cap \hat{E}_{i,k}$ with $x_i \in \hat{U}_{k-1} \setminus \hat{U}_k$ and $\tilde{x}^i \in (\hat{U}_{k-1})^3 \setminus (\hat{U}_{3k-3})^3$. In this case $A_{i,k}$ is a constant function in x_i-direction and therefore

$$|D_i S_t(x)| = |t'(x_i)| \leq \frac{a_{k-1} - a_k}{b_{k-1} - b_k} \leq C\frac{2^{k\beta}}{k^{\alpha+1}}.$$

(2) Let us next assume that $x \in L_i \cap \hat{E}_{i,k}$ with $x_i \in \hat{U}_{k-1} \setminus \hat{U}_k$ and $\tilde{x}^i \in (\hat{U}_{3k-3})^3$. We recall that the maps $H^{3,i}_{3k-3}$ and $H^{3,i}_{3k}$ are independent on x_i which implies

$$D_i H^{3,i}_{3k-3}(x) = 0 \text{ and } D_i H^{3,i}_{3k}(x) = 0$$

and by the construction of mappings H_{3k} we easily obtain

$$\|H^{3,i}_{3k-3} - H^{3,i}_{3k}\| \leq 2^{-3k+4}.$$

On the other hand by applying (4.3) we may conclude that

$$|D_i d_{i,k}(x)| \leq \frac{4}{\hat{\ell}_{k-1} - 2\hat{\ell}_k} \leq \frac{C}{\frac{1}{2}\hat{r}_{k-1} - \hat{r}_k}. \tag{4.5}$$

By combining these facts we get

$$\begin{aligned} |D_i S_t(x)| &\leq |D_i A_{i,k}(x)| + |t'(x_i)| \\ &\leq |D_i d_{i,k}(x)| |H^{3,i}_{3k-3}(x) - H^{3,i}_{3k}(x)| + |t'(x_i)| \\ &\leq C \frac{\|H^{3,i}_{3k-3} - H^{3,i}_{3k}\|}{\frac{1}{2}\hat{r}_{k-1} - \hat{r}_k} + \frac{a_{k-1} - a_k}{b_{k-1} - b_k} \\ &\leq C \frac{2^{\beta k}}{2^{3k-4}} + C \frac{2^{k\beta}}{k^{\alpha+1}} \leq C \frac{2^{k\beta}}{k^{\alpha+1}}, \end{aligned}$$

as we wanted. Now (1) and (2) together will give us (iii).

Step 2B: Proving the condition (iv). Let us next assume that $x \in (\hat{U}_j \setminus \hat{U}_{j+1}) \times \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big)$ with $k \leq j \leq 3k+2$. The case $j = k$ is easy to deal with and therefore we may assume that $j > k$. In this case we have that $x \in \hat{E}_{i,k}$ for some $i \neq 1$. With the help of (4.5) we easily obtain

$$|Dd_{i,k}(x)| \leq C \max\{2^{\beta k}, 2^{\beta j}\} \text{ and } |d_{i,k}(x)| \leq 1 \quad \text{for every } x.$$

Moreover, we also have $\max\{H^{3,i}_{3k-2}(x), H^{3,i}_{3k}(x)\} \leq 1$ for every x. As $x \notin (\hat{U}_{j+1})^4$ we easily obtain $H^{3,i}_{3k}(x) = H^{3,i}_{j+1}(x)$ and thus using (2.12) that

$$|DH^{3,i}_{3k}(x)| \leq C 2^{\beta j}.$$

Thus, it follows from (4.4) and (2.12) that for every $l \neq i$

$$\begin{aligned} |D_l S_t(x)| &\leq |Dd_{i,k}(x)| \big(|H^{3,i}_{3k-2}(x)| + |H^{3,i}_{3k}(x)|\big) + |d_{i,k}(x)| \big(|D_l H^{3,i}_{3k-2}(x)| + |D_l H^{3,i}_{3k}(x)|\big) \\ &\leq C \max\{2^{\beta k}, 2^{\beta j}\} + |D_l H^{3,i}_{3k-2}(x)| + |D_l H^{3,i}_{3k}(x)| \\ &\leq C \max\{2^{\beta k}, 2^{\beta j}\} + C 2^{\beta j} \leq C 2^{\beta j}. \end{aligned}$$

On the other hand, it follows from the step 2A that $|D_i S_t(x)| \leq C \frac{2^{\beta k}}{k^{\alpha+1}}$. Thus, because $j > k$ we may estimate

$$|DS_t(x)| \leq C 2^{\beta j}.$$

There is no difference in the proof for $x \in \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big) \times (\hat{U}_j \setminus \hat{U}_{j+1})$ and also for two other permutations of coordinates, and thus this ends the proof of (iv).

Step 2C: Proving the condition (v). Finally, assume that $x \in \hat{U}_{3k+3} \times \big((\hat{U}_k)^3 \setminus (\hat{U}_{k+1})^3\big)$. Again, we have that $x \in \hat{E}_{i,k}$ for some $i \neq 1$. By applying (4.4) and (2.14) we have

$$|D_l S_t(x)| \leq C \max\{2^{\beta(3k+3)}, 2^{\beta k}\} \leq C 2^{\beta(3k+3)} \quad \text{for every } l \neq i.$$

On the other hand, it follows from the step 2A that $|D_i S_t(x)| \leq C \frac{2^{\beta k}}{k^{\alpha+1}}$, and thus we conclude

$$|DS_t(x)| \leq C 2^{\beta(3k+3)}.$$

There is no difference in the proof for the other permutations of coordinates, and thus this ends the proof of (v).

□

5. Proof of Theorem 1.2 for $n = 4$

We will now define the mapping $f : (-1,1)^4 \to (-1,1)^4$ by

$$f = S_t \circ F \circ S_q\,.$$

Let us first remark that as a composition of three sense-preserving homeomorphisms S_q, F and S_t the mapping f is obviously a sense-preserving homeomorphism.

5.1. **The sign of the Jacobian:** We need to show that $J_f > 0$ on a set of positive measure and $J_f < 0$ on a set of positive measure. We know that S_q and F are Lipschitz maps, and by Lemma 4.1 (ii) that S_t is locally Lipschitz outside the set $C_B = (F \circ S_q)(C_A)$ and hence f is locally Lipschitz outside of C_A. Therefore f is a sense-preserving homeomorphism which is locally Lipschitz there and hence $J_f \geq 0$ outside of C_A (see e.g. [24]). We may also require that J_f is not identically zero on $\{x : J_f \geq 0\}$ because otherwise by [22] f would not satisfy Lusin's condition (N) on this set which cannot happen for a locally Lipschitz map, see e.g. [23, Theorem 4.2]. Hence $\mathcal{L}^4(\{x : J_f > 0\}) > 0$.

Now we show that $J_f(x) < 0$ for almost every $x \in C_A$. For this let us fix $x \in C_A$. If q and t are the functions in the definitions of homeomorphisms S_q and S_t we may observe that for every $x \in C_A$ we have

$$\begin{aligned} f(x) &= (S_t \circ F \circ S_q)(x) = (S_t \circ F)(q(x_1), q(x_2), q(x_3), q(x_4)) \\ &= S_t\big(q(x_1), q(x_2), q(x_3), -q(x_4)\big) = S_t\big(q(x_1), q(x_2), q(x_3), q(-x_4)\big) \\ &= \big(t(q(x_1)), t(q(x_2)), t(q(x_3)), t(q(-x_4))\big) = (x_1, x_2, x_3, -x_4)\ . \end{aligned} \tag{5.1}$$

Here we have used the following facts in the given order:

(i) $S_q(x) \in C_B$ for every $x \in C_A$,
(ii) $F(z) = (z_1, z_2, z_3, -z_4)$ for every $z \in C_B$,
(iii) the function q is odd, i.e. $q(-s) = -q(s)$ for every $s \in (-1,1)$,
(iv) if $x_4 \in \mathcal{C}_B$ then also $-x_4 \in \mathcal{C}_B$,
(v) $S_t(x) = (t(x_1, t(x_2), t(x_3), t(x_4))$ on C_B by Lemma 4.1 (i), and
(vi) $t = q^{-1}$.

It follows that at the points of density of C_A we know that the approximative derivate equals to the reflection in the last coordinate and hence the determinant of this matrix is -1. Once we show that f is Sobolev mapping we will know that its distributional derivative equals to approximative derivative a.e. and hence $J_f(x) = -1$ a.e. on C_A.

5.2. **ACL condition:** To verify the ACL-condition for f let us suppose that L is a line segment parallel to x_i-axis and consider the following cases:

Case 1: Suppose first that $L \cap C_A = \emptyset$. We know that both mappings S_q and F are Lipschitz maps, and by Lemma 4.1 (ii) that S_t is locally Lipschitz outside the set $C_B = (F \circ S_q)(C_A)$. Thus, the mapping $f = S_t \circ F \circ S_q$ is locally Lipschitz outside the set $\mathcal{K}_A$. It follows that f is Lipschitz and hence also absolutely continuous on the segment L.

Case 2: Suppose next that $L \cap C_A \neq \emptyset$, which means that $L \subset \mathcal{K}_A$. The line L decomposes into the part of L in C_A, and segments, which are mapped by f onto segments. On the parts of lines L intersecting C_A we use (5.1) to see that f is in fact 1-Lipschitz continuous on $L \cap C_A$. Now it remains to consider $L \setminus C_A$.

We fix $k \in \mathbb{N}$ and use the fact that

$$L \cap \big((U_k)^4 \setminus (U_{k+1})^4\big) = \bigcup_{\boldsymbol{v}(k) \in \mathbb{V}^k} L \cap (Q'_{\boldsymbol{v}(k)} \setminus Q_{\boldsymbol{v}(k)}).$$

Further, $L \cap (Q'_{\boldsymbol{v}(k)} \setminus Q_{\boldsymbol{v}(k)})$ is either empty or made up of two segments $L^1_{\boldsymbol{v}(k)}, L^2_{\boldsymbol{v}(k)}$ (recall that we assume now $L \cap C_A \neq \emptyset$). Each of these segments has length $\frac{1}{2}r_{k-1} - r_k$, which is squeezed by S_q into a segment parallel to x_i of length $\frac{1}{2}\hat{r}_{k-1} - \hat{r}_k$. We then apply the mapping F, which merely reflects the segment in the last variable (see (3.1)). Finally we apply the mapping S_t which maps each of the segments $F(S_q(L^1_{\boldsymbol{v}(k)}))$ and $F(S_q(L^2_{\boldsymbol{v}(k)}))$ onto a segment. Since $D_i f$ is constant on $L^1_{\boldsymbol{v}(k)}$ and $L^2_{\boldsymbol{v}(k)}$, we have that f maps each segment to a segment at constant speed. Therefore the restriction of f to each segment is Lipschitz. Then we can estimate the length of the image of the segment using Lemma 4.1 (iii) as follows

$$\begin{aligned}\mathcal{H}^1(f(L^1_{\boldsymbol{v}(k)})) = \mathcal{H}^1(f(L^2_{\boldsymbol{v}(k)})) &= |D_i S_t(x)|\big(\tfrac{1}{2}\hat{r}_{k-1} - \hat{r}_k\big)\\ &\leq C\frac{2^{\beta k}}{k^{\alpha+1}}(2^{-k}2^{-\beta k}) \leq C\big(\tfrac{1}{2}r_{k-1} - r_k\big).\end{aligned}$$

The length of each segment has increased by no more than a factor of C. Thus we see that the restriction of f to $L \setminus C_A$ is Lipschitz continuous and hence it is Lipschitz on the whole L and therefore absolutely continuous on L.

5.3. **Sobolev regularity of the mapping:** We would like to estimate

$$\int_{(-1,1)^4} |Df(x)|^p \,\mathrm{d}x \leq C\sum_{i=1}^4 \int_{(-1,1)^4} |D_i f(x)|^p \,\mathrm{d}x,$$

where $D_i f$ denotes the derivative with respect to x_i coordinate. Without loss of generality it is enough to estimate

$$\int_{(-1,1)^4} |D_1 f(x)|^p \,\mathrm{d}x = \int_{(-1,1)^3}\int_{-1}^1 |D_1 f(x_1, \tilde{x})|^p \,\mathrm{d}x_1 \,\mathrm{d}\tilde{x}, \tag{5.2}$$

where $\tilde{x} = (x_2, x_3, x_4)$ (derivatives in the other directions can be estimated analogously). For this, let us recall that the Cantor type sets C_A and C_B were constructed as the intersections of the sets

$$\bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k} Q_{\boldsymbol{v}(k)} = U_k \times U_k \times U_k \times U_k \quad \text{and} \quad \bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k} \hat{Q}_{\boldsymbol{v}(k)} = \hat{U}_k \times \hat{U}_k \times \hat{U}_k \times \hat{U}_k,$$

where $U_k = \bigcup_{i=1}^{2^k} I_{i,k}$ and $\hat{U}_k = \bigcup_{i=1}^{2^k} \hat{I}_{i,k}$. Moreover, recall also that

$$P_k := U_k \times U_k \times U_k \quad \text{and} \quad \hat{P}_k := \hat{U}_k \times \hat{U}_k \times \hat{U}_k.$$

Then the sets P_k and $\hat{P}_k$ are formed by 2^{3k} cubes.

Let us consider several possibilities. If $\tilde{x} \in \mathcal{C}_A \times \mathcal{C}_A \times \mathcal{C}_A$ then it is easy to see that f restricted to the line $[-1,1] \times \{\tilde{x}\}$ is in fact Lipschitz as it was explained at the end of subsection 5.2. It thus remains to estimate the integral (5.2) for $\tilde{x}$ in the sets $P_k \setminus P_{k+1}$. Because the mapping f is locally Lipschitz on $[-1,1]^4 \setminus C_A$ it suffices to analyze the mapping only near the set C_A, i.e. on the set $U_{k_0}^4$. For this fix now the exponent $p \in [1,2)$ and put

$$\alpha = \frac{2p}{2-p},$$

and β large enough for Theorem 3.1. Then we may find an index $k_0 \geq 4N_F + 5$, where $N_F \in \mathbb{N}$ is from Theorem 3.1, large enough so that

$$\max\{2^{-kp\beta/2} k^{(p-1)(\alpha+1)}, 2^{-p\beta}(\tfrac{k+1}{k})^{\alpha}\} < 1 \quad \text{for all } k \geq k_0. \tag{5.3}$$

Let us then fix $k \geq k_0$ and suppose that $\tilde{x} \in P_k \setminus P_{k+1}$. We will define the following divisions of the segment $L(\tilde{x}) = L := [-1,1] \times \{\tilde{x}\}$ according to $x_1 \in [-1,1]$ into the following sets

$$L_j = \{(x_1, \tilde{x}) : x_1 \in U_j \setminus U_{j+1}\} \text{ and } L_0 = \{(x_1, \tilde{x}) : x_1 \in \mathcal{C}_A\}.$$

The aim of the following calculations is to prove the estimate (5.16) below.

Case 1: Consider first those parts of the line segment L which are far away from the set C_A. More precisely, suppose that $k \geq k_0$ and

$$x \in L_j \quad \text{with } j = 1, \dots, k - 2N_F - 3.$$

First we observe that S_q maps the line segment L_j which is parallel to x_1-axis to a line segment L_j^1 which is also parallel to x_1-axis and lies inside the set $(\hat{U}_j \setminus \hat{U}_{j+1}) \times (\hat{P}_k \setminus \hat{P}_{k+1})$. Furthermore, we may estimate the derivative of S_q in the x_1-direction as

$$|D_1 S_q(x)| \leq C 2^{-\beta j} j^{\alpha+1}, \tag{5.4}$$

(see subsection 2.7).

Next we observe that

$$\begin{aligned} L_j^1 \subset ([-1,1] \setminus \hat{U}_{(j+N_F+2)-N_F-1}) \times \big((\hat{U}_{(k-N_F)+N_F})^3 \setminus (\hat{U}_{(k-N_F+1)+N_F})^3\big) \\ \subset A_{1,(j+N_F+2)-N_F-1,(k-N_F+1)+N_F}, \end{aligned}$$

where $N_F < j + N_F + 2 \leq k - N_F + 1$, and thus it follows from Theorem 3.1 that the bi-Lipschitz map F maps L_j^1 to a line segment L_j^2 parallel to x_1-axis such that

$$L_j^2 \subset A_{1,(j+N_F+2)-1,k-N_F}.$$

Moreover, because F is a bi-Lipschitz map, we have

$$|D_1 F(S_q(x))| \leq \operatorname{Lip}(F) \quad \text{for a.e. } x \in L_j, \tag{5.5}$$

where $\mathrm{Lip}(F)$ stands for the Lipschitz constant of the mapping F.

Finally, because $F(S_q(L_j)) = L_j^2$ is a line segment parallel to x_1-axis which is contained in set $[-1,1]^4 \setminus (\hat{U}_{j+N_F+1})^4$ it follows from Lemma 4.1 (iii) that

$$|D_1 S_t(F(S_q(x)))| \leq C \max_{1\leq l\leq j+N_F+1} 2^{\beta l} l^{-(\alpha+1)} \leq C 2^{\beta j} j^{-(\alpha+1)} \tag{5.6}$$

with C independent of j, k.

If we now put together the estimates (5.4), (5.5) and (5.6) the chain rule gives us

$$|D_1 f(x)| \leq C \quad \text{for a.e. } x \in L_j, \tag{5.7}$$

where C is an absolute constant.

Case 2: Let us next assume that

$$x \in L_j \quad \text{with } k - 2N_F - 3 < j \leq 3k - 3N_F - 3.$$

Again S_q maps the line segment L_j which is parallel to the x_1-axis to a line segment L_j^1 which is also parallel to x_1-axis and lies inside the set $(\hat{U}_j \setminus \hat{U}_{j+1}) \times (\hat{P}_k \setminus \hat{P}_{k+1})$. Furthermore, we have

$$|D_1 S_q(x)| \leq C 2^{-\beta j} j^{\alpha+1}, \tag{5.8}$$

with C independent of j, k.

Next, we recall again that

$$|D_1 F(S_q(x))| \leq \mathrm{Lip}(F) \quad \text{for a.e. } x \in L_j. \tag{5.9}$$

Moreover, it follows from the assumption $j \geq k - 2N_F - 2$ that

$$\begin{aligned}
L_j^1 &\subset \{x \in [-1,1]^4 : \mathrm{dist}(x, \mathcal{K}_B) > \min\{\hat{r}_j - \hat{r}_{j+1}, \hat{r}_k - \hat{r}_{k+1}\}\} \\
&\subset \big\{x \in [-1,1]^4 : \mathrm{dist}(x, \mathcal{K}_B) > \min\{2^{-\beta(j+1)-(j+1)}, 2^{-\beta(k+1)-(k+1)}\}\big\} \\
&\subset \big\{x \in [-1,1]^4 : \mathrm{dist}(x, \mathcal{K}_B) > \min\{2^{-\beta(j+1)-(j+1)}, 2^{-\beta(j+2N_F+3)-(j+2N_F+3)}\}\big\} \\
&\subset \big\{x \in [-1,1]^4 : \mathrm{dist}(x, \mathcal{K}_B) > 2^{-\beta(j+2N_F+3)-(j+2N_F+3)}\big\}.
\end{aligned}$$

Suppose now that $\tilde{C} > 0$ is the constant given by Lemma 3.5.

We may assume that $N_F \in \mathbb{N}$ is so large that $\tilde{C}^{-1} > 2^{3\beta+1} 2^{-\beta N_F - N_F}$. We may assume this because if Theorem 3.1 holds for a certain N_F, then it immediately holds for any $\tilde{N}_F \geq N_F$.

Then it follows from Lemma 3.5 and from the inclusion above that

$$\begin{aligned}
F(L_j^1) &\Subset \big\{x \in [-1,1]^4 : \mathrm{dist}(x, \mathcal{K}_B) > \tilde{C}^{-1} 2^{-\beta(j+2N_F+3)-(j+2N_F+3)}\big\} \\
&\subset \big\{x \in [-1,1]^4 : \mathrm{dist}(x, \mathcal{K}_B) > 2^{-\beta(j+3N_F)-(j+3N_F)}\big\}.
\end{aligned}$$

Thus, we have that $F(L_j^1)$ is contained in the following union of four sets

$$\begin{aligned}
F(L_j^1) \subset \Bigg(([-1,1] \setminus \hat{U}_{j+3N_F}) \times \big([-1,1]^3 \setminus (\hat{U}_{j+3N_F})^3\big) \Bigg) \cup \\
\cdots \cup \Bigg(\big([-1,1]^3 \setminus (\hat{U}_{j+3N_F})^3\big) \times ([-1,1] \setminus \hat{U}_{j+3N_F}) \Bigg).
\end{aligned}$$

Without loss of generality suppose that

$$F(L_j^1) \subset ([-1,1] \setminus \hat{U}_{j+3N_F}) \times \big([-1,1]^3 \setminus (\hat{U}_{j+3N_F})^3\big).$$

Then by Lemma 4.1 (iv) it follows that

$$|DS_t(F(S_q(x)))| \le C2^{\beta j} \quad \text{for every } x \in L_j. \tag{5.10}$$

If we now combine the estimates (5.8), (5.9) and (5.10) the chain rule gives us

$$|D_1 f(x)| \le C j^{\alpha+1} \quad \text{for a.e. } x \in L_j. \tag{5.11}$$

Case 3: Let us now assume that

$$x \in L_j \quad \text{with } j > 3k - 3N_F - 3.$$

Also in this case S_q maps the line segment L_j to a line segment L_j^1 parallel to x_1-axis and inside the set $(\hat{U}_j \setminus \hat{U}_{j+1}) \times (\hat{P}_k \setminus \hat{P}_{k+1})$, and we have

$$|D_1 S_q(x)| \le C2^{-\beta j} j^{\alpha+1}. \tag{5.12}$$

Also the derivative of F can be estimated again by

$$|D_1 F(S_q(x))| \le \mathrm{Lip}(F) \quad \text{for a.e. } x \in L_j. \tag{5.13}$$

Moreover, as $x \in [-1,1]^4 \setminus (\hat{U}_{k+1})^4$ we have

$$\begin{aligned}[-1,1]^4 \setminus (\hat{U}_{k+1})^4 &\subset \{y \in [-1,1]^4 : \mathrm{dist}(y, C_B) > \hat{r}_{k+1} - \hat{r}_{k+2}\} \\ &\subset \{y \in [-1,1]^4 : \mathrm{dist}(y, C_B) > 2^{-\beta(k+2)-(k+2)}\}.\end{aligned}$$

If we then assume that $\tilde{C} > 0$ is the constant in Lemma 3.5 we may again assume that $\tilde{C}^{-1} > 2^{\beta+1} 2^{-\beta N_F - N_F}$ (see case 2). Then it follows from Lemma 3.5 and from the inclusion above that

$$\begin{aligned}F(S_q(x)) &\in \{z \in [-1,1]^4 : \mathrm{dist}(z, C_B) > \tilde{C}^{-1} 2^{-\beta(k+2)+(k+2)}\} \\ &\subset \{z \in [-1,1]^4 : \mathrm{dist}(z, C_B) > 2^{-\beta(k+N_F+2)-(k+N_F+2)}\} \\ &\subset [-1,1]^4 \setminus (\hat{U}_{k+N_F+2})^4.\end{aligned}$$

Thus, it follows from Lemma 4.1 (iv) and (v) that we may estimate

$$|DS_t(F(S_q(x)))| \le C2^{\beta(3k-3N_F-3)}. \tag{5.14}$$

If we now combine (5.12), (5.13) and (5.14) the chain rule gives us

$$|D_1 f(x)| \le C j^{\alpha+1} 2^{-\beta(j-3k)} \quad \text{for a.e. } x \in L_j. \tag{5.15}$$

Estimating the Sobolev norm of f: The above estimates (5.7), (5.11) and (5.15) can be summarized as follows. Suppose that $k \ge k_0$ and let $x \in L_j := (U_j \setminus U_{j+1}) \times \{\tilde{x}\}$ with $\tilde{x} \in P_k \setminus P_{k+1}$. Then

$$|D_1 f(x)| \le \begin{cases} C & \text{if } 1 \le j \le k - 2N_F - 3 \\ C j^{\alpha+1} & \text{if } k - 2N_F - 3 < j \le 3k - 3N_F - 3 \\ C j^{\alpha+1} 2^{-\beta(j-3k)} & \text{if } j > 3k - 3N_F - 3, \end{cases} \tag{5.16}$$

where the constant C does not depend on k or j.

Also note that S_q maps $\mathcal{C}_A \times \mathbb{R}^3$ onto $\mathcal{C}_B \times \mathbb{R}^3$ and using $|\mathcal{C}_A| > 0$ and $|\mathcal{C}_B| = 0$ we easily obtain $|D_1 S_q| = 0$ on $\mathcal{C}_A \times \mathbb{R}^3$. As F is just a reflection on $\mathcal{C}_B \times \mathbb{R}^3$ and S_t is locally Lipschitz on $[-1,1]^4 \setminus C_B$, we easily obtain that

$$|D_1 f| = 0 \quad \text{on } (\mathcal{C}_A \times \mathbb{R}^3) \setminus C_A.$$

Therefore, for $\tilde{x} \in P_k \setminus P_{k+1}$ we can calculate

$$\int_{(-1,1)} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1 = \int_{(-1,1)\setminus \mathcal{C}_A} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1$$
$$= \sum_{j=1}^{\infty} \int_{U_j \setminus U_{j+1}} |Df(x_1, \tilde{x})|^p \, \mathrm{d}x_1.$$

We use the fact that f is Lipschitz on $[-1,1]^4 \setminus (U_{k_0})^4$ for every fixed k_0 to see that

$$\int_{(-1,1)} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1 \leq C + \sum_{j=k_0}^{\infty} \int_{U_j \setminus U_{j+1}} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1$$

for every $\tilde{x} \in U_k^3 \setminus U_{k+1}^3$ with $k \geq k_0$.

Let us next estimate the measure of the set $\{x_1 \in [-1,1] : x_1 \in U_j \setminus U_{j+1}\}$. For every given j this set contains 2^j line segments each having length which can be approximated above by $2^{-j}\big(1 + \frac{1}{(j+1)^\alpha} - 1 - \frac{1}{(j+2)^\alpha}\big)$. Thus the measure of the set can be approximated as

$$\mathcal{L}^1(U_j \setminus U_{j+1}) \leq C 2^j 2^{-j} \Big(1 + \frac{1}{(j+1)^\alpha} - 1 - \frac{1}{(j+2)^\alpha}\Big) \leq \frac{C}{j^{\alpha+1}}.$$

Therefore, for the line segment $L = [-1,1] \times \{\tilde{x}\}$ we have using (5.16)

$$\int_L |D_1 f|^p dx_1 = C + \sum_{j=k_0}^{\infty} \int_{L_j} |Df(x_1, \tilde{x})|^p dx_1$$
$$\leq C\Bigg(1 + \sum_{j=k_0}^{k-2N_F-3} \frac{1}{j^{\alpha+1}} + \sum_{j=k-2N_F-2}^{4k-3N_F-3} \frac{j^{p(\alpha+1)}}{j^{\alpha+1}} + \sum_{j=4k-3N_F-2}^{\infty} 2^{-p\beta(j-3k)} \frac{j^{p(\alpha+1)}}{j^{\alpha+1}}\Bigg).$$

The first sum converges even if we sum to infinity, the second sum will be estimated simply by taking an estimate of the largest summand and multiplying by an estimate of the total number of summands. We will use (5.3) to estimate the final sum by a convergent geometric sum ($\sum_{l=k}^{\infty} 2^{-pl\beta/2}$). Continuing the calculation and using $k \geq 4N_F + 5$ we have

$$\int_L |D_1 f|^p \, dx_1 \leq C + C4k(4k)^{(p-1)(\alpha+1)} + \frac{C}{1 - 2^{-kp\beta/2}}$$
$$\leq C + C\frac{k^{p\alpha+p}}{k^\alpha}. \tag{5.17}$$

The estimate (5.17) holds for all lines $L = [-1,1] \times \{\tilde{x}\}$ such that $\tilde{x}$ in $P_k \setminus P_{k+1}$ with $k \geq k_0$. Furthermore, since f is Lipschitz on $[-1,1]^n \setminus U_{k_0}^n$, we may estimate

$$\int_L |D_1 f|^p \, \mathrm{d}x_1 \leq C \quad \text{for all } \tilde{x} \in P_{k+1} \setminus P_k \text{ with } k < k_0,$$

which proves the validity of (5.17) for all $k \in \mathbb{N}$ (not only for $k \geq k_0$). If $\tilde{x} \in \mathcal{C}_A^3$ then we will again use the fact that

$$|D_1 f(x)| \leq C \text{ for a.e. } x \in L. \tag{5.18}$$

Now we integrate (5.17) over $\tilde{x} \in [-1,1]^3$. By (2.4) we know that

$$\mathcal{L}^3(P_k \setminus P_{k+1}) \leq \frac{C}{k^{\alpha+1}}$$

and we continue by multiplying this with (5.17) and summing over k plus (5.18) multiplied by the measure $\mathcal{L}^3(\mathcal{C}_A^3) = 1$. Since $\alpha \geq 2$ we have

$$\begin{aligned} \int_{(-1,1)^4} |D_1 f(x)|^p dx &\leq \sum_{k=1}^{\infty} C k^{-\alpha-1} + C \sum_{k=1}^{\infty} \frac{k^{p\alpha+p}}{k^{2\alpha+1}} + C \\ &\leq C + C \sum_{k=1}^{\infty} \frac{k^p}{k^{(2-p)\alpha}} = C \sum_{k=k_0}^{\infty} \frac{1}{k^p} < \infty \end{aligned} \tag{5.19}$$

by our choice of $\alpha = \frac{2p}{2-p}$ at the start of the proof. This ends the proof of Theorem 1.2 when $n = 4$. Taking our mapping f in 4 dimensions and using it to define a mapping $f^* : \mathbb{R}^5 \to \mathbb{R}^5$ as follows

$$f^*(x_1, x_2, x_3, x_4, x_5) = (f(x_1, x_2, x_3, x_4), x_5) \tag{5.20}$$

proves Theorem 1.2 when $n = 5$. □

6. The higher dimensional case $n \geq 6$

Let $\mathcal{M}(o, n)$ be the set of all linear subspaces of $\mathbb{R}^n$ of dimension o, parallel to the coordinate axes (i.e. $M \in \mathcal{M}(o, n)$ if and only if there exists a basis of M of o vectors chosen from the canonical basis). Where there is no danger of confusion we will omit n and write simply $\mathcal{M}(o)$. Previously we defined $\mathcal{K}_A$ as $\bigcup_{L \in \mathcal{M}(1,4)} C_A + L$. From now on we take $n \geq 6$ even, $m = n/2 - 1$ and define

$$\mathcal{K}_A = \bigcup_{L \in \mathcal{M}(m,n)} C_A + L \quad \text{and} \quad \mathcal{K}_B = \bigcup_{L \in \mathcal{M}(m,n)} C_B + L,$$

where $C_A = \bigcap_k U_k^n$ and $C_B = \bigcap_k \hat{U}_k^n$. Of course C_B and $\mathcal{K}_B$ depend on the parameter β. During our proof we show that if β is large enough then our mapping exists and we show how to construct the mapping for any β sufficiently large. Let us note that for n odd we can define our mapping analogously to (5.20) by using identity in the last coordinate.

We make a further explicitation to the notation used above and that is the sets $P_k = U_k^{n-1}$. It is more or less obvious how to generalize the notion from subsections 2.5 and 2.6 to the higher dimensional case, see e.g. [23, Proof of Theorem 4.9]. In this section we will show that if we fix $1 \leq p < [n/2]$ then by choosing the parameters $\alpha > 0$ and $\beta > 0$ large enough we may construct the mapping $f \in W^{1,p}$ which we have in mind in Theorem 1.2.

6.1. **Mapping F in higher dimensions:** We will introduce some sets that will aid notation for Theorem 6.1. Let $L \in \mathcal{M}(o, n)$, $1 \leq o \leq m$ then call

$$N_L = \{e_j \in \mathbb{R}^n : e_j \in L^\perp\}$$

and let M_L be the set of all subsets of N_L with $n-m$ elements. Let $k,l\in\mathbb{N}$ then call

$$A_{L,k,l}=\bigcup_{W\in M_L}\Big(\{x\in\mathbb{R}^n: x_i\in[-1,1]\setminus\hat{U}_k, e_i\in L\}\cap\{x\in\mathbb{R}^n: x_j\in\hat{U}_l, e_j\in W\}\Big). \tag{6.1}$$

This is the set where informally speaking we are far away from our Cantor set C_B in o directions and close in some $n-m$ directions (perpendicular to the given o directions) and in the remaining $n-o-(n-m)$ directions x_i could be arbitrary.

Theorem 6.1. *Let $m\in\mathbb{N}$ and $n=2m+2$. There exists a mapping F which is a sense-preserving bi-Lipschitz extension of the map*

$$F(x_1,x_2,\dots,x_{n-1},x_n)=(x_1,x_2,\dots,x_{n-1},-x_n)\quad x\in\mathcal{K}_B. \tag{6.2}$$

There exists an $N_F\in\mathbb{N}$ such that for each $k\in\mathbb{N}$, $k>N_F$, $1\le o\le m$, $L\in\mathcal{M}(o,n)$ we have

$$F\big((x+L)\cap A_{L,j-N_F,k+N_F+1}\big)\subset\big(F(x)+L\big)\cap A_{L,j,k+1} \tag{6.3}$$

for any given $x\in A_{L,j-N_F,j+N_F+1}$.

The inclusion (6.3) basically means that the image of those parts of affine spaces $x+L$ which are much closer to $\mathcal{C}_B$ in $n-m$ directions from $L^\perp$ than it is in directions from L in the map F is part of an affine space $F(x)+L$ and the distance of the affine space from $\mathcal{K}_B$ is roughly maintained.

The proof of Theorem 6.1 is similar to that of Theorem 3.1. We find vectors v and u, a Lipschitz extension g onto $X=\mathbb{R}^{n-1}\times\{0\}$ of $g(P_v(x))=-x_n$ and then $F=F_{g,u}\circ F_{g,v}$. The following lemma, corresponds to Lemma 3.4

Lemma 6.2. *Let $m\in\mathbb{N}$ and $n=2m+2$. Let $v=(2^{-n},2^{1-n},\dots,\frac{1}{4},1)$ and $u=(-2^{-n},-2^{1-n},\dots,-\frac{1}{4},1)$. Then there exists $\beta>0$ and a corresponding set $\mathcal{K}_B$ such that P_v is one-to-one on $\mathcal{K}_B$ and the function g defined on $P_v(\mathcal{K}_B)$ as $g(P_v(x))=-x_n$ can be extended onto X as a Lipschitz function. Furthermore, it is possible to find a Lipschitz extension of the function g which guarantees that*

$$D_iF_{g,u}\circ F_{g,v}(x)=\begin{cases} e_i & \text{if } i=1,2,\dots,n-1\\ -e_i & \text{if } i=n\end{cases} \tag{6.4}$$

whenever $x_i\in[-1,1]\setminus\hat{U}_k$ and we can find a set of $n-m$ indexes $\{j_1,j_2,\dots j_{n-m}\}$ such that $x_{j_1},x_{j_2},\dots,x_{j_{n-m}}\in\hat{U}_{k+2}$ and $j_l\neq i$ for every $l=1,2,\dots,n-m$.

Proof. With some small modifications the proof will mainly follow the proof of Lemma 3.4.

Step 1: The projection P_v is one-to-one on C_B. Step 1 here is the same as in the previous lemma. The reader can somewhat laboriously but easily check that $P_v(a)\neq P_v(b)$ whenever a,b are distinct vertices of $Q(0,\frac{1}{2})$. This gives us a set of 2^n distinct points and so (using Q^{n-1} to denote cubes in $\mathbb{R}^{n-1}$) there exists a $d_0>0$ such that $Q^{n-1}(P_v(a),d)\cup Q^{n-1}(b,d)=\emptyset$ whenever a and b are distinct vertices of $Q(0,\frac{1}{2})$ and $0<d\le d_0$. By the continuity of P_v there exists a $d_1>0$ such that the sets $P_v(Q(a,d))$ are pairwise disjoint for distinct vertices a of the cube $Q(0,\frac{1}{2})$ and $0<d\le d_1$. Thus we have proved that whenever we construct the cantor set C_B using $\beta=\log_2(d)-1$ for any $0<d\le d_1$ we have

$$P_v(\hat{Q}_{\boldsymbol{v}(1)})\cap P_v(\hat{Q}_{\boldsymbol{v}'(1)})=\emptyset \text{ whenever } \boldsymbol{v}(1)\neq\boldsymbol{v}'(1).$$

The self-similarity argument applied in Lemma 3.4 applies here too and so we see that the image of the collection of all k-th generational cubes $\hat{Q}_{\boldsymbol{v}(k)}$ are pairwise disjoint and this holds for all k. This implies that P_v is one-to-one on C_B for $\beta > \beta_0$. In fact this is a special case of the next step for $o = 0$.

Step 2: The projection P_v is one-to-one on $\mathcal{K}_B$. We would like to prove that P_v is one-to-one on $\mathcal{K}_B$. Let $\boldsymbol{v}(k) \in \mathbb{V}^k$ and let $M \in \mathcal{M}(o, n)$, $1 \le o \le m$, then we will prove that the projection of any distinct pair of k-"bars" $\hat{S}^M_{\boldsymbol{v}(k)}$ where

$$\hat{S}^M_{\boldsymbol{v}(k)} = \Big(Q(z_{\boldsymbol{v}(k)}, \hat{r}_k) + M\Big) \setminus \Big(\bigcup_{\boldsymbol{w} \in \mathbb{V}^k} \bigcup_{L \in \mathcal{M}(o-1)} Q(\hat{z}_{\boldsymbol{w}}, \hat{r}_{k-2}) + L\Big), \tag{6.5}$$

is disjoint. Similarly to before we achieve this by projecting them into disjoint sets

$$\begin{aligned}\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)} := \bigg(Q^{n-1}\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(M)\bigg) \\ \setminus \bigg(\bigcup_{\boldsymbol{w} \in \mathbb{V}^k} \bigcup_{L \in \mathcal{M}(o-1)} Q^{n-1}\big(P_v(\hat{z}_{\boldsymbol{w}}), \hat{r}_{k-1}\big) + P_v(L)\bigg).\end{aligned} \tag{6.6}$$

Let us note that in dimension $n = 4$ our definition was slightly different as we used $w \in \mathbb{V}^{k-1}$ (or $w \in \mathbb{V}^{k-2}$) in previous definitions. However, this is not a big change as the union of cubes over all $w \in \mathbb{V}^{k-1}$ or $w \in \mathbb{V}^k$ is similar (up to some multiple of radius) and from technical reasons this is better here.

Step 2A: The projection P_v is one-to-one on every $M \in \mathcal{M}(n-1)$. Recall the corresponding notation from Lemma 3.4,

$$\tilde{v} = (2^{-n}, 2^{1-n}, \dots, \tfrac{1}{4}).$$

The definition of P_v (3.2) immediately yields that

$$P_v(e_l) = \begin{cases} e_l & \text{if } 1 \le l \le n-1 \\ -\tilde{v} & \text{if } l = n. \end{cases} \tag{6.7}$$

We take $M \in \mathcal{M}(n-1)$ and solve $P_v(u) = 0$, $u \in M$. First we will assume that $e_n \in M^{\perp}$. Then (6.7) says that $P_v(u) = u$ for all $u \in M$ and the only solution to $P_v(u) = 0$ is $u = 0$ and thus P_v is one-to-one on M. Now we assume that $e_n \in M$ and we find j such that $\operatorname{span}\{e_j\} = M^{\perp}$. Using (6.7) we obtain

$$0 = P_v(u) = P_v\Big(\sum_{\substack{i=1\\ i \ne j}}^{n} \lambda_i e_i\Big) = \sum_{\substack{i=1\\ i \ne j}}^{n-1} \lambda_i e_i - \lambda_n \tilde{v}.$$

Thus the j-th coordinate of the last expression must be zero, which implies $\lambda_n \tilde{v}_j = 0$ and hence $\lambda_n = 0$. Thus we have reduced to the first case which has been proved already.

Step 2B: Finding β such that $\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ are disjoint sets. We defined sets $\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ in (6.6) and now we would like to show that if one chooses $\beta > \beta_2$ that these sets are pairwise disjoint. Exactly the same arguments from Lemma 3.4, Step 2B, Claim (1) can be applied here to see that the two contesting definitions from Lemma 3.4 of $\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ which we could generalize are equivalent.

Let A denote the set of vertices of $Q(0, \frac{1}{2})$. We define the "sliced" affine sets for $M \in \mathcal{M}(o)$ and $a \in A$

$$W^M_a = (a + M) \setminus \Big(\bigcup_{b\in A} \bigcup_{L\in\mathcal{M}(o-1)} Q(b, \tfrac{4}{5}) + L \Big). \tag{6.8}$$

The sets $W^{M_1}_{a_1}$ and $W^{M_2}_{a_2}$ are equal if and only if $M_1 = M_2$ and $a_1 - a_2 \in M_1$.

Let us make the following useful observations on P_v. Since $\|\tilde{v}\| = \frac{1}{4}$ we obtain the simple observation

$$\|P_v(x)\| \leq \tilde{x} + \tfrac{1}{4}|x_n| \leq \tfrac{5}{4}\|x\|, \tag{6.9}$$

recall that $\|\cdot\|$ denotes the maximum norm. Also we denote the distance with respect to this norm as $\operatorname{dist}_\infty$. Further we use the fact that P_v is one-to-one whenever restricted to any $M \in \mathcal{M}(n-1)$ and especially P_v is one-to-one on each $a + M$ for $M \in \mathcal{M}(o)$ and $a \in A$. There are a finite number of such affine spaces and P_v is one-to-one on each of them. This implies that there is a $\lambda > 0$ such that whenever we choose $M \in \mathcal{M}(o)$ and $x \in M$ that $\|P_v(x)\| \geq \lambda^{-1}\|x\|$. The fact that $\|P_v(x)\| \leq \lambda\|x\|$ is shown in (6.9) if $\lambda \geq \frac{5}{4}$. Now we choose any $M \in \mathcal{M}(o)$, any $L \in \mathcal{M}(o-1)$, $a \in A$ and any $x \in W^M_a$ and conclude that

$$\lambda^{-1} \operatorname{dist}_\infty(x, W^L_a) \leq \operatorname{dist}_\infty(P_v(x), P_v(W^L_a)) \leq \lambda \operatorname{dist}_\infty(x, W^L_a) \tag{6.10}$$

as the distance of x to W^L_a is attained in some direction in $M \setminus L$.

Let us recall that the sets W^M_a are defined in (6.8). In the following we will be interested in pairs of distinct W^M_a. Another fact that is clear is if we have a pair of distinct $W^{M_1}_{a_1}$ and $W^{M_2}_{a_2}$ (with $M_1, M_2 \in \mathcal{M}(o)$) then either $M_1 = M_2$ and $(a_1 + M_1) \cap (a_2 + M_2) = \emptyset$ or $\dim(M_1 \cap M_2) \leq o - 1$ and there exists an $L \in \mathcal{M}(\tilde{o})$, $\tilde{o} \leq o - 1$, and an $a \in A$ such that

$$(a_1 + M_1) \cap (a_2 + M_2) \subset a + L \subset \bigcup_{b\in A} \bigcup_{L\in\mathcal{M}(o-1)} Q(b, \tfrac{4}{5}) + L.$$

This means that our pair $W^{M_1}_{a_1}$ and $W^{M_2}_{a_2}$ are disjoint if distinct in both cases. We can easily calculate that

$$\operatorname{dist}_\infty(W^{M_1}_{a_1}, W^{M_2}_{a_2}) \geq \frac{4}{5}$$

and so (6.10) gives that

$$\operatorname{dist}_\infty\big(P_v(W^{M_1}_{a_1}), P_v(W^{M_2}_{a_2})\big) \geq \frac{4}{5\lambda}. \tag{6.11}$$

This however immediately implies that $\{P_v(W^M_a) + Q^{n-1}(0, \frac{2}{5\lambda})\}$ is a finite family of closed pairwise disjoint sets.

Let $\delta > 0$. Assuming that $a \in A$, $M \in \mathcal{M}(o)$, $\boldsymbol{v}(1) = 2a \in \mathbb{V}$, $\hat{r}_1 < \frac{\delta}{q}$, it is simple to observe that

$$\begin{aligned} P_v(a+M) + Q(0,\delta) &= P_v(\hat{z}_{\boldsymbol{v}(1)}) + P_v(M) + Q(0,\delta) \\ &\supset Q^{n-1}(P_v(\hat{z}_{\boldsymbol{v}(1)}), q\hat{r}_1) + P_v(M). \end{aligned} \tag{6.12}$$

We will again use the fact that P_v is one-to-one on all $M \in \mathcal{M}(o)$ to see that

$$P_v\big(W_a^M\big) = P_v\big(a+M\big) \setminus P_v\big((a+M) \cap (\bigcup_{b\in A} \bigcup_{L\in\mathcal{M}(o-1)} Q(b, \tfrac{4}{5}) + L)\big)$$

and so for any $L \in \mathcal{M}(o-1)$ and for the $b \in A$ such that $\boldsymbol{w}(1) = 2b \in \mathbb{V}$

$$\begin{aligned} P_v\big((a+M) \cap (Q(b,\tfrac{4}{5}) + L)\big) &\subset P_v\big(Q(b,\tfrac{4}{5}) + L\big) \\ &= P_v\big(Q(b,\tfrac{4}{5})\big) + P_v\big(L\big) \\ &\subset Q^{n-1}(P_v(\hat{z}_w), 1) + P_v(L). \end{aligned} \tag{6.13}$$

The definition of $\hat{\mathcal{S}}^M_{\boldsymbol{v}(1)}$ (6.6) in combination with (6.12) and (6.13) show that

$$\hat{\mathcal{S}}^M_{\boldsymbol{v}(1)} \subset P_v(W_a^M) + Q^{n-1}(0,\delta), \text{ whenever } \hat{z}_{\boldsymbol{v}(1)} = a.$$

Further by applying (6.11) and assuming $\delta < \frac{1}{5\lambda}$ (and $\hat{r}_1 < \frac{\delta}{q}$) we see that the sets $P_v(W_a^M) + Q^{n-1}(0,\delta)$ are pairwise disjoint and in fact

$$\mathrm{dist}_\infty(P_v(W_{a_1}^{M_1}) + Q^{n-1}(0,\delta), P_v(W_{a_2}^{M_2}) + Q^{n-1}(0,\delta)) \geq \tfrac{1}{5\lambda},$$

whenever the pair is distinct. This implies that the sets $\hat{\mathcal{S}}^M_{\boldsymbol{v}(1)}$ satisfy

$$\mathrm{dist}(\hat{\mathcal{S}}^{M_1}_{\boldsymbol{v}(1)}, \hat{\mathcal{S}}^{M_2}_{\boldsymbol{w}(1)}) \geq \tfrac{C(n)}{5\lambda},$$

whenever distinct. Further by self similarity we get the same for all k, i.e.

$$\mathrm{dist}(\hat{\mathcal{S}}^{M_1}_{\boldsymbol{v}(k)}, \hat{\mathcal{S}}^{M_2}_{\boldsymbol{w}(k)}) \geq \tfrac{C(n)}{5\lambda}\hat{r}_{k-1}$$

whenever distinct.

Step 2C: Proving the inclusion $P_v(\hat{S}^M_{\boldsymbol{v}(k)}) \subset \hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$. We will prove the inclusion $P_v(\hat{S}^M_{\boldsymbol{v}(2)}) \subset \hat{\mathcal{S}}^M_{\boldsymbol{v}(2)}$ and for other k it will hold by self-similarity. Again we will employ (6.9) in the following to calculate that

$$\begin{aligned} P_v\big(Q(z_{\boldsymbol{v}(2)}, \hat{r}_2) + M\big) &\subset Q^{n-1}\big(P_v(\hat{z}_{\boldsymbol{v}(2)}), \tfrac{5}{4}\hat{r}_2\big) + P_v(M) \\ &\subset Q^{n-1}\big(P_v(\hat{z}_{\boldsymbol{v}(2)}), q\hat{r}_2\big) + P_v(M) \end{aligned} \tag{6.14}$$

whenever $q \geq \frac{5}{4}$. The remainder of what we need to prove is that for each $\boldsymbol{w}(2) \in \mathbb{V}^2$ and for each $L \in \mathcal{M}(o-1)$

$$Q^{n-1}\big(P_v(\hat{z}_{\boldsymbol{w}(2)}), \hat{r}_1\big) + P_v(L) \subset P_v\big(Q(\hat{z}_{\boldsymbol{w}(2)}, \hat{r}_0) + L\big)$$

which can easily be achieved by selecting $\hat{r}_1$ small enough (i.e. β large enough) as $\hat{r}_0 = 1$. This step is analogous to the proof in dimension $n = 4$ and therefore we skip the details.

Step 2D: Conclusion of Step 2. By definition

$$\mathcal{K}_B = \bigcup_{M\in\mathcal{M}(m)} C_B + M.$$

Let us consider the sets

$$K_o = \Big(\bigcup_{M\in\mathcal{M}(o)} C_B + M\Big) \setminus \Big(\bigcup_{L\in\mathcal{M}(o-1)} C_B + L\Big).$$

It is easy to see that

$$K_o = \bigcup_{k_0\geq 1}\bigcap_{k\geq k_0}\bigcup_{M\in\mathcal{M}(o)}\bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k}\hat{S}^M_{\boldsymbol{v}(k)}$$

and since $\mathcal{K}_B = \bigcup_{o=1}^m K_o$ we obtain

$$\mathcal{K}_B = \bigcup_{0\leq o\leq m}\bigcup_{k_0\geq 1}\bigcap_{k\geq k_0}\bigcap_{k\geq 1}\bigcup_{M\in\mathcal{M}(o)}\bigcup_{\boldsymbol{v}(k)\in\mathbb{V}^k}\hat{S}^M_{\boldsymbol{v}(k)}.$$

We have proven that for any fixed k the images of $\hat{S}^M_{\boldsymbol{v}(k)}$ in P_v are pairwise disjoint, whenever the pair of sets in question are distinct. Take any pair of distinct points $x, y \in \mathcal{K}_B$. If there exists $k, M_1, \boldsymbol{v}(k)$ and $M_2, \boldsymbol{w}(k)$ such that $x \in \hat{S}^{M_1}_{\boldsymbol{v}(k)} \neq \hat{S}^{M_2}_{\boldsymbol{w}(k)} \ni y$ then P_v maps x and y onto distinct points in X because as we have proven

$$P_v(\hat{S}^{M_1}_{\boldsymbol{v}(k)}) \cap P_v(\hat{S}^{M_2}_{\boldsymbol{w}(k)}) = \emptyset.$$

If for almost every k we have $x, y \in \hat{S}^M_{\boldsymbol{v}(k)}$, then $x - y \in M$ and P_v is one-to-one on M and so maps x and y to distinct points.

Step 3: Defining the function g on X. Now we expound how to perform step 3 of the proof, that is how to define g on X. In steps 3A, 3B and 3C we assume always that $e_n \notin M$ and $M \in \mathcal{M}(o)$ for some $1 \leq o \leq m$. The case $e_n \in M$ is dealt with in 3D. Step 3E then proves that g has the desired properties. We will make use of the sets

$$\hat{H}^M_{\boldsymbol{v}(k)} := \partial_X\Big(Q^{n-1}\big(P_v(\hat{z}_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(M)\Big) \setminus \Big(\bigcup_{\boldsymbol{w}\in\mathbb{V}^{k-1}}\ \bigcup_{\substack{L\in\mathcal{M}(m-1)\\ L\subset M}} Q\big(\hat{z}_{\boldsymbol{w}}, \hat{r}_{k-1}\big) + P_v(L)\Big)$$

where Q^{n-1} is a cube in $\mathbb{R}^{n-1}$ (specifically in $X = \mathbb{R}^{n-1}\times\{0\}$) and $\partial_X U$ is the relative boundary of a set U with respect to X.

Step 3A. First we take a "pipe" $\hat{H}^M_{\boldsymbol{v}(k)}$ with $e_n \notin M$ and $k \geq 2$ and define

$$g(x) = -(\hat{z}_{\boldsymbol{v}(k)})_n \text{ for all } x \in \hat{H}^M_{\boldsymbol{v}(k)}.$$

Again, first we remark that if $\hat{H}^M_{\boldsymbol{v}(k)} = \hat{H}^M_{\tilde{\boldsymbol{v}}(k)}$ then $(\hat{z}_{\boldsymbol{v}(k)})_n = (\hat{z}_{\tilde{\boldsymbol{v}}(k)})_n$ because $e_n \notin M$ and therefore g is well-defined at these points. It is easy to see that if we have two pipes, both parallel to M, one inside another (that is $\hat{z}_{\boldsymbol{v}(k+1)} + M$ intersects $\hat{Q}_{\boldsymbol{v}(k)}$) then

$$\operatorname{dist}(\hat{H}^M_{\boldsymbol{v}(k)}, \hat{H}^M_{\boldsymbol{v}(k+1)}) \geq C\hat{r}_k \text{ for a suitable } C > 0 \text{ independent of } k.$$

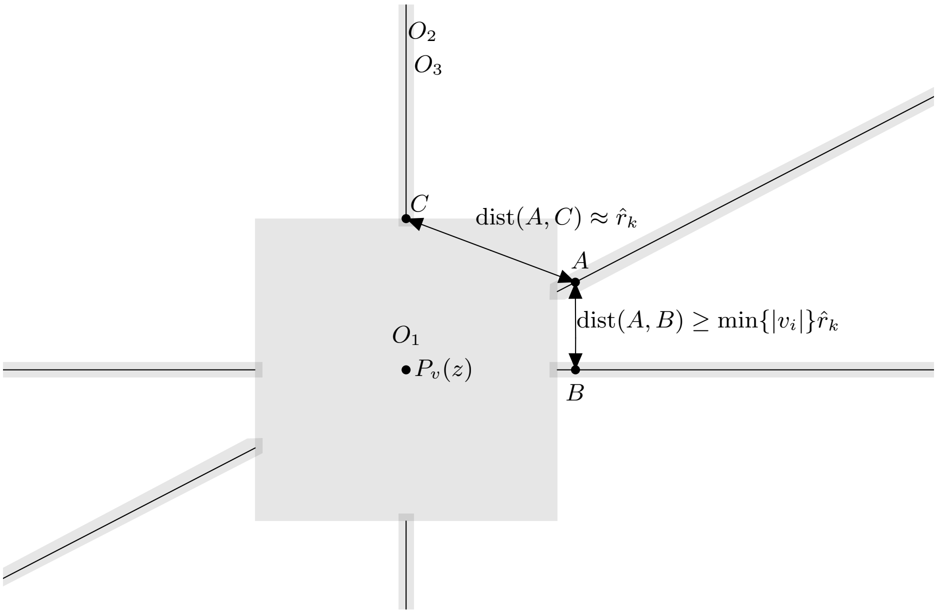

FIGURE 10. Measuring the distances between sliced projected affine spaces reduces to the case dealt with in Lemma 3.4 where we measured the distance between sliced lines. The thickness of the 'bars' is $\hat{r}_{k+1}$ which can be made much smaller than the distance between them which is comparable to $\hat{r}_k$.

Further considering $x \in \hat{H}^M_{\boldsymbol{v}(k)}$ and $y \in \hat{H}^M_{\boldsymbol{v}(k+1)}$ we have

$$|g(x) - g(y)| = |-(\hat{z}_{\boldsymbol{v}(k)})_n + (\hat{z}_{\boldsymbol{v}(k+1)})_n| = \tfrac{1}{2}\hat{r}_k.$$

Considering two distinct pipes $\hat{H}^M_{\hat{\boldsymbol{v}}(k+1)}$ and $\hat{H}^M_{\boldsymbol{v}(k+1)}$ of the same generation, both inside $\hat{H}^M_{\boldsymbol{v}(k)}$ we see that

$$\operatorname{dist}(\hat{H}^M_{\hat{\boldsymbol{v}}(k+1)}, \hat{H}^M_{\boldsymbol{v}(k+1)}) \geq C\hat{r}_k \text{ for a suitable } C > 0 \text{ independent of } k.$$

Furthermore, for $x \in \hat{H}^M_{\boldsymbol{v}(k+1)}$ and $y \in \hat{H}^M_{\boldsymbol{v}(k+1)}$ we have

$$|g(x) - g(y)| = |-(z_{\hat{\boldsymbol{v}}(k+1)})_n + (z_{\boldsymbol{v}(k+1)})_n| \leq \hat{r}_k\,.$$

This proves that g, thus defined, on the pipes $\hat{H}^M_{\boldsymbol{v}(k)}$ with $e_n \notin M$ is Lipschitz with respect to parallel pipes, i.e. pipes given by the same subspace M.

Step 3B. Similarly, for $M \in \mathcal{M}(o)$, $1 \leq o \leq m$, and $e_n \notin M$ we define

$$g(P_v(\hat{z}_{\boldsymbol{v}(k)} + M)) = -(\hat{z}_{\boldsymbol{v}(k)})_n.$$

Also for every $x \in C_B$ we define

$$g(P_v(x + M)) = -x_n.$$

Note that by step 2 we know that these sets are pairwise disjoint whenever distinct and P_v is one-to-one on $\mathcal{K}_B$ and therefore this definition is correct. The estimates from Lemma 3.4, step 3B easily generalize to this setting showing that our definition of g is Lipschitz on the collection of all pipes, i.e., all sets of type $P_v(\hat{z}_{\boldsymbol{v}(k)} + M)$ and $P_v((C_B \cap \hat{Q}_{\boldsymbol{v}(k)}) + M)$.

Step 3C. Now we will fix $k \geq 2$, $\boldsymbol{v}(k) \in \mathbb{V}^k$, $1 \leq o \leq m$ and $M \in \mathcal{M}(o)$ (we still assume that $e_n \notin M$) and define g on

$$L^M_{\boldsymbol{v}(k)} := \hat{\mathcal{S}}^M_{\boldsymbol{v}(k)} \setminus \bigcup_{\boldsymbol{v}(k+1) \in \mathbb{V}^{k+1}} \hat{\mathcal{S}}^M_{\boldsymbol{v}(k+1)}.$$

Call $Y_M = M^\perp \cap X$ and denote π_{Y_M} the orthogonal projection onto this subspace. In general one can only claim that the projection of a pair of sets does not increase the distance between them. Here however we consider sets parallel to a given vector space M and project them onto Y_M, which is perpendicular to M. In this case the projection does not decrease the distance between the sets either. That is to say (in the following we use $P_v(M) = M$, see (6.7))

$$\operatorname{dist}(\pi_{Y_M}(\hat{H}^M_{\boldsymbol{v}(k)}), \pi_{Y_M}(\hat{H}^M_{\boldsymbol{v}(k+1)})) = \operatorname{dist}(\hat{H}^M_{\boldsymbol{v}(k)}, \hat{H}^M_{\boldsymbol{v}(k+1)}).$$

Similarly

$$\operatorname{dist}(\pi_{Y_M}(\hat{H}^M_{\boldsymbol{v}(k)}), \pi_{Y_M}(\hat{z}_{\boldsymbol{v}(k)} + M)) = \operatorname{dist}(\hat{H}^M_{\boldsymbol{v}(k)}, \hat{z}_{\boldsymbol{v}(k)} + M)$$

and

$$\operatorname{dist}(\pi_{Y_M}(\hat{H}^M_{\boldsymbol{v}(k)}), \pi_{Y_M}(x + M)) = \operatorname{dist}(\hat{H}^M_{\boldsymbol{v}(k)}, x + M)$$

for $x \in P_v(C_B \cap \hat{Q}_{\boldsymbol{v}(k)})$. We defined g as constant on sets $\hat{H}^M_{\boldsymbol{v}(k)}$, therefore we may define a function $\tilde{g}$ on Y_M as $\tilde{g}(\pi_{Y_M}(x)) = g(x)$ for any $x \in \hat{H}^M_{\boldsymbol{v}(k)}$ and this definition is correct. The above estimates on the distances of the sets projected onto Y_M shows that $\tilde{g}$ is Lipschitz with respect to the projection of those sets. Therefore we may use the McShane extension theorem to get a Lipschitz $\tilde{g}$ defined on Y_M. For $x \in L^M_{\boldsymbol{v}(k)}$ we define $g(x) = \tilde{g}(\pi_{Y_M}(x))$ and so get a function g, which is constant on the intersection of any affine space parallel to M with the set $L^M_{\boldsymbol{v}(k)}$.

Once we have defined g on all $\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ for all $k, \boldsymbol{v}(k)$ and M we still need to fill in certain "gaps", where we transition from $M \in \mathcal{M}(o)$ to $L \in \mathcal{M}(o-1)$. Considering Figure 10 we need to define g on sets corresponding to O_3. Specifically, for $M \in \mathcal{M}(o)$ we need to define g on

$$\begin{aligned} \hat{T}^M_{\boldsymbol{v}(k)} = &\Big(Q^{n-1}\big(P_v(z_{\boldsymbol{v}(k)}), q\hat{r}_k\big) + P_v(M)\Big) \\ &\setminus \bigg(\bigcup_{\substack{\tilde{L} \in \mathcal{M}(o-2) \\ \tilde{L} \subset M}} \Big(Q^{n-1}\big(P_v(z_{\boldsymbol{v}(k)}), \hat{r}_{k-1}\big) + P_v(\tilde{L})\Big) \cup \hat{\mathcal{S}}^M_{\boldsymbol{v}(k)} \bigg). \end{aligned}$$

These gaps were necessary as they made the sets $\mathcal{S}^M_{\boldsymbol{v}(k)}$ disjoint (whenever distinct) and this made the definition of g in step 3A and 3B correct. Note that it was not possible to define g as constant on entire m-dimensional subspaces (without removing $m-1$ dimensional subspaces) because they intersect other m-dimensional subspaces where g has a different value. Informally speaking, in each $o-1$ dimensional gap we project onto a corresponding perpendicular $n-1-(o-1)$ dimensional subspace on which, g is already defined in some points by step 3A and 3B. We keep those values and extend them as a Lipschitz function on the perpendicular subspace and then, by projecting along the $o-1$ dimensional subspace we define g everywhere in the gap.

If $x \in \hat{T}^M_{\boldsymbol{v}(k)}$ and $o \geq 2$ then there exists exactly one coordinate, let us say the i-th coordinate where $e_i \in M$,

$$|x_i - \big(P_v(z_{\boldsymbol{v}(k)})\big)_i| \leq \hat{r}_{k-1}$$

but for all $j \neq i$ such that $e_j \in M$ we have

$$|x_j - \big(P_v(z_{\boldsymbol{v}(k)})\big)_j| > \hat{r}_{k-1}.$$

So, set $M \in \mathcal{M}(o)$, $L \in \mathcal{M}(o-1)$ and e_i such that $\operatorname{span}\{L, e_i\} = M$. Recall that $M \subset X$ and P_v is identity on X to see that a point $x \in \hat{T}^M_{\boldsymbol{v}(k)}$ can be expressed as

$$x = P_v(z_{\boldsymbol{v}(k)}) + \sum_{e_j \in L} \lambda_j e_j + t e_i + \sum_{e_l \in M^\perp \cap X} \tilde{\lambda}_l e_l$$

where $\lambda_j > \hat{r}_{k-1}$, $\tilde{\lambda}_l < q\hat{r}_k$, $t < \hat{r}_{k-1}$. We project $\hat{T}^M_{\boldsymbol{v}(k)}$ onto $Y_L = L^\perp \cap X$. Since g is constant on affine subsets contained in $\mathcal{S}^{\hat{M}}_{\boldsymbol{v}(k)}$ parallel to L it is also constant on affine subsets of $\partial\hat{T}^M_{\boldsymbol{v}(k)} \cap \partial\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ parallel to L (note that these affine sets on boundaries have dimension $o-1$). Thus the following definition is correct

$$\tilde{g}(y) = g(x) \text{ whenever } y = \pi_{Y_L}(x)$$

and $x \in \partial\hat{T}^M_{\boldsymbol{v}(k)} \cap \partial\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$. The function $\tilde{g}$ is Lipschitz and can be Lipschitz extended onto Y_L by the McShane Theorem. For $x \in \hat{T}^M_{\boldsymbol{v}(k)}$ we define

$$g(x) = \tilde{g}(\pi_{Y_L}(x)).$$

In this manner we extend g for all $M, L, \hat{z}_{\boldsymbol{v}(k)}$ and all k in the case where $e_n \notin M$.

Step 3D. Next we will define g on pipes $\hat{H}^M_{\boldsymbol{v}(k)}$ with $e_n \in M$ and $k \geq 2$. For this, let us next denote $\tilde{v} := \big(2^{-n}, 2^{1-n}, \dots, \frac{1}{4}\big)$ and define

$$Y_n := \{w \in \mathbb{R}^{n-1} : \langle w, \tilde{v}\rangle = 0\}.$$

Then we separate $\mathbb{R}^{n-1}$ into the direct sum $\mathbb{R}\tilde{v} \oplus Y_n$. Suppose now that $\lambda_0 \in \mathbb{R}$ and $w_0 \in Y_n$ are such that

$$P_v(\hat{z}_{\boldsymbol{v}(k)}) = w_0 + \lambda_0 \tilde{v}. \tag{6.15}$$

Then, if $\tilde{x} \in \hat{H}^M_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + M)$ we may find $\lambda \in \mathbb{R}$ such that $\tilde{x} = w + \lambda\tilde{v}$ with $w \in Y_n$ which leads us to define

$$g(\tilde{x}) = \lambda - \lambda_0 - (\hat{z}_{\boldsymbol{v}(k)})_n \quad \text{for every } \tilde{x} \in \hat{H}^M_{\boldsymbol{v}(k)} \cup P_v(\hat{z}_{\boldsymbol{v}(k)} + M). \tag{6.16}$$

This means that g has been defined as constant on the intersections of the sets in question with hyperplanes in X parallel to $\tilde{v}^\perp$.

We claim that the definition in (6.16) and the definition $g(P_v(x)) = -x_n$ for $x \in \mathcal{K}_B$ gives us a g Lipschitz on the collection of sets $\hat{H}^M_{\boldsymbol{v}(k)}$, $P_v(\hat{z}_{\boldsymbol{v}(k)} + M)$ and $P_v(\mathcal{K}_B)$. The proof of this is just a repetition of step 3D from Lemma 3.4. Once again we extend our map by creating a Lipschitz extension on Y_M and by using $g(x) = \tilde{g}(\pi_{Y_M}(x))$.

Where not yet defined we may extend g Lipschitz arbitrarily, for example by the McShane extension theorem.

Step 3E: verifying the condition (6.4)**.** Now we define the spaghetti strand map $F_{g,v}$ again as

$$F_{g,v}(x) = x + vg(P_v(x)).$$

Analogously to the proof in dimension $n = 4$ it is possible to show that

$$D_i\big(F_{g,u} \circ F_{g,v}\big)(x) = \begin{cases} e_i & \text{if } i = 1, 2, \dots, n-1 \\ -e_i & \text{if } i = n \end{cases}$$

whenever $x_i \in [-1, 1] \setminus \hat{U}_k$ and we can find a set of $n - m$ indexes $\{j_1, j_2, \dots j_{n-m}\}$ such that $x_{j_1}, x_{j_2}, \dots, x_{j_{n-m}} \in \hat{U}_{k+2}$. There are two possibilities. Either $i \neq n$ and g is constant on $P_v(x+\mathbb{R}e_i)$ or $i = n$ and $g(P_v(x+te_n)) = c-t$. This is true because all x such that $x_i \in [-1, 1] \setminus \hat{U}_k$ and $x_{j_1}, x_{j_2}, \dots, x_{j_{n-m}} \in \hat{U}_{k+2}$ belong to some $\hat{S}^M_{\boldsymbol{v}(k)}$ which is projected into $\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ and we defined g on $\hat{\mathcal{S}}^M_{\boldsymbol{v}(k)}$ to have precisely these qualities. The rest of the calculations are just a repetition of step 3E of Lemma 3.4. □

Proof of Theorem 6.1. It suffices to repeat the proof from Theorem 3.1 to show that the mapping $F = F_{g,u} \circ F_{g,v}$ with g, v, u from Lemma 6.2 satisfies (6.2). Also we see that F is sense preserving for the same reason as before. The proof of the behavior of F on m-dimensional planes close to $\mathcal{K}_B$ is the same as it was for lines in the previous too. The choice of N_F follows from the same arguments as in Theorem 3.1 and an adaption of Lemma 3.5, where we replace lines with affine spaces and the proof remains the same. □

6.2. **Mapping f in higher dimensions.** We define our mapping in much the same way as in the 4-dimensional case. We set

$$f = S_t \circ F \circ S_q,$$

where $S_q = (q(x_1), q(x_2), \dots, q(x_n))$, F is the mapping from Theorem 6.1 and S_t is defined exactly as before. More precisely, we define S_t by

$$S_t(x) = d_{i,k}(x)H^{n-1,i}_{3k-3}(x) + (1 - d_{i,k}(x))H^{n-1,i}_{3k}(x) + t(x_i)e_i \qquad \text{for } x \in \hat{E}_{i,k}, \tag{6.17}$$

where the mappings $H^{n-1,i}_{3k-3}$ and $H^{n-1,i}_{3k}$ are the obvious higher dimensional generalizations of the mappings $H^{3,i}_{3k-3}$ and $H^{3,i}_{3k}$ in Section 4, and

$$\hat{E}_{i,k} = \{x \in (\hat{U}_{k-1})^n \setminus (\hat{U}_k)^n : d_{i,k}(x) \geq d_{j,k}(x), j \neq i\}.$$

By following the arguments in Section 4 the reader may generalize Lemma 4.1 in all dimensions:

Lemma 6.3. *Suppose that $S_t : (-1, 1)^n \to (-1, 1)^n$ is defined as in* (6.17) *where $n \geq 4$. Then S_t is a sense-preserving homeomorphisms which satisfies the following conditions:*

(i) *S_t maps C_B onto C_A and $S_t = S_q^{-1}$ on C_B.*
(ii) *Mapping S_t is locally Lipschitz on $(-1, 1)^n \setminus C_B$.*

(iii) *If L_i is a line parallel to x_i-axis with $L_i \cap (\hat{U}_k)^n \neq \emptyset$ then*

$$|D_i S_t(x)| \leq C \frac{2^{\beta k}}{k^{\alpha+1}}$$

for every $x \in L_i \cap \big((\hat{U}_{k-1})^n \setminus (\hat{U}_k)^n\big)$.

(iv) *If $k \leq j \leq 3k+2$ and $z \in (\hat{U}_j \setminus \hat{U}_{j+1}) \times \big((\hat{U}_k)^{n-1} \setminus (\hat{U}_{k+1})^{n-1}\big)$ then*

$$|DS_t(x)| \leq C 2^{\beta j}.$$

The same holds for $x \in \big((\hat{U}_k)^{n-1} \setminus (\hat{U}_{k+1})^{n-1}\big) \times (\hat{U}_j \setminus \hat{U}_{j+1})$ and also for $n-2$ other permutations of coordinates.

(v) *If $x \in \hat{U}_{3k+3} \times \big((\hat{U}_k)^{n-1} \setminus (\hat{U}_{k+1})^{n-1}\big)$ then*

$$|DS_t(x)| \leq C 2^{\beta(3k+3)}.$$

The same holds for $z \in \big((\hat{U}_k)^{n-1} \setminus (\hat{U}_{k+1})^{n-1}\big) \times (\hat{U}_{3k+3})$ and also for $n-2$ other permutations of coordinates.

Sobolev regularity of f. Just as before f, as the composition of homeomorphisms, is a homeomorphism. Our aim is to prove that if $1 \leq p < [\frac{n}{2}]$ then for an aptly chosen $\alpha > 0$ in the definition of C_A the corresponding mapping f belongs to $W^{1,p}$. Therefore we are interested in calculating the integral

$$\int_{(1,1)^{n-1}} \int_{(-1,1)} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1 \mathrm{d}\tilde{x}.$$

The integrals over lines in other directions can all be estimated in the same way as the reader may easily check. Recall that n is even and we start by fixing the exponent $1 \leq p < [n/2]$ and

$$\alpha = \frac{2p}{n/2 - p}$$

and the index $k_0 \geq 4N_F + 5$, where $N_F \in \mathbb{N}$ is from Lemma 3.5, large enough so that

$$\max\{2^{-kp\beta/2} k^{(p-1)(\alpha+1)}, 2^{-p\beta} (\tfrac{k+1}{k})^{\alpha}\} < 1 \quad \text{for all } k \geq k_0. \tag{6.18}$$

The reasoning in the arguments in section 5 for the ACL condition and the use of the chain rule hold here too. Both S_q and F are Lipschitz maps. By Lemma 6.3 (*ii*) we see that S_t is $C(k_0)$-Lipschitz on $[-1,1]^n \setminus \hat{U}^n_{k_0+N_F}$. Therefore it follows that f is Lipschitz on $[-1,1]^n \setminus U^n_{k_0}$, and it remains to consider the set $U^n_{k_0}$.

We use the ACL property of f to make the following estimates on the derivatives of $|D_1 f|$. For convenience sake we will denote $x = (x_1, \tilde{x})$. Now we will fix $k \in \mathbb{N}$ with $k \geq k_0 \geq 4N_F + 5$ and in the further we assume that $\tilde{x} \in (U_k)^{n-1} \setminus (U_{k+1})^{n-1}$. We define the following divisions of a segment $L = [-1,1] \times \{\tilde{x}\}$:

$$L_j(\tilde{x}) = L_j = \{(x_1, \tilde{x}) : x_1 \in (U_j \setminus U_{j+1})\}.$$

In the following we use the simpler notation L_j to aid readability. The aim of the following calculations is to prove the estimate (6.31) below.

Case 1: Let us first assume that $k \geq k_0$ and

$$x \in L_j \quad \text{with } j = 1, \dots, k - 2N_F - 3.$$

Then S_q maps the line segment L_j to a line segment L_j^1 which is parallel to x_1-axis and lies inside the set $(\hat{U}_j \setminus \hat{U}_{j+1}) \times (\hat{P}_k \setminus \hat{P}_{k+1})$. Furthermore, we have

$$|D_1 S_q(x)| \leq C 2^{-\beta j} j^{\alpha+1}, \tag{6.19}$$

with C independent of j, k.

Recall that the sets $A_{L,j,k}$ are defined in (6.1). Observe that

$$L_j^1 \subset A_{\mathbb{R}e_1,(j+N_F+2)-N_F-1,(k-N_F+1)+N_F},$$

where $N_F < j + N_F + 2 \leq k - N_F + 1$, and thus it follows from Theorem 6.1 that the bi-Lipschitz map F maps L_j^1 to a line segment L_j^2 parallel to x_1-axis and such that

$$L_j^2 \subset A_{\mathbb{R}e_1,(j+N_F+2)-1,k-N_F+1}.$$

Moreover, we have

$$|DF(S_q(x))| \leq \operatorname{Lip}(F) \quad \text{for a.e. } x \in L_j. \tag{6.20}$$

Finally, because L_j^2 is a line segment parallel to the x_1-axis which is contained in $[-1,1]^n \setminus (\hat{U}_{j+N_F+1})^n$ it follows from Lemma 6.3 (iii) that

$$|D_1 S_t(F(S_q(x)))| \leq C 2^{\beta j} j^{-(\alpha+1)}, \tag{6.21}$$

with C independent of j, k.

If we now combine (6.19), (6.20) and (6.21) we get

$$|D_1 f(x)| \leq C \quad \text{for a.e. } x \in L_j, \tag{6.22}$$

with C independent of j, k.

Case 2: Let us next assume that

$$x \in L_j \quad \text{with } k - 2N_F - 3 < j \leq 3k - 3N_F - 3.$$

Again S_q maps the line segment L_j to a line segment L_j^1 which is parallel to x_1-axis and lies inside the set $(\hat{U}_j \setminus \hat{U}_{j+1}) \times (\hat{P}_k \setminus \hat{P}_{k+1})$. Furthermore, we have

$$|D_1 S_q(x)| \leq C 2^{-\beta j} j^{\alpha+1}, \tag{6.23}$$

with C independent of j, k.

Next, we recall that

$$|D_1 F(S_q(x))| \leq \operatorname{Lip}(F) \quad \text{for a.e. } x \in L_j. \tag{6.24}$$

Moreover, it follows from the assumption $j \geq k - 2N_F - 2$ that

$$\begin{aligned}
L_j^1 &\subset \{x \in [-1,1]^4 : \operatorname{dist}(x, \mathcal{K}_B) > \min\{\hat{r}_j - \hat{r}_{j+1}, \hat{r}_k - \hat{r}_{k+1}\}\} \\
&\subset \big\{x \in [-1,1]^4 : \operatorname{dist}(x, \mathcal{K}_B) > \min\{2^{-\beta(j+1)-(j+1)}, 2^{-\beta(k+1)-(k+1)}\}\big\} \\
&\subset \big\{x \in [-1,1]^4 : \operatorname{dist}(x, \mathcal{K}_B) > \min\{2^{-\beta(j+1)-(j+1)}, 2^{-\beta(j+2N_F+3)-(j+2N_F+3)}\}\big\} \\
&\subset \big\{x \in [-1,1]^4 : \operatorname{dist}(x, \mathcal{K}_B) > 2^{-\beta(j+2N_F+3)-(j+2N_F+3)}\big\}.
\end{aligned}$$

Suppose now that $\tilde{C} > 0$ is the constant given by Lemma 3.5. Then we may choose $N_F \in \mathbb{N}$ to be so large that $\tilde{C}^{-1} > 2^{3\beta+3}2^{-\beta N_F - N_F}$. Then it follows from Lemma 3.5 and from the inclusion above that

$$\begin{aligned} F(L_j^1) &\subset \big\{x \in [-1,1]^4 : \operatorname{dist}(x, \mathcal{K}_B) > \tilde{C}^{-1}2^{-\beta(j+2N_F+3)-(j+2N_F+3)}\big\} \\ &\subset \big\{x \in [-1,1]^4 : \operatorname{dist}(x, \mathcal{K}_B) > 2^{-\beta(j+3N_F)-(j+3N_F)}\big\}. \end{aligned}$$

Thus, we have that $F(L_j^1)$ is contained in the following union of n sets

$$\begin{aligned} F(L_j^1) \subset \Bigg(&([-1,1] \setminus \hat{U}_{j+3N_F}) \times \big([-1,1]^{n-1} \setminus (\hat{U}_{j+3N_F})^{n-1}\big)\Bigg) \cup \\ &\cdots \cup \Bigg(\big([-1,1]^{n-1} \setminus (\hat{U}_{j+3N_F})^{n-1}\big) \times ([-1,1] \setminus \hat{U}_{j+3N_F})\Bigg). \end{aligned}$$

Without loss of generality suppose that

$$F(L_j^1) \subset ([-1,1] \setminus \hat{U}_{j+3N_F}) \times \big([-1,1]^{n-1} \setminus (\hat{U}_{j+3N_F})^{n-1}\big).$$

Then by Lemma 6.3 (iv) it follows that

$$|DS_t(F(S_q(x)))| \leq C2^{\beta j} \quad \text{for a.e. } x \in L_j. \tag{6.25}$$

If we now combine the estimates (6.23), (6.24) and (6.25) the chain rule gives us

$$|D_1 f(x)| \leq Cj^{\alpha+1} \quad \text{for a.e. } x \in L_j, \tag{6.26}$$

with C independent of j, k.

Case 3: Let us now assume that

$$x \in L_j \quad \text{with } j > 3k - 3N_F - 3.$$

Also in this case S_q maps the line segment L_j to a line segment L_j^1 parallel to x_1-axis and inside the set $(\hat{U}_j \setminus \hat{U}_{j+1}) \times (\hat{P}_k \setminus \hat{P}_{k+1})$, and we have

$$|D_1 S_q(x)| \leq C2^{-\beta j} j^{\alpha+1}, \tag{6.27}$$

with C independent of j, k. Furthermore, also the derivative of F can be estimated again by

$$|D_1 F(S_q(x))| \leq \operatorname{Lip}(F) \quad \text{for a.e. } x \in L_j. \tag{6.28}$$

Moreover, as $x \in [-1,1]^n \setminus (\hat{U}_{k+1})^n$ we have

$$\begin{aligned} [-1,1]^n \setminus (\hat{U}_{k+1})^n &\subset \{y \in [-1,1]^n : \operatorname{dist}(y, C_B) > \hat{r}_{k+1} - \hat{r}_{k+2}\} \\ &\subset \{y \in [-1,1]^n : \operatorname{dist}(y, C_B) > 2^{-\beta(k+2)-(k+2)}\}. \end{aligned}$$

If we then assume that $\tilde{C} > 0$ is the constant in Lemma 3.5 we may again assume that $\tilde{C}^{-1} > 2^{\beta+1}2^{-\beta N_F - N_F}$ (see case 2). Then it follows from Lemma 3.5 and from the inclusion above that

$$\begin{aligned} F(S_q(x)) &\in \{z \in [-1,1]^n : \operatorname{dist}(z, C_B) > \tilde{C}^{-1}2^{-\beta(k+2)+(k+2)}\} \\ &\subset \{z \in [-1,1]^n : \operatorname{dist}(z, C_B) > 2^{-\beta(k+N_F+2)-(k+N_F+2)}\} \\ &\subset [-1,1]^n \setminus (\hat{U}_{k+N_F+2})^n. \end{aligned}$$

Thus, it follows from Lemma 6.3 (iv) and (v) that we may estimate

$$|DS_t(F(S_q(x)))| \leq C2^{\beta(3k-3N_F-3)}, \tag{6.29}$$

with C independent of j, k.

If we now combine (6.27), (6.28) and (6.29) the chain rule gives us

$$|D_1 f(x)| \leq Cj^{\alpha+1}2^{-\beta(j-3k)} \quad \text{for a.e. } x \in L_j, \tag{6.30}$$

with C independent of j, k.

Estimating the Sobolev norm of f: The above estimates (6.22), (6.26) and (6.30) can be summarized as follows. Suppose that $k \geq k_0$ and let $x \in L_j := (U_j \setminus U_{j+1}) \times \{\tilde{x}\}$ with $\tilde{x} \in P_k \setminus P_{k+1}$. Then

$$|D_1 f(x)| \leq \begin{cases} C & \text{if } 1 \leq j \leq k - 2N_F - 3 \\ Cj^{\alpha+1} & \text{if } k - 2N_F - 3 < j \leq 3k - 3N_F - 3 \\ Cj^{\alpha+1}2^{-\beta(j-3k)} & \text{if } j > 3k - 3N_F - 3, \end{cases} \tag{6.31}$$

where the constant $C := C(n, \alpha, \beta, N_F, \mathrm{Lip}(F))$ does not depend on k or j.

Also note that S_q maps $\mathcal{C}_A \times \mathbb{R}^{n-1}$ onto $\mathcal{C}_B \times \mathbb{R}^{n-1}$ and using $|\mathcal{C}_A| > 0$ and $|\mathcal{C}_B| = 0$ we easily obtain $|D_1 S_q| = 0$ on $\mathcal{C}_A \times \mathbb{R}^{n-1}$. As F is just a reflection on $\mathcal{C}_B \times \mathbb{R}^{n-1}$ and S_t is locally Lipschitz on $[-1, 1]^n \setminus C_B$, we easily obtain that

$$|D_1 f| = 0 \quad \text{on } (\mathcal{C}_A \times \mathbb{R}^{n-1}) \setminus C_A.$$

Therefore, for $\tilde{x} \in P_k \setminus P_{k+1}$ we can calculate

$$\begin{aligned} \int_{(-1,1)} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1 &= \int_{(-1,1)\setminus\mathcal{C}_A} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1 \\ &= \sum_{j=1}^{\infty} \int_{U_j \setminus U_{j+1}} |Df(x_1, \tilde{x})|^p \, \mathrm{d}x_1. \end{aligned}$$

We use the fact that f is C-Lipschitz on $[-1, 1]^n \setminus (U_{k_0})^n$ (k_0 fixed in (6.18)) to see that

$$\int_{(-1,1)} |D_1 f(x_1, \tilde{x})|^p \, \mathrm{d}x_1 \leq C^p$$

for every $\tilde{x} \in [-1, 1]^{n-1} \setminus (U_{k_0})^{n-1}$.

Therefore we may now restrict to the case that $\tilde{x} \in P_k \setminus P_{k+1}$ for $k \geq k_0$. By the same reasoning as in Section 5 we have

$$\mathcal{L}^1(U_j \setminus U_{j+1}) \leq \frac{C}{j^{\alpha+1}}.$$

Therefore, for the line segment $L = [-1, 1] \times \{\tilde{x}\}$ we have using (6.31)

$$\begin{aligned} &\int_L |D_1 f|^p dx_1 = C + \sum_{j=k_0}^{\infty} \int_{L_j} |Df(x_1, \tilde{x})|^p dx_1 \\ &\leq C\biggl(1 + \sum_{j=k_0}^{k-2N_F-3} \frac{1}{j^{\alpha+1}} + \sum_{j=k-2N_F-2}^{4k-3N_F-3} \frac{j^{p(\alpha+1)}}{j^{\alpha+1}} + \sum_{j=4k-3N_F-2}^{\infty} 2^{-p\beta(j-3k)} \frac{j^{p(\alpha+1)}}{j^{\alpha+1}}\biggr). \end{aligned}$$

The first sum converges even if we sum to infinity, the second sum will be estimated simply by taking an estimate of the largest summand and multiplying by an estimate of the total number of summands. We will use (6.18) to estimate the final sum by a convergent geometric sum ($\sum_{l=k}^{\infty} 2^{-pl\beta/2}$). Continuing the calculation and using $k \geq 4N_F + 5$ we have

$$\begin{aligned}\int_L |D_1 f|^p \, dx_1 &\leq C + C4k(4k)^{(p-1)(\alpha+1)} + \frac{C}{1-2^{-kp\beta/2}} \\ &\leq C + C\frac{k^{p\alpha+p}}{k^{\alpha}}.\end{aligned} \tag{6.32}$$

The estimate (6.32) holds for all lines $L = [-1,1] \times \{\tilde{x}\}$ such that $\tilde{x}$ in $P_k \setminus P_{k+1}$ with $k \geq k_0$. Furthermore, since f is Lipschitz on $[-1,1]^n \setminus U_{k_0}^n$, we may estimate

$$\int_L |D_1 f|^p \, \mathrm{d}x_1 \leq C \quad \text{for all } \tilde{x} \in P_k \setminus P_{k+1} \text{ with } k < k_0, \tag{6.33}$$

which proves the validity of (6.32) for all $k \in \mathbb{N}$ (not only for $k \geq k_0$). We will use the estimate (6.32) on those lines which are not entirely contained in $\mathcal{K}_A$ and on lines which are entirely contained in $\mathcal{K}_A$ we will use again the fact that

$$|D_1 f(x)| \leq C \text{ for a.e. } x \in L. \tag{6.34}$$

Now we integrate the above estimates over $\tilde{x} \in [-1,1]^{n-1}$. Calling $\tilde{K}_A$ the set of those $\tilde{x}$ such that $L(\tilde{x})$ is contained entirely in $\mathcal{K}_A$ we claim that

$$\mathcal{L}^{n-1}\big(U_k^{n-1} \setminus (U_{k+1}^{n-1} \cup \tilde{\mathcal{K}}_A)\big) < \mathcal{L}^{n-1}\big(U_k^{n-1} \setminus \tilde{\mathcal{K}}_A\big) \leq Ck^{-m\alpha} \tag{6.35}$$

where $m := \frac{n}{2} - 1$. Once we will have established this estimate the rest of the proof follows quickly from the following calculations. We continue by multiplying (6.32) by the measure estimate (6.35) and summing over k plus (6.34) multiplied by the measure $\mathcal{L}^{n-1}([-1,1]^{n-1}) > \mathcal{L}^{n-1}(\tilde{\mathcal{K}}_B \cap [-1,1]^{n-1})$. Assuming (6.35) we have (it holds that $\alpha m > 1$)

$$\begin{aligned}\int_{(-1,1)^n} |D_1 f(x)|^p dx &\leq \sum_{k=1}^{\infty} Ck^{-m\alpha} + C\sum_{k=1}^{\infty} \frac{k^{p(\alpha+1)}}{k^{\alpha(m+1)}} + C \\ &\leq C + C\sum_{k=1}^{\infty} \frac{k^p}{k^{(m+1-p)\alpha}} = C\sum_{k=k_0}^{\infty} \frac{k^p}{k^{2p}} < \infty\end{aligned} \tag{6.36}$$

by our choice $\alpha = \frac{2p}{n/2-p}$ at the beginning of the proof.

Therefore it remains to prove (6.35). We notice that a line segment $L := [-1,1] \times \{\tilde{x}\}$ parallel to x_1-axis is contained in $\mathcal{K}_A$ if and only if $L \subset C_A + M$ for some $M \in \mathcal{M}(m)$ containing e_1. We write all such subspaces M as $\mathbb{R}e_1 + \tilde{M}$ for some $\tilde{M} \in \mathcal{M}(m-1, n-1)$. Hereby we see that

$$L \subset \mathcal{K}_A \quad \text{if and only if} \quad \tilde{x} \in \tilde{\mathcal{K}}_A := \bigcup_{\tilde{M} \in \mathcal{M}(m-1,n-1)} \Big(\mathcal{C}_A^{n-1} + \tilde{M}\Big).$$

Next, we estimate

$$\begin{aligned}\mathcal{L}^{n-1}\big(U_k^{n-1} \backslash (U_{k+1}^{n-1} \cup \tilde{\mathcal{K}}_A)\big) &< \mathcal{L}^{n-1}\big(U_k^{n-1} \setminus \tilde{\mathcal{K}}_A\big) \\ &= 2^{k(n-1)}\big(2^{-k}(1+k^{-\alpha})\big)^{n-1} - \mathcal{L}^{n-1}\big((U_k)^{n-1} \cap \tilde{\mathcal{K}}_A\big).\end{aligned} \tag{6.37}$$

and therefore it suffices to calculate $\mathcal{L}^{n-1}\big((U_k)^{n-1}\cap\tilde{\mathcal{K}}_A\big)$. We do this by decomposing the set $U_k^{n-1}\cap\tilde{\mathcal{K}}_A$ into the disjoint union of m sets $E_0^k, E_1^k,\dots,E_{m-1}^k$ and each E_i^k is the disjoint union of $\binom{n-1}{i}$ measurable rectangles. For simplicity of the notation call $G_k = U_k\setminus\mathcal{C}_A$. We denote

$$
\begin{aligned}
&E_0^k := \mathcal{C}_A^{n-1}\\
&E_1^k := (G_k\times\mathcal{C}_A^{n-2})\cup(\mathcal{C}_A\times G_k\times\mathcal{C}_A^{n-3})\cup\dots\cup(\mathcal{C}_A^{n-3}\times G_k\times\mathcal{C}_A)\cup(\mathcal{C}_A^{n-2}\times G_k)\\
&\vdots\\
&E_{m-1}^k := (G_k^{m-1}\times\mathcal{C}_A^{m+3})\cup\dots\cup(\mathcal{C}_A^{m+3}\times G_k^{m-1}).
\end{aligned}
$$

So each E_j^k is a union of $\binom{n-1}{j}$ sets $F_l(j,k)$, $l=1,2,\dots,\binom{n-1}{j}$. Each $F_l(j,k)$ is a measurable rectangle and G_k appears in its product j times and the set $\mathcal{C}_A$ appears $n-1-j$ times. Each $F_l(j,k)$ is uniquely determined by the sequence of sets in its product. So if $F_l(j,k)\neq F_{l'}(j',k')$ then there is a direction such that one of the sets is projected onto $\mathcal{C}_A$ and the other is projected onto G_k and $\mathcal{C}_A\cap G_k=\emptyset$.

Now simple calculation gives

$$\mathcal{L}^1(\tilde{\mathcal{C}}_A)=1\quad\text{and}\quad\mathcal{L}^1(G_k)=\frac{1}{(k+1)^\alpha}$$

and so by definition

$$
\begin{aligned}
\mathcal{L}^{n-1}(E_j^k) &= \binom{n-1}{j}\mathcal{L}^{n-1}\big((G_k)^j\times(\tilde{\mathcal{C}}_A)^{n-1-j}\big)\\
&=\binom{n-1}{j}\big[\mathcal{L}^1(G_k)\big]^j\big[\mathcal{L}^1(\tilde{\mathcal{C}}_A)\big]^{n-1-j}=\binom{n-1}{j}\frac{1}{(k+1)^{\alpha j}}
\end{aligned}
$$

for each $j=0,1,\dots,m-1$. Therefore, we see that

(6.38)

$$\mathcal{L}^{n-1}\big((U_k)^{n-1}\cap\tilde{\mathcal{K}}_A\big)=\mathcal{L}^{n-1}\bigg(\bigcup_{j=0}^{m-1}E_j^k\bigg)=\sum_{j=0}^{m-1}\mathcal{L}^{n-1}\big(E_j^k\big)=\sum_{j=0}^{m-1}\binom{n-1}{j}\frac{1}{(k+1)^{\alpha j}}.$$

When we now combine (6.37) and (6.38) we get

$$
\begin{aligned}
&\mathcal{L}^{n-1}\big(U_k^{n-1}\setminus(U_{k+1}^{n-1}\cup\tilde{\mathcal{K}}_A)\big)\\
&\qquad\le\Big(1+\frac{1}{(k+1)^\alpha}\Big)^{n-1}-\sum_{j=0}^{m-1}\binom{n-1}{j}\frac{1}{(k+1)^\alpha}\le\frac{C}{k^{m\alpha}},
\end{aligned}
$$

which is exactly what we claimed in (6.35). As shown in (6.36) This ends the proof of Theorem 1.2 for all $n\ge 4$. □

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Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic
E-mail address: `daniel.campbell@uhk.cz; hencl@karlin.mff.cuni.cz`

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FIN-40014, Jyväskylä, Finland
E-mail address: `ville.tengvall@jyu.fi`